Bayesian Hierarchical Spatial-temporal Models

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Abstract

Spatial-temporal processes are prevalent especially in environmental sciences where, under most circumstances, the processes are non-stationary in time so that their temporal-variability must be captured in traditional spatial models for better estimation and prediction. We propose a Bayesian hierarchical spatial-temporal model to describe the dependence of extreme data on spatial locations as well as temporal effects. The first layer of the hierarchical model specifies a measurement process for the observed extreme data. The second layer characterizes the latent spatial process and temporal process. The hierarchical formulation concludes with a third layer of priors on parameters. A key idea is to model spatial and temporal dependence simultaneously. Statistical inference is performed by Markov chain Monte Carlo methods which involve filtering and smoothing of parameters. The methodology is applied to simulated data and real environmental data.

Keywords: Bayesian analysis; environmental data; hierarchical models; spatial-temporal processes.

1 Introduction

Studies on the pattern of extreme climatological events play crucial roles in environmental risk analysis and policy making, where the non-stationarity of spatial processes render the spatio-temporal models as powerful quantitative tools for estimation and prediction. Many studies in the literature have concerned themselves with the effects on extreme behaviors by space and time. Wikle et al. (1998) used underlying state processes to characterize the measurement-error process for which they allowed site-specific time series models including large-scale variability and small-scale space-time dynamic effects. Casson and Coles (1999) and Cooley et al. (2007) focused on the spatial structure of extremes and captured its effect via respectively point process representation and threshold exceedance approach. Sang and Gelfand (2009) generalized their spatial structures and adopted multivariate Markov random field models with temporal dependence to cater for the whole space-time characterization. Our work shares the spirit of this space-time structure but differs from the coregionalization approach of Sang et al. (2009) by modeling spatial and temporal dependence simultaneously. Both spatial and temporal dependence as well as possible dependence among model parameters can be captured by our model.

2 Hierarchical Spatio-Temporal Formulation

Let $x_1, ..., x_d$ denote the d locations and let $Z_t(x_i)$ denote the spatial data observed at location x_i and at time t, $t = 1, ..., n$. As with typical hierarchical modeling, we specify our model with the *Data level*, *Process level* and the *Prior level*. The data level depends on the type of data of our interest and usually realizes on the *conditionally independent* assumption. For example, if our target is to model daily temperature distribution, specifically the mean and standard deviation, we can easily specify the distribution of $Z_t(x_i)$ given θ_1 as a Gaussian process, where θ_1 is a process variable. However, as we are mainly interested in modeling spatial extremes,

there are primarily three choices of the data process, namely the generalized extreme value distribution (GEV) like Sang and Gelfand (2009), the generalized Pareto distribution (GPD) like Cooley et al. (2007), and the point process representation like Casson and Coles (1999), among which the first two characterization are of our primary interest.

1. Data level

   - Let $Z_t(x_i)$ represent threshold exceedances given thresholds $u_t(x_i)$.
     In this case, $Z_t(x_i)$ follows the GPD below.

   $$p(z_t(x_i)|\theta_1) = \frac{1}{\exp \delta_t(x_i)} \left(1 + \frac{\xi_t(x_i)z_t(x_i)}{\exp \delta_t(x_i)}\right)^{-1/\xi(x_i)-1},$$

   where θ_1 is a vector of $\xi_t(x_i)$ and $\delta_t(x_i)$, i=1,..., d and t=1,..., n.

   - Let $Z_t(x_i)$ represent block maxima.
     In this case, $Z_t(x_i)$ follows the GEV below.

   $$p(z_t(x_i)|\theta_1) = \exp \left\{ - \left[1 + \xi_t(x_i)\frac{z_t(x_i) - \mu_t(x_i)}{\exp \delta_t(x_i)}\right]^{-1/\xi(x_i)}\right\},$$

   where θ_1 is a vector of $\mu_t(x_i)$, $\xi_t(x_i)$ and $\delta_t(x_i)$, i=1,..., d and t=1,..., n.

2. Process level

   The process level specifies most of the spatial-temporal dependence structure, where we characterize the spatial and temporal variability simultaneously. Take the GEV as an example here: the model parameters $\mu_t(x_i), \delta_t(x_i)$ and $\xi_t(x_i)$ conditionally follow Gaussian distribution with constant variance. Their means, however, are formulated as an additive model consisting of a fixed effect component $X_i^T\beta(x_i)$ and a random effect component $U_t(x_i)$. The fixed effect component varies only with location while the random effect component varies with both time and location. The random effect component can be further decomposed into a spatial component $\psi(x_i)$ and a temporal component R_t. The spatial component $\psi(x_i)$ follows independent Gaussian spatial process with mean zero and covariance structure corresponding to exponential variogram models with no nugget effect. The temporal component R_t follows a three-dimensional AR(1) process. An advantage of the above specification is that any possible dependence $\mu_t(x_i)$, $\delta_t(x_i)$ and $\xi_t(x_i)$ can be captured by the covariance structure Σ_η of the innovations η_t in the multivariate autoregressive process. This idea is different from the coregionalization idea of Sang and Gelfand (2009). The above process level specification allows us to consider ordinary spatial observations, spatial exceedances and spatial block maxima.

3 Posterier Inference

The posterior inference is implemented by Markov chain Monte Carlo (MCMC) methods. All parameters are assigned with normal or uniform priors except Σ_η. Embedded in a Gibbs sampler, parameters are generally sampled by the Metropolis-Hastings algorithm with the exception of the temporal component R_t. The parameters $(\mu_t(x_i), \delta_t(x_i), \xi_t(x_i))$ are sampled independently for each t and i. $\beta(x_i)$ is then sampled directly from its full conditional distribution. The spatial component $\psi(x_i)$ and its corresponding covariance parameters are updated subsequently. The temporal component R_t is sampled by the forward filtering backward sampling (FFBS) algorithm: We sequentially sample R_t forward from $t = 0$ up to $t = n$, computing the underlying

parameters at each stage. At time n, this shall give us a full time series vector. We then sequence backwards through time for smoothing by sampling R_t from the conditional distribution involving the sampled value of R_{t+1} and the underlying parameters computed in the forward procedure. An Inverse-Wishart prior is assigned to Σ_η so that its posterior can be easily sampled from an Inverse-Wishart distribution with more degrees of freedom. Finally, the AR parameters can be sampled from its conditional distribution on R_t and Σ_η.

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