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## Abstract

The gauge/string duality, a.k.a. the holographic principle is a profound assertion that emerged from string theory. It relates strongly-coupled gauge theories to weakly coupled string theories living in a higher-dimensional curved geometry. Nevertheless, it is a conjecture, and only a few instances of its more concrete form, the AdS/CFT correspondence, are well-understood. The most well-studied example is the duality between  $N=4$  SYM, which is a CFT, and type IIB string theory in  $AdS_5 \times S^5$  background. Generalization to less symmetric cases is a must, and the next logical step is to add a mass scale to  $N=4$  SYM, therefore breaking its conformal symmetry and leading to  $N=2^*$  SYM, the theory we study in this thesis. It is supersymmetric enough to employ the powerful localization method that reduces its partition function to a matrix model. We will see that the mass scale causes non-trivial phase structures in its vacuum configuration, visible in the holographic regime. We will probe them using Wilson loops in different representations of the gauge group. On the other hand, the dual supergravity background was derived by Pilch-Warner, making  $N=2^*$  theory an explicitly testable non-conformal holographic case, which is a rare example. We will prove that the duality works for the dual observables (string action, D-branes) we managed to compute, even at a quantum-level.

# List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I X. Chen-Lin, J. Gordon, and K. Zarembo, “ $\mathcal{N} = 2^*$  super-Yang-Mills theory at strong coupling,” *JHEP* **1411** (2014) 057, arXiv:1408.6040 [hep-th].
- II X. Chen-Lin and K. Zarembo, “Higher Rank Wilson Loops in  $N = 2^*$  Super-Yang-Mills Theory,” *JHEP* **1503** (2015) 147, arXiv:1502.01942 [hep-th].
- III X. Chen-Lin, A. Dekel, and K. Zarembo, “Holographic Wilson loops in symmetric representations in  $\mathcal{N} = 2^*$  super-Yang-Mills theory,” *JHEP* **02** (2016) 109, arXiv:1512.06420 [hep-th].
- IV X. Chen-Lin, “Symmetric Wilson Loops beyond leading order,” *SciPost Phys.* **1** no. 2, (2016) 013, arXiv:1610.02914 [hep-th].
- V X. Chen-Lin, D. Medina-Rincon, K. Zarembo, “Quantum String Test of Nonconformal Holography,” *JHEP* **04** (2017) 095, arXiv:1702.07954 [hep-th].

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# 1. Introduction

The universe must be self-consistent. This is the basic premise physicists assume in the quest for the ultimate laws that govern our universe. Without it, a theory would be a mere description of a collection of observed facts.

Despite highly precise experimental corroboration of the Standard Model of particle physics, which relies on Quantum Field Theory (QFT) with gauge symmetry, and Einstein's General Relativity (GR) that describes gravity, these two frameworks are mutually incompatible. That is without mentioning the foundational problems of QFT, that the Clay Mathematics Institute urges to solve with a million-dollar prize incentive<sup>1</sup>.

Classical physics is elegantly formulated in terms of the least action principle. The standard procedure is to encode the dynamics in a quantity called action, minimize it with respect to the physical variables which gives a set of differential equations (of motion) to be solved. In quantum physics, we relax this principle. When quantizing a classical theory in Feynman path integral formulation, paths that do not minimize the action also contribute to the dynamics. In strong analogy with statistics, physical quantities are weighted observables  $O$  by the exponential of the classical action<sup>2</sup>:

$$\langle O \rangle = \int D\phi O e^{-iS[\phi]/\hbar},$$

where we used the shorthand notation  $\phi$  for the collection of quantum fields<sup>3</sup>. The problem is, however, the lack of a rigorous definition of the measure  $D\phi$ . Very often, we only know how to compute quantum observables in certain limits, when we can rely on the saddle-point approximation to the integral. For example, in the classical limit (vanishing Plank constant  $\hbar$ ), the saddle-point method for the partition function gives us the least action principle:

$$Z = \langle 1 \rangle \approx e^{-iS[\phi_c]/\hbar}; \quad \delta S[\phi_c] = 0.$$

Other examples are the weak coupling limit, such as in Quantum Electrodynamics (QED), or the thermodynamic limit in quantum many-body systems. By contrast, we have no established analytical tool to handle strongly-coupled

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<sup>1</sup> See the link: <http://www.claymath.org/millennium-problems/yang%E2%80%9993mills-and-mass-gap>

<sup>2</sup> The analogy is even closer when we analytically continue the time to pure imaginary time, i.e. Wick rotate  $t \rightarrow -it_E$ , then the action becomes purely imaginary, and the exponential decays.

<sup>3</sup> Quantum mechanics can be formulated as a (0+1)-dimensional QFT.

QFTs, which would explain quark confinement in Quantum Chromodynamics (QCD), and phases of high-Tc superconductors.

Only very few classes of path integrals are solvable. Besides free theories, path integrals of topological quantum field theories and some supersymmetric theories localize *exactly* at their saddle-points. This highly non-trivial phenomenon is called the localization of path integrals, and it is one of the pillars our work is based on.

Generalizing point-particles to strings, String Theory might be the best candidate for a consistent quantum gravity theory, which would be a unifying framework for QFT and GR. It is not free of mathematical problems though, for example the string action in a curved background is not fully known, and path integral measures are not rigorously defined either. Nevertheless, it is mathematically rich, with a web of dualities that connect different perturbative string theories (the underlying theory is known as the M-theory). It is offering us many insights otherwise unsuspected and a very important realization is the AdS/CFT correspondence. The most well-studied instance of the correspondence is the one between weakly-coupled superstring theory living on  $AdS_5 \times S^5$  space, and a strongly-coupled supersymmetric conformal field theory (CFT) called  $\mathcal{N} = 4$  SYM. Despite much success (and still some issues concerning quantum stringy corrections), the correspondence remains an unproven conjecture.

An even more general correspondence is the gauge/string (or gauge/gravity) duality, also known as the holographic principle (so named because the gauge theory lives in a lower dimensional space than the string theory). If it were true, it would be revolutionary in many aspects. From a practical point of view, on one hand we would have a toolbox to solve strongly-coupled gauge theories using essentially GR, and on the other hand, experiments on quantum gravity could be done in laboratories by handling systems like cold atoms. From a conceptual point of view, it could indicate that gravity, and hence spacetime, are emergent from lower dimensional quantum systems.

We lack the mathematical constructs to even formulate the conjecture in a precise manner. However, we can generalize what we learned from AdS/CFT to settings where we still hold some analytical control. A perfect toy model to investigate *non-conformal* gauge/string duality is the unique massive deformation of  $\mathcal{N} = 4$  SYM, called  $\mathcal{N} = 2^*$  SYM. On one hand, like  $\mathcal{N} = 4$  SYM, supersymmetric localization is applicable and reduces the complicated path integrals to a manageable matrix model, allowing us to access its strongly-coupled regime (actually any finite coupling regime). On the other hand,  $\mathcal{N} = 2^*$  SYM is conjectured to be dual to a supergravity solution known as the Pilch-Warner background. Figure 1.1 shows a graph on the relationship between different theories that we consider.



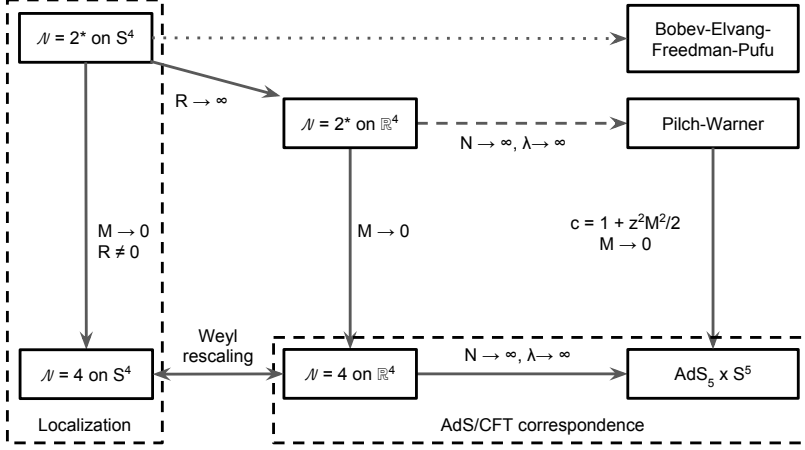


Figure 1.1. Localization applies to theories on the sphere, indicated by a dashed box. Since the gravity dual of  $\mathcal{N} = 2^*$  on  $S^4$  is only partially known (indicated by the dotted line arrow), we take the decompactification limit ( $R \rightarrow \infty$ ) to obtain  $\mathcal{N} = 2^*$  on  $\mathbb{R}^4$ . Our research focuses on the dashed arrow, which generalizes the well-established AdS/CFT correspondence shown in the dashed box below it. The limits will be explained through out the thesis.

## Outline

This thesis has two parts, both aimed at introducing and reviewing some background material in order to help the readers follow the appended papers. Part I focuses on the gauge theory side. We start off with the basics of the supersymmetric localization technique, then we go on to discuss the action of  $\mathcal{N} = 4$  SYM on  $\mathbb{R}^4$  and on  $S^4$ . We then extend it to  $\mathcal{N} = 2^*$  SYM on  $S^4$ , and focus on the partition function after localization. We show the common technique to solve a matrix model, and conclude with the phase diagram of the theory. The last chapter of this part introduces the Wilson loop observables, and we review some known results in  $\mathcal{N} = 4$  case, and finally, write down the results we computed for  $\mathcal{N} = 2^*$  case. Part II starts with supergravity and some of its solutions, including  $AdS_5 \times S^5$  and Pilch-Warner. Then, we review the AdS/CFT correspondence and the logic behind its original derivation. In the end, we talk about holographic Wilson loops and connect our holographic solutions to the gauge theory results.

*Physics is really nothing more than a search for ultimate simplicity, but so far all we have is a kind of elegant messiness. — Bill Bryson*



**Part I:**  
**Supersymmetric gauge theories**



## 2. Localization

Localization of an integral happens when the integral is exactly equal to its saddle-point approximation. A trivial example is therefore the Gaussian integration formula, which the saddle-point method is based on:

$$\int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2} x^T A x} = \sqrt{\frac{(2\pi)^n}{\det A}}, \quad (2.1)$$

where  $A$  is a positive-definite  $n \times n$  matrix.

In the case of path integrals in quantum field theories, which are of infinite dimension, localization, if applicable, reduces them to finite dimensional integrals. This is an exact approach, in contrast to perturbative methods such as Feynman diagrams, valid only in the weakly-coupled regime. Unfortunately, very few path integrals are solvable, but there are very specific classes of theories where localization does apply. These are supersymmetric theories and topological quantum field theories, from which we distinguish two types of localization: supersymmetric localization and equivariant localization.

Many results of this thesis are based on the supersymmetric localization. We will review its basic idea and illustrate it explicitly in an example with an ordinary integral. We refer the reader to [1] and [2] for reviews of the topic.

### 2.1 Supersymmetric localization

In the path integral formulation of quantum theories, physical observables are expectation values of operators. Let us consider the path integral (in the Euclidean signature and set  $\hbar = 1$ ):

$$\langle O \rangle = \int D\phi O e^{-S[\phi]}, \quad (2.2)$$

where  $O$  is an operator, built out of the fields of the theory, represented by  $\phi$ .

Assume we can deform the action  $S$  in such a way that it does not affect the physical quantity, namely

$$\langle O \rangle_t = \int D\phi e^{-S[\phi] - t Q V[\phi]}, \quad (2.3)$$

where  $Q$  is a fermionic symmetry of the theory, and require

$$\frac{d \langle O \rangle_t}{dt} = 0. \quad (2.4)$$

Integrating by parts and assuming vanishing boundary terms, we obtain the following necessary conditions:

$$Q^2 = 0, \quad QS = 0, \quad QO = 0, \quad (2.5)$$

and that the measure  $D\phi$  is also invariant (the symmetry must not be anomalous).

If this deformation were possible, then the deformation parameter  $t$  could be of any value. The most useful case is to take it to be infinitely large, in order to use the saddle-point approximation. Since by construction the integral is independent of  $t$ , the saddle-point approximation must be exact, hence the path integral localizes to a set of loci. These are determined by<sup>1</sup>

$$(QV)_{\text{Bosonic}} = 0 \quad (2.6)$$

whose solutions—let us denote them by  $\phi_*$ —belong to the moduli space of vacua of the theory. This means that some fields acquire a non-zero vacuum expectation value. Accounting the fluctuations around the vacuum configuration, the localized integral can be formally written as:

$$\langle O \rangle = \int_{\text{Moduli}} D\phi_* O(\phi_*) e^{-S(\phi_*)} Z_{1\text{-loop}}, \quad (2.7)$$

where the integral over  $\phi_*$  is analogous to the sum over all the saddle-points, and  $Z_{1\text{-loop}}$  is the functional generalization of the Gaussian integral (2.1) for the fluctuations around the saddle-points. It is thus a functional determinant, which is divergent in general. In order to obtain a finite result, the theory is defined in a compact manifold, like a sphere, such that the spectrum of the operator is discrete, and supersymmetry guarantees the cancellation of divergences between bosons and fermions. Despite conceptual simplicity, the main challenge of this technique is to find the localization action  $QV$ , since no general recipe exists.

## 2.2 A toy example

Consider an ordinary integral

$$I = \int_m^M dx p g'(x) e^{pg(x)}, \quad (2.8)$$

with  $p$  being a constant. It can be solved exactly as a total derivative:

$$I = \int_m^M dx \frac{d}{dx} e^{pg(x)} = e^{pg(M)} - e^{pg(m)}. \quad (2.9)$$

---

<sup>1</sup> If the bosonic part is positive-definite. The fermionic contribution of the localization action is subleading.

If we were to solve it using the saddle-point approximation for  $p \rightarrow \infty$ , then the saddle-points must be the endpoints  $x_* = \{m, M\}$ . Let us consider this case and expand around the saddle-points:

$$g(x_* + \xi) = g(x_*) + \frac{1}{2}g''(x_*)\xi^2 + \mathcal{O}(\xi^3). \quad (2.10)$$

Each saddle-point contributes to the integral as following:

$$I_m = e^{pg(m)} \int_0^\infty d\xi p g''(m) \xi e^{p \frac{1}{2} g''(m) \xi^2} = -e^{pg(m)} \quad (2.11)$$

$$I_M = e^{pg(M)} \int_{-\infty}^0 d\xi p g''(M) \xi e^{p \frac{1}{2} g''(M) \xi^2} = e^{pg(M)} \quad (2.12)$$

valid for  $\Re(pg'') < 0$ . The sum  $I_m + I_M$  indeed gives the exact result.

Now, let us see how we can use supersymmetric localization to solve it. The integral  $I$  can be rewritten in the supersymmetric form:

$$I = \int_m^M dx \int da \int db e^{p(g(x) - abg'(x))}, \quad (2.13)$$

where  $a$  and  $b$  are Grassmann numbers (our fermions), which means they satisfy  $ab = -ba$  and  $a^2 = b^2 = 0$ , hence

$$f(x) = \int da \int db e^{-a f(x) b}. \quad (2.14)$$

The action  $S = p(g(x) - abg'(x))$  is invariant under the supersymmetry transformation:

$$\delta_\varepsilon x = \varepsilon x, \quad \delta_\varepsilon a = 0, \quad \delta_\varepsilon b = \varepsilon, \quad (2.15)$$

where  $\varepsilon$  is a Grassmann number. The supersymmetric operator, defined by  $\varepsilon Q = \delta_\varepsilon$ , is explicitly

$$Q = a \frac{\partial}{\partial x} + \frac{\partial}{\partial b}. \quad (2.16)$$

We can check that, indeed, (2.5) are fulfilled.

Let us deform the integral with a  $Q$ -exact term

$$I(t) = \int_m^M dx \int da \int db e^{p(g(x) - abg'(x)) + tQV}, \quad (2.17)$$

where

$$V(x, a, b) = f(x)b \quad (2.18)$$

that leads to a non-trivial localization action

$$QV = f(x) - abf'(x). \quad (2.19)$$

This is actually the most general supersymmetric action we can write for this case. Thus,  $S$  is not just  $Q$ -closed (i.e.  $QS = 0$ ), but also  $Q$ -exact (i.e.  $S = QV_s$ ).

We require the deformed integral to be independent of the deformation parameter  $t$ :

$$I'(t) = 0. \quad (2.20)$$

An explicit and straightforward computation shows

$$I'(t) = \int_m^M dx \int da \int db e^{S+tQV} QV \quad (2.21)$$

$$= \int_m^M dx \int da \int db Q(e^{S+tQV} V) \quad (2.22)$$

$$= e^{pg(x)+tf(x)} f(x) \Big|_m^M \quad (2.23)$$

where we used the supersymmetry condition and nilpotency. In this case, we must additionally require vanishing boundary conditions

$$f(m) = f(M) = 0. \quad (2.24)$$

Now we can take large  $t$  to solve the integral using the saddle-point method. The boundary points here must be either global maxima or global minima of  $f(x)$ , for  $t > 0$  or  $t < 0$ , in order for the saddle-point integral to be convergent. In other words, for  $t$  negative (positive), the bosonic part of the localization action is positive (negative) definite, and it vanishes at the saddle-points:

$$(QV)_{\text{Bosonic}} = 0. \quad (2.25)$$

The saddle-point approximation is exact, in the same fashion as for the large  $p$  case we studied before. This is to be expected, since the deformed integral is still a total derivative when we integrate out the fermions:

$$I(t) = \int_m^M dx \frac{d}{dx} e^{pg(x)+tf(x)} = e^{pg(M)+tf(M)} - e^{pg(m)+tf(m)}. \quad (2.26)$$

When  $g(x) = \cos x$  and the integration region is extended to a sphere parametrized by the polar angle  $x \in [0, \pi]$  and the azimuthal angle  $\varphi \in [0, 2\pi]$ , this becomes a particular example of the Duistermaat-Heckman integration formula, which is the precursor of the localization of path integrals.



### 3. $\mathcal{N} = 4$ SYM

Consider the action of a  $d$ -dimensional Yang-Mills (YM) theory with a massless spin- $\frac{1}{2}$  field  $\Psi$  in the adjoint representation of the gauge group  $U(N)$ :

$$S = -\frac{1}{g_{YM}^2} \int d^d x \operatorname{tr} \left( \frac{1}{2} F_{MN} F^{MN} - \bar{\Psi} \Gamma^M D_M \Psi \right), \quad (3.1)$$

where  $F_{MN} = \sum_{a=1}^{N^2} F_{MN}^a T_{ij}^a$ , so that the trace is over the matrix indices  $i, j = 1, \dots, N$ .  $T^a$  are the generators of the gauge group. More explicitly, the field-strength and the covariant derivative are

$$F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N], \quad (3.2)$$

$$D_M = \partial_M + [A_M, \cdot]. \quad (3.3)$$

In Minkowski spacetime  $\mathbb{R}^{9,1}$ , this action turns out to be invariant under the supersymmetry transformation:

$$\delta_\varepsilon A_M = \varepsilon \Gamma_M \Psi, \quad (3.4)$$

$$\delta_\varepsilon \Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon, \quad (3.5)$$

where  $\varepsilon$  is a constant Majorana-Weyl spinor that parametrizes the transformation.

Lower dimensional supersymmetric theories can actually be obtained by dimensional reduction from the above theory [3]. Let us review how we can derive the action for the maximally supersymmetric  $\mathcal{N} = 4$  SYM on  $\mathbb{R}^4$ .

#### 3.1 Action on $\mathbb{R}^4$

The dimensional reduction consists of restricting the dependence of the fields only to 4 dimensions:  $(x_1, \dots, x_4)$ . The original Lorentz symmetry group  $Spin(9, 1)$  is then broken to  $Spin(4) \times Spin(5, 1)^R$ , that is the Lorentz group in 4d and an internal symmetry group called R-symmetry (hence the superindex  $R$ ). These groups are isomorphic to:

$$Spin(4) = SU(2)_L \times SU(2)_R \quad (3.6)$$

$$Spin(5, 1)^R = Spin(4)^R \times SO(1, 1)^R \quad (3.7)$$

$$= SU(2)_L^R \times SU(2)_R^R \times SO(1, 1)^R. \quad (3.8)$$

The gauge field  $A_M$  is reduced to the 4d gauge field and scalars:

$$Spin(4) : \quad A_\mu, \quad \mu = 1, 2, 3, 4 \quad (3.9)$$

$$Spin(4)^R : \quad \Phi_I, \quad I = 5, 6, 7, 8 \quad (3.10)$$

$$SO(1, 1)^R : \quad \Phi_I, \quad I = 0, 9 \quad (3.11)$$

where we wrote down the group they transform under.

The fermionic field, which is a Majorana-Weyl spinor, can be decomposed to 4 Majorana spinors (each having 2 degrees of freedom):

$$\Psi = \begin{bmatrix} \psi_L \\ \chi_R \\ \psi_R \\ \chi_L \end{bmatrix} \quad (3.12)$$

where the spinors with the subindex  $L$  (or  $R$ ) transform in the spin- $\frac{1}{2}$  representation of  $SU(2)_L$  (or  $SU(2)_R$ ). Also, spinors with the name  $\psi$  and  $\chi$  transform in the spin- $\frac{1}{2}$  representation of  $SU(2)_L^R$  and  $SU(2)_R^R$  from the R-symmetry subgroup, respectively.

The  $\mathcal{N} = 4$  SYM action explicitly written in terms of the 4d bosonic fields is:

$$S_{\mathbb{R}^4} = -\frac{1}{g_{YM}^2} \int d^4x \text{tr} \left( \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi_I D^\mu \Phi^I + \frac{1}{2} [\Phi_I, \Phi_J] [\Phi^I, \Phi^J] \right. \\ \left. - \bar{\Psi} \Gamma^\mu D_\mu \Psi - \bar{\Psi} \Gamma^I [\Phi_I, \Psi] \right), \quad (3.13)$$

where we decomposed the 10d gamma matrices as  $\Gamma^M = \Gamma^\mu \otimes \Gamma^I$ .

The action has no mass scale, and it is in fact conformal invariant even at quantum level [4, 5]. The conformal group extends the Poincaré group (translation, rotations and boosts) to include scaling (dilatation), and special conformal transformation, which is a composition of inversion, translation and inversion. The conformal symmetry, the four copies of supersymmetry and the internal R-symmetry are part of the larger  $\mathcal{N} = 4$  superconformal group  $PSU(2, 2|4)$ . Its Lie algebra is generated by the generators of the conformal algebra, 16 supercharges (that commute with momentum generators), and 16 superconformal charges (that commute with special conformal generators). Details of the algebra can be found for example in [6].

## 3.2 Action on $S^4$

In order to apply supersymmetric localization to  $\mathcal{N} = 4$  SYM, we shall put this theory on a hypersphere  $S^4$ . Conformal invariance implies an additional conformal coupling of the scalars to the scalar curvature  $\mathcal{R}$ , namely,

$$\frac{\mathcal{R}}{6} \Phi^I \Phi_I, \quad I = 0, 5, 6, 7, 8, 9 \quad (3.14)$$

where for  $S^d$  with radius  $R$ , the scalar curvature is  $\mathcal{R} = d(d-1)/R^2$ . Then, there will be also a metric factor  $\sqrt{g}$  coming from the curved background. The action on  $S^4$  is hence

$$S_{S^4} = -\frac{1}{g_{YM}^2} \int d^4x \sqrt{g} \text{tr} \left( \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi_I D^\mu \Phi^I + \frac{1}{2} [\Phi_I, \Phi_J] [\Phi^I, \Phi^J] \right. \\ \left. - \bar{\Psi} \Gamma^\mu D_\mu \Psi - \bar{\Psi} \Gamma^I [\Phi_I, \Psi] + \frac{2}{R^2} \Phi^I \Phi_I \right). \quad (3.15)$$

Localization also requires the existence of an off-shell supersymmetry. To fulfill this condition, additional auxiliary field terms are added to the above action, see [7, 8]. Then, on the localization locus, the action will effectively be 3/2 times the conformal coupling (3.14), see also [9]. Naturally, at the decompactification limit  $R \rightarrow \infty$ , we recover the flat space version.

## 4. $\mathcal{N} = 2^*$ SYM

The fields of  $\mathcal{N} = 4$  SYM form a  $\mathcal{N} = 4$  vector (or gauge) multiplet, but the latter can be decomposed into two  $\mathcal{N} = 2$  massless supermultiplets, namely

$$\text{Vector multiplet:} \quad \{A_1, A_2, A_3, A_4, \Phi_0, \Phi_9, \psi_L, \psi_R\}, \quad (4.1)$$

$$\text{Matter hypermultiplet:} \quad \{\Phi_5, \Phi_6, \Phi_7, \Phi_8, \chi_L, \chi_R\}. \quad (4.2)$$

$\mathcal{N} = 2^*$  SYM is the unique massive deformation of  $\mathcal{N} = 4$  SYM that breaks half of its supersymmetries. This is achieved by giving mass to the matter hypermultiplet, either via the  $\mathcal{N} = 1$  superpotential [10, 11], or using Scherk-Schwarz reduction of  $\mathcal{N} = 1$  SYM [7, 12]. The latter prescribes the following replacement rules in the action (3.13):

$$D_0 \Phi_i \rightarrow D_0 \Phi_i + M_{ij} \Phi_j, \quad i, j = 5, \dots, 8 \quad (4.3)$$

$$D_0 \chi \rightarrow D_0 \chi + \frac{1}{4} \Gamma_{ij} M_{ij} \chi, \quad (4.4)$$

where  $M_{ij}$ , a  $4 \times 4$  matrix, is a generator of  $SU(2)_R^R$ , and is normalized as  $M_{ij} M^{ij} = 4M^2$ , where  $M$  is the mass scale. All the repeated indices are summed over. These replacements will give the standard mass term to the bosons and the fermions, and a cubic coupling term for the scalars. In its infinite-mass limit, the matter hypermultiplet can be integrated out, and the resulting theory is the pure  $\mathcal{N} = 2$  SYM.

In order to apply the localization method to the partition function of  $\mathcal{N} = 2^*$  SYM, we need to put it on  $S^4$ . Since the theory is no longer conformal due to the hypermultiplet mass scale, an additional curvature correction term to the mass is required in order to preserve supersymmetry, besides the conformal coupling discussed in (3.15). The full action can be found in [7], with the details of the localization procedure therein. We are interested in the final localized result, that we will discuss next.

### 4.1 Partition function

Theories with an action can be quantized using a path integral. The partition function in Euclidean signature is defined as

$$Z = \int D\phi e^{-S[\phi]}, \quad (4.5)$$

which is an infinite-dimensional integral, over all possible field configurations (represented by the measure  $D\phi$ ) on all of spacetime.

For  $\mathcal{N} = 2^*$  on  $S^4$  and its limiting cases  $\mathcal{N} = 4$  (massless hypermultiplet) and pure  $\mathcal{N} = 2$  (infinitely-heavy hypermultiplet that is integrated out), there is one localization locus. It is the moduli space of Coulomb vacua, parametrized by the vacuum expectation value of a scalar of the vector multiplet:

$$\langle \Phi_0 \rangle = \text{diag}(a_1, \dots, a_N), \quad (4.6)$$

where  $a_i$  are real numbers. The Coulomb phase of the theory is a consequence of the spontaneous symmetry breaking of the bosonic quartic potential  $\sim [\Phi_I, \Phi_J]^2$ , that breaks the original gauge group  $U(N)$  to  $U(1)^N$ .

The result for the localized partition function is of the form:

$$Z = \int d^N a \prod_{i < j} (a_i - a_j)^2 Z_{1\text{-loop}}(a) |Z_{\text{inst}}(a)|^2 e^{-\frac{8\pi^2 N}{\lambda} \sum_k a_k^2}. \quad (4.7)$$

The classical action comes from the curvature coupling of the scalars in (3.15).

The 1-loop correction, for the different theories are

$$\mathcal{N} = 2^* \text{ SYM:} \quad Z_{1\text{-loop}} = \prod_{i < j} \frac{H^2(a_i - a_j)}{H(a_i - a_j - MR)H(a_i - a_j + MR)} \quad (4.8)$$

$$\mathcal{N} = 2 \text{ SYM:} \quad Z_{1\text{-loop}} = \prod_{i < j} H^2(a_i - a_j) \quad (4.9)$$

$$\mathcal{N} = 4 \text{ SYM:} \quad Z_{1\text{-loop}} = 1 \quad (4.10)$$

where

$$H(x) \equiv \prod_{n=1}^{\infty} \left( 1 + \frac{x^2}{n^2} \right)^n e^{-\frac{x^2}{n}}. \quad (4.11)$$

We used the 't Hooft coupling  $\lambda = g_{YM}^2 N$ , and  $R$  is again the radius of the hypersphere.

The instanton partition function is the generating function of instantons of topological charge  $k$ :

$$Z_{\text{inst}} = \sum_{k=0}^{\infty} (e^{i2\pi\tau})^k Z_k, \quad (4.12)$$

where  $Z_0 = 1$  and

$$\tau = i \frac{4\pi}{g_{YM}^2} + \frac{\theta}{2\pi} \quad (4.13)$$

is the complexified Yang-Mills coupling<sup>1</sup>.

<sup>1</sup> The instanton action is the pure Yang-Mills action with an additional topological term. For an instanton of charge  $k$ , the action is given by:

$$S_{YM}(k) = -\frac{1}{2g_{YM}^2} \text{tr} \int d^4 x \sqrt{g} F_{\mu\nu} F^{\mu\nu} - i \frac{\theta}{8\pi^2} \text{tr} \int F \wedge F = \left( \frac{8\pi^2}{g_{YM}^2} - i\theta \right) k = -i2\pi\tau k.$$

For the  $\mathcal{N} = 4$  case,  $Z_{\text{inst}} = 1$ , [13]. For the  $\mathcal{N} = 2$  cases, it is non-trivial, [14]. However, the regime of interest is the large  $N$  limit:

$$N \rightarrow \infty \quad \text{and} \quad \lambda = g_{\text{YM}}^2 N \quad \text{is kept finite,} \quad (4.14)$$

then the expansion parameter becomes

$$e^{i2\pi\tau} = e^{-\frac{8\pi^2 N}{\lambda} + i\theta},$$

hence the instanton contributions are expected to be exponentially suppressed. This is checked in [15].

## 4.2 Large- $N$ matrix model

The localized partition function (4.7) is a matrix model. For the  $\mathcal{N} = 4$  case, the model is particularly simple, it is the well-known Gaussian unitary ensemble (GUE) in the random matrix theory. The product of the eigenvalue differences is the Vandermonde determinant squared, which is essentially the Jacobian factor after diagonalizing the Hermitean matrix in (4.6). Writing it in terms of the effective action:

$$Z_{\text{GUE}} = \int d^N a e^{-S[a]}, \quad S[a] = -\sum_{i \neq j}^N \log(|a_i - a_j|) + \frac{8\pi^2 N}{\lambda} \sum_k^N a_k^2 \quad (4.15)$$

we see that the Vandermonde determinant gives the 2d Coulomb potential. This partition function is indeed identified with that of a 2d Coulomb gas confined on a line, with the repulsive electrostatic force and an attractive harmonic force [16].

For  $\mathcal{N} = 2^*$ , the matrix model is highly non-trivial, but we can solve it systematically in the large  $N$  limit, where the saddle-point approximation and the continuous approximation in principle apply. The only relevant quantity to compute then is the distribution of the eigenvalues:

$$\rho(x) = \frac{1}{N} \sum_{i=1}^N \delta(x - a_i). \quad (4.16)$$

It must have a compact support, i.e.  $\rho(\pm\mu) = 0$ , where the endpoint  $\mu$  determines the scale of the spontaneous symmetry breaking. By definition, it is also unit-normalized:

$$\int_{-\mu}^{\mu} dx \rho(x) = 1. \quad (4.17)$$

The saddle-point equation, from extremizing the effective action (4.15) in terms of (4.16), is a singular (principal value) integral equation:

$$\frac{\delta S[\rho]}{\delta \rho(x)} = 0 \quad \Rightarrow \quad \oint_{-\mu}^{\mu} dy K(x-y) \rho(y) = \frac{8\pi^2}{\lambda} x. \quad (4.18)$$

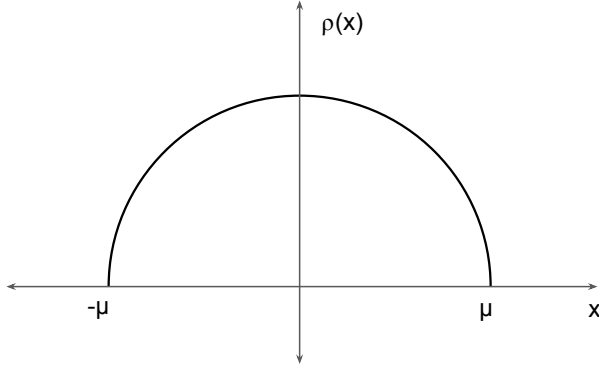


Figure 4.1. Wigner's semicircle distribution, solution to the Gaussian unitary ensemble (GUE).

The singular kernel for the GUE case is just the Hilbert kernel<sup>2</sup>, that is

$$K_{\text{Hilbert}}(x) = \frac{1}{x}. \quad (4.19)$$

In the Coulomb gas picture, the saddle-point equation determines the equilibrium distribution due to the force balance. The result is the well-known Wigner's semicircle distribution

$$\rho(x) = \frac{2}{\pi\mu^2} \sqrt{\mu^2 - x^2}, \quad \mu = \frac{\sqrt{\lambda}}{2\pi}, \quad (4.20)$$

shown in figure 4.1.

For the  $\mathcal{N} = 2^*$  SYM, the kernel is

$$K(x) = \frac{1}{x} - \mathcal{H}(x) + \frac{1}{2} \mathcal{H}(x + MR) + \frac{1}{2} \mathcal{H}(x - MR), \quad (4.21)$$

where

$$\mathcal{H}(x) \equiv -\frac{H'(x)}{H(x)} = 2x \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{n}{n^2 + x^2} \right), \quad (4.22)$$

and much more interesting features show up in the saddle-point solution, as shown in figure 4.2. Let us briefly review these results.

In the strong-coupling regime, the bulk of the distribution is also a semicircle but with a rescaled endpoint:

$$\rho(x) = \frac{2}{\pi\mu^2} \sqrt{\mu^2 - x^2}, \quad \mu = \frac{\sqrt{\lambda(1 + (MR)^2)}}{2\pi}, \quad (\lambda \rightarrow \infty). \quad (4.23)$$

<sup>2</sup> The name is in relation with the Hilbert transform. It is also known as Cauchy kernel in the literature.

This is due to the fact that the kernel is approximately a Hilbert kernel [17]:

$$K(x) \approx \frac{1 + (MR)^2}{x}, \quad (\lambda \rightarrow \infty). \quad (4.24)$$

Close to the edge-points, this approximation no longer holds, and it was the goal of Paper I to find the endpoint distribution in the strong coupling regime.

We computed the endpoint distribution exactly using the so-called Wiener-Hopf method, which is essentially the Fourier transform of the convolution integral in (4.18) in a semi-infinite interval (zero being at the endpoint), that relies on a certain factorization of the kernel.

The distribution exhibits oscillatory behavior with a period proportional to the scale  $MR$ . In the decompactification limit  $MR \rightarrow \infty$  (but  $\mu \gg MR$  because we remain in the strong coupling limit), the peaks of the oscillation diverge, see figure 4.2. The analytical endpoint distribution at strictly infinite coupling and flat space, with  $\xi \equiv \mu - x$ , is summarized below:

$$\rho(\xi) = \frac{2^{3/2}}{\pi\mu^{3/2}} \begin{cases} MR\sqrt{\xi}, & (\xi \sim 1) \\ \frac{\sqrt{MR}}{2} \sum_{k=0}^{\left[\frac{\xi}{MR}\right]} \left(\left\{\frac{\xi}{MR}\right\} + k\right)^{-1/2} & (\xi \sim MR), \end{cases} \quad (4.25)$$

where  $[\cdot]$  and  $\{\cdot\}$  denote the integer and the fractional part, respectively, and  $\mu = MR\sqrt{\lambda}/(2\pi)$ , from (4.23).

Physically, the cusps appear due to a resonance phenomena

$$m_{ij} = |a_i - a_j \pm MR| \approx 0 \quad (4.26)$$

of very light states in the hypermultiplet sector. Similar features were already observed for finite couplings in the flat space limit [9], and it was numerically shown that there are infinite-many critical couplings defined by

$$\mu = g(\lambda_c^{(n)})MR, \quad g(\lambda_c^{(n)}) = \frac{n}{2}, \quad n = 1, \dots \quad (4.27)$$

This means there are infinitely many phase transitions, where the phases are distinguished by the number of cusps of the distribution, and the coupling  $\lambda$  is the order parameter. At strong coupling, the critical behavior persists and matches with the one obtained from the decompactification limit, hence these two limits commute [18].

Such phase transitions are common among large- $N$  matrix models, and have been observed in e.g. ABJM models [19], 5d  $\mathcal{N} = 1$  SYM with massive matter multiplets [20]. In our case, the gauge/string duality in principle gives us an opportunity to understand them from the point of view of gravity. The physical observables we use to probe the infinite-coupling phase are Wilson loops, the topic of the next chapter.



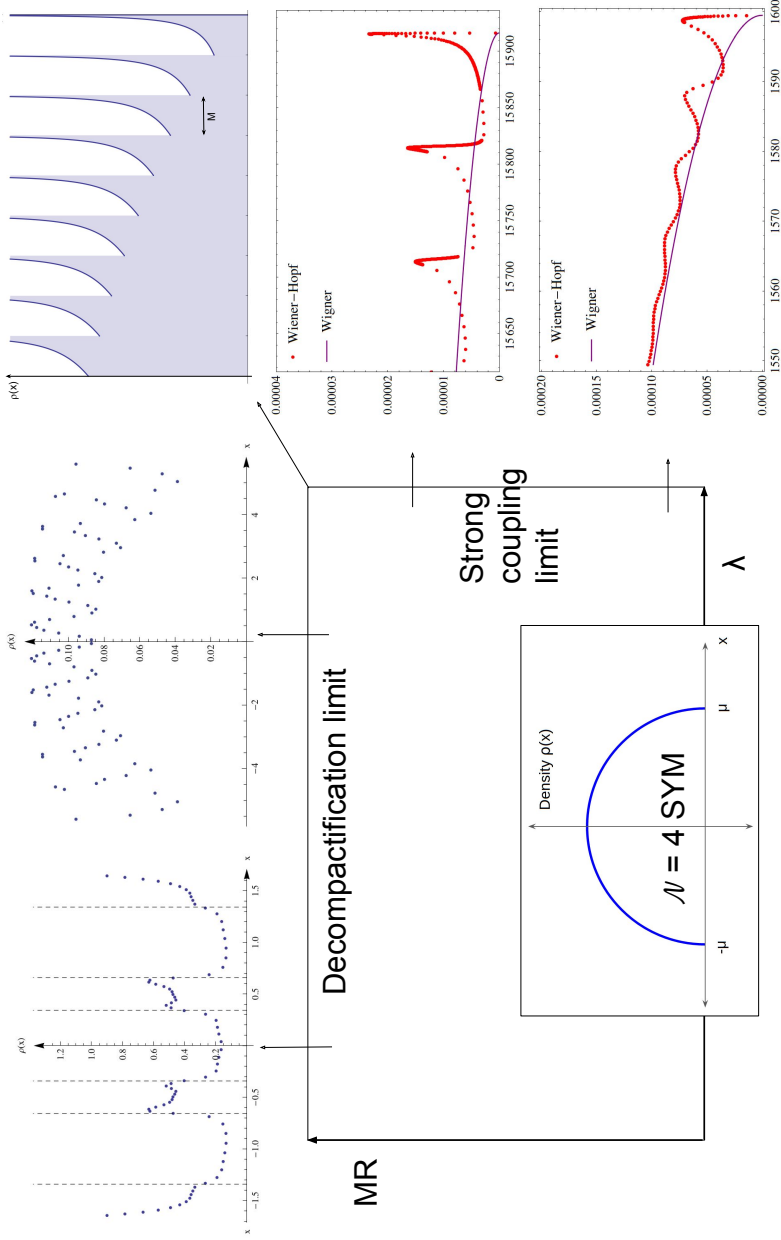


Figure 4.2. Phase diagram for the partition function of  $\mathcal{N} = 2^* \text{ SYM}$  on  $S^4$ , in the large  $N$  limit. The plots at the decompactification limit are taken from [9], and the ones at strong coupling limit are from Paper I, where only close to the endpoint is shown and  $R = 1$ . In the zero mass limit, we have  $\mathcal{N} = 4 \text{ SYM}$ , where the distribution is the Wigner semicircle for any coupling.

## 5. Wilson loops

A Wilson loop is a gauge-invariant observable, defined as the expectation value of the character of the representation  $\mathcal{R}$  of the gauge group  $U(N)$  in our case):

$$W_{\mathcal{R}}(C) = \langle \text{tr}_{\mathcal{R}} U \rangle, \quad U \in U(N) \quad (5.1)$$

where  $U$  is a path-ordered exponential of the gauge connection  $A_{\mu}$ , transported along an arbitrary curve  $C$  parametrized by  $s$ :

$$U = \text{Pexp} \left[ i \int_C ds \dot{x}^{\mu} A_{\mu} \right], \quad (5.2)$$

where the dot denotes derivative respect  $s$ .

Physically, the Wilson loop operator measures the phase associated with moving a probe particle with charge  $\mathcal{R}$  around a curve  $C$  in spacetime. In particular, the long rectangular Wilson loop in the fundamental representation, the path shown in the figure 5.1, determines the static quark and antiquark potential:

$$V_{q\bar{q}}(L) = -\frac{1}{T} \lim_{T \rightarrow \infty} \log W_1(C). \quad (5.3)$$

In confined theories such as QCD, the potential is linear, which is referred as the *area law* in terms of the Wilson loop,  $\log W \propto L \times T$ , while in the deconfined phase, the Wilson loop follows the *perimeter law*,  $\log W \propto L$ .

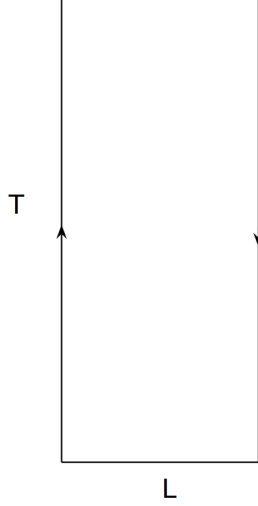
We are interested in a supersymmetric extension of the Wilson loop, such that it is computable through localization, see conditions (2.5). We will study the so-called Maldacena-Wilson loop [21], where we add a coupling to the scalars of the vector multiplet:

$$U = \text{Pexp} \left[ \int_C ds \left( i \dot{x}^{\mu} A_{\mu} + |\dot{x}| n^I \Phi_I \right) \right], \quad I = 0, 9. \quad (5.4)$$

Let us start by reviewing some known results in  $\mathcal{N} = 4$  SYM and then generalize them to  $\mathcal{N} = 2^*$ .

### 5.1 Wilson loops in $\mathcal{N} = 4$ SYM

The simplest Wilson loop is an infinite straight line. It is a half-BPS object, meaning it commutes with half of the 32 supercharges of  $\mathcal{N} = 4$  SYM. This



*Figure 5.1.* Trajectory of a probe quark and antiquark separated by distance  $L$ , and travel  $T$  distance in time.

fact protects the Wilson line from quantum corrections and its value is simply one:

$$\langle W_{\text{line}} \rangle = 1. \quad (5.5)$$

By conformal transformation, the Wilson line can be mapped to a circular Wilson loop. The result, however, is not the same. This is often referred to as a conformal anomaly, and it is due to the fact that large conformal transformations such as inversion are not symmetries in the flat space (infinity is not a point of  $\mathbb{R}^d$ ). On the sphere, these are symmetries, hence there is no distinction between a circle and a line, and the expectation value of either is the same as for a circle on  $\mathbb{R}^4$  [22]. The circular Wilson loop, which is also half-BPS, is exactly computable using the localized partition function for the theory on  $S^4$ , where the path is the equator of sphere, see figure 5.2.

### 5.1.1 Fundamental representation

The circular Wilson loop in the fundamental representation is mapped to a matrix model expectation value:

$$W_1 = \left\langle \frac{1}{N} \sum_{i=1}^N e^{2\pi a_i} \right\rangle_{\text{matrix model}}, \quad (5.6)$$

which can be solved exactly for GUE [22]:

$$W_1 = \frac{1}{N} L_{N-1}^1(-\lambda/(4N)) e^{\frac{\lambda}{8N}}, \quad (5.7)$$

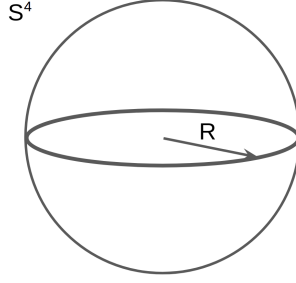


Figure 5.2. The contour of the circular Wilson loop we study is the equator of the hypersphere the theory is defined on.

where  $L$  is the generalized Laguerre polynomial:

$$L_n^m(x) = \frac{x^{-m} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+m}). \quad (5.8)$$

In the large  $N$  limit and fixed 't Hooft coupling, (5.6) can be written as:

$$W_1 = \int_{-\mu}^{\mu} dx \rho(x) e^{2\pi x} \quad (5.9)$$

$$= \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}), \quad (N \rightarrow \infty \quad \text{and} \quad \lambda - \text{fixed}) \quad (5.10)$$

where for the last equality, we used the semicircle distribution (4.20), since the Wilson loop insertion to the partition function is subleading in  $N$ .  $I_1(x)$  is the modified Bessel function:

$$I_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}. \quad (5.11)$$

Historically, the large- $N$  result was initially obtained by Erickson-Semenoff-Zarembo [23], by summing over rainbow diagrams in perturbation theory, and they conjectured the Gaussian matrix model structure for  $\mathcal{N} = 4$  SYM. Eventually Pestun's work in localization [7] proved it.

In the 't Hooft limit, which is the holographic regime,

$$W_1 = \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}}, \quad (N \rightarrow \infty \quad \text{and} \quad \lambda \rightarrow \infty). \quad (5.12)$$

### 5.1.2 Higher rank representations

Exact results for arbitrary representation of  $U(N)$  can also be obtained [24]. These are very generic, though, but the generating function of  $k$ -antisymmetric

representation

$$\langle G_{A_k}(t) \rangle = \sum_{k=0}^N t^k W_{A_k} \quad (5.13)$$

has a nice compact form:

$$\langle G_{A_k}(t) \rangle = \det \left( t \delta_{ij} + L_{i-1}^{j-i} (-\lambda / (4N)) e^{\frac{\lambda}{8N}} \right). \quad (5.14)$$

We study the 't Hooft limit for the symmetric (+) and the antisymmetric (-) representations. The generating functions<sup>1</sup> for the character of these representations are explicitly known:

$$G^\pm(v) = \prod_{i=1}^N (1 \mp e^{a_i - v})^\mp. \quad (5.15)$$

Notice that these are also the Bose (+) and Fermi (-) distributions, in terms of the eigenvalues  $a_i$ .

The standard procedure is to derive the character by inverting the generating function using Cauchy's integral formula:

$$\chi_k^\pm \equiv \text{tr}_\pm U = \int_{C-i\pi}^{C+i\pi} \frac{dv}{2\pi i} e^{v k} G^\pm(v), \quad (5.16)$$

where  $C > a_i, \forall i$ , for the symmetric case, and  $C$  is arbitrary for the antisymmetric case.

All we need to do now is to compute the expectation value of the above integral. We can still employ the semicircle distribution (4.20), and we further take the large representation limit  $k \sim N$ , which allows us to use the saddle-point method in (5.16), [25]. The saddle-point equation to solve for  $v_*$  is

$$\frac{k}{N} = \int_{-\mu}^{\mu} dx \frac{\rho(x)}{e^{L(v_* - x)} \mp 1}, \quad (N \rightarrow \infty \quad \text{and} \quad \frac{k}{N} - \text{fixed}), \quad (5.17)$$

which is analogous to the particle density equation in a Bose/Fermi system.

The final leading solutions for the antisymmetric representation is

$$\log W_k^- = N \frac{2\sqrt{\lambda}}{3\pi} \sin^3 \theta, \quad (5.18)$$

where  $\cos \theta \equiv v_*/\mu$  satisfies the transcendental equation

$$\theta - \frac{1}{2} \sin 2\theta = \pi \frac{k}{N}, \quad (5.19)$$

resulting from (5.17) after the step-function approximation of the Fermi distribution.

---

<sup>1</sup> Here, unlike in (5.13), we use the expansion parameter  $e^{-v}$  instead of  $t$ .

For the symmetric representation, however, there is no saddle-point solution for (5.17). This is the same phenomenon as the Bose-Einstein condensation. It is possible to analytically continue the solution to the second Riemann sheet, though, as done in [25], and the final result is:

$$\log W_k^+ = 2Nf(\kappa), \quad \kappa \equiv \frac{\sqrt{\lambda}k}{4N} \quad (5.20)$$

where

$$f(x) = x\sqrt{1+x^2} + \operatorname{arcsinh} x. \quad (5.21)$$

The drawback of the analytic continuation was that computing large- $N$  corrections became less clear, as attempted in [26]. In Paper II and Paper IV, we used a more systematic approach. Paper IV focused exclusively on the symmetric Wilson loop in  $\mathcal{N} = 4$  SYM, where subleading corrections in  $N$  were computed. This result is consistent with the expansion of the known exact result for the multiply wrapped fundamental Wilson loop [27], which helped to clarify the apparent mismatch observed in [26], and agrees with the analysis by [28] that symmetric representations and the multiply-wound fundamental ones differ by exponentially-suppressed terms in strong coupling. Paper IV also derived the strong-coupling corrections, in response to the strong-coupling expansion done for the antisymmetric case in [29], where the Sommerfeld expansion of the Fermi distribution was used.

## 5.2 Wilson loops in $\mathcal{N} = 2^*$ SYM

The story can be extended to  $\mathcal{N} = 2^*$  SYM on  $S^4$ . Here we have an extra parameter: the scale  $MR$ . We will take the decompactification limit  $MR \rightarrow \infty$ , where interesting phase transitions were seen, and also, the dual theory is fully known on  $\mathbb{R}^4$  [30].

### 5.2.1 Fundamental representation

Since the fundamental Wilson loop is basically an exponentially-weighted integral (5.9), its value in the strong coupling limit is determined by the largest eigenvalue  $\mu$  (recall  $\mu \sim \sqrt{\lambda}$ , see (4.23)). Thus, we do not expect its strong coupling corrections to probe the cusps region. In Paper I, we computed the subleading correction to the endpoint  $\mu$ , which lead to the same correction to the Wilson loop (in terms of the perimeter  $l = 2\pi R$ ):

$$\log W_1 = P(\lambda)Ml, \quad P(\lambda) = \frac{\sqrt{\lambda}}{2\pi} - \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \quad (MR \rightarrow \infty). \quad (5.22)$$

The leading order term is the same as its homologous case in  $\mathcal{N} = 4$ , only rescaled by  $MR$  [17], as a direct consequence of the semicircle behavior of

the bulk distribution (4.23). As a consistent check, when  $M \rightarrow 0$ , the Wilson loop goes to 1, as expected for the  $\mathcal{N} = 4$  case. Moreover, we clearly see the perimeter law here since the theory is not confining neither conformal.

### 5.2.2 Symmetric and antisymmetric representations

In Paper II, result for symmetric and antisymmetric representations were computed, up to the next-to-leading order in the strong coupling expansion. The decompactified results at the leading order in  $N$  are also the same as the ones in the  $\mathcal{N} = 4$  case, but rescaled differently:

$$\log W_k^- = NMR \frac{2\sqrt{\lambda}}{3\pi} \sin^3 \theta, \quad (5.23)$$

$$\log W_k^+ = 2N(MR)^2 f \left( \frac{\kappa}{MR} \right), \quad (5.24)$$

where  $\theta$  and  $f$  satisfy the same equation as in (5.19) and (5.21), respectively.

Now, the interesting part lays in the subleading terms. Unlike the fundamental representation, the higher rank representations do probe the endpoint distribution of the eigenvalues, which has periodic cusps with period  $MR$ , see figure 4.2. The results are:

$$\delta \log W_k^- = -\frac{2\pi^2}{3} \frac{NMR}{\lambda^{3/4}} \begin{cases} 4\tilde{f}^3 & 0 < \tilde{f} \leq 1 \\ \tilde{f}^3 + 6\tilde{f} - \frac{3}{\tilde{f}} & 1 < \tilde{f} \leq 1 + \sqrt{2}, \\ \vdots & \end{cases} \quad \tilde{f} = \frac{\lambda^{3/4} k}{4\sqrt{\pi} N} \quad (5.25)$$

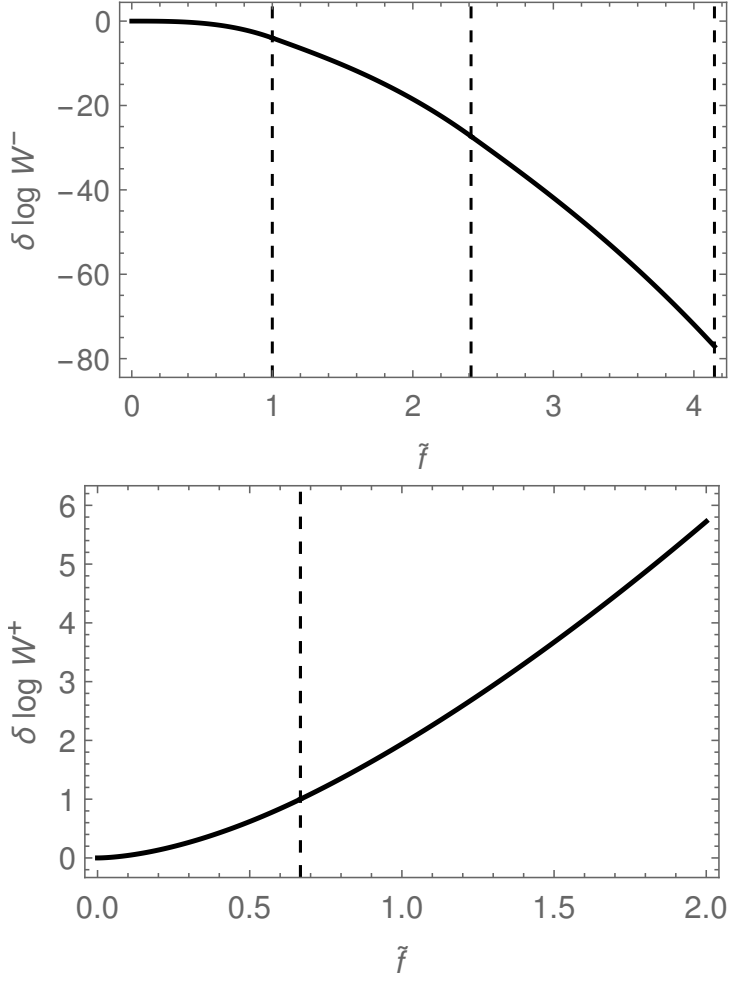
and

$$\delta \log W_k^+ = \frac{2^5 \pi^{3/2}}{5} \frac{N(MR)^2}{\lambda^{3/4}} \left( v^{5/2} + \Theta(v-1) \left( v + \frac{2}{3} \right) (v-1)^{3/2} \right), \quad (5.26)$$

where  $\Theta(x)$  is the Heaviside function and  $v$  is written in terms of the scaling parameter  $\tilde{f}$  as

$$\frac{3}{2}\tilde{f} = v^{3/2} - \Theta(v-1)(v-1)^{3/2}, \quad \tilde{f} = \frac{\lambda^{3/4} k}{8\sqrt{\pi} MRN}. \quad (5.27)$$

The solutions are plotted in figure 5.3. The phase transitions are of second and third order for the antisymmetric and symmetric representations, respectively, in the sense that the derivatives of the free energy  $F = -\frac{1}{N} \log W$  with respect to  $\tilde{f}$  exhibit discontinuity at these orders.



*Figure 5.3.* Strong coupling correction for (rescaled) log of Wilson loops in antisymmetric representation (up) with the critical points at  $\{1, 1 + \sqrt{2}, 1 + \sqrt{2} + \sqrt{3}, \dots\}$ , and in symmetric representation (down) with the critical point at  $2/3$ .



**Part II:**  
**Gauge/string duality**



## 6. Supergravity

Quantum gravity theories must contain massless spin 2 particles, that are gravitons. In the same fashion as we built super-Yang-Mills (SYM) theories, with maximum spin 1, we can build supergravity (SUGRA) theories by requiring the supermultiplet to contain gravitons (and no higher spin particles). This constrains the number of super-Poincaré charges to a maximum of  $32^1$ .

For one copy of supersymmetry, i.e.  $\mathcal{N} = 1$ , we have a unique supergravity theory in 11 dimensions [32], from which many lower dimensional supergravity theories can be obtained by the means of Kaluza-Klein compactification and dimensional reduction. This theory contains a metric, an antisymmetric rank 3 tensor and a Majorana gravitino field.

Compactifying it on a circle and taking its radius to be zero, we obtain type IIA  $\mathcal{N} = 2$  SUGRA in 10 dimensions, which is also a low energy effective action of type IIA superstring theory. Using T-duality<sup>2</sup>, we can obtain type IIB  $\mathcal{N} = 2$  SUGRA, which is of our interest to study the AdS/CFT duality. Let us list its field content:

rank 2 symmetric tensor: metric	$g$
complex scalar: axion-dilaton	$C_{(0)} + i e^{-\Phi}$
rank 2 antisymmetric tensor	$C_{(2)} + i B_{(2)}$
rank 4 antisymmetric tensor	$C_{(4)}$
Majorana-Weyl 3/2-spinor: gravitinos	$\psi_M^I, I = 1, 2$
Majorana-Weyl 1/2-spinor: dilatinos	$\lambda^I, I = 1, 2$

The gauge potentials  $C_{(i)}$  are called the Ramond-Ramond potentials. The fields above must satisfy the supergravity equations that consist of [33]:

- Einstein's equations
- Maxwell's equations
- The dilaton equation

<sup>1</sup> In the unitary massless spinor representation  $S$ , half of the supercharges annihilates the highest weight state. Only half of the remaining supercharges are spin-raising operators (the other half lowers spin). On the hand, we can have at most 8 raising operators from spin -2 to 2 by steps of 1/2. Then (see e.g. [31]), for  $\mathcal{N}$  copies of supersymmetries:

$$1/4 \times \mathcal{N} \times \dim S = 8 \quad \Rightarrow \quad \mathcal{N} \dim S = 32.$$

<sup>2</sup> It states that strings compactified on a torus with radius  $R$  is equivalent to strings compactified on a torus with radius proportional to  $1/R$ .

- (Hodge) self-duality equation:  $*F_{(5)} = F_{(5)}$ , where

$$F_{(5)} = dC_{(4)} - \frac{1}{2}C_{(2)} \wedge dB_{(2)} + \frac{1}{2}B_{(2)} \wedge dC_{(2)}. \quad (6.1)$$

To avoid introducing more notation, we refer the reader to e.g. [10] for explicit expressions of the above equations.

The supersymmetry condition imposes the variation of dilatinos and of gravitinos to be zero [33]. These are Killing equations and by solving them we obtain the Killing spinors of the background geometry. For  $AdS_5 \times S^5$ , the solutions can be found in [34], and for the Pilch-Warner background, see [35] and the appendix D of Paper III.

## 6.1 $AdS_5 \times S^5$

The simplest solution to the supergravity equations is  $AdS_5 \times S^5$  geometry, where the dilaton  $\Phi$ , the  $B$ -field and the gauge potentials  $C_{(0)}$  and  $C_{(2)}$  are trivial.

Since spheres are more familiar, let us review only the metric of the anti-de-Sitter space ( $AdS$ ). It is a hyperboloid with Minkowski signature, embedded in the flat space of one dimension higher, i.e.:

$$ds^2 = -dX_0^2 - dX_D^2 + \sum_{i=1}^{d-1} X_i^2 \quad (6.2)$$

with the constraint

$$-X_0^2 - X_d^2 + \sum_{i=1}^{d-1} X_i^2 = -L^2. \quad (6.3)$$

Figure 6.1 shows an example for  $AdS_2$ . The isometry group is clearly  $SO(2, d)$ .

By solving the constraint, the induced metric (in global coordinates) for  $AdS_{d+1}$  becomes:

$$ds^2 = L^2(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2). \quad (6.4)$$

where  $d\Omega_d^2$  is the metric of  $S^d$ ,  $\rho \geq 0$  and  $2\pi \geq \tau \geq 0$ .

Another set of solutions that solve the constraint (6.3) is the Poincaré coordinates. These are preferred for holographic studies, since the metric in these coordinates

$$ds^2 = \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) \quad (6.5)$$

is manifestly conformal invariant, with flat space slicing for any  $z > 0$  (see figure 6.1), despite it covers only half of the geometry. The conformal boundary<sup>3</sup> corresponds to  $z \rightarrow 0$ , and  $z$ , called radial coordinate, is interpreted as the energy scale of the boundary theory in AdS/CFT.

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<sup>3</sup> The boundary of  $AdS$  is infinitely far away from the bulk, but it can be mapped to a finite distance using a conformal transformation.

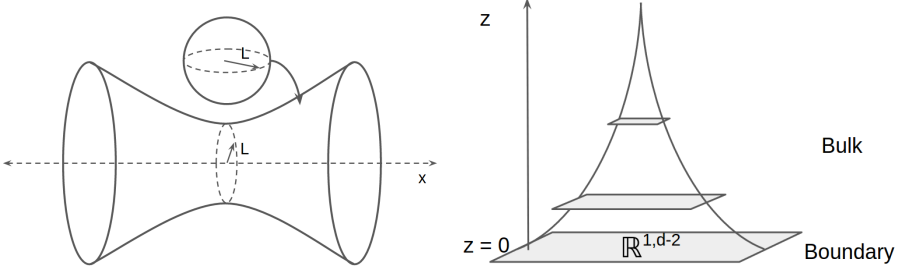


Figure 6.1. Left:  $AdS_2 \times S^2$ , where  $AdS_2$  is defined by  $x^2 - y^2 - t^2 = -L^2$ , and at all its surface points we find the sphere. Right: Flat space slicing for  $AdS_d$  in Poincaré coordinates.

## 6.2 D-branes

D-branes are soliton-like solutions of the supergravity equations that preserve half of the 32 background supercharges (BPS solutions), see textbook [36]:

$$ds^2 = H_p(r)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_p(r)^{1/2} \delta_{ij} dy^i dy^j \quad (6.6)$$

$$e^\Phi = g_s H_p(r)^{(3-p)/4} \quad (6.7)$$

$$C_{(p+1)} = (H_p(r)^{-1} - 1) dx^0 \wedge \cdots \wedge dx^p \quad (6.8)$$

$$B_{(2)} = 0, \quad (6.9)$$

where  $x^\mu, \mu = 1, \dots, p$  are the D-brane worldvolume coordinates,  $y^i, i = p+1, \dots, 9$  are the transversal directions to the brane, and the harmonic function is

$$H_p(r) = 1 + \left( \frac{L_p}{r} \right)^{7-p}. \quad (6.10)$$

It is intuitive to think of these solutions as charged point-like particles from the  $(9-p)$ -dimensional transverse space point of view. For  $N$  coincident D-branes, their total charge is proportional to  $N$ , and from the Gauss law, we obtain

$$N \propto \int_{S^{9-p-1}} *F_{(p+2)}, \quad (6.11)$$

quantizing therefore the (Hodge dual) of the flux of the Ramond-Ramond field  $C_{(p+1)}$  through the hypersphere,  $F_{(p+2)}$ . The above relation also determines the characteristic length scale

$$L_p^{7-p} = (4\pi)^{(5-p)/2} \Gamma\left(\frac{7-p}{2}\right) g_s N \alpha'^{(1-p)/2}, \quad (6.12)$$

where  $g_s$  is the string length and  $\alpha'$  is the Regge slope. We will comment on these parameters in chapter 7, where we will also see that  $AdS_5 \times S^5$  geometry

can be obtained as the near horizon geometry of a stack of  $N$  coincident D3-branes.

To conclude this section, let us write down the generic action of a single D-brane in Minkowski signature, which will be used later on in the thesis. The action consists of the Dirac-Born-Infeld (DBI) term:

$$S_{\text{DBI}} = -T_{\text{Dp}} \int_M d^{p+1} \xi e^{-\Phi} \sqrt{-\det \left( P[g] + P[B] + \frac{1}{T_{\text{F1}}} F \right)}, \quad (6.13)$$

and the Wess-Zumino (WZ) term with the Ramond-Ramond potentials pulled back to the worldvolume:

$$S_{\text{WZ}} = T_{\text{Dp}} \int_M \exp \left( P[B] + \frac{1}{T_{\text{F1}}} F \right) \wedge P[C]. \quad (6.14)$$

The couplings are

$$T_{\text{F1}} = \frac{1}{2\pi\alpha'}, \quad T_{\text{D1}} = \frac{1}{g_s} T_{\text{F1}}, \quad T_{\text{Dp}} = T_{\text{D1}} \left( 2\pi\sqrt{\alpha'} \right)^{1-p}. \quad (6.15)$$

### 6.3 Pilch-Warner geometry

Another solution that preserves  $\mathcal{N} = 2$  supersymmetric flow is found by Pilch-Warner [30]. The geometry is a deformed and warped  $AdS_5 \times S^5$ , and all the fields are non-trivial in this case. The metric (in Einstein frame) is

$$ds^2 = \frac{(cX_1X_2)^{1/4}}{\sqrt{A}} L^2 \left( \frac{M^2 A}{c^2 - 1} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{A(c^2 - 1)^2} dc^2 \right. \\ \left. + \left[ \frac{1}{c} d\theta^2 + \frac{\sin^2 \theta}{X_2} d\phi^2 + \cos^2 \theta \left( \frac{A}{X_1} (\sigma_1^2 + \sigma_2^2) + \frac{A}{cX_2} \sigma_3 \right) \right] \right), \quad (6.16)$$

where  $\sigma_i, i = 1, 2, 3$  are the Maurer-Cartan forms for  $SU(2)$  and

$$X_1(c, \theta) = \cos^2 \theta + cA(c) \sin^2 \theta, \\ X_2(c, \theta) = c \cos^2 \theta + A(c) \sin^2 \theta, \\ A(c) = c + \frac{1}{2}(c^2 - 1) \log \left( \frac{c-1}{c+1} \right).$$

It reduces to  $AdS_5 \times S^5$  near the boundary, i.e. expand the metric for  $c \approx 1 + z^2 M^2/2$ , where  $z$  is the radial coordinate in (6.5), then set  $M = 0$ .

The Pilch-Warner solution is dual to  $\mathcal{N} = 2^*$  SYM on  $\mathbb{R}^4$ . Similarly the dual for  $\mathcal{N} = 2^*$  on  $S^4$  has been studied in [11], but the solution is partially known only. We are interested in the flat space regime, where interesting physics happen such as the phase-transitions discussed in Part I (see figure 4.2). Hence, the dual computations will be done in the Pilch-Warner background.

## 7. AdS/CFT correspondence

We have seen that a Yang-Mills theory with gauge group  $U(N)$  is characterized by two parameters: the Yang-Mills coupling  $g_{\text{YM}}$  and the gauge group rank  $N$ . Before stating the gauge/string duality, also known as the holographic principle, let us discuss different perturbative limits in string theory.

In string theory, there are two parameters used for expansion: the string coupling constant  $g_s$  and the string length<sup>1</sup>  $l_s = \sqrt{\alpha'}$ . When the string coupling is small, it leads to classical strings, where only tree level diagrams are taken into account. This means strings do not merge and split, which would give different worldsheet topologies, see figure 7.1. If the strings propagate in a curved background with a characteristic length scale  $L$ , the classical gravity limit is obtained by requiring additionally  $l_s \ll L$ . In other words, strings can be approximated by point particles (their center mass).

The AdS/CFT correspondence was originally conjectured by Maldacena [37], and stated the below theories are dynamically equivalent:

- $\mathcal{N} = 4$  SYM in 4 dimensions with gauge group  $SU(N)$
- Type IIB superstring theory on  $AdS_5 \times S^5$  (both with the same radius  $L$ ), where the  $F_{(5)}$  has integer flux  $N$  on  $S^5$ .

The parameters of these two theories are related as:

$$g_{\text{YM}}^2 = 4\pi g_s, \quad \lambda \equiv g_{\text{YM}}^2 N = \left( \frac{L}{l_s} \right)^4. \quad (7.1)$$

The conjecture was further developed by [38, 39], and since its original inception, other examples have been found such as a lower dimensional correspondence between a 3d CFT called ABJM theory and type IIA superstring on  $AdS_4 \times CP^3$  [40], known as  $AdS_4/CFT_3$  correspondence. It is also an active field of research, but here, we will focus only on the  $\mathcal{N} = 4$  case.

We see the first relation in (7.1) implies the YM's coupling to be small in the classical strings limit. Moreover, from the second relation,  $L/l_s$  is arbitrary in this limit, hence  $N$  must be large to compensate the smallness of  $g_{\text{YM}}$ . The 't Hooft coupling  $\lambda$  is then kept fixed when  $N$  is large. This limit gives planar Feynman diagrams, see figure 7.2, in the leading order on the gauge theory side, and corresponds to the genus expansion in string worldsheet, see figure 7.1. This was actually noticed by 't Hooft long before the advent of AdS/CFT correspondence [41], and suggested planar diagrams as triangulations of the string worldsheet. Notice that, in the planar limit, the Lie groups  $U(N)$  and  $SU(N)$  are indistinguishable.

<sup>1</sup> It is a fundamental scale in string theory, and not literally the length of the strings.

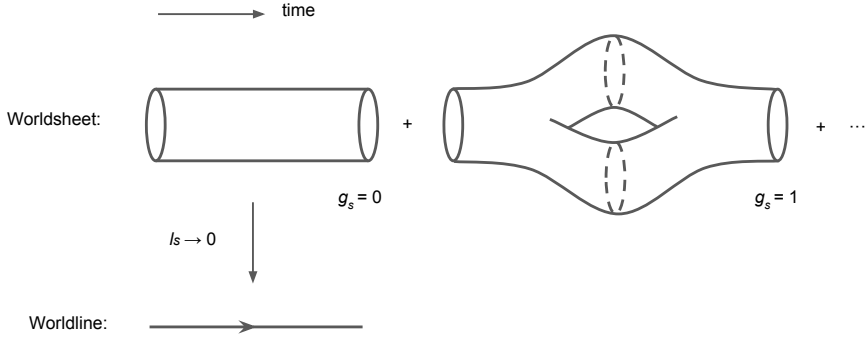


Figure 7.1. Genus expansion of the closed string worldsheet and the point-particle approximation.

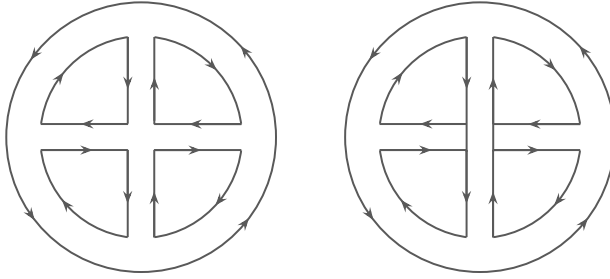


Figure 7.2. A planar diagram (left) and a non-planar diagram (right).

If we further impose classical gravity limit, then the 't Hooft coupling must be large too. Hence, the correspondence becomes a form of strong/weak duality. This is the 't Hooft limit and it is the most computationally accessible one, since our understanding of string theory beyond the above-mentioned perturbative regimes is poor. Moreover, it is potentially seen as a tool to solve strongly coupled quantum field theory using supergravity. Let us summarize the different limits in the table below:

$\mathcal{N} = 4$ SYM	Type IIB strings on $AdS_5 \times S^5$
$N, \quad \lambda = g_{YM}^2 N$	$g_s = \frac{\lambda}{4\pi N}, \quad T = \frac{\sqrt{\lambda}}{2\pi}$
$N \rightarrow \infty, \quad \lambda \text{ fixed}$	$g_s = 0, \quad T \text{ fixed}$
$N \rightarrow \infty, \quad \lambda \rightarrow \infty$	$g_s = 0, \quad T \rightarrow \infty$

Symmetry-wise, the conjecture is consistent:

$\mathcal{N} = 4$ SYM	Type IIB strings on $AdS_5 \times S^5$
conformal symmetry $SO(4, 2)$	isometry of $AdS_5$ : $SO(4, 2)$
R-symmetry $SO(6)$	isometry of $S^5$ : $SO(6)$
SUSY: 32 supercharges	SUSY: 32 supercharges



As for local operators in CFT, the AdS/CFT statement is that<sup>2</sup>

$$\int_{\phi \sim \phi_{(0)}} D\phi e^{-S_{\text{string}}[\phi]} = \langle e^{-\int_{\partial \text{AdS}} \phi_{(0)} O} \rangle_{\text{CFT}}, \quad (7.2)$$

where  $\phi_{(0)}$  represents the boundary values of fields  $\phi$  living in the bulk of  $\text{AdS}$ . In the supergravity limit, the on-shell supergravity action corresponds to the generating function of connected graphs in the field theory. In other words, bulk fields with non-trivial boundary values are sources of gauge invariant operators in CFT. Since the volume of  $\text{AdS}$  is infinite (due to the second order pole in the radial coordinate), the divergences are removed using *holographic renormalization*, which is analogous to the renormalization of correlation functions. Holographic renormalization consists of expanding bulk fields close to the boundary, identifying the divergent terms and remove them by counterterms, as in the usual renormalization procedure, see e.g. [42] for a review.

A formal proof of AdS/CFT correspondence is yet elusive, despite numerous explicit checks. Next, we will review the original argument from [37].

## 7.1 N D3-branes

The origin of this correspondence lies on the two perspectives of D-branes in superstring theory.

D-branes are non-perturbative objects where open strings end (with Dirichlet conditions, i.e. fixed endpoint with zero momentum). In the weak coupling limit  $g_s \ll 1$ , open strings can be thought as excitations of D-branes, see e.g. figure 7.3. In the low energy limit  $E \ll 1/l_s$ , the massive string excitations can be ignored.

Consider a single D3-brane and let us expand its action (the DBI part) for  $l_s \ll 1$ , which gives the low energy open (bosonic) string action:

$$S = -\frac{1}{2\pi g_s} \int d^4x \left( 1 + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \partial_\mu \Phi_I \partial^\mu \Phi^I + \dots \right). \quad (7.3)$$

These fields are the gauge field  $F_{\mu\nu}$  that transforms in the unbroken Lorentz symmetry of the 4-dimensional worldvolume, and massless scalar fields  $\Phi^I$  that are Goldstone bosons of the broken translation symmetry in the six transverse directions of the target space, due to the presence of the brane. The resulting effective theory (excluding the term with no fields) is a  $U(1)$  gauge theory living in the worldvolume of the D3-brane. This identifies the string coupling constant with the Yang-Mills coupling:  $4\pi g_s = g_{\text{YM}}^2$ . The gauge group is  $U(1)$  for a single D-brane, and for  $N$  D-branes, it gets enhanced from

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<sup>2</sup>It is generalized to asymptotic  $\text{AdS}$  spaces, such as the Pilch-Warner geometry.

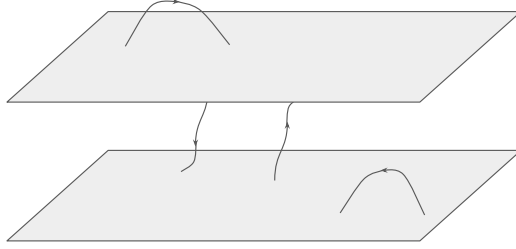


Figure 7.3. All the possible (oriented) open string excitations in two D-branes. If we have  $N$  non-coincident D-branes, then there are  $N^2$  possible open string configurations.

$U(1)^N$  to  $U(N)$  if they are coincident, see explanations in [43]. The effective coupling constant is then given by  $g_s N$ . The full spectrum also contains closed string modes and interaction terms, but by carefully taking the low energy limit, these sectors decouple and the interaction term vanishes.

On the other hand, D-branes are also solutions of supergravity field equations, as discussed in chapter 6. Hence, as gravitational objects, they can deform the surrounding spacetime, and closed strings can propagate there. For a stack of  $N$  coincident D3-branes, the metric is

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \eta_{ij} dx^i dx^j + \left(1 + \frac{L^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2). \quad (7.4)$$

Notice the two limits for this metric:

- $r \gg L$ : leading to the 10-dimensional flat Minkowski metric.
- $r \ll L$ : redefining  $z \equiv L^2/r$ , we get the Poincaré patch for  $AdS_5 \times S^5$ .

The characteristic length scale (6.12) in this case is

$$\frac{L^4}{l_s^4} = 4\pi g_s N. \quad (7.5)$$

By taking the low energy (Maldacena) limit:  $l_s \rightarrow 0$  while  $L/l_s$  fixed, we can decouple the two sectors, where the flat space part gets canceled exactly with the closed string modes in the open string analysis. We arrive now the conclusion of the AdS/CFT correspondence. However, instead of  $U(N)$ , we have  $SU(N)$ , because there is an overall  $U(1)$  phase that decouples, and this degree of freedom corresponds to a boundary field that cannot propagate into the bulk of  $AdS_5$ .

## 8. Holographic Wilson loops

As discussed in chapter 5, Wilson loops are important gauge invariant observables that can play the role of order parameters of the different phases of the gauge theory. Let us describe the basic idea that lead to find the holographic dual of Wilson loops, in the context of AdS/CFT correspondence. We then summarize the results obtained for the Pilch-Warner supergravity.

### 8.1 In the $AdS_5 \times S^5$ background

#### 8.1.1 Fundamental representation

In the fundamental representation, the (Maldacena-)Wilson loop (5.4) describes the phase of a trajectory of an external quark. A way to introduce the massive quark is to consider  $\mathcal{N} = 4$  SYM theory with all the fields in the adjoint representation of  $U(N+1)$  instead. Then, we break spontaneously the gauge group  $U(N+1) \rightarrow U(N) \times U(1)$ . In this way, the off-diagonal states of the scalars that were in the adjoint of  $U(N+1)$  become fundamental quarks and anti-quarks in  $U(N)$ , which are massive due to the Higgs mechanism. This is a useful picture because, in string theory, it is equivalent to separating one D-brane from the original stack of  $N+1$  coincident D3-branes. This produces excited open strings that stretch along the stack and the individual brane, with the mass proportional to the separation distance. Since we consider probe quarks (non-dynamical), the brane must be infinitely far away from the stack. The stretched strings not only source the gauge fields, but by pulling the  $N$  branes, they cause deformation on the branes that are described by the scalar fields in (5.4). The details of the derivation can be found in the appendix of [44].

Now, let us consider the dual gravitational picture. The stack gravitates and the near-horizon geometry is  $AdS_5 \times S^5$ . Then the position of the single D-brane lays on the conformal boundary of  $AdS_5$ , i.e.  $z \rightarrow 0$ , and sits at a point on  $S^5$ . The probe particle moves on the single D-brane. The Wilson loop operator is then dual to the partition function of fundamental strings in  $AdS_5 \times S^5$  whose worldsheets end on the same curve  $C$  that defines the Wilson loop at the boundary, [21]:

$$W(C) = \int_{C=\partial\Sigma} DX e^{-S_{\text{string}}[X]}. \quad (8.1)$$

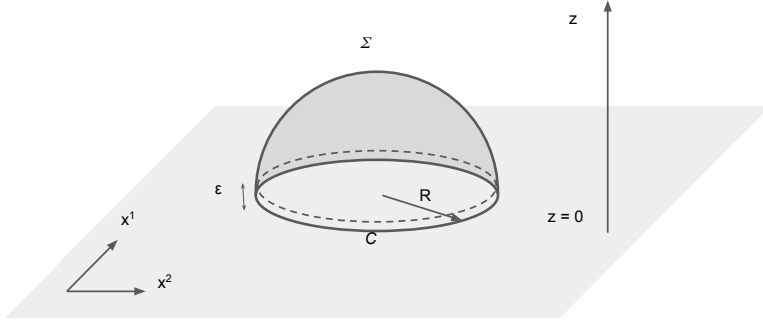


Figure 8.1. Circular Wilson loop as the minimal worldsheet area  $\Sigma$  drawn by string ending in the contour  $C$  at the boundary of  $AdS_5$ .

In the 't Hooft limit, which corresponds to classical supergravity limit, minimizing the bosonic string action is sufficient for the leading order, which is essentially the minimal worldsheet area. Nevertheless, the area is infinite in  $AdS$ , hence we must regularize it. For example, the minimal surface that is dual to the circular Wilson loop in the fundamental representation is parametrized as

$$z(r) = \sqrt{R^2 - r^2}, \quad r \in [0, R], \quad \phi \in [0, 2\pi]. \quad (8.2)$$

This result can be obtained either by minimizing the string Nambu-Goto action [44], or by exploiting the conformal symmetry, i.e. mapping the special conformal transformation of the straight line solution [45]. The induced worldsheet metric, that is the pullback of the background metric  $g$  (6.5) in polar coordinates for some  $x^i$ , is:

$$ds^2 = \frac{L^2}{z^2} ((1 + z'^2)dr^2 + r^2 d\phi^2). \quad (8.3)$$

Then the on-shell (Nambu-Goto) action gives:

$$S = T_{F1} \int \sqrt{\det P[g]} \quad (8.4)$$

$$= \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\phi \int_0^{\sqrt{R^2 - \varepsilon^2}} dr \frac{r}{z^2} \sqrt{1 + z'^2} \quad (8.5)$$

$$= \sqrt{\lambda} \left( \frac{R}{\varepsilon} - 1 \right). \quad (8.6)$$

The correct prescription to regularize the action is to set the boundary cut-off at  $z = \varepsilon$ , then, we remove the perimeter divergence [44]. This regularization scheme will be used for other cases of Wilson loop dual computations. The finite remnant in (8.6) matches with the leading order field theory result (5.12), i.e.

$$W_1 = e^{\sqrt{\lambda}}, \quad (N \rightarrow \infty \quad \text{and} \quad \lambda \rightarrow \infty). \quad (8.7)$$

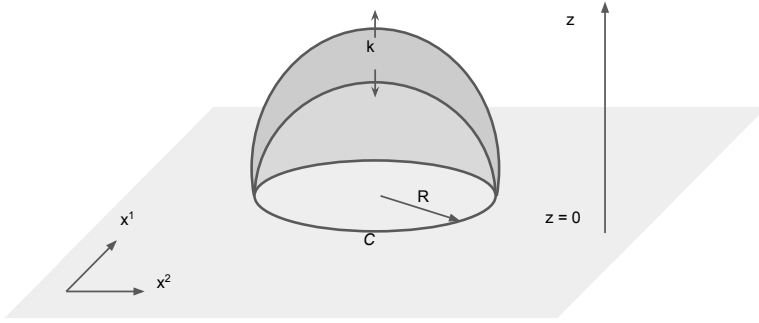


Figure 8.2. Circular Wilson loop of rank  $k$  as a D-brane with  $k$  string charges dissolved in it and the worldvolume ends in the contour  $C$  at the boundary of  $AdS_5$ .

In order to compute the subleading correction in (5.12), the fermionic contribution must be taken into account. The full string action to use would be the Green-Schwarz action. Although it is not fully known in curved background, its quadratic order is known [46], which suffices for 1-loop corrections. Its quartic order was also derived in [47]. Many efforts have been put into finding the subleading correction [48, 49, 50, 51, 52, 53] but no full matching has been achieved. The ambiguity in the path integral measure hindered this computation.

### 8.1.2 Higher rank representations

The lesson from the fundamental Wilson loop suggests that the string dual of rank- $k$  representations must be an object that carries  $k$  units of the string charge. Intuitively, if we consider a Wilson loop wrapped  $k$  times around the contour (the  $k$ -fundamental representation), we would expect the worldsheet to puff up, due to repulsive charges from multiple coincident fundamental strings. We see that the individual string action is no longer useful in this case. Instead, we can describe the new object in terms of D-branes with fundamental string charges dissolve in it. The worldvolume must also pinch off at the boundary of  $AdS_5$ , ending along the curve defined by the Wilson loop, see figure 8.1. Supersymmetry will then guide us which D-brane configurations are allowed.

More generally, [54] related supersymmetric Wilson loops of any representation of the gauge group to a stack of bulk D3-branes or a stack of bulk D5-branes, see figure 8.3. They proved the correspondence by explicitly integrating out the physics on the D-branes, which results to a half-BPS Wilson loop insertion in the desired representation in the  $\mathcal{N} = 4$  SYM path integral.

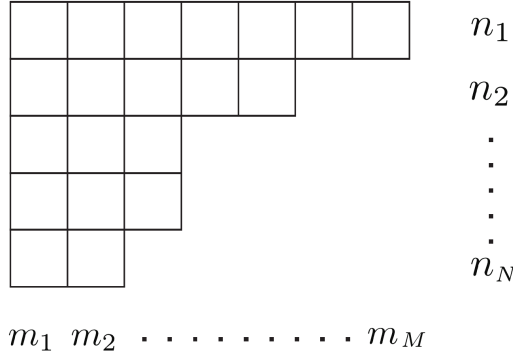


Figure 8.3. A generic Young table, taken from [54]. Row  $i$  corresponds to a D3-brane with  $n_i$  fundamental string charge dissolved in it. Column  $j$  corresponds to a D5-brane with  $m_j$  fundamental string charge dissolved in it. Thus, for symmetric (one row) and antisymmetric (one column) representations, their rank  $k$  corresponds to the string charges a D3-brane and a D5-brane contains.

Let us focus on a single D3-brane and a single D5-brane. Consider first the line element for  $AdS_5 \times S^5$  is this convenient form:

$$ds^2 = L^2 \left( du^2 + \cosh^2 u d\check{\Omega}^2 + \sinh^2 u d\Omega_2^2 + d\theta^2 + \sin^2 \theta d\Omega_4^2 \right), \quad (8.8)$$

where  $u \geq 0$ , and  $\pi \geq \theta \geq 0$ .  $d\check{\Omega}_n^2$  and  $d\Omega_n^2$  indicate the line element for  $AdS_n$  and  $S^n$ , respectively. Using the supersymmetry condition<sup>1</sup>

$$\Gamma \varepsilon = \varepsilon, \quad (8.9)$$

where  $\Gamma$  is the kappa symmetry<sup>2</sup> projector for the D-brane and  $\varepsilon$  is the Killing spinor of the background geometry, we can find the D-brane embeddings that preserve half of the original supersymmetries. Note that the supersymmetry condition, which gives first order differential equations, imply the D-brane equations of motions, which are of second order, up to some integration constants. The list below shows the half-BPS D-brane embeddings with their worldvolume geometries induced from the target space, and the worldvolume gauge fields  $F$  [55]:

$$\text{D3-brane:} \quad AdS_2 \times S^2, \quad u = \text{constant}, \quad \theta = 0 \quad (8.10)$$

$$\frac{1}{T_{F1}} F = L^2 \sqrt{1 + \kappa^2} e^0 e^1, \quad \kappa \equiv \sinh u \quad (8.11)$$

$$\text{D5-brane:} \quad AdS_2 \times S^4, \quad u = 0, \quad \theta = \text{constant} \quad (8.12)$$

$$\frac{1}{T_{F1}} F = L^2 \cos \theta e^0 e^1 \quad (8.13)$$

<sup>1</sup> May be up to a sign depending on the conventions.

<sup>2</sup> It is a fermionic local gauge symmetry that is present also in particles and strings. Its full definition can be found in Paper III.

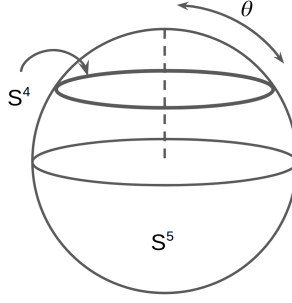


Figure 8.4.  $S^4$  that is  $\theta$  angle away from the pole of  $S^5$ . The angle depends on the amount of string charges  $k$  that D5-brane contains.

where  $e^0$  and  $e^1$  are the vielbeins<sup>3</sup> of  $d\check{\Omega}^2$ . We see that the D3-brane sits on a fixed point on the  $S^5$ , while the D5-brane sits on  $S^4$  that is  $\theta$  angle away from the pole of  $S^5$ , see figure 8.4.

The dual counterparts of the symmetric/antisymmetric representation in the leading order 't Hooft limit are the on-shell action DBI (6.13) + WZ (6.14) of the D3-brane/D5-brane configuration aforementioned:

$$\log W_k^+ = -S_{D3}, \quad \log W_k^- = -S_{D5}. \quad (8.14)$$

Moreover, we must ensure the string charge  $k$  constraint, which can be added as a Lagrange multiplier term to the D-brane action:

$$S_k = -k \int_{\Sigma} F \quad \Rightarrow \quad k = \frac{1}{T_{F1}} \frac{\delta S_{\text{DBI}}}{\delta F}. \quad (8.15)$$

The couplings in terms of  $\lambda$  and  $N$  are:

$$T_{F1} = \frac{\sqrt{\lambda}}{2\pi L^2}, \quad T_{D3} = \frac{N}{2\pi^2 L^4}, \quad T_{D5} = \frac{N\sqrt{\lambda}}{8\pi^4 L^6}, \quad (8.16)$$

which are derived from (6.15) using (7.1).

Finally, the regularized actions are [56, 55, 57]:

$$S_{D3} = -2N(\kappa\sqrt{1+\kappa^2} + \text{arcsinh } \kappa) \quad (8.17)$$

$$S_{D5} = -N \frac{2\sqrt{\lambda}}{3\pi} \sin^3 \theta, \quad (8.18)$$

with the  $\theta$  satisfying the string charge constraint (8.15) that gives (5.19), i.e.

$$\theta - \frac{1}{2} \sin 2\theta = \pi \frac{k}{N}. \quad (8.19)$$

<sup>3</sup> Vielbeins are defined by  $ds^2 = \eta_{mn} e^m e^n$ , which is often referred as the *local frame*. In terms of components:  $e^m = e_M^m dx^M$ , thus, we can relate them to the metric as  $g_{MN} = \eta_{mn} e_M^m e_N^n$ .

Thus the latitude angle  $\theta$  depends on the amount of string charges  $k$  dissolved in the D5-brane. In conclusion, there is an exact agreement with (5.20) and (5.18), according to (8.14).

## 8.2 In Pilch-Warner background

In  $\mathcal{N} = 2^*$  SYM on  $S^4$ , we took the decompactification limit for circular Wilson loops in order to have results on  $\mathbb{R}^4$ . This means that in the Pilch-Warner background, the contour is a straight line of length  $l \rightarrow \infty$ .

### 8.2.1 Fundamental representation

The minimal surface of a straight Wilson line is a wall, with the worldsheet coordinates  $(\tau, \sigma)$  induced from the Pilch-Warner geometry in the following way:

$$\tau = x^1, \quad \sigma = c. \quad (8.20)$$

The regularized on-shell action was computed in [17], which agrees with the leading order result in (5.22), that is:

$$\log W_1 = \sqrt{\lambda} M \frac{l}{2\pi}, \quad (l \rightarrow \infty). \quad (8.21)$$

The goal of Paper V was to compute the loop corrections to the minimal surface by using string perturbation around the above classical solution. There are two contributions, which turned out to be of equal weight in our case. The first one comes from the dilaton coupling to the worldsheet curvature called the Fradkin-Tseytlin term [58]:

$$S_{FT} = \frac{1}{4\pi} \int d^2\sigma \sqrt{h} R^{(2)} \Phi, \quad (8.22)$$

which is actually a bit controversial, with some authors arguing it is not needed [59, 60, 46]. Besides, it is zero for the familiar  $AdS_5 \times S^5$  background due to a vanishing dilaton. The other one comes from 1-loop stringy corrections, which are computed by expanding the Green-Schwarz action up to quadratic fluctuations [46]. These contribute as functional determinants, from generalizing the Gaussian integration formula (2.1) for the bosons, and the Grassmann integration for the fermions (2.14). Thus the semiclassical partition function is schematically of the form

$$W = e^{-S_{\text{cl}} - S_{\text{FT}}} \frac{\det F}{\sqrt{\det B}}, \quad (8.23)$$

where  $S_{\text{cl}}$  is the classical on-shell action in (8.21),  $B$  and  $F$  here just represent the bosonic and fermionic operators.



Supersymmetry simplifies our problem by cancelling exactly the sector of bosonic and fermionic operators that are asymptotically massless (far away from the boundary  $\sigma = 1$ ). The remaining sector is asymptotically massive, and the operators (after several manipulations) look enticingly similar:

$$H_B = \begin{pmatrix} 1 + \frac{A}{\sigma} & \mathcal{L} \\ \mathcal{L}^\dagger & -1 \end{pmatrix}, \quad H_F = \begin{pmatrix} -1 & \mathcal{L} \\ \mathcal{L}^\dagger & 1 + \frac{A}{\sigma} \end{pmatrix}, \quad (8.24)$$

with

$$\begin{aligned} \mathcal{L} &= A\sqrt{\sigma^2 - 1} \partial_\sigma + \frac{A(2\sigma^2 + 1)}{2\sigma\sqrt{\sigma^2 - 1}}, \\ \mathcal{L}^\dagger &= -A\sqrt{\sigma^2 - 1} \partial_\sigma - \frac{A(4\sigma^2 - 1)}{2\sigma\sqrt{\sigma^2 - 1}} + \frac{2}{\sqrt{\sigma^2 - 1}}. \end{aligned} \quad (8.25)$$

The final semiclassical partition function to compute is then:

$$W = e^{-S_{\text{cl}} - S_{\text{FT}}} \frac{\det^2(\partial_\tau - H_F)}{\det^2(\partial_\tau - H_B)}, \quad (8.26)$$

Instead of using the heat kernel technique or the Gelfand-Yaglom method, both commonly used in the computation of functional determinants, we used the phase-shift method from quantum mechanical scattering problems, see e.g. [61]. The method requires the operators to be asymptotically free, hence it works for (semi-)infinite intervals. Our operators (8.24) share the same asymptotics in large  $\sigma$ , and they are defined in a semi-infinite interval  $\sigma > 1$ , so the method applies. Note that for  $\tau$  variable, we do a usual Fourier transformation. The exponential of the ratio of the determinants of the operators in (8.26) can be written in terms of the phase-shift  $\delta(p)$  between the wave function and the asymptotic plane wave, see figure 8.5, which is

$$-2 \int_0^\infty \frac{dp}{2\pi} \frac{4p}{9\sqrt{\frac{4}{9}p^2 + 1}} (\delta_F^+(p) + \delta_F^-(p) - \delta_B^+(p) - \delta_B^-(p)) = -\frac{1}{4}, \quad (8.27)$$

where  $\pm$  distinguish the two eigenvectors (particles and holes) of the Dirac Hamiltonians (8.24). The last equality is backed by our numerics. Thus, together with the Fradkin-Tseytlin contribution, the total correction is  $-1/2$ . In conclusion, we do have a perfect matching with the field theory result (5.22). Moreover, the existence of Fradkin-Tseytlin term is necessary for this agreement.

### 8.2.2 Symmetric representation

As for higher rank Wilson loops, Paper III found the D3-brane embedding dual to the straight Wilson line of length  $l$  in symmetric representation, with

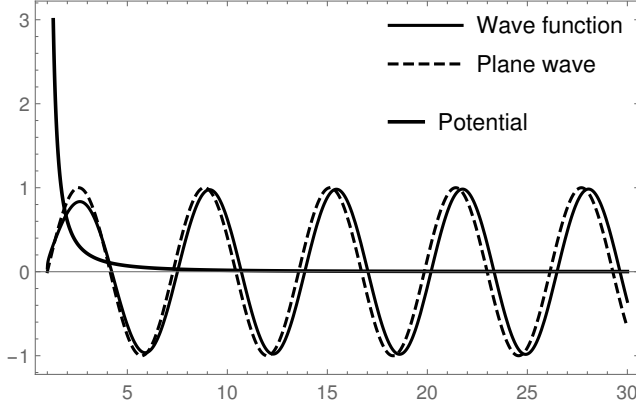


Figure 8.5. Phase-shift  $\delta(p)$  of the wave function in the presence of a typical potential wall that many of our operators show, versus the free wave, for certain momentum  $p$ .

the help of the supersymmetric condition (8.9). The worldvolume metric is induced from the deformed  $AdS$  part of (6.16) in string frame<sup>4</sup>:

$$ds^2 = \frac{AM^2L^2}{c^2 - 1} (dx^2 - \rho(c)^2 d\Omega_2^2) - L^2 \left( \frac{1}{A(c^2 - 1)^2} + \frac{AM^2\rho'(c)^2}{c^2 - 1} \right) dc^2, \quad (8.28)$$

where

$$\rho(c) = \kappa \sqrt{c^2 - 1}, \quad (8.29)$$

since the deformed sphere shrinks to  $\theta = \pi/2$  and  $\phi = 0$ . The worldvolume gauge field is given by

$$\frac{1}{T_{F1}} F(c) = -\frac{ML^2}{(c^2 - 1)^{3/2}} dx \wedge dc. \quad (8.30)$$

As expected, it also reduces to the analogous case in  $AdS_5 \times S^5$  close to the boundary.

The regularized on-shell action reduces to

$$S_{D3} = -\sqrt{\lambda} k M \frac{l}{2\pi}. \quad (8.31)$$

This agrees with the matrix model result (5.26) according to (8.14), only in the low rank limit

$$\kappa \ll Ml. \quad (8.32)$$

Furthermore, for  $k = 1$ , we recover the fundamental case (8.21). We can conclude that this D3-brane configuration cannot probe the entire matrix model region, and a full dual object is still to be understood.

<sup>4</sup> We changed to mostly minus signature in order to follow Paper III. The string metric  $g$  and the Einstein metric  $G$  are related by  $g = e^{-\frac{4\Phi}{D-2}} G$ , where  $\Phi$  is the dilaton.

For the antisymmetric case, it is technically challenging to find the D5-brane solution. So far, we have not succeeded. We do expect it though to match the leading order field theory result, because the matrix model result (5.25) is proportional to  $Ml$ , the same as in the D-brane action. It would be definitely interesting to compute the subleading order to determine the phase-transitions observed.

## 9. Conclusions

In this thesis, we explored a very special class of supersymmetric gauge theories whose path integrals are directly reducible to finite-dimensional integrals, i.e. 0-dimensional matrix models, by the means of the supersymmetric localization method. In particular, we studied the vacuum partition function and the expectation values of BPS Wilson loop observables of  $\mathcal{N} = 2^*$  SYM on  $S^4$ .

Our main motivation was to rigorously test the gauge/string duality conjecture in a slightly more generalized setting than the well-known case of  $\mathcal{N} = 4$  SYM and type IIB strings on  $AdS_5 \times S^5$ , by introducing a mass scale. The total access to the strong coupling phase of  $\mathcal{N} = 2^*$  SYM gives us a golden opportunity to explicitly test it against its presumed dual theory of type IIB strings on the Pilch-Warner background.

Computations are highly non-trivial, on the gauge theory side, as well as on the string theory side. From our work, we can conclude the following, supplemented with possible future work:

- Paper I and [18] showed that the decompactification limit of  $\mathcal{N} = 2^*$  SYM on  $S^4$  commutes with the strong coupling limit. This allows us to do holographic studies, since the dual of  $\mathcal{N} = 2^*$  SYM on  $\mathbb{R}^4$  is known. For the sake of completeness, it would be interesting to extend the holographic study for the theory on  $S^4$ .
- We used BPS Wilson loops in symmetric and antisymmetric representations of the gauge group to probe the strong-coupling phase of  $\mathcal{N} = 2^*$  SYM on  $S^4$ , especially in the decompactification limit, where infinite cusps appear. Paper II showed that these cusps induce phase transitions seen in the subleading order in strong coupling for these Wilson loops. In order to understand the nature of these phase transitions in the string theory side, Paper III computed the D3-brane configuration in the Pilch-Warner background dual to the symmetric Wilson loop. Only the leading result was derived, and it does not probe the full field theory result regime. From scaling arguments, we believe the D5-brane dual to the antisymmetric representation can be a promising case for a full matching, and hopefully even at 1-loop level, where phase transitions happen.
- The gauge/string duality for  $\mathcal{N} = 2^*$  case works with the fundamental Wilson line at least up to 1-loop quantum corrections, shown numerically by Paper V. Given the simplicity of the field theory prediction, an analytical result in the string theory side might be possible and desirable. Moreover, the matching requires the existence of the controversial

Fradkin-Tseytlin term in the string action. It would be clarifying to prove its existence independently, for example by requiring the beta function of the Green-Schwarz action to be zero.

- It is known that Wilson loops in the symmetric representation and the multiply-wrapped representation differ by exponentially suppressed terms for strong coupling [28]. Paper IV provided a non-planar correction to this statement for the  $\mathcal{N} = 4$  case, and helped to clarify a mismatch in an open problem concerning 1-loop corrections in the holographic dual computation [26, 62]. Solving this problem is definitely a further non-trivial quantum rigorous test, and we would gain a better understanding of divergences in stringy corrections. There is a similar open problem concerning 1-loop matching for circular fundamental Wilson loop in  $\mathcal{N} = 4$  SYM [48, 49, 50, 51, 52, 53].

Despite technical challenges, further studies on this  $\mathcal{N} = 2^*$  SYM case are necessary for better rigorous understanding of the generic gauge/string duality.

## 10. Sammanfattning

Gauge/sträng-dualiteten (eller den holografiska principen) är en anmärkningsvärd och djup förmodan som kommer från strängteorins mångfasetterade ramverket. Den relaterar starkt kopplade gauge teorier till svagt kopplad strängteori i en högre-dimensionell krökt geometri. Från en konceptuell synvinkel antyder den att gravitation, vilken är strängteorins lågenergigräns, och sålunda rumtiden, är ett emergent fenomen från en lägre-dimensionell gauge teori. Från en praktisk synvinkel möjliggör dualiteten att familjära verktyg inom gravitationsteorier kan ge svar till beryktat svåra frågeställningar inom gauge teori vid stark koppling. Eftersom att gauge teorier är så vanligt förekommande inom fysik – från beskrivningen av fundamentala interaktioner i partikelfysik till modellering av kondenserad materia-system – är ett sådant analytiskt verktyg potentiellt väldigt lovande. Exempel på ännu olösta problem som skulle kunna lösas genom dualiteten är confinement i kvantkromodynamik, eller att förstå faserna i en okonventionell supraleddare.

Brasklappen hos denna "för-bra-för-att-vara-sann"-dualitet är att väldigt lite är känt om den, förutom i vissa specialfall. Den bäst förstådda versionen är AdS/CFT-korrespondensen, vilken hävdar att  $\mathcal{N} = 4$  SYM, som är en gauge teori med supersymmetri och konform symmetri, är ekvivalent med typ IIB slutna strängteori i en 10-dimensionell  $AdS_5 \times S^5$ -bakgrund. Under de senaste 20 åren sedan Maldacena först formulerade detta antagande har många rigorösa tester utförts. Få experter inom fältet, om någon, betvivlar numer riktigheten av just denna korrespondens. Denna framgång inspirerar till generaliseringar till mindre symmetriska teorier. När allt kommer omkring så är  $\mathcal{N} = 4$  SYM så begränsad av sina symmetrier att det är troligt att den är tillfullo kvantintegrabel.

Nästa naturliga steg är att lägga till massiva tillstånd till  $\mathcal{N} = 4$  SYM. Den unika massdeformation vilken bevarar hälften av supersymmetrin men bryter den konforma symmetrin är den så kallade  $\mathcal{N} = 2^*$  SYM. Det är denna teori som är studieämnet för denna avhandling. Den är tillräckligt supersymmetrisk för att den kraftfulla metoden supersymmetrisk lokalisering skall kunna användas, genom vilken  $\mathcal{N} = 2^*$ -teorins partitionsfunktion över en sfär reduceras till en effektiv matrismodell. På så vis är beräkningskomplexiteten kraftigt reducerad så att standardmetoder för slumpmässiga matriser kan användas. Å andra sidan orsakar de introducerade massskalorna icke-triviala fasstrukturer i vakuumfältkonfigurationerna, i gränsen av platt rum. Detta saknar uppenbarligen motstycke i fallet med  $\mathcal{N} = 4$  SYM då den senare är masslös. Att förstå denna nya egenskap, både inom fältteorin och framför allt i den duala gravitationsteorin, är ett av avhandlingens huvudmotiv.

Gravitationsdualen till  $\mathcal{N} = 2^*$  SYM utan krökning är den så kallade Pilch-Warner-bakgrunden. Denna geometri är en deformerad  $AdS_5 \times S^5$ , och den reduceras asymptotiskt till den senare nära den konforma randen. Alla supergravitationsfält är icke-triviala, till skillnad från den enklaste situationen i  $AdS_5 \times S^5$ -bakgrunden. Tyvärr försvårar detta beräkningar betydligt. Icke desto mindre har vi lyckats att rigoröst och framgångsrikt testa denna icke-konforma version av AdS/CFT-korrespondensen för vissa observabler.

Observablerna som vi studerar är supersymmetriska Wilson-loopar i olika representationer av gaugegruppen. För den fundamentala Wilson-loopen, dual till strängdiskpartitionfunktionen, lyckades vi med överensstämmelse jämföra fältteoriförutsägelsen genom att beräkna strängkorrektionerna till den minimala arean. Detta är det första testet på kvantnivå som har gjorts i denna icke-konforma sättning. Å andra sidan bevisade vi att Wilson-loopar med representationer av högre rang uppvisar de fasövergångarna som observerats i vakuumpartitionfunktionen vid andra ordningens stark koppling. Deras duala objekt är supersymmetriska, elektriskt laddade D-bran. Vi lyckades finna D-branskonfigurationen som är dual till Wilson loopen i den symmetriska representationen och visade att korrespondensen håller till första ordningen, åtminstone i en specifik gräns av det fältteoretiska resultatet. Loopkorrektionerna kan därmed inte ha de nämnda fasövergångarna. Det senare kvarstår som ett öppet problem och vi förväntar oss fortsatt arbete kring den anti-symmetriska representationens D-branskonfiguration för att vidare belysa detta.

# 11. Acknowledgement

First of all, I would like to thank Kostya for being my supervisor, teacher and a model physicist. Much of the success of this thesis is due to him, as well as to the hard-work and perseverance of my other collaborators: Amit, Daniel and Jamie. I very much appreciate having you to discuss our projects and work together.

I also thank Kostya and Amit for proofreading parts of my manuscript. Thanks goes to Erik for the Swedish translation of the summary.

I thank Nordita for a nice working environment, and fellow students Erik, Daniel and Janina, who literally added light, music and chocolate to the office.

I would like to thank the GATIS network for funding my first three years of research and training my soft skills.

Last but not least, let me finally acknowledge my dearest Stephane, for his more-than-a-thousand times speedup of the numerical code used in my first paper, and his careful proofreading of many of my texts, including parts of this thesis.



# References

- [1] R. J. Szabo, “Equivariant localization of path integrals,” [arXiv:hep-th/9608068](#) [hep-th].
- [2] V. Pestun *et al.*, “Localization techniques in quantum field theories,” [arXiv:1608.02952](#) [hep-th].
- [3] L. Brink, J. H. Schwarz, and J. Scherk, “Supersymmetric Yang-Mills Theories,” *Nucl. Phys.* **B121** (1977) 77–92.
- [4] S. Mandelstam, “Light-cone superspace and the ultraviolet finiteness of the  $n=4$  model,” *Nuclear Physics B* **213** no. 1, (1983) 149 – 168. <http://www.sciencedirect.com/science/article/pii/0550321383901797>.
- [5] L. Brink, O. Lindgren, and B. E. Nilsson, “The ultra-violet finiteness of the  $n=4$  yang-mills theory,” *Physics Letters B* **123** no. 5, (1983) 323 – 328. <http://www.sciencedirect.com/science/article/pii/0370269383912108>.
- [6] J. A. Minahan, “Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in  $N=4$  Super Yang-Mills,” *Lett. Math. Phys.* **99** (2012) 33–58, [arXiv:1012.3983](#) [hep-th].
- [7] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” *Commun. Math. Phys.* **313** (2012) 71–129, [arXiv:0712.2824](#) [hep-th].
- [8] G. Festuccia and N. Seiberg, “Rigid Supersymmetric Theories in Curved Superspace,” *JHEP* **06** (2011) 114, [arXiv:1105.0689](#) [hep-th].
- [9] J. G. Russo and K. Zarembo, “Evidence for Large- $N$  Phase Transitions in  $N=2^*$  Theory,” *JHEP* **04** (2013) 065, [arXiv:1302.6968](#) [hep-th].
- [10] A. Buchel, A. W. Peet, and J. Polchinski, “Gauge dual and noncommutative extension of an  $N=2$  supergravity solution,” *Phys. Rev.* **D63** (2001) 044009, [arXiv:hep-th/0008076](#) [hep-th].
- [11] N. Bobev, H. Elvang, D. Z. Freedman, and S. S. Pufu, “Holography for  $N = 2^*$  on  $S^4$ ,” *JHEP* **1407** (2014) 001, [arXiv:1311.1508](#) [hep-th].
- [12] J. Scherk and J. H. Schwarz, “How to Get Masses from Extra Dimensions,” *Nucl. Phys.* **B153** (1979) 61–88.
- [13] T. Okuda and V. Pestun, “On the instantons and the hypermultiplet mass of  $N=2^*$  super Yang-Mills on  $S^4$ ,” *JHEP* **03** (2012) 017, [arXiv:1004.1222](#) [hep-th].
- [14] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” *Adv. Theor. Math. Phys.* **7** no. 5, (2003) 831–864, [arXiv:hep-th/0206161](#) [hep-th].
- [15] J. G. Russo and K. Zarembo, “Massive  $N=2$  Gauge Theories at Large  $N$ ,” *JHEP* **11** (2013) 130, [arXiv:1309.1004](#) [hep-th].
- [16] F. J. Dyson, “Statistical theory of the energy levels of complex systems. I,” *J. Math. Phys.* **3** (1962) 140–156.

- [17] A. Buchel, J. G. Russo, and K. Zarembo, “Rigorous Test of Non-conformal Holography: Wilson Loops in  $N=2^*$  Theory,” *JHEP* **1303** (2013) 062, arXiv:1301.1597 [hep-th].
- [18] K. Zarembo, “Strong-Coupling Phases of Planar  $N=2^*$  Super-Yang-Mills Theory,” *Theor.Math.Phys.* **181** no. 3, (2014) 1522–1530, arXiv:1410.6114 [hep-th].
- [19] L. Anderson and K. Zarembo, “Quantum Phase Transitions in Mass-Deformed ABJM Matrix Model,” *JHEP* **1409** (2014) 021, arXiv:1406.3366 [hep-th].
- [20] A. Nedelin, “Phase transitions in 5D super Yang-Mills theory,” *JHEP* **07** (2015) 004, arXiv:1502.07275 [hep-th].
- [21] J. M. Maldacena, “Wilson loops in large  $N$  field theories,” *Phys. Rev. Lett.* **80** (1998) 4859–4862, arXiv:hep-th/9803002 [hep-th].
- [22] N. Drukker and D. J. Gross, “An Exact prediction of  $N=4$  SUSYM theory for string theory,” *J. Math. Phys.* **42** (2001) 2896–2914, arXiv:hep-th/0010274 [hep-th].
- [23] J. K. Erickson, G. W. Semenoff, and K. Zarembo, “Wilson loops in  $N=4$  supersymmetric Yang-Mills theory,” *Nucl. Phys.* **B582** (2000) 155–175, arXiv:hep-th/0003055 [hep-th].
- [24] B. Fiol and G. Torrents, “Exact results for Wilson loops in arbitrary representations,” *JHEP* **1401** (2014) 020, arXiv:1311.2058 [hep-th].
- [25] S. A. Hartnoll and S. P. Kumar, “Higher rank Wilson loops from a matrix model,” *JHEP* **08** (2006) 026, arXiv:hep-th/0605027 [hep-th].
- [26] A. Faraggi, J. T. Liu, L. A. Pando Zayas, and G. Zhang, “One-loop structure of higher rank Wilson loops in AdS/CFT,” *Phys.Lett.* **B740** (2015) 218–221, arXiv:1409.3187 [hep-th].
- [27] S. Kawamoto, T. Kuroki, and A. Miwa, “Boundary condition for D-brane from Wilson loop, and gravitational interpretation of eigenvalue in matrix model in AdS/CFT correspondence,” *Phys.Rev.* **D79** (2009) 126010, arXiv:0812.4229 [hep-th].
- [28] S. Yamaguchi, “Semi-classical open string corrections and symmetric Wilson loops,” *JHEP* **0706** (2007) 073, arXiv:hep-th/0701052 [hep-th].
- [29] M. Horikoshi and K. Okuyama, “ $\alpha'$ -expansion of Anti-Symmetric Wilson Loops in  $\mathcal{N} = 4$  SYM from Fermi Gas,” arXiv:1607.01498 [hep-th].
- [30] K. Pilch and N. P. Warner, “ $N=2$  supersymmetric RG flows and the IIB dilaton,” *Nucl. Phys.* **B594** (2001) 209–228, arXiv:hep-th/0004063 [hep-th].
- [31] E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS / CFT correspondence,” in *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*, pp. 3–158. 2002. arXiv:hep-th/0201253 [hep-th].
- [32] E. Cremmer, B. Julia, and J. Scherk, “Supergravity Theory in Eleven-Dimensions,” *Phys. Lett.* **B76** (1978) 409–412.
- [33] J. H. Schwarz, “Covariant Field Equations of Chiral  $N=2$   $D=10$  Supergravity,” *Nucl. Phys.* **B226** (1983) 269.
- [34] K. Skenderis and M. Taylor, “Branes in AdS and p p wave space-times,” *JHEP* **06** (2002) 025, arXiv:hep-th/0204054 [hep-th].
- [35] K. Pilch and N. P. Warner, “Generalizing the  $N=2$  supersymmetric RG flow solution of IIB supergravity,” *Nucl. Phys.* **B675** (2003) 99–121, arXiv:hep-th/0306098 [hep-th].

- [36] M. Ammon and J. Erdmenger, *Gauge/gravity duality*. Cambridge Univ. Pr., Cambridge, UK, 2015.  
<http://www.cambridge.org/de/academic/subjects/physics/theoretical-physics-and-mathematical-physics/gaugegravity-duality-foundations-and-applications>.
- [37] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [arXiv:hep-th/9711200 \[hep-th\]](#). [*Adv. Theor. Math. Phys.* 2,231(1998)].
- [38] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” *Phys. Lett.* **B428** (1998) 105–114, [arXiv:hep-th/9802109 \[hep-th\]](#).
- [39] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [arXiv:hep-th/9802150 \[hep-th\]](#).
- [40] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” *JHEP* **10** (2008) 091, [arXiv:0806.1218 \[hep-th\]](#).
- [41] G. ’t Hooft, “A Planar Diagram Theory for Strong Interactions,” *Nucl. Phys.* **B72** (1974) 461.
- [42] K. Skenderis, “Lecture notes on holographic renormalization,” *Class. Quant. Grav.* **19** (2002) 5849–5876, [arXiv:hep-th/0209067 \[hep-th\]](#).
- [43] L. Wulff, *Strings, boundary fermions and coincident D-branes*. PhD thesis, Stockholm U., 2007. [arXiv:hep-th/0701129 \[hep-th\]](#).
- [44] N. Drukker, D. J. Gross, and H. Ooguri, “Wilson loops and minimal surfaces,” *Phys. Rev.* **D60** (1999) 125006, [arXiv:hep-th/9904191 \[hep-th\]](#).
- [45] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, “The Operator product expansion for Wilson loops and surfaces in the large N limit,” *Phys. Rev.* **D59** (1999) 105023, [arXiv:hep-th/9809188 \[hep-th\]](#).
- [46] M. Cvetič, H. Lu, C. N. Pope, and K. S. Stelle, “T duality in the Green-Schwarz formalism, and the massless / massive IIA duality map,” *Nucl. Phys.* **B573** (2000) 149–176, [arXiv:hep-th/9907202 \[hep-th\]](#).
- [47] L. Wulff, “The type II superstring to order  $\theta^4$ ,” *JHEP* **07** (2013) 123, [arXiv:1304.6422 \[hep-th\]](#).
- [48] M. Kruczenski and A. Tirziu, “Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling,” *JHEP* **05** (2008) 064, [arXiv:0803.0315 \[hep-th\]](#).
- [49] C. Kristjansen and Y. Makeenko, “More about One-Loop Effective Action of Open Superstring in  $AdS_5 \times S^5$ ,” *JHEP* **09** (2012) 053, [arXiv:1206.5660 \[hep-th\]](#).
- [50] R. Bergamin and A. A. Tseytlin, “Heat kernels on cone of  $AdS_2$  and  $k$ -wound circular Wilson loop in  $AdS_5 \times S^5$  superstring,” [arXiv:1510.06894 \[hep-th\]](#).
- [51] V. Forini, V. Giangreco M. Puletti, L. Griguolo, D. Seminara, and E. Vescovi, “Precision calculation of 1/4-BPS Wilson loops in  $AdS_5 \times S^5$ ,” *JHEP* **02** (2016) 105, [arXiv:1512.00841 \[hep-th\]](#).
- [52] A. Faraggi, L. A. Pando Zayas, G. A. Silva, and D. Trancanelli, “Toward precision holography with supersymmetric Wilson loops,” *JHEP* **04** (2016) 053, [arXiv:1601.04708 \[hep-th\]](#).

- [53] V. Forini, A. A. Tseytlin, and E. Vescovi, “Perturbative computation of string one-loop corrections to Wilson loop minimal surfaces in  $\text{AdS}_5 \times S^5$ ,” *JHEP* **03** (2017) 003, [arXiv:1702.02164 \[hep-th\]](#).
- [54] J. Gomis and F. Passerini, “Holographic Wilson Loops,” *JHEP* **08** (2006) 074, [arXiv:hep-th/0604007 \[hep-th\]](#).
- [55] S. Yamaguchi, “Wilson loops of anti-symmetric representation and D5-branes,” *JHEP* **05** (2006) 037, [arXiv:hep-th/0603208 \[hep-th\]](#).
- [56] N. Drukker and B. Fiol, “All-genus calculation of Wilson loops using D-branes,” *JHEP* **0502** (2005) 010, [arXiv:hep-th/0501109 \[hep-th\]](#).
- [57] K. Zarembo, “Localization and AdS/CFT Correspondence,” 2016. [arXiv:1608.02963 \[hep-th\]](#).
- [58] E. S. Fradkin and A. A. Tseytlin, “One Loop Infinities in Dimensionally Reduced Supergravities,” *Phys. Lett.* **B137** (1984) 357–362.
- [59] M. Grisaru, H. Nishino, and D. Zanon, “ $\beta$ -function approach to the green-schwarz superstring,” *Physics Letters B* **206** no. 4, (1988) 625 – 630. <http://www.sciencedirect.com/science/article/pii/0370269388907083>.
- [60] M. Grisaru, P. Howe, L. Mezincescu, B. Nilsson, and P. Townsend, “ $N = 2$  superstrings in a supergravity background,” *Physics Letters B* **162** no. 1, (1985) 116 – 120. <http://www.sciencedirect.com/science/article/pii/0370269385910718>.
- [61] R. F. Dashen, B. Hasslacher, and A. Neveu, “Nonperturbative methods and extended-hadron models in field theory. ii. two-dimensional models and extended hadrons,” *Phys. Rev. D* **10** (Dec, 1974) 4130–4138. <https://link.aps.org/doi/10.1103/PhysRevD.10.4130>.
- [62] E. I. Buchbinder and A. A. Tseytlin, “ $1/N$  correction in the D3-brane description of a circular Wilson loop at strong coupling,” *Phys. Rev.* **D89** no. 12, (2014) 126008, [arXiv:1404.4952 \[hep-th\]](#).