Time series Analysis
Lecture 3 Time Series Models

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Outline

回归模型 自回归模型

- Regression and Autoregression
- Autoregressive (AR) models自回归模型
- Moving average (MA) models移动平均模型
- Autoregressive moving average (ARMA) models;
  自回归滑动平均模型

- Regressive model:

$$y_i = \alpha + \delta x_i + e_i \tag{1}$$

- Autoregressive (AR) model

自回归模型和历史数据中的时间有关

$$y_t = \alpha + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + e_t \tag{2}$$

- Distributed Lag (DL) model

分布滞后模型（DL）是专门处理“自变量历史值对因变量有持续影响”**的工具**

$$y_t = \alpha + \delta_0 x_t + \delta_1 x_{t-1} + \cdots + \delta_q x_{t-q} + e_t \tag{3}$$

- Autoregressive Distributed Lag (ADL) model:

$$y_t = \alpha + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \delta_0 x_t + \delta_1 x_{t-1} + \cdots + \delta_q x_{t-q} + e_t \tag{4}$$

自回归分布滞后模型（ADL）是时间序列分析中处理“因变量自身惯性 + 外部变量滞后影响”**的强大工具**

AR(1)

- $y_t = \phi_0 + \phi_1 y_{t-1} + e_t,$ where $e_t \sim i.i.d.(0, \sigma^2)$.
- A compact form: $(1 - \phi_1 B) y_t = \phi_0 + e_t$.
- B: back-shift operator:

$$\begin{aligned} By_t &= y_{t-1}, \\ B^k y_t &= y_{t-k}, \\ B^0 y_t &= y_t, \\ B^k c &= c. \end{aligned}$$

- Characteristic equation: $\phi(x) = 1 - \phi_1 x = 0$.
- Stationary condition: the root of $1 - \phi_1 x = 0$ is outside of the unit circle, then AR(1) is stationary iff $|\phi_1| < 1$.

AR(1)

Given the stationarity of y_t :

- Mean: $\mu = Ey_t = \dfrac{\phi_0}{1-\phi_1}$,
- Variance:

$$\begin{aligned} E[(y_t-\mu)^2] &= \phi_1^2 E[(y_{t-1}-\mu)^2] + \sigma^2, \\ \gamma_0 &= \phi_1^2\gamma_0 + \sigma^2 \Rightarrow \gamma_0 = \frac{\sigma^2}{1-\phi_1^2}. \end{aligned}$$

- The autocovariance function (ACVF) at lag j:

$$\begin{aligned} \gamma_j &= Cov(y_{t-j}, y_t) \\ &= \phi_1\gamma_{j-1} = \frac{\phi_1^j\sigma^2}{1-\phi_1^2}, \text{ for all } j \geq 1. \end{aligned}$$

- The autocorrelation function (ACF) at lag j:

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \phi_1^j.$$

ACF of AR(1)

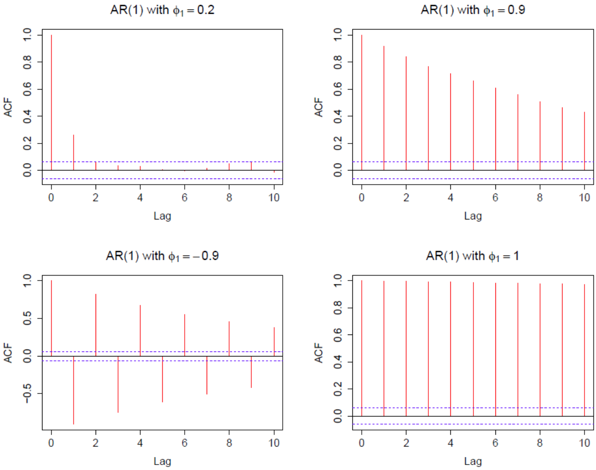


Figure 1: ACF of AR(1) processes

$|\phi_1| < 1$

- Case 1: Stationary.
  When $|\phi_1| < 1$, the series $\{r_t\}$ exhibits mean-reverting behavior.
  Let $y_t = y_t - \mu$, then $y_t = \phi_1 y_{t-1} + e_t$ and
  $\Delta y_t = y_t - y_{t-1} = (\phi_1 - 1)y_{t-1} + e_t$,

$$\begin{cases} E(\Delta y_t | I_{t-1}) < 0, & \text{if } y_{t-1} > 0; \\ E(\Delta y_t | I_{t-1}) > 0, & \text{if } y_{t-1} < 0. \end{cases}$$

$y_t = 0.5y_{t-1} + e_t$

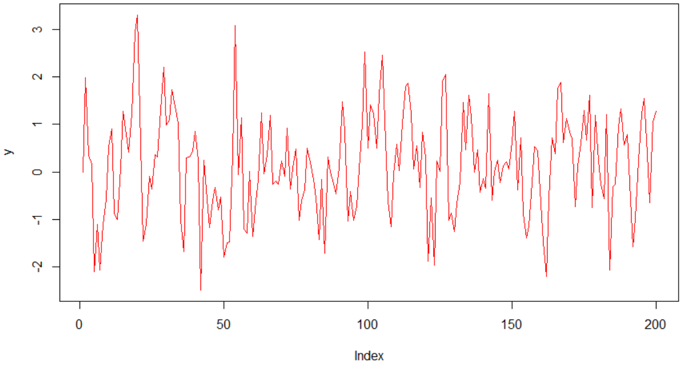


Figure 2: Stationary AR(1)

- Case 2: Non-stationary.
  $\phi_1 = 1$, it is not covariance stationary.
  - Assuming $\phi_0 = 0$, AR(1) model can be rewritten as the random walk process:
  $$y_t = e_t + e_{t-1} + \cdots + e_1 + y_0.$$
  - $\gamma_{0,t} \equiv E(y_t - y_0)^2 = t\sigma^2$
    $\gamma_{j,t} \equiv E(y_t - y_0)(y_{t-j} - r_0) = (t-j)\sigma^2$ and
    $\rho_{j,t} = (t-j)/t$ for all $k \geq 0$.
- Similarly, when $\phi_1 = -1$, it is not covariance stationary.

$y_t = y_{t-1} + e_t$

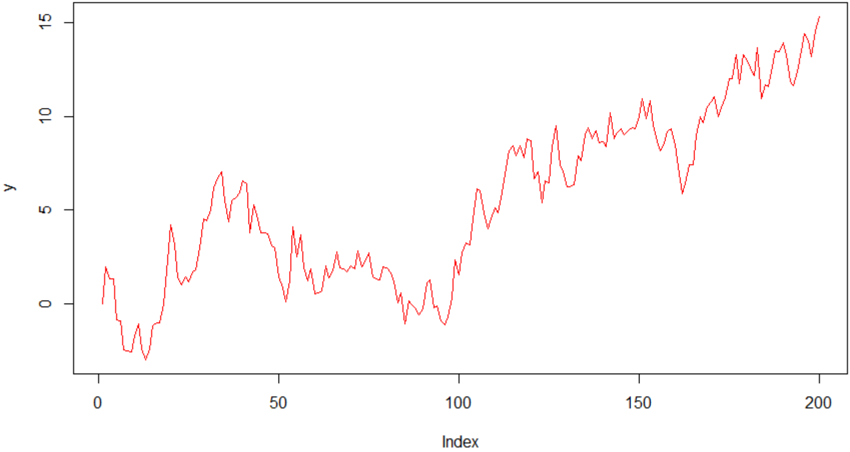


Figure 3: Unit-root nonstationary

$|\phi_1| > 1$

- Case 3: Explosive. When $|\phi_1| > 1$, e.g. $\phi_1 = 3$ and assume $\phi_0 = 0$,

$$y_t = e_t + 3e_{t-1} + 3^2 e_{t-2} + \cdots + 3^{t-1} e_1 + 3^t y_0.$$

In this case, r_t is called the explosive process, in the sense that r_t diverges to ∞.

$y_t = 1.2 * y_{t-1} + e_t$

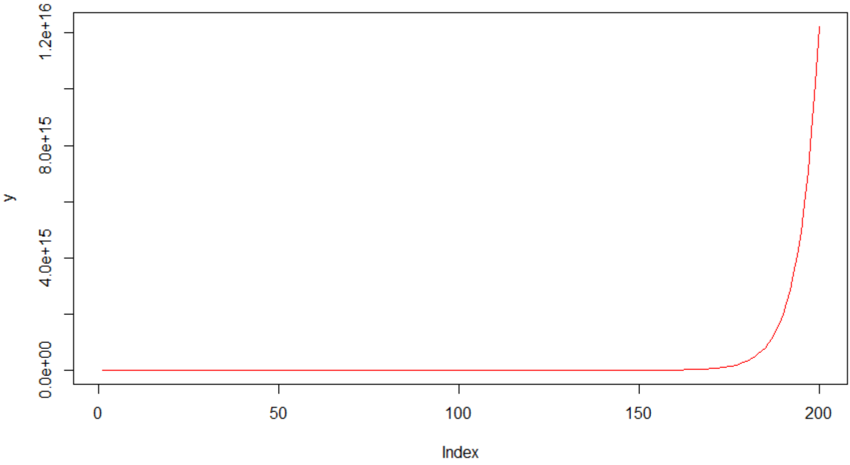


Figure 4: Explosive AR(1)

- $y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + e_t,$ where $e_t \sim i.i.d.(0, \sigma^2)$.
- $(1 - \phi_1 B - \phi_2 B^2) y_t = \phi_0 + e_t$.
- Characteristic equation: $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = 0$.
- Stationary condition: the roots of $1 - \phi_1 x - \phi_2 x^2 = 0$ are outside of the unit circle.
- Stationarity condition for AR(2) model is equivalent to

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1. \end{cases}$$

Stationary region for AR(2) model

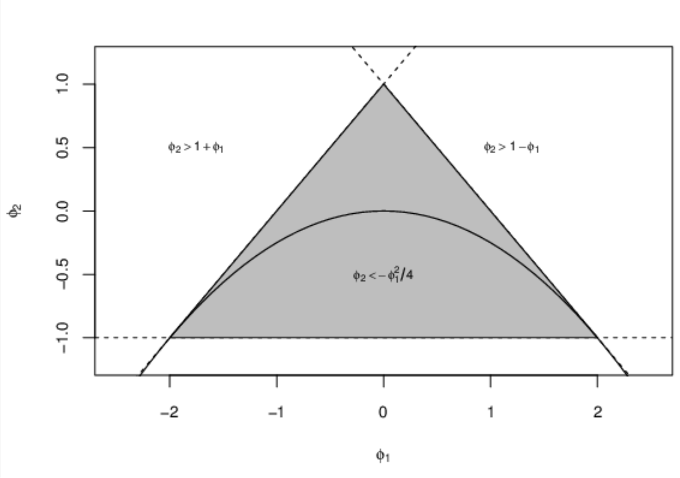


Figure 5: Stationary region for AR(2) model.

Mean of AR(2) process

Given y_t is a stationary AR(2) process

- Mean: $E(y_t) = \phi_0 + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2}) + E(e_t)$,

$$\mu = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

- $$y_t - \mu = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + e_t.$$

- ACVF: Multiplying $(Y_{t-j} - \mu)$ on both sides and taking expectation,

$$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2}, \text{ for all } j = 1, 2, 3, \ldots.$$

  - $j = 1$,

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_{-1} = \phi_1 \gamma_0 + \phi_2 \gamma_1.$$

  - $j = 2$,

$$\gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0;$$

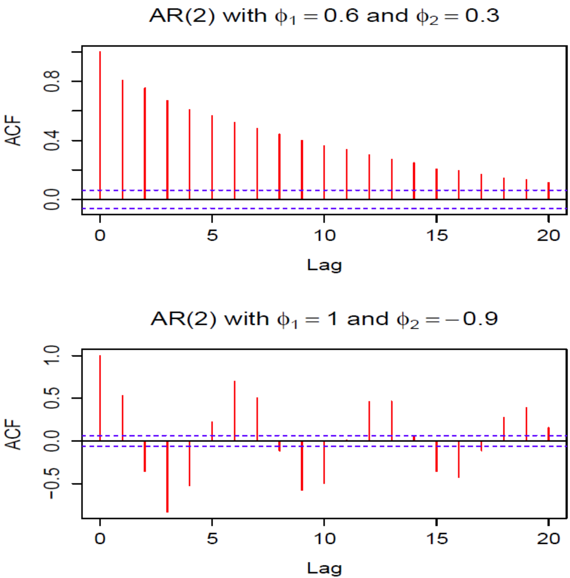
Variance:

$$\begin{aligned}
\gamma_0 &= E(y_t - \mu)^2 \\
&= \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2 \\
&= \phi_1\rho_1\gamma_0 + \phi_2\rho_2\gamma_0 + \sigma^2; \\
&= \frac{(1-\phi_2)\sigma^2}{(1+\phi_2)((1-\phi_2)^2 - \phi_1^2)}.
\end{aligned}$$

ACF:

$$\begin{aligned}
\rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{\phi_1}{1-\phi_2}; \\
\rho_2 &= \phi_1\rho_1 + \phi_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2; \\
\rho_j &= \phi_1\rho_{j-1} + \phi_2\rho_{j-2}, \quad \text{for} \quad j \geq 3.
\end{aligned}$$

ACF pattens of AR(2)



- $y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + e_t$
- $\phi(B) y_t = \phi_0 + e_t$, where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$.
- Condition for Stationarity: the roots of $\phi_p(z) = 0$ lie outside the unit circle.
- ACF of AR(p) model:

$$\gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_p \gamma_{k-p}, \quad k > 0.$$

$$\rho_k = \phi_1 \rho_{k-1} + \cdots + \phi_p \rho_{k-p}, \quad k > 0.$$

(the so-called Yule-Walker equations of ρ_k.)

Cont'd for AR(p)

- Solve the following sets of equations to obtain $\rho_1, \cdots, \rho_p$.

$$\begin{cases} \rho_1 - \phi_1 \rho_0 - \cdots - \phi_p \rho_{p-1} = 0, \\ \cdots\cdots \\ \rho_p - \phi_1 \rho_{p-1} - \cdots - \phi_p \rho_0 = 0. \end{cases}$$

- Using the following recursive equations to get calculate ρ_k when $k \geq p+1$.

$$\rho_k = \phi_1 \rho_{k-1} + \cdots + \phi_p \rho_{k-p}.$$

PACF for AR(p) Processes

- The PACF at lag k (denoted by π_k) measures the correlation between y_t and y_{t-k} regardless of their linear relationship with the intermediate variables $\{y_{t-1}, \cdots, y_{t-k+1}\}$.
- If y_t is a normally distributed times series, then

$$\pi_k = Corr(y_t, y_{t-k} | y_{t-1}, \cdots, y_{t-k+1})$$

- This definition is equivalent to say that

$$\begin{aligned}\pi_k &= Corr(y_t - E(y_t | y_{t+1}, \ldots, y_{t+k-1}), y_{t+k} - E(y_{t+k} | y_{t+1}, \ldots, y_{t+k-1}) \\ &= Corr(y_t - \hat{y}_t, y_{t+k} - \hat{y}_{t+k}).\end{aligned}$$

- On linear regression theory, $\hat{y}_t$ and $\hat{y}_{t+k}$ are the best linear estimates of y_t and y_{t+k} (respectively) based on the values of $y_{t+1}, \ldots, y_{t+k-1}$.

Computing the PACF

- According to the above definitions, the partial autocorrelation coefficient of order k is computed as the least squares estimator of the coefficient ϕ_{kk} in

$$y_t = \phi_{k1} y_{t-1} + \cdots + \underbrace{\phi_{kk}}_{\pi_k} y_{t-k} + e_t \tag{5}$$

where y_t is assumed to be zero mean.

- We will only include a further lagged variable y_{t-k} in the model for y_t if y_{t-k} makes a genuine and additional contribution to y_t in addition to those from $y_{t-1}, \cdots, y_{t-k+1}$.

How to obtain ϕ_{kk}?

- Multiplying both sides of Eq. (5) by y_{t-j} ($j > 0$), and taking expectation, we can get

$$\gamma_j - \phi_{k1}\gamma_{j-1} - \phi_{k2}\gamma_{j-2} - \cdots - \phi_{kk}\gamma_{j-k} = 0.$$

Consider the above (moment) equations jointly for $j = 1, 2, \ldots, k$, we have

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_k \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{k-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1} & \gamma_{k-2} & \cdots & \gamma_0 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix}.$$

How to obtain ϕ_{kk}?

- Similarly, we can replace all γs with ρs, the corresponding result is

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix} = \begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_0 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_0 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix}.$$

By the Cramer rule, we have

$$\phi_{kk} = \frac{\det \begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & \rho_0 & \cdots & \rho_{k-3} & \rho_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & \rho_k \end{bmatrix}}{\det \begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & \rho_0 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & \rho_0 \end{bmatrix}}$$

The ACF and PACF of AR Models

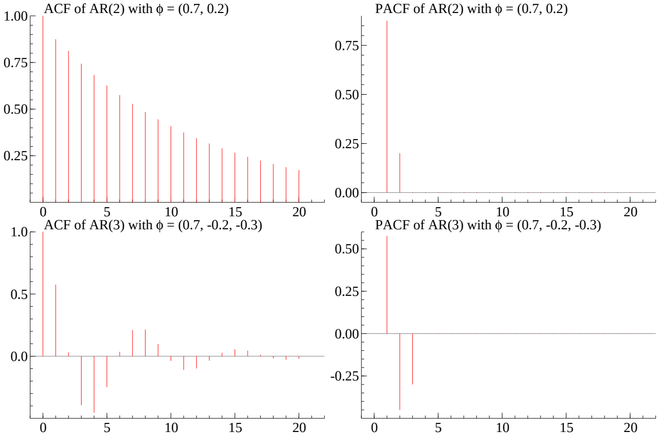


Figure 7: The PACF of AR(p) Models

Moving average (MA) Model

- MA(q):

$$y_t = \theta_0 + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q},$$

  where $q \geq 0$ is a finite integer and $\{e_t\} \sim i.i.d.(0, \sigma_e^2)$.
- MA(0) is actually a i.i.d. sequence if $\theta_0 = 0$.
- First proposed by E. Slutsky in 1927 to explain some cycle phenomena in economic data etc.
- In some textbooks, they use the following definition equation:

$$y_t = \theta_0 + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q},$$

MA(1) process

- $y_t = \theta_0 + e_t - \theta_1 e_{t-1}$.
- $\mu = E(y_t) = \theta_0$
- Variance $\gamma_0 = Var(y_t) = \sigma_e^2(1+\theta_1^2)$.
- ACVF at lag 1 is

$$\begin{aligned}\gamma_1 &= Cov(y_t, y_{t-1}) \\ &= Cov(e_t - \theta_1 e_{t-1}, e_{t-1} - \theta_2 e_{t-2}) \\ &= Cov(-\theta_1 e_{t-1}, e_{t-1}) = -\theta_1 \sigma_e^2,\end{aligned}$$

- ACF at lag 1 is

$$\rho_1 = \frac{-\theta_1}{1+\theta_1^2}.$$

MA(1) process

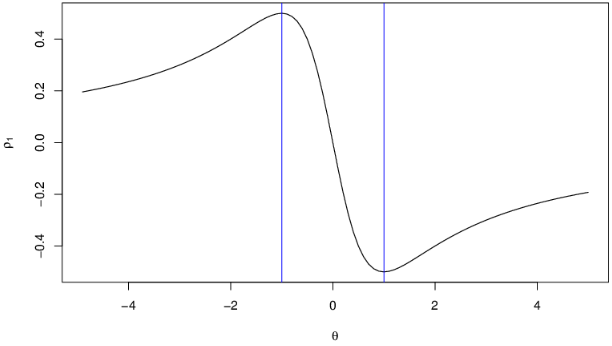
- The ACVF at lag 2 is

$$\begin{aligned}\gamma_2 &= Cov(y_t, y_{t-2}) \\ &= Cov(e_t - \theta_1 e_{t-1}, e_{t-2} - \theta_1 e_{t-3}) \\ &= 0,\end{aligned}$$

- Similarly, $\gamma_k = Cov(y_t, y_{t-k}) = 0$, and $\rho_k = 0$, whenever $k \geq 2$;
- That is, the process has no correlation beyond lag 1.

ACF of MA(1)

$$\begin{cases} E(y_t) = \theta_0, \\ \gamma_0 = \sigma_e^2(1+\theta_1^2), \\ \rho_1 = \dfrac{-\theta_1}{1+\theta_1^2}, \\ \rho_k = 0, \quad \forall\ k \geq 2. \end{cases}$$



ACF pattern of MA(1) model

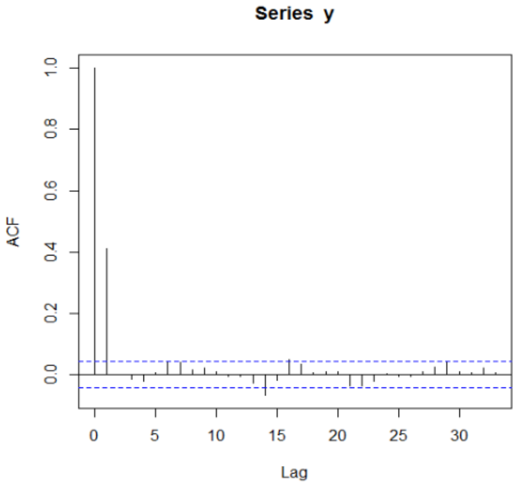


Figure 8: The ACFs of MA(1) model cuts off from lag 2.

MA(2) process

Consider a MA(2) process,

$$y_t = \theta_0 + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}.$$

- Mean: $\mu = E(y_t) = \theta_0$.
- Variance:

$$\begin{aligned} \gamma_0 &= Var(y_t) = Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\ &= (1 + \theta_1^2 + \theta_2^2)\sigma_e^2, \end{aligned}$$

MA(2) process

- ACVF at lag 1:

$$\begin{aligned}\gamma_1 &= Cov(y_t, y_{t-1}) \\ &= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\ &= (-\theta_1 + \theta_1\theta_2)\sigma_e^2,\end{aligned}$$

- ACF at lag 2:

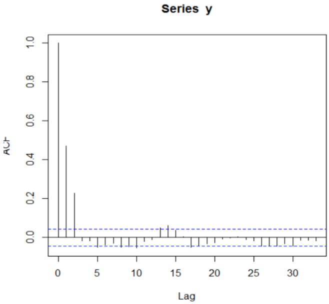
$$\begin{aligned}\gamma_2 &= Cov(y_t, y_{t-2}) \\ &= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) \\ &= Cov(-\theta_2 e_{t-2}, e_{t-2}) = -\theta_2\sigma_e^2.\end{aligned}$$

- $\gamma_k = 0$, for all $k \geq 3$.

ACF of MA(2) Model

The ACF of the MA(2) model is

$$\begin{cases} \rho_1 = \dfrac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}, \\ \rho_2 = \dfrac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}, \\ \rho_k = 0, \quad \forall k \geq 3. \end{cases}$$



MA(q) process

- $y_t = \theta_0 + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - ... - \theta_q e_{t-q}, e_t \sim i.i.d.(0, \sigma^2)$.
- Lag form: $y_t = \theta_0 + (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)e_t$.
- MA(q) model is always weakly stationary. (Why?)
- Mean: $\mu = E(y_t) = \theta_0$;
- Variance:$\gamma_0 = (1 + \theta_1^2 + \cdots + \theta_q^2)\sigma^2$;
- ACF at lag j:

$$\rho_j = \frac{\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + ... + \theta_q\theta_{q-j}}{1 + \theta_1^2 + ... + \theta_q^2}, \text{for } j = 1, 2, ..., q;$$

  and $\rho_j = 0$ for $j > q$.
- Invertibility: All roots of $1 - \theta_1 x \theta_2 x^2 - \cdots \theta_q x^q = 0$ lie out of unit circle.

MA(q) process

Summary:

- For MA(q) process, $\rho_k = 0$ for any $k > q$. (ACF cuts off after q lags.)
- The cut-off property of the ACF is a special property which can be used to identify the value of q if a MA(q) model is specified to the data.
- $\sqrt{T}(\hat{\rho}_k) \xrightarrow{d} N(0, 1 + 2\sum_{j=1}^{k-1} \rho_j^2)$.

Autoregressive moving average (ARMA) model

ARMA(1,1):

- $y_t = \phi_1 y_{t-1} + \phi_0 + e_t - \theta_1 e_{t-1}$.
  or Lag form: $(1 - \phi_1 B)y_t = \phi_0 + (1 - \theta_1 B)e_t$, where $e_t \sim i.i.d.(0, \sigma_e^2)$.
- Stationary condition: same as AR(1)
- Invertible condition: same as MA(1)
- Mean:$\mu = E(y_t) = \frac{\phi_0}{1-\phi_1}$ (same as AR(1))
- Variance: $\gamma_0 = Var(y_t) = \dfrac{(1 - 2\phi_1\theta_1 + \theta_1^2)\sigma_e^2}{1 - \phi_1^2}$
- ACF: $\rho_k = \phi_1 \rho_{k-1}$ for $k > 1$ and $\rho_1 = \phi_1 - \dfrac{\theta_1 \sigma_e^2}{\gamma_0}$.)

ARMA(p, q)

- A general ARMA(p,q) model is in the form:

$$\begin{aligned} y_t &= \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} \\ &\quad + e_t - \theta_1 e_{t-1} - \cdots - \theta_q e_{t-q}. \end{aligned}$$

- Lag form:

$$\phi(B) y_t = \phi_0 + \theta(B) e_t,$$

  where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$.

- It is assumed that the polynomials $\phi(B)$ and $\theta(B)$ can not have common factors.

- Again, for a stationary process, we can rewrite the model as

$$\phi(B)(y_t - \mu) = \theta(B) e_t.$$

Properties of ARMA(p, q) models

- Stationarity: The ARMA process is stationary *iff* all roots of $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0$ lie outside the complex unit circle.
- Invertibility: All roots of $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q = 0$ lie outside the unit circle.
- AR representation: $\pi(B)y_t = \dfrac{\phi_0}{\theta(1)} + e_t$, where $\pi(B) = \dfrac{\phi(B)}{\theta(B)}$. The π-weight π_i can be obtained by equating the coefficients of B^i in $\pi(B)\theta(B) = \phi(B)$.
- MA representation: $y_t = \dfrac{\phi_0}{\phi(1)} + \psi(B)e_t$, where $\psi(B) = \frac{\theta(B)}{\phi(B)}$. Again, the ψ-weight can be obtained by equating the coefficients.
- The MA representation is particularly useful in computing variances of forecast errors.

Properties of ARMA(p, q)

Given stationarity:

- Mean: $\mu = \dfrac{\phi_0}{1 - \phi_1 - \cdots - \phi_p}$.
- ACVF: (Centralized by μ) Using the result

$$E(y_t e_{t-j}) = \begin{cases} \sigma_e^2, & \text{for } j = 0; \\ \psi_j \sigma_e^2, & \text{for } j > 0; \\ 0, & \text{for } j < 0. \end{cases}$$

Then

$$\begin{aligned} \gamma_j & -\phi_1 \gamma_{j-1} - \cdots - \phi_p \gamma_{j-p} \\ = & \begin{cases} (1 + \theta_1 \psi_1 + \cdots + \theta_q \psi_q) \sigma_e^2, & \text{for } j = 0; \\ (\theta_j + \theta_{j+1} \psi_1 + \cdots + \theta_q \psi_{q-j}) \sigma_e^2, & \text{for } j = 1, \ldots, q; \\ 0, & \text{for } j > q, \end{cases} \end{aligned}$$

where $\psi_0 = 1$ and $\theta_j = 0$ for $j > q$.

Cont'd

- ACF: the correlation coefficient ρ_j satisfies that

$$\rho_j - \phi_1 \rho_{j-1} - \cdots - \phi_p \rho_{j-p} = 0, \quad \text{for } j > q,$$

then the ACF satisfy the difference equation $\phi(B)\rho_j = 0$ for $j > q$ with $\rho_1, \ldots, \rho_q$ as initial conditions.

Homework

1. Write the Yule-Walker equations for the AR(3) model:

   $y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_3 y_{t-3} + e_t$, where $e_t \sim i.i.d.(0, \sigma^2)$.

2. Simulate the data generating process from the MA(1), MA(2) and MA(3) models respectively and plot their sample ACF. Observe the empirical patterns.

3. $$y_t = e_t + \theta e_{t-1} + \theta^2 e_{t-2} + \theta^3 e_{t-3} + \cdots ,$$

   where $|\theta| < 1$, and $\{e_t\} \sim i.i.d.(0, \sigma_e^2)$. Calculate the mean, variance and autocorrelations of y_t.