Time Series Analysis
Lecture 2 Basic Concepts

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Basic concepts

自协方差函数 自相关函数

1. Mean, Autocovariance function (ACVF) and Autocorrelation function (ACF)
   平稳性
2. Stationarity
   鞅差分序列 白噪声序列
3. I.I.D. sequence, Martingale difference sequence (MDS), White noise sequence
   时间序列分解
4. Time Series Decomposition

Mean, Autocovariance and Autocorrelation

Suppose $\{Y_t\}$ is a time series

- Mean function:

$$\mu_t = E(Y_t).$$

- Autocovariance function:

$$\begin{aligned}\gamma_Y(s,t) &= Cov(Y_s, Y_t) \\ &= E[(Y_s - \mu_s)(Y_t - \mu_t)].\end{aligned}$$

- Autocorrelation function (ACF):

$$\rho_Y(s,t) = \frac{Cov(Y_s, Y_t)}{\sqrt{Var(Y_t, Y_t) Var(Y_s, Y_s)}}$$

1. 独立同分布
$Y_t = \mu + \varepsilon_t$ where ε_t is a i.i.d $N(0, \sigma^2)$.
Solution:

$$\begin{aligned} \mu_t &= E(Y_t) = E(\mu + \varepsilon_t) = \mu \\ \gamma_Y(s,t) &= Cov(Y_s, Y_t) = E[(Y_s - \mu)(Y_t - \mu)] \\ &= E(\varepsilon_s \varepsilon_t) = \begin{cases} 0, & \text{if } s \neq t \\ \sigma^2, & \text{if } s = t \end{cases} \end{aligned}$$

因为独立同分布.
所以 $E(\varepsilon_s \varepsilon_t)$ $s \neq t \Longrightarrow E(\varepsilon_s)E(\varepsilon_t)$

2. How about $Y_t = \beta t + \varepsilon_t$?

$E(Y_t) = \beta t$ $Cov(Y_t, Y_s) = \begin{cases} 0, & s \neq t \\ \sigma^2, & s = t \end{cases}$

协方差平稳　弱平稳

- A time series $\{Y_t\}$ is Covariance stationary or weakly stationary if

$$
\begin{aligned}
E(Y_t) &= \mu_t = \mu \quad \text{for all } t \\
Var(Y_t) &= \sigma^2 < \infty \\
Cov(Y_{t+j}, Y_t) &= \gamma_Y(t+j, t) = \gamma_j \quad \text{for all } t \text{ and } j.
\end{aligned}
$$

它的 "平均水平""波动幅度""不同时刻的关联程度" 都具有时间平移不变性

- In other words, both μ_t and $\gamma_Y(t+j, t)$ are time invariant.
- For a weakly stationary process,

t与t+j的相关程度和t+j与t的相关程度是一样的

$$
\begin{aligned}
Corr(Y_{t+j}, Y_t) &= \frac{Cov(Y_{t+j}, Y_t)}{\sqrt{Var(Y_{t+j})}\sqrt{Var(Y_t)}} \\
&= \rho_Y(j) = \rho_Y(-j) \quad \text{symmetric and time invariant}
\end{aligned}
$$

Example 1

- White Noise (WN): Y_t is defined as a white noise process if

$$E(Y_t) = 0 \quad \text{and} \quad \gamma_Y(t-j, t) = \begin{cases} 0, & \text{if } j \neq 0 \\ \sigma^2, & \text{if } j = 0 \end{cases}$$

- In a short, white noise process is a uncorrelated sequence with zero mean.
- Obviously, White Noise process is weakly stationary.

Example 2

- Random Walk: $Y_t = \sum_{i=1}^{t} e_i$, where e_i is a i.i.d. process with mean 0 and variance σ^2. ($Y_t = Y_{t-1} + e_t$)
- $$\begin{aligned} E(Y_t) &= 0 \\ E(Y_t^2) &= t\sigma^2 \\ \gamma_Y(t-j,t) &= Cov(Y_{t-j}, Y_t) \\ &= Cov(Y_{t-j}, Y_{t-j}) + Cov(Y_{t-j}, \sum_{i=t-j+1}^{t} e_i) \\ &= Cov(Y_{t-j}, Y_{t-j}) \\ &= (t-j)\sigma^2. \end{aligned}$$
- Thus, Random Walk is not weakly stationary.

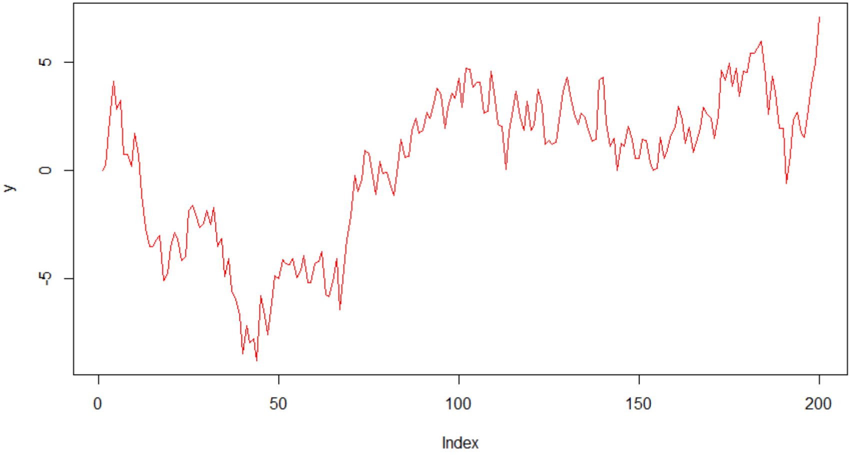

Figure: Random walk

- Y_t is strictly stationary if for all $k, t_1, ..t_k, y_1, ..., y_k$ and h, the probability

$$\Pr(Y_{t_1} \leq y_1, ..., Y_{t_k} \leq y_k) = \Pr(Y_{t_1+h} \leq y_1, ..., Y_{t_k+h} \leq y_k);$$

- In other words, the joint distribution $F(Y_{t_1}, ..., Y_{t_k})$ depends only on the intervals separating the dates $(t_1, ..t_k)$, not on the date itself.

只依赖时间的间隔，不依赖时间数据其本身

weakly stationary$\neq$ strictly stationary

不是弱平稳，但是在合适的时刻是弱平稳的

- Strictly stationary does not imply weakly stationary, but a strict stationary process with *finite second moments* is weakly stationary;
- Weakly stationary does not imply strictly stationary because the third order moment could be time-varying, even the first and second moments are constant.
- An i.i.d. sequence is definitely strictly stationary.
- **Question:** Is an i.i.d. sequence definitely weakly stationary?

weakly stationary= strictly stationary?

- Gaussian process: Y_t is a Gaussian process, if the joint distribution of $(Y_{t_1}, ..., Y_{t_k})$ for any $k, t_1, ..t_k$, is joint Gaussian distribution.
- If $\{X_t\}$ is a Gaussian process, under this situation,

$$\text{weakly stationary} \Leftrightarrow \text{strictly stationary.}$$

高斯过程由“任意有限维联合分布都是高斯”

Weakly Stationary/Strictly Stationary

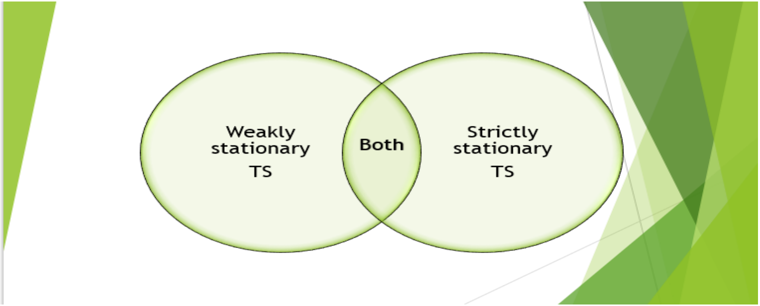

Figure: The relationship between weak stationarity and strict stationarity

From now on, stationarity in this course means weakly stationary.

Dependent structure

过去的事儿再怎么分析，也没法预判这次意外是 "该赚还是该亏"

Assume innovations $\{\varepsilon_t\}$. Four situations:

1. independent and identically distributed, $\varepsilon_t \sim i.i.d.(0, \sigma^2)$;
2. Martingale difference sequence (MDS):

$$E(\varepsilon_t | \mathfrak{F}_{t-1}) = 0.$$

3. White noise: $E\varepsilon_t = 0$, $Var(\varepsilon_t) = \sigma^2 < \infty$ and $Corr(\varepsilon_t, \varepsilon_s) = 0$ for all $t \neq s$.
4. Stationary process: $E(\varepsilon_t) = \mu$ and $\rho_\varepsilon(t+j, t) = \rho_\varepsilon(j)$.

Graphic demonstration

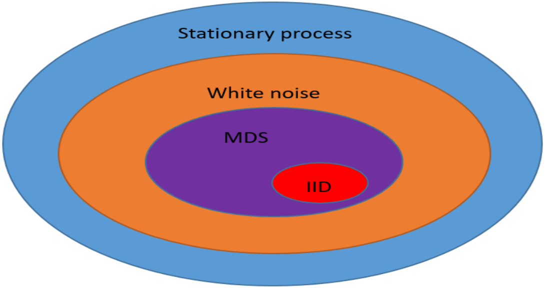

Figure: The classification of stochastic process

Time Series Component

- Time series: a systematic part +a non-systematic part.
- The systematic part : Level, Trend, Seasonality.
  - 等级 Level:describes the average value of the series;
  - 趋势 Trend: the change in the series from one period to the next (commonly approximated by linear, exponential and other mathematical functions);
  - 季节性 Seasonality: describes a short-term cyclical behavior that can be observed several times within the given series.
- The nonsystematic part: Noise.

Time Series Decomposition

- Two decompositions: either additive or multiplicative.
- Additive: $Y_t = Level + Trend + Seasonality + Noise$
- Multiplicative: $Y_t = Level \times Trend \times Seasonality \times Noise$
- The systematic part: for generating point forecasts The noise level : for assessing the uncertainty associated with the point forecasts).

Time series patterns

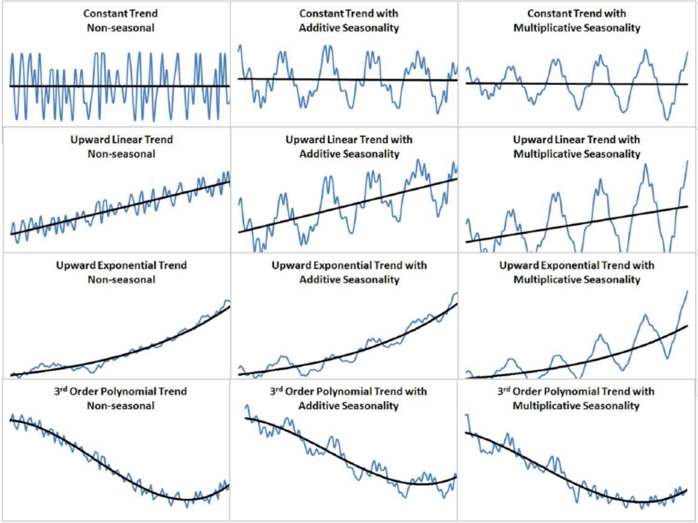

Figure: Illustrations of common trend and seasonality patterns

Homework

Discuss the stationarity of the following time series:

(1) $\{X_t : t \in \mathcal{T}\} = \{\varepsilon, \eta, \varepsilon, \eta, ...\}$, i.e., $X_{2t-1} = \varepsilon$, $X_{2t} = \eta$, where $\varepsilon \sim \mathcal{N}(0,1)$, $\eta \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$, and ε and η are independent.

(2) $\{Y_t : t \in T\}$ is an i.i.d. standard Cauchy distributed sequence with density $f(x) = \dfrac{1}{\pi(1+x^2)}$.

(3) $\{Y_t : t \in \mathcal{T}\}$ is i.i.d. with $Y_t \sim \mathcal{N}(0,1)$.

(4) $\{\varepsilon, \eta, \xi_3, \xi_4, ...\}$, where $\varepsilon \sim \mathcal{N}(0,1)$, $\eta \sim t_1$(Student $t-$ distribution with degree of freedom 1), and $\{\xi_j : j = 3, 4, ...\}$ is i.i.d. $\sim Exp(1)$.