Time Series Analysis
Lecture 4 ARIMA modelling

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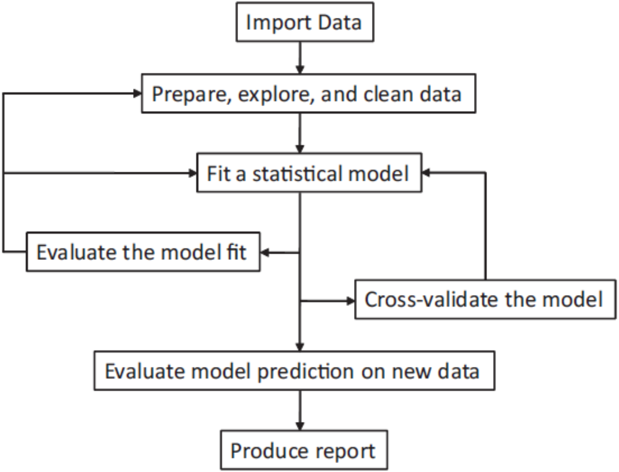
Outline

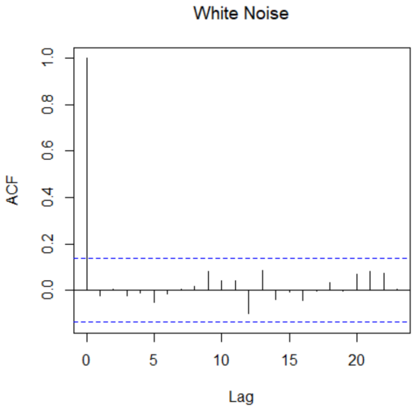
- Model Steps
- Model Specification
- Model Estimation
- Model Checking
- Model Selection

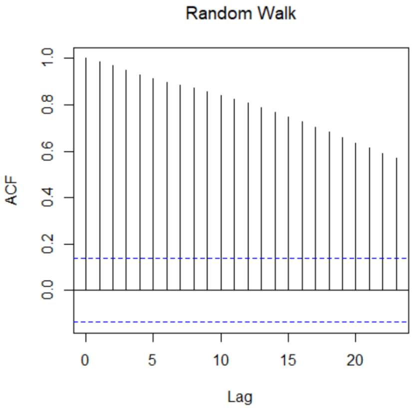
Modeling Steps

All models are wrong, but some are useful—George, E.P.Box



Illustrations of Simulated Time Series





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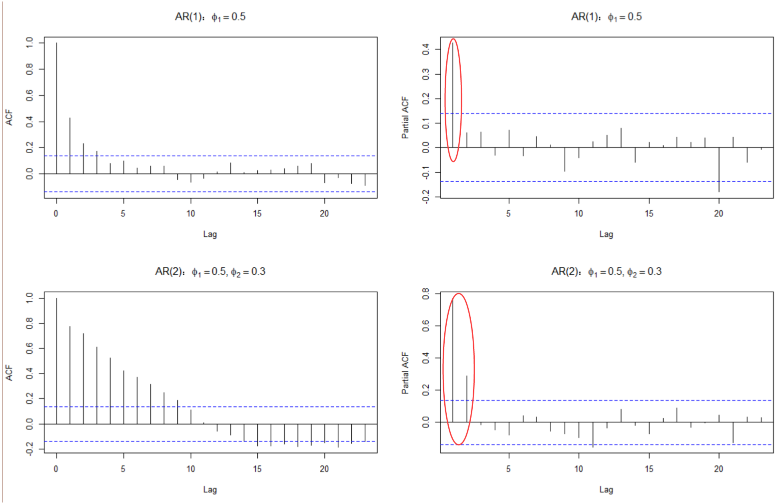

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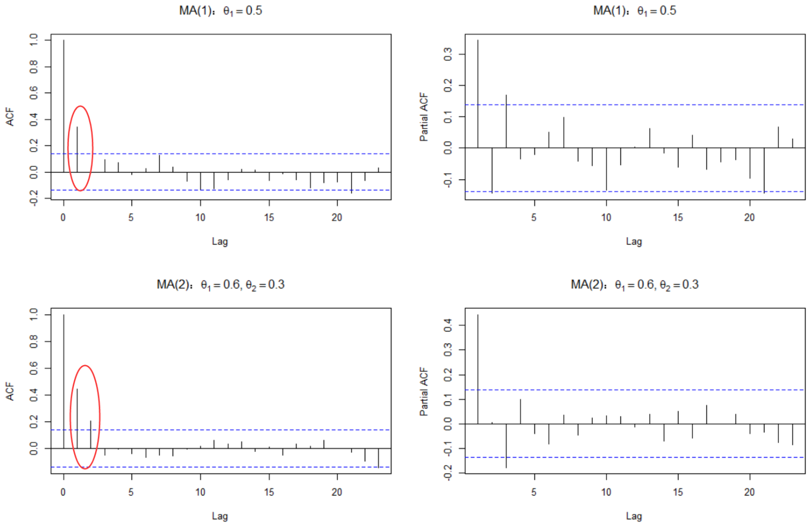

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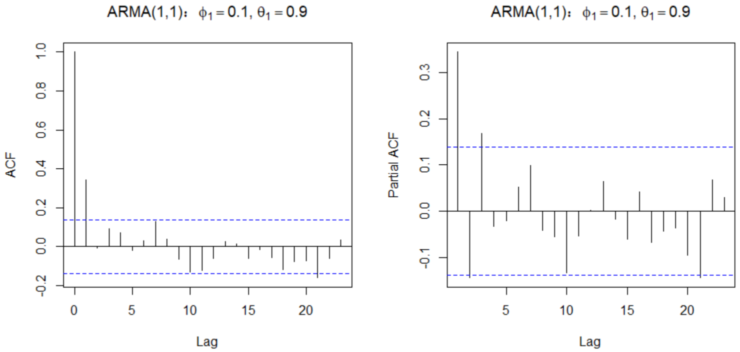

Model Specification

1. If the sample ACF $\hat{\rho}_k$ are close to 1 for all small k and decay slowly, difference the data first.存在非平稳结构
2. For the sample ACF $\hat{\rho}_k$, we have

$$\sqrt{T}\hat{\rho}_k \xrightarrow{d} N(0, 1 + 2\sum_{j=1}^{k-1} \rho_j^2),$$

   hence if it is bounded by $\frac{1.96}{\sqrt{T}}\left[1 + 2\sum_{j=1}^{k-1} \hat{\rho}_j^2\right]^{1/2}$ for any $k > q$, we can try an MA(q) model to the data.
3. For the sample PACF $\hat{\pi}_k$,

$$\sqrt{T}\hat{\pi}_k \xrightarrow{d} N(0, 1),$$

   hence, if it is bounded by $1.96/\sqrt{T}$ for any $k > p$, we can try an AR(p) model to the data.
4. In general, the correct (p, q) is not unique. We can select the best one by AIC/BIC.

Ordinary Least Square of AR(p) Model

- **Consider an AR(p) model,**

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + e_t.$$

The parameter space $\theta = (\phi', \sigma^2), \phi = (\phi_0, \phi_1, \cdots, \phi_p)'$.

- **The OLS estimates of coefficients:**

$$\begin{aligned}(\widehat{\phi}_0, \widehat{\phi}_1, ..., \widehat{\phi}_p) &= argmin \sum_{t=p+1}^{T} (y_t - \phi_0 - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p})^2 \\ &= argmin S_C(\phi_0, \phi_1, ..., \phi_p)\end{aligned}$$

OLS of AR(p)

- Let $x_t = (1, y_{t-1}, y_{t-2}, ..., y_{t-p})'$ and $\phi = (\phi_0, \phi_1, ..., \phi_p)'$. Then,

$$\begin{aligned}\widehat{\phi}_{OLS} &= \left(\sum_{t=p+1}^{T} x_t x_t'\right)^{-1} \sum_{t=p+1}^{T} x_t y_t \\ &= \phi + \left(\sum_{t=p+1}^{T} x_t x_t'\right)^{-1} \sum_{t=p+1}^{T} x_t e_t.\end{aligned}$$

- $$\widehat{\sigma}_e^2 = \frac{1}{T-p} S_C(\widehat{\phi}_0, \widehat{\phi}_1, ..., \widehat{\phi}_p).$$

MLE for AR(p) Model

- Consider the following $AR(p)$ process:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t,$$

  where ε_t is i.i.d $N(0, \sigma^2)$.
- The parameter vector is $\theta = \{\phi_0, , \phi_1, \phi_2, ..., \phi_p, \sigma^2\}$.
- For Y_t, $t > p$, the conditional distribution is

$$(Y_t | Y_{t-1}, Y_{t-2}, ..., Y_{t-p}) \sim N(\phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} ... + \phi_p y_{t-p}, \sigma^2)$$

- the conditional density is

$$\begin{aligned} & f_{Y_t|Y_{t-1},Y_{t-2},...,Y_{t-p}}(y_t|y_{t-1}, y_{t-2}, ...y_{t-p}, \theta) \\ = \quad & \frac{1}{\sqrt{2\pi}\sigma} \exp(\frac{-(y_t - \phi_0 - \phi_1 y_{t-1} - \phi_2 y_{t-2} ... - \phi_p y_{t-p})^2}{2\sigma^2}). \end{aligned}$$

MLE for AR(p) process

- Assume $\mathbf{y}_p = (y_1, y_2, \cdots, y_p)$.
- Conditional MLE: assume $\mathbf{y}_p$ to be deterministic.
- The conditional likelihood function

$$L(\theta) = \sum_{t=p+1}^{T} \log(f_{Y_t|Y_{t-1},Y_{t-2},\ldots,Y_{t-p}}(y_t|y_{t-1}, y_{t-2}, \ldots y_{t-p}, \theta))$$

$$= -(T-p)/2 \log(2\pi\sigma^2) - \sum_{t=p+1}^{T} \frac{(y_t - \phi_0 - \phi_1 y_{t-1} - \ldots - \phi_p y_{t-p})^2}{2\sigma^2}.$$

MLE of AR(p) model

- Unconditional MLE: assume $\mathbf{y}_p$ to be random vector
- the log likelihood function is

$$\begin{aligned} L(\theta) &= \log(f_{Y_T, Y_{T-1}, Y_{T-2}, \ldots, Y_1}(y_T, y_{T-1}, \ldots, y_1, \theta)) \\ &= -T/2 \log(2\pi) - T/2 \log(\sigma^2) + \\ &\quad \frac{1}{2\sigma^2} \log|V_p^{-1}| - \frac{1}{2\sigma^2}(\mathbf{y}_P - \mu_p)' V_p^{-1}(\mathbf{y}_P - \mu_p) \\ &\quad - \sum_{t=p+1}^{T} \frac{(y_t - \phi_0 - \phi_1 y_{t-1} - \phi_2 y_{t-2} \ldots - \phi_p y_{t-p})^2}{2\sigma^2} \end{aligned}$$

Conditional MLE for AR(1)

Consider AR(1) model: $y_t = \phi_0 + \phi_1 y_{t-1} + e_t$, $e_t \sim i.i.d.(0, \sigma^2)$.

- Assume y_1 be given as deterministic.

$$y_2 | y_1 = y_1 \sim N(\phi_0 + \phi_1 y_1, \sigma^2)$$

- The conditional density function is

$$f(y_2 | y_1, \phi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y_2 - \phi_0 - \phi_1 y_1)^2}{2\sigma^2}\right).$$

- The conditional joint density is

$$f(y_T, y_{T-1}, ..., y_2 | y_1, \theta) = \Pi_{t=2}^{T} f(y_t | y_{t-1}, \theta)$$

Conditional MLE for AR(1)

- The conditional log likelihood function

$$
\begin{aligned}
L(\theta) &= \sum_{t=2}^{T} \log(f(y_t|y_{t-1}, \theta)) \\
&= -(\frac{T-1}{2}) \log\left(2\pi\sigma^2\right) - \sum_{t=2}^{T} \frac{(y_t - \phi_0 - \phi y_{t-1})^2}{2\sigma^2}.
\end{aligned}
$$

- Theconditional MLE for θ is defined as

$$
\widehat{\theta} = \arg\max_{\theta} L(\theta).
$$

Conditional MLE for AR(1)

For $\widehat{\sigma}^2$, the first order condition is

$$-\frac{T-1}{2\sigma^2}+\sum_{t=2}^{T}\frac{(y_t-\phi_0-\phi y_{t-1})^2}{2\sigma^4}=0.$$

and the solution is

$$\widehat{\sigma}^2=\sum_{t=2}^{T}\frac{(y_t-\widehat{\phi_0}-\widehat{\phi}y_{t-1})^2}{T-1}.$$

Model Checking

Testing for residual autocorrelations

- $$H_0 : \rho_1 = \rho_2 = \cdots = 0 \leftrightarrow H_1 : \exists \quad j, \text{ such that} \rho_j \neq 0.$$
- Joint significance test of the first m residual autocorrelations.
- Ljung-Box typed test statistics:

$$T_m = n(n+2) \sum_{k=1}^{m} \frac{\rho_k^2(\hat{e})}{n-k} \sim \chi^2_{m-K}.$$

where

$$\rho_k(\hat{e}) = \frac{\sum_{t=k+1}^{n} \hat{e}_t \hat{e}_{t-k}}{\sum_{t=k+1}^{n} \hat{e}_t^2}$$

and K is the number of unknown parameters.

Testing for Normality

- The skewness and kurtosis of $\hat{e}_t$ can be calculated as

$$\widehat{SK}_{\hat{e}} = \frac{\hat{m}_3}{\sqrt{\hat{m}_2^3}}, \quad \text{and} \quad \widehat{K}_{\hat{e}} = \frac{\hat{m}_4}{\hat{m}_2^2}.$$

  where $\hat{m}_j = \frac{1}{n}\sum_{t=1}^{n} \hat{e}_t^j$,

- Under the null hypothesis of normality,

$$\sqrt{n/6} \cdot \widehat{SK}_{\hat{e}} \sim \mathcal{N}(0,1), \sqrt{n/24} \cdot (\widehat{K}_{\hat{e}} - 3) \sim \mathcal{N}(0,1).$$

- A joint test for normality (Jarque and Bera, 1987):

$$\text{JB} = \frac{n}{6}\widehat{SK}_{\hat{e}}^2 + \frac{n}{24}(\widehat{K}_{\hat{e}} - 3)^2 \sim \chi_2^2.$$

Model Diagnostic Checking

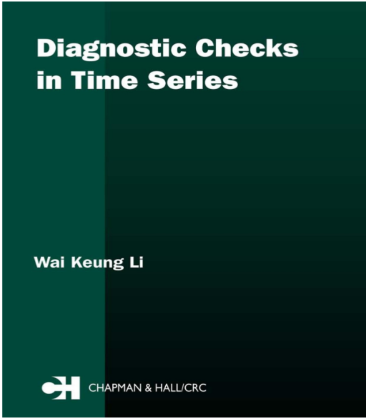

Figure 1: Li,WaiKeung.(2004). Diagnostic Checks in Time Series. Chapman& Hall/CRC

Model Selection: Order Determination

- Akaike information criterion (AIC):$-2L(\widehat{\theta}) + 2k$
- Bayesian information criterion (BIC):$-2L(\widehat{\theta}) + \ln(T)k$
- Compared to the AIC, the BIC counteracts the overfitting tendency of the AIC.
- Other information criterions.