<< Introduction to Robotics >>. 4th edition

1. The Mechanics and Control of Mechanical Manipulators

1.1. Description of Position and Orientation

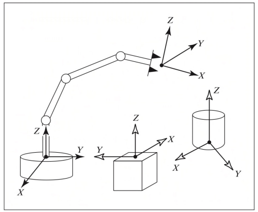


FIGURE 1.5: Coordinate systems or "frames" are attached to the manipulator and to objects in the environment.

Any frame can serve as a reference system within which to express the position and orientation of a rigid body in 3D space. We will learn the mathematics of manipulating these quantities to transform from one frame to another.

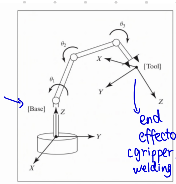
1.2. Forward Kinematics of Manipulators

Kinematics is the science of motion that treats motion without regard to the forces which cause it.

⇒ the geometrical and time-based properties of the motion (position, velocity, acceleration, all higher order derivatives of the variables)

The Degrees of Freedom that a manipulator prossesses is the number of independent position variables that would have to be specified to locate ALL parts of the mechanism

fixed frame



end effector (gripper, welding torch ..)

FIGURE 1.6: Kinematic equations describe the tool frame relative to the base frame as a function of the joint variables.

- Because a manipulator is usually an open kinematic chain, and because each joint position is usually defined with a single variable, the number of joints equals the number of degrees of freedom.

⇒ Forward kinematics is the static geometrical problem of computing the (pos, orient) of {tool} relative to {base}

Ch2. Spatial Descriptions and Transformations

2.1. Introduction

Robotic Manipulation, by definition, implies that parts and tools will be moved around in space by some sort of mechanism. To define and manipulate mathematical quantities that represent position and orientation we must define coordinate systems and develop conventions for representation.

2.2. Descriptions of Positions and Orientation

- Position :

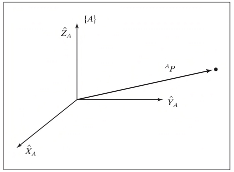


FIGURE 2.1: Vector relative to frame (example).

$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

position vector

- Orientation :

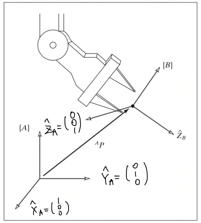


FIGURE 2.2: Locating an object in position and orientation.

To desribe the orientation of a rigid body in 3D space, we will attach a coordinate system to the body and then give a description of this coordinate system relative to the reference system.

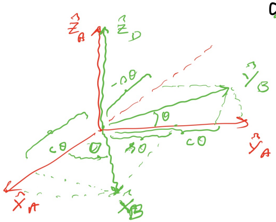
One way to describe the body-attached coordinate system, {B}, is to write the unit vectors of its 3 principal directions as $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$, when written in terms of {A}'

$[\,^{A}\hat{X}_B \;\; ^{A}\hat{Y}_B \;\; ^{A}\hat{Z}_B\,] = {}^{A}_{B}R$, this is a 3x3 Rotation Matrix

Theorem from Mechanics: Every rigid body motion can be represented as a Rotation and a Translation.

Example: $\{A\} \xrightarrow[\text{around } \hat{Z}_A]{\text{Rotate } \theta} \{B\}$.

$${}^A\hat{X}_A = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad {}^A\hat{Y}_A = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad {}^A\hat{Z}_A = \begin{pmatrix}0\\0\\1\end{pmatrix}$$



$${}^A_BR = [\,{}^A\hat{X}_B \;\; {}^A\hat{Y}_B \;\; {}^A\hat{Z}_B\,] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

"direction cosine"
- the dot product of two unit vectors yields the cosine angle between them

Note 1: the components of any vector are simply the projections of that vector onto the unit directions of its reference frame:

$${}^A_BR = [\,{}^A\hat{X}_B \;\; {}^A\hat{Y}_B \;\; {}^A\hat{Z}_B\,] = \begin{pmatrix} \hat{X}_B\cdot\hat{X}_A & \hat{Y}_B\cdot\hat{X}_A & \hat{Z}_B\cdot\hat{X}_A \\ \hat{X}_B\cdot\hat{Y}_A & \hat{Y}_B\cdot\hat{Y}_A & \hat{Z}_B\cdot\hat{Y}_A \\ \hat{X}_B\cdot\hat{Z}_A & \hat{Y}_B\cdot\hat{Z}_A & \hat{Z}_B\cdot\hat{Z}_A \end{pmatrix}$$

Note 2. The Rows of the matrix are the Unit Vectors of {A} expressed in {B}

$$= \begin{bmatrix} {}^B\hat{X}_A^T \\ {}^B\hat{Y}_A^T \\ {}^B\hat{Z}_A^T \end{bmatrix} = \begin{bmatrix} {}^B\hat{X}_A & {}^B\hat{Y}_A & {}^B\hat{Z}_A \end{bmatrix}^T$$

$$\Rightarrow {}^A_BR = {}^B_AR^T$$

This suggests that the inverse of a rotation matrix is equal to its transpose: ${}^A_BR^{-1} = {}^A_BR^T = {}^B_AR$

$$\Rightarrow {}^A_BR^T \cdot {}^A_BR = \begin{bmatrix} {}^A\hat{X}_B^T \\ {}^A\hat{Y}_B^T \\ {}^A\hat{Z}_B^T \end{bmatrix} [\,{}^A\hat{X}_B \;\; {}^A\hat{Y}_B \;\; {}^A\hat{Z}_B\,] = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The space of Orientations of a rigid body is only 3 dimensional, but we have 9 numbers in a Rotation Matrix

- 9 entries of the matrix must be subject to 6 constraints
  - $\Rightarrow$ 3 column vectors are unit vectors: $|{}^A\hat{X}_B| = |{}^A\hat{Y}_B| = |{}^A\hat{Z}_B| = 1$
  - $\Rightarrow$ 3 vectors are orthogonal to each other.
    - dot products of any two of unit vectors are 0
- These constraints ensures $\det(R) = 1$ for R-H frame

The set of all 3x3 Rotation Matrix is called the
- Special Orthogonal Group − SO(3), satisfying

$\Rightarrow R^TR = I$ and $\det(R) = 1$

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5 Properties of Rotation Matrices:

1. Inverse: $R^{-1} = R^T \in SO(3)$
2. Closure: $R_1R_2 \in SO(3)$
3. Associative: $(R_1R_2)R_3 = R_1(R_2R_3)$
4. NOT commutative. $R_1R_2 \neq R_2R_1$
5. $\vec{x} \in \mathbb{R}^3$, $\|R\vec{x}\| = \|x\|$

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3 Common Uses of Rotation Matrices:

① Represent an Orientation ② Change reference frame ③ Rotate a Vector or Frame

2.3. Mappings: Change Descriptions from frame to frame

1. Description of a Frame

① The situation of [a position and an orientation] pair arises so often in Robotics that we define an entity called Frame,

- which is a set of 4 vectors giving position and orientation information (or a position vector and a Rotation Matrix)

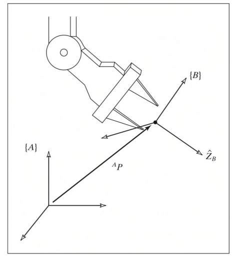


FIGURE 2.2: Locating an object in position and orientation.

- Figure 2.2: 1 vector locates the fingertip position, and 3 more describe its orientation.

② A frame is a coordinate system where, in addition to the orientation, we give a position vector which locates its origin relative to some other embedding frame.

- Figure 2.2: Frame {B} is described by ${}^A_B R$ and $^A P_{BORG}$ (the vector that locates the origin of the frame {B}) $\Rightarrow \{B\} = \{{}^A_B R, {}^A P_{BORG}\}$.
- Figure 2.3: There are 3 frames: {A} & {B} are relative to the Universe Coordinate System, and {C} is relative to frame {A}.

③ A Frame encompasses 2 ideas by representing both position and orientation:

- Positions could be represented by a frame whose Rotation-Matrix part is the Identity Matrix and whose position-vector part locates the point being described.
- An orientation could be represented by a frame whose Position-Vector part was the Zero Vector.

2. Mappings Involving Translated Frames

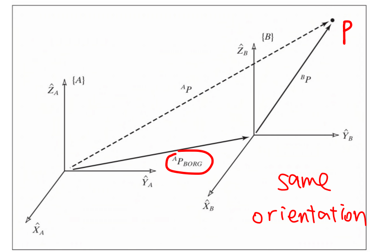


FIGURE 2.4: Translational mapping. of a vector $^B P$ from one frame {B} to another ($^A P$)

$^A P = {}^A P_{BORG} + {}^B P$, when A has the same orientation as {B}, in this case, {B} differs from {A} ONLY by a Translation, so we can add vectors that are defined in terms of different frames.

- vector $^A P_{BORG}$ defines this mapping, because ALL the INFO needed to perform the change in description is contained in $^A P_{BORG}$ (along with the knowledge that the frame had equi. orient.)
- The quantity (point described by $^B P$) is NOT changed (translated) and instead we have computed a new description of the same point, w.r.t. system {A}

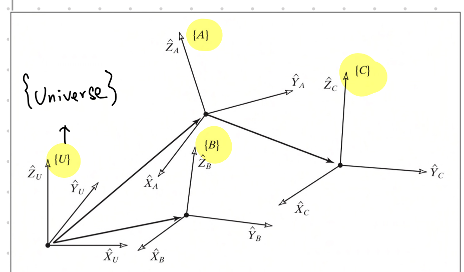


FIGURE 2.3: Example of several frames.

3. Mappings Involving Rotated Frames.

① Recall that Rotation Matrix

$$ {}^{A}_{B}R = [\,{}^{A}\hat{X}_B \;\; {}^{A}\hat{Y}_B \;\; {}^{A}\hat{Z}_B\,] = \begin{bmatrix} {}^{B}\hat{X}_A^{T} \\ {}^{B}\hat{Y}_A^{T} \\ {}^{B}\hat{Z}_A^{T} \end{bmatrix} $$

$$ = {}^{B}_{A}R^{T} = {}^{B}_{A}R^{-1} $$

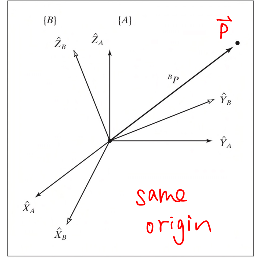


FIGURE 2.5: Rotating the description of a vector.

② Figure 2.5: Given ${}^{B}P$, to calculate ${}^{A}P$,

⇒ the components of any vector are simply the projections of that vector onto the unit directions of its frame.

⇒ The Projection is calculated as the vector dot product.

the components of ${}^{A}P$

$$ \begin{cases} {}^{A}P_x = {}^{B}\hat{X}_A \cdot {}^{B}P \\ {}^{A}P_y = {}^{B}\hat{Y}_A \cdot {}^{B}P \\ {}^{A}P_z = {}^{B}\hat{Z}_A \cdot {}^{B}P \end{cases} \Rightarrow {}^{A}P = {}^{A}_{B}R \cdot {}^{B}P. $$

"${}^{A}_{B}R$ acts as a mapping changes the description of a vector - from {B} into {A} (${}^{B}P \to {}^{A}P$)"

③ We can think of a mapping from one frame to another as an operator in matrix form. ⇒ ${}^{A}P' = {}^{A}_{B}T\,{}^{B}P'$.

• define ${}^{A}P' = \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}$, ${}^{B}P' = \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$ 4x1 position vectors

$$ {}^{A}_{B}T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right] \quad 4\times 4 $$

Homogeneous Transformation Matrix

This can cast the Rotation and Translation of the general transform into a single matrix form

• Just as we used Rotation Matrices to specify an orientation, we will use Homogeneous Transform Matrices to specify a frame. The description of Frame {B} relative to {A} is ${}^{A}_{B}T$.

4. Mapping Involving General Frames.

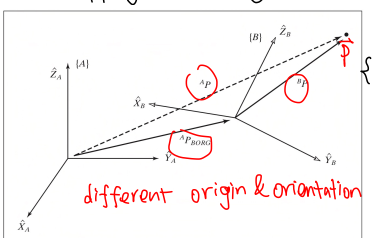


FIGURE 2.7: General transform of a vector.

① General Case: The Origin of frame {B} has a general vector offset, (${}^{A}P_{BORG}$), and {B} is rotated w.r.t {A}. (${}^{A}_{B}R$)

② Given ${}^{B}P$, to compute ${}^{A}P$, we can first change ${}^{B}P$ to its description relative to an frame that has the same orientation as {A}, by premultiplying by ${}^{A}_{B}R$, and then account for the Translation between origins by simple vector addition.

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$

"A general transformation mapping of $\vec{P}$ from its description in {B} → {A}"

Example: A frame {B}, which is Rotated relative to {A} about $\hat{Z}_A$ by 30°, translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$. Given ${}^{B}P = [3.0, 7.0, 0.0]^T$.

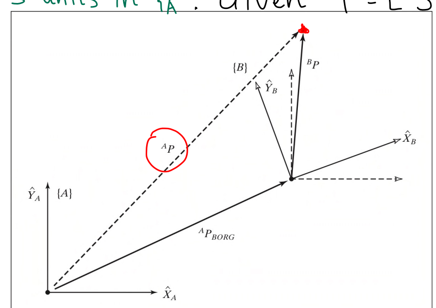


FIGURE 2.8: Frame {B} rotated and translated.

The definition of frame {B} is

$$ {}^{A}_{B}T = \left[\begin{array}{ccc|c} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}P = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \end{bmatrix} $$

⇓

Exercise 2.27–30, ${}^A_BT,\ {}^A_CT,\ {}^B_CT,\ {}^C_AT$.

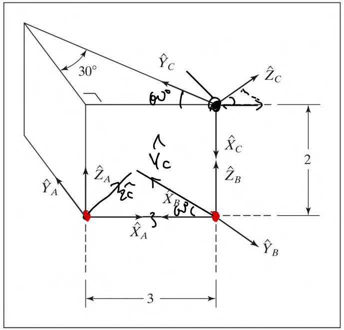


FIGURE 2.25: Frames at the corners of a wedge.

$$ {}^A_BT = \left[\begin{array}{c|c} {}^A_BR & {}^AP_{BORG} \\ \hline 000 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^A_BR = \left[{}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B \right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

rotate {A} → {B} relative to {A}

$$ {}^A_CT = \left[\begin{array}{c|c} {}^A_CR & {}^AP_{BORG} \\ \hline 000 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} {}^A\hat{X}_C & {}^A\hat{Y}_C & {}^A\hat{Z}_C & {}^AP_{CORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ = \left[\begin{array}{ccc|c} 0 & -\sin 30^\circ & \cos 30^\circ & 3 \\ 0 & \cos 30^\circ & \sin 30^\circ & 0 \\ -1 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^B_CT = \left[\begin{array}{ccc|c} {}^B\hat{X}_C & {}^B\hat{Y}_C & {}^B\hat{Z}_C & {}^BP_{CORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ = \left[\begin{array}{ccc|c} 0 & \sin 30^\circ & -\cos 30^\circ & 0 \\ 0 & -\cos 30^\circ & -\sin 30^\circ & 0 \\ -1 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^C_AT = \left[\begin{array}{ccc|c} {}^C\hat{X}_A & {}^C\hat{Y}_A & {}^C\hat{Z}_A & {}^CP_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & -1 & 2 \\ -\sin 30^\circ & \cos 30^\circ & 0 & 3\cdot\cos 60^\circ \\ \cos 30^\circ & \sin 30^\circ & 0 & -3\cdot\sin 60^\circ \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Exercise 2.31–34 : ${}^A_BT,\ {}^A_CT,\ {}^B_CT,\ {}^C_AT$.

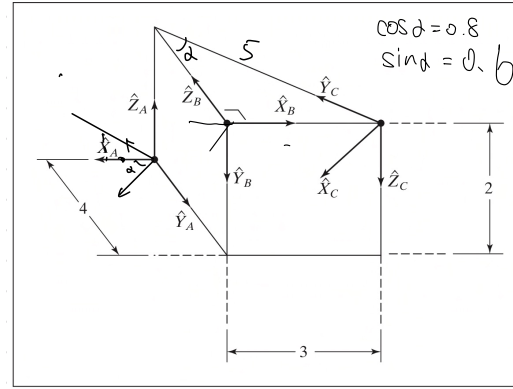


FIGURE 2.26: Frames at the corners of a wedge.

$$ {}^A_BT = \left[\begin{array}{ccc|c} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B & {}^AP_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & -1 & 0 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^A_CT = \left[\begin{array}{ccc|c} {}^A\hat{X}_C & {}^A\hat{Y}_C & {}^A\hat{Z}_C & {}^AP_{CORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} \cos\theta & \sin\theta & 0 & -3 \\ \sin\theta & -\cos\theta & 0 & 4 \\ 0 & 0 & -1 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^B_CT = \left[\begin{array}{ccc|c} {}^B\hat{X}_C & {}^B\hat{Y}_C & {}^B\hat{Z}_C & \begin{matrix}3\\0\\0\end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} -\cos\theta & -\sin\theta & 0 & 3 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

A1

$$ {}^C_AT = \left[\begin{array}{ccc|c} {}^C\hat{X}_A & {}^C\hat{Y}_A & {}^C\hat{Z}_A & {}^CP_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} \cos\theta & \sin\theta & 0 & 0 \\ \sin\theta & -\cos\theta & 0 & 5 \\ 0 & 0 & -1 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

2.4. Operators: Translations, Rotations & Transformations

1. Translational Operators

① A translation moves a point in space a finite distance along a given vector direction.

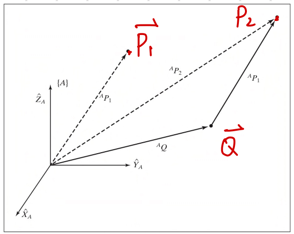


FIGURE 2.9: Translation operator.

⇒ The vector ^AQ gives the information needed to perform the translation of $^AP_1$: $^AP_2 = {^AP_1} + {^AQ}$.

⇒ $^AP_2 = D_Q(q) \cdot {^AP_1}$.

components of the translation vector Q

homogeneous transform ← $D_Q(q) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$

$q = \sqrt{q_x^2 + q_y^2 + q_z^2}$

2. Rotational Operators $^AP_2 = R_K(\theta)\, {^AP_1}$ $(\Leftrightarrow {^AP} = {^A_BR}\, {^BP})$

- $R_K(\theta)$ is a Rotational Operator that performs a rotation about the axis $\hat{K}$ by θ degrees.

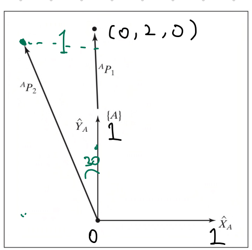


FIGURE 2.10: The vector $^AP_1$ rotated 30 degrees about $\hat{Z}$.

Example: To Rotate $^AP_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ by 30° about $\hat{Z}$, the rotational operator is

$$R_Z(30) = \left[\begin{array}{ccc|c} \cos 30° & -\sin 30° & 0 & 0 \\ \sin 30° & \cos 30° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$^AP_2 = R_Z(30) \cdot {^AP_1} = \begin{bmatrix} -1 \\ \sqrt{3} \\ 0 \end{bmatrix}$$

3. Transformation Operators: $^AP_2 = T\, {^AP_1}$

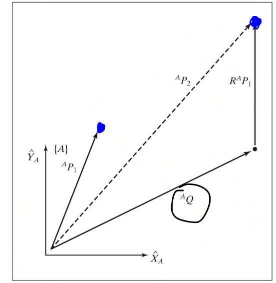


FIGURE 2.11: The vector $^AP_1$ rotated and translated to form $^AP_2$.

- A transform is usually thought of as being in the form of a homogeneous transform with general rotation-matrix and position-vector parts.

Example: $T = \left[\begin{array}{ccc|c} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$

$^AP_1 = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$

$^AP_2 = T\, {^AP_1} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \end{bmatrix}$

2.5. Summary of Interpretations of Homogeneous Transform

Homogeneous Transform is a 4×4 matrix containing rientation and position information:

- Interpretation 1: it is a description of a frame.
  - $^A_BT = \left[\begin{array}{c|c} ^A_BR & ^AP_{BORG} \\ \hline 000 & 0 \end{array}\right]$ describes the frame {B} relative to the frame {A}
  - the Columns of ^A_BR are unit vectors defining the directions of the principal axes of {B}.
  - $^AP_{BORG}$ locates the position of the origin of {B}
- Interpretation 2: it is a Transform Mapping
  ⇒ ^A_BT maps $^BP \to {^AP}$.
- Interpretation 3: it is a Transform Operator
  ⇒ T operates on $^AP_1$ to create $^AP_2$

2.6. Transformation Arithmetic (Mutiplication & Inversion)

1. Compound Transformations: Given ${}^CP$, to compute ${}^AP$.

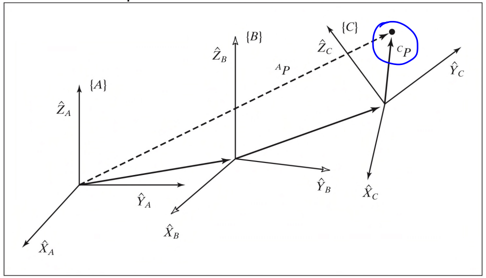


FIGURE 2.12: Compound frames: each is known relative to the previous one

$$\Rightarrow {}^AP = \underbrace{{}^A_BT\,{}^B_CT}\,{}^CP.$$

$$\Downarrow$$

$${}^A_CT$$

$$\| $$

$$\left[\begin{array}{c|c} {}^A_BR\,{}^B_CR & {}^A_BR\,{}^BP_{CORG} + {}^AP_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

2. Inverting a Transform

- Consider ${}^A_BT$ describes a frame {B} with respect to {A}, to get ${}^B_AT$, a description of {A} relative to {B} we must compute ${}^B_AR$ and ${}^BP_{AORG}$ from ${}^A_BR$ and ${}^AP_{BORG}$
- Recall ${}^B_AR = {}^A_BR^T$, change the description of ${}^AP_{BORG}$ into {B}.

$$\underbrace{{}^B({}^AP_{BORG})}_{=0} = {}^B_AR\,{}^AP_{BORG} + {}^BP_{AORG} \Leftrightarrow {}^BP_{AORG} = -{}^B_AR\,{}^AP_{BORG}$$

$$= {}^A_BR^T\,{}^AP_{BORG}$$

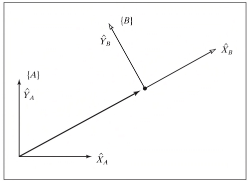


FIGURE 2.13: {B} relative to {A}.

$${}^B_AT = {}^A_BT^{-1} = \left[\begin{array}{c|c} {}^A_BR^T & -{}^A_BR^T\,{}^AP_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

2.7. Transform Equations: ${}^U_DT = {}^U_AT\,{}^A_DT = {}^U_BT\,{}^B_CT\,{}^C_DT$

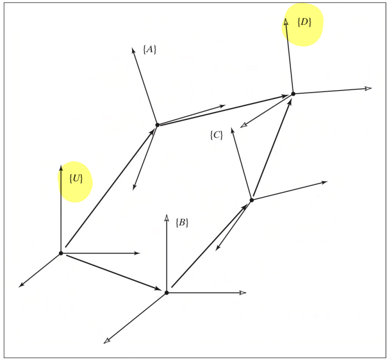


FIGURE 2.14: Set of transforms forming a loop.

transform equations can be used to solve for transforms in the case of n unknown transforms and n transform equations.

- If ${}^B_CT$ unknown, n=1

$${}^B_CT = {}^U_BT^{-1}\,{}^U_AT\,{}^A_DT\,{}^C_DT^{-1}$$

Example: Given ${}^B_TT$, which describes the frame at the manipulator's fingertips {T} relative to the base of the manipulator {B}, and ${}^B_ST$, where the tabletop is located in space relative to the manipulator's base. And ${}^S_GT$, the location of frame attached to the bolt relative to the table frame:

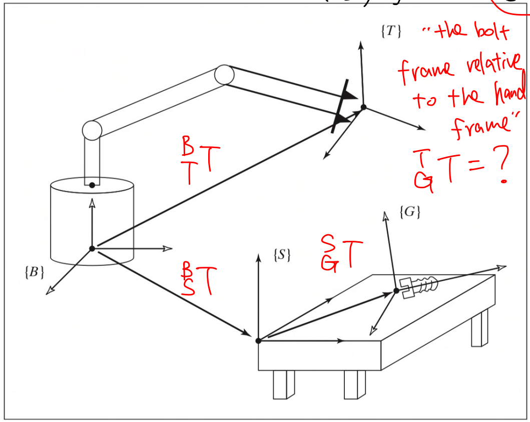


FIGURE 2.16: Manipulator reaching for a bolt.

$${}^T_GT = {}^B_TT^{-1}\,{}^B_ST\,{}^S_GT$$

2.8. More on Representation of Orientation

Recall: So far, a 3x3 Rotation Matrix with all columns are mutually orthogonal and have unit magnitude could represent an orientation.

- Rotation Matrices may also be called Proper Orthonormal Matrices, where "proper" $\Leftrightarrow \det(R) = +1$

- Question: Is it possible to describe an orientation with fewer than 9 elements?

$\Rightarrow$ Cayley's Formula: (with 6 constraints)

For any Proper Orthonormal Matrix R, there exists a skew-symmetric matrix S, $\Rightarrow R = (\underbrace{I_3}_{3\times3 \text{ Unit Matrix}} - S)^{-1}(I_3 + S)$

$$\Rightarrow S = \begin{bmatrix} 0 & -S_z & S_y \\ S_z & 0 & -S_x \\ -S_y & S_x & 0 \end{bmatrix} = -S^T$$

$\Rightarrow$ Therefore, any 3x3 Rotation Matrix can be specified by just 3 parameters. (S_x, S_y, S_z)

- Question: Is it possible for representations of orientation to be devised such that the representation is conveniently specified with 3 parameters?

Representation 1: X − Y − Z Fixed Angles.

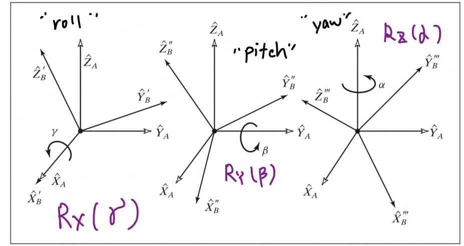


FIGURE 2.17: X–Y–Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

To describe the orientation of frame {B} w.r.t. {A}.

Rotate {B} about $\hat{X}_A$ by an angle γ, then about $\hat{Y}_A$ by an angle β, and about $\hat{Z}_A$ by an angle α. (fixed frame {A})

"$c = \cos$, $s = \sin$"

· The derivation of the equivalent Rotation Matrix,

$$ {}^A_B R_{XYZ}(\gamma,\beta,\alpha) = \underbrace{\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_Z(\alpha)} \leftarrow \underbrace{\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}}_{R_Y(\beta)} \leftarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}}_{R_X(\gamma)}$$

$$= \begin{bmatrix} c\alpha \cdot c\beta & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot c\gamma + s\alpha \cdot s\gamma \\ s\alpha \cdot c\beta & s\alpha \cdot s\beta \cdot s\gamma - c\alpha \cdot c\gamma & s\alpha\, s\beta \cdot c\gamma - c\alpha\, s\gamma \\ -s\beta & c\beta \cdot s\gamma & c\beta \cdot c\gamma \end{bmatrix}$$

Inverse Problem: Given ${}^A_B R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ to extract the fixed angles:

$$\Rightarrow \begin{cases} \beta = \text{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \\ \alpha = \text{Atan2}\left(r_{21}/c\beta,\ r_{11}/c\beta\right) \\ \gamma = \text{Atan2}\left(r_{32}/c\beta,\ r_{33}/c\beta\right) \end{cases}$$

$-90^\circ \le \beta \le 90^\circ$.

If $\beta = 90^\circ \rightarrow \alpha = 0$, $\gamma = \text{Atan2}(r_{12}, r_{22})$

If $\beta = -90^\circ \rightarrow \alpha = 0$, $\gamma = -\text{Atan2}(r_{12}, r_{22})$

$\text{Atan2}(y, x) = \tan^{-1}(y/x)$

"2-argument arc tangent function"

Representation 2: Z-Y-X Euler Angles

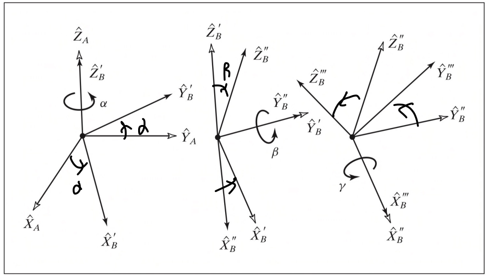


FIGURE 2.18: Z–Y–X Euler angles.

Description of a Frame {B}:

- Rotate {B} first about $\hat{Z}_B$, by an angle α, then about $\hat{Y}_B$ by an angle β, then about $\hat{X}_B$ by an angle γ, Such sets of 3 rotations are called ⇒ Euler Angles.

In this representation, each rotation is performed about an axis of the moving system {B}, rather than fixed reference {A}. Each rotation takes place about an axis whose location depends upon the preceding rotations

$$R_Z(\alpha) \longrightarrow R_Y(\beta) \longrightarrow R_X(\gamma)$$

$$ {}^A_B R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ = \begin{bmatrix} c\alpha \cdot c\beta & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot c\gamma + s\alpha \cdot s\gamma \\ s\alpha \cdot c\beta & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma \\ -s\beta & c\beta\, s\gamma & c\beta \cdot c\gamma \end{bmatrix} $$

$= {}^A_B R_{XYZ}(\gamma, \beta, \alpha)$ ⇒ Three Rotations taken about fixed axes yield the same final orientation as the same 3 rotations taken in opposite order about the axes of the moving frame

Representation 3: Z-Y-Z Euler Angles

Start with the frame coincident with a known frame {A}. Rotate {B} first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β, and about $\hat{Z}_B$ by an angle γ.

$$\Rightarrow {}^A_B R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$ = \begin{bmatrix} c\alpha \cdot c\beta \cdot c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha \cdot c\gamma & c\alpha \cdot s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha \cdot s\beta \\ -s\beta c\gamma & s\beta \cdot s\gamma & c\beta \end{bmatrix} $$

$$\beta = \operatorname{Atan2}\left(\sqrt{r_{31}^2 + r_{32}^2},\ r_{33}\right) \quad (\text{if } s\beta \neq 0)$$

$$\alpha = \operatorname{Atan2}(r_{23}/s\beta,\ r_{13}/s\beta)$$

$$\gamma = \operatorname{Atan2}(r_{32}/s\beta,\ -r_{31}/s\beta)$$

if $\sin\beta = 0$, $\beta = 0$ or $\beta = 180°$ ⇒ $\alpha = 0$

$\gamma = \operatorname{Atan2}(-r_{12}, r_{11})$ $\qquad$ $\gamma = \operatorname{Atan2}(r_{12}, -r_{11})$

There are 24 Angle-Set Conventions: 12 are for fixed-angle sets ⟷ 12 are for Euler angle sets. Because of the duality, there are only 12 unique parameterizations of a rotation matrix.

Representation 4: Equivalent Angle-Axis Representation.

Consider the following description of a frame {B}:

" Start with the frame coincident with a known frame {A}; then rotate {B} about the vector ${}^{A}\hat{K}$ by an angle θ according to the right-hand rule."

- $\hat{K}$: the equivalent axis of a finite rotation
- A general orientation of {B} relative to {A} may be written as ${}^{A}_{B}R(\hat{K},\theta)$ or $R_K(\theta)$
- The specification of the vector ${}^{A}\hat{K}$ requires ONLY 2 parameters, because its length is always taken to be 1 (unit). The angle specifies the 3rd parameter

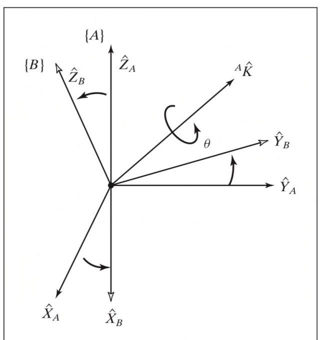


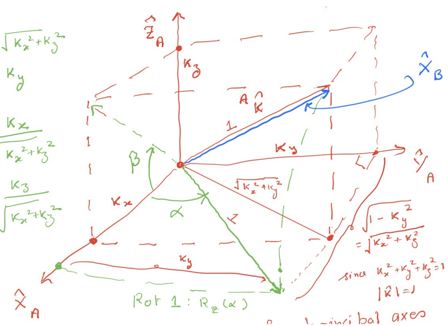
FIGURE 2.19: Equivalent angle–axis representation.

$c\alpha = \sqrt{k_x^2+k_z^2}$

$s\alpha = k_y$

$c\beta = \dfrac{k_x}{\sqrt{k_x^2+k_z^2}}$

$s\beta = \dfrac{k_z}{\sqrt{k_x^2+k_z^2}}$



If the axis of Rotation is a general axis $\hat{K}$, the equivalent rotation matrix is [w.r.t {A}]

$$R_K(\theta) = \begin{bmatrix} k_xk_xv\theta + c\theta & k_xk_yv\theta - k_zs\theta & k_xk_zv\theta + k_ys\theta \\ k_xk_yv\theta + k_zs\theta & k_yk_yv\theta + c\theta & k_yk_zv\theta - k_xs\theta \\ k_xk_zv\theta - k_ys\theta & k_yk_zv\theta + k_xs\theta & k_zk_zv\theta + c\theta \end{bmatrix}$$

$${}^{A}\hat{K} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}, \quad s\theta = \sin\theta, \quad c\theta = \cos\theta, \quad v\theta = 1-\cos\theta$$

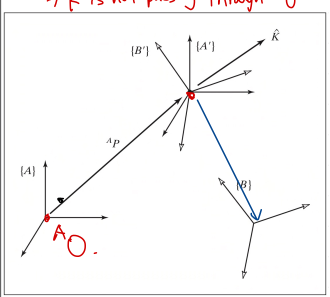
Inverse Problem: Given ${}^{A}_{B}R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

$$\Rightarrow \theta = A\cos\left(\frac{r_{11}\, r_{22}\, r_{33} - 1}{2}\right), \quad 0 < \theta < 180^\circ$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

Example: Rotate {B} about the vector ${}^{A}\hat{K} = [0.707\ 0.707\ 0]^T$ (passing through ${}^{A}P = [1\ 2\ 3]$) by an amount $\theta = 30^\circ$.

$\Rightarrow \hat{K}$ is not passing through ${}^{A}O$



① Translate ${}^{A}O \to {}^{A}P$, ${}^{B}O \to {}^{B}P$, define

$${}^{A}_{A'}T = \left[\begin{array}{c|c} I_3 & \begin{matrix}1\\2\\3\end{matrix} \\ \hline 0\,0\,0 & 1 \end{array}\right] \quad {}^{B'}_{B}T = \left[\begin{array}{c|c} I_3 & \begin{matrix}-1\\-2\\-3\end{matrix} \\ \hline 0\,0\,0 & 1 \end{array}\right]$$

② Rotate {B′} w.r.t {A′} about ${}^{A'}\hat{K}$ that pass through the origin by using formula

$${}^{A'}_{B'}T = \left[\begin{array}{c|c} {}^{A'}_{B'}R_{\hat{K}}(\theta) & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0\,0\,0 & 1 \end{array}\right]$$

$${}^{A}_{B}T = \left[\begin{array}{ccc|c} 0.933 & 0.067 & 0.354 & -1.13 \\ 0.067 & 0.933 & -0.354 & 1.13 \\ -0.354 & 0.354 & 0.866 & 0.05 \\ \hline 0.000 & 0.000 & 0.000 & 1.00 \end{array}\right].$$

③ Transform equation: ${}^{A}_{B}T = {}^{A}_{A'}T\ {}^{A'}_{B'}T\ {}^{B'}_{B}T$

2.9. Transformation of Free Vectors

In Mechanics, one makes a distinction between the Equality and the Equivalence of vectors.

- Two vectors are Equal if they have the same dimensions, magnitude, and direction.

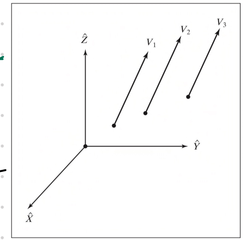


FIGURE 2.21: Equal velocity vectors.

- Two vectors are Equivalent in a certain capacity if each produces the very same effect in this capacity.
  - if the criterion in Figure 2.21 is Distance Traveled

$\Rightarrow$ all three vectors give the same result and are thus equivalent in this capacity.

  - if the criterion is height above the XY plane

$\Rightarrow$ V_1, V_2, V_3 are NOT equivalent despite their equality.

Two basic classes of vector quantities:

① Line Vector: a vector that is dependent on its line of action, along with direction and magnitude, for causing its effects. often, the effects of a force vector depend upon its line of action, so it is a line vector

② Free Vector: a vector that may be positioned anywhere in space w/o loss or change of meaning, provided that magnitude and direction are preserved.

- e.g. A pure Moment Vector is always a free vector.

$\Rightarrow$ If we have a Moment Vector ${}^{B}N$ that is known in terms of $\{B\}$, then we can calculate the same moment in terms of frame $\{A\}$ as ${}^{A}N = {}^{A}_{B}R\,{}^{B}N$.

- Likewise, a velocity vector written in $\{B\}$, ${}^{B}V$, is written in $\{A\}$ as ${}^{A}V = {}^{A}_{B}R\,{}^{B}V$

$\Rightarrow$ the velocity of a point is a free vector, so all that counts is the magnitude and direction

$\Rightarrow$ Note: ${}^{A}\cancel{P_{BORG}}$, in a position-vector transformation does NOT appear in a velocity transform

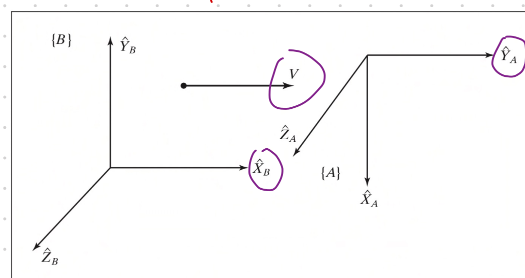


FIGURE 2.22: Transforming velocities.

${}^{B}V = 5\hat{X}_B$

${}^{A}V = 5\hat{Y}_A$

2.19. ① An object is rotated about its $\hat{X}$-axis, by ϕ, ② then it is rotated about its NEW $\hat{Y}$. by ψ. X-Y Euler: ${}^{A}_{B}R_{X'Y'}(\phi, \psi) = R_X(\phi) R_Y(\psi)$

If ① & ② rotated about fixed reference:

X-Y fixed angles:

${}^{A}_{B}R_{YX}(\psi, \phi) = R_Y(\psi) R_X(\phi)$

In the case of Specifying a rotation about an axis of the frame being moved, we are specifying a rotation in the Fixed system given by: $R_X(\phi) R_Y(\psi) R_X^{-1}(\phi)$

§3.1. Intro to Manipulator Kinematics