Constraint Satisfaction Problems

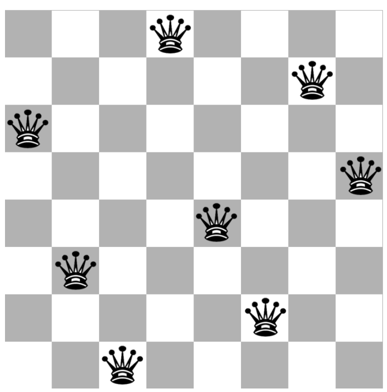
Chapter 6.1 – 6.4

Derived from slides by S. Russell and P. Norvig, A. Moore, and R. Khoury

Constraint Satisfaction Problems (CSPs)

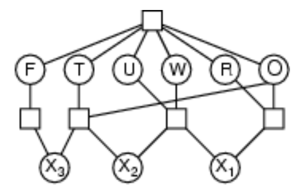
- Standard search problem:
  - **state** is a "black box" – any data structure that supports successor function, heuristic function, and goal test
- CSP:
  - **state** is defined by **variables** X_i with **values** from **domain** D_i
  - **goal test** is a set of **constraints** specifying allowable combinations of values for subsets of variables
  - Use *variable-based model*
    - Solution is not a path but an *assignment of values for a set of variables that satisfy all constraints*

Example: 8-Queens



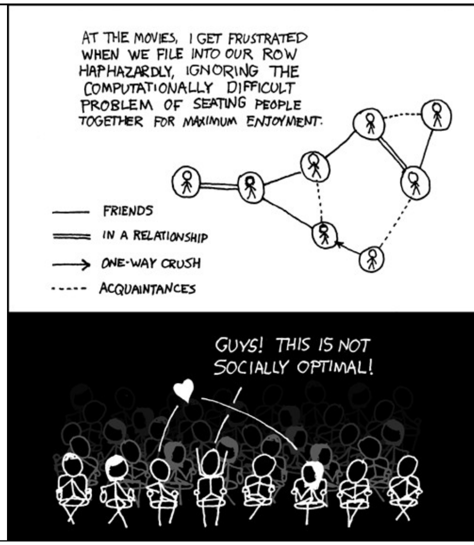
Example: Cryptarithmetic

```
  T W O
+ T W O
F O U R
```

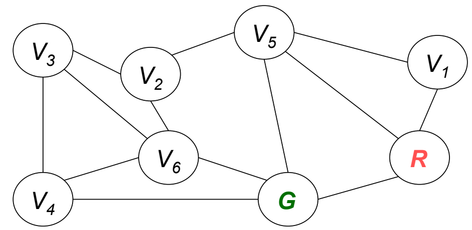


- **Variables**: $F, T, U, W, R, O, X_1, X_2, X_3$
- **Domains**: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
- **Constraints**: *Alldiff* (F, T, U, W, R, O)
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F, T \neq 0, F \neq 0$

Example: Movie Seating



Example: Graph Coloring



- Each circle marked V_1 .. V_6 we must assign *R*, *G* or *B*
- No two adjacent circles may be assigned the same value
- Note: 2 variables have already been assigned a color in this example

Other Applications of CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetable problems
  - e.g., which class is offered when and where?
- Scheduling problems
- VLSI or PCB layout problems
- Boolean satisfiability
- N-Queens
- Graph coloring
- Games: Minesweeper, Magic Squares, Sudoku, Crosswords
- Line-drawing labeling

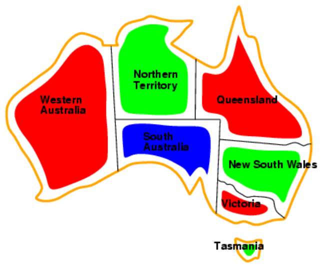
Note: many problems require real-valued variables

Example: Map-Coloring

Note: In general, 4 colors are necessary

- **Variables**: *WA, NT, Q, NSW, V, SA, T*
- **Domains**: D_i = {red,green,blue}
- **Constraints**: adjacent regions must have different colors
  e.g., WA ≠ NT, or (WA,NT) in {(red,green), (red,blue), (green,red), (green,blue), (blue,red), (blue,green)}

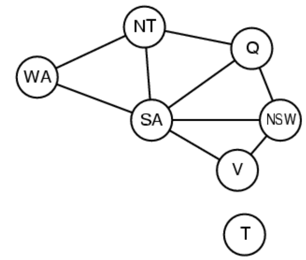
Example: Map-Coloring



Solutions are **complete** (i.e., all variables are assigned values) and **consistent** (i.e., does not violate any constraints) assignments, e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Constraint Graph

- Binary CSP: each constraint relates ***two*** variables
- Constraint graph: nodes are **variables**, arcs are **constraints**



Varieties of CSPs

- Discrete variables
  - finite domains:
    - n variables, domain size $d \rightarrow O(d^n)$ complete assignments
    - e.g., Boolean CSPs, Boolean satisfiability
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end times for each job
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Kinds of Constraints

- **Unary** constraints involve a single variable
  - e.g., SA ≠ green
- **Binary** constraints involve pairs of variables
  - e.g., SA ≠ WA
- **Higher-order** constraints involve 3 or more variables
  - e.g., cryptarithmetic column constraints

Local Search for CSPs

- Hill-climbing, simulated annealing, genetic algorithms typically work with "complete" states, i.e., *all* variables have values at every step
- To apply to CSPs:
  - allow states with some *unsatisfied* constraints
  - operators assign a value to a variable
- Variable selection: randomly select any conflicted variable
- Value selection by **min-conflicts heuristic**:
  - choose value that *violates the fewest constraints*, i.e., hill-climb by minimizing $f(n)$ = total number of violated constraints

Local Search

Min-Conflicts Algorithm:

1. Assign to each variable a random value, defining the initial state

2. **while** state not consistent **do**

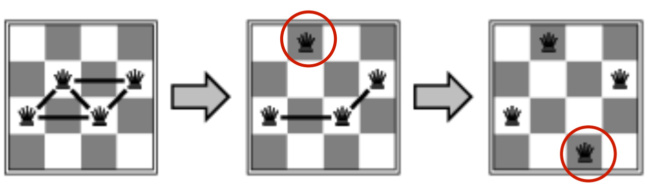
   2.1 Pick a variable, *var*, that has constraint(s) violated

   2.2 Find value, *v*, for *var* that minimizes the *total* number of violated constraints (over all variables)

   2.3 *var* = *v*

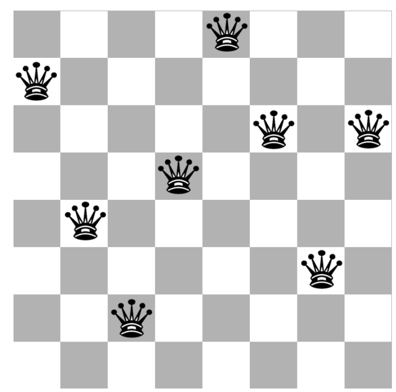
Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen to new row in its column
- Goal test: no attacks
- Evaluation function: $f(n)$ = total number of attacks



$f = 5$ $f = 2$ $f = 0$

Min-Conflicts Algorithm



Min-Conflicts Algorithm

- Advantages
  - Simple and Fast: Given random initial state, can solve *n*-Queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 1,000,000 can be solved on average in about 50 steps!)
- Disadvantages
  - Only searches states that are reachable from the initial state
    - Might not search entire state space
  - Does not allow worse moves (but can move to a neighbor with the *same* cost)
    - Might get stuck in a local optimum
  - Not complete
    - Might not find a solution even if one exists

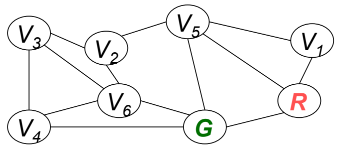
Standard Tree Search Formulation

States are defined by all the values *assigned so far*

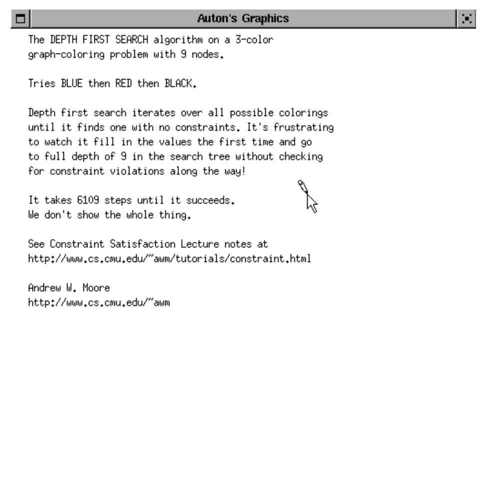
- **Initial state**: the empty assignment { }
- **Successor function**: assign a value to an unassigned variable
- **Goal test**: the current assignment is *complete* and *consistent*, i.e., all variables assigned a value and all constraints satisfied

- Goal: Find ***any*** solution, so cost is not important
- Every solution appears at depth *n* with *n* variables
  → use depth-first search

DFS for CSPs



- Variable assignments are **commutative**}, i.e.,
  [WA=R then NT=G] same as [NT=G then WA=R]
- What happens if we do DFS with the order of assignments as *B* tried first, then *G*, then *R*?
- **Generate-and-test strategy**: Generate candidate solution, then test if it satisfies all the constraints
- This makes DFS look very stupid!
- Example: http://www.cs.cmu.edu/~awm/animations/constraint/9d.html



Improved DFS: Backtracking w/ Consistency Checking

- Don't generate a successor that creates an inconsistency with any *existing* assignment, i.e., perform **consistency checking** *when node is generated*
- Successor function assigns a value to an unassigned variable that does ***not*** conflict with *all* current assignments
  - Deadend if no legal assignments (i.e., no successors)
- Backtracking (DFS) search is the basic uninformed algorithm for CSPs
- Can solve *n*-Queens for $n \approx 25$

Backtracking w/ Consistency Checking

Start with empty state

while not all vars in state assigned a value **do**

  Pick a variable (randomly or with a heuristic)

  if it has a value that does not violate any constraints

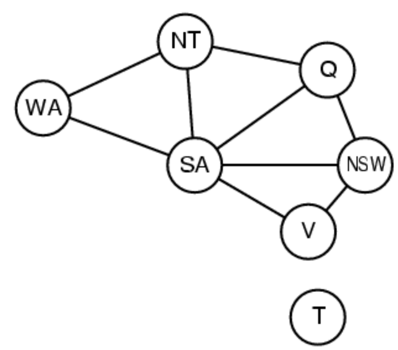
    then Assign that value

  else

    Go back to previous variable and assign it another value

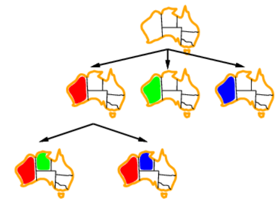
Backtracking Example

Australia Constraint Graph

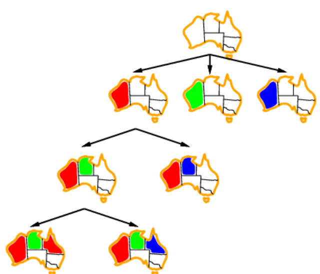


Backtracking Example

Backtracking Example



Backtracking Example



Backtracking Search

- Depth-first search algorithm
  - Goes down one variable at a time
  - At a deadend, backs up to *last* variable whose value can be changed without violating any constraints, and changes it
  - If you backed up to the root and tried all values, then there is *no* solution
- Algorithm is *complete*
  - Will find a solution if one exists
  - Will expand the entire (finite) search space if necessary
- Depth-limited search with depth limit = *n*

Auton's Graphics

The BACKTRACKING algorithm on a 3-color graph-coloring problem with 9 nodes.

Tries BLUE then RED then BLACK.

This prunes parts of the depth first search as soon as it notices a violation. Beats the heck out of DFS, though it still backtracks a little bit.

It takes 15 steps until it succeeds.

See Constraint Satisfaction Lecture notes at http://www.cs.cmu.edu/~awm/tutorials/constraint.html

Andrew W. Moore
http://www.cs.cmu.edu/~awm

Auton's Graphics

The BACKTRACKING algorithm on a 3-color graph-coloring problem with 27 nodes.

Tries BLUE then RED then BLACK.

This prunes parts of the depth first search as soon as it notices a violation. But notice how early decisions mean that no matter what it tries, for a long time nothing will work up in the top left node.

It takes 65448 steps until it succeeds.

See Constraint Satisfaction Lecture notes at http://www.cs.cmu.edu/~awm/tutorials/constraint.html

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Top-left node is hard to label!

Improving Backtracking Efficiency

- **Heuristics** can give huge gains in speed
  - Which *variable* should be assigned next?
  - In what order should its *values* be tried?
  - Can we detect inevitable failure early?

Which Variable Next? **Most-Constrained Variable**

- ***Most-constrained variable***
  - Choose the variable with the *fewest* legal values

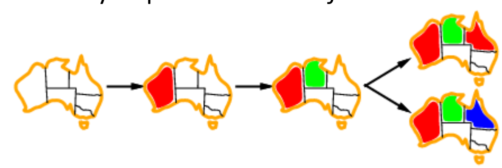
- Called the **minimum remaining values (MRV)** heuristic
- Minimizes branching factor; Tries to cut off search ASAP

Which Variable Next?
Most-Constraining Variable

- Tie-breaker among most-constrained variables
- ***Most-constraining variable***
  - Choose the variable with the *most* constraints on the *remaining variables*
- Called the **degree heuristic**
- Tries to cut off search ASAP

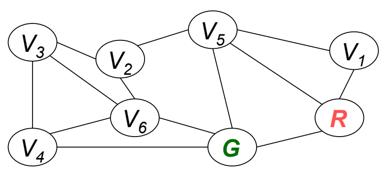
Which Value Next?
Least-Constraining Value

- Given a variable, choose the ***least-constraining value***:
  - i.e., the one that rules out the *fewest values* in the remaining variables
  - try to pick values *best first*



- Combining these heuristics makes 1000-Queens feasible

Forward Checking Algorithm



- Initially, for each variable, record the **set of *all* possible legal values for it**
- When you assign a value to a variable in the search, *update the set of legal values for **all** unassigned variables. Backtrack immediately if you **empty** a variable's set of possible values.*

Forward Checking Algorithm

- Keep track of **remaining legal values** for all variables
- Deadend when any variable has ***no*** legal values

Example: Map-Coloring

- **Variables**: *WA, NT, Q, NSW, V, SA, T*
- **Domains**: D_i = {red,green,blue}
- **Constraints**: adjacent regions must have different colors
  e.g., WA ≠ NT, or (WA,NT) in {(red,green), (red,blue), (green,red), (green,blue), (blue,red), (blue,green)}

Constraint Graph

- Binary CSP: each constraint relates ***two*** variables
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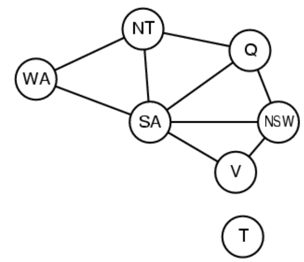
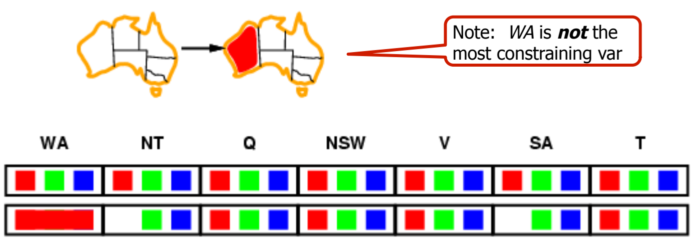


Forward Checking

– Keep track of **remaining legal values** for all unassigned variables
– Deadend when any variable has ***no*** legal values



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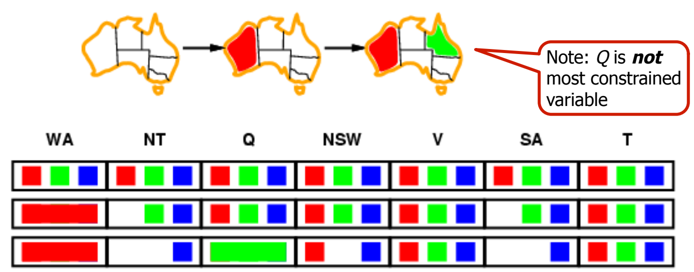

Forward Checking

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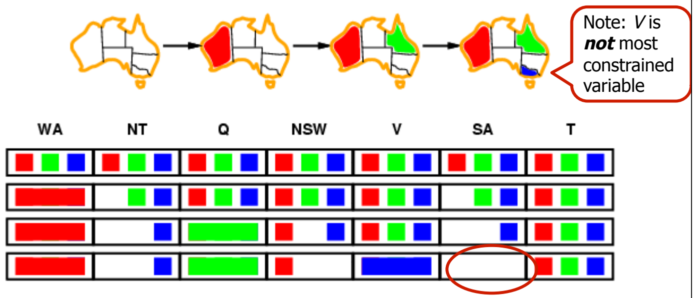

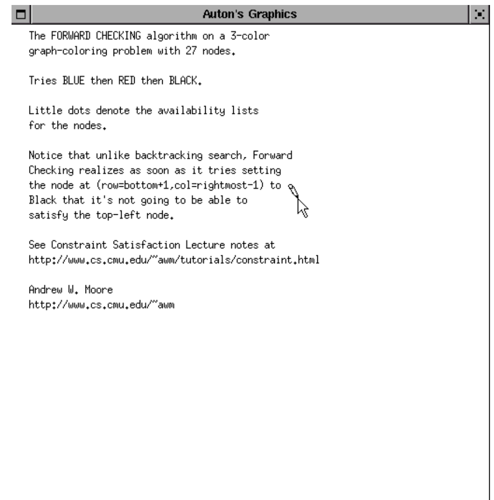

Auton's Graphics

The FORWARD CHECKING algorithm on a 3-color
graph-coloring problem with 27 nodes.

Tries BLUE then RED then BLACK.

Little dots denote the availability lists
for the nodes.

Notice that unlike backtracking search, Forward
Checking realizes as soon as it tries setting
the node at (row=bottom+1,col=rightmost-1) to
Black that it's not going to be able to
satisfy the top-left node.

See Constraint Satisfaction Lecture notes at
http://www.cs.cmu.edu/~awm/tutorials/constraint.html

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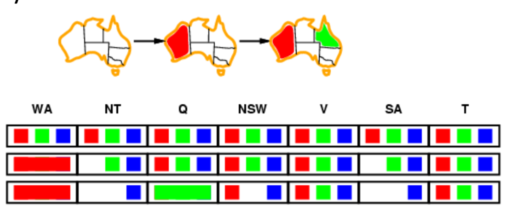
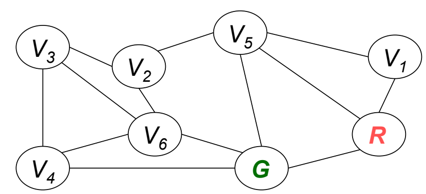
Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for ***all*** failures:



- NT and SA cannot *both* be blue!
- **Constraint propagation** *repeatedly (recursively) enforces constraints* for *all* variables

Constraint Propagation



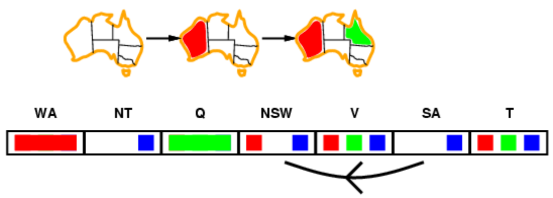
Main idea: When you delete a value from a variable's domain, check all variables connected to *it*. If any of them change, delete all inconsistent values connected to *them*, etc.

Note: In the above example, nothing changes

Arc Consistency

- Simplest form of propagation makes each **arc** (i.e., each binary constraint) **consistent**
- *X* → *Y* is consistent if

  for every value *x* at var *X* there is *some* allowed *y*, i.e., there is at least 1 value of *Y* that is consistent with *x* at *X*

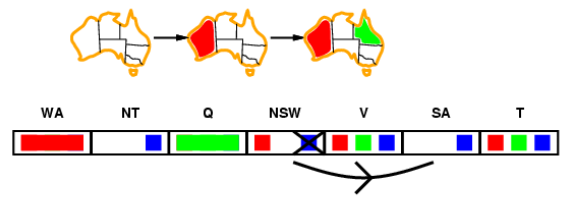


X = SA
Y = NSW

Arc Consistency

- Simplest form of propagation makes each **arc consistent**
- *X* → *Y* is consistent if

  for every value *x* at *X* there is some allowed *y*;
  if not, delete *x*

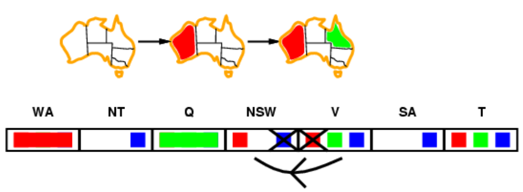


X = NSW
Y = SA

Arc Consistency

- Simplest form of propagation makes each **arc consistent**
- *X* → *Y* is consistent if

  for every value *x* at *X* there is some allowed *y*; if not, delete *x*



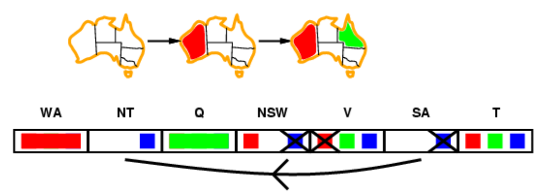
X = V
Y = NSW

- If *X* loses a value, *all* neighbors of *X* must be rechecked

Arc Consistency

- Simplest form of propagation makes each **arc consistent**
- *X* → *Y* is consistent if

  for every value *x* at *X* there is some allowed *y*; if not, delete *x*



X = SA
Y = NT

- If *X* loses a value, all neighbors of *X* must be rechecked
- Arc consistency detects failure *earlier* than forward checking
- Use as a preprocessor and after each assignment during search

Auton's Graphics

The CONSTRAINT PROPAGATION algorithm on a 3-color graph-coloring problem with 27 nodes.

Tries BLUE then RED then BLACK.

Little dots denote the availability lists for the nodes.

Notice that unlike forward checking search, Constraint Propagation realizes very early on (on its third step) that (row=bottom+1,col=rightmost-1) must not be black and so (row=bottom,col=4) must not be red. It does better than forward checking and MUCH better than backtracking!

See Constraint Satisfaction Lecture notes at http://www.cs.cmu.edu/~awm/tutorials/constraint.html

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Bottom-right node must *not* be red because node to upper-left must *not* be black

Arc Consistency Algorithm “**AC-3**”

function AC-3(*csp*) // returns false if inconsistency is found and true otherwise

// input: *csp*, a binary CSP with components (*X*, *D*, *C*)

// local variables: *queue*, a queue of arcs; initially all arcs in *csp*

while *queue* not empty **do** {

  (X_i , X_j) = Remove-First(*queue*); // check if $X_i \rightarrow X_j$ consistent

  if Revise(*csp*, X_i , X_j) **then** { // make arc consistent

    if size of D_i = 0 **then return** *false*

    foreach X_k in X_i.Neighbors – { X_j } **do** // propagate changes to neighbors

      add (X_k , X_i) to *queue*

  }

}

return *true*

Arc Consistency Algorithm “**AC-3**”

function Revise(*csp*, X_i , X_j) // returns *true* if we revise the domain of X_i

  revised = *false*;

  foreach *x* **in** D_i **do** { // check if $X_i \rightarrow X_j$ consistent

    if no value *y* in D_j allows (*x*, *y*) to satisfy the constraints between X_i and X_j **then** {

      delete *x* from D_i ;

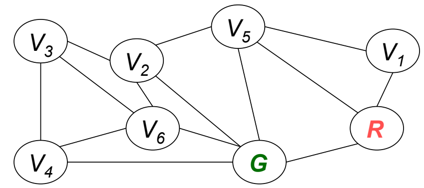
      revised = *true*;

    }

  }

  return *revised*

Constraint Propagation



- In this example, constraint propagation solves the problem without search … But not always that lucky!
- Constraint propagation can be done as a **preprocessing step**
- *And* it can be performed ***during*** search
  - Note: when you backtrack, you must *undo* some of your additional constraints

Combining Search with CSP

- Idea: Interleave search and CSP inference
- Perform DFS
  - At each node assign a selected value to a selected variable
  - Run CSP to reduce variables' domains and check if any inconsistencies arise as a result of this assignment

Combining Backtracking Search with CSP

```
function BACKTRACKING-SEARCH(csp) returns a solution or failure
  return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution or failure
  if assignment is complete then return assignment;
  var = SELECT-UNASSIGNED-VARIABLE(csp);
  foreach value in ORDER-DOMAIN-VALUES(var, assignment, csp) do {
    if value is consistent with assignment then {
      add {var = value} to assignment;
      inferences = AC-3(csp, var, value);
      if inferences != failure then {
        add inferences to assignment;
        result = BACKTRACK(assignment, csp);
        if result != failure then return result; }
      }
    remove {var = value} and inferences from assignment;
  }
  return failure
```

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node plus consistency checking
- Variable ordering and value selection heuristics help significantly
- **Forward checking** prevents assignments that guarantee later failure
- Constraint propagation (e.g., **arc consistency**) does additional work to constrain values and detect inconsistencies earlier
- Iterative **min-conflicts** is usually effective in practice