Fields on Lattice

April 16, 2016

The gauge fields conventions should follow *GaugeTheoryConvention.pdf*.

1 General SU(2) Vector

$$U \equiv U_0 - \frac{i}{2}\tau^a U_a \; ; \text{ with } U_0^2 + \left(\frac{U_1}{2}\right)^2 + \left(\frac{U_2}{2}\right)^2 + \left(\frac{U_3}{2}\right)^2 = 1 \tag{1}$$

Compontents are given by

$$\begin{aligned} U_0 &= \frac{1}{2}\text{Tr } U \\ U_a &= \text{Tr } i\tau^a U \end{aligned} \tag{2}$$

I want to change to the below definition. Hope everything is still consistent. I think 5 will be changed.

$$U \equiv U_0 + i\tau^a U_a \; ; \text{ with } U_0^2 + U_1^2 + U_2^2 + U_3^2 = 1 \tag{3}$$

Compontents are given by

$$\begin{aligned} U_0 &= \frac{1}{2}\text{Tr } U \\ U_a &= \frac{1}{2}\text{Tr } - i\tau^a U \end{aligned} \tag{4}$$

2 Link Fields and Plaquette Fields

Conventions In the program, when choosing Weyl gauge, $W_0 = 0$, all the quantities of gauge fields are in Euclidean metric, i.e., $W_i = W^i \equiv W^{\mu(=1,2,3)}$.

The evolution equations are derived solely from Kogut-Susskind Hamiltonian, so it is unaffected with these conventions.

In this section, without specification, all quantities are in Euclidean metric.

The electric field is

$$E_i \equiv F_{0i}^M = \partial_0 A_i^M = -\partial_0 A_i$$

The magnetic field is

$$B_i \equiv -\frac{1}{2}\epsilon_{ijk}F_{jk}^M = \frac{1}{2}\epsilon_{ijk}F_{jk}$$

Initial condition: what I assign are the Euclidean components, W_i, of the gauge fields. The evolution and field strength F_{ij} are well-defined.

EM fields: when converting to E and B fields, I should use the above two equations.

Boundary condition:

$$\vec{E_T} = -\vec{n} \times \vec{B}$$

and can be rewritten as $E_i^T = -n_j F_{ij}$. Now the code implementation is correct.

Link fields defined in terms of SU(2) and U(1) fields:

$$U_i(t,x) \equiv \exp\left(-\frac{i}{2} g \Delta x W_i^a \tau^a\right) \tag{5}$$

Expansion:

$$\begin{aligned} U_i(x) &= \exp\left(-\frac{i}{2} g \Delta x \tau^a W_i^a(x)\right) \\ &\simeq 1 - \frac{i}{2} g \Delta x \tau^a W_i^a(x) - \frac{1}{8} g^2 \Delta x^2 \left(W_i^a(x)\right)^2 + \mathcal{O}\left(g\Delta x\right)^3 \end{aligned}$$

$$V_i(t,x) \equiv \exp\left(-\frac{i}{2} g' \Delta x Y_i\right) \tag{6}$$

$U_i(x)$ means parallel-transports a field, say $\phi(x+i)$, back to site x, denoted as $\phi_x(x+i)$, i.e. $\phi_x(x+i) = U_i(x)\phi(x+i)$.

The forward transport is given by $U_i^\dagger(x)$, i.e. $\phi_{x+i}(x) = U_i^\dagger(x)\phi(x)$.

Basic plaquette fields[3]

$$U_{ij}(t,x) \equiv U_j(x) U_i(x+j) U_j^\dagger(x+i) U_i^\dagger(x) \approx \exp\left(\frac{i}{2} g \Delta x^2 \tau^a W_{ij}^a(t,x)\right) \tag{7}$$

$$W_{ij}^a(t,x) = \frac{1}{g\Delta x^2} \mathrm{Tr}\ \ - i\tau^a U_{ij}(t,x)$$

$$V_{ij}(t,x) = \exp\left(\frac{i}{2} g' \Delta x^2 B_{ij}(t,x)\right) \tag{8}$$

$$B_{ij}(t,x) = \frac{2}{g'\Delta x^2} \mathrm{Im}\ V_{ij}(t,x)$$

Expansion:
only repeated SU2 indice a, b, c are summed.

$$
\begin{aligned}
U_{ij}(x) \quad &\simeq \quad \left\{1 - \frac{i}{2} g\Delta x \tau^a W_j^a(x) - \frac{1}{8} g^2 \Delta x^2 W_j^a(x)^2 + \mathcal{O}\left(g\Delta x\right)^3\right\} \\
&\quad \times \left\{1 - \frac{i}{2} g\Delta x \tau^b W_i^b(x+j) - \frac{1}{8} g^2 \Delta x^2 W_i^b(x+j)^2 + \mathcal{O}\left(g\Delta x\right)^3\right\} \\
&\quad \times \left\{1 + \frac{i}{2} g\Delta x \tau^c W_j^c(x+i) - \frac{1}{8} g^2 \Delta x^2 W_j^c(x+i)^2 + \mathcal{O}\left(g\Delta x\right)^3\right\} \\
&\quad \times \left\{1 + \frac{i}{2} g\Delta x \tau^d W_i^d(x) - \frac{1}{8} g^2 \Delta x^2 W_i^d(x)^2 + \mathcal{O}\left(g\Delta x\right)^3\right\} \\
&\simeq \quad 1 + \frac{i}{2} g\Delta x^2 \tau^a \left[\frac{W_j^a(x+i) - W_j^a(x)}{\Delta x} - \frac{W_i^a(x+j) - W_i^a(x)}{\Delta x}\right] \\
&\quad -\frac{i}{4} g^2 \Delta x^2 \tau^a \epsilon^{abc} \left[W_j^b(x) W_i^c(x+j) - W_j^b(x) W_j^c(x+i) - W_j^b(x) W_i^c(x)\right. \\
&\quad \left. -W_i^b(x+j) W_j^c(x+i) - W_i^b(x+j) W_i^c(x) + W_j^b(x+i) W_i^c(x)\right] \\
&\quad -\frac{1}{8} g^2 \Delta x^2 \left[W_j^a(x)^2 + W_i^a(x+j)^2 + W_j^a(x+i)^2 + W_i^a(x)^2\right] \\
&\quad -\frac{1}{4} g^2 \Delta x^2 \left[W_j^a(x) W_i^a(x+j) - W_j^a(x) W_j^a(x+i) - W_j^a(x) W_i^a(x)\right. \\
&\quad \left. -W_i^a(x+j) W_j^a(x+i) - W_i^a(x+j) W_i^a(x) + W_j^a(x+i) W_i^a(x)\right]
\end{aligned}
$$

Electric link fields

$$\tilde{E}_i(x, t + \frac{\Delta t}{2}) \equiv \frac{1}{g\Delta x \Delta t} U_{0i}(x,t) = \frac{1}{g\Delta x \Delta t} U_0(x,t) U_i(x, t+\Delta t) U_0^\dagger(x+i, t) U_i^\dagger(x,t) \tag{9}$$

In temperal gauge, this reduces to

$$\tilde{E}_i(x, t + \frac{\Delta t}{2}) = \frac{1}{g\Delta x \Delta t} U_i(x, t+\Delta t) U_i^\dagger(x,t)$$

or,

$$U_i(x, t+\Delta t) = g\Delta x \Delta t \tilde{E}_i(x, t + \frac{\Delta t}{2}) U_i(x,t)$$

Relations with continuum electric fields:

$$\tilde{E}_i\left(x, t + \frac{\Delta t}{2}\right) = \frac{1}{g\Delta x \Delta t} \exp\left(\frac{i}{2} g\Delta x \Delta t \tau^a E_i^a(x, t + \frac{\Delta t}{2})\right)$$

$$E_i^a(x, t + \frac{\Delta t}{2}) = \mathrm{Tr}\ -i\tau^a \tilde{E}_i\left(x, t + \frac{\Delta t}{2}\right)$$

$$\tilde{F}_i(x, t + \frac{\Delta t}{2}) = \frac{2}{g'\Delta x \Delta t} \exp\left(\frac{i}{2} g' \Delta x \Delta t E_i(x, t + \frac{\Delta t}{2})\right)$$

$$E_i(x, t + \frac{\Delta t}{2}) = \mathrm{Im}\ \tilde{F}_i(x, t + \frac{\Delta t}{2})$$

Rectangular plaquette fields I added this explicit definition.

$$U_{rect,ij} \equiv U_j(x) U_j(x+j) U_i(x+j+j) U_j^\dagger(x+i+j) U_j^\dagger(x+i) U_i^\dagger(x) \tag{10}$$

3 Kogut-Susskind Hamiltonian

$$\pi(x, t + \frac{\Delta t}{2}) \equiv D_0\phi(x) = \frac{1}{\Delta t}\left[\phi(x, t + \Delta t) - \phi(x)\right] \tag{11}$$

$$D_i\phi(x) = \frac{1}{\Delta x}\left[U_i(x)\phi(x+i) - \phi(x)\right] \tag{12}$$

$$D_i^2\phi(x) = \frac{1}{\Delta x^2}\left[U_i(x)\phi(x+i) - 2\phi(x) + U_i^\dagger(x-i)\phi(x-i)\right] \tag{13}$$

$$\begin{aligned} S \;=\; & \sum_{x,t}\Delta t\Delta x^3\left\{(D_0\phi)^\dagger(D_0\phi) - \sum_i (D_i\phi)^\dagger(D_i\phi) - V(\phi)\right. \\ & + \left(\frac{2}{g\Delta t\Delta x}\right)^2\sum_i\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{0i}\right) + \left(\frac{2}{g'\Delta t\Delta x}\right)^2\sum_i(1 - \mathrm{Re}\ V_{0i}) \\ & \left. - \frac{2}{g^2\Delta x^4}\sum_{i,j}\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{ij}\right) - \frac{2}{g'^2\Delta x^4}\sum_{i,j}(1 - \mathrm{Re}\ V_{ij})\right. \end{aligned} \tag{14}$$

Convention transform from [3], for SU2 part:

1.an overall factor $\left(\frac{2}{g\Delta x^2}\right)^2$ for Hamiltonian, note when summing over i and j, the energy is double counted, and should be divided by 2.

2.$\Delta t \to \frac{\Delta t}{\Delta x}$.

3. For all electric and magnetic fields, the pre-factor $\frac{1}{2}$ is modified.

Equation of motion: (Ref. Eq.(19) in[3])

$$\mathrm{Tr}\ -i\tau^m\tilde{E}_i(x, t + \frac{\Delta t}{2}) = \mathrm{Tr}\ -i\tau^m\tilde{E}_i(x, t - \frac{\Delta t}{2}) - \frac{\Delta t}{g\Delta x^3}DF_i^m(x,t)$$

$$DF_i^m(x,t) \equiv \sum_{4\square}\mathrm{Tr}\ -i\tau^m U_\square \tag{15}$$

where the 4 plaquettes contains the (x, i) link, oriented to contain U_i. (They should begin and end at x.

Improved gauge field Hamiltonian: $\mathcal{O}(a^4)$

$$S = S_E - S_U$$

$$\begin{aligned} S_E \;=\; & \left(\frac{2}{g\Delta t\Delta x}\right)^2\left[\frac{5}{3}\sum_i\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{i0}\right) - \frac{1}{12}\sum_i\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{rect,i0}\right) - \frac{1}{12}\sum_i\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{rect,0i}\right)\right] \\ \to\; & \left(\frac{2}{g\Delta t\Delta x}\right)^2\left[\frac{4}{3}\sum_i\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{i0}\right) - \frac{1}{12}\sum_i\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{rect,0i}\right)\right] \end{aligned}$$

to prevent unstable behaviour.

$$S_U = \left(\frac{2}{g\Delta x^2}\right)^2\left[\frac{5}{3}\sum_{i>j}\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{ij}\right) - \frac{1}{12}\sum_{i>j}\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{rect,ij}\right) - \frac{1}{12}\sum_{i>j}\left(1 - \frac{1}{2}\mathrm{Tr}\ U_{rect,ji}\right)\right]$$

Where U_{rect} are 1×2 rectangular plaquettes.

$$DF_i^m(x) = \frac{5}{3}\left(\sum_{4\Box} \text{Tr } i\tau^m U_\Box\right) - \frac{1}{12}\left(\sum_{12\ rect} \text{Tr } i\tau^m U_{rect}\right) \tag{16}$$

where the 12 rectangular plaquettes contain the $U_i(x)$ link, beginning and ending at x.
Equation of motion:

$$\begin{aligned}
& \frac{4}{3}\text{Tr } i\tau^m E_i(x,t+\frac{\Delta t}{2}) - \frac{g\Delta x\Delta t}{12}\left[\text{Tr } i\tau^m E_i(x,t+\frac{\Delta t}{2})E_i(x+i,t+\frac{\Delta t}{2}) + \text{Tr } i\tau^m E_i(x-i,t+\frac{\Delta t}{2})E_i(x,t+\frac{\Delta t}{2})\right] \\
= & \frac{4}{3}\text{Tr } i\tau^m E_i(x,t-\frac{\Delta t}{2}) - \frac{g\Delta x\Delta t}{12}\left[\text{Tr } i\tau^m E_i(x+i,t-\frac{\Delta t}{2})E_i(x,t-\frac{\Delta t}{2}) + \text{Tr } i\tau^m E_i(x,t-\frac{\Delta t}{2})E_i(x-i,t-\frac{\Delta t}{2})\right] \\
& -\frac{\Delta t}{g\Delta x^3}DF_i^m(x,t)
\end{aligned} \tag{17}$$

where $DF_i^m(x,t)$ should be replaced by the improved version, Eq.(16).
This equation should be solved iteratively.

$$\begin{aligned}
\frac{7}{6}\text{Tr } i\tau^m E_i(x) \quad = \quad & \text{RHS of Eq.(17)} \\
& +\frac{g\Delta x\Delta t}{12}\left[\text{Tr } i\tau^m E_i(x)E_i(x+i) + \text{Tr } i\tau^m E_i(x-i)E_i(x)\right] - \frac{1}{6}\text{Tr } i\tau^m E_i(x)
\end{aligned} \tag{18}$$

According to [3], using $E_i(t-\Delta t/2)$ as the initial guess, it will take about 4 iterations to get the roundoff error.

Improved Scalar part:$\mathcal{O}(a^4)$ Lagrangian

$$\mathcal{L} = |D_t\phi(x)|^2 - \sum_i |\nabla_i\phi(x)|^2 - V(\phi) \tag{19}$$

with

$$|D_t\phi(x)|^2 = \frac{1}{\Delta t^2}\left[\phi(x,t+\Delta t) - \phi(x,t)\right]^\dagger \left[\phi(x,t+\Delta t) - \phi(x,t)\right] \tag{20}$$

$$\begin{aligned}
-|\nabla_i\phi(x)|^2 \quad = \quad & -\frac{1}{\Delta x^2}\left[\phi(x) - U_i^\dagger(x-i)\phi(x-i)\right]^\dagger \\
& \times\left[-\frac{1}{12}U_i(x)\phi(x+i) + \frac{5}{4}\phi(x) - \frac{5}{4}U_i^\dagger(x-i)\phi(x-i) + \frac{1}{12}U_i^\dagger(x-i)U_i^\dagger(x-2i)\phi(x-2i)\right]
\end{aligned} \tag{21}$$

Equation of motion

$$\begin{aligned}
\nabla^2_{Lat}\phi(x) \quad = \quad & \frac{1}{\Delta x^2}\sum_i\left\{\frac{4}{3}\left[U_i(x)\phi(x+i) + U_i^\dagger(x-i)\phi(x-i)\right]\right. \\
& \left.-\frac{1}{12}\left[U_i(x)U_i(x+i)\phi(x+2i) + U_i^\dagger(x-i)U_i^\dagger(x-2i)\phi(x-2i)\right] - \frac{5}{2}\phi(x)\right\}
\end{aligned}$$

$$\pi(x,t+\frac{\Delta t}{2}) = \pi(x,t-\frac{\Delta t}{2}) + \Delta t\left[\nabla^2_{Lat}\phi(x) - \frac{\partial V(\phi)}{\partial\phi^\dagger}\right] \tag{22}$$

4 Implementation

Summary: Lagrangian

$$
\begin{aligned}
\mathcal{L} &= |D_t\phi(x)|^2 - \sum_i |\nabla_i \phi(x)|^2 - V(\phi) \\
&+ \left(\frac{2}{g\Delta t\Delta x}\right)^2 \left[\frac{4}{3}\sum_i \left(1-\frac{1}{2}\mathrm{Tr}\ U_{0i}\right) - \frac{1}{12}\sum_i \left(1-\frac{1}{2}\mathrm{Tr}\ U_{rect,0i}\right)\right] \\
&- \left(\frac{2}{g\Delta x^2}\right)^2 \left[\frac{5}{3}\sum_{i>j} \left(1-\frac{1}{2}\mathrm{Tr}\ U_{ij}\right) - \frac{1}{12}\sum_{i>j} \left(1-\frac{1}{2}\mathrm{Tr}\ U_{rect,ij}\right) - \frac{1}{12}\sum_{i>j} \left(1-\frac{1}{2}\mathrm{Tr}\ U_{rect,ji}\right)\right] \\
&+ \left(\frac{2}{g'\Delta t\Delta x}\right)^2 \left[\frac{4}{3}\sum_i (1-\mathrm{Re}\ V_{0i}) - \frac{1}{12}\sum_i (1-\mathrm{Re}\ V_{rect,0i})\right] \\
&- \left(\frac{2}{g'\Delta x^2}\right)^2 \left[\frac{5}{3}\sum_{i>j} (1-\mathrm{Re}\ V_{ij}) - \frac{1}{12}\sum_{i>j} (1-\mathrm{Re}\ V_{rect,ij}) - \frac{1}{12}\sum_{i>j} (1-\mathrm{Re}\ V_{rect,ji})\right]
\end{aligned}
$$

where details are given in Eq.(20), (21), (7), (10).
The evolution for scalar field is given by Eq.(22).
The SU2 field evolution is given by Eq.(17) and (18). The contribution needs to be added.
For U1 part, the additional contribution due to scalar field is

$$
\begin{aligned}
\frac{g'\Delta t\Delta x}{-2i} V_i(t,x)\frac{\delta\mathcal{L}}{\delta V_i(t,x)} &= (?) \quad \Delta t\frac{g'}{\Delta x} \times \left\{-\frac{1}{12}\mathrm{Im}\ \left[\phi^\dagger(x+2i)V_i^\dagger(x+i)V_i^\dagger(x)\phi(x)\right] + \frac{4}{3}\mathrm{Im}\ \left[\phi^\dagger(x+i)V_i^\dagger(x)\phi(x)\right]\right. \\
&\left. -\frac{1}{12}\mathrm{Im}\ \left[\phi^\dagger(x+i)V_i^\dagger(x)V_i^\dagger(x-i)\phi(x-i)\right]\right\}
\end{aligned}
$$

For SU2 part, the additional contribution should be (I guess)

$$
\begin{aligned}
\frac{\delta\mathcal{L}}{\delta U_i^a(t,x)} &= \frac{-2}{\Delta x^2} \times \left\{-\frac{1}{12}\mathrm{Re}\ \left[\phi^\dagger(x+2i)U_i^\dagger(x+i)U_i^\dagger(x)i\sigma^a\phi(x)\right] + \frac{4}{3}\mathrm{Re}\ \left[\phi^\dagger(x+i)U_i^\dagger(x)i\sigma\phi(x)\right]\right. \\
&\left. -\frac{1}{12}\mathrm{Re}\ \left[\phi^\dagger(x+i)U_i^\dagger(x)i\sigma^a U_i^\dagger(x-i)\phi(x-i)\right]\right\}
\end{aligned}
$$

which should be added to the RHS of the evolution equation 18.
(?) sign of the $-i$, different from my calculation.

Assignment of gauge fields Normally, for initial conditions, we will have the analytical expressions. It is easy to calculate the distance of each lattice site and therefore the field values on each lattice site. However, the link fields are defined on the links between neighboring sites. It is reasonable to take the field value on the mid-point of two neighboring sites as the link field value.

Calculate the W field at the mid-point first, then directly get the $U_i(x)$.

Below gives an error estimate of on-site assignment and mid-point assignment.

Here x_0 denotes the site we consider, x_c denotes the center point the plaquette encloses, at which point the magnetic field is defined.

ON-SITE assignment

$$
\begin{aligned}
U_{ij}^O &= \exp\left[-iA_j(x_0)a\right]\exp\left[-iA_i(x_0+j)a\right]\exp\left[+iA_j(x_0+i)a\right]\exp\left[+iA_i(x_0)a\right] \\
&= \exp\left[ia^2F_{ij}(x_c) + i\frac{a^3}{2}\partial_i\partial_j\left(A_i - A_j\right)(x_c) + O(a^4)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{a^4}\left(1-\mathrm{Tr}\ U^O_{ij}\right) &\approx \frac{1}{2}\left[F_{ij}(x_c)+\frac{a}{2}\partial_i\partial_j\left(A_i-A_j\right)(x_c)+O(a^2)\right]^2 \\
&= \frac{1}{2}F_{ij}^2+\frac{a}{2}F_{ij}\partial_i\partial_j\left(A_i-A_j\right)+\frac{1}{2}\frac{a^2}{4}\left[\partial_i\partial_j\left(A_i-A_j\right)\right]^2+F_{ij}O(a^2)
\end{aligned}
$$

MID-POINT assignment

$$
\begin{aligned}
U^M_{ij} &= \exp\left[-iA_j(x_0+\frac{j}{2})a\right]\exp\left[-iA_i(x_0+j+\frac{i}{2})a\right]\exp\left[+iA_j(x_0+i+\frac{j}{2})a\right]\exp\left[+iA_i(x_0+\frac{i}{2})a\right] \\
&= \exp\left[ia^2F_{ij}(x_c)+i\frac{a^4}{24}\left(\partial_i^3A_j-\partial_j^3A_i\right)(x_c)+O(a^6)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{a^4}\left(1-\mathrm{Tr}\ U^M_{ij}\right) &\approx \frac{1}{2}\left[F_{ij}(x_c)+\frac{a^2}{24}\left(\partial_i^3A_j-\partial_j^3A_i\right)(x_c)+O(a^4)\right]^2 \\
&= \frac{1}{2}F_{ij}^2+\frac{a^2}{24}F_{ij}\left(\partial_i^3A_j-\partial_j^3A_i\right)+\frac{1}{2}\frac{a^4}{24^2}\left(\partial_i^3A_j-\partial_j^3A_i\right)^2+F_{ij}O(a^4)
\end{aligned}
$$

Plaquette fields The basic plaquette fields physically represents the (magnetic) field value at the center point of the four sites surrounding it. To get the on-site magnetic field, we need to average four basic plaquette fields.

$$
\bar{U_{ij}}(t,x)=\frac{1}{4}\left[U_{ij}(t,x)+U_{ij}(t,x-i)+U_{ij}(t,x-j)+U_{ij}(t,x-i-j)\right]
$$

Similarly, for on-site electric field,

$$
\bar{E_i}(t,x)=\frac{1}{2}\left[E_i(t,x)+E_i(t,x-i)\right]
$$

Energy and other observables The electric fields are defined on half-timesteps, say $t+\frac{\Delta t}{2}$, while the magnetic fields are defined on integer-timesteps. To calculate some observables, e.g. energy, then we need to average two timesteps for one of the two quantities.[2] Note the magnetic fields need not to be averaged (they are same).

$$
\begin{aligned}
E_{SU2}(t) &= \frac{1}{\Delta t^2\Delta x^2 g^2}\sum_i\left[2-\mathrm{Tr}\ U_{0i}(x,t+\Delta t/2)\right]+\left[2-\mathrm{Tr}\ U_{0i}(x,t-\Delta t/2)\right] \\
&+\frac{1}{\Delta x^4 g^2}\sum_{ij}\left[2-\mathrm{Tr}\ U_{ij}(t,x)\right]
\end{aligned}
$$

In [3], the author averaged 8 magnetic fields to get its value on links, then calculate the change in CS number.

Boundaries With the improved evolution equation, the needed sites to update one specific site x is $x\pm 2$, increased by 1. So the outmost 2 layers in the lattice should be treated as boundaries. I think we can update the second outmost layer by the original KS solution, while the outmost layer should be updated by the Absorbing Boundary Condition.

The lowest-order absorbing boundary conditions are adopted from [5].

$$
n_iD_i\phi=-D_t\phi \tag{23}
$$

$$
\vec{E}_T=-\vec{n}\times\vec{B} \tag{24}
$$

The ABC for scalar field can be improved,

$$\left(1+\frac{\Delta t}{2\Delta x}\sum_i \frac{n_i}{\|n\|}\right)\pi(t+\frac{\Delta t}{2},x) = \pi(t-\frac{\Delta t}{2},x)+\frac{\Delta t}{2\Delta x}\sum_i \frac{n_i}{\|n\|}\pi(t+\frac{\Delta t}{2},x\mp i) + \frac{\Delta t}{2}\left[\nabla_\perp^2\phi(t,x)-\frac{\partial V}{\partial\phi^\dagger}\right]-\frac{\Delta t}{2\Delta x}\sum_i\frac{n_i}{\|n\|}\left[\pi(t-\frac{\Delta t}{2},x)-\pi(t-\frac{\Delta t}{2},x\mp i)\right] \tag{25}$$

where, n_i is the (unnormalized) normal vector for the boundary. For right boundary, $n_i = 1$; for left boundary, $n_i = -1.\|n\|$ is the module of n_i. The choice of $x \mp i$ is such that the site $x \mp i$ is inside the lattice. The above equation is derived from [6] and [7], and is for pure scalar fields. The extension to coupling with gauge fields is intuitive and not rigorous, i.e. just replace the ordinary derivative with covariant derivative. (In the previouly code, I didn't put the corresponding link fields to the $\pi(t,x)$ fields, which I think should.)

For the gauge field,

$$\begin{aligned} E_{Ti} &= -\epsilon_{ijk}n_jB_k = -\frac{1}{2}\epsilon_{ijk}\epsilon_{kpq}n_jF_{pq} \\ &= -n_jF_{ij} \end{aligned}$$

(this is different from my previous derivation, with a sign, and should be checked in code)

$$\text{Im } F_i^T(x,t+\frac{\Delta t}{2}) = -n_j\frac{2}{g'\Delta x^2}\text{Im } V_{ij}(t,x)$$

$$\text{Tr } i\tau^aE_i^T(x,t+\frac{\Delta t}{2}) = n_j\frac{1}{g\Delta x^2}\text{Tr } -i\tau^aU_{ij}(t,x)$$

EM electric field

$$\begin{aligned} E_i^{em} &\equiv \sin\theta_w n^aE_i^a+\cos\theta_wE_i - i\frac{2g}{|\phi|^2}\left[\pi^\dagger D_i\phi-\left(D_i\phi\right)^\dagger\pi\right] \\ &= \sin\theta_w n^a\text{Tr}\left[-i\sigma^aE_i(x,t+\frac{\Delta t}{2})\right]+\cos\theta_w\text{Im }\left[F_i\left(x,t+\frac{\Delta t}{2}\right)\right]-i\frac{2g}{|\phi|^2} \\ &\quad \left[\pi(t+\frac{\Delta t}{2})^\dagger D_i\phi-\left(D_i\phi\right)^\dagger\pi(t+\frac{\Delta t}{2})\right] \end{aligned}$$

E_i^{em} is defined at $t+\frac{\Delta t}{2}$ moments, so ϕ field should take apropriate time average.

5 Conversions

For SU(2) fields, in temporal gauge $W^0 = 0$,

$$U = \exp\left(-ig\Delta x\frac{\tau^a}{2}W^a\right) = U_0 - \frac{i}{2}\tau^aU_a$$

In terms of W fields,

$$U = \cos\frac{g\Delta x|W|}{2} - i\tau^a\tilde{W}^a\sin\frac{g\Delta x|W|}{2}$$

with $|W|^2 = \sum_a\left(W^a\right)^2$, $\tilde{W}^a = \frac{W^a}{|W|}$.

Transformation from W fields to link fields:

$$U_0 = \cos\frac{g\Delta x|W|}{2} \quad ; \quad \frac{U_a}{2} = \tilde{W}^a\sin\frac{g\Delta x|W|}{2}$$

Transformation from link fields to W fields:

$$\frac{g\Delta x|W|}{2} = \arccos U_0 \text{ or } \frac{g\Delta x|W|}{2} = \arctan\left(\sqrt{\sum_a \left(\frac{U_a}{2}\right)^2}/U_0\right)$$

$$\frac{g\Delta x}{2}W^a = \tilde{W^a}|W| = \frac{U_a/2}{\sin\left(g\Delta x|W|/2\right)}\arctan\left(\sqrt{\sum_a \left(\frac{U_a}{2}\right)^2}/U_0\right)$$

For U(1) fields, in temporal gauge $B^0 = 0$,
Transformation between Y fields and link fields:

$$V = \exp\left(-i\frac{g'\Delta x}{2}Y\right) = \cos\left(\frac{g'\Delta x}{2}Y\right) - i\sin\left(\frac{g'\Delta x}{2}Y\right)$$

$$\frac{g'\Delta x}{2}Y = -\arctan\frac{\mathrm{Im}\ V}{\mathrm{Re}\ V}$$

References

[1] G. D. Moore, Phys. Rev. D 59, 014503 (1998).

[2] G. D. Moore, Nucl. Phys. B 480, 689 (1996).

[3] [1] G. D. Moore, 480, 657 (1996).

[4] a. Rajantie, P. M. Saffin, and E. J. Copeland, Phys. Rev. D 63, 1 (2001).

[5] Olum, K. D., & Blanco-Pillado, J. J. (1999). Radiation from cosmic string standing waves, 4. doi:10.1103/PhysRevLett.84.4288

[6] Engquist, B., & Majda, A. (1977). Absorbing Bounday Conditions for the Numerical Simulation of Waves. Mathematics of Computation, 31(139), 629–651.

[7] Szeftel, J. (2006). A NONLINEAR APPROACH TO ABSORBING BOUNDARY CONDITIONS FOR THE SEMILINEAR WAVE EQUATION. Mathematics of Computation, 75(254), 565–594.