Maths concepts revision

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Revision of matrices

$A_{n \times m}$: matrix with n rows and m columns

$\mathbb{A}$: normally written in bold or capital letter is the matrix

a_{ij} : is an element of the matrix at row i and column j

$$\mathbb{A} = \begin{bmatrix} 2 & 45 \\ 13 & 4 \end{bmatrix} \quad ; \quad a_{11} = 2 \quad ; \quad a_{12} = 45 \quad ; \quad a_{21} = 13$$

$$a_{22} = 4$$

$$\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & \cdots & a_{nm} \end{bmatrix}$$

In class $\mathbb{T}$ is the matrix with elements T_{ij}

Operations with matrices:

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ \vdots & & & \\ a_{n1}+b_{n1} & \cdots & \cdots & a_{nm}+b_{nm} \end{bmatrix}$$

the same holds for $A - B$ with $-$ sign

need to have the same dimensions

Multiplication: e.g. 3 x 3 matrices

$$A \times B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$c_{11} = a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31}$$

$$\boxed{c_{12}} = a_{11} \times b_{12} + a_{12} \times b_{22} + a_{13} \times b_{32}$$

$$\boxed{c_{21}} = a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31}$$

$$\Rightarrow A \times B \neq B \times A$$

Matrices do not need to be the same size

$$A_{n \times m} \times B_{m \times k} = C_{n \times k}$$

e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \times b_1 + a_{12} \times b_2 \\ a_{21} \times b_1 + a_{22} \times b_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

What happened? $\Rightarrow$ Transformation of vector

B and C are vectors

Vectors:

In general, and in particular for the networks section, vectors contain characteristics of places, populations, etc.

For example, if we want to encode socio-demographic characteristics, such as:

v_1: the number of individuals in extreme poverty

v_2: " " " " with education at primary level

v_3: " " " " " access to clean water

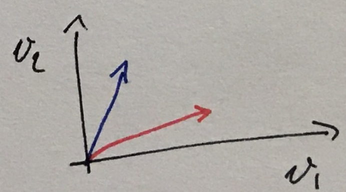
etc.

These can be encoded in a vector $\vec{v}$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix}$$

For simplicity, imagine a situation of only 2 characteristics

$$\vec{v} = \begin{bmatrix} \text{n. of people in poverty} \\ \text{" " " in high skilled jobs} \end{bmatrix}$$



$\vec{v}$: city A

$\vec{v}$: " B

We would like to introduce an intervention to reduce v_1 and we do this through an education programme to increase v_2.

Unfortunately, due to greed, ~~part~~ and economic crises, part of the population in v_2 loses their jobs and become part of v_1.

P_e : percentage of population from $v_1 \to v_2$ after 1 year

P_c : " " " " $v_2 \to v_1$ " " "

Imagine at t_0 : v_1 and v_2 represent those populations.

We want to know what happens at t = future.

After 1 year $t_1 = t_0 + 1$:

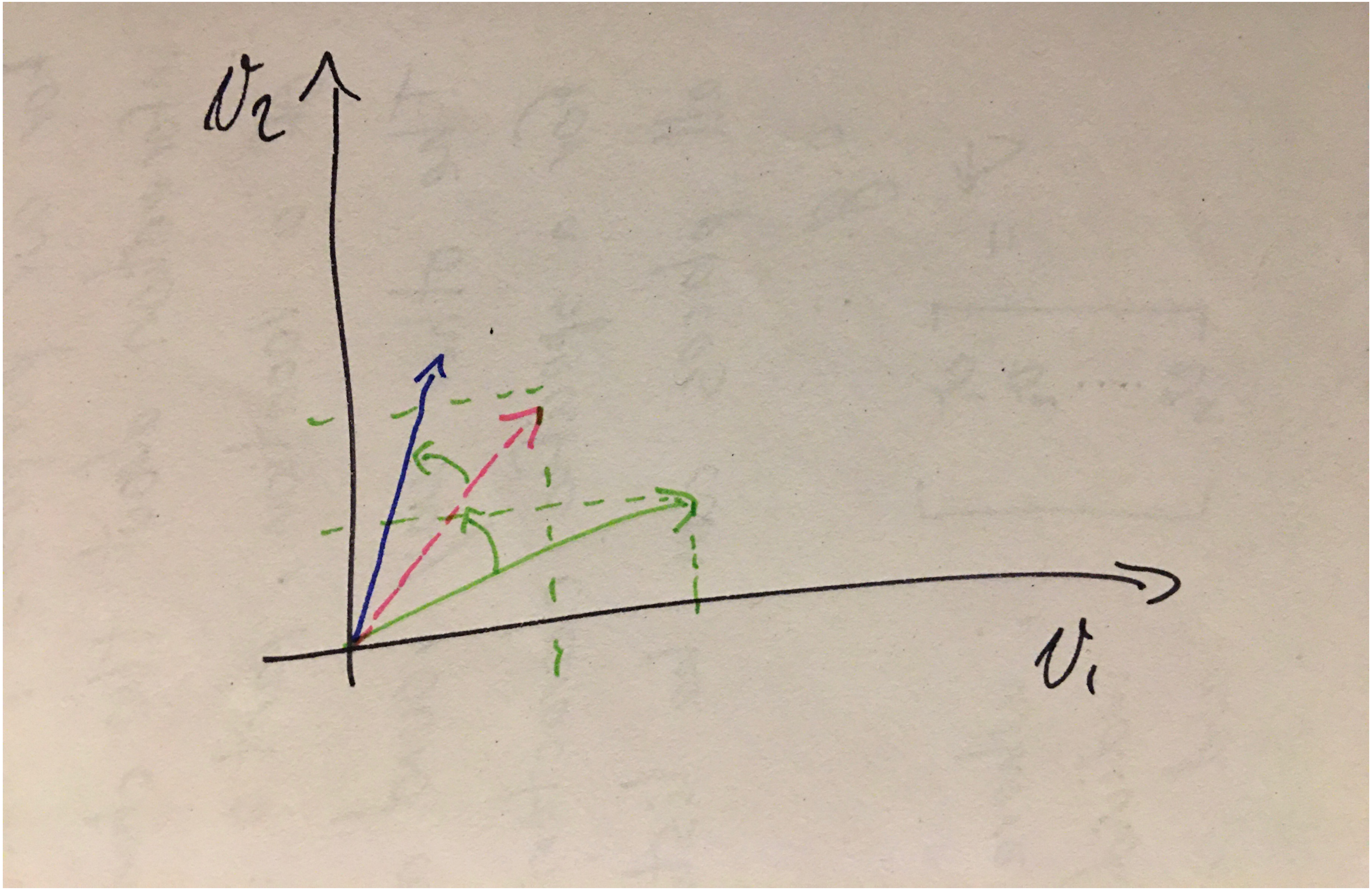
$$v_1(t_1) = v_1(t_0) - p_e \cdot v_1(t_0) + p_c \cdot v_2(t_0)$$
$$v_2(t_1) = p_e \cdot v_1(t_0) + v_2(t_0) - p_c \cdot v_2(t_0)$$

$$\Rightarrow \begin{array}{l} (1-p_e) \cdot v_1(t_0) + p_c \cdot v_2(t_0) = v_1(t_1) \\ p_e \cdot v_1(t_0) + (1-p_c)\, v_2(t_0) = v_2(t_1) \end{array}$$

$$\begin{pmatrix} 1-p_e & p_c \\ p_e & 1-p_c \end{pmatrix} \begin{pmatrix} v_1(t_0) \\ v_2(t_0) \end{pmatrix} = \begin{pmatrix} v_1(t_1) \\ v_2(t_1) \end{pmatrix}$$

Markov matrix

IMPORTANT! $v_1(t_0) + v_2(t_0) = v_1(t_1) + v_2(t_1)$ in this example


v_2
v_1

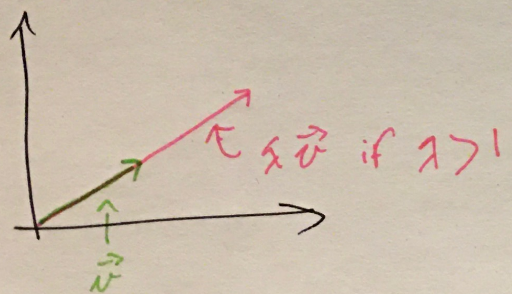
scalar multiplication:

Let λ be just a scalar, that is, a number.

$$\lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1m} \\ \vdots & & & \\ \lambda a_{n1} & \cdots & \cdots & \lambda a_{nm} \end{bmatrix}$$

$$\lambda \vec{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

each component is multiplied by a number



We saw that matrices "transform" vectors. They can translate them, rotate them, etc

BUT, there are cases where for a matrix, the vectors are only "scaled":

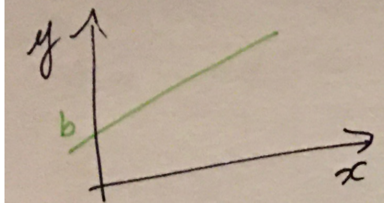
$A \vec{v} = \lambda \vec{v}$, in such cases, we call $\vec{v}$ the eigenvector of A

Exponential and logarithmic functions:

We hear a lot linear relation, or exponential growth, etc.

what do we mean?

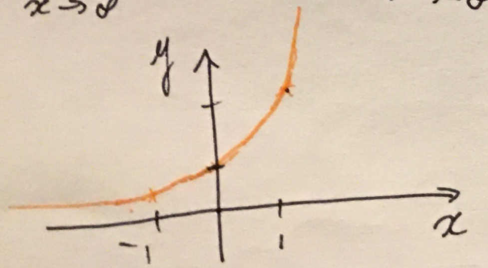
$y = ax + b$: linear regression relates to finding how a variable y changes as variable x is updated.



$y = \kappa e^{x}$: We could take *x as* the population, and this could be growth

$e^{-x} = (e^{-1})^{x} = \frac{1}{e^{x}}$; $e^{0} = 1$; $e^{1} \sim 2,718$

$\lim_{x \to \infty} e^{x} = \infty$; $\lim_{x \to -\infty} e^{x} = \lim_{x \to \infty} \frac{1}{e^{x}} = 0$



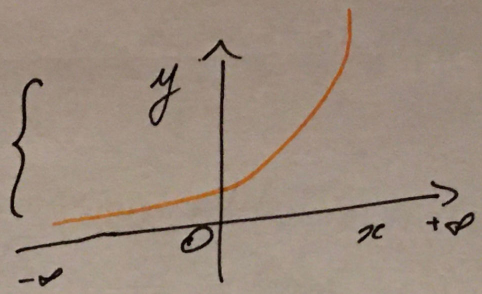
grows <u>very</u> fast

Logarithm : Inverse Function of exp.

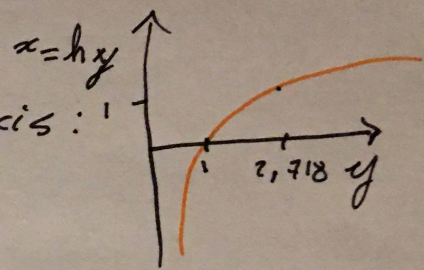
$y = e^x$

$\ln y = x$

Recall $y > 0$ {



So the inverse function can only have values in + horizontal axis :



Recall that $e^0 = 1 \Rightarrow$ for $x = 0$; $y = 1$

$\Rightarrow \boxed{\ln 1 = 0}$

Also recall : $\ln e = 1$ from $e^1 = e$, and $\lim_{x \to -\infty} e^x = 0 \Rightarrow y > 0$

e^x grows very fast, its inverse $\ln y$ grows very slow. If we transform very large numbers we can reduce their value and "see" them in the plot.

Also : If $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$

$$\ln(y_1 y_2) = \ln(e^{x_1} e^{x_2}) = \ln(e^{x_1 + x_2}) = x_1 + x_2 = \ln y_1 + \ln y_2 \Rightarrow \boxed{\ln(y_1 y_2) = \ln y_1 + \ln y_2}$$

$$
\begin{aligned}
\ln\left(k y^{a}\right) &= \ln k + \ln y^{a} \\
&= \ln k + \ln\left(e^{ax}\right) \\
&= \ln k + ax \\
&= \ln k + a \ln y
\end{aligned}
$$

$$
\boxed{\ln\left(k y^{a}\right) = \ln k + a \ln y}
$$

Potential:

$$V_i = \sum_j t_{ij} = \sum_j P_i P_j d_{ij}^{-2}$$

only summing over j

$\Downarrow$

P_i is the population at chosen location i

Example:

$V_i = V_1$ for $i=1$

$$V_1 = \sum_j P_1 P_j d_{1j}^{-2}$$

$$V_1 = P_1 \sum_j P_j d_{1j}^{-2}$$

$$= P_1 \left(P_2 d_{12}^{-2} + P_3 d_{13}^{-2} + \cdots + P_n d_{1n}^{-2} \right)$$

If there are n locations

· If d_{1j} is small $\Rightarrow$ location $i=1$ very close to most locations $\Rightarrow$ $d_{1j}^{-2} \gg$ and V_1 will be big $\Rightarrow$ highly accessible