A MODEL OF METROPOLIS

Ira S. Lowry

MEMORANDUM
RM-4035-RC
AUGUST 1964

A MODEL OF METROPOLIS

Ira S. Lowry

The RAND *Corporation*

1700 MAIN ST · SANTA MONICA · CALIFORNIA · 90406

PREFACE & ACKNOWLEDGMENTS

models during the course of my research. These workshops have greatly
facilitated the interchange of ideas among the scattered professionals
engaged in the theoretical analysis of metropolitan structure and
the design of computer models of urban form.

Section V of this report describes experimental runs of the model,
using data for Pittsburgh, Pennsylvania. The results of these runs
are interpreted in the text as "quasi-predictions" of the emerging
spatial structure of Pittsburgh. I bear sole responsibility for this
interpretation as well as for related comment on Pittsburgh's
future, and my views are not necessarily endorsed by the sponsoring
agencies, nor by the individuals who are mentioned in this
acknowledgment.

SUMMARY

This report describes a computer model of the spatial organization of human activities within a metropolitan area. The model is intended for eventual use as 1) a device for evaluating the impact of public decisions (e.g., concerning urban renewal, tax policies, land-use controls, transportation investments) on metropolitan form; and 2) a device for predicting changes in metropolitan form which will follow over time as a consequence of currently visible or anticipated changes in key variables such as the pattern of "basic" employment, the efficiency of the transportation system, or the growth of population.

The development of the model to the stage at which it becomes a practical tool for planning and decision-making will require more time and effort. The first-generation model described in this report has been fitted to data for Pittsburgh, Pennsylvania, and enough experimental computer-runs have been completed to allow an appraisal of the practicability of my approach, and to indicate the strong and weak points of the model and also of the available data.

The findings reported here offer guidance both to the development of a second-generation model and to data-collection programs. As examples of the eventual usefulness of model-oriented analysis of metropolitan form, I discuss the fitted model's quasi-predictions of the emerging spatial structure of Pittsburgh.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

I. INTRODUCTION

The responsibilities of local governments for comprehensive civic planning have increased greatly during the past several decades. To meet these new responsibilities, planners are searching for new tools which will enable them to deal with the complex interrelationships among land uses, traffic flows, population characteristics, economic activities, tax revenues, and demands for public services. Recently planners have shown considerable interest in computer models of the urban environment -- an interest greatly stimulated by the apparent success of traffic models developed in the early 1950's for highway planning.

The potential advantage of a computer model over ad hoc planning procedures derives mainly from the computer's ability to deal simultaneously with the large numbers of variables intrinsic to comprehensive planning: to provide forecasts of need, normative solutions to problems, or tests of alternative policies which take into account a greater range of relationships, ramifications, and feed-backs than could be managed by the more traditional tools of project-planning. Since 1960, a number of local agencies have sponsored model-development programs; only "pure" traffic models have actually been applied to planning problems.[1]

---

[1]The purposes and scope of these projects vary considerably. One of the first and most ambitious is the land-use and traffic interaction model begun in 1960 at the Penn-Jersey Transportation Study. More recent ventures include the land-use model of the San Francisco Department of City Planning (Arthur D. Little, Inc.); the Boston Regional Planning Project's Program Polimetric (Traffic Research Corporation); the Pittsburgh Department of City Planning's Urban Renewal Simulation (Center for Regional Economic Studies of the University of Pittsburgh, and the CONSAD Research Corporation); and the Regional Systems Study prepared for the Southeastern Wisconsin Regional Planning Commission (College of Engineering, Marquette University).

The conceptual framework for what is now the standard traffic model is largely the creation of J. Douglas Carroll, Jr., who directed or supervised major traffic studies in Detroit, Chicago, and Pittsburgh, and is presently Director of the Tri-State Transportation Committee.

I began work on the location model described in this report while on the staff of the Economic Study of the Pittsburgh Region, a three-year project sponsored by the Pittsburgh Regional Planning Association and the Regional Industrial Development Corporation. The model-building effort was frankly experimental; neither I nor the study sponsors felt any certainty that it would succeed, so this work went on in tandem with a more conventional analysis of the spatial structure of the regional economy.[2] In January 1963, the model was moved to The RAND Corporation, where work has continued as part of a research program in urban transportation. During this period, a number of experimental computer runs have been made, using data for the Pittsburgh area.

The object of this research has been the development of an analytical model capable of assigning urban activities to sub-areas of a bounded region in accordance with those principles of locational interdependence that could be reduced to quantitative form. The model is not designed to project regional aggregates, such as total employment or population, but rather to allocate such aggregates to locations within the region. Properly adapted, it should be useful for the projection of future patterns of land development and for the testing of public policies in the fields of transportation planning, land-use controls, taxation, and urban renewal.

For treatment by the model, activities occupying space within the designated area are divided into three broad groups:[3]

    1) A <u>basic sector</u>, including industrial, business, and administrative establishments whose clients are predominantly non-local. These "export" industries

---

[2] The original plan of the model was published as "Design for an Intra-Regional Locational Model," Working Paper No. 6, Economic Study of the Pittsburgh Region, Pittsburgh Regional Planning Association, 1960. The "more conventional analysis" was published as <u>Portrait of a Region</u>, University of Pittsburgh Press, Pittsburgh, 1963.

[3] My choice of names for these groups will not please readers with a strong sense of descriptive taxonomy. It has been suggested to me that "site-oriented" and "residence-oriented" would be more exact labels than "basic" and "retail," but the brief names are much more convenient in discourse.

are relatively unconstrained in local site-selection by
problems of access to local markets, and their employ-
ment levels are primarily dependent on events outside
the local economy. Consequently, they have been treated
as exogenous to the model, as activities whose locations
and employment levels must be assumed as "given."

2) A retail sector, including those business, administrative,
and other establishments which deal predominantly and
directly with the local residential population. Because
these establishments have local clients, site-selection
is assumed to be powerfully constrained by problems of
access to local residents, and employment levels are
assumed to be closely tied to local growth of popula-
tion. The locations and levels of employment of estab-
lishments in this sector (which include most schools
and local government agencies as well as retail trades
and services) are treated as endogenous variables whose
values are determined within the model.

3) A household sector, consisting of the resident popula-
tion. It is assumed that the level of employment in
the retail sector depends directly on the number of
resident households, and that the number of resident
households in turn depends upon the number of basic and
retail jobs available at any given time. Furthermore,
it is assumed that residential site-selection is power-
fully influenced by the location of the resident's place
of work. Thus, the number and locations of households
are also determined within the model.

Briefly stated, the model is designed to generate estimates of re-
tail employment, residential population (number of households), and land
use for sub-areas of a bounded region. (These sub-areas are nominally
one mile square, defined by a coordinate grid whose systematic proper-
ties are extremely important to the computational methods used.) The
estimates are derived from assumptions or from actual data regarding the
geographic distribution of basic employment within the region, the
amounts of site space occupied by basic establishments, and constraints
imposed on land use by the physical characteristics of sites and by legal
restrictions such as zoning laws.

Given these inputs, the distributions of residential population
and retail employment are generated by means of algebraic functions
which relate places of residence to places of work and relate the
locations of various types of retail activity to the accessible market
of consumers. These functions were developed from an analysis of

work- and shopping-trip links based on traffic-study data,[4] and may be interpreted as roughly expressing the outcome of a competition for sites accessible to the relevant activity. Thus, rather than simulating detailed market processes in which individual establishments (households, business firms, and other activities) compete for sites, the model summarizes these processes by calculating the "potential" of each location as a residential and/or retail site, given the pre-existing distributions of linked activities. The method can best be clarified by a sketch of the computer algorithm through which the model generates its spatial distributions. (See Fig. 1.)

We start with the given distribution of "basic" workplaces (e.g., steel mills) by mile-square tract, and with certain specifications as to the land available for residential settlement in each tract. The computer distributes around each cluster of workplaces a residential population which can supply an appropriate labor force; most residential areas thus receive populations which are linked by employment to a number of workplaces. Some tracts adjacent to major employment centers are unable to accommodate all the households assigned to them because much of the space has been preempted for non-residential uses; the overflow is reassigned to other tracts.

This spatially distributed residential population is then available as a base for the location of population-serving activities, from department stores to elementary schools.[5] The market potential of each location (in terms of accessible customers) is evaluated; and retail employment is spatially distributed in proportion to these potentials.

---

[4] See Ira S. Lowry, "Location Parameters in the Pittsburgh Model," Papers and Proceedings of the Regional Science Association, Vol. 11, forthcoming.

[5] Most success to date has been achieved by grouping population-serving activities according to typical cluster-types--e.g., "neighborhood" clusters consisting of food, drug, and gasoline dealers, personal-service establishments, and elementary schools; "local" clusters containing a wider range of retail businesses, repair and professional services, municipal administration, churches, etc.; and "metropolitan" clusters, consisting predominantly of financial institutions, large department stores, public lodgings, business services, etc.
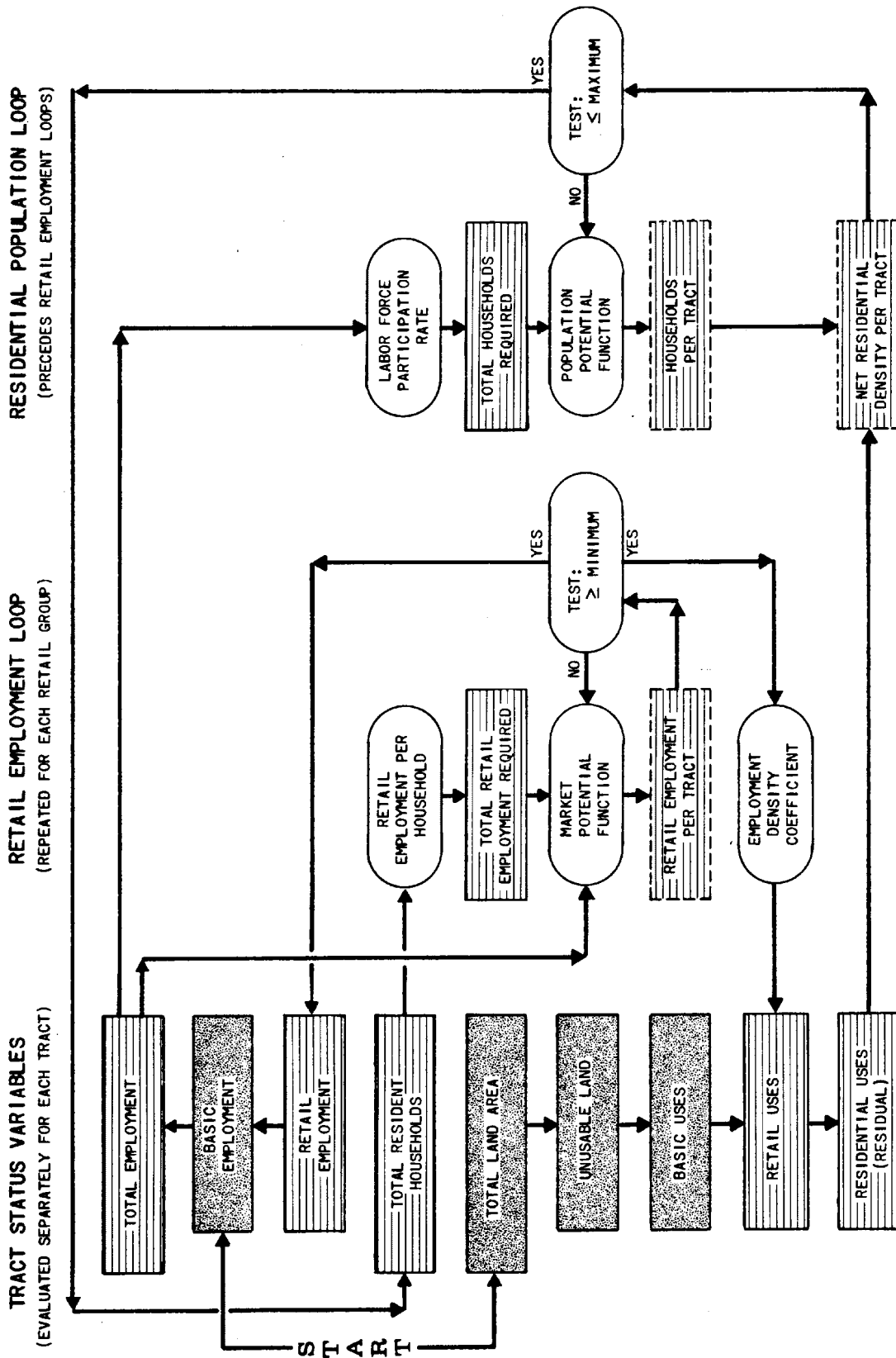
Fig. 1 -- Information Flows in the Pittsburgh Model

In a new round of calculations, the residences of retail and
service employees are located. This event changes the distribution
of residential population, so that market potentials must be recal-
culated. Iterations proceed in this manner until a stable co-distribution
of all employment and all residences is achieved within the constraints
of available land, efficient scales of operation for enterprises, and
density ceilings for residential population.

The model has been fitted to a region of 420 square miles centering
on the City of Pittsburgh -- the area defined by the Pittsburgh Area
Transportation Study as the probable "commutershed" of travel into the
central city for as far into the future as 1980. Estimates of struc-
tural parameters were made from data for this area circa 1958-1960;
and the required input variables for individual tracts were taken from
surveys conducted by the Transportation Study in 1958. In its present
form, the final output of the model's iterative processes -- spatial
distributions of residential population and land use, retail employ-
ment and land use -- are logically "equilibrium" values based on 1958
patterns of basic employment. Since there is every reason to expect
substantial time-lags in the mutual locational adaptations of users of
urban space, these equilibrium output values cannot be presumed to
coincide with 1958 observations for the corresponding real-world
variables; and since the model calls for input data which are not avail-
able either prior to or subsequent to 1958, it was not possible to con-
struct an empirical base for systematic tests of the dynamic properties
of the model.

This shortage of data thus eliminates the possibility of rigorous
tests of validity for the model as a whole; we can only assert that
the spatial patterns of residential and retail activity generated by
the model bear a strong resemblance to those existing in 1958, and
that many (not all) of the discrepancies are plausible when treated
as forecasts of the next decade's changes. They reflect trends al-
ready visible in the Pittsburgh area and in other large urban areas
of the United States.

As one who has worked closely with small-area data for Pittsburgh,
I believe that the model's short-cut simulation of the broad processes

of urban development is sufficiently promising to warrant a second-generation effort of revision, elaboration, and improvement of data within the general structure of the present model. (This effort is already under way as part of the Urban Renewal Simulation Model sponsored by the Pittsburgh Department of City Planning and the Housing and Home Finance Agency.) Besides providing evidence in support of this judgment, the experimental work to date reveals a good deal about the sensitivity of the model to changes in its various parameters; in some cases the findings were distinctly at odds with my expectations, and with those of others working in this field. Such findings will be helpful in planning further efforts, since they suggest priorities for data-gathering and analysis, as well as providing guides for improvement in the logical and computational structure of the model itself.

This report is addressed primarily to persons and agencies seriously interested in the development of computer models of urban environments. It describes the logical structure of the Pittsburgh Model, the computational methods used, the derivation of parameters and variables, and the results of test runs and experiments performed to date. As similar reports will doubtless issue from other model-building projects currently under way, the next generation of researchers in this field will be in a position to pick and choose among strategies, techniques, and components on the basis of our collective experience.

II. THE FORMAL MODEL

The Pittsburgh Model is at best a half-way house on the road to
a general model of urban form. Its map of the metropolis is filled
in partly by hand, and it offers a minimum of detail about the char-
acteristics of land use, population, and economic activity allocated
to the various sub-areas. Its properties as a model of change over
time are not altogether clear, although it is structurally well-
adapted to dealing with incremental changes and lag variables.

At its present stage of development, the model is designed as a
set of simultaneous equations whose solution represents an "equilibrium"
in the pattern of land use and in the distribution of employment and
population. As indicated in the preceding section, the amounts and
distributions of basic employment and basic land use are determined
outside the model; given this information, the model generates appro-
priate amounts of retail employment and residential population, and
distributes these employees and households among the sub-areas of the
metropolitan region, assigning land for each use.

The model has no normative interpretation. It is meant to
simulate -- roughly, to be sure -- the actual behavior of households
and enterprises in a given institutional setting, when they are faced
with given circumstances outside their control, and other given cir-
cumstances within their control. It is possible, nevertheless, to
conduct policy experiments by altering various explicit parameters of
the model and evaluating the desirability of the resultant changes in
solution values.

In the following section, I present the formal logic of the model
as a set of simultaneous equations. Subsequently I describe the com-
putational steps used to solve the system. The general reader may
prefer to skip to Section III, where the model's structure is explained
and interpreted.

THE MODEL AS A SYSTEM OF EQUATIONS

The logical structure of the model can be expressed in nine simultaneous equations and three inequalities. These standard components are replicated many times in the complete system. The following notation will be used:

A = area of land (thousand square feet)

E = employment (number of persons)

N = population (number of households)

T = index of trip distribution

Z = constraints

In conjunction with these symbols, the reader will find the following superscripts and subscripts:

U = unusable (land)

B = basic sector

R = retail sector

H = household sector

k = class of establishments within the retail sector; also defines related class of "shopping" trips

m = number of classes of retail establishments (k=1,...m)

i,j = sub-areas of a bounded region, called <u>tracts</u>

n = number of tracts (i=1,...n; j=1,...n)

Unspecified functions and coefficients are represented by lower-case letters: a, b, c, d, e, f, g.

Land Use

We are given the area of each tract, and the amount of land therein which is not usable by any of the activities with which we are concerned. The remainder of the land in each tract is available for use by basic establishments, retail establishments, and households. All land not otherwise assigned is treated as available for residential use.

$$(1) \qquad A_j = A_j^U + A_j^B + A_j^R + A_j^H .$$

Basic Sector

For each tract, we are given exogenously the quantity of land used by basic establishments (A_j^B) and the employment opportunities provided by these establishments (E_j^B).

Retail Sector

Retail establishments are divided into m groups, each of which has a characteristic production function; the elements of this production function which enter directly into the model are: minimum efficient size of establishment,[6] number of clients required to support one employee, and number of square feet of space per employee. Since local consumer demand provides the market for establishments of this sector, we may treat employment in each line of retail trade as roughly a function of the number of households in the region:

$$(2) \qquad\qquad E^k = a^k N$$

The distribution of this retail employment among the square-mile tracts depends on the strength of the market at each location. Assuming that shopping trips originate either from homes or from workplaces, the market potential of any given location can be defined as a weighted index of the numbers of households in the surrounding areas, and the number of persons employed nearby.

$$(3) \qquad\qquad E_j^k = b^k \left[\sum_{i=1}^{n} \left(\frac{c^k N_i}{T_{ij}^k} \right) + d^k E_j \right]$$

This equation could easily be made more general; however, we have assumed that none but short-range pedestrian trips originate from workplaces, so that the only relevant origins are those in Tract j. Those originating from home are often longer vehicular trips, but the likelihood of a shopping trip from i to j diminishes with intervening

---

[6]Actually the minimum number of employees per tract; these employees may represent more than one establishment of the same type.

distance. (The variable T_{ij}^K is a positive function of this distance, fitted from an analysis of home-based vehicular shopping trips.) The coefficients c^k and d^k measure the relative importance of homes and workplaces as origins for a particular type of shopping. Finally, b^k is a scale factor which adjusts the retail employment in each tract to the regional total determined in Eq. 2.

$$(4) \qquad E^k = \sum_{j=1}^{n} E_j^k$$

In this way we determine the amount of employment in any tract for each line of retail trade. The sum of these employment figures, plus the quantity of basic employment allocated to the tract is total employment for that tract.

$$(5) \qquad E_j = E_j^B + \sum_{k=1}^{m} E_j^k$$

Finally, with the aid of exogenously-determined employment-density coefficients (e^k) for each line of trade, we can determine the amount of land in each tract which will be occupied by retail establishments

$$(6) \qquad A_j^R = \sum_{k=1}^{m} e^k E_j^k$$

Household Sector

The region's population of households may be regarded as a function of total employment.

$$(7) \qquad N = f \sum_{j=1}^{n} E_j$$

The number of households in each tract is a function of that tract's accessibility to employment opportunities.

$$(8) \qquad N_j = g \sum_{i=1}^{n} \frac{E_i}{T_{ij}}$$

The coefficient g is a scale factor whose value is determined by the requirement that the sum of tract populations must equal the total population of the region as determined in Eq. 7.

$$(9) \qquad N = \sum_{j=1}^{n} N_j$$

Constraints

In order to limit the dispersion of retail employment, we impose a minimum-size constraint (Z^k), expressed in terms of employment. If the market potential of a particular location does not justify an establishment above this minimum size, the "customers" are sent elsewhere.

(10) $$E^k_j \geq Z^k, \quad \text{or else} \quad E^k_j = 0$$

In order to prevent the system from generating excessive population densities in locations with high accessibility indices, we impose a maximum-density constraint (Z^H_j). The value of this constraint (number of households permitted per 1,000 square feet of residential space) may vary from tract to tract, as would be the case under zoning ordinances.

(11) $$N_j \leq Z^H_j A^H_j$$

Finally, the amount of land set aside for retail establishments by Eq. 6 must not exceed the amount available.

(12) $$A^R_j \leq A_j - A^U_j - A^B_j$$

Taken together with the accounting relationships expressed in Eq. 1, this constraint also prevents the assignment of negative values to residential land.

SOLUTION OF THE SYSTEM

Ignoring for the moment the three inequalities, one can show that the nine structural equations form an adequately-determined system, whose solution (if it exists) describes an "equilibrium" distribution of retail activities, and a coordinate "equilibrium" distribution of residential population. The formal adequacy of this structure is demonstrated below by a count of equations and unknowns in the expanded system.

Our model contains the following unknowns:

Type	Symbol	Number in Expanded System	Number Exogenously Determined
VARIABLES			
Land Use	A_j	n	n
	A_j^U	n	n
	A_j^B	n	n
	A_j^R	n	-
	A_j^H	n	-
Employment	E_j	n	-
	E_j^B	n	n
	E_j^k	mn	-
	E^k	m	-
Population	N_j	n	-
	N	1 (one)	-
Trip-distribution indices	T_{ij}	n^2	n^2
	T_{ij}^k	mn^2	mn^2
STRUCTURAL PARAMETERS			
Retail employment coefficients[*]	a^k	m	m
Retail employment scale factor	b^k	m	-
Shopping trip weight factors	c^k, d^k	$2m$	$2m$
Retail employment density ratio	e^k	m	m
Labor force participation rate[**]	f	1 (one)	1 (one)
Population scale factor	g	1 (one)	1 (one)

---

[*]Retail employment per household.

[**]In reciprocal form, 1/number of employed persons per household.

Altogether, the expanded system contains $(m+1)n^2 + 8n + mn + 6m + 3$ unknowns. However, values are obtained exogenously for all but $4n + mn + 2m + 2$ of these unknowns. We may compare this remainder to a count of independent equations:

Equation Number	Number of Times It Appears in the Expanded System
1	n
2	m
3	mn
4	m
5	n
6	n
7	1 (one)
8	n
9	1 (one)

We thus have a total of $4n + mn + 2m + 2$ independent equations, equal to the number of endogenous unknowns, a necessary but not sufficient condition of solution.

The addition of three inequalities as constraints on this system changes the problem considerably. Although it is possible that there may be simultaneous solutions in which none of the three constraints is binding for any tract, such good fortune cannot be expected. We need, therefore, a solution method which will allow us to apply these constraints where necessary, yet retain the "regular" solution values offered for most of the tracts. Furthermore, this method (or at least its consequences) should preferably have some interpretation as an analogue to real-world events.

One such method of solution is described below. The treatment of the three constraints is not parallel, but as we shall see in Section III, each has an interpretation.

Step One

The technique best adapted to machine computation is an iterative method, beginning with Eqs. 1, 7, 8, and 9. As a first approximation, we assign the exogenously-determined value of E_i^B to the variable E_i (or E_j), and set $A_j^R = 0$. These four equations can then be partitioned from the rest of the system and solved for g, N_j, and N by an inner iteration. First we obtain a value for A_j^H from Eq. 1.

$$(1.1) \qquad A_j^H = A_j - A_j^U - A_j^B - A_j^R$$

The value of N is then established in Eq. 7.

$$(7.1) \qquad N = f \sum_{j=1}^{n} E_j$$

Or, in order to speed convergence, we can anticipate the final value of N by allowing for the labor-force requirements of retail establishments as well; where E^B represents all basic employment,

$$(7.2) \qquad N = f \frac{E^B}{1 - f \sum_{i=1}^{m} a^k}$$

This short-cut, which would not be appropriate in all contexts, has been used in my experimental runs to date.

We then compute population potentials.

$$(8.1) \qquad \frac{1}{N_j} = \sum_{i=1}^{n} \frac{E_i}{T_{ij}}$$

The value of the scale factor (g) is determined by reference to the total population to be allocated, as follows:

$$\frac{1}{g} = \frac{N}{\sum_{j=1}^{n} \frac{1}{N_j}}$$

This scale factor is used to reduce the population potentials to a second approximation.

$$(8.2) \qquad \overset{2}{N}_j = g \overset{1}{N}_j$$

The left-hand term is then tested against the maximum-density constraint (Z_j^H); for all cases in which

$$\overset{2}{N}_j \geq Z_j^H A_j^H \,,$$

we set

$$\overset{3}{N}_j = Z_j^H A_j^H \,.$$

The excess population of Tract j (i.e., $\overset{2}{N}_j - \overset{3}{N}_j$) is distributed among all other tracts in proportion to their population potentials by revising the scale factor once more.[7]

For all other cases, $\overset{3}{N}_j = \overset{2}{g} \overset{2}{N}_j$, and we close the system with:

$$(9.1) \qquad N = \sum_{i=1}^{n} \overset{3}{N}_j$$

Step Two

We can now partition Eqs. 2, 3, and 4, and again solve by repeated approximations. The solution of Step One gives us values for N and N_j, which are now fed into Eqs. 2 and 3, respectively. Once more we use the exogenous E_j^B as a first approximation for E_j in Eq. 3. We are

---

[7] I have experimented with an alternative to this method: distributing the excess population of Tract j among the eight adjacent tracts in proportion to their available capacity; the method described above, however, gives better empirical results and is much more stable in the context of successive iterations.

then prepared to calculate employment potentials in retail trade k for each tract.

$$(3.1) \qquad {}^1_{}E^k_j = \sum_{i=1}^{n} \left(\frac{c^k N_i}{T^k_{ij}} \right) + d^k E_j$$

These potentials are rescaled so that they sum to the total employment determined from Eq. 2.

$$(2.1) \qquad E^k = a^k N$$

$$(3.2) \qquad {}^2_{}E^k_j = b^k \, {}^1_{}E^k_j \quad ,$$

where

$$b^k = \frac{E^k}{\displaystyle\sum_{j=1}^{n} {}^1_{}E^k_j} \quad .$$

This provisional solution is then tested against the minimum-size constraint. We use a search routine to locate the smallest ${}^2_{}E^k_j$, set it equal to zero, and rescale (increase) employment in all other tracts. This process[8] is repeated until there are no instances in which ${}^2_{}E^k_j$ is less than z^k, and Eq. 4 is satisfied.

$$(4.1) \qquad E^k = \sum_{j=1}^{n} E^k_j$$

---

[8]Where z^k is large, a more efficient method is to zero ${}^2_{}E^k_j$ in batches (e.g., all values below .01 z^k; all values below .02 z^k; etc.) rather than one tract at a time. I have used both methods at various stages in the development of the model.

Step Three

When Step Two has been completed for each of the m retail trades, Eqs. 5 and 6 may be solved by substitution. The retail-land variable (A_j^R) generated by Eq. 6 is then tested against the amount of space actually available in each tract; i.e., where $A_j^R > A_j - A_j^U - A_j^B$, we set $A_j^R = A_j - A_j^U - A_j^B$. In other words, if there is not enough space to accommodate retail employment at average densities, we allow overcrowding. Note that retail uses still have priority over residential uses; if population has been allocated to such a tract by Step One, it will be removed by the residential-density constraint on the next grand loop.

The values of E_j yielded by Eq. 5 will be equal to or greater than the assumed E_j used in the first trial of Eq. 8, and the same is true of the values of A_j^R yielded by Eq. 6. Our solution method thus slowly feeds in retail employment and land use as determinants of population distribution. We may now return with these new values E_j and A_j^R to the system composed of Eqs. 1, 7, 8, and 9, beginning a second iteration of the entire model. We continue in this fashion until the iterations converge on a stable set of values for all the variables.

SOME FEATURES OF THE ITERATIVE SOLUTION

As retail employment and land use are fed into the model on successive iterations, they have a double impact on population densities, $\frac{N_j}{A_j^H}$: the presence of retail employment in Tract j increases the population potentials of that tract and surrounding tracts, while the land absorbed by the retail establishments diminishes the amount available for residential use. In some cases, these events will bring the maximum-density constraint into play, forcing a non-homogeneous redistribution of population among tracts. If such shifts involve many tracts or large numbers, there is a distinct possibility that the system will never stabilize.

Similar non-homogeneous shifts of retail employment may occur by virtue of the minimum-size constraint. As population shifts, a tract previously below the minimum market-potential crosses the boundary; it is now entitled not only to have retail establishments, but to recover patronage earlier distributed elsewhere.

In experimental runs, however, I have found that such shifts in population and retail employment do not ordinarily disturb the system enough to prevent convergence. Since these variables are distance-weighted geographical averages of other variables, the impact of a local change tends, in turn, to be localized.

Even though these shifts do not prevent convergence on a set of solution values, it is fairly clear from the nature of our constraints that the set may not be unique. If the same data were fed into the program in a different sequence, it is conceivable that a somewhat different distribution of retail employment and of population would be generated by the program without violating the system of equations into which the solution values must fit.

Such a formally indeterminate system may still have a unique meaningful solution, provided we can give a dynamic interpretation to the system which enables us to rationalize the order in which data are fed into the computational sequence. This topic is further discussed in Sec. III.

III. INTERPRETATION OF THE MODEL

Analysts build models of real-world systems in order to run tests and experiments which would otherwise be impossible or inconveniently expensive. To be useful, the model must replicate those features of the real world which are relevant to the experiment and abstract from those that are irrelevant or of only minor import.

The wish to experiment clearly bespeaks incomplete knowledge of the real-world environment; so we cannot know for sure which abstractions are safe and which are not. The first principle of model-building is internal coherence; beyond that, the choice of abstractions is guided in part by the experience of others who have worked in the field, in part by "hunch" (primitive theory), and in part by a sense of analytical style (say, a preference for mathematical elegance, or massive generalization, or intricate mechanism).

The strategy of model-building is thus less science than art. And since it is quite difficult to devise a clear-cut test of validity for a complex model, the value of the product is easily disputable. So it is only fair to tell the reader, not just how the Pittsburgh Model works, but why it was built that way.

MARKET MODEL OR GRAVITY MODEL?

Two analytic traditions offer guidance to the builder of a location model. One is economic location theory; the other is sometimes called social physics.

The literature of location theory includes a number of fragmentary analyses of the problem of locational choice from the viewpoint of a business enterprise or a household; we are told what considerations should, on a priori grounds, enter into a rational choice, and how they should be weighed against each other. There have also been a few notable attempts to deduce from these principles of rational choice the conditions of a general locational equilibrium, an equilibrium emerging from competition in the market for urban land. The abstract problem

has yet to be solved for a system as complex as a modern metropolis.[9]

The social physicists are interested in empirical laws of social interaction and mass behavior, rather than in reasoning from principles of individual behavior. They offer evidence of a number of interesting statistical regularities. The one that concerns us here is their finding that a variety of relationships among groups of actors (people, business enterprises) can be summarized by a simple analogue to Newton's law of gravity: The level of interaction is directly proportional to the mass of the interacting bodies (size of the groups of actors) and inversely proportional to the distance between them.[10]

---

[9]The bibliography of economic location theory is extensive. The most comprehensive review is contained in Walter Isard, Location and Space Economy, Cambridge, Technology Press, 1956. Except for analyses of agricultural land use, economists have mostly focused on problems with such a broad geographic grain that site-space and position rent were ignored as constraints on location. For intra-metropolitan locational analyses, these aspects of the location problem are crucial, but serious work on them has only recently begun. For example, R. U. Ratcliff's widely-used textbook, Urban Land Economics, McGraw-Hill, New York, 1949, while commenting that "Residential land use accounts for the major area of all cities," devotes only six of its 500 pages to the subject; and Walter Isard (op. cit., p. 144) once declared that residential location was not even an appropriate subject for economic analysis.

The most important of recent contributions to intra-metropolitan location theory are: William Alonso, Location and Land Use, Cambridge, Harvard University Press, 1964; Lowden Wingo, Transportation and Urban Land, Resources for the Future, Inc., Washington, D.C., 1961; and Benjamin H. Stevens, "Linear Programming and Location Rent," Journal of Regional Science, Vol. 3, No. 2, Winter 1961, pp. 15-26.

[10]For a general background to the work of the "social physicists," see George Kingsley Zipf, Human Behavior and the Principle of Least Effort, Cambridge, Addison Wesley, 1949; and Gerald A. P. Carrothers, "An Historical Review of the Gravity and Potential Concepts of Human Interaction," Journal of the American Institute of Planners, Vol. 22, No. 2, pp. 94-102. Among the most relevant of recent applications are Walter G. Hansen, Accessibility and Residential Growth, M. A. thesis, Massachusetts Institute of Technology, 1959; Willard B. Hansen, "An Approach to the Analysis of Residential Extension," Journal of Regional Science, Vol. 3, No. 1, Summer 1961; William C. Pendleton, "The Value of Highway Accessibility," unpublished Ph.D. thesis, University of Chicago, 1963; and David L. Huff, Determination of Intra-Urban Retail Trade Areas, Real Estate Research Program, University of California at Los Angeles, 1961.

The traditions are not necessarily antithetical in content, although they certainly are in style. If there are dependable "laws" of mass behavior, they certainly must represent the aggregate outcome of myriad individual acts; given time, and saved from logical errors, the location theorists ought to come out at the same place as the social physicists.

There is, however, an important difference which bears upon the relative reliability of models constructed along these alternative lines. The statistical regularities on which a gravity model is based represent the outcome of myriad forces operating in an unspecified technological and institutional environment. We can be fairly sure that both the environment and the forces at play are subject to change, but the gravity principle offers few clues as to the impact of a specific change on the parameters of the model. In other words, if fitted to the current environment, the model is subject to obsolescence at an unknown rate; and if used to test the impact of radically new public policies (or major changes in transport technologies, or changing standards of living), the parameters fitted from current data may be quite irrelevant.

It has been forcefully argued that a location model carefully constructed on a theory of rational choice would be less likely to run out of date and would have a wider range of relevance, because it would apply invariant parameters of choice to whatever concatenation of circumstances faced the actors at any given moment.[11] The argument,

_____

At the annual meeting of the Regional Science Association in November, 1963, Brian J. L. Berry presented a remarkable paper ("Cities as Systems Within Systems of Cities") which integrates many of the findings of social physics and related them to current thinking about stochastic processes.

[11] "Any consistent model of the interaction of decision-units in a field, to be used for predictive purposes, presupposes an analysis which has identified fundamental behavioral patterns, and their changes over time, such that by using them as inputs to the model, a sound prediction will be obtained. In principle, it would appear desirable to find absolutely invariant behavioral patterns and parameters. Any projection of trends in such parameters suggests the likelihood that the analysis has overlooked some observable variable which is influencing behavior and

if not pushed too far, is sound: The more variables we can take into account, the less likely we are to be surprised. But to identify in-variant parameters of locational choice is not so easy!

In planning the design of the Pittsburgh Model, I have followed the guidance of the social physicists more than that of the location theorists. This choice of strategies was partly a matter of circum-stance: the recently-completed Pittsburgh Area Transportation Study had collected data that seemed nearly adequate to fit a gravity model, while the evaluation of locators' preference functions called for field work clearly beyond the resources of the Economic Study. Data problems aside, a gravity model promised to be much easier to build and cheaper to operate.

Thus the model presented in Section II bypasses virtually all of the building-blocks of micro-economic location theory, and deals in very broad aggregates of locators, distributing them over the urban space with the aid of mathematical probability-functions. Since pro-positions relating to individual decision-makers do not appear expli-citly in the system of equations, the behavioral underpinnings of the model are rather obscure. In the remainder of this Section, I present the postulates and parameters of individual (i.e., household and enterprise) behavior which seem necessary to reconcile at least the general structure of the model with the economic theory of choice, and offer interpretations of some of the more ambiguous results which emerge from this structure.

LOCATION OF RETAIL ENTERPRISES

Equations 2 through 6 of the model describe the behavior of retail enterprises, aggregated by tract. We may assume that such enterprises are governed by the principle of profit-maximization. The model

which itself has a time trend. The exclusion of this variable from the model reduces its predictive power and increases the probability of imperfect results." Britton Harris, "Some Problems in the Theory of Intra-Urban Location," (mimeograph) Penn Jersey Transportation Study, April 1961, p. 16. The paper was presented at a seminar sponsored by the Committee on Urban Economics of Resources for the Future.

explicitly defines the production function as requiring fixed proportions of labor and site-space; above minimum efficient size, there are neither internal nor external economies of scale.[12] Customers bear the burden of "delivery costs," which include all economic costs that vary with distance between seller and buyer.

Given these institutional and technical parameters, the locational choice-problem of the individual firm is fairly straightforward: since profits vary directly with the volume of sales, the firm will seek to locate where it can attract the maximum patronage.

According to our potential formula, the number of customers attracted to any given Tract j (containing one or more retail outlets) depends on the spatial distribution of residence and employment with respect to Tract j, and also on this same distribution with respect to all other tracts containing retail outlets. The model thus incorporates both competition and distance as determinants of the probability that a particular buyer will do his shopping at j. The system is in equilibrium when establishments are so located that this competition reaches a stalemate.

A clustering of establishments of the same type is quite consistent with our model, which does not specify the number of establishments in Tract j, only aggregate employment in each type of establishment by tract. Just as competition between locations is a constant-sum game in which the players strive for shares of the total regional market, competition between establishments at a given location is a constant sum-game in which the players strive for shares of the market allocated to that location. Since the outcome of the latter competition is not particularly relevant to a locational problem, it remains unspecified in our solution.

MARKET POTENTIAL AND CONSUMER BEHAVIOR

From the point of view of the sellers, our potential formula

---

[12]These production-function postulates are not empirically defensible, but they greatly simplify the structure of the model, the derivation of parameters, and the computation of solutions. One can only hope that errors from this source are tolerable.

(Eq. 3) describes market areas. It is worth noting that these market areas are interpenetrating: the establishments at all locations are given some share, however small, of the patronage of customers located in Tract i. It follows that the buyer does not choose between sellers simply to minimize the distance covered by his shopping trips; if so, each buyer would patronize the nearest outlet, and there would be no interpenetration of market areas.

It will help us to analyze the implied behavior of our consumers if we shift perspective from the market area of the seller to the shopping area of the buyer. Consider a special case in which the outlets for a particular type of retail merchandise are distributed evenly throughout a region. At a central point, i, within the region, there is a residential cluster. How would these residents distribute their patronage among the various retail outlets?

The gravity model of the social physicists offers a solution of appealing simplicity: interaction between consumers located at i and retail outlets located at j will vary directly with the size of each population (say, number of households, number of retail outlets) and inversely with some power (usually the square) of the distance from i to j.

If we assume that retail outlets are undifferentiated except by location, and are evenly and ubiquitously distributed through the region, it follows that each unit increment to radial travel distance from Tract i brings within reach a larger increment of retailers; more precisely, the number of sellers encompassed in an added ring increase with respect to the length of the radius at the rate $dA/dr = 2\pi r$. At the same time, the attraction of the shopping opportunities in each ring is geometrically diminished by a power of distance, r^x. In this context, the classical gravity model yields a frequency distribution of trip-ends (i.e., of patronage) by distance from the origin which can be expressed as

$$\frac{dP}{dr} = \frac{2\pi r}{r^x} = 2\pi r^{1-x} .$$

The increment to radial distance, dr, defines the annulus

dA = 2πr dr. Trip-ends falling within this annulus, if evenly distri-
buted, would have a point-density of G, where

$$G = \frac{dP/dr}{2\pi r} = \frac{2\pi r^{1-x}}{2\pi r} = r^{-x} .$$

The market potential created by N_i consumers living in Tract i can thus
be distributed over the entire region in a continuous surface, $N_i G$. A
similar surface can be created around each other cluster of households
in the region; summing the potentials at each point, we obtain a map of
the total market potentials corresponding to that particular spatial
distribution of consumers.

Such a surface is approximated by Eq. 3 of the Pittsburgh Model,
although we use a discrete (lumpy) rather than a continuous geography.
In principle, our trip distribution index (T_{ij}) is the reciprocal of the
point-density function (G); if we were to adopt the classical gravity
model, therefore,

$$T_{ij} = r^x .$$

But what is the rationale for this particular transformation of
distance? The social physicists who have used gravity models of this
type in the analysis of spatial patterns of activity seem to have in-
terpreted them as representing random behavior of some sort; yet I
have not discovered any serious attempt to derive the parameters of a
gravity model from theoretical probability distributions prior to an
article by Morton Schneider published in 1959.[13]

Schneider's model, and several other alternatives are explored
in the Appendix to this section. Each of the alternatives suggests a
somewhat different transformation of distance for the denominator of
our potential function, and no one of the alternatives is so closely
attuned to the real world that it may be selected on a priori grounds.

---

[13]"Gravity Models and Trip Distribution Theory," Papers and Pro-
ceedings of the Regional Science Association, Vol. 5, 1959, pp. 51-56.

So the formal structure of the Pittsburgh Model simply names a function of distance, T_{ij}, leaving the function itself unspecified.

In Section IV, the method used to fit this function to empirical data is described in some detail; I should say here, however, that the best fit was obtained for all retail trade groups considered using quadratic expressions of the form:

$$\frac{dP}{dr} = (a - br + cr^2)^{-1},$$

or

$$T_{ij} = \frac{a - br + cr^2}{2\pi r}.$$

This quadratic yields a distribution of trip ends which peaks within a mile or two of the origin in most cases; the parameters a, b, and c vary with the kind of retail trade considered.

I have included the distribution of total employment as a determinant of the market-potential for retail establishments. As Roland Artle has pointed out, the distribution of households is not directly relevant to the optimal location for restaurants catering to business-men, but the distribution of jobs is highly relevant.[14] If, for example, we were to segregate "Businessmen's Luncheon Restaurants" from restaurants in general, the numerator of the access formula (Eq. 3) for this kind of establishment might be modified by giving no weight to the distribution of households--i.e., by setting $c^k = 0$ and $d^k = 1$. In other cases, both weights might be positive--e.g., for drug and variety stores.

I am inclined to doubt that the employees of manufacturing and wholesaling establishments generate a significant market-potential in

---

[14]*Studies in the Structure of the Stockholm Economy*, The Business Research Institute at the Stockholm School of Economics, Stockholm, 1959, pp. 122-124. Artle presents a linear regression of the number of retail establishments of various types by tract, with population by tract and employment by tract as determining variables. This is really a simplified access-model, in which ($T_{ij} = \infty$) for (i $\neq$ j).

the vicinities of their jobs. The problem is therefore one of genera-
ting commercial districts with little or no residential population, but
with concentrations of retail establishments; and for this purpose, the
market-potential weighting factors should be very useful. The minimum-
size constraint should also help to "build" business districts by
limiting specialty goods establishments to locations with particularly
high market potentials.

Although the model postulates standards of space-consumption for
retail establishments, it does not impose a "maximum-density" constraint
on the distribution of retail employment. It is thus possible in the
model for an indefinitely large number of such establishments to be
located in a tract already completely filled by "exogenous" establish-
ments--provided that the market-potential of the location is suffi-
ciently great to justify the presence of these retail and service
enterprises.

There is a logical incongruity in placing space-using establish-
ments in a bounded area whose space is already fully occupied--but
only because we have abstracted from the third dimension of space in
our model in order to simplify the set of constraints. The circumstance
described in the preceding paragraph may be interpreted as implying
mixed uses in multi-story buildings (e.g., ground-floor retail combined
with upper-floor wholesale or central office use), thereby discreetly
admitting the third dimension into our scheme through the back door of
interpretation.

With this interpretation, the lack of any capacity-constraint on
the locations of retail establishments is an implicit statement that
there is no absolute economic or political ceiling on the height of
commercial buildings. This is hardly an accurate statement, but it is
probably an adequate one considering the broad scale of the model,

---

Unfortunately, he does not indicate the extent to which the fit was
improved by the addition of the employment variable. In the Pitts-
burgh Model, the retail market-potential of Tract j is influenced by
the residential population of all other tracts, but Artle's convention
was adopted with respect to work-places: only those in Tract j con-
tribute to its market potential.

and the rather small proportion of metropolitan land outside the CBD which is devoted to retail or mixed commercial use.

RESIDENTIAL LOCATION AND THE JOURNEY TO WORK

Equations 7, 8, and 9 of the model are devoted to the determination of the pattern of residential location. This pattern is described as a function of the spatial distribution of employment opportunities, constrained by the availability of land suitable for residential development. Doubtless this is a considerable simplification, but the model undertakes only to solve for the number of households in each sub-area, and to calculate the gross residential density of the sub-area.[15]

From Eq. 8 of the model, it is apparent that the residential population of Tract i is to be made up of "delegations" of employees (with their dependents) from each nearby employment center. The greater the distance between Tract i and an employment center in Tract j, the smaller will be the number of those employed at j who settle in i. The maximum-density constraint sets a ceiling on the residential population of Tract i, forcing surplus households who would "like" to live there to settle in other tracts.

By shifting our perspective from the employment distribution of the residents of Tract i to the residential distribution of those employed in Tract j, we can get a clearer picture of the pattern generated by this model. Let us assume an isolated employment center, surrounded by land all of which is available for residential use. Our population-potential formula would then distribute the residences of the work-force symmetrically about the employment center at j.

This distribution may be conveniently described in either of two ways: as a relative frequency distribution of households by distance of residence from the central point j, or as a schedule of average residential densities, varying with distance of residence from j.

---

[15]"Gross density" here has the sense of "number of households per thousand square feet of available land." Unusable land and land already allocated to other uses are excluded from this calculation, but no allowance is made for land held off the market.

Even supposing a rectangular frequency distribution of households by distance, the residential density would decline with distance from j because each increment of radial distance encompasses a larger increment of space for residential development.

The exact nature of the frequency distribution--hence of the density schedule--depends, from the formal viewpoint, on the relationship between the radial distance (r) and the trip-distribution index (T_{ij}) which forms the denominator of the right-hand term of Eq. 8. As in the case of our market-potential formula, the interpretive problem is mainly one of rationalizing an empirically satisfactory transformation of r. But there are significant differences between the two contexts with respect to the nature of buyer-seller relationships-- differences which should be reflected in the parameters of the potential function, and perhaps even in its form.[16]

---

[16]Experimental work with gravity functions has led to the general abandonment of the earlier hope that a single transformation of distance would be applicable to all or to many forms of social interaction. Cf. Walter G. Hansen, "Accessibility and Residential Growth," unpublished M.A. thesis, Massachusetts Institute of Technology (June 1959), summarized in an article, "How Accessibility Shapes Land Use," in _Journal of the American Institute of Planners_, Vol. 25, May 1959, pp. 73-76. Traffic studies in Baltimore and Detroit separated work, social, and shopping trips in fitting classical gravity functions from trip-data. The following exponents of travel-time (not distance) were obtained:

	Work	Social	Shopping
Detroit	1.20	1.0	2.60
Baltimore	.900	1.0	2.00

Traffic data in four smaller cities, fitted to the same model, yielded exponents for work trips between .680 and .805. (Figures from Appendix A of Hansen's M.A. thesis.) Hansen believes that a better fit is achievable using a function of the form:

$$T_{ij} = a + r^x$$

where _a_ is a constant representing terminal time. Graphic methods of curve-fitting to the Baltimore traffic data cited above gave him x-values of 2.20, 2.35, and 3.00 for work, social, and shopping trips, respectively.

In the labor market, first of all, the seller ordinarily absorbs the cost of delivering the product to the buyer.[17] Furthermore, although this delivery takes place almost daily, the seller is ordinarily bound to a single buyer for long periods of time, by contract or custom.

It follows that the household, as a seller of labor, has a strong incentive to treat commuting costs as an important factor in the choice of residential location, and that it is ordinarily feasible for the household's residential choice to be made with respect to a particular job-location (not, as some discussions of the subject appear to imply, to the greatest concentration of jobs).

I do not propose a model of household behavior in which the sole criterion of locational choice is proximity to the place of employment, hence a minimization of commuting cost. Of equal concern to the household is adequate site-space for its domestic activities. Both site-rents and commuting cost vary according to location, and the problem of the individual household is to find a combination of these outlays best suited to its particular desires and circumstances--yet within its budget.

By an application of Ricardian rent-theory, it can be shown that, when households compete for residential space in the neighborhood of an employment center, position-rents should systematically decline with distance from this central point--a proposition that seems to be reflected in the general pattern of metropolitan land prices. At the same time, of course, commuting costs increase.[18]

---

[17]A distinction must be made between the employee's commuting costs as a variable influencing his locational decision, and the long-run incidence of aggregate transportation costs. Under conditions of full employment and mobility of resources, the transportation bill (including both out-of-pocket costs and time lost en route) is a charge against total product. (See Wingo, op. cit., pp. 52-62.) But with trivial exceptions, such as the bus fare paid to domestic servants, the individual employee pays his own transportation costs out of a standard wage; and he is able to modify the amount of this payment by his choice of residential location much easier than by his choice of job.

[18]Position-rent is that portion of the market rent (or its

Under any reasonable assumptions as to the behavior of commuting costs (e.g., $\frac{dC}{dr} \leq 1$), it can be demonstrated that the combined outlay for rent on a site of given size plus daily commuting cost will decrease as we consider locations at greater distances from the employment center.[19] Thus a household willing to settle for a suburban location could afford a larger plot than a household choosing a central location, even though they were subject to identical budgetary restrictions. Differentiating among income groups, it can also be shown that prosperous households, who would in any case allocate a larger sum to site-rent, have a positive incentive to settle "further out" than those with meagre budgets.[20]

In sum, a micro-analysis of the residential land market leads us to the conclusion that residential densities should diminish with distance from employment centers, a conclusion that fits so well with

---

capitalized equivalent, the market price) which is attributable to the advantageous location of the site, as distinguished from its other characteristics (slope, view, soil structure, existing improvements, etc.).

The reader has his choice of at least four independently-conceived models of such a residential rent-surface; all start from the same behavioral premises, differing mainly in their simplifying assumptions and in the technique of aggregation. The citations are: Wingo, op. cit.; Alonso, op. cit.; Ira S. Lowry, "Residential Location in Urban Areas," unpublished Ph.D. thesis, University of California at Berkeley, 1959, Appendix D; and John F. Kain, "The Journey to Work as a Determinant of Residential Location," unpublished Ph.D. thesis, University of California at Berkeley, 1961. A paper based on the last-named work was presented at the 1962 meetings of the Regional Science Association, and is scheduled for publication in Papers and Proceedings, Vol. 11.

[19] Cf. Lowry, op. cit. The demonstration relies on the assumption that the area A available for residential settlement increases faster than the distance r from the central place, with an upper limit, $dA/dr = 2\pi$. Ordinarily, geographical constraints and pre-emption of land by non-residential users must hold this derivative well below its theoretical maximum; it may sometimes be less than unity over parts of the range of r. The paradigm case, $dA/dr \sim 1$, would be a narrow residential corridor.

[20] Cf. Lowry, op. cit., pp. 37-41. The last proposition is familiar to economists as the "income effect" in budget allocation; it requires the assumption that transportation costs (including the value of time en route) per unit distance do not increase proportionately with the income of the traveller.

common experience that it hardly needs elaboration. But we cannot establish the _rate_ of decline in density on purely _a priori_ grounds; we would need to know the transportation cost schedule, the topographical and institutional obstacles to residential land use, the number of households, their incomes, and the marginal value of site-space to each household.

The much more feasible alternative adopted for this study was to estimate the workplace/residence co-distribution directly. (The way in which this was done is explained in Section IV.) I found that the classical gravity function ($T_{ij} = r^x$) could be fit nicely to Pittsburgh data; it sufficed to describe the relationship between places of residence and places of work for each of four broad occupational groups. As anticipated, the parameter x varied consistently with the socio-economic status of these occupational groups.

The reader will perhaps recall that the model does not identify the occupational characteristics of employment opportunities or of households. This identification could easily be added to the formal structure of the model. Employment opportunities in each tract could be broken down by occupation, and separate surfaces of population potential could be calculated for each occupation. In view of the empirical evidence of occupational differences in the workplace/residence relationship, such an elaboration of the model is attractive.

However, the addition of an occupational dimension to the employment surface would be too great a strain on our present data-resources. The model would be required to specify an occupational mix for each industrial group (i.e., for the "basic" sector and for each "retail" sector); the industrial mix of employment assigned to each tract could then be converted to an occupational mix for that tract. But the key assumption of small-area constancy of the occupational mix within each industrial group is quite unrealistic for the very broad industrial categories of the present model.

Rather than over-extend the model by disaggregation, it seemed wiser to use a single trip-distribution function covering all occupations and industries. The implicit assumption underlying this procedure is that employment in every tract has the same occupational mix,

regardless of its industrial composition--or at least that variations in mix are unimportant for our purposes. While this assumption, in turn, is far from comfortable, the resulting model is simpler in structure and considerably cheaper to operate than the more elaborate version sketched above.

So far, my interpretation of the section of the model devoted to residential location has been in terms of the distribution of households about a single employment center. Generalizing to a more complex situation, we encounter a difficulty which is hard to handle within the framework of the otherwise tractable and versatile potential-formula.

Consider the case of more than one employment center. Each such center is surrounded by a radially-symmetrical "access-field" which governs the placement of the residences of its employees, and these access fields are mutually independent. Consequently, where two or more such fields overlap, we can expect residential densities to build up, the area within the overlap receiving a "normal" delegation of employees (and their dependents) from each center.

Under these circumstances, there will probably be variable residential densities within annular zones concentric on each employment center, since portions of any such zone may lie within areas of overlapping access fields while other portions do not. If this is the case, it follows that a householder in a high-density sector of the band could obtain more site-space at no sacrifice in access to his job simply by moving to a less-crowded section of the same annular zone.

The maximum-density constraint serves only as a rough corrective; while it prevents the development of excessive densities in individual tracts, there is no special provision made for the redistribution of the "overflow" population from these tracts. The overflow is allocated among all tracts which are still "open," in proportion to the access-potentials of each; whereas it would seem more in keeping with our interpretation to send this surplus exclusively to tracts whose access-characteristics are similar to those of the tract from which the households were turned away.

A similar problem arises when we allow the intrusion of pre-emptive non-residential land users. Our potential-formula distributes

residences in a pattern which is radially-symmetrical with respect to workplaces; if the space available for residential use is radially asymmetrical, residential densities would vary without compensating variations in accessibility to the relevant workplace.

A problem of a different sort appears when we examine our premise that residential location is always oriented to the breadwinner's place of work. In 1960, for instance, about nine per cent of all families in the Pittsburgh SMSA reported no members in the labor force; these cases include normal retirements, fragmentary families living on public assistance or contributions from a non-resident relative, etc. Nearly 33 per cent of all families reported two or more labor-force members.

In the formal model, no special arrangements have been made for either of these groups. Implicitly, I have assumed that households with no working members are distributed over the urban space exactly as households with working members; the locational behavior of households with more than one worker is only indirectly and poorly taken into account in the empirical evaluation of the access variable (T_{ij}) for all households.

THE MEASUREMENT OF ACCESSIBILITY

Distance imposes a cost on social interaction. In the Pittsburgh Model, this cost is represented by airline distance between home and workplace, or between home and retail outlet.

Airline distance, however, is an exceedingly suspect measure of the mutual accessibility of pairs of points, for it takes no account of the variable characteristics of the network of transportation facilities: the pattern of roadways, the degree of congestion, public transit routes and schedules, etc. No urban area is internally homogeneous in these respects; common experience indicates that radial travel is usually easier than circumferential travel, that topographical features such as rivers introduce sharp discontinuities in the pattern of point-to-point accessibility, that grade-separated arterials offer distinct advantages to travel within specific sectors of the typical urban area.

The characteristics of the traveller also seem to bear upon the costs of movement. Individuals without private vehicles face a very different cost map from their auto-owning neighbors. Those who travel at rush hours must allow more time to cover a given route than off-peak travellers. And it has been cogently argued that the more prosperous the traveller the greater valuation he is likely to put on his time en route.

As a practical matter, it is easy to measure the out-of-pocket costs of movement for a particular individual over a particular route at a particular time of day. The valuation of his time is not so easy; little credence can be given to his own estimate in an interview, and it is hard to set up an experimental environment which fairly evaluates his trade-off function as between time and money.

The value-of-time problem aside, building a point-to-point matrix of the cost of movement which reflects variations in the transportation network is a major job. Metropolitan traffic analysts have created such matrices for automobile travellers. They begin with map measurements of over-the-road distances along segments of a selected network of major arterials. These measurements are then fed into an ingenious computer routine which identifies minimum-distance paths over the arterial network between pairs of traffic zones. A variant of this method calls for field measurement of peak-hour travel time over each arterial segment, so that minimum-time paths between zones can be identified.

A few systematic comparisons have been made between these measures.[21] Oddly enough, both time en route and over-the-road distance

---

[21]The Pittsburgh Area Transportation Study tested the linear relationship between airline and over-the-road distance for 70 randomly-selected pairs of traffic zones, obtaining $R^2 = .939$. (Louis E. Keefer, "Vehicle Miles of Travel Accuracy Check," in Interim Technical Report, No. II, November, 1959.) The significance of the test may be disputed on the grounds that a random sample overweights the long-distance comparisons, for which the correlation is expectably high. Most trips are made between nearby zones, for which the correlation between airline and over-the-road distances would be lowest. (I am indebted for this point to Howard Bevis.)

For Washington, D.C., a test has been made of the correlation

appear to be simple linear transformations of airline distance; measures of goodness of fit are high, with $R^2 \simeq .90$. While more tests are needed, the present evidence is that there is very little to be gained by using the more refined measures of accessibility.

The costs of the refined measures, on the other hand, would be high. Airline distance between the "tracts" of our model can be measured analytically without field work because the tracts are laid out along a systematic coordinate grid; in fact a single matrix of tract-to-tract distances can be used with any origin, since this kind of "distance" is a property of Euclidean space. In order to use the alternatives mentioned above, we must either store a separate matrix of tract-to-tract distances (or travel-times) for each origin, or else recalculate these values for each origin every time the access-variable comes into play.[22]

RESIDENTIAL DENSITY AND HOUSEHOLD CHARACTERISTICS

The distribution of households which is generated by the model is converted to a residential-density map by dividing the number of households assigned to Tract i by the residual acreage available after all "higher" uses are satisfied. This density-figure is peculiarly ambiguous. It does not differentiate between densities effected by

---

between airline distance and travel time from the White House to each of a random sample of 104 traffic zones. Off-peak driving times, estimated by the minimum-path method, were fitted to airline distance with $R^2 = .905$; peak-hour driving times, reported directly by commuters fitted nearly as well ($R^2 = .862$). There were only minor directional differences in the relationships. (William Pendleton, "The Value of Highway Accessibility," unpublished Ph.D. thesis, University of Chicago, 1963, pp. 17-23.)

[22]For the Pittsburgh Model, the airline distance matrix (r) is transformed into an accessibility index (T_{ij}) for work trips and for each type of shopping trip. Each T-matrix requires 650 core storage locations, and is instantly available for repeated use in the iterative program. If either over-the-road distance or travel-time were substituted for airline distance, we would need 650^2 (or 422,500) storage locations, well over ten times the core storage capacity of the more advanced general-purpose computers.

high-rise building as opposed to densities created by "overcrowding" in low-rise buildings. Because of the assumption that all residual acreage is residentially developed, it would be misleading to think of the figure as a _net_ density in fringe areas, where part of the residual acreage may in fact remain undeveloped.

This ambiguity is the price which must be paid for keeping the model simple--e.g., excluding such factors as building costs and land values as explicit variables. For many potential applications, this density-measurement is adequate; the output of the model for individual tracts can be interpreted on the basis of collateral evidence as to the probable social characteristics of the population (or in a dynamic context, initial structural types in the existing development).

For example, my own experiments with Detroit and Oakland (California) data show that there is a rather remarkable correlation between net residential density (in this case, persons per acre actually in residential use), per capita income, and age distribution of census tract populations. I have not seen precisely comparable correlations for other cities, but I feel sure the relationship will generalize. If variations in net residential density can be systematically accounted for by these two variables--income and age distribution--it should be possible to specify within reasonable limits the social characteristics of the inhabitants of the various tracts on our density map by means of a separate model drawing on the tradition of "social area analysis."[23]

Assuming that these collateral techniques will provide us with information as to the social characteristics of each tract, the density-figures generated by the model can then be interpreted in

---

[23] My work on this topic is reported in "Residential Location in Urban Areas," unpublished Ph.D. thesis, University of California, 1959. For a more general background, see Eshref Shevsky and Wendell C. Bell, Social Area Analysis, Palo Alto, Stanford University Press, 1955; and Robert C. Tryon, Identification of Social Areas by Cluster Analysis, Berkeley, University of California Press, 1955. A latent structure analysis of household and area characteristics undertaken by Vladimir Almendinger for the Penn Jersey Transportation Study should also be helpful; his prospectus is given in a mimeographed paper, "Topics in the Regional Growth Model: I," April 5, 1961.

terms of structural types—e.g., high-rise luxury apartments, low-rent tenements, row houses.

So far as the fringe-density problem is concerned, it could be handled by a minimum-density constraint built into the model; I would prefer, however, to evaluate fringe-density outputs as they occur without such a constraint. For many planning and policy purposes, it is not really important whether we find that outlying Tract i will contain ten households living on one-acre lots adjacent to 90 acres of undeveloped land, or (as this model would report), ten households living at an average density whose base is 100 acres. Indeed the distinction between gross and net density on the fringe is so ambiguous that arbitrariness is the rule even in empirical measurement.

THE PROBLEM OF TIME

The model as presented in Section II and as actually programmed and operated to date has no time dimension. Its iterative sequences are simply convenient substitutes for an analytical solution; they generate an "instant metropolis."

The reader is entitled to ask what relationship there could be between such an abstraction and a real-world metropolis whose history is embodied in individual structures and even in whole subdivisions dating from the 19th century. Given that the present form of each metropolis is an outgrowth of its unique past, can this form be approximated by a simultaneous system without lag variables?

Experimental results indicate that the answer is a qualified "Yes." The model performs this replicative function correctly in broad outline, but is unreliable in detail. Its failings are quite consistent with what we can observe of the processes and pace of change. Despite the 19th century remnants which are visible in today's metropoli, it is clear that these cities are yet functioning systems whose parts are in some way mutually adapted. There is a constant and equally visible process of land-use succession and functional reorganization of activities within our great metropolitan areas, operating both by demolition of existing structures and street patterns, and by adaptation of old facilities to new uses.

Ideally, a model of metropolis should be a dynamic system with variables whose values continuously change under the impact of external forces and internal momentum. Such models are on the horizon; in the meantime, I believe, experimentation with static models is a necessary preliminary to dynamics. The Pittsburgh Model, moreover, is designed so that it can easily be adapted to semi-dynamic form. Thus, its iterative solution has an interesting resemblance to temporal processes of urban change, although the analogy is not exact.

The "initial conditions" of a dynamic version of the model might consist of the existing distribution of employment, population, and land use. A change in the level and/or distribution of basic employment, or a change in the structural parameters of the model[24] would provide the motive force for subsequent events. To accommodate such a change, the model re-estimates and redistributes population, with due regard for employment-access and land-use constraints. The next step is the redistribution of sufficient retail establishments to serve the new arrangement of households; this event in turn shifts retail employment locations, calling for a further redistribution of population in subsequent rounds, until an equilibrium is reached. Thus the solution to each successive iteration of the system of equations can be interpreted as representing changes over time. And the model is constructed so that it can easily be taken off the computer between iterations for insertion of further exogenous changes, these changes in turn redirecting the system toward a new equilibrium solution.

Another way in which the model can be adapted to dynamic problems is through incorporating lag variables. There are three parameters which can be used quite effectively in this way:

Available-Land Constraint: Equation 1 lists four land-use variables and a control total. Two of these variables, A_j^U and A_j^B, are data inputs to the model; the values of A_j^R and A_j^H are part of the solution. We may, if we desire, hold

---

[24]E.g., parameters representing labor-force participation rates, zoning laws or other limitations on land use, the efficiency of the transportation system, land-use coefficients for households and enterprises. The possibilities are discussed further below.

a quantity of land "off the market" (as A_j^U) for x iterations, then either assign it to A_j^B in order to accommodate a postulated growth or redistribution of the basic sector, or make it available for retail and residential use. Marginal (e.g., steep-slope) land may be withheld from the market until population-potential reaches a certain level, then released for development.

Minimum-Size Constraint: Equation 10 is a constraint which prevents the location of a retail establishment in a tract whose market-potential is below the level necessary to support an establishment of minimum efficient size (Z^k). If early iterations in the solution-sequence develop retail uses in a given tract, we may argue that the sunk cost in this retail development voids the minimum-efficient-size constraint for x iterations thereafter—so that, even though market-potential for that location declines during the later stages of the iteration process, the now "unprofitable" retail outlet remains as a competitor, inhibiting the development of other locations.

Maximum-Density Constraint: We may argue that the original development pattern of residential land inhibits density charges as population potential increases. For example, if Tract i is first developed in detached single-family houses, we might set the maximum-density constraint for that tract at a level appropriate to that type of development, for a period of x iterations.

Tract j, on the other hand, is currently developed in very versatile structures which can easily be converted into efficiency apartments and rooming houses; therefore the maximum-density constraint is set high enough to allow for such conversion at any time that access-potential justifies increasing density.

These are only a few of the possibilities, but sufficient to demonstrate that these parameters are built into the model in such a way that, through them, we can substantially alter the dynamic sequence,

and/or the final solution. Moreover, they are adaptable to changes: no functional relations need be rewritten; the bulk of the input data is undisturbed. A change of parameters can be a general change affecting all tracts equally; or the change can apply only to a specific tract.

Moreover, through these same parameters we may apply alternative public policies to particular areas, and observe their consequences--e.g., establish the maximum residential density of Tract j by a zoning ordinance or exclude commercial development from Tract i. It should even be possible to simulate a planned redevelopment project through the alteration of these parameters--all in the context of this iterative series.

Appendix

MARKET POTENTIAL AND CONSUMER BEHAVIOR

The text of Section III describes the general form of the market potential functions used in the Pittsburgh Model to allocate retail employment among the square-mile tracts of the study area. In Section IV, the method used to fit these potential functions to empirical data is described. In this Appendix, I propose to explore the logical basis of the concept of market potential with the aid of several models of consumer behavior.

The reader will recall from Section III the application of the classical gravity principle to a standard situation: a group of consumers, based at i, are free to choose among a number of retail outlets as destinations for their shopping trips; the retail outlets are evenly and ubiquitously distributed throughout the surrounding region. The gravity principle would imply a frequency distribution of trip-ends (i.e., of patronage) by distance from i which can be written:

$$\frac{dP}{dr} = 2\pi r^{1-x} .$$

This distribution has been standardized and plotted in Fig. 2 for various values of x; for all x greater than unity, the distribution decays toward zero frequency as we move away from the origin, a general pattern indicated by systematic evidence as well as by common sense.

However, when we inquire more closely into the behavior of the shoppers, it becomes difficult to rationalize the precise shape of this trip-distribution curve, the negative power function offered by the social physicists as typical of this and other types of interaction-at-a-distance. Let me illustrate by a paradigm of consumer behavior.

Suppose a person residing at i sets out in search of a pack of cigarettes, travelling on a randomly-chosen radial from i until he encounters a vending machine. If vending machines are spaced at random throughout the area, the probability-distribution of trip-ends by distance from the origin is an inverse exponential function of that
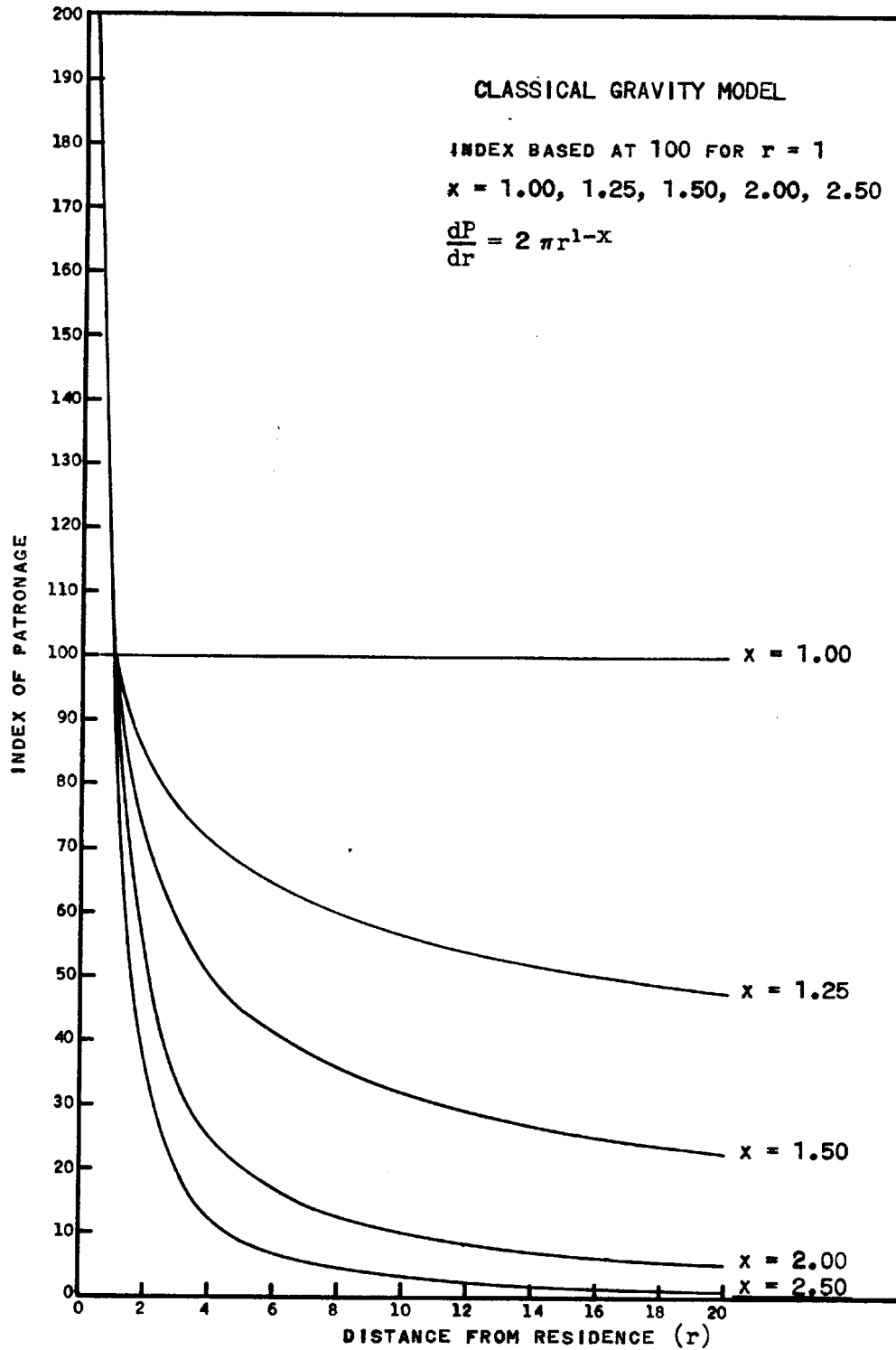
Fig. 2 — Theoretical Distribution of Retail Patronage by
Distance from Residence, Standardized Conditions:
Classical Gravity Model

distance,

$$\frac{dP}{dr} = pe^{-pr}$$

where p represents the point-density of vending machines.[25] Figure 3 shows this function in standard form for several alternative values of p.

Perhaps it is simply the force of professional tradition, but an economist would much prefer that his implicit model of individual behavior be cast in terms of intelligent choice under known conditions-- e.g., that our smoker be aware of the locations of all vending machines, and choose among them so as to maximize or minimize some magnitude. The obvious magnitude to be minimized is total cost (including "delivery") of his purchase. But if distance is given a "cost" interpretation, it is difficult to explain why anyone should venture beyond the closest vending machine.

---

[25]Given a random distribution of vending machines, the probability of encountering a machine while travelling any unit of distance is a constant, p. It follows that the probability of encountering a machine while crossing any annular zone of unit width is also p, and that the probability-distribution of trip-ends by successive zones is:

Prob (Trip-end in Zone 1) = p
Prob (Trip-end in Zone 2) = $p(1-p)$
Prob (Trip-end in Zone 3) = $p(1-p)^2$
Prob (Trip-end in Zone j) = $p(1-p)^{j-1}$

This distribution can be more conveniently treated as continuous:

Let p = Point density of trip-terminals along any radial r.
    P = Probability of a trip-end within distance r of the origin.
Then dP = Probability of a trip-end within an element dr of the radial r. (Since the direction of r is random, a trip-end within the element dr could as well be described as lying within the annular element dA of the region A.)

It can then be shown by a derivation paralleling that of Schneider (op. cit.) that:

$$P = 1 - e^{-pr},$$

or $$\frac{dP}{dr} = pe^{-pr}.$$
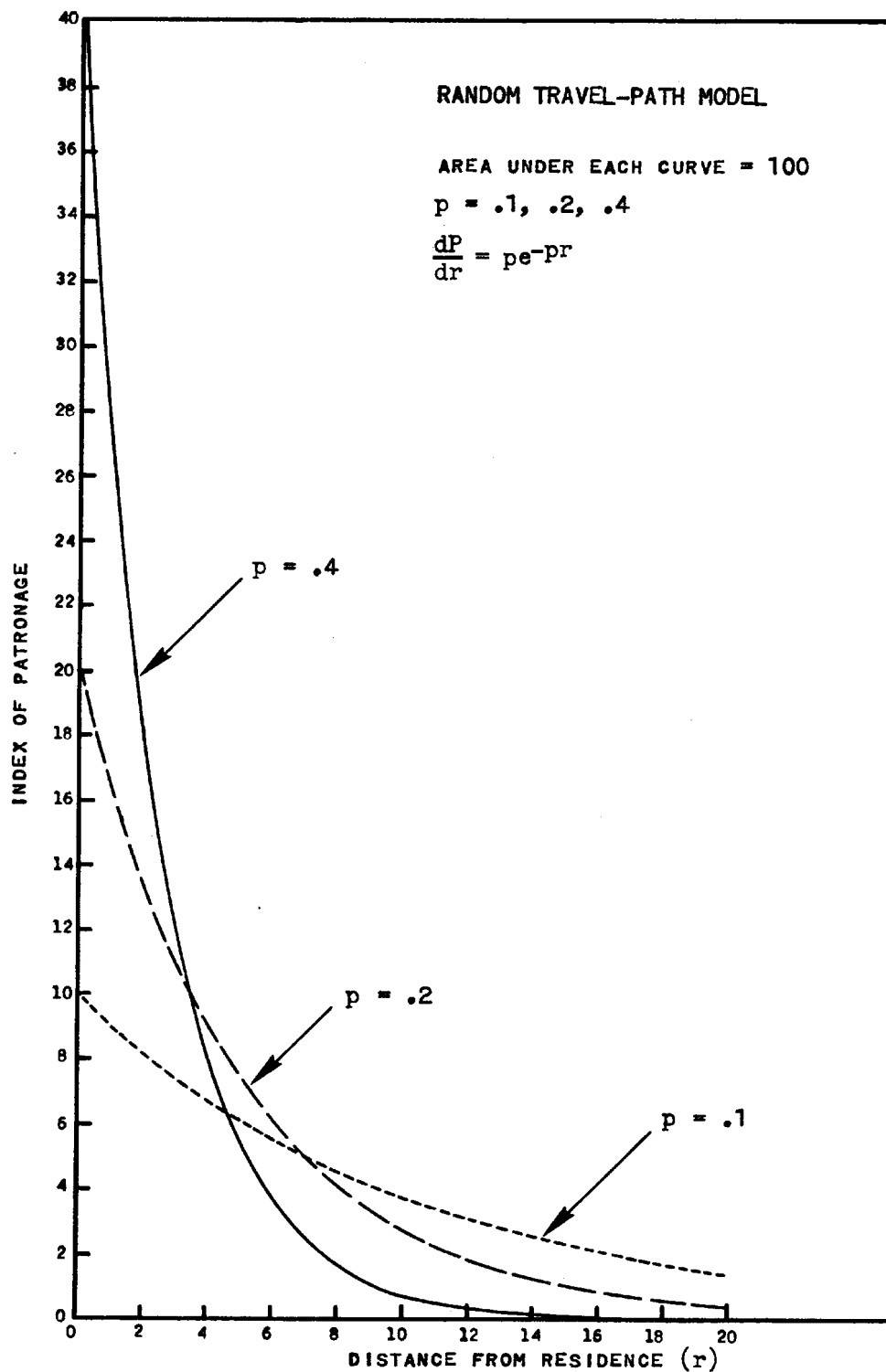
Fig. 3 — Theoretical Distribution of Retail Patronage by
Distance from Residence, Standardized Conditions:
Random Travel-Path Model

The figure shows a graph titled "RANDOM TRAVEL-PATH MODEL" with "AREA UNDER EACH CURVE = 100", $p = .1, .2, .4$, and $\frac{dP}{dr} = pe^{-pr}$. The y-axis is labeled "INDEX OF PATRONAGE" ranging from 0 to 40, and the x-axis is labeled "DISTANCE FROM RESIDENCE (r)" ranging from 0 to 20. Three curves are shown labeled $p = .4$, $p = .2$, and $p = .1$.

This is really only a comment on the limitations of our paradigm. While cigarette machines may be all alike to the shopper, grocery stores are not; in fact, there are very few lines of retail trade in which customers are really indifferent as between sellers. In other words, we are forced into a model of choice in a context of product differentiation. This is an unfortunate resort, because once we allow differentiated products, we lose most of the conventional constraints on consumer choice which are necessary for a determinate solution to the market process. These constraints must be replaced by others which have a low heuristic value--or which only lack the appeal of conventional usage.

Schneider's model mentioned in Section III fits the case of product differentiation, although I am not entirely sure that this is the interpretation he would give to it:

> The assumptions...are these: that the probability of a trip finding a terminal in any element of a region is proportional to the number of terminal opportunities contained in that element; that a trip prefers to be as short as possible, lengthening only as it fails to find a terminal...(as a simplification) the trip-receiving region is regarded as an unbounded plane surface over which trip terminal opportunities are evenly distributed.[26]

An "element of a region" is defined as an annular zone, dA, where $A = \pi r^2$. Schneider proposes that the probability of a trip's terminating in dA--having gotten past previous terminal opportunities--is proportional to the area dA. This contrasts with my reasoning in the case of the cigarette machines that this same probability is proportional to the length of the trip-path across dA. Otherwise, the structure of the two models is identical.

In his article, Schneider goes no further by way of illustration of the behavior to which his probability distribution,

$$\frac{dP}{dA} = pe^{-p\pi r^2}$$

(where p = probability-density of trip-terminals throughout the region),

is relevant; but in a later unpublished paper, it appears that the

---

[26] Op. cit., pp. 52-53.

trip-maker is to choose his destination prior to setting out, considering each "terminal opportunity" in order of radial distance from the origin, until he encounters one which is "acceptable."

If we consider only the set of "acceptable destinations" for a given individual, it is clear that any trip he makes must terminate at the nearest acceptable destination. But if each member of a group of trip-makers constructs his own set of "acceptable destinations"--and these sets[27] are random sub-sets of total "terminal opportunities"--we can then construct a probability function whose derivative gives us the relative frequency distribution of first acceptable destinations by radial distance from the origin. The function is:

$$P = 1 - e^{-p\pi r^2},$$

and its derivative is an inverse exponential function of r^2:

$$\frac{dP}{dr} = 2\pi rpe^{-p\pi r^2}.$$

This is the Schneider function, differentiated with respect to radial distance from the origin, rather than with respect to area. Its graph is shown in Fig. 4, for various values of p, which is the point-density of retail outlets.

Considering this model as a statistical aggregation of individual decisions under conditions of product differentiation, we can see at least one major simplification which cannot be derived from randomization principles: product differentiation is a two-valued matter; to each consumer, a particular retail outlet is either "acceptable" or "unacceptable," without regard for the "delivery cost" associated with shopping at that outlet.

We might profitably explore a model which allows for a somewhat greater range of calculation and choice under conditions of product

---

[27]Schneider never discusses the formation of the sets of "acceptable destinations"; in fact, it was some time before I realized that more than one such set was intended.

Fig. 4 — Theoretical Distribution of Retail Patronage by
Distance from Residence, Standardized Conditions:
Differentiated Product Model (Schneider)

differentiation. Let us assume that each retail outlet in the region sells a slightly different version of the product, and that consumers are able to rank these versions in order of preference. This ranking by preference, however, is random as between consumers. In the absence of delivery cost, the households located at i would distribute their patronage at random over the region--each to his first-choice retailer. If for some reason each consumer were to buy from his second-choice retailer, the distribution of patronage would continue to be random with respect to the locations of sellers.

Now let us inject delivery cost as a parameter of choice, supposing this cost to be a linear function of radial distance from the origin, and that it applies to each unit of the product purchased. A given difference in delivery cost will persuade a consumer to substitute his second choice for his first, and so on. Substitution will thus occur only when it has the effect of decreasing the length of the shopping trip sufficiently to compensate for the inferiority of the more accessible product.

Following these rules, we can construct a frequency-distribution of trip-ends for annular zones centered on i. Although the formula for this distribution is rather complex, the probability that a trip-end will fall in the jth zone can be generally described as the product of two terms, A_j and B_j. The value of each term varies with the location of the jth zone, i.e., with respect to the distance of the jth zone from the origin i and also from the outer boundary of the region.[28] The graph of this function is shown in Fig. 5, for a region of 20 such zones.

––––––––––

[28]Unlike the probability models presented earlier, the logic of this model requires a step-function rather than a continuous distribution.

> Let P_j = probability that a trip-end will fall within the jth annular zone (j=1,n) of a bounded region, where
>
> Δr = the width of each zone and the unit of radial distance which makes a perceptible difference in delivery costs, and

In the preceding pages, I have sketched three alternative models of consumer behavior which are relevant to the mapping of retail

---

$R = n\Delta r$ = the length of the radius from the origin i to a finite boundary of the region, beyond which shoppers will not travel no matter how attractive the shopping opportunities may be.

Assuming as before that retailers are evenly distributed throughout the region, the probability that a given retailer is located in the jth zone is proportional to its area; but the probability that a shopper will give his patronage to this retailer depends on the travel-distance (or delivery cost) associated with each store in the region and each store's position in the shopper's system of preferences. Thus, the probability that a trip-end will fall in Zone j=1 can be written as follows:

$P_{j=1}$ = Prob (1st-choice store is located in Zone j=1)
+ Prob (1st-choice store is located in Zone j≥2).
 . Prob (2nd-choice store is located in Zone j=1)
+ Prob (1st-choice store is located in Zone j≥3)
 . Prob (2nd-choice store is located in Zone j≥2)
 . Prob (3rd-choice store is located in Zone j=1
+ etc.

Each of the subsidiary probabilities may be expressed in terms of the areas of the zones described. Two examples are given below.

Prob (1st-choice store is located in Zone j=1) =

$$\frac{\pi(\Delta r)^2}{\pi R^2}$$

Prob (1st-choice store is located in Zone j≥2) =

$$1 - \frac{\pi(2\Delta r)^2}{\pi R^2}$$

Taking advantage of the relationship $R = n\Delta r$, we can simplify these expressions and collect them into a general expression for P_j, the probability that a trip-end will fall within the jth zone.

market potentials. The trip-distribution functions derived from each[29] are given below, along with that of the classical gravity model:

1) Classical Gravity Model

$$dP/dr = r^{-x}$$

2) Random Travel-Path Model

$$dP/dr = pe^{-pr}$$

3) Differentiated Product Model (Schneider)

$$dP/dr = 2\pi rpe^{-p\pi r^2}$$

4) Differentiated Product Model (Lowry)

$$\Delta P/\Delta r = A(r,n) \cdot B(r,n)$$

$$P_j = A_j B_j$$

$$A_j = \left[\prod_{k=1}^{j-1} \left(\frac{n^2 - k^2}{n^2} \right) \right] \left[\frac{2j - 1}{n^2} \right]$$

$$B_j = n^2 \left[1 + \sum_{t=2}^{n-j} \left\{ \prod_{s=t}^{n-j} (2sn - s^2) \right\} \right] + \left[\prod_{s=1}^{n-j} (2sn - s^2) \right]$$

The various zone-counters in these equations--j,k,t, and s--are integral multiples of the unit of radial distance, Δr; consequently, P_j could also be written as $\Delta P/\Delta r$ evaluated at $r = j\Delta r$. This expression is the discrete analogue to the continuous probabilities, dP/dr, offered by the other, more tractable models of trip distribution.

[29]My exposition of each model of consumer behavior terminated with the derivation of the function dP/dr, the relative frequency of trip-ends (patronage) by radial distance from the origin. I stopped at this point because dP/dr can be plotted as a recognizable "curve of decay," as in Figs. 2-5. As explained on p. 26 above, this function relates to the point-density of trip-ends (G) and to the trip-distribution index (T_{ij}) as follows:
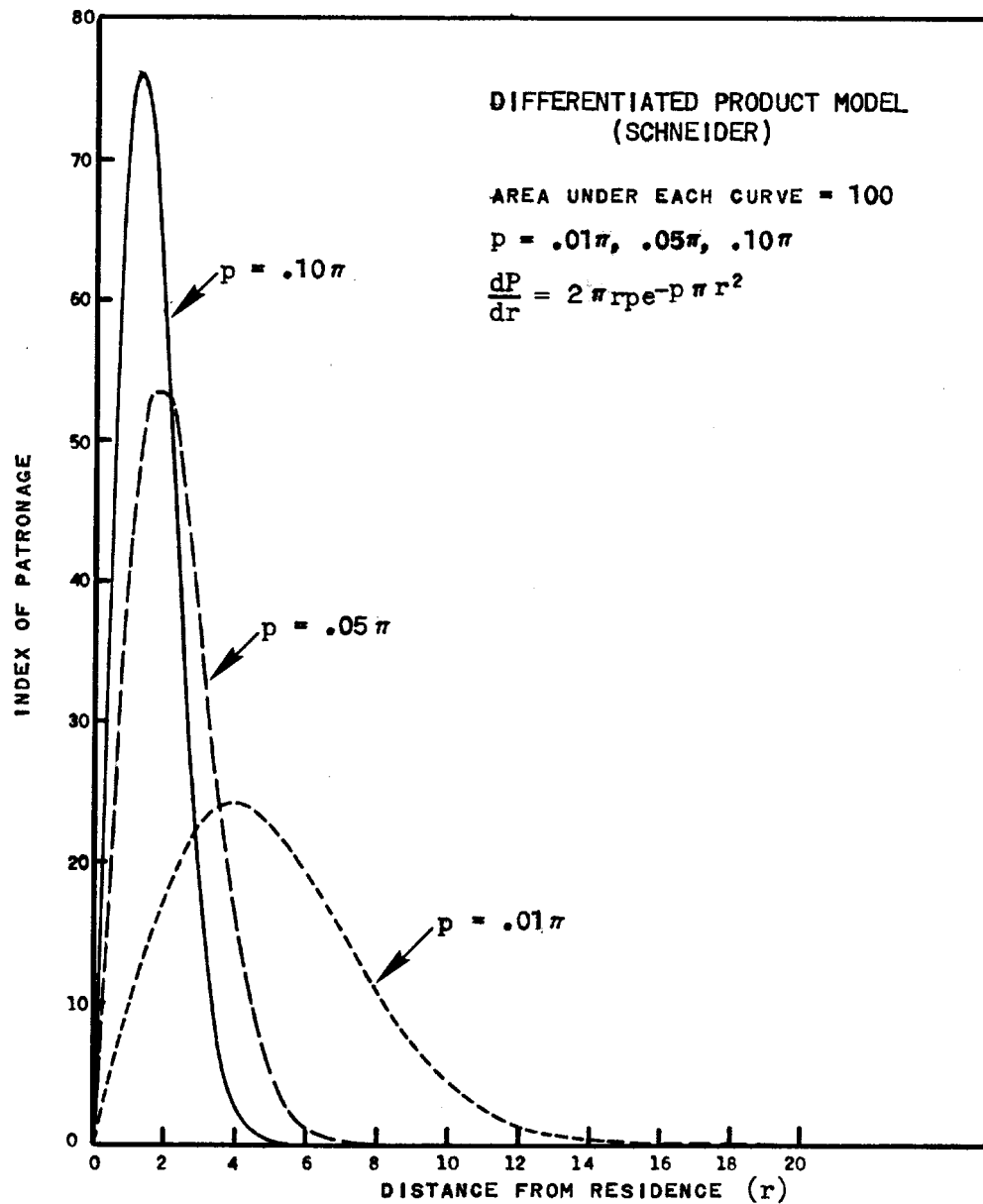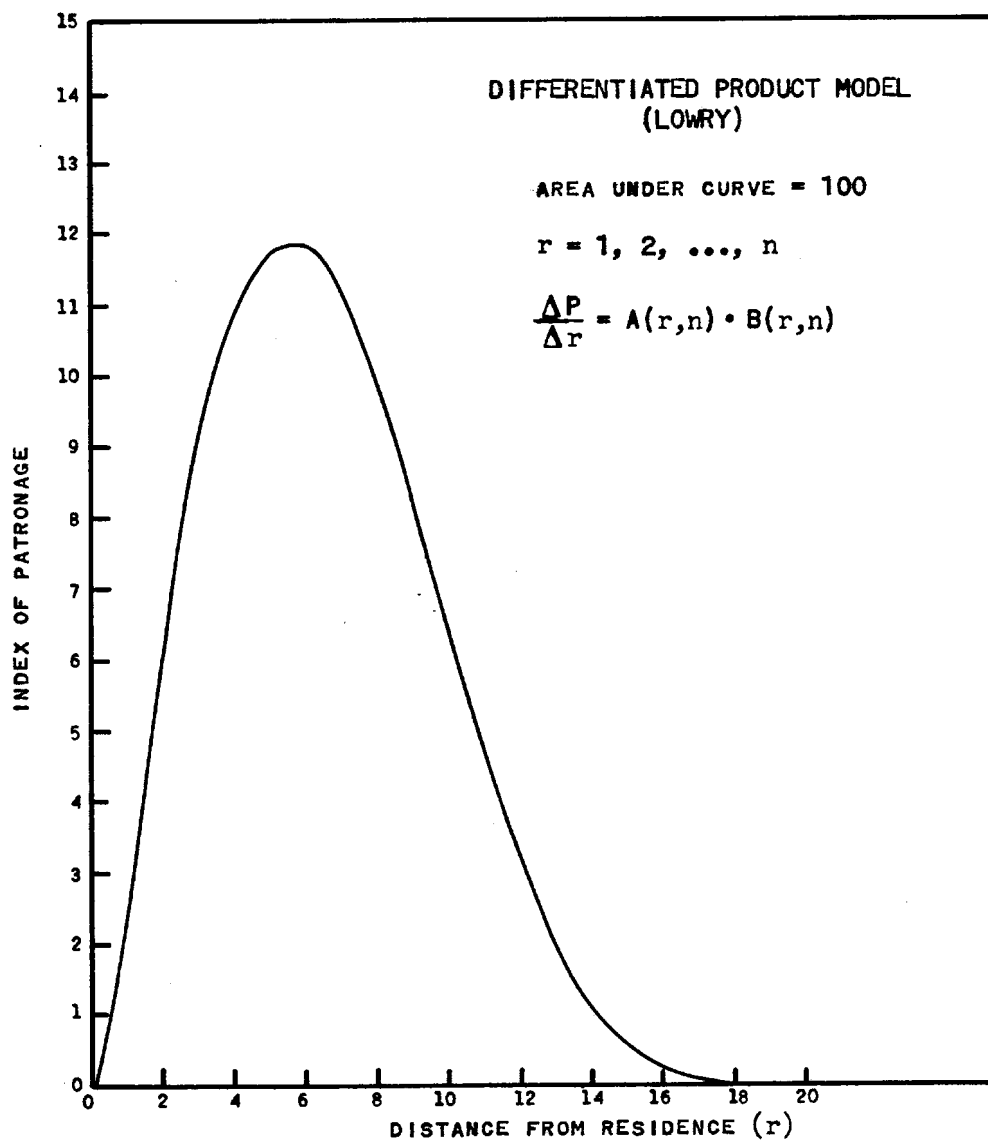
Fig. 5 — Theoretical Distribution of Retail Patronage by
Distance from Residence, Standardized Conditions:
Differential Product Model (Lowry)

None of the models of consumer behavior generated the precise function called for by the classical gravity model, but there is a strong family resemblance among the alternatives shown above. The second and third versions call for an additional parameter, p, representing the density in space of retail outlets. (The counterpart of this parameter in the Pittsburgh Model would be the minimum-size constraint for retail establishments, which effectively controls density.) The fourth requires a finite boundary for the region. And the reader can doubtless think of other plausible variants. So the formal structure of the Pittsburgh Model simply names a function of distance, T_{ij}, leaving the function itself to be defined by an analysis of the shopping habits of Pittsburgh residents. (See Sec. IV.)

$$T_{ij} = G^{-1} = \left(\frac{dP/dr}{2\pi r} \right)^{-1}$$

In empirical work, since the "points" i and j are substantial units of area (square-mile tracts, in the Pittsburgh Model), this formulation can only be approximated. See Section IV, "Trip-Distribution Functions."

IV. FITTING THE MODEL

From the very beginning, the design of the Pittsburgh Model was constrained by the resources available for fitting its parameters and for validating its overall structure. Since original field work was out of the question, the model had to be accommodated to existing data-files.

In this respect, we were most fortunate in that the Pittsburgh Area Transportation Study (PATS) had assembled a massive file of small-area data pertaining to land use, household characteristics, and travel behavior for 1958. A supplementary home interview survey was done by PATS for the Bureau of Public Roads in the spring of 1960; it offered additional detail with respect to certain relationships between household characteristics and trip-making. The U.S. Censuses of Population and Housing for 1960 provided independent controls for certain of the sample data in the PATS surveys. Studies of the central business district of Pittsburgh by the City Planning Department and the Regional Planning Association gave important information about this concentration of employment and shopping facilities.

Because the PATS data files were so central to our plans, the model was fit to the PATS study area -- 420 square miles centered on Pittsburgh. (See Fig. 6.) This area encompassed about 1.5 million inhabitants and 550,000 jobs, including all of the Pittsburgh Urbanized Area as defined by the 1960 Census except for three narrow corridors extending into neighboring counties. It also included some 225 square miles of usable vacant or agricultural land -- enough space to accommodate Pittsburgh's growth for several decades.

The process of fitting the model involved a great deal of trial and error. Between the first experimental run and the version presented here, there were five major revisions, each involving alterations in computational routines, changes in structural parameters, and different treatments of input data. For example, hospitals and colleges, as places of employment, were shifted back and forth between the basic and retail sectors; alternative groupings of establishments within the
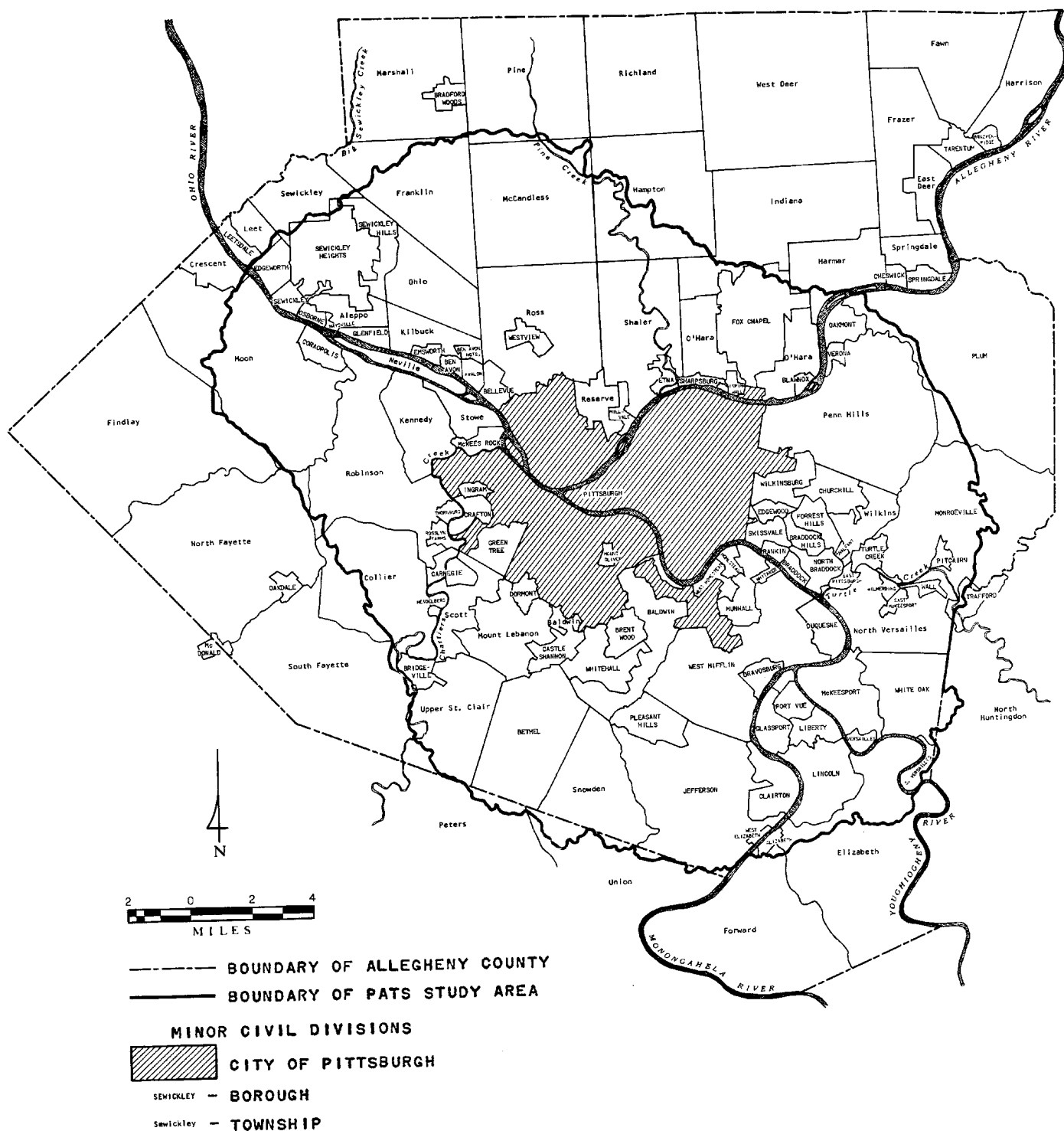
Fig. 6 — Minor Civil Divisions of the Pittsburgh Area

retail sector were tried, each necessitating revision of location parameters and minimum-size constraints; alternative solutions to the boundary problem were explored. In the following pages, I have reported primarily on the present version of the model, but some of its features can only be explained by reference to earlier experiments.

I feel rather keenly the inadequacies of empirical work reported here. Somehow, one's theoretical structures always seem to demand more data and more elaborate analysis of data than can be supplied. In the end, one must look back on a record of rough and dirty estimates, of compromises with the principle of parametric independence,[30] of discoveries made too late and of opportunities foregone. It is for the reader to judge the extent to which the performance of the model, reported in Section V, is vitiated by these shortcomings.

For the reader's convenience, the parametric values and input variables actually used in the present version of the model are summarized in an appendix to this section.

---

[30]Structural parameters -- constants which control the behavior of a mathematical system -- can be fit either in the context of the system itself, or independently. Following the first method, one chooses parametric values which optimize the performance of the system as a whole in the context of a given set of data; this is typically the case of multiple-regression models. To test the system as a model with more general applicability, one must operate it with new input data, independent of that used for the original parametric fit. If the model still works -- in the sense of yielding good estimates of output variables -- one gains confidence that the parameters are truly structural.

But for a model too complex for analytical solution, such bestfit parameters can only be derived by trial-and-error, an expensive and usually impractical method. Moreover, where independent sets of data are not available for subsequent testing of the fitted model, there is no way of confirming the generality of the fitted parameters.

The alternative, used in the present case, is to fit the parameters independently of each other, outside the context of the model. The fitted parameters may then be plugged into the system, and the system applied to whatever data are available. There is no guarantee that such an assemblage of independently-fitted components will function smoothly together; but if they do, at least the performance of the system cannot be dismissed as merely reflecting built-in circularity.

TERRITORIAL UNITS

All PATS data files -- on land use, household characteristics, trip origins and destinations -- were keyed to a plane-coordinate grid. The smallest unit of area for which data were recorded was the city block (or its analogue, in outlying areas), identified by the coordinates of its centroid. Contiguous blocks were aggregated into small tracts, accommodated as nearly as possible to the cells of a coordinate grid with half-mile intervals. The modified grid cells, nominally one quarter of a square mile in area, were then used as building blocks for traffic analysis zones, rings, and sectors.

For my purposes, the larger PATS units were overly-specialized as to traffic analysis, and the quarter-square-mile tracts were too numerous, and also too small for use in the analysis of sample data. By systematic aggregation of these tracts, I created 456 larger units, fitted to a coordinate grid with one-mile intervals. (See Fig. 7.) To simplify computations, dummy tracts were added to fill out a rectangular matrix 26 by 25 miles in dimension.

For both input and output data these square-mile (nominal) tracts are the smallest geographical unit recognized by the model. While the actual tract boundaries follow those of constituent city blocks, the abstract grid takes no account of the boundaries of other natural areas; individual tracts may be functionally heterogeneous, divided by topography or discontinuities in land use. The main advantage of this geographic coding system is computational flexibility, especially in the calculation of internal spatial relationships. As a fringe benefit, reciprocal to the advantages of a natural-area geography, the systematic grid is neutral with respect to theoretical patterns of urban structure.

LAND USE

PATS conducted a parcel land-use survey, classifying each property in terms of its ground-floor use. Forty-five categories of use were distinguished; except for "central offices," those uses related to economic activities could be grouped into nine major categories corresponding to the industry codes used in recording employment data.
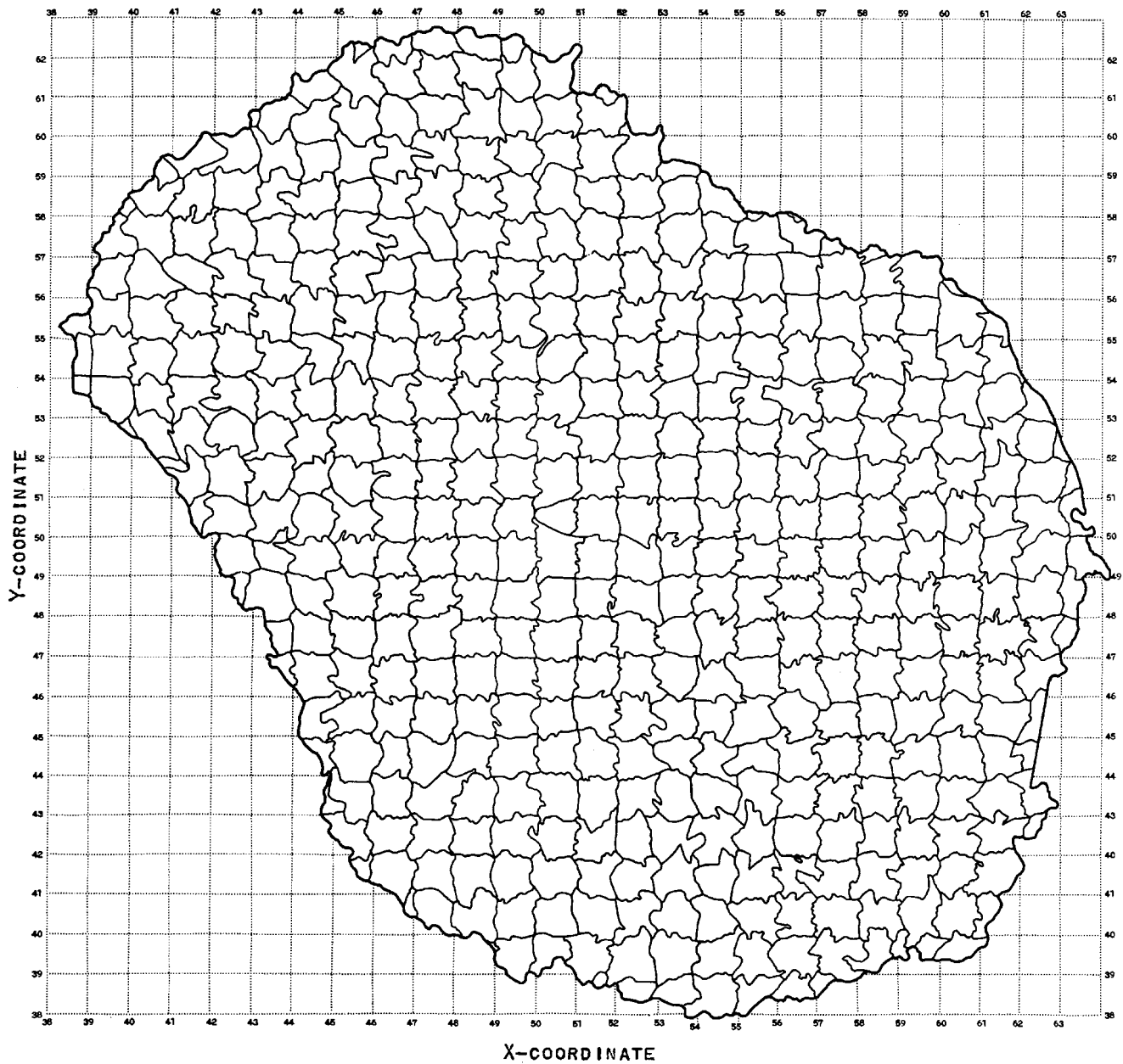
NOTE: TRACTS ARE IDENTIFIED
BY COORDINATES OF THE
LOWER LEFT CORNER.

Fig. 7 -- Division of the Pittsburgh Area into Tracts Based
on One-Mile Coordinate Grid

For the Pittsburgh Model, these 45 land uses were reduced to
five: basic, retail, residential, unusable, and agricultural or vacant.
The basic group included all uses related to the economic activities
treated exogenously by the model, as well as public thoroughfares and
parks. The retail group included those uses associated with the retail,
service, and public facilities treated endogenously by the model. Un-
usable land included slopes in excess of 25 per cent, water and swamp
areas, abandoned strip mines, etc. (Details of this classification
are given in the appendix to this section.)

RESIDENTIAL POPULATION

The home interview survey conducted by PATS in 1958 included an
area-wide sample of households, with an average sampling rate of about
four per cent. The data recorded included: the type of residential
structure; the numbers of persons under 5, from 5 to 16, and over 16
years of age; race, sex, occupation, and industry of the head of the
household; and the number of automobiles available to each household.
For each person over 5 years of age, a record was made of vehicular
trips on the preceding day: origin and destination, mode of travel,
time en route, purposes[31] and land uses at origins and destinations.

On the basis of the PATS sample, I estimated that there were
about 448,000 separate households in the study area in 1958, including
the residences of primary individuals, but excluding the institutional
population. Detail concerning household characteristics other than
travel patterns was processed, but its use awaits further refinement
and expansion of the model.

EMPLOYMENT

While the U.S. Census of Population records employment by the
employee's place of residence, there is no direct source of small-area

---

[31]Trip purposes identify the kind of activity carried on at
origin and destination--e.g., home to work, work to eat meal, eat meal
to social-recreation.

data for employment by place of work.[32] Consequently I undertook to
generate an employment surface from work-trip records of the transpor-
tation study. This project was not as simple as it might sound. Some
persons employed in the study area live outside its boundaries, and
were thus not included in the home interview survey. Work trips re-
corded by the home interview do not adequately represent the universe
of employed residents, because many employed persons may not go to work
on a given day, and the irregularity of work schedules varies consid-
erably among occupations. Finally, the spatial distribution of work-
places must be deduced from the destinations of work-trips; and since
I was dealing with sample data, a great many of the territorial units
received too few work trips to serve as reliable estimators of the
numbers of persons employed there, much less as estimates of their
occupations or industrial affiliations.[33]

After a complicated series of manipulations, adjustments, and
reconciliations, I arrived at some reasonable estimates of total
employment in the study area, broken down by occupation and by in-
dustry (but not cross-classified), and was able to allocate these jobs
among the 456 territorial units.[34] There was no cure, however, for the
sampling variability associated with small numbers; an estimate of 90
jobs in a particular tract, for instance, would ordinarily be based on
only three or four actual work-trip records. Since the employment
surface is fundamental to the distributions generated by the model,
the unreliability of such figures infects all results for small areas,

---

[32]State agencies dealing with workmen's compensation and unemploy-
ment insurance maintain files of their clients which list the employers'
business addresses. Coverage ordinarily extends to about two-thirds of
the labor force. These files have been exploited in some studies as a
source of small-area employment data, but only after considerable
negotiation with the state agency.

[33]There are also intrinsic ambiguities in the concept of "workplace"
for a substantial portion of the labor force whose work is peripatetic.
Where does a postman work? An outside salesman? A bricklayer?

[34]For a detailed account of the problems of generating the employ-
ment surface, and the methods finally used, see Ira S. Lowry, Employment
by Place of Work in the Regional Core, a preliminary report of the Re-
gional Economic Study, Pittsburgh Regional Planning Association, June,
1962.

and makes it exceedingly difficult to appraise the appropriateness of related small-area values generated by the model.

The next problem was dividing this spatially-distributed employment into basic and retail components, the former to serve as inputs to the model, the latter to be compared with the output of the model. The endogenous retail component was to include establishments serving the local population, except where special locational factors were likely to obscure the more general requirement of accessibility to this local market. All other kinds of establishment were to be relegated to the exogenous (basic) sector -- with the hope that in due course ways might be found to expand the former sector at the expense of the latter.

Of the kinds of business identifiable as "retail" under the PATS coding scheme, many clearly contain specialized components whose locational requirements are not those of the general run of consumer-oriented establishments. Some, in fact, probably do not deal with retail customers at all. In the end, I found only ten categories which were plausible candidates for endogenous treatment by the model:[35]

Kind of Business	Number of Employees
Food and drug stores	17,327
Eating and drinking places	13,117
Department, furniture and appliance stores	16,731
Other retail stores	29,046
Financial, insurance, and real estate offices	23,634
Personal service establishments	9,907
Medical, dental, and legal offices	7,069
Other service establishments	33,207
Government offices and public buildings	23,507
Schools (except colleges)	18,096
Total	191,641

[35]In some of the earlier experimental runs, I included hospitals and sanitaria (22,606) and colleges (4,048) in this list; they were removed when it became apparent that their massive and specialized site-requirements outweighed normal considerations of each access to patients and students.

As work with the model progressed, it became clear that it was too expensive in terms of machine time to work with so many categories of retail trade, so groups with similar market-patterns were combined. This step led finally to the choice of the retail cluster (containing assorted kinds of business) as the locating unit. With the assistance of Census of Business reports for Allegheny County, the City of Pittsburgh, its central business district, and several secondary business districts, I was able to define three fairly distinct types of retail clusters, and the employment listed above was distributed among these cluster types as carefully as the data allowed. Most kinds of business were represented in at least two of the three cluster types, but in varying proportions. The clusters were designated as follows:

Cluster Type	Number of Employees
Neighborhood facilities: Food stores; drug stores; gasoline service stations; personal services (part); elementary and secondary schools; domestic services.	50,000
Local facilities: Parts of the following: Eating and drinking places; medical and health services; welfare and religious services; personal services; finance, insurance and real estate services; automotive dealers and repair services; department, general merchandise, and variety stores; amusement and recreation facilities; public administration; miscellaneous retail and service trades not listed above.	85,000
Metropolitan facilities: Parts of most groups listed under "local facilities," with large shares of department stores, financial services and public lodgings, business services, and public administration.	56,700
Total	191,700

Thus each of these groups was treated as a separate "retail trade" whose employment was to be distributed by a separate computational routine. Employment from more than one group might be assigned to a given tract -- e.g., Tract j could have both a neighborhood cluster and a local cluster. Minimum-size constraints would apply to the

cluster rather than to the individual establishment, implying for each cluster-type a minimum number of households necessary for its support.

The remaining "basic" employment amounted to 360,900 jobs, the bulk of which were in manufacturing industries. (See appendix for details.) These jobs, whose spatial distribution was inferred from the PATS data concerning work-trip destinations, are really the locational "anchors" for the entire model.

TRIP-DISTRIBUTION FUNCTIONS

The Pittsburgh Model relies heavily on parameters which define the spatial distribution of the residential population around employment centers, and on other parameters which define the potential patronage offered by the dispersed residential population to retail establishments at various locations. In Sec. III it was demonstrated that these parameters could be interpreted as trip-distribution functions, relating the numbers of trips made by a given residential population positively to the number of terminal opportunities and negatively to intervening distances. These functions were fitted by means of an analysis of data gathered by PATS for the Bureau of Public Roads in the spring of 1960.[36]

For the BPR survey, home interviews similar to those of the original PATS survey were conducted in thirteen residential neighborhoods scattered throughout the study area. My analysis of trip-making was based on this 1960 survey (rather than on the more comprehensive one of 1958) primarily because it had the advantage of a much higher sampling rate. In each of the selected neighborhoods, at least 10 per cent of the resident households were interviewed, and the rate rose to 33 per cent for low-density neighborhoods. Trip data were collected for almost 11,500 persons over five years of age, representing an estimated 94,700 persons in the universe of thirteen areas.

---

[36]For a detailed discussion of this analysis, see Ira S. Lowry, "Location Parameters in the Pittsburgh Model," Papers and Proceedings of the Regional Science Association, Vol. 11, 1962 (forthcoming).

Work Trips

The sample yielded nearly 4,000 work-trip records, representing an estimated 32,400 first work trips by residents of the thirteen areas. Taking each residence area separately, I compared the destinations of work trips with the locations of employment opportunities, as defined by the 1958 employment surface described above. The method used was the computational equivalent of drawing concentric circles around each residence zone at one-mile intervals and aggregating both work-trip destinations and employment opportunities for each annulus.[37]

By standardizing for the uneven distribution of employment opportunities around the residence zone, we achieve the statistical equivalent of the theoretical models presented in Sec. III, i.e., estimates of the relative frequency of trip-terminals by distance from residence, given an area in which the point-density of employment is constant.[38] (The point-estimates yielded by this empirical model can be identified as values along the theoretical function dP/dr.)

A successful statistical model, in this context, would be one in which the values of P for a particular annulus were stable over the universe of residential neighborhoods. Although there were definite central tendencies, the range of these values for a given annulus was large.[39] Sample sizes for individual residential neighborhoods were

---

[37] The annuli, however, were built from the square-mile cells of a rectangular matrix, with consequent irregularities.

[38] The standardized measures are as follows

$$P_{ij} = k \left[\frac{W_{ij}}{\sum_j W_{ij}} \div \frac{E_{ij}}{\sum_j E_{ij}} \right],$$

where W is the number of work trips originating from the residence zone i, and E is employment in a given annulus j. A scale-factor, k, standardizes the measure so that

$$\sum_j P_{ij} = 100.$$

[39] See Lowry, "Location Parameters," for details. Notation has been revised for clarity.

too small to permit the introduction of additional dimensions of standardization which might have reduced this variability.

However, a weighted average of the values for each annulus can be calculated by aggregating the work-trip and employment variables for all thirteen residence areas. These average values are plotted in Fig. 8. They may be fitted rather well with a negative power function,

$$\frac{dP}{dr} = ar^{-x} ,$$

which is also shown in the figure. The distance variable r was evaluated at one-mile increments of airline distance rather than at the midpoints of the annuli.

Since employment opportunities relevant to a member of the labor force include only those to which that person's training give him entry, employment opportunities and work trips were stratified into four broad occupational groups and the procedures described above were repeated for each group. The resulting standardized frequencies of work trips by distance from residence are plotted in Fig. 8, each fitted with a negative power function.

It is interesting to note that the exponents of these power functions vary inversely with the generally-accepted rankings of the occupational groups, by social status or by income. The indices thus indicate that upper-income families have a more dispersed residential pattern in relation to the relevant places of work than lower-income families. The finding fits very neatly with several independently-conceived models of the trade-off between site-rents and transportation costs, which predicate such a variation as an "income effect" on locational choice. (See Section III above.)

Shopping Trips

For the estimation of the trip-distribution indices which link the residential population with population-serving establishments, our sample offered about 5,000 trip records, representing an estimated
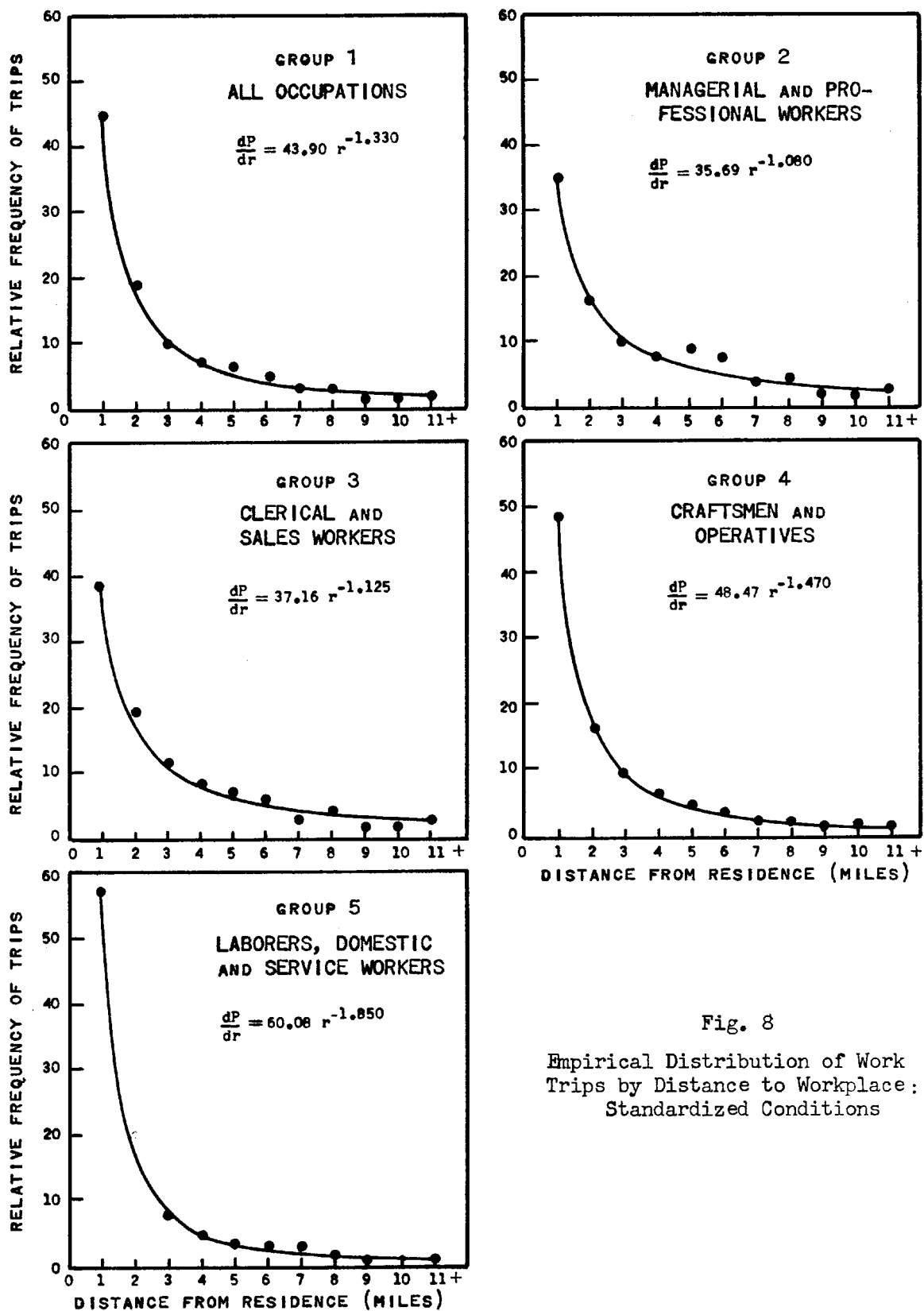
Fig. 8

Empirical Distribution of Work
Trips by Distance to Workplace :
Standardized Conditions

39,000 trips to ten distinguishable categories of establishment.[40]
The distribution of trip terminals for each type of shopping trip
originating in each of the 13 residential areas was standardized, as
in the case of work trips, by adjusting for variations in the fre-
quency of terminal opportunities. These terminal opportunities were
measured on the basis of the number of persons employed in the rele-
vant category of establishments within each annulus surrounding the
residence area.

Since only a small number of trip records from any one residence
area pertained to a particular type of shopping trip, the trip-
distribution indices, specific both as to residence area and to type
of trip, were expectably unstable. However, when the index values
for a specific type of trip are weighted and averaged over all 13
residential areas, they, like the corresponding indices for the work
trips, can be plotted as negative functions of airline distance. The
sequence of index values is not usually as regular as those found for
work trips, and they could not be fit either with power or exponential
functions. The best approximations were found to be reciprocals of
quadratic functions,

$$dP/dr = (a - br + cr^2)^{-1} ,$$

usually fit specifically to the first three values in the sequence.[41]

Experimental work with the computer model, however, soon dis-
closed that dealing with this level of retail disaggregation was
indefensibly expensive. Accordingly, the ten retail trades were

---

[40]There are three qualifications worth noting about the shopping
trip data. While work trip records were made for both vehicular and
pedestrian modes of travel, shopping trip records included only vehicular
modes. Thus an unknown number of pedestrian shopping trips were not
included in the analysis. Moreover, shopping trips are frequently multi-
purpose, and the trip records did not include any but the primary des-
tination unless the shopper moved by vehicle to subsequent destinations.
Finally, several of the ten distinguishable types of shopping trips re-
lated to rather heterogeneous categories of establishment.

[41]Trip-distribution functions for each of ten retail trade groups
are shown in Lowry, "Location Parameters."

combined into four broader groups on the basis of similarities between trip-distribution functions, and new indices were calculated and fit for these groups. (See Fig. 9.) Finally, when other considerations led us to settle on retail clusters (containing an assortment of retail trades) as the units to be located by the model, three of these functions, with slight modifications, were chosen as most relevant to the cluster types, neighborhood, local, and metropolitan facilities.[42]

The fitted curves shown in Fig. 9 have rather startling peaks, each of them occurring between one and two miles from the origin. One would certainly feel more comfortable with the functions chosen if more observations were available in the region of maximum values. However, the models of consumer behavior presented in the Appendix to Section III lead us to expect trip-distributions peaking, as these do, to the right of the origin. In any event, these functions, as used in the Pittsburgh Model, are evaluated only at one-mile increments, so that the interpolated portions of the curves are, for practical purposes, irrelevant.[43]

Supplementing these indices for home-based shopping trips is an index for work-based shopping trips. I assume that the latter are pedestrian trips with a very limited range, so that the market potential of a particular tract is affected only by work places within that tract. (See Section II, Eq. 3.)

---

[42]The contents of each cluster type are described above, p. 61. Parameters of the exact functions used in experimental runs of the model are given in the appendix to this section.

[43]The computational routine associated with these location parameters in the model observes, so far as possible, the conventions used in deriving them. Thus, in the calculation of market potentials generated by the population of Tract i, airline distances to surrounding tracts are rounded upward to integral miles, and the market potential, dP/dr, of each one-mile annulus is divided equally among the tracts falling within that annulus.

The computational convention enables us to avoid the use of the interpolated portion of the empirical trip-distribution functions; it also circumvents a problem created by the model's rough geographical grain: the increments of area enclosed by each annulus do not form a smooth function, $dA/dr = 2\pi r$, because the annuli are built of rectangular elements. Fig. 10, below, will give the reader an idea of the resulting distortions of the annuli.
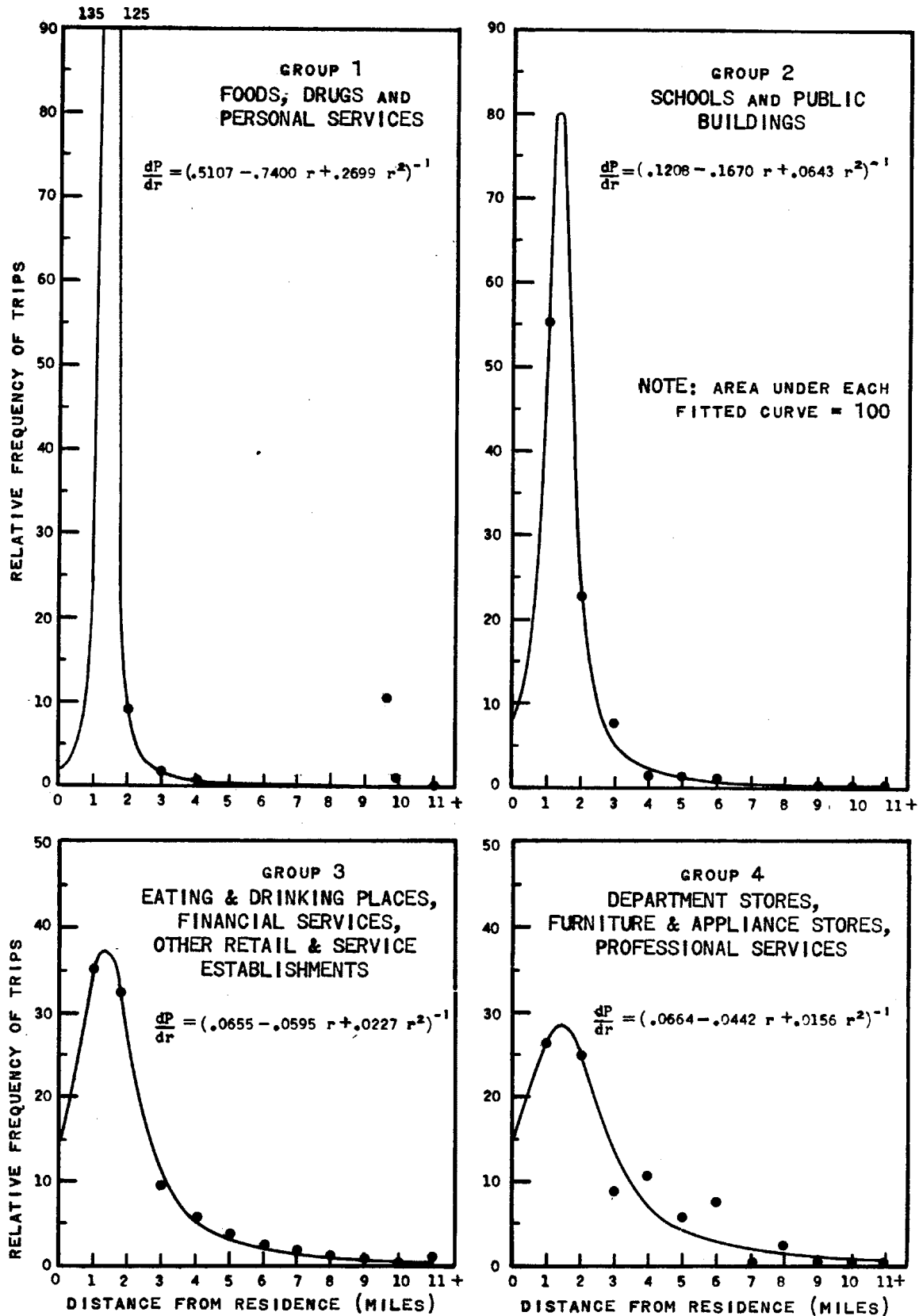
Fig. 9 — Empirical Distribution of Retail Patronage by Distance
from Residence :   Standardized Conditions

There remains, however, the problem of determining the influence
on market potential of these work places as compared to nearby resi-
dences. We cannot estimate from the traffic study data the relative
importance of these places as trip-origins, because traffic studies
have not concerned themselves with pedestrian shopping trips. It is
generally believed that virtually all trips to establishments of the
types included in our "neighborhood" clusters are home-based,[44] while
the usual downtown shopping district owes a substantial part of its
patronage to the surrounding concentration of work-places. But there
are virtually no data which provide a basis for estimating the rela-
tive importance to retailers of home-based and work-based shopping.[45]

Consequently the parameters chosen for experimental work with the
Pittsburgh Model are little better than informed guesses. For neighbor-
hood clusters, market potential contributed by residences is weighted
by .90, that contributed by local employment by .10; for local clusters,
the weights are .70 and .30 respectively; and for metropolitan clusters,
the weights are .50 and .50 respectively.

MINIMUM-SIZE CONSTRAINTS

Controlling the distribution of retail employment was anticipated
as a major problem in model design. The difficulty, essentially, is

[44]Even though some employed persons may buy groceries on the way
home from work, they are most likely to shop at a market near home to
minimize parking problems and bundle-carrying.

[45]Larry Smith and Company interviewed pedestrians in Pittsburgh's
central business district in 1961, inquiring about the respondent's
activities while downtown, including his expenditures for merchandise
and services. About 40 per cent of the reported dollar value of depart-
ment store purchases was attributed to persons whose primary reason for
being downtown was work-connected; but the sample design was such that
we cannot tell whether interviewees were really a representative cross-
section of the population of downtown shoppers. (Golden Triangle
Market Survey, Supplemental Report, Pittsburgh Regional Planning
Association, 1961.)

Robert L. Morris of the National Capital Downtown Committee reports
that about 25 per cent of all downtown shoppers in Washington, D.C.,
work in the area; about 25 per cent have come downtown on other errands,
but incidentally to shop; and about 50 per cent have come specifically
to shop. (Letter to the author, dated February 6, 1963.)

that the potential functions do not allow for those external economies of scale which, in the real world, encourage the clustering of retail establishments. The control device originally chosen was a minimum-size constraint, imposed on the distribution of employment for each kind of business. No tract would be allowed, for instance, to have a three-man department store. But it was otherwise left entirely to the model to build retail clusters by assigning employment in the various kinds of business to a single tract.

Experiments demonstrated that the model was not clever enough to do this. Consequently we shifted to a different approach, using the retail cluster itself as the unit of activity to be located. The constraint, in this context, defines the minimum size (number of employees) of a cluster composed of various kinds of retail business, service establishments, public agencies, etc., in this way:

Type of Cluster	Minimum Number of Employees
Neighborhood facilities	30
Local facilities	250
Metropolitan facilities	20,000

The precise values assigned to the constraints are not crucial. In a metropolitan geography whose territorial units are square miles, the retail employment surface tends to be continuous, though with many local peaks. Raising a minimum-size constraint ordinarily eliminates only locations on the fringe of development. The proper value for the constraints was taken to be the largest value that would not eliminate the isolated retail centers in the further reaches of suburbia.

Fixing the minimum size of a cluster implicitly sets a limit on the number of clusters which can be generated by the model. In each case, this maximum is considerably greater than the number of clusters of that type observable in the 1958 employment surface.[46] In particular,

---

[46]Admittedly, the identification of clusters in the 1958 employment surface calls for a good deal of judgment, since the system of tracts does not follow the boundaries of "natural" clusters, and there is no exact dividing line in size between cluster types. First-hand familiarity with the study area helps a good deal.

the minimum size for a metropolitan cluster was chosen so that, if market potentials were strongly peaked in more than one place, it would be possible for the model to create two central business districts.

MAXIMUM-DENSITY CONSTRAINTS

In order to prevent the model from generating excessive population densities in the vicinity of major employment centers, limiting values are determined outside the model, and imposed as constraints on the allocation process. These constraints are expressed as maximum numbers of households per thousand square feet of land available for residential use, and may vary from tract to tract, as would be the case under zoning ordinances. For most uses of the model, the zoning-ordinance approach to the density-ceiling seems to be the appropriate one, although there may be purposes for which the existing density of settlement is the more relevant figure for fully-developed areas. In either case, the maximum-density constraint would be quite powerful in shaping the distribution of residential population, imposing on individual tracts limitations derived from public policy or from the actual history of that tract.

Since experimental runs to date have had the sole object of validating the structure of the model and exploring its parameters, it seemed wisest to use this constraint only to prevent extreme densities. In our experimental runs, its value was finally set for all tracts at 1.5 households per thousand square feet, a site-density which would ordinarily be achieved in a four- to six-unit apartment structure. In 1958 there were only four tracts in Pittsburgh whose average densities exceeded 1.0 households per thousand square feet of residential land actually in use; average density exceeded 1.5 only in the central business district, which housed an insignificant number of families and primary individuals.

LABOR FORCE PARTICIPATION RATE

The number of households to be located in the study area is determined inside the model as a function of the number of jobs available.

(See Section II, Eq. 7.) This function may be interpreted roughly as a labor-force participation rate: the average number of workers per household.[47]

The concept is straightforward, and there is an abundance of relevant data; there are, however, some minor difficulties. One is the boundary problem: not all persons working in the study area actually live there, and some employed residents work elsewhere. What is the appropriate universe for calculating this rate? Also, should we include the entire labor force, or only those actually employed?

The limits within which our LFP rate might plausibly vary are given by the following figures:

PATS Cordon Area, 1958

Total employment	552,547	
Resident households	447,734	
LFP rate		1.234
Employed residents	526,342	
Resident households	447,734	
LFP rate		1.176

Allegheny County, 1960

Resident labor force	619,221	
Resident households	483,893	
LFP rate		1.280
Employed residents	578,834	
Resident households	483,893	
LFP rate		1.196

A comparison of total and resident employment in the PATS cordon area makes it evident that the net inflow of commuters from the surrounding area is about 26,000. If the minimum LFP rate of 1.176 is applied to total employment in the study area proper, the relevant population of households numbers about 470,000, or roughly 22,000 more than the number actually located within the PATS cordon line.[48]

---

[47]Actually, Eq. 7 calls for the reciprocal of this rate, or the average number of households required to supply one worker.

[48]552,547/1.176 = 469,853.

The model is designed so that persons working in the study area are "free" to settle outside its boundaries -- i.e., in any of the 144 dummy tracts which complete the 25 by 26-mile geographical matrix. Since there are no input data (land use, employment, etc.) for these dummy tracts, the model has a minimum of control over the allocation of households beyond the boundary of the study area proper, and we can only evaluate the output in a most general way against the 1960 census tract enumerations of households in the outer portion of Allegheny County.

While the LFP rate of 1.176 is the best alternative in the sense of generating the appropriate number of households, it takes no account of variable unemployment. Thus if the number of jobs were to increase by one or two per cent, new positions could doubtless be filled from the existing labor force, with virtually no recruitment from outside the area, and no increase in the number of resident households. Our parameter has a built-in unemployment rate of 6.5 per cent, a fact which must be considered in any subsequent application of the model.

Another source of instability in the LFP rate is its implicit allowance for households with no working members. In 1960, nearly 9 per cent of all families in the Pittsburgh SMSA were of this type, but the proportion of retirees can be expected to vary as the age structure of the population shifts over time.

RETAIL EMPLOYMENT COEFFICIENTS

After the model has evaluated the market potential of each tract, it allocates employment in the appropriate category of retail trade among all tracts in proportion to their potentials. The amount of employment to be allocated is a function of the aggregate market, i.e., the number of households to be served.

The relevant population of households numbers 470,000, of which about 448,000 should settle within the study area proper and the remaining 22,000 should be relegated to the dummy tracts of the fringe. Analysis of the PATS data for 1958 indicates that the study area proper contains the work places of about 191,700 retail employees,

divided among neighborhood shops, local business districts, and metropolitan-level establishments. (See above, "Employment.")

Retail employment coefficients -- the ratio of clerks to customers -- for each type of cluster can thus be calculated by simple division:

Type of Cluster		Retail Employment Coefficients
Neighborhood facilities:	50,000/470,000 =	.1064
Local facilities:	85,000/470,000 =	.1808
Metropolitan facilities:	56,700/470,000 =	.1206

Again, the boundary problem is a source of some discomfort. It is well known that shoppers come even from neighboring counties to patronize Pittsburgh department stores, and in 1958 there were nearly 40,000 persons employed by retail establishments in Allegheny County but outside the PATS cordon line.

Except for its insistence on clusters, the model has no structural barriers to the allocation of retail employment among dummy tracts outside the study area proper; but in point of fact, most of the outlying retail trade is in a few tight clusters around such industrial centers as Brackenridge-Natrona-New Kensington; and since we lack the basic employment inputs for the dummy tracts on the fringe, there is no direct way to build market potentials that can compete with those inside the cordon line.[49]

The most convenient solution is to set the retail employment coefficients so that they will generate the amount of retail employment required for the study area proper, as was done in the present version of the model.

---

[49]In one experimental run, basic inputs were invented for a few of the most important outlying industrial centers. While the model then assigned plausible retail and residential contingents to these areas, the impact upon the employment and population surfaces of the study area proper was negligible. The experiment was not pursued further. Obviously the boundary problem can be postponed in this way, but there is not much to be gained by extending the analysis to areas for which small-area data are simply not extant.

RETAIL LAND-USE COEFFICIENTS

After retail employment has been distributed among tracts, the model sets aside space in each tract to accommodate its retail establishments. The amount of space allocated for each kind of retail activity is a function of the numbers of employees assigned to the tract.

From the PATS land-use survey of 1958, I have made estimates of land actually in use by categories of retail establishments that are equivalents of the categories used in distinguishing retail employment. It is thus possible to derive space-use standards for various kinds of retail trade:

PATS Land-Use Code	Kind of Business	Square Feet per Employee
11	Food and drug stores	878
12	Eating and drinking places	874
13	Department, furniture, and appliance stores	204
19	Other retail stores	1,258
21	Financial, insurance, and real estate offices	137
22	Personal service establishments	656
23	Medical, dental, and legal offices	411
29	Other service establishments	1,158
61	Government offices and public buildings	518
62	Schools (except colleges)	3,666

However, these averages are not really very helpful: within each kind of activity listed, there is in the patterns of space-utilization a considerable variety, which is related in part to the location of the establishment. The land-use figures include open space and parking lots together with structural uses, and make no distinction between single- and multi-story uses. Thus a one-story suburban medical office with lawns and a parking apron may use several times as much site-space per employee as a high-rise medical building in Oakland. Public buildings include densely-populated downtown offices and such notorious space-users as memorials and museums.

There are ways in which the Pittsburgh Model might be elaborated to take account of variable land-use coefficients (see Section VI); but for the present, we can take comfort in the fact that there are

no more than a handful of tracts in which the space occupied by retail activities is a significant part of the total. Overall, the retail uses reported by the PATS survey amounted to 2 per cent of all usable land in the study area.

The shift from land-use coefficients for particular kinds of business to those for clusters containing a mixture of kinds offered some thorny problems. However, since experimental work with the model had already indicated that retail land-use coefficients were merely nominal parameters, whose values need only be such that the land-allocation routine does not interfere with more important business, it did not seem sensible to expend much effort on the conversion problem. Values were chosen so as to satisfy several conditions of significance to the model, without much expectation that the resulting spatial distribution of retail land use would have a close resemblance to reality. The conditions were:

1) The space allocated to metropolitan facilities is equal to the actual retail space in use in Pittsburgh's Golden Triangle in 1958; I refer, of course, to site-space, not floor-space. Note, however, that the model is by no means compelled to locate its central business district in the Triangle.

2) The space allocated to neighborhood facilities is approximately equal to that actually in use by the corresponding activities;[50] the largest component (85 per cent of the total) is elementary and secondary schools.

3) The space allocated to local facilities is sufficient to bring total retail space generated by the model to approximate equality with the total in use in 1958.

_____

[50] An error in land-use classification was discovered in the course of analysis of the final series of experimental runs. About twelve per cent too much space was allocated to neighborhood facilities, about eight per cent too much retail space overall. The error involved the misclassification of custodial institutions, which should have been treated as "basic."

The land-use coefficients finally used in the experimental runs reported in Section V were the following:

Type of Cluster	Square Feet Per Employee
Neighborhood facilities	1,900
Local facilities	1,300
Metropolitan facilities	80

The effect, in the model, is to generate about 210 million square feet of retail space for the study area as a whole.

APPENDIX

CONTROL TOTALS AND STRUCTURAL PARAMETERS
USED IN THE PITTSBURGH MODEL

LAND USES

	Thousands of Square Feet	
I. BASIC SECTOR LAND USES, PATS CORDON AREA		
TOTAL - ALL BASIC USES	2,615,813	
2 - Services	2,602	
24 - Offices - Central		2,602
3 - Heavy Commercial	53,005	
31 - Wholesalers and Distributors		23,290
32 - Junk and Salvage Yards		11,783
38 - Vacant Commercial		550
39 - Other Heavy Commercial		17,382
4 - Manufacturing	228,012	
41 - Primary Metals		113,522
48 - Vacant Manufacturing		1,385
49 - Other Manufacturing		113,105
5 - Utilities and Communication	127,970	
51 - Utilities - Communication		31,512
52 - Trucking Terminals and Warehouses		45,362
53 - Railroad Stations and Bus Depots		46,543
58 - Vacant		340
59 - Other		4,213
6 - Public Buildings	192,347	
61 - (Part) County Farm and Bureau of Mines		28,234
62 - (Part) Colleges and Custodial		18,857
63 - Hospitals and Sanitaria		104,665
64 - Military Installations		16,556
68 - Vacant		421
69 - Other		23,614
7 - Public Open Space	577,671	
71 - Stadiums, Drive-in Theaters, Race Tracks		18,212
72 - Golf Courses		134,181
73 - Parks, Beaches		251,136
74 - Cemeteries		138,417
79 - Other		35,725
8 - Airports, Streets, and Railroad Land	1,434,206	
81 - Streets and Alleys		1,269,150
82 - Airports		54,753
83 - Railroad Rights-of-way		110,303
II. UNUSABLE LAND, PATS CORDON AREA		
TOTAL - ALL UNUSABLE LAND	1,931,236	
91 - Water or Swamp Areas		312,453
92 - Mines and Quarries		35,363
94 - Slope 25 per cent +		1,583,420

III. RESIDUAL AVAILABLE FOR RESIDENTIAL AND
       "RETAIL" USES, PATS CORDON AREA

TOTAL BOUNDED AREA	11,698,786
BASIC USES	- 2,615,813
UNUSABLE	- 1,931,236
RESIDUAL	7,151,737

Source: Pittsburgh Area Transportation Study

EMPLOYMENT

		No. of Employees
I. "BASIC" SECTOR EMPLOYMENT, PATS CORDON AREA		360,948

PATS Industry Code

Code	Description	No. of Employees
3	Wholesale and Heavy Commercial	53,123
4	Manufacturing - Primary Metals	71,580
5	Manufacturing - Other	113,829
6	Utilities, Communications, Transportation	45,592
7	Hospitals,Colleges, and Institutions	25,470
8	Outdoor Public Services	915
9	Mining and Agriculture	1,421
(1 (2 (7	(Employees with "Retail" Sector (Industry Code, but "Basic" Sector (Land-Use Code	49,018

II. "RETAIL" SECTOR EMPLOYMENT, PATS CORDON AREA		191,599

Industry Code	Land-Use Code	No. of Employees
1 - Retail Trade	11 - Food and Drugs	17,327
	12 - Eating and Drinking	13,117
	13 - Department Stores, Furniture, Appliances	16,731
	19 - Other Retail	29,046
2 - Personal and Business Services	21 - Finance, Insurance, Real Estate	23,634
	22 - Personal Services	9,907
	23 - Medical, Dental, Legal	7,069
	29 - Other Services	33,207
7 - Government, Institutional and Religious	61 - Government and Public Buildings	23,465
	62 - (Part) Elementary and Secondary Schools	18,096

Source: Pittsburgh Area Transportation Study and Regional Economic Study.

POPULATION AND LABOR FORCE

I. RESIDENT HOUSEHOLDS, PATS CORDON AREA

Total permanent dwelling places	468,326
Vacant dwelling places	-19,075
Institutional dwelling places	- 1,517
Normal households (including primary individuals)	447,734

II. LABOR FORCE PARTICIPATION RATE

Employed residents, PATS Cordon Area	526,342
Resident households, PATS Cordon Area	447,734
Labor-force members per household	1.176

III. POPULATION REQUIRED TO FILL PATS CORDON AREA JOBS

Total employment, PATS Cordon Area	552,547
Labor-force participation rate	1.176
Total households required (employment/LFP)	469,853
Residing in PATS Cordon Area	447,734
Residing outside PATS Cordon Area	22,119

Sources: Pittsburgh Area Transportation Study, Regional Economic Study, and U. S. Census of Population.

PARAMETERS FOR RETAIL FACILITIES

	Neighborhood Facilities	Local Facilities	Metropolitan Facilities
Total employment, PATS Cordon Area	50,000	85,000	56,700
Minimum employment per cluster (Tract)	30	200	20,000
Number of households necessary to support one employee	9.40	5.53	8.29
Number of households necessary to support one retail cluster	282	1,106	165,800
Square feet of site-space per retail employee	1,900	1,300	80
Per cent of shopping trips originating from home	90	70	50

Sources: Calculated from data in previous tables, or estimated by the author.

TRIP DISTRIBUTION PARAMETERS

Type of Trip and Origin	Distribution of Trip-Ends by Air-line Distance (r) from Origin[*]
Work trips, all occupations:	
From workplace to home	$r^{-1.33}$
Neighborhood shopping trips:	
From home to retail establishment	$(.5107 - .7400\ r + .2699\ r^2)^{-1}$
From workplace to retail establishment	All trips terminate in work-place tract.
Local shopping trips:	
From home to retail establishment	$(.0116 - .0012\ r + .0202\ r^2)^{-1}$
From workplace to retail establishment	All trips terminate in work-place tract.
Metropolitan shopping trips:	
From home to retail establishment	$(.0664 - .0442\ r + .0156\ r^2)^{-1}$
From workplace to retail establishment	All trips terminate in work-place tract.

_____

[*]The parameters shown are fitted values of dP/dr, as defined in Section III, above. Empirically, square-mile tracts within the study area were grouped into annuli which are nominally concentric on the origin, and one mile in width. The functions shown were evaluated at one-mile intervals, and trips from the origin were allocated among annuli in proportion to these values. The share of all trips received by each annulus was then divided equally among all tracts contained in that annulus.

Sources: Pittsburgh Area Transportation Study, Bureau of Public Roads, and calculations by the author.

V. RESULTS OF EXPERIMENTAL RUNS

Experiments with the Pittsburgh Model were run on RAND's IBM 7090 over a period of about nine months in 1963. The process was perhaps typical of experimental work with computer models: the bulk of the effort went into the search for bugs in the program and for ways of increasing computational efficiency.[51] As experience accumulated, certain weaknesses in data became apparent, and parts of the model were redesigned to reduce the rôle of the less reliable inputs. Alternatives in logical structure were explored, and a few were adopted -- although the end product is remarkably close to the description given in the original prospectus. The parameters, as we have seen, were fit independently of the model; but some of them were varied in order to test the sensitivity of output to such changes.

The final version of the model has been run in three ways:

Experiment A. Given as initial inputs the 1958 distribution of basic employment and related land uses, the model was run through three complete iterations, generating interdependent distributions of resident population and retail employment by the method explained in Section III, above.

Experiment B. Given as initial inputs the 1958 distribution of total employment (including retail) and related land uses, the model was used to estimate population potentials for each tract and to generate a distribution of residential population.

Experiment C. Given as initial inputs the 1958 distribution of resident population and of total employment and related land uses, the model was used to estimate retail market potentials and to redistribute retail employment and land use on this basis.

The first experiment exposes to our scrutiny the iterative processes of the model: the rate and path of convergence towards a stable co-distribution of employment, population, and related land uses. The outputs of the last iteration can also be compared to observed values

---

[51] The basic computer algorithm was programmed to the author's specifications by John F. Muth of Carnegie Institute of Technology. Revisions were programmed by the author, with assistance from Gary Brown of The RAND Corporation.

for the endogenous variables (i.e., actual data for 1958) so that we may mark the similarities and differences between our "instant metropolis" and that which has resulted from a century and a half of urban development.

In Experiments B and C, we partition the model into its two major components so that inputs to each are uncontaminated by failures in the other component. This procedure allows us further insight into the perdestined convergence of the iterative sequence, and into the replicative competence of the model.

The statistical output of these experiments is much too voluminous to present in full geographical detail. Consequently, I have summarized spatial distributions of employment, population, and land use by annular zone (Fig. 10), and alternatively by ring and sector (Fig. 16). Although these summaries are concentric to the existing central business district of Pittsburgh, the model itself has no such structural orientation.

The interpretation of these experiments, and the conclusions I draw from them will naturally be much more accessible to readers who are familiar with Pittsburgh, although I have tried to make this section intelligible to a wider audience. A few maps (Figs. 6, 7, and 13) are included to orient the reader to Pittsburgh's geography and the many official and unofficial place-names that appear in the text. Volume One (Study Findings) of the PATS Final Report and Volume Two (Portrait of a Region) of the Economic Study Series may be usefully consulted for further orientation and background material.

ITERATIVE PROCESSES

While the running time of the Pittsburgh Model is substantial,[52] convergence is rapid. Figure 11 shows a profile of the spatial

---

[52] On RAND's IBM 7090 (equipped with off-line 1401 input-output hardware), a complete iteration of the model requires nearly 17 minutes. Running time is quite sensitive to the size of the geographical matrix and to the number of types of retail trade. With 650 units of area, each of the three loops for the distribution of employment in a retail trade requires nearly four minutes. The population loop requires about three minutes. The remainder of the time is devoted to housekeeping functions and the manipulation of parameters.
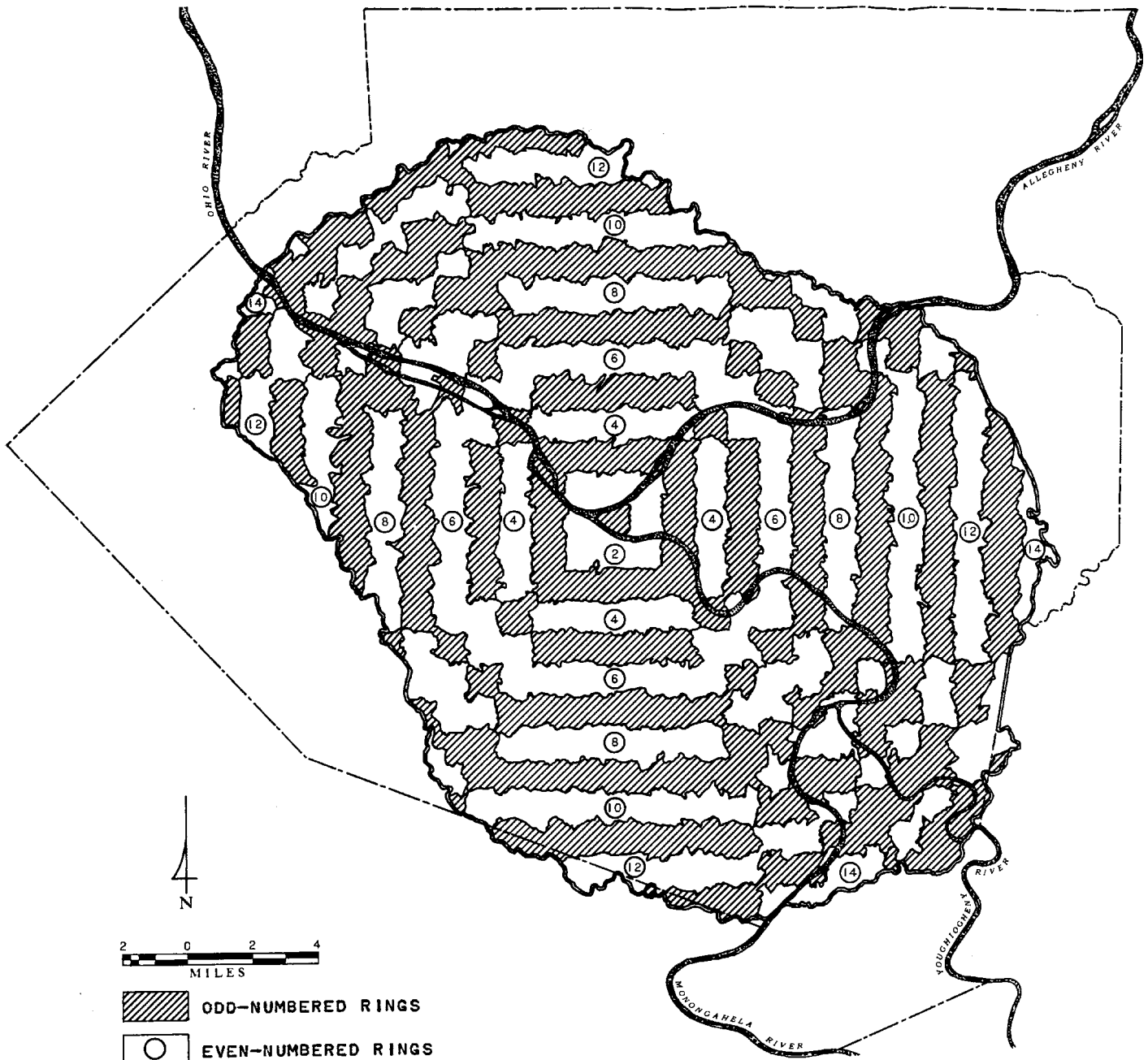
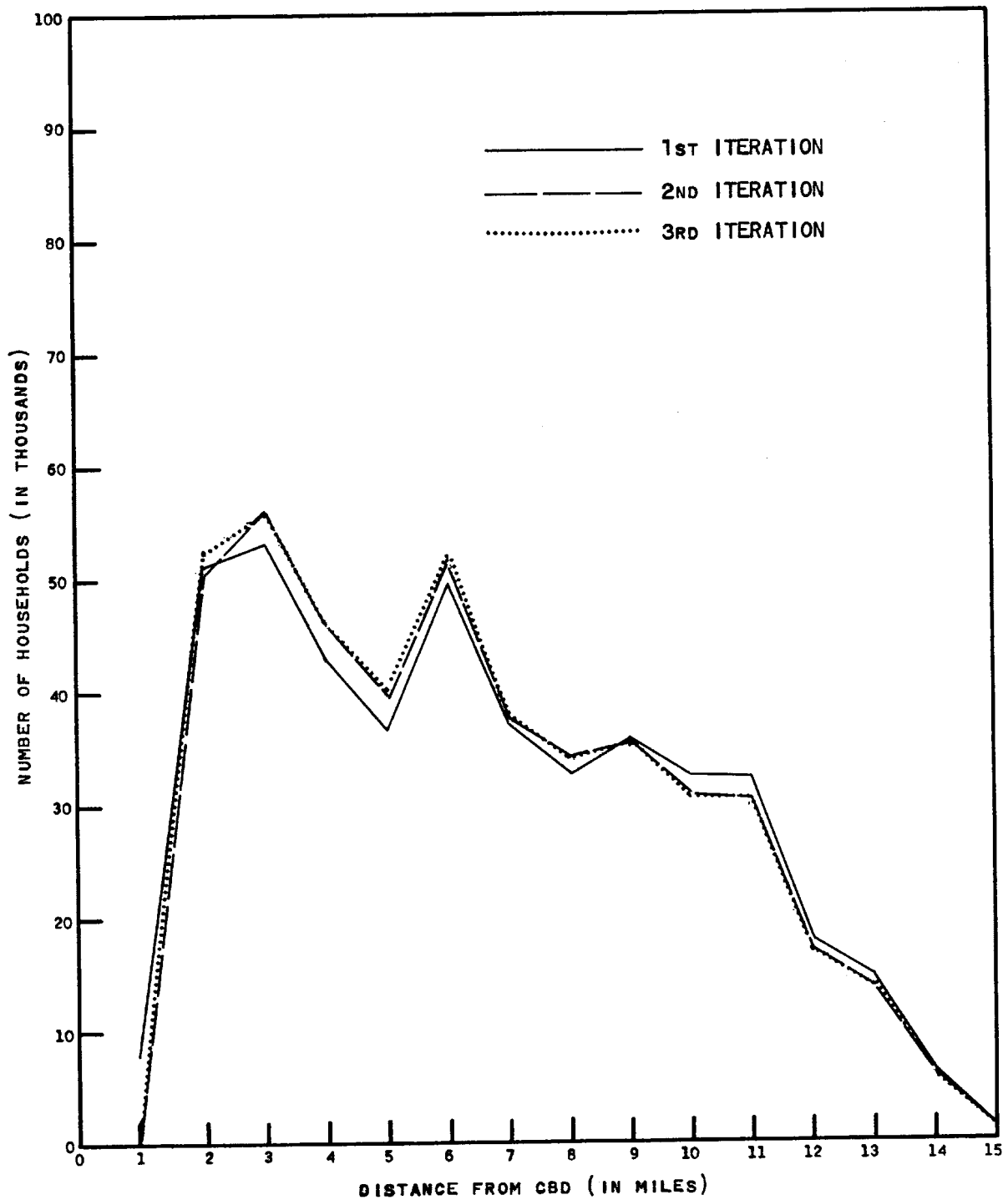Fig. 10 -- Division of the Pittsburgh Area into Annular Rings

Fig. 11 — Number of Households by Distance from CBD, as Distributed
by Successive Iterations of the Pittsburgh Model

distribution of resident households (by annular ring centered on the
business district) as generated by each of the three iterations of
Experiment A. The first distribution is predicated on the initial input
of basic employment, although the total number of households located
exceeds the number required to man this initial complement of jobs.[53]

Prior to the second iteration of the population loop, retail
establishments have been introduced, changing the employment surface sub-
stantially. Since they are less dispersed than basic establishments,
these retail workplaces exert a centripetal pull on the distribution of
population. At the same time, retail land use in the core of the area
pre-empts residential space, forcing core population outward.

These mutual readjustments between retail employment opportunities,
land-use constraints, and residential location continue in subsequent
iterations, but the net effects are small. As between the second and
third population distributions, the largest shift comes in Ring 2, an
increase of four per cent. With one exception, changes in the popula-
tions of individual tracts are less than three per cent.[54] Linear
regression of the individual tract populations generated by the second
and third iterations provides a convenient test of their similarity;
if the two sets of estimates were identical, the regression function
would be ($\hat{Y} = X$) with $R^2 = 1.00$. In the present case, the function takes
the form, ($\hat{Y} = 2.04 + .998X$), with $R^2 = .991$. (See Table 1 for complete
regression statistics.)

---

[53] Parameters supplied to the model make it possible to anticipate
consistent final totals of basic employment, retail employment, and
residential population. See Section II, p. 15, above. Convergence
of the iterative system would, of course, be slower if we limited each
iteration's population to that justified by employment already "in
place." There are some contexts, however, in which it might be appro-
priate to follow this method, treating the iterations as a recursive
sequence over time.

[54] The exception, which accounts for Ring 2's growth, is Tract
51-51 (the Allegheny Strip). The second iteration assigned 4,700
households to this tract, and the third iteration assigned 7,200; but
the subsequent build-up of retail uses (at the end of the third itera-
tion) would have the consequence in a fourth iteration of reducing
this tract's population to 5,700 households, by way of the maximum-
density constraint.

Table 1

REGRESSION STATISTICS FOR NUMBER OF HOUSEHOLDS BY TRACT:
COMPARISONS OF MODEL SOLUTIONS WITH 1958 INVENTORY

Sources of Regression Variables	Means of Regression Variables		Regression Parameters		Goodness of Fit		Size of Sample
	x (Independent Variable)	y (Dependent Variable)	a (Intercept)	b (Slope)	Sy.x (Standard Error of Estimate)	R^2 (Coefficient of Determination)	
Convergence of Iterative Sequence x = Iteration 2 y = Iteration 3	985	985	2.0	.998	121	.991	456
Partial Equilibrium and Iterative Solutions x = Partial Equilibrium y = Iteration 3	985	985	83.2	.916	217	.972	456
Iterative Solution and 1958 Inventory x = Iteration 3 y = 1958 Inventory	985	982	81.8	.913	919	.621	456
Partial Equilibrium Solution and 1958 Inventory x = Partial Equilibrium y = 1958 Inventory	985	982	109.7	.885	850	.676	456

The distribution of retail employment also stabilizes in the course of three iterations, as evidenced by Fig. 12. There is a qualification here, for the model's choices of locations for the central business district were modified "by hand." The reader will recall from Section III the problems raised by our shift from the classification of retail trades according to SIC kind-of-business categories to less rigorous kind-of-cluster categories. In the end, about 57,000 jobs, or 30 per cent of all the study area's retail employment, were assigned to "metropolitan-level" clusters, and minimum efficient size was set high enough (20,000 jobs) so that no more than two such clusters could be generated by the model.

In the first iteration of the retail loop, nearly equal market potentials were generated by the model for tracts adjacent to and including the actual CBD. Had the minimum-size constraint been set at 10,000 jobs, the "CBD" would have been divided by the model nearly equally between the Golden Triangle, the North Side, the Allegheny Strip, Lower and Upper Hill, and Oakland. (See Fig. 13.) In the event, the model assigned half of this retail employment to the Golden Triangle and the other half to Oakland.

In order to keep the model on the track, I altered these results before feeding them into the second iteration. All metropolitan-level retail employment was assigned to the Golden Triangle where it "belonged," and Oakland was left with its complement of local and neighborhood retail functions (2,837 retail jobs) as generated by the model.

The model, however, was persistent. In the second iteration, despite the fact that the Golden Triangle's work force had doubled (providing an additional market of noon-hour shoppers), the population readjustments described above pulled peak market potentials eastward. The shift was assisted by a loss of residents in the Triangle and Allegheny Strip -- households displaced by retail establishments which pre-empted most of the available space. The second iteration of the retail loops divided metropolitan-level establishments between the Hill District and Oakland. The North Side had the third highest market potential for this type of trade, with the Golden Triangle now lagging well behind.
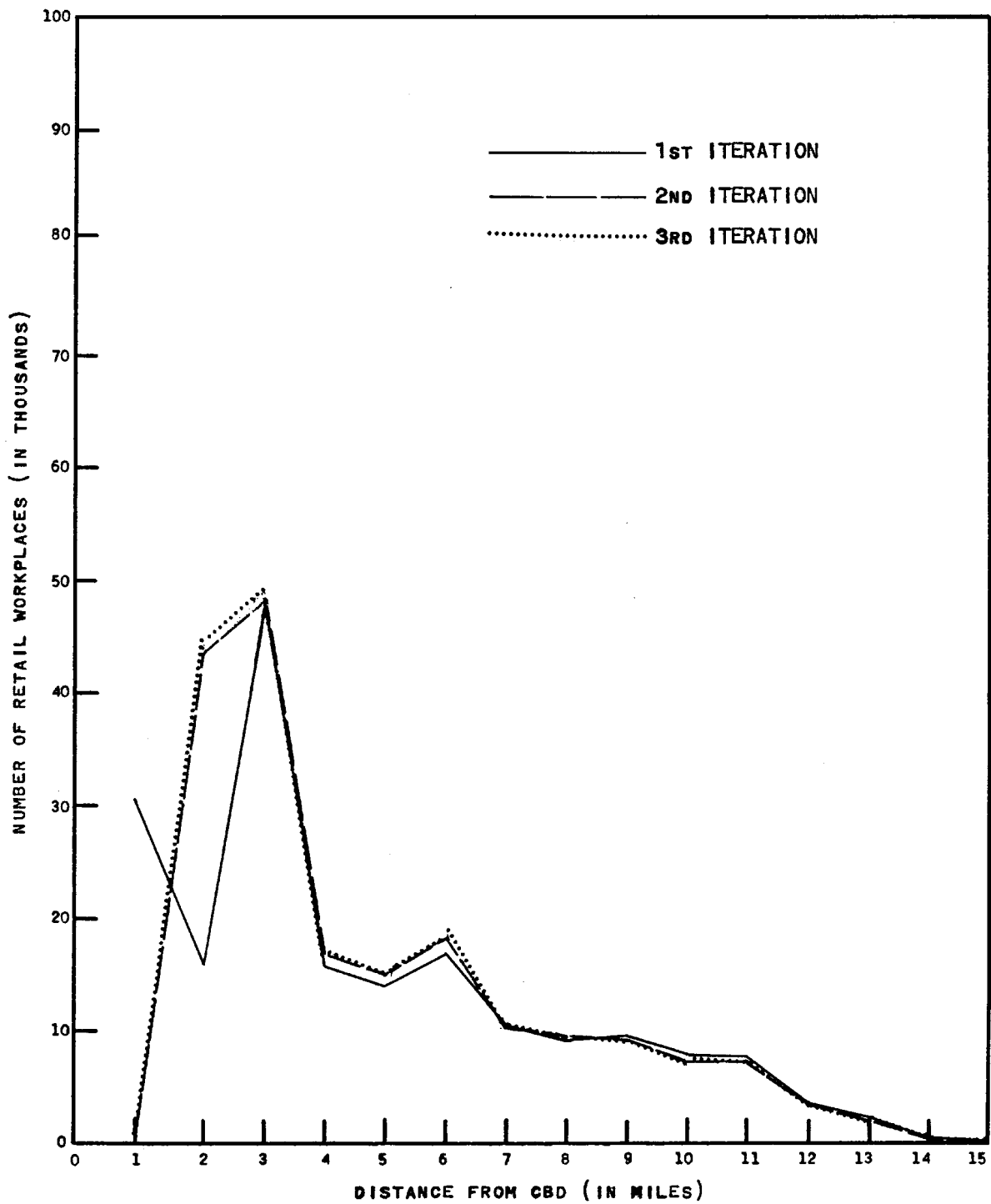
Fig. 12 — Number of Retail Workplaces by Distance from CBD, as Distributed
by Successive Iterations of the Pittsburgh Model
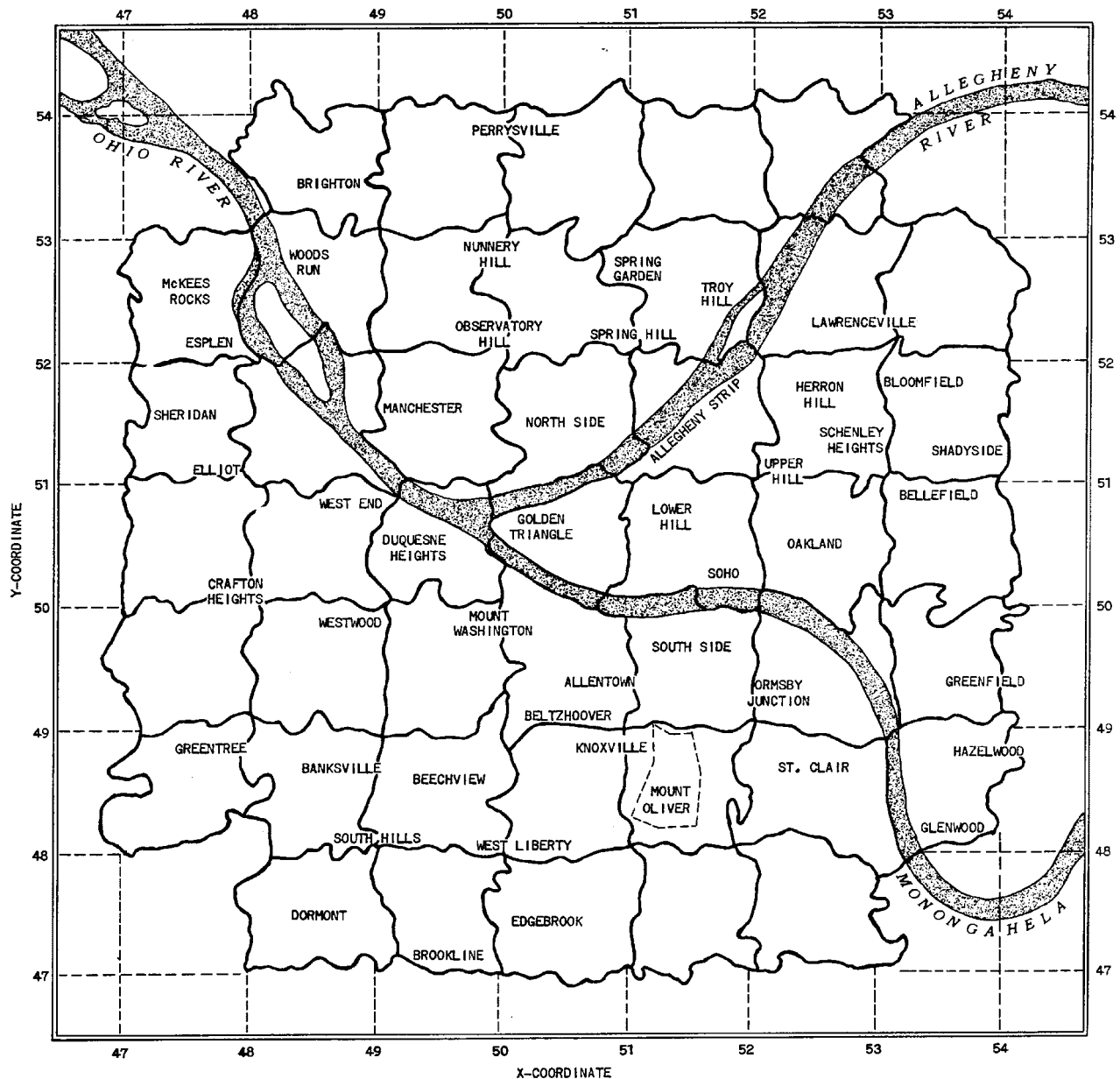
Fig. 13 — Tracts and Place Names in the Metropolitan Core

However, I again shifted these retail jobs by hand to the Golden
Triangle, and once more the model refused to accept this location,
dividing metropolitan shopping facilities as before, between the Hill
District and Oakland.

Postponing the question of the real-world appropriateness of this
location, it is quite clear that the model has its own track in this
iterative sequence, and cannot easily be diverted from its goal of a
stable co-distribution of employment and population. The rapid con-
vergence illustrated in the three iterations of Experiment A is ex-
tremely satisfying, as is the evidence of stability accumulated from
a number of other runs in which the relevant parameters -- trip dis-
tribution functions, residential/work-place weights, minimum-size and
maximum-density constraints -- were varied slightly. Because these
parameters are highly generalized, and based on limited evidence, it
is good to know that the model is not freakishly sensitive to such
tinkering.

ITERATIVE AND PARTIAL SOLUTIONS

In Experiment B, the model was run so as to generate a population
distribution, given the actual (1958) locations of all work places.
In Fig. 14 this distribution of residences is compared to that gen-
erated by the iterative sequence of Experiment A, in which the emergent
population distribution is influenced by an endogenous allocation of
retail work places. (Note, however, that we are not here comparing
the model's output with the actual (1958) distribution of residences;
such comparisons are made later in this section.)

The comparison clearly establishes that the iterative sequence of
population distribution is not significantly misdirected by whatever
flaws may exist in the model's choices of location for retail employ-
ment; by the third iteration, solution values are quite close to those
of the partial equilibrium represented by Experiment B. Linear regres-
sion of the individual tract populations generated by these two experi-
ments yields R^2 = .972. (See Table 1, p. 92.)
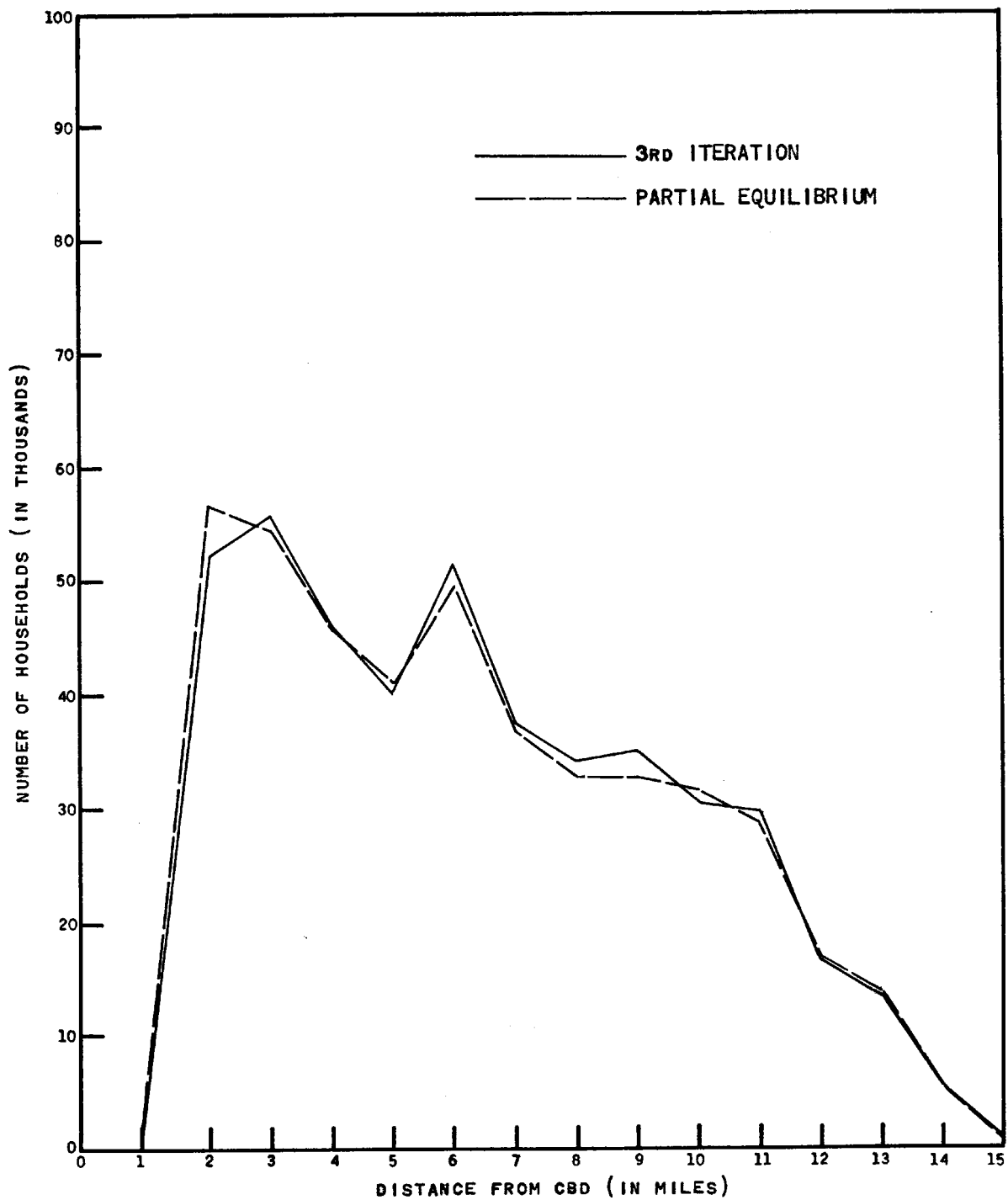
Fig. 14 -- Number of Households by Distance from CBD: Iterative Solution
Compared with Partial Equilibrium Solution

A similar comparison may be made between the iterative solution for retail employment and the partial equilibrium solution of Experiment C. The main difference in these solutions is that the former divides metropolitan-level retail establishments between the Hill District and Oakland while the latter divides these establishments between the Hill District and the North Side. Had the minimum-size constraint been set at 17,000 instead of 20,000, both Experiments A and C would have divided these establishments between the same three tracts (Hill District, North Side, and Oakland).

For purposes of comparison, solutions of Experiments A and C were recalculated with the lower minimum-size constraint for metropolitan-level retail clusters; the modified solutions, plotted in annular profile, are shown in Fig. 15. (Beyond Ring 3, of course, the data plotted include only local and neighborhood retail employment.) The two profiles are close, though not so nearly congruent as the corresponding population profiles (Fig. 14).

Omitting the three tracts containing metropolitan-level retail clusters, I also ran a regression analysis on retail employment for individual tracts, as generated by Experiments A and C.[55] The iterative solution "accounts for" nearly three-fourths of the variance in the tract values of the partial equilibrium solution. (See Table 2.)

The appropriate conclusion is that the model's endogenous allocation of population has more impact on its distribution of retail employment than the reverse. The finding is quite reasonable; the distribution of basic employment, which is exogenous to both solutions, enters more prominently into the determination of population potentials than into the determination of retail market potentials, leaving more scope in the latter for endogenous influences. Moreover, the minimum-size constraint on retail clusters introduces discontinuities into the distribution of

---

[55]Because the metropolitan-level clusters are so large, they quite dominate any regression analysis in which they are included, so that the results of such a regression could be quite misleading.
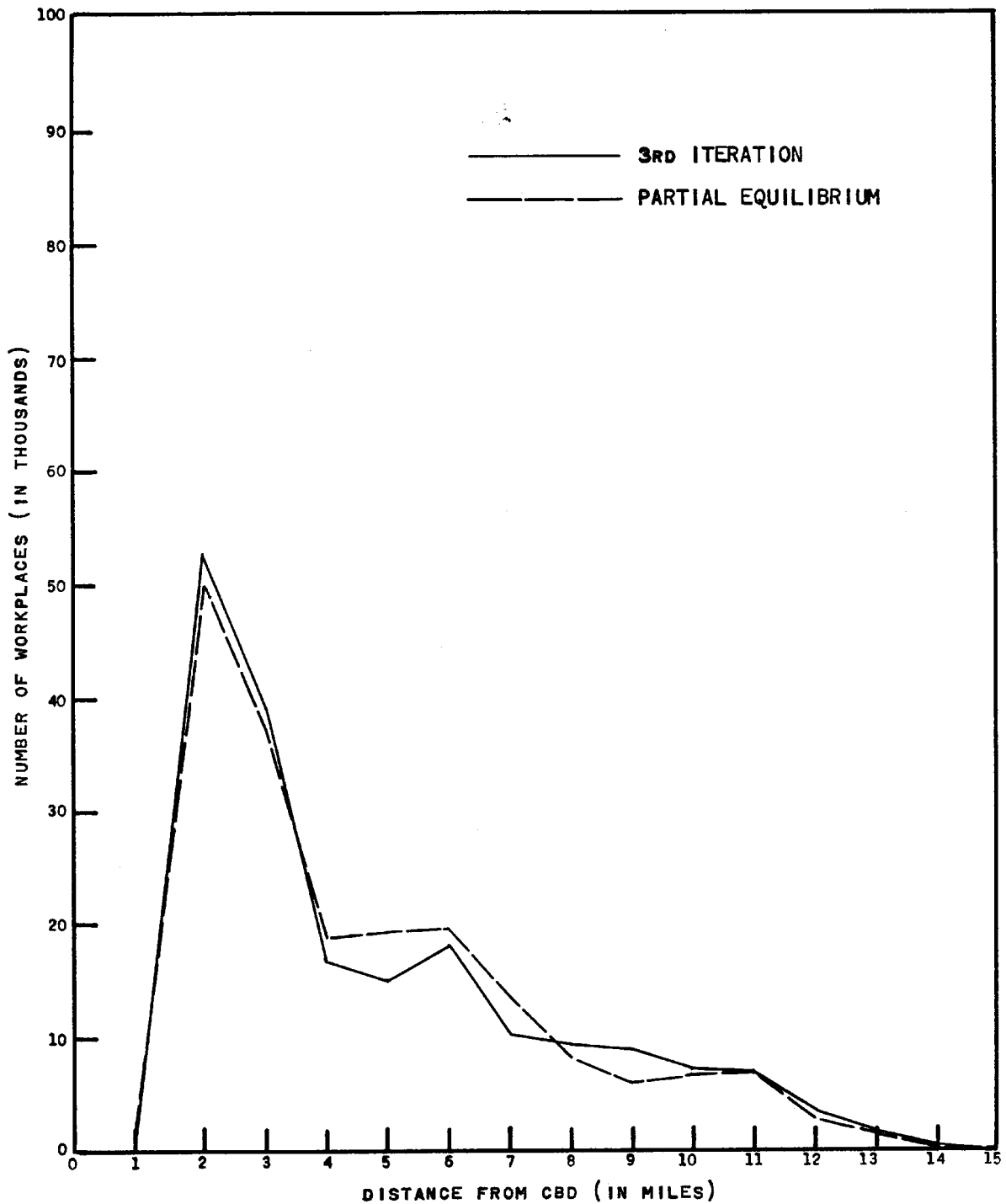
Fig. 15 -- Number of Retail Workplaces by Distance from CBD: Iterative
Solution Compared with Partial Equilibrium Solution

Table 2

REGRESSION STATISTICS FOR NUMBER OF RETAIL WORKPLACES[a] BY TRACT:
COMPARISONS OF MODEL SOLUTIONS WITH 1958 INVENTORY

Sources of Regression Variables	Means of Regression Variables		Regression Parameters		Goodness of Fit		Size of Sample[b]
	x (Independent Variable)	y (Dependent Variable)	a (Intercept)	b (Slope)	Sy.x (Standard Error of Estimate)	R^2 (Coefficient of Determination)	
Convergence of Iterative Sequence x = Iteration 2 y = Iteration 3	294	294	- 0.4	1.001	26	.996	455
Partial Equilibrium and Iterative Solutions x = Partial Equilibrium y = Iteration 3	281	282	21.8	.925	226	.725	453
Iterative Solution and 1958 Inventory x = Iteration 3 y = 1958 Inventory	296	297	-80.8	1.277	571	.482	455
Partial Equilibrium Solution and 1958 Inventory x = Partial Equilibrium y = 1958 Inventory	294	297	-88.2	1.312	517	.577	455

[a]Does not include Metropolitan-Level retail trade.

[b]Tracts containing Metropolitan-Level retail trade were dropped from sample.

retail employment so that, as we have seen, relatively small shifts in
market potentials may cause significant movements of major clusters.

THE ITERATIVE SOLUTION AND THE 1958 INVENTORY

Since we have complete data for 1958, it is instructive to compare
the iterative solution of our model with inventory values for the
endogenous variables. There are many reasons to anticipate considerable
divergence between the model's output and these observations. The
model's solution, after all, is anchored to the contemporary (1958) dis-
tribution of basic employment and related land uses; most of the area's
residential neighborhoods and retail districts, on the other hand, were
laid out decades ago and have changed but slowly. In other words, the
"instant metropolis" of the model makes no provision for time lags in
the process of mutual adjustment between competing and complementary
users of urban space--lags which are enforced by sunk costs in structural
improvements, by difficulties in the assembly of land for large-scale
redevelopment, by the hesitation of individual households or retail en-
terprises to break away from an outmoded but familiar neighborhood or
shopping district.

The simplicity of the Pittsburgh Model also militates against any
expectation of precision. Our aggregative treatment of decision-
processes, location parameters, and constraints suppresses many local
peculiarities in residential development and in the organization of
retail trade which are, in the real world, of considerable importance.
We take no account of ethnic cohesiveness, prestige of locations, scenery,
local climatic conditions, etc., in the distribution of populations; the
present version of the model does not even make use of the clear evi-
dence of class differences in residential location parameters. Market
potentials are calculated for each of our three types of retail clusters
simply on the basis of numbers of accessible customers, without regard
for variations in purchasing power among these customers, orientation to
major transportation routes, problems of commercial site-selection under
restrictive zoning ordinances, etc.

Finally, the model measures accessibility in terms of airline

distance, whereas the actual transportation network, both because of topographical constraints and historical precedent, is not necessarily homogeneous in all directions from any given point. We should, therefore, expect to find more asymmetry in the real world than in the world of the model.

My purpose in making comparisons between the model's solution-values and 1958 inventory-values for the same variables is not to discount the expectable discrepancies, but to measure them and to explore their probable sources. This preamble is intended to alert the reader to questions that I will try to answer in the final section of this paper. How much might it help to convert the model to its implicit dynamic form? How much to elaborate functions and disaggregate variables? What new variables might profitably be added? Could fitted parameters be significantly improved by better data?

The Distribution of Resident Households

The reader will recall that the study area is an irregular oval of 420 square miles, set for computational convenience in a rectangle, 25 by 26 miles. The aggregate number of households to be located in this rectangular area is predetermined by our choice of parameters, but the locations themselves are generated endogenously, and are not restricted by the cordon line of the study area proper. According to our estimate, 470,000 households would be required to supply manpower for the jobs, both basic and retail, provided in 1958 by establishments within this cordon line. Of this number, the model allocated 449,000 to residential sites in the study area proper, and the remaining 20,000 to the dummy tracts that fill out the rectangle. The PATS survey of 1958 estimated the resident population of the study area at 448,000 households. So far, so good.

Proceeding to more detailed comparisons, we may divide the study area into four sectors radiating from the Golden Triangle. (See Fig.16.) Three of the sector-lines follow the major rivers from their confluence just below the Triangle; the fourth, running overland to the southwest, separates the mill towns along Chartiers Creek from the residential

NORTHERN SECTOR

3

2

WESTERN

SECTOR

3

2   1

1

0   1

1

EASTERN SECTOR

2   3

2

2

3

SOUTHERN SECTOR

N

2   0   2   4
MILES

SECTOR BOUNDARIES

ZONE 0

ZONE 1

ZONE 2

ZONE 3

REMAINDER OF ALLEGHENY COUNTY
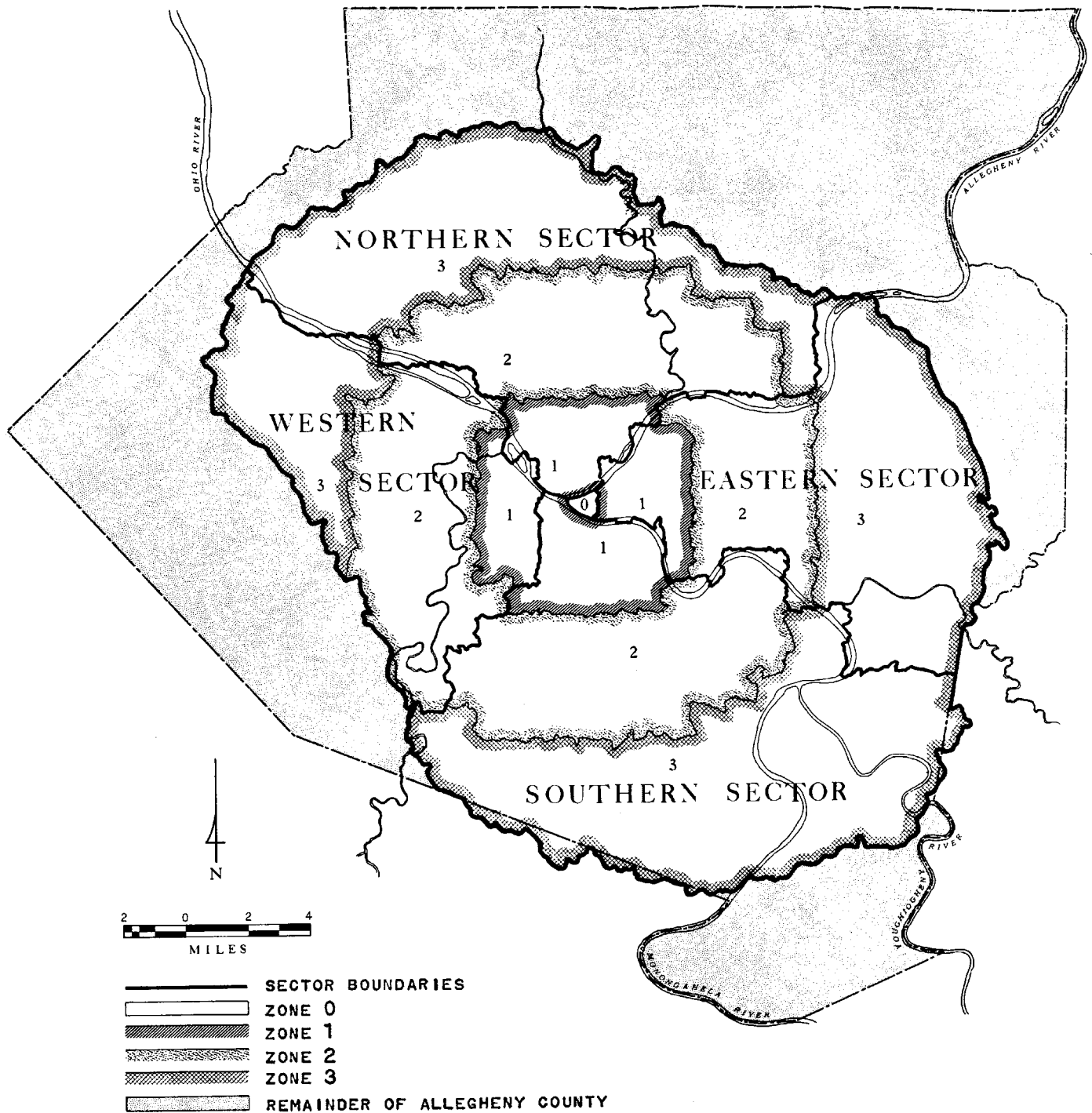
Fig. 16 -- Division of the Pittsburgh Area into Zones and Sectors

suburbs of the South Hills. Each sector has been subdivided into three zones. Zone 1 is three miles wide, Zone 2 is four miles wide, and Zone 3 includes the remainder of the study area, as much as ten miles in some directions. (The Golden Triangle, which is not included in any of the sectors, may be thought of as Zone 0.)

For the CBD (where space constraints are binding) and for the northern and southern sectors, the population assigned by the model is quite close to that reported in the 1958 survey. For the eastern sector, however, the model "underestimates" the 1958 figure by 21,000 households, and for the western sector, overestimates by 16,000. Broadly speaking, the model generates a more symmetrical metropolis than actually exists.

Looking at the zonal detail in Table 3, it is apparent that the overestimate is general throughout the western sector, while the underestimate for the eastern sector is attributable entirely to Zone 2, which extends roughly from East Liberty and Squirrel Hill to Wilkinsburg and Braddock. In the northern and southern sectors, the model calls for somewhat more residential dispersion than existed in 1958.

The "extra" residents assigned by the model to the western sector of the metropolis seem to be the households of CBD workers. This sector is in fact unique in its self-containment, generating few trips to the Golden Triangle as compared with the other three sectors.[56] The most plausible explanation is the historical weakness of its radial transportation system. Until the extension of the Penn-Lincoln Parkway during the 1950's, there was no major highway serving the area between the north bank of the Ohio and the Chartiers Valley; the Parkway link with the Golden Triangle was not completed until late 1960, when the Fort Pitt Tunnel was opened.[57]

---

[56] In Vol. 2 (Forecasts and Plans) of the Pittsburgh Area Transportation Study's Final Report, Figs. 12-15 show configurations of travel movements within the study area.

[57] From another perspective, it can be argued that access to the western sector would have been improved earlier had there been an overload of existing facilities, excess demand being transformed into political pressure. This view is presented in the PATS Final Report, Vol. 2, pp. 25-26. As a matter of fact, the crucial impetus to the

In the few years since the completion of this high-speed link between the Golden Triangle and the Greater Pittsburgh Airport, the growth of the western sector has been rapid, both in employment and population. The Greentree-Carnegie interchange, especially, has become the locus of an important cluster of suburban offices, and formerly unsettled stretches of Parkway West are now lined with light-industrial and research establishments. Tract-housing development is also proceeding apace, with considerable advertising emphasis on the ease of access to downtown.

The question for the future is not so much whether the asymmetry of the metropolis will diminish, but how much and how soon. On the basis of the 1958 pattern of employment, the model calls for 16,000 additional households in the western sector. Such a change could only be plausible on the time scale of a decade or so, given the relatively slow growth of the metropolis as a whole.

Although the population of Allegheny County--more or less equivalent to that of the study area--grew at the rate of 7 per cent per decade, 1940-60, all this growth was outside the central city. Pittsburgh's population did not change significantly between 1940 and 1950, and the city lost 72,000 inhabitants between 1950 and 1960. A similar fate befell most of the streetcar suburbs (dating from the early part of the century) that ring the city.

That part of Pittsburgh included in our eastern sector lost 49,000 inhabitants (about 16,000 households) between 1950 and 1960. The band of minor civil divisions at the eastern city limit[58] lost more than 8,000 persons. Thus there is precedent for further decrease in the population of this sector, although the number suggested by the model (21,000 fewer households than in 1958) seems exorbitant.

Table 3 reveals that the decrease suggested by the model for the

---

development of Parkway West and its extension into Moon Township came from the selection of one of the region's rare plateaus there as the site for the Greater Pittsburgh Airport.

[58]Wilkinsburgh, Edgewood, Swissvale, Rankin, Braddock, and Braddock Hills.

Table 3

DISTRIBUTION OF RESIDENT HOUSEHOLDS BY SECTOR AND ZONE:
1958 INVENTORY COMPARED WITH ITERATIVE SOLUTION

Sector and Zone	Thousands of Square Feet		
	1958 Inventory	Iterative Solution	Difference (Col. 2 - Col. 1)
Golden Triangle	1,369	-	- 1,369
Northern Sector	82,332	87,911	5,579
Zone 1	40,060	39,049	- 1,011
Zone 2	30,376	32,048	1,672
Zone 3	11,896	16,814	4,918
Eastern Sector	168,726	147,377	-21,349
Zone 1	49,179	52,253	3,074
Zone 2	80,821	51,993	-28,828
Zone 3	38,726	43,131	4,405
Southern Sector	152,414	155,082	2,668
Zone 1	44,205	44,055	- 150
Zone 2	57,504	48,639	- 8,865
Zone 3	50,705	62,388	11,683
Western Sector	42,893	58,963	16,070
Zone 1	14,971	18,744	3,773
Zone 2	22,036	30,259	8,223
Zone 3	5,886	9,960	4,074

eastern sector is confined to Zone 2. Increases are indicated both for the distant suburbs (Zone 3) and for the area adjacent to the central business district (Zone 1). Between 1950 and 1960, the latter area lost quite heavily--about 32,000 persons, including at least 10,000 displaced by slum clearance and urban renewal. As the projects are completed, certainly, some population will flow back into the area.

Another perspective on the model solution is offered in Table 4, where the core area, corresponding to Zone 1, is divided into annular rings, nominally a mile in width. The model calls for a significant increase in population in the two rings adjacent to the central business district; the increase is not symmetrically distributed, for the northern sector of these rings actually shows a deficit. The model calls for an additional 4,600 households in the Hill District (eastern sector of Ring 1), and an additional 6,100 south of the Monongahela: Duquesne Heights, Mt. Washington, and the South Side flood-plain.[59] These areas do in fact seem likely candidates for high-rise apartments, a phenomenon peculiarly scarce in Pittsburgh. Such structures are included in redevelopment plans for the Hill District, and a planning study for the Golden Triangle recommended the construction of 6,000 apartment units at the fringes of the business district. Redevelopment now under way on the North Side (Allegheny Center, Chateau West) also includes high-rise apartments, but is expected to result in an over-all decrease in residential density, since much space formerly occupied by high-density row-houses and flats has been committed to non-residential uses.

Except on the flood-plains along the rivers, it does not appear that an increase in the residential population of Rings 2 and 3 has really been forestalled by space shortages, even given Pittsburghers' traditional avoidance of apartment houses. In 1958, these two rings contained 412 acres and 1,570 acres, respectively, of vacant but

---

[59]The western sector, as I have laid it out, has no territory in Ring 1.

Table 4

DISTRIBUTION OF RESIDENT HOUSEHOLDS OF ZONE 1
BY RING AND SECTOR: 1958 INVENTORY
COMPARED WITH ITERATIVE SOLUTION

Ring and Sector	Thousands of Square Feet		
	1958 Inventory	Iterative Solution	Difference (Col. 2 - Col. 1)
CBD	1,369	-	-1,369
Ring 2	43,098	52,224	9,126
North	17,778	16,236	-1,542
East	10,993	15,558	4,565
South	14,327	20,430	6,103
West	-	-	
Ring 3	50,127	55,733	5,606
North	12,371	12,404	33
East	16,867	18,186	1,319
South	15,328	15,568	240
West	5,561	9,575	4,014
Ring 4	55,190	46,144	-9,046
North	9,911	10,409	498
East	21,319	18,509	-2,810
South	14,550	8,057	-6,493
West	9,410	9,169	- 241

usable land, enough to accommodate the extra 13,000 households
assigned to them by the model, at prevailing residential densities.[60]

We need not examine the outer portions of the study area in as
much detail as we have presented for the core. Perhaps the most in-
formative perspective is that of the annular profile, used earlier in
this section. Figure 17 shows such profiles for the residential dis-
tribution generated by the model and for that reported by PATS in
1958. While this perspective suppresses the asymmetrical features of
the metropolis, it does give a good view of the general gradient of
population as we move away from the central business district. The
dotted line in the figure is a profile of the actual workplace dis-
tribution, also derived from PATS data for 1958.

It appears that residential location is more closely associated
with workplace location in the model than in the real world. In the
first two rings, of course, the density of workplaces is such that
there is little room to accommodate places of residence. Thereafter,
the model's residential profile parallels the workplace profile quite
closely, diverging only at Rings 8,9, and 10. These rings, at present
sparsely settled, lie for the most part beyond the older suburban
belt which predates the automobile; they include thousands of acres
of rolling highland whose development was not seriously begun until
the end of World War II, and where virtually all of Pittsburgh's
postwar growth has taken place. The model assigns some 18,000 more
households to this belt than were found there in 1958. Such an incre-
ment seems to me not unreasonable as a prediction for 1970.[61]

---

[60] I.e., allowing 1,400 square feet per household in Ring 2 and
2,500 square feet in Ring 3. "Vacant but usable" land, as reported
by PATS,is meant to be space actually available for development; it
does not include parcels with untenanted structures (these are listed
according to the purpose of the structure), dedicated public open
space, or slopes in excess of 25 per cent.

Inspection of the areas in question indicates that a good deal of
this vacant but usable land is hard to reach. Many small parcels are
isolated from the channels of entry to the Triangle by the rugged topo-
graphy of the area.

[61] In case the reader is puzzled by the secondary peak in the
workplace profile in the neighborhood of Ring 10, I should explain
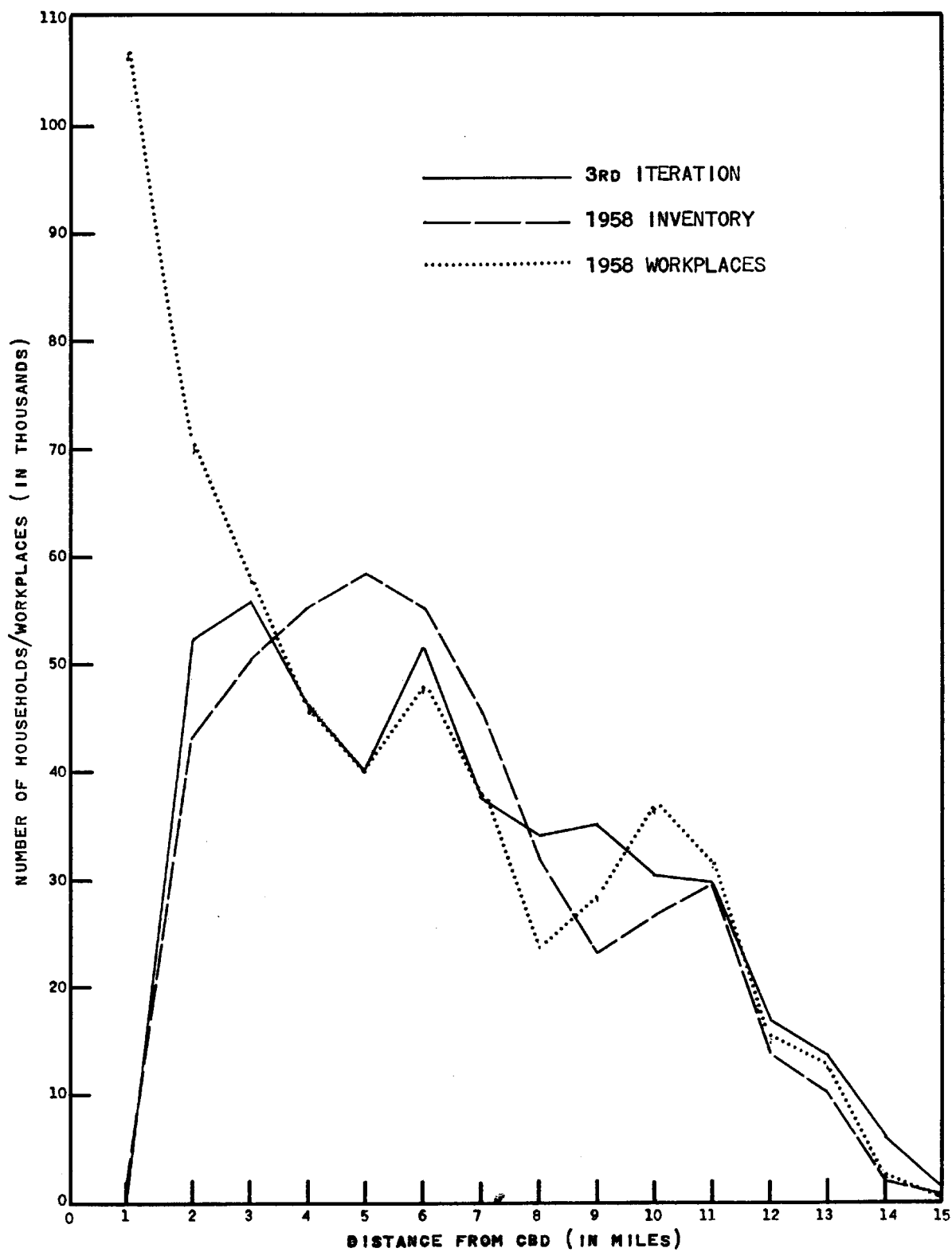
Fig. 17 -- Number of Households by Distance from CBD: Iterative Solution
          Compared with 1958 Inventory and with 1958 Workplace
          Distribution

The model solutions for rings, sectors, and zones can, I think, be profitably compared to 1958 data for similar areas, in the sense that most of the discrepancies noted can be identified with lag phenomena or with major transportation biases. For the smallest geographical units recognized by the model--the 456 tracts, nominally one mile square--discrepancies are proportionately greater and less easily accountable; at this level of detail we encounter local peculiarities of geography and history that the model is not equipped to handle. Yet linear regression yields a respectable measure of correlation between the two sets of population data; the model solution "accounts for" 62 per cent of the variance in 1958 tract populations. (Complete regression statistics are given in Table 1, p. 92.)

The Distribution of Retail Workplaces

Both the numbers and the spatial distributions of retail workplaces are determined endogenously by the Pittsburgh Model. The total number of retail workplaces of each type--neighborhood, local, and metropolitan facilities--is determined by the total number of households generated by the model. In the version reported in this paper, 9.4 households are required to support one neighborhood-retail employee; 5.5 to support one local-retail employee; and 8.3 to support one metropolitan-retail employee. The 470,000 households generated by the model thus called up, in turn, 191,600 retail employees.

Although the model, in its iterative solution, allocated about 20,000 households to dummy tracts outside the study area proper, these households were so dispersed that they were unable to attract even neighborhood retail clusters (minimum size = 30 employees). With one trivial exception, all retail workplaces fell within the boundary of the study area.

For each type of retail cluster, moreover, the workplace surface generated by the model was continuous in the sense that there were

---

that is accountable to several major industrial complexes southeast of Pittsburgh--at Turtle Creek, East Pittsburgh, Duquesne, and McKeesport--employing altogether about 30,000 persons.

virtually no isolated clusters.[62] Moreover, and rather to my surprise, the quite vivid differences in trip distribution functions associated with each type of trade had relatively little impact on the final maps of market potentials; these were so similar that it is fair to say that most of the differences in the final configurations of retail clusters of each type were determined by the minimum-size constraints.

Essentially, then, our map of retail workplaces consists of a base of widespread--almost ubiquitous--neighborhood clusters, varying in size from 30 employees (the predetermined minimum) at the fringes of the populated area to more than 800 in the densely-populated tracts of the East End. Local clusters form an overlay on this base, covering only its central portion, but with roughly the same gradients and the same peaks and valleys; they ranged in size from the minimum of 200 employees to more than 1,600. Finally, metropolitan-level retail facilities are assigned to the few tracts of maximum market potential, peaks which coincide with those for the other types of retail trade.

"Market potential" for a given tract is defined by the model as a linear combination of the number of workplaces and the number of households in that tract and adjoining tracts. Inverse distance-weights are attached to the tract populations, so that a household at a distance of several miles contributes considerably less to market potential than a household in or adjacent to the subject tract. In the case of workplaces, the weights are, in effect, infinite for all distances greater than one mile.

Since each retail "trade" had a different set of distance weights and a different set of linear coefficients attached to workplaces and residences as sources of "shopping trips," the market

---

[62]The two exceptions are "local" clusters assigned to outlying tracts containing the Greater Pittsburgh Airport and the City of Coraopolis; but when all retail workplaces are mapped as a single surface, these locations are connected to the main body of the metropolis by a series of neighborhood clusters along the Ohio Valley.

potentials for a given tract are different for each retail trade.
What is surprising is how slight these differences are. In Table 5,
I have listed these market potentials as generated by the iterative
solution for eleven tracts in the core of the metropolis. The rank
order of market potentials is identical for neighborhood and local
trade; for metropolitan trade, there are several reversals of rank
positions as shown for the other trades, but on the whole, little
significant difference in pattern.

Tract 50-50, which is the actual central business district,
stands far down the lists with relatively low market potentials for
all three kinds of trade. This outcome is to be expected for neigh-
borhood and local retail potential, since the CBD has an insignificant
residential population and the surrounding tracts have large popula-
tions. But in spite of my efforts to constrain the location of metro-
politan trade (see above, pp. 93-96), the model was insistent in
ranking adjacent tracts higher than the Golden Triangle in retail
potential of this type.

In the mechanical sense, it will be fairly easy to persuade the
model to choose the Golden Triangle as the site of a metropolitan
business district. All that need be done is to decrease the linear
coefficient assigned to households in the market potential function,
and increase that assigned to workplaces.[63] Since the Golden Triangle
has a larger complement of basic employment than any other tract,
this method is bound to work. At present, with both coefficients set
at .5, market potentials in the core of the metropolis are still
dominated by the numerically-larger residential population of the
central city.[64]

Some adjustment in this direction probably should be made; as I
have indicated, the values assigned to these coefficients have no

---

[63] These coefficients are designated in Section II, above by the
symbols c and d respectively, with c + d = 1.

[64] For instance, the residential component accounts for about
70 per cent of the value of the potential index cited in Table 5 for
the Golden Triangle.

Table 5

RETAIL MARKET POTENTIALS IN THE METROPOLITAN
CORE, BY TRACT AND TYPE OF TRADE

Tract Coordinates		Index of Market Potential			Rank Order of Tracts		
x	y	Neighbor-hood	Local	Metropol-itan	Neighbor-hood	Local	Metropol-itan
49	49	112,534	132,333	124,871	12	12	12
49	50	143,533	162,373	153,057	10	10	11
49	51	217,241	240,841	195,794	4	4	5
50	49	158,632	180,340	164,445	9	9	10
50	50	70,410	106,291	174,260	11	11	9
50	51	224,083	247,579	211,269	3	3	3
51	49	188,788	211,517	188,568	6	6	7
51	50	240,732	263,864	229,609	2	2	2
51	51	210,091	233,609	207,968	5	5	4
52	49	185,246	207,838	188,958	7	7	6
52	50	244,693	269,006	232,180	1	1	1
52	51	168,343	188,584	182,669	8	8	8

firm base in independent empirical evidence, for there are only limited data which bear on the problem. But it is nonetheless important to note what the model is telling us: so far as residentially-based shopping trips are concerned, the Triangle is not the most accessible location. This result emerges whether market potentials are based on actual or on endogenous population distributions, actual or endogenous workplace distributions. Since the same pattern of market potentials emerges for metropolitan, local and neighborhood trade even though these potentials are based on quite different trip-distribution functions, it is clear that our finding is not sensitive to plausible variations in these functions. The Triangle does have certain transportation advantages not considered by the model, which measures accessibility in terms of airline distance; and it has a certain "going-concern" value, deriving from an era in which population was more compactly clustered about the commercial district.

The model's "conclusion" about Pittsburgh's Triangle is certainly in line with the trend of retail trade in most American metropoli, at least in pointing up the weakness of the traditional central business district in an age in which the automobile has robbed the CBD of its unique position in the transportation network. It would hardly do, however, to suppose that the Triangle's retail facilities will shortly move to Oakland or the North Side, as the model suggests. It is more probable that in the future fewer types of retail establishment will be confined to the downtown shopping district. An increasing share of trade in apparel, furniture, appliances, and specialized services will shift to smaller shopping "centers" with local rather than metropolitan markets.[65]

In distributing neighborhood- and local-retail centers within the study area, the model of course responds to its endogenous distribution of resident population. We have already seen that the latter distribution differs significantly from that reported in 1958; so we

---

[65]This "prediction" points up the defect of treating, as the model does, retail facilities in terms of cluster-types rather than in terms of lines of merchandise. This point is discussed further in Section VI.

may expect commensurate differences in the loci of retail clusters. Table 6 shows the pattern of such retail workplaces by sector and zone.

The model's estimate of retail workplaces is close to the 1958 inventory only for the northern sector. As with resident population, the model calls for a considerable reduction in the retail workforce of the eastern sector, and an increase in that of the western sector. Although the model and the 1958 inventory were in substantial agreement as to the population of the southern sector, the model solution calls for a 20 per cent increase in the number of retail workplaces there.

Some light is thrown on these discrepancies by comparing the ratios of neighborhood and local retail employment to resident population in each sector and zone. The first column of Table 7 gives such ratios for 1958; those shown in the second column are based on the output of the model, for both retail employment and population.

In 1958 the number of retail workplaces per resident household was considerably higher in the eastern sector than in the other three. The difference is attributable to Zone 1 of that sector, where the ratio is .552; examination of the original data shows that 14,000 of that zone's 27,100 retail employees work in only two tracts, corresponding roughly to the Hill District and Oakland. It is clear that these are far too many to be supported by neighborhood and local retail trade; and, in fact, at least half are definitely identifiable (in retrospective examination of source documents) as employed by establishments whose market is much wider in scope.[66]

Neighborhood and local retail employment assigned to this zone by the model falls short of the 1958 inventory by 10,300 workers.

_____

[66] Tract 51-50 (the Hill District) includes the metropolitan post office, which is located near the edge of the Golden Triangle; the 1958 inventory lists 4,600 government employees for that tract. In Tract 52-50 (Oakland), the inventory lists 1,800 workers in "other service" establishments, clearly part of the large medical community there; and 1,000 insurance workers, most of whom are connected with two insurance company headquarters rather than with local agencies.

Table 6

DISTRIBUTION OF NEIGHBORHOOD AND LOCAL RETAIL WORKPLACES
BY SECTOR AND ZONE: 1958 OBSERVATIONS COMPARED
WITH ITERATIVE SOLUTION

Sector and Zone	Thousands of Square Feet		
	1958 Observations	Iterative Solution	Difference (Col. 2 - Col. 1)
Golden Triangle	1,029	903	- 126
Northern Sector	22,659	23,813	1,154
Zone 1	13,056	13,007	- 49
Zone 2	6,661	9,550	2,889
Zone 3	2,942	1,256	- 1,686
Eastern Sector	63,005	46,853	-16,152
Zone 1	27,146	16,878	-10,268
Zone 2	25,338	18,936	- 6,402
Zone 3	10,521	11,039	518
Southern Sector	39,211	47,195	7,984
Zone 1	11,694	15,045	3,351
Zone 2	13,545	16,249	2,704
Zone 3	13,972	15,901	1,929
Western Sector	10,407	16,016	5,609
Zone 1	3,940	6,820	2,880
Zone 2	4,241	8,199	3,958
Zone 3	2,226	997	- 1,229

Table 7

RATIO OF NEIGHBORHOOD AND LOCAL RETAIL WORKPLACES TO
RESIDENT HOUSEHOLDS BY SECTOR AND ZONE: 1958
INVENTORY COMPARED WITH ITERATIVE SOLUTION

Sector and Zone	1958 Inventory	Iterative Solution
Golden Triangle	.752	(a)
Northern Sector	.275	.270
Zone 1	.326	.333
Zone 2	.219	.297
Zone 3	.247	.075
Eastern Sector	.373 [b]	.318
Zone 1	.552	.323
Zone 2	.313	.364
Zone 3	.271	.256
Southern Sector	.257	.304
Zone 1	.264	.341
Zone 2	.235	.334
Zone 3	.276	.255
Western Sector	.243	.272
Zone 1	.263	.364
Zone 2	.192	.271
Zone 3	.378	.100

[a]No residential population was allocated to Golden
Triangle.

[b]Based on retail employment of 27,000. If alternative
figure of 19,700 is used, the ratio is .402, and that
for the Eastern Sector as a whole is .330. See text
for explanation of alternative figure.

But it is worth noting that the model chose the two tracts mentioned in the preceding paragraph as the site of its metropolitan-level retail employment. In other words, the difficulty here seems to pertain at least in part to our classification of retail establishments rather than to the loci of employment.

For Zone 2 of the eastern sector, the model reports 6,400 fewer retail employees than the 1958 inventory; but this discrepancy is quite clearly associated with the model's "prediction" of a very substantial decline in the population of this sector. (See Table 2, p. 100.)

In the southern sector, the model proposes a more dispersed residential population, but little change from 1958 in total number of households. For local and neighborhood retail trade, however, the model proposes an over-all increase of 20 per cent, with a tendency toward greater concentration in Zones 1 and 2. The ratio of retail workers to resident households for the sector as a whole would be increased to a level comparable to that for the eastern sector.

For the western sector, the model calls for an increase in retail employment of 54 per cent over 1958, as compared with an increase in the number of resident households of 37 per cent. The resulting ratio of retail employment to resident population (.272) is still well below either existing or projected ratios for the eastern sector, but comparable to those of the northern and southern sectors. (See Table 7.) For the northern sector, there is relatively little difference between the model solution and the 1958 inventory for either population or retail employment.

Perhaps the most striking feature of the model solution for the northern and western sectors is that the population increases projected for Zone 3 of those sectors are not matched by increases in retail employment; indeed, the model calls for a substantially smaller retail workforce than was revealed by the 1958 inventory. I am inclined to attribute this result partly to the unavoidable boundary problem and partly to a possible remediable weakness in the model's measure of accessibility. Retail employment allocated by the model to places such as Bridgeville (Tracts 44-44 and 45-44) and Sewickley (Tract 41-57) was far under that reported in 1958, and nothing corresponding

to several large suburban shopping centers can be located in the
model's solution.

All of these places are retail clusters whose trade areas are
much larger than those of the more traditional commercial center im-
bedded in a residential community of relatively high density. The
geographically larger market is possible because of the ease of travel
offered to a dispersed population by uncongested highways; the model,
whose measure of accessibility is based on airline distance, makes no
allowance for congestion in defining the effective size of trade areas.
Moreover, these places depend in part on trade originating from resi-
dential communities beyond the boundaries of our study area, and the
model has no information about these communities.

We may also look at the distribution of retail trade in terms of
annular rings centered on the Golden Triangle. Such annular profiles
are shown in Fig. 18 for local and neighborhood trade as inventoried
in 1958 and as allocated by the model. From this perspective, it is
apparent that the discrepancies between model and reality are less in
terms of concentration or dispersion than in terms of symmetry: the
aberrations visible in our sectoral analysis above largely disappear
in the annular profile. The two instances of substantial disagreement
come in Rings 2 and 9-10; the former instance we have already dis-
cussed as reflecting an unfortunate ambiguity in the classification
of metropolitan-level retail trade; in the latter case, since the model
failed to generate the tight population clusters of Ring 10, it also
failed to generate tight clusters of retail trade for the outlying
industrial centers like McKeesport.[67]

The clustering pattern of retail trade of course makes tract-by-
tract comparison of the 1958 inventory values and the model solution
a rather hit-or-miss affair, and regression analysis does not give any

---

[67]For instance, the 1958 inventory reports 5,000 retail employ-
ees in Tract 57-44 (McKeesport), while the model solution gives
1,160. But for the eight surrounding tracts, the 1958 inventory
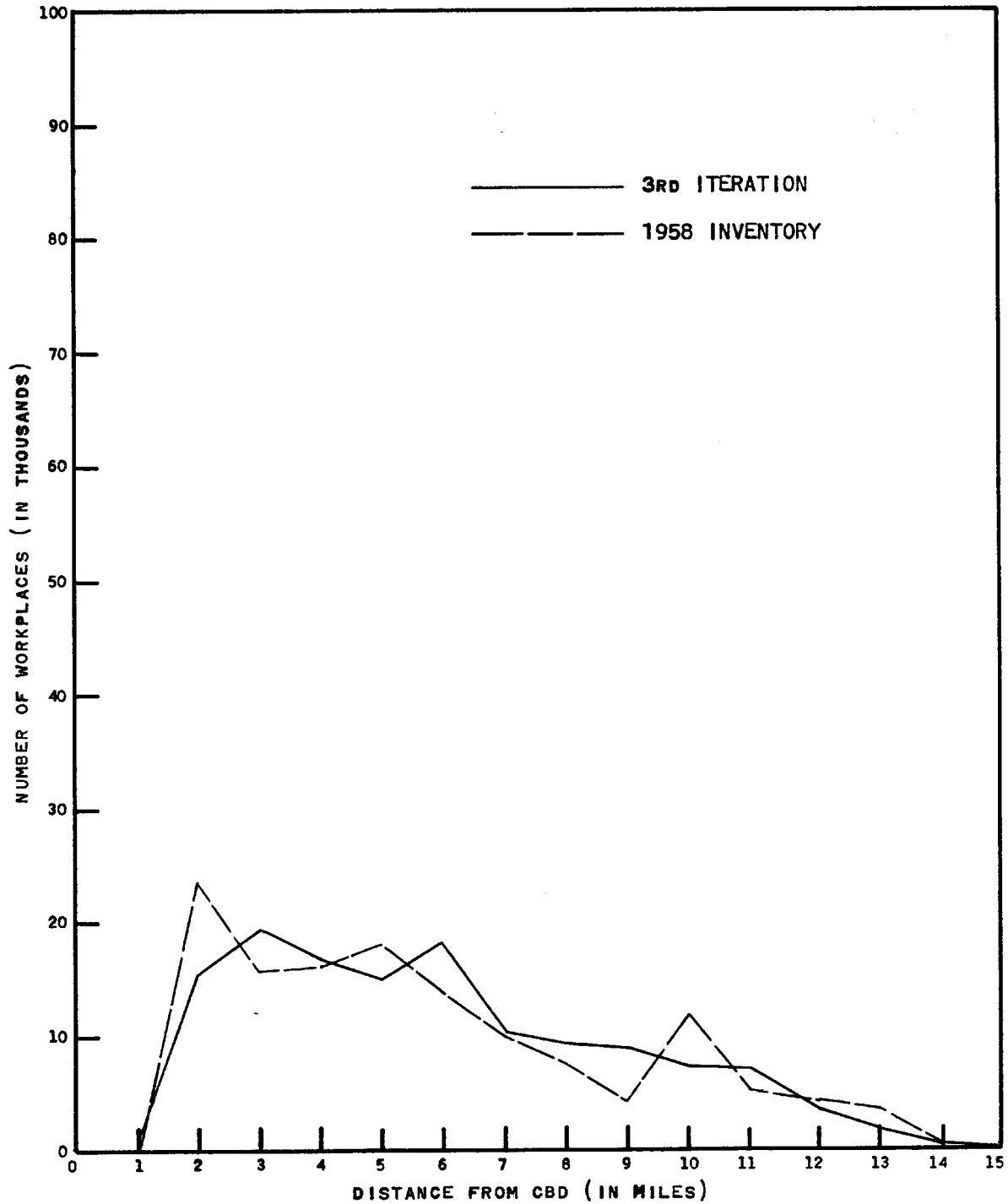reports a total of 3,400 retail employees, while the model solution
gives 4,800.

Fig. 18 -- Number of Neighborhood and Local Retail Workplaces by
Distance from CBD: Iterative Solution Compared with
1958 Inventory

credit for near misses in the geographical sense. Nevertheless, when we compare the results of Experiment C (in which retail location is based on actual rather than on endogenously-generated patterns of residence) to the 1958 inventory of employment in local and neighborhood trade, linear regression of individual tract values yields a quite respectable Coefficient of Determination, $R^2 = .577$. (Complete regression statistics are given in Table 2.)

Patterns of Land Use

In the present version of the Pittsburgh Model, land-use variables are uncomplicated and rather unsophisticated. Land is classified as unusable, basic, retail, or residential. Data from the 1958 inventory are fed to the model as initial information: the total area of each tract, the amount of unusable space, and the amount of space in use by "basic" activities.[68] As the model generates its estimates of retail employment by tract, these are transformed to estimates of retail land use by the application of employment density coefficients for each type of retail use. All remaining land is treated as available for residential use by the model's endogenously-generated residential population. So far as the model is concerned, the only functional significance of land use information is its rôle in determining residential density, which occasionally affects the distribution of population through the maximum-density constraint.

Residential Land. Since the model assumes that all space not otherwise allocated is "residential" (i.e., available for residential use), the values thus generated are not directly comparable to the data for residential land recorded in 1958; the latter are limited to parcels (building sites and associated grounds) actually in residential use at the time of the inventory, or vacant but with habitable residential structures.

---

[68]The "basic" category consists mainly of eonomic activities, but includes dedicated public open space, institutional campuses and grounds, highways, streets, and public parking. Private parking lots are included with the activity to which they are attached.

Table 8 thus shows, for the 1958 inventory, not only the amounts of residential land in each sector and zone, but also residential and vacant land combined. The difference between the latter, figure and the "residential" figure generated by the model is entirely attributable to discrepancies between the 1958 inventory and the model solution with respect to the distributions of retail space. (Compare Table 9, below.)

There are no surprises in Table 8, except perhaps the formidable amounts of usable vacant land in the core of the metropolis (Zone 1). In the outer zones, the amount of space available for residential (or any other) use is adequate for any plausible forecast of growth; the pattern of future development depends rather on the shifts in the loci of workplaces and on the pioneering influences of major highway construction.

For the experiments reported in this paper, the maximum-density constraint was set at 667 square feet of site-space per household,[69] roughly the density of generous apartments in a four-story building. In 1958, the only tracts which approached such a residential density were the Golden Triangle (which had, however, a negligible population), the North Side, and the Hill District. Although the Oakland-Bellefield-Shadyside area contains the most prominent collection of high-rise apartments in the city, these are so interspersed with large single-family residences that the average site-density for that tract is nearly 1,400 square feet per household.

In the iterative solution of the model, residential densities are in general much lower than those reported for 1958, because no allowance is made by the model for vacant land; all available space in each tract is divided among the households assigned to that tract. The maximum-density constraint is binding only in the Golden Triangle (which consistently has the highest population _potential_ in the study

---

[69]For the reader's convenience, I will present density statistics in inverse form. Properly speaking, "density" means units of population per unit of space, and the figure cited in the text is the reciprocal of 1.5 households per thousand square feet.

Table 8

DISTRIBUTION OF SPACE AVAILABLE FOR RESIDENTIAL USE, BY RING
AND SECTOR: 1958 INVENTORY COMPARED
WITH ITERATIVE SOLUTION

Sector and Zone	Thousands of Square Feet			
	1958 Inventory		Iterative Solution	Difference (Col. 3 - Col. 2)
	Residential	Residential or Vacant		
Golden Triangle	215	696	3,950	3,254
Northern Sector	508,514	2,060,761	2,061,782	1,021
Zone 1	80,988	133,950	125,397	- 8,553
Zone 2	232,371	716,546	719,667	3,121
Zone 3	195,155	1,210,265	1,216,718	6,453
Eastern Sector	589,900	1,207,430	1,198,429	- 9,001
Zone 1	83,491	98,047	84,488	-13,559
Zone 2	212,747	287,540	286,420	- 1,120
Zone 3	293,662	821,843	827,521	5,678
Southern Sector	859,461	2,447,547	2,443,255	- 4,292
Zone 1	131,631	189,738	180,128	- 9,610
Zone 2	380,709	800,959	806,730	5,771
Zone 3	347,121	1,456,850	1,456,397	- 453
Western Sector	298,100	1,239,457	1,233,444	- 6,013
Zone 1	63,065	130,952	125,974	- 4,978
Zone 2	168,102	687,685	684,671	- 3,014
Zone 3	66,933	420,820	422,799	1,979

area) and one adjacent tract, which includes the wholesaling district and railyards of the Allegheny Strip.

I should remind the reader, however, that the model is designed to allow for much greater sophistication in the treatment of residential land. Vacant land may be held off the market by placing it temporarily in the category of unusable space, and the density-constraint need not be general: each tract may be assigned a maximum density based on existing structural capacity, zoning laws, redevelopment plans, etc.

Retail Land. Since the allocation of retail space is derived from the prior allocation of retail employment, the employment-density co-efficients are key parameters in determining the distribution of retail space. It is well known that retail employment densities vary greatly within any given line of trade; an establishment choosing an expensive site will economize by full site-coverage, high-rise construction, and close interior planning. No explicit recognition is given to such variable "production functions" in the model, although our classification of retail trade in terms of neighborhood, local, and metropolitan clusters provides a plausible basis for graduating employment-density coefficients from low-cost to high-cost sites. The parameters actually used in the present experiments are as follow:

Type of Retail Cluster	Square Feet per Employee
Neighborhood facilities	1,900
Local facilities	1,300
Metropolitan facilities	80

These figures refer to site-area rather than to floor-space, and reflect the unique importance of high-rise structures in the metropolitan business district, and of integral parking lots elsewhere. The parameters were chosen as nearly as possible to represent average employment-densities for each kind of cluster in 1958; as a consequence, neighborhood retail facilities in the core of the metropolis use as much space per employee as those on the fringe, and the local business districts of major suburbs are as densely-used as those of the North Side or East Liberty.

In 1958, about 195 million square feet of space was in use by the establishments classified as "retail" for the purposes of this study. Although this seems at first glance to be a formidable figure, it shrinks to insignificance when placed in context. Retail uses accounted for 2 per cent of the usable land of the study area, as compared to 23 per cent devoted to residential uses. Excluding vacant and agricultural land from the total, retail space accounts for less than 4 per cent of all "used" space. There were only a handful of tracts in which retail uses occupied as much as 10 per cent of the total area; the Golden Triangle, the North Side, the Hill District, East Liberty, and Wilkinsburg are the conspicuous examples.

The fact is that retail land use, in the sense of quantity, is a relatively unimportant variable in the world of the model. (Nor is it of much consequence in the world of metropolitan planning; much more significant than the quantity of retail space is its precise location, for a retail cluster is usually the focus of local traffic patterns.) Even so, the highly-generalized parameters of the Pittsburgh Model do not perform satisfactorily. Table 9, which compares the model solution to 1958 inventory values, by sector and zone, reveals that the model solution has a persistent bias. Too much retail space is allocated to the core of the metropolis and too little is allocated to the fringe.[70]

It may be possible to correct this bias by increasing the space-allowance for neighborhood facilities and decreasing it for local facilities, so that, as the mix of retail activities varies over the study area, there will be a more gradual transition in over-all retail densities from fringe to core. But if precision is desired, it is fairly clear that the number of classes of retail trade must be increased. (See Section VI, below.)

---

[70] Over-all, the retail space allocated by the model is about 8 per cent greater than that listed by the 1958 inventory. This is because of certain ambiguities in the 1958 land-use data; about 15 million square feet of institutional uses which should have been classified for our purposes as "basic" were listed as "retail" at the time the parameters were fit. The error was not discovered soon enough to revise the parameters, and the resulting discrepancy did not justify a special re-run of the model.

Table 9

DISTRIBUTION OF RETAIL LAND BY RING AND SECTOR:
1958 INVENTORY COMPARED WITH ITERATIVE SOLUTION

Sector and Zone	Thousands of Square Feet		
	1958 Inventory	Iterative Solution	Difference (Col. 2 - Col. 1)
Golden Triangle	4,577	1,323	- 3,254
Northern Sector	37,496	36,475	- 1,021
Zone 1	10,942	19,495	8,553
Zone 2	17,846	14,725	- 3,121
Zone 3	8,708	2,255	- 6,453
Eastern Sector	66,440	75,441	9,001
Zone 1	16,347	29,906	13,559
Zone 2	27,150	28,270	1,120
Zone 3	22,943	17,265	- 5,678
Southern Sector	67,407	71,699	4,292
Zone 1	12,910	22,520	9,610
Zone 2	30,335	24,564	- 5,771
Zone 3	24,162	24,615	453
Western Sector	18,730	24,743	6,013
Zone 1	5,210	10,188	4,978
Zone 2	9,783	12,797	3,014
Zone 3	3,737	1,758	- 1,979

VI. PERSPECTIVES AND PROBLEMS

The preparation of this report offers my first opportunity, in three years of intermittent work on the Pittsburgh Model, to consolidate findings and regain perspective. In retrospect, I can see much fruitless endeavor and many things that should have been done differently. There is also a residue of accomplishment--less than I hoped for, to be sure, but still enough to be of service to others working on problems of metropolitan spatial organization and its development over time.

AN APPRAISAL OF THE MODEL

I should make it quite clear that I do not consider the Pittsburgh Model a finished product, which is usable at this point for any serious practical purpose. It is at best a prototype with a promising future. Its present value lies mainly in the guide-lines it offers for further research. But even from its present ambiguities, there emerge some valuable insights into the spatial structure of metropolis and the trends of change.

The Pittsburgh Model was designed for eventual use as a tool for metropolitan and regional planning. I have assumed that there is a logic to the spatial arrangement of human activities, a logic which is obscured partly by the intrinsic complexity of the relationships among these activities but even more by the large numbers of interacting elements. My aim has been to simplify and generalize the relationships among the myriad locators, while preserving enough spatial detail to be of use in the broader problems of land-use and transportation planning. The resulting model is exceedingly simple in its structure and data-requirements; even so, it strains against the available data-base and the economic limits of computer use.

In the preceding sections of the report, I have offered for the reader's inspection the logical structure of the model, reviewed the steps followed in applying that structure to a specific body of data, and reported on the behavior of the model in actual operation.

Limitations of the data-base forestall rigorous tests of the model's ability to replicate the processes of urban development over time; but the experiments herein reported at least demonstrate that the model responds intelligibly to such data as are available.

Perhaps the best service I can offer to the reader by way of summary is to review what seem to me the most salient findings and their implications:

1. The gravity principle seems to have enough flexibility to comprehend the spatial interactions of a variety of locators, and it requires much less of a data-base than any alternative so far proposed. Given only the barest specifications of the properties of urban space and of the units to be located therein, the model was able to generate quite plausible co-distributions of employment and residential population.

2. The model's distributions of resident households and retail facilities can most plausibly be interpreted as a "forward" solution; i.e., as the end-point of a presently-incomplete process of relocation which is itself a response to past changes in the location of workplaces and to improvements in personal transportation. If the model is to be practically useful, these implicit dynamics must be formalized and assigned a time-scale.

3. The solution of the model calls for more residents than in 1958 in selected locations near the Golden Triangle. For other parts of the central city--the East End, in particular - -the model projects population losses, with commensurate gains for the low-density fringe. As a continuation of the 1950-1960 trend, such dispersion is plausible, although one might question the magnitude of the redistribution called for by the model.

Judgments on this score must take account of the fact that the post-war years have witnessed a considerable dispersion of workplaces. While the Golden Triangle appears to have held its own (replacing retail with administrative

employment), the number of jobs elsewhere in the central
city has declined substantially. The pattern of residence
generated by the model is closely tied to the 1958 distri-
bution of workplaces--much more closely than the pattern of
residence actually prevailing at that date. This difference
may reflect a failure of the model to take into account
other factors conditioning residential location, even in the
"long run." But a strong argument can be made in favor of
the model solution as indicative of the residential pattern
of the future.

4. Although the model is anchored to an existing pattern of
"basic" workplaces, it projects a more symmetrical pattern
of residential distribution than was apparent in 1958.
Pittsburgh's past asymmetry is largely attributable to the
importance of the rivers to its major employers and to the
topographical irregularities which constrained the routing
of overland transportation. Because of its topography,
Pittsburgh is perhaps the least appropriate metropolis in
the nation for a model which measures accessibility in terms
of airline distance.

But in the age of the automobile, the achievements of
highway engineers are nullifying the constraints formerly
imposed by topographical irregularity, and the logic of
spatial symmetry is asserting itself. The opening of the
Fort Pitt Tunnel (1960) as a gateway to the western sector
of Allegheny County was a major step in balancing radial
accessibility to the metropolitan core, and the implementation
of the PATS freeway plans will complete this task.

Granted the likelihood that the projection of greater
symmetry will thus be realized, it is nonetheless clear that
the Pittsburgh Model's distributive routines are inadequately
sensitive to features of the transportation system which are
more appropriately viewed as exogenous planning variables
than as automatic responses to demand pressures. There exist
well-developed techniques for tracing minimum travel-times

and distances over a specified transportation network, and such a routine could easily be integrated into the Pittsburgh Model--provided that the user were prepared to bear a formidable increase in computational costs.

5. The Pittsburgh Model has a structural capacity for generating output detail for territorial units as small as a square mile. In experimental work to date, I judge the minimum grain of usable output to be at least four times as large; but the model has been constrained only by the barest specifications of the natural and historical properties of these small areas.

   The urban environment is well supplied with unique local features of terrain and with legacies of development (e.g., structures and lot-sizes) which condition future uses. Many of these peculiarities defy generalization. However, they may be imposed, tract by tract, as constraints on the model's projected distributions of population and retail employment.

   The most pressing need is for greater specification of the characteristics of vacant land and its availability for development, and for density constraints which reflect existing patterns of structural development.

6. The most interesting result of the model's treatment of retail trade was the similarity of market-potential surfaces based on a variety of trip-distribution functions. This similarity suggests a major economy in programming which would allow greater disaggregation of retail trades than has so far been feasible. Instead of calculating a separate matrix of market-potentials for each retail sub-sector, one could calculate a general matrix of retail market-potentials and use other constraints, specific to each retail sub-sector, to enforce different distributions of employment in these sub-sectors. The minimum-size parameters of the present model could be used in this way as constraints on the sub-sector employment distribution.

   Moreover, if these repetitions in the calculation of

market potentials (and population potentials) were eliminated, it would be more practical to base the calculation on a fully-specified transportation network as suggested above, rather than on airline distance.

7. The identification of homogeneous sub-sectors of retail trade is difficult under any circumstances, but particularly so when the original data-base is crude, and at a time when merchandising techniques are rapdily changing. I have experimented with broad kind-of-business classifications and also with typical-cluster classifications. Using the former method, I found that the model was reluctant to generate focal clusters (i.e., concentrated shopping districts); better results were obtained by the latter method, but the model was thereby prevented from reflecting an important current trend, the selective dispersion of kinds of business which were once unique to the metropolitan central business district. It would be profitable, I think, to re-examine the classification of retail trade with the aid of a stronger data-base.

8. Model-builders nearly always begin by assuming a data-base of better quality than is in fact available. The original design of the Pittsburgh Model was tailored directly to the data-bank of the Pittsburgh Area Transportation Study, which offered as much small-area detail (particularly about employment) as was available anywhere in the nation. Even so, I was forced to assume an implausible level of reliability for small-area samples, and was severely limited as to the disaggregation of variables either for input purposes or for use in the fitting of parameters. The experience leads me to wonder whether any metropolitan model designed for periodic projections of small-area detail can in practice be furnished with the necessary base of current input data.

THE FUTURE OF METROPOLITAN MODELS

The Pittsburgh Model is only one of a number of current attempts

to develop quantitative models of metropolitan spatial structures--
perhaps the simplest and least ambitious. Its siblings vary in style,
including such diverse approaches as a loosely-articulated system
study, an exceedingly formal set of simultaneous equations, an elab-
orate multi-stage mixture of linear programming and computer simula-
tion, and an accounting system for reconciling detailed judgments
about the development prospects of small areas.

While it is clear that the present state of the arts is one of
experimentation, there is no reason to anticipate eventual agreement
on a single all-purpose model of metropolis. These models are
oriented to particular planning problems, ranging from urban renewal
to the design of a regional transportation system; each has its
unique informational requirements. If nothing else, differences in
the relevant time-scales, in the need for geographical detail, and
in the available data-base must be reflected in the grand strategy
of model-building.

The simulation of complex physical, biological, and social sys-
tems has become a very active field of research in the past decade;
the most advanced models relating to urban systems are those
developed by traffic engineers for network assignments. It is be-
coming increasingly clear that the development and testing of such
models is a long-term process, and that their validity will always
be ambiguous. Thus, no one can really assert, after ten years of
experiment, that traffic-assignment models provide a mechanically
reliable guide for transportation planning. But there can be little
doubt that the use of these models as an element in the planning
process has enormously increased the planners' understanding of metro-
politan transportation problems and forced them to deal explicitly
and rigorously with many relationships within the system that were
previously glossed over.

We can reasonably anticipate a similar future for the current
fumbling attempts to submit other aspects of metropolitan planning
to the discipline of the computer. Granted that the model-builders
will never be able to simulate accurately all of the relevant
features of the urban environment, they can at least go far beyond

our present inability to trace system-wide and recursive impacts of major changes in environmental conditions or in public policy.

Perhaps even more important is the fact that in communicating with an industrious but simple-minded computer, all questions and instructions must be meticulously framed. In the process, false issues are unmercifully exposed, and others assume hitherto unsuspected importance. In the development of public policy, as in scientific research, the proper formulation of a question is the most important step in reaching an effective answer.

BIBLIOGRAPHY

Almendinger, Vladimir, "Topics in the Regional Growth Model: I," Penn Jersey Transportation Study, April 5, 1961.

Alonso, William, Location and Land Use, Cambridge, Harvard University Press, 1964.

Artle, Robert, Studies in the Structure of the Stockholm Economy, The Business Research Institute at the Stockholm School of Economics, Stockholm, 1959.

Berry, Brian J. L., "Cities as Systems Within Systems of Cities," paper presented at a meeting of the Regional Science Association, November 1963.

Carrothers, Gerald A. P., "An Historical Review of the Gravity and Potential Concepts of Human Interaction," Journal of the American Institute of Planners, Vol. 22, No. 2, pp. 94-102.

Golden Triangle Market Survey Supplemental Report, Pittsburgh Regional Planning Association, 1961.

Hansen, Walter G., Accessibility and Residential Growth, M. A. thesis, Massachusetts Institute of Technology, 1959.

-----, "How Accessibility Shapes Land Use," Journal of the American Institute of Planners, Vol. 25, May 1959, pp. 73-76.

Hansen, Willard B., "An Approach to the Analysis of Residential Extension," Journal of Regional Science, Vol. 3, No. 1, Summer 1961.

Harris, Britton, "Some Problems in the Theory of Intra-Urban Location," paper presented at a seminar sponsored by the Committee on Urban Economics of Resources for the Future, April 1961.

Huff, David L., Determination of Intra-Urban Retail Trade Areas, Real Estate Research Program, University of California at Los Angeles, 1961.

Isard, Walter, Location and Space Economy, Cambridge, Technology Press, 1956.

Kain, John F., "The Journey to Work as a Determinant of Residential Location," unpublished Ph.D. thesis, University of California at Berkeley, 1961.

Keefer, Louis E., "Vehicle Miles of Travel Accuracy Check," Interim Technical Report, No. II, Pittsburgh Area Transportation Study, 1959.

Lowry, Ira S., "Design for an Intra-Regional Locational Model," Working Paper No. 6, Economic Study of the Pittsburgh Region, Pittsburgh Regional Planning Association, 1960.

-----, Employment by Place of Work in the Regional Core, Pittsburgh Regional Planning Association, June 1962.

-----, "Locational Parameters in the Pittsburgh Model," Papers and Proceedings of the Regional Science Association, Vol. 11, forthcoming

-----, Portrait of a Region, University of Pittsburgh Press, Pittsburgh, 1963.

-----, "Residential Location in Urban Areas," unpublished Ph.D. thesis, University of California at Berkeley, 1959.

Pendleton, William C., "The Value of Highway Accessibility," unpublished Ph.D. thesis, University of Chicago, 1963.

Pittsburgh Area Transportation Study, Final Report, Vol. 1, Study Findings, and Vol. 2, Forecasts and Plans, Pittsburgh, 1961-1962.

Ratcliff, R. U., Urban Land Economics, McGraw-Hill, New York, 1949.

Schneider, Morton, "Gravity Models and Trip-Distribution Theory," Papers and Proceedings of the Regional Science Association, Vol. 5, 1959, pp. 51-56.

Shevsky, Eshref, and Wendell C. Bell, Social Area Analysis, Palo Alto, Stanford University Press, 1955.

Stevens, Benjamin H., "Linear Programming and Location Rent," Journal Of Regional Science, Vol. 3, No. 2, Winter 1961, pp. 15-26.

Tryon, Robert C., Identification of Social Areas by Cluster Analysis, Berkeley, University of California Press, 1955.

Wingo, Lowden, Transportation and Urban Land, Resources for the Future, Inc., Washington, D. C., 1961.

Zipf, George Kingsley, Human Behavior and the Principle of Least Effort, Cambridge, Addison Wesley, 1949.