Mathematics Methods for Computer Science

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Lecture

Norms, Sensitivity and Conditioning

Questions

Gaussian elimination works in theory, but what about floating point precision?

How much can we trust $\vec{x}_0$ if

$$0 < ||A\vec{x}_0 - \vec{b}|| \ll 1$$

Recall: Backward Error

Backward Error

The amount a problem statement would have to change to realize a given approximation of its solution

Example 1: $\sqrt{x}$

Example 2: $A\vec{x} = \vec{b}$

Perturbation Analysis

How does change if we solve
$$(A + \delta A)\vec{x} = \vec{b} + \delta\vec{b}?$$

Two viewpoints:

- Thanks to floating point precision, A and $\vec{b}$ are approximate
- If $\vec{x}_0$ isn't the exact solution, what is the backward error?

What is "Small"?

What does it mean for a statement to hold for small $\delta\vec{x}$?

Vector norm

A function $||\cdot|| : \mathbb{R}^n \to [0, \infty)$ satisfying:

1. $||\vec{x}|| = 0$ iff $\vec{x} = 0$
2. $||c\vec{x}|| = |c|||\vec{x}||\forall c \in R, \vec{x} \in \mathbb{R}^n$
3. $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||\forall \vec{x}, \vec{y} \in \mathbb{R}^n$

How are Norms Different?

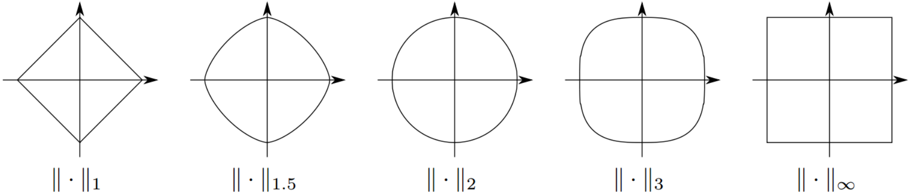

Figure 4.7 The set $\{\vec{x} \in \mathbb{R}^2 : \|\vec{x}\| = 1\}$ for different vector norms $\|\cdot\|$.

How are Norms the Same?

Equivalent norms

Two norms $||\cdot||$ and $||\cdot||'$ are equivalent if there exist constants c_{low} and c_{high} such that $c_{low}||\vec{x}|| \leq ||x||' \leq c_{high}||\vec{x}||$ for all $\vec{x} \in \mathbb{R}^n$.

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Theorem

All norms on $\mathbb{R}^n$ are equivalent.

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Theorem

All norms on $\mathbb{R}^n$ are equivalent.

$$(10000, 1000, 1000) \ vs. \ (10000, 0, 0)?$$

Matrix Norms: "Unrolled" Construction

Convert to vector, and use vector p-norm:

$$A \in \mathbb{R}^{m\times n} \leftrightarrow \mathrm{a}[:] \in \mathbb{R}^{mn}$$

- Achieved by $vecnorm(A, p)$ in Julia.

Special Case: Frobenius norm $(p = 2)$:

$$\|A\|_{\mathrm{Fro}} \equiv \sqrt{\sum_{ij} a_{ij}^2}$$

Matrix Norms: "Induced" Construction

Maximum stretching of a unit vector by A:

$$\|A\| \equiv \max\{\|A\vec{x}\| : \|\vec{x}\| = 1\}$$

Matrix Norms: "Induced" Construction

Maximum stretching of a unit vector by A:

$$||A|| \equiv \max\{||A\vec{x}|| : ||\vec{x}|| = 1\}$$

Different matrix norms induced by different vector p-norms.

Case $p = 2$: What is the norm induced by $||\cdot||_2$?

Matrix Norms: $||A||_2$ Visualization

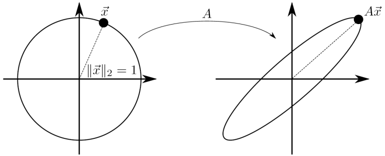

Figure 4.8 The norm $\|\cdot\|_2$ induces a matrix norm measuring the largest distortion of any point on the unit circle after applying A.

Induced two-norm, or spectral norm, of $A \in \mathbb{R}^{n\times n}$ is the square root of the largest eigenvalue of $A^T A$:

$$||A||_2^2 = \max\{\lambda\}$$

There exists $\vec{x} \in \mathbb{R}^n$ with $A^T A\vec{x} = \lambda\vec{x}$

Other Induced Norms

$$\|A\|_1 \equiv \max_j \sum_i |a_{ij}|$$
$$\|A\|_\infty \equiv \max_i \sum_j |a_{ij}|$$

Question

Are all matrix norms equivalent?

Recall: Condition Number

Condition number

Ratio of forward to backward error

Root-finding example:

$$\frac{1}{f'(x^*)}$$

Model Problem

$$(A + \varepsilon\delta A)\vec{x}(\varepsilon) = \vec{b} + \varepsilon\delta\vec{b}$$

Simplification (on the board!)

$$\left.\frac{d\vec{x}}{d\varepsilon}\right|_{\varepsilon=0} = A^{-1}(\delta\vec{b} - \delta A\vec{x}(0))$$

$$\frac{\|\vec{x}(\varepsilon) - \vec{x}(0)\|}{\|\vec{x}(0)\|} \leq |\varepsilon| \, \|A^{-1}\| \, \|A\| \left(\frac{\|\delta\vec{b}\|}{\|\vec{b}\|} + \frac{\|\delta A\|}{\|A\|} \right) +$$

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$$\frac{||\vec{x}(\varepsilon) - \vec{x}(0)||}{||\vec{x}(0)||} \leq \varepsilon \cdot D \cdot \kappa + O\left(\varepsilon^2\right)$$

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Invariant to scaling (unlike determinant!); equals one for the identity.

Condition Number of Induced Norm

$$\operatorname{cond} A = \frac{\max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|}{\|\vec{x}\|}}{\min_{\vec{y} \neq \vec{0}} \frac{\|A\vec{y}\|}{\|\vec{y}\|}} = \frac{\max_{\|\vec{x}\|=1} \|A\vec{x}\|}{\min_{\|\vec{y}\|=1} \|A\vec{y}\|}$$

Condition Number: Visualization

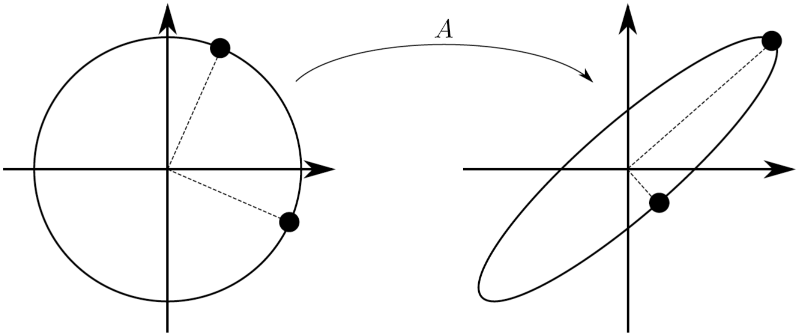

Figure 4.9 The condition number of A measures the ratio of the largest to s
distortion of any two points on the unit circle mapped under A.

Experiments with an ill-conditioned Vandermonde

Chicken $\Leftrightarrow$ Egg

$$\operatorname{cond} A \equiv \|A\| \, \|A^{-1}\|$$

Computing $\|A^{-1}\|$ typically requires solving $A\vec{x} = \vec{b}$, but how do we know the reliability of $\vec{x}$?

To Avoid...

What is the condition number of computing the condition number of A?

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Instead

Bound the condition number.

- Below: Problem is at least this hard
- Above: Problem is at most this hard

Potential for Approximation

$$\|A^{-1}\vec{x}\| \leq \|A^{-1}\| \, \|\vec{x}\|$$

$$\Downarrow$$

$$\text{cond } A = \|A\| \, \|A^{-1}\| \geq \frac{\|A\| \, \|A^{-1}\vec{x}\|}{\|\vec{x}\|}$$