Mathematics Methods for Computer Science

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SJTU-SE DALAB

Lecture

Eigenproblems I

Setup

Given: Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate

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Find: Correlations between different dimensions

Simplest Model

One-dimensional subspace

$\vec{x}_i \approx c_i \vec{v}, \vec{v}$ unknown

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$\vec{x}_i \approx c_i \vec{v}, \vec{v}$ unknown

Equivalently:

$$\vec{x}_i \approx c_i \hat{v}$$

$\hat{v}$ unknown with $||\hat{v}||_2 = 1$

Variational Idea

$$minimize_{\hat{v}} \sum_i ||\vec{x}_i - proj_{\hat{v}} \vec{x}_i||_2^2$$
$$\text{such that } ||\hat{v}||_2 = 1$$

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What does the constraint do?

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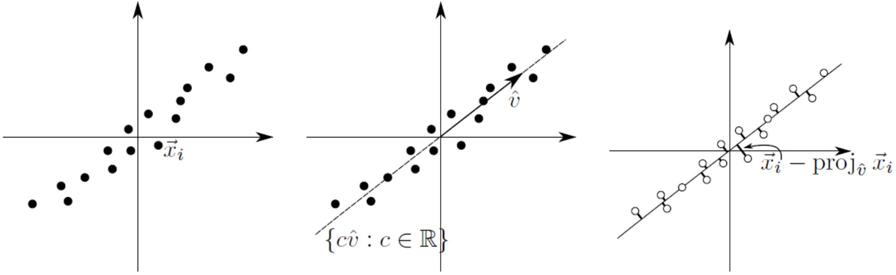
What does the constraint do?

- Does not affect optimal $\hat{v}$
- Removes scaling ambiguity

Geometric Interpretation



(a) Input data (b) Principal axis (c) Projection error

Review from Last Lecture

$$min_{c_i}||\vec{x}_i - c_i\hat{v}||_2$$

What is c_i ?

Review from Last Lecture

$$min_{c_i}||\vec{x}_i - c_i\hat{v}||_2$$

What is c_i ?

$$c_i = \vec{x}_i \cdot \hat{v}$$

Equivalent Optimization

$$\text{maximize } ||X^T \hat{v}||_2^2$$
$$\text{such that } ||\hat{v}||_2^2 = 1$$

End Goal

Eigenvector of XX^T with largest eigenvalue.

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"First principal component"

More after SVD!

Definitions

Eigenvalue and eigenvector

An eigenvector $\vec{x} \neq \vec{0}$ of $A \in \mathbb{R}^{n\times n}$ satisfies $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$; λ is an eigenvalue. Complex eigenvalues and eigenvectors instead have $\lambda \in \mathbb{C}$ and $\vec{x} \in \mathbb{C}^n$.

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Scale doesn't matter!

$\rightarrow$ can constrain $||\vec{x}||_2 \equiv 1$

Eigenproblems in the Wild

- Optimize $||A\vec{x}||_2$ such that $||\vec{x}||_2 = 1$ (important!)

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Eigenproblems in the Wild

- Optimize $||A\vec{x}||_2$ such that $||\vec{x}||_2 = 1$ (important!)
- ODE/PDE problems: Closed solutions and approximations for $\vec{y}' = B\vec{y}$
- Critical points of Rayleigh quotient:
$$\frac{\vec{x}^T A\vec{x}}{||\vec{x}||_2^2}$$

Two Basic Properties

Proved in textbook

Lemma

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

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$\rightarrow$ at most n eigenvalues

Diagonalizability

Nondefective

$A \in \mathbb{R}^{n\times n}$ is nondefective or diagonalizable if its eigenvectors span $\mathbb{R}^n$.

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$$D = X^{-1} A X$$

A is diagonalized by a similarity transformation $A \to X^{-1} A X$

Definitions

Spectrum and spectral radius

The spectrum of A is the set of eigenvalues of A. The spectral radius $\rho(A)$ is the eigenvalue λ maximizing $|\lambda|$.

Extending to $\mathbb{C}^{n\times n}$

Complex conjugate

The complex conjugate of a number $z = a + bi \in \mathbb{C}$ is $\bar{z} \equiv a - bi$.

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Complex transpose

The conjugate transpose of $A \in \mathbb{C}^{m\times n}$ is $A^H \equiv \bar{A}^T$.

Hermitian Matrix

$$A = A^H$$

Properties

Lemma

All eigenvalues of Hermitian matrices are real.

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Lemma

Eigenvectors corresponding to distinct eigenvalues of Hermitian matrices must be orthogonal.

Spectral Theorem

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Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian (if $A \in \mathbb{R}^{n \times n}$, suppose it is symmetric). Then, A has exactly n orthonormal eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \cdots, \lambda_n$.

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Full set: $D = X^T A X$

Matrix Inverse

$$\vec{b} = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$$
$$A\vec{x} = \vec{b}$$

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$$A = XDX^{-1} \Rightarrow A^{-1} = XD^{-1}X^{-1}$$

Matrix Square Root

- Given symmetric positive semi-definite (PSD) matrix, U
- Can compute matrix square root, $U^{1/2}$

Application: Polar decomposition

- Given real $n - by - n$ matrix, A
- There exists a unique factorization called the Polar Decomposition
$$A = RU$$
where R is an $n - by - n$ orthogonal matrix, and U is an $n - by - n$ symmetric PSD right "stretch" matrix.
- Also a left stretch matrix, W, such that $A = WR$.
- Geometric interpretation.

Application: Shape Matching

- Fast Lattice Shape Matching (Fast LSM)
- SIGGRAPH 2007 [Rivers and James 2007]
- http://www.alecrivers.com/fastlsm
- Need to compute orientation, R, of local particle groups
- Millions of polar decompositions (and eigenvalue decomps) per second

Physics (in one slide)

Newton:

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2}$$

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Newton:

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2}$$

Hooke:

$$\vec{F}_s = k(\vec{x} - \vec{y})$$

First-Order System

$$M\vec{X}'' = K\vec{X}$$

$$\longrightarrow \frac{d}{dt}\begin{pmatrix} \vec{X} \\ \vec{V} \end{pmatrix} = \begin{pmatrix} 0 & I_{3n\times 3n} \\ M^{-1}K & 0 \end{pmatrix}\begin{pmatrix} \vec{X} \\ \vec{V} \end{pmatrix}$$

General ODE

$$\vec{Y}' = B\vec{Y}$$

Eigenvector Solution

$$\vec{y}' = B\vec{y}$$
$$B\vec{y}_i = \lambda_i \vec{y}_i$$
$$\vec{y}(0) = c_1\vec{y}_1 + \cdots + c_k\vec{y}_k$$

Eigenvector Solution

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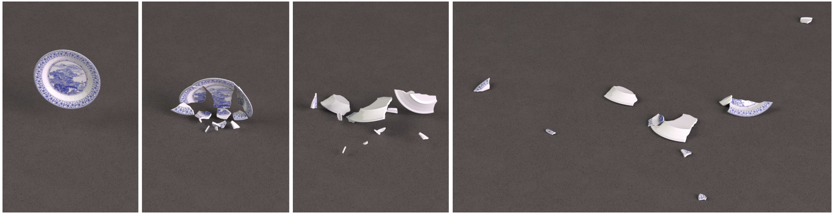
$$B\vec{y}_i = \lambda_i \vec{y}_i$$
$$\vec{y}(0) = c_1\vec{y}_1 + \cdots + c_k\vec{y}_k$$

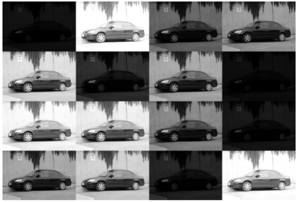
$$\rightarrow \vec{y}(t) = c_1 e^{\lambda_1 t}\vec{y}_1 + \cdots + c_k e^{\lambda_k t}\vec{y}_k$$

Application: Modal Sound Synthesis

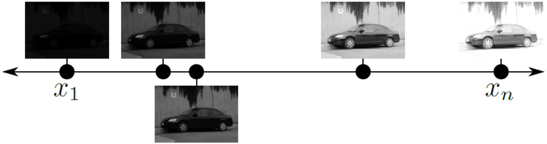


Major role in physics-based sound synthesis
https://www.youtube.com/watch?v=dMUHp8i6E5E

Organizing a Collection



(a) Database of photos



(b) Spectral embedding

Setup

Have: n items in a dataset

$w_{ij} \geq 0$ similarity of items i and j

$w_{ij} = w_{ji}$

Want: x_i embedding on $\mathbb{R}$

Quadratic Energy

$$E(\vec{x}) = \sum_{ij} w_{ij} \left(x_i - x_j\right)^2$$

Optimization

$$\text{minimize } E(\vec{x})$$

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$$\text{minimize } E(\vec{x})$$
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$$\vec{1}\vec{x} = 0$$

Simplification

$$E(\vec{x}) = 2\vec{x}^T(A - W)\vec{x}$$

Desired

Eigenvector of $A - W$ with **second** smallest eigenvalue.