Mathematical Foundation of Computer Sciences VII

Pushdown Systems

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Pushdown Systems

Pushdown System (PDS)

Definition

A pushdown system (PDS) is a triplet (P, Γ, Δ) where

- P is a finite set of states,
- Γ is finite stack alphabet, and
- $\Delta \subseteq P \times \Gamma^{\leq 2} \times P \times \Gamma^{\leq 2}$ is a finite set of transitions, where $(p, v, q, w) \in \Delta$ is denoted by $(p, v) \rightarrow (q, w)$.

Pushdown Systems

$$\frac{(p, \gamma) \to (p', \gamma') \in \Delta}{\langle p, \gamma w \rangle \hookrightarrow \langle p', \gamma' w \rangle}\ \text{Inter} \qquad \frac{(p, \gamma) \to (p', \alpha\beta) \in \Delta}{\langle p, \gamma w \rangle \hookrightarrow \langle p', \alpha\beta w \rangle}\ \text{Push}$$

$$\frac{(p, \gamma) \to (p', \epsilon) \in \Delta}{\langle p, \gamma w \rangle \hookrightarrow \langle p', w \rangle}\ \text{Pop}$$

Transition Rules

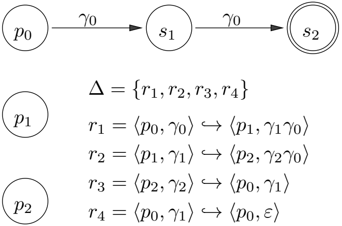
$$\frac{(p,\gamma) \to (p',\gamma') \in \Delta}{\langle p, \gamma w\rangle \hookrightarrow \langle p', \gamma' w\rangle}\ \text{Inter} \qquad \frac{(p,\gamma) \to (p',\alpha\beta) \in \Delta}{\langle p, \gamma w\rangle \hookrightarrow \langle p', \alpha\beta w\rangle}\ \text{Push}$$

$$\frac{(p,\gamma) \to (p',\epsilon) \in \Delta}{\langle p, \gamma w\rangle \hookrightarrow \langle p', w\rangle}\ \text{Pop}$$

$$\frac{(p,\epsilon) \to (p',\alpha) \in \Delta}{\langle p, w\rangle \hookrightarrow \langle p', \alpha w\rangle}\ \text{Simple-Push}$$

$$\frac{(p,\alpha\beta) \to (p',\gamma) \in \Delta}{\langle p, \alpha\beta w\rangle \hookrightarrow \langle p', \gamma w\rangle}\ \text{Nonstandard-Pop}$$

Reachability of PDS: *Pre**



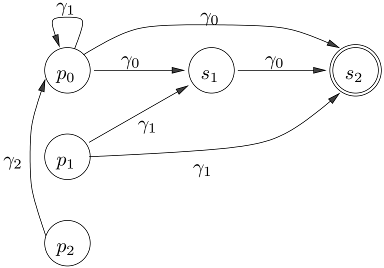
$\Delta = \{r_1, r_2, r_3, r_4\}$

$r_1 = \langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_1, \gamma_1\gamma_0 \rangle$

$r_2 = \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_2, \gamma_2\gamma_0 \rangle$

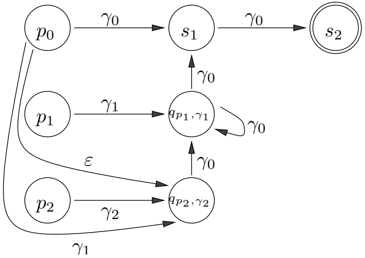
$r_3 = \langle p_2, \gamma_2 \rangle \hookrightarrow \langle p_0, \gamma_1 \rangle$

$r_4 = \langle p_0, \gamma_1 \rangle \hookrightarrow \langle p_0, \varepsilon \rangle$



Pre*: If $\langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle$, and $p' \xrightarrow{w}{}^{*} q$, then (p, γ, q).

Reachability of PDS: *Post**



- If $\langle p, \gamma \rangle \hookrightarrow \langle p', \varepsilon \rangle$ and $p' \xrightarrow{\gamma}^* q$, then (p', ε, q)
- If $\langle p, \gamma \rangle \hookrightarrow \langle p', \gamma' \rangle$ and $p' \xrightarrow{\gamma}^* q$, then (p', γ', q)
- If $\langle p, \gamma \rangle \hookrightarrow \langle p', \gamma'\gamma'' \rangle$ and $p' \xrightarrow{\gamma}^* q$, then $(p', \gamma', q_{p',\gamma'})$ and $(q_{p',\gamma'}, \gamma'', q)$

Well-Structured Transition Systems

Well-Quasi-Order (WQO)

Definition

A quasi-order $(D, \leq)$ is a reflexive transitive binary relation on D. It is a well-quasi-order (WQO) if, for each infinite sequence $a_1, a_2, \ldots$ in D, there exist i, j with $i < j$ and $a_i \leq a_j$.

Well-Structured Transition Systmes (WSTS)

Definition

A well-structured transition system (WSTS) is a triplet $M = \langle (P, \preceq), \rightarrow \rangle$ where $(P, \preceq)$ is a WQO, and $\rightarrow \subseteq P \times P$ is monotonic, i.e., for each $p_1, p_2, q_1 \in P$, $p_1 \rightarrow q_1$ and $p_1 \preceq p_2$ imply that there exists q_2 with $p_2 \rightarrow q_2 \wedge q_1 \preceq q_2$.

Well-Structured Pushdown Systems

WSPDS is a triplet $\langle (P, \preceq), (\Gamma, \leq), \Delta \rangle$, where

- $(P, \preceq), (\Gamma, \leq)$ are well-quasi-orders,
- Δ is monotonic.

Two subclasses

- pushdown VAS $\langle (P, \preceq), \Gamma, \Delta \rangle$
- vector pushdown systems $\langle P, (\Gamma, \leq), \Delta \rangle$

However...

- pushdown VAS $\subset$ WSPDS
- Vector pushdown systems $=$ WSPDS

A Final Quiz

Do *pre*$*$ and *post*$*$ work in WSPDS?