MIE 1603 / 1653 - Integer Programming
Winter 2020
Assignment #4

Due Date: March 20, Friday, no later than 11:59PM.

- You may work in groups of two students (from the same course code) or individually. In the former case, turn in **only one** solution set with both group members listed on it.
- Groups must work independently.
- You **must** cite any references (texts, papers, websites, etc.) you have used to help you solve these problems.
- Submit your written/typed solutions via Quercus (pdf file only). For Question 2, additionally submit your code along with your solutions in Quercus.

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1. [25pts] Solve the following problem using Benders Decomposition. Clearly state your master problem and subproblem, their solutions as well as the optimality cuts iteration by iteration. Solve the problem to optimality and show all your iterations (by hand).

$$\begin{aligned} \min\ & 2x_1 + 3x_2 + 2y_1 + y_2 \\ \text{s.t. } & x_1 + 2x_2 + y_1 + 2y_2 \geq 5 \\ & 2x_1 - x_2 + 3y_1 + y_2 \geq 4 \\ & x_1, x_2 \geq 0 \\ & y_1, y_2 \in \{0, 1\} \end{aligned}$$

2. [55pts] A company needs to build warehouses to supply their retail stores located in a set of C cities. The set of candidate locations to build warehouses to supply the retail stores is W, and there is a cost f_w to build each warehouse $w \in W$. Finally, there is a transportation cost t_{wr} to ship a unit from a warehouse to a retail store. The warehouses are located in three regions, R_1, R_2, R_3, and at most N_i warehouse can be built in each location $i \in \{1, 2, 3\}$. We define binary decision variables y_w equal to 1 if warehouse $w \in W$ is built, and x_{wr} equal to the fraction of the demand of store r to met by warehouse w.

   A possible model for this problem is below.

$$
\begin{aligned}
\min \quad & \sum_{w \in W} \sum_{r \in C} t_{wr} x_{wr} + \sum_{w \in W} f_w y_w && && (1) \\
s.t. \quad & \sum_{w \in W} x_{wr} = 1 && \forall r \in C && (2) \\
& \sum_{r \in C} x_{wr} \leq |C| y_w && \forall w \in W && (3) \\
& \sum_{w \in R_i} y_w \leq N_i && \forall i \in \{1, 2, 3\} && (4) \\
& y_w \in \{0, 1\} && \forall w \in W && (5) \\
& x_{wr} \geq 0 && \forall w \in W, r \in C && (6)
\end{aligned}
$$

The constraints (3), which ensure no store is supplied by an unopened warehouse, can also be modelled as follows:

$$x_{wr} \leq y_w \quad \forall w \in W, r \in C \tag{7}$$

(a) Prove that either the *formulation* with constraints (3) or the *formulation* with constraints (7) is better, or show the two *formulations* are incomparable.

(b) If we were to solve this problem using Benders decomposition and constraints (3), what would be the master problem before any cuts are added, what is the subproblem?

(c) Implement Benders decomposition for this problem in Gurobi and solve the instance `HW4_Q2_Data.py` using:

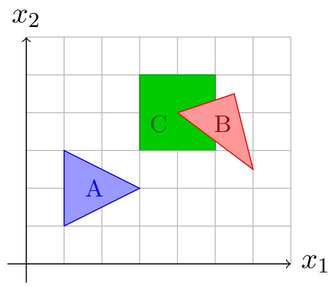
  i. cutting plane

  ii. branch-and-cut

  iii. **(BONUS)** [5pts] In branch-and-cut, instead of creating a new subproblem (i.e., a new Gurobi model) every time you wish to generate a cut, create only one subproblem, i.e., only one Gurobi model for it to be used through the algorithm, and modify it. If you choose this version of the implementation, you do not need to turn in the other version, i.e., either create a new subproblem every time or modify it, not both.

(d) Compare the results and the solution times of solving this model with Benders decomposition using a cutting plane algorithm and using branch-and-cut. Which algorithm has better performance? Why?

3. [20pts] Consider region X that is given as the **union** of the shaded regions (red, blue and green) in the following figure:



That is, $X = A \cup B \cup C$. The triangles A and B are the polytopes defined as follows:

$$A = \left\{(x, y) \in \mathbb{R}^2 : x \geq 1,\ -x + 2y \geq 1,\ x + 2y \leq 7\right\}$$
$$B = \left\{(x, y) \in \mathbb{R}^2 : 3x + 4y \geq 28,\ -x + 3y \leq 8,\ 8x + 2y \leq 53\right\},$$

while region C is the polytope with extreme points (3,3), (3,5), (5,5) and (5,3). Let $c \in \mathbb{R}^2$ be a given objective vector.

(a) Write down an *ideal* integer programming formulation for the problem $\min\{c^\top x : x \in X\}$.

(b) Write down an alternative integer programming formulation for the problem $\min\{c^\top x : x \in X\}$.