

Supplementary Materials: A Novel Local Search Algorithm for the Vertex Bisection Minimization Problem

1 Deep Analysis of the Two-Manner Based Local Search

In this section, we present a deep analysis of the two-manner-based local search. Specifically, we evaluate a total of nine combinations of swap operations. First, we present the execution condition for the swap operation (Line 5 in Algorithm 5) in CELS and another two different execution conditions for the swap operation as follows:

- $|VM(D)| \leq VM(D_{\text{best}})$,
- $|VM(D)| \leq VM(D_{\text{best}}) + 1$,
- $|VM(D)| \leq VM(D_{\text{best}}) + 2$.

Additionally, we present the swap rule utilized by CELS and another two alternate swap rules, where the swap rules become gradually greedy:

- **Swap Rule:** Vertices with the lowest $\Delta_{\text{drop}}(D, v)$ from $B \setminus V_{\text{lock}}$ are stored in S_B , and vertices with the lowest $\Delta_{\text{add}}(D, v)$ from $B' \setminus V_{\text{lock}}$ are stored in $S_{B'}$. If either set contains more than K vertices, we randomly retain K vertices in the corresponding set, where K is a pre-defined parameter. Then, we select a pair of vertices $(v, u) := \arg \min_{v \in S_B, u \in S_{B'}} \Delta_{\text{swap}}(D, v, u)$, breaking ties randomly.
- **Swap Rule 1:** Vertices with the lowest $\Delta_{\text{drop}}(D, v)$ from $B \setminus V_{\text{lock}}$ are stored in S_B , and vertices with the lowest $\Delta_{\text{add}}(D, v)$ from $B' \setminus V_{\text{lock}}$ are stored in $S_{B'}$. Then, we directly select a pair of vertices $(v, u) := \arg \min_{v \in S_B, u \in S_{B'}} \Delta_{\text{swap}}(D, v, u)$, breaking ties randomly.
- **Swap Rule 2:** A pair of vertices is selected as $(v, u) := \arg \min_{v \in B \setminus V_{\text{lock}}, u \in B' \setminus V_{\text{lock}}} \Delta_{\text{swap}}(D, v, u)$, breaking ties randomly.

By combining the three execution conditions with the three swap rules, we generate nine types of swap operations, one of which is adopted by CELS. We then implement the other eight modified versions of the swap operation in CELS under the same settings, and compare the best solutions obtained by CELS with those produced by each corresponding algorithm. The results are summarized in Table 1. From Table 1, it is clear that the swap operation used in CELS outperforms the other swap operations, indicating the effective design of the swap operation in CELS.

2 Detailed Results of All Algorithms

We present the results of CELS and the comparative algorithms in Tables 2–5. For each instance, min represents the smallest size achieved by each algorithm, while avg indicates the average size obtained across 10 runs. If min equals avg, the average value is omitted. The Pre_best column refers to the best results previously reported in the literature. In the tables, bold values highlight the best solutions achieved among all algorithms.

References

Execution Condition	$ VM(D) \leq VM(D_{\text{best}})$	$ VM(D) \leq VM(D_{\text{best}}) + 1$	$ VM(D) \leq VM(D_{\text{best}}) + 2$
Swap Rule	*	29 (10)	42 (17)
Swap Rule 1	19 (10)	24 (13)	39 (5)
Swap Rule 2	23 (11)	25 (12)	47 (1)

Table 1: Comparison of CELS with eight modified versions. Each cell contains two numbers: the first represents the number of instances where CELS achieves a better minimal solution, and the second indicates the number of instances where it obtain a worse minimal solution.

Instance	Pre_best	CELS max(avg)	BVNSBucket max(avg)	BVNSBucket2 max(avg)	CLHUS max(avg)	BVNS max(avg)
g1	398	384(384.9)	399(399.7)	399(399.6)	387	399(399.3)
g2	399	387(387.9)	399(399.5)	398(399.8)	388	398(399.4)
g3	399	384(384.9)	399(399.6)	399(399.8)	386	398(399.2)
g4	398	387(388.2)	399(399.7)	399(399.6)	385	398(399.4)
g5	398	384(384.8)	399(399.8)	399(399.5)	386	399(399.7)
g6	398	384(384.9)	399(399.8)	399(399.6)	387	399(399.5)
g7	398	387(387.9)	399(399.5)	398(399.8)	388	399(399.5)
g8	399	384(384.9)	399(399.6)	399(399.8)	386	399(399.7)
g9	398	387(388.2)	399(399.7)	399(399.6)	385	399(399.6)
g10	398	384(384.8)	399(399.8)	399(399.5)	386	399(399.6)
g11	16	16	16	16	16	16
g12	32	32	32	32	32	32
g13	50	50	50	50	50	50
g14	188	184(186)	187(190.4)	187(188.1)	185(186.3)	188(192)
g15	184	183(184.7)	185(186.4)	185(186.2)	183(184)	185(188.4)
g16	188	185(185.7)	188(190)	188(189.9)	185(185.7)	188(189.8)
g17	182	180(181.7)	182(183.4)	182(183.7)	181(181.8)	181(184.9)
g18	187	184(186)	187(190.4)	187(188.1)	185(186.3)	189(192)
g19	184	183(184.7)	185(186.4)	185(186.2)	183(184)	186(187.7)
g20	187	185(185.7)	188(190)	188(189.9)	185(185.7)	187(189.8)
g21	180	180(181.7)	182(183.4)	182(183.7)	181(181.8)	182(184.2)
g22	875	860(861.6)	873(874.8)	868(873.4)	862(864.5)	889(895.2)
g23	877	863(864.8)	874(877.9)	874(876.8)	866(868.9)	890(894.2)
g24	874	861(862.4)	874(876.4)	873(875.5)	865(866.4)	880(890.3)
g25	876	861(863.7)	875(877.1)	873(875.5)	865(868.5)	892(899.8)
g26	877	860(862.2)	875(876.2)	873(876)	864(866.8)	885(897)
g27	875	860(861.6)	873(874.8)	868(873.4)	862(864.6)	879(889.6)
g28	878	863(866)	877(879.6)	876(878)	864(867)	896(900.7)
g29	874	861(862.4)	874(876.4)	873(875.5)	865(866.4)	883(894.4)
g30	876	861(863.7)	875(877.1)	873(875.5)	865(868.5)	890(894.8)
g31	877	860(862.2)	875(876.2)	873(876)	864(866.7)	887(893.7)
g32	40	40	40	40	40	40
g33	50	50	50	50	50	50
g34	80	80	80	80	80	80
g35	470	454(456.3)	465(471.1)	466(472)	456(458.1)	475(485.4)
g36	471	457(462)	470(477.4)	464(474.6)	459(462.5)	480(485.7)
g37	463	453(455.9)	463(471.9)	463(469.1)	456(458.2)	478(489.8)
g38	474	455(458.4)	465(473.4)	465(472.9)	457(461.9)	474(482.5)
g39	470	454(456.3)	465(471.1)	466(472)	456(458.1)	478(488.5)
g40	471	457(462)	470(477.4)	464(474.6)	459(462.5)	475(485.1)
g41	463	453(455.9)	463(471.9)	463(469.1)	456(458.2)	478(487.5)
g42	474	455(458.4)	465(473.4)	465(472.9)	457(461.9)	468(484.5)
g43	437	430(431.1)	436(437.9)	436(437.1)	430(431.1)	436(438)
g44	437	430(431.2)	437(437.9)	437(437.5)	430(431.1)	437(439.6)
g45	434	429(430)	435(436.6)	434(436.2)	429(430)	437(437.9)
g46	437	431(431.4)	436(438.2)	436(438)	430(431.8)	438(439.7)
g47	438	430(431.6)	436(438.7)	437(438.2)	430(431.5)	438(438.9)
g48	100	100	100	100	100	100
g49	60	60	60	60	60	60
g50	50	50	50(55)	50	50	50(55)
g51	234	233(233.7)	235(237)	232(236.1)	234(234.6)	235(238.3)
g52	233	231(232.2)	230(236.7)	232(236.4)	233(233.9)	236(237.7)
g53	232	229(230.1)	231(234.5)	230(234.7)	230(231.5)	234(236.6)
g54	229	228(228.4)	229(231.2)	228(230.8)	228(229.3)	229(232.6)
g55	1052	1014(1023)	1047(1065.5)	1056(1071.7)	1052(1060)	1095(1145.8)
g56	1052	1014(1023)	1047(1065.5)	1056(1071.7)	1052(1059.6)	1115(1175.3)
g57	100	100	100	100	100	100(109.6)
g58	1180	1146(1156.6)	1190(1219.2)	1195(1213.7)	1160(1173.9)	1239(1314.5)
g59	1180	1146(1156.6)	1190(1219.2)	1195(1213.7)	1160(1173.4)	1249(1292.1)
g60	1455	1402(1417.2)	1465(1480)	1462(1474.8)	1475(1486.4)	1626(1666.4)
g61	1455	1402(1417.2)	1465(1480)	1462(1474.8)	1475(1487)	1602(1663.2)
g62	140	140	140	140	140	142(274.1)
g63	1659	1613(1624.4)	1680(1700.1)	1659(1696.6)	1688(1702)	1808(1859.7)
g64	1659	1613(1624.4)	1680(1700.1)	1659(1696.6)	1688(1697.1)	1826(1867)
g65	160	160	160	160	160	213(347.8)
g66	180	200	180(182)	180	200	257(431.8)
g67	199	200	199(199.7)	199(199.5)	200	360(522.8)
g70	426	357(364.1)	380(392.5)	366(379.1)	457(467.9)	515(530)
g72	199	200	199(199.7)	199(199.6)	200	413(526.7)
g77	200	200	200	200	200	885(1072)
g81	200	200	200(346.3)	200(287.6)	200	1950(2110.1)

Table 2: Results for all algorithms on Large benchmark

Instance	Pre.best	CELS max(avg)	BVNSBucket max(avg)	BVNSBucket2 max(avg)	CLHUS max(avg)	BVNS max(avg)
494_bus	6	6	6	6	6	6
arc130	8	8	8	8	8	8
ash292	9	9	9	9	9	9(9.2)
ash85	7	7	7	7	7	7
bcsprw01	3	3	3	3	3	3
bcsprw02	2	2	2	2	2	2
bcsprw03	4	4	4	4	4	4
bcsprw04	7	7	7(7.1)	7	7	7(7.2)
bcsprw05	7	7	7	7	7	7
bcsstk01	12	12	12	12	12	12
bcsstk04	24	24	24	24	24	24
bcsstk05	15	15	15(15.6)	15(15.4)	15	16
bcsstk06	36	36	36	36	36	36(36.2)
bcsstk07	36	36	36	36	36	36(37)
bcsstk20	7	10(12.5)	7(7.4)	7	10(13.6)	7
bcsstk22	4	4	4	4	4	4
bcsstm07	36	36	36(37.8)	36	36	36(39.2)
can...24	4	4	4	4	4	4
can...61	5	5	5	5	5	5
can...62	3	3	3	3	3	3
can...73	8	8	8	8	8	8
can...144	6	6	6	6	6	6
can...161	16	16	16	16	16	16
can...292	18	18	18	18	18	18(18.3)
can...445	38	38(38.8)	38(39.6)	40	38	38(39.4)
curtis54	4	4	4	4	4	4
dwt...162	7	7	7	7	7	7
dwt...193	23	23	23	23	23	23
dwt...209	15	15	15	15	15	15
dwt...221	7	8(8.3)	8	8	8	8
dwt...234	4	4	4	4	8	4
dwt...245	8	8	8(8.6)	8(8.5)	8(8.4)	8(8.4)
dwt...310	8	8	8	8	8	8
dwt...361	14	14	14	14	14	14
dwt...419	16	16	16	16	16	16
fs...183_1	15	15	15	15	15	15
fs...183_3	15	15	15	15	15	15
fs...183_4	15	15	15	15	15	15
fs...183_6	15	15	15	15	15	15
gent113	13	13	13	13	13	13
gre...115	18	18	18	18	18	18
gre...185	20	20	20	20	20	20
gre...343	28	28	28	28	28	28
gre...216a	21	21	21	21	21	21
gre...216b	21	21	21	21	21	21
grid_5.Changed	5	5	5	5	5	5
grid_6.Changed	6	6	6	6	6	6
grid_7.Changed	7	7	7	7	7	7
Grid3x3.Changed	3	3	3	3	3	3
hor...131	33	33(33.4)	33	33	33(33.4)	33(33.1)
hypercube_4.16	6	6	6	6	6	6
hypercube_5.32	10	10	10	10	10	10
hypercube_6.64	20	20	20	20	24	20
hypercube_7.128	35	35	35	35	40(40.3)	35
hypercube_8.256	70	70	70	70	78(85)	70
ibm32	9	9	9	9	9	9
impcol.a	20	20	20	20	20	20(20.4)
impcol.b	15	15	15	15	15	15
impcol.c	21	21	21	21	21	21
impcol.d	17	17(17.9)	17(17.7)	17(17.7)	17	17(17.5)
impcol.e	30	30	30	30	30	30
lms...131	11	11	11	11	11	11
lund.a	20	20	20	20	20	20
lund.b	20	20	20	20	20	20
mbeacxc	187	187(187.5)	231(235.6)	232(235)	187	206(210)
mbeafw	187	187(187.5)	231(235.6)	232(235)	187	207(207.5)
mbeause	178	178	206(221.8)	215(228.1)	178	180(198.5)
mcca	18	18	18	18	18	18
nnc261	11	11	11	11	11	11
nos1	3	3	3	3	3	3
nos4	7	7	7	7	7	7
p100_24_34	3	3	3	3	3	3
p17_16_24	3	3	3	3	4	3
p18_16_21	2	2	2	2	3	2
p19_16_19	2	2	2	2	2(2.2)	2
p20_16_18	2	2	2	2	2	2
p21_17_20	2	2	2	2	2	2
p22_17_19	2	2	2	2	2	2
p23_17_23	2	2	2	2	2	2
p24_17_29	3	3	3	3	3(3.6)	3
p25_17_20	2	2	2	2	2	2
p26_17_19	2	2	2	2	3	2
p27_17_19	2	2	2	2	2	2
p28_17_18	2	2	2	2	2	2
p29_17_18	1	1	1	1	2	1
p30_17_19	2	2	2	2	2	2
p31_18_21	2	2	2	2	3	2
p32_18_20	2	2	2	2	2	2
p33_18_21	3	3	3	3	3	3

Table 3: Results for all algorithms on classic_medium benchmark I

Instance	Pre_best	CELS max(avg)	BVNSBucket max(avg)	BVNSBucket2 max(avg)	CLHUS max(avg)	BVNS max(avg)
p34.18.21	2	2	2	2	2	2
p35.18.19	2	2	2	2	2	2
p36.18.20	2	2	2	2	2(2.1)	2
p37.18.20	2	2	2	2	2	2
p38.18.19	2	2	2	2	2	2
p39.18.19	2	2	2	2	2	2
p40.18.32	4	4	4	4	4	4
p41.19.20	1	1	1	1	2	1
p42.19.24	3	3	3	3	4	3
p43.19.22	2	2	2	2	2	2
p44.19.25	3	3	3	3	3	3
p45.19.25	2	2	2	2	3	2
p46.19.20	2	2	2	2	2	2
p47.19.21	2	2	2	2	2	2
p48.19.21	2	2	2	2	2	2
p49.19.22	2	2	2	2	2	2
p50.19.25	2	2	2	2	2(2.9)	2
p51.20.28	4	4	4	4	4	4
p52.20.27	2	2	2	2	2	2
p53.20.22	2	2	2	2	2	2
p54.20.28	3	3	3	3	3(3.3)	3
p55.20.24	2	2	2	2	2	2
p56.20.23	3	3	3	3	3	3
p57.20.24	2	2	2	2	2	2
p58.20.21	2	2	2	2	2	2
p59.20.23	2	2	2	2	2(2.2)	2
p60.20.22	2	2	2	2	3	2
p61.21.22	2	2	2	2	2	2
p62.21.30	3	3	3	3	3(3.3)	3
p63.21.42	5	5	5	5	5	5
p64.21.22	2	2	2	2	2	2
p65.21.24	2	2	2	2	2	2
p66.21.28	3	3	3	3	3	3
p67.21.22	2	2	2	2	2	2
p68.21.27	3	3	3	3	3	3
p69.21.23	2	2	2	2	2	2
p70.21.25	3	3	3	3	3	3
p71.22.29	3	3	3	3	3	3
p72.22.49	5	5	5	5	5	5
p73.22.29	2	2	2	2	2	2
p74.22.30	3	3	3	3	3	3
p75.22.25	2	2	2	2	2	2
p76.22.30	2	2	2	2	2	2
p77.22.37	4	4	4	4	4	4
p78.22.31	3	3	3	3	3	3
p79.22.29	3	3	3	3	3	3
p80.22.30	3	3	3	3	3	3
p81.23.46	6	6	6	6	6	6
p82.23.24	2	2	2	2	2	2
p83.23.24	1	1	1	1	1	1
p84.23.26	2	2	2	2	2	2
p85.23.26	1	1	1	1	1	1
p86.23.24	2	2	2	2	2	2
p87.23.30	3	3	3	3	3	3
p88.23.26	2	2	2	2	2	2
p89.23.27	3	3	3	3	3	3
p90.23.35	3	3	3	3	3	3
p91.24.33	3	3	3	3	3	3
p92.24.26	2	2	2	2	2	2
p93.24.27	2	2	2	2	2	2
p94.24.31	3	3	3	3	3	3
p95.24.27	2	2	2	2	2	2
p96.24.27	2	2	2	2	3	2
p97.24.26	2	2	2	2	2	2
p98.24.29	2	2	2	2	2	2
p99.24.27	2	2	2	2	2	2
plat362	27	27	27	27	27	27
plskz362	10	10	10	10	10	10
pores.1	7	7	7	7	7	7
pores.3	12	12	12	12	12	12
saylr1	14	14	14	14	14(15.9)	14
steam1	39	39	39	39	39	39
steam3	4	4	4	4	4	4
str.....0	81	81	83(84.8)	83(84.9)	81(81.4)	82(85)
str...200	95	94(94.6)	95(95.8)	95(95.7)	95	95(96.3)
str...600	102	101(101.2)	103(104.3)	102(103.4)	102(102.8)	102(103.9)
TREE.22.3_rot1.Changed	2	2	2	2	2	2
TREE.22.3_rot2.Changed	2	2	2	2	2	2
TREE.22.3_rot3.Changed	2	2	2	2	2	2
TREE.22.3_rot4.Changed	2	2	2	2	2	2
TREE.22.3_rot5.Changed	2	2	2	2	2	2
west0132	18	18	18	18	18	18
west0156	26	26	26	26	26	26
west0167	19	19	19	19	19	19
west0381	110	110	111(112.1)	111(111.8)	110(110.2)	111(112.3)
west0479	75	75(75.1)	76(76.2)	75(75.8)	75	76(76.4)
west0497	44	44(47.6)	44	44	44(44.7)	44(47.5)
will199	52	52	52	52	52	52
will57	3	3	3	3	3	3

Table 4: Results for all algorithms on classic_medium benchmark II

Instance	Pre_best	CELS max(avg)	BVNSBucket max(avg)	BVNSBucket2 max(avg)	CLHUS max(avg)	BVNS max(avg)
685_bus	8	8	8 (8.8)	8 (8.5)	8 (8.6)	8 (8.5)
bcsstk08	60	60	60 (60.9)	60 (61)	60 (61.5)	60 (60.9)
bcsstk09	61	61	61	61	61	61
bcsstk11	36	36	36 (85.3)	36 (75.9)	36	92(160.2)
bcsstk12	36	36	36 (85.3)	36 (75.9)	36	91(173.3)
bcsstk19	4	4	4	4	4	4 (8.2)
bcsstk27	41	41	45(92.1)	41 (92.7)	41 (43)	425(524.2)
bcsstm27	41	41	45(92.1)	41 (92.7)	41 (43)	483(532.2)
blkhole	61	61	62(62.9)	62(62.9)	61 (62.3)	62(63)
bp_----0	153	150 (150.9)	155(156.5)	152(156.5)	153(153.6)	155(157.2)
bp_---200	175	174 (175.3)	181(182.3)	181(183.5)	175(176.1)	182(186.2)
bp_--400	184	183 (185.5)	188(193.6)	186(192.3)	185(185.3)	186(193.9)
bp_-600	195	193(194)	195(201.1)	191 (199.2)	196(197)	197(202)
bp_800	199	198 (198.3)	202(205.6)	200(204.9)	199(199.6)	203(206.7)
bp_1000	203	202 (202.1)	204(209.7)	204(210.2)	203(203.7)	208(210.4)
bp_1200	207	207	211(215.4)	209(213.5)	208(208.9)	212(216.2)
bp_1400	211	210	211(217.6)	212(216.1)	211(211.9)	216(219.8)
bp_1600	211	209 (209.7)	213(217.1)	213(217.2)	211(211.8)	215(218.1)
can_715	35	35	35(35.6)	35(37.4)	35(35.6)	35 (38)
can_838	34	34 (36.3)	34 (34.3)	34(34.4)	34 (34.8)	34 (35)
can_1054	28	28 (35.1)	28 (30)	28 (29.2)	30(32.2)	28 (38.6)
can_1072	30	30 (30.8)	30 (30.6)	30 (30.6)	30 (31.1)	30 (33.7)
dwt_503	26	26	26(32.2)	26(31)	26	26(31.2)
dwt_592	22	22 (22.2)	22 (22.2)	22 (22.4)	22	22 (22.4)
dwt_878	18	18	18	18	18	18
dwt_918	22	22 (22.8)	22	22	22	22
dwt_992	34	34	34	34	34	34 (34.2)
dwt_1005	33	33	33	33	33	33
dwt_2680	29	29 (49)	29 (37.7)	29 (34.2)	29 (39.4)	29 (82.6)
fs_541_1	19	19	19	19	19	19
fs_541_2	19	19	19	19	19	19
fs_541_3	19	19	19	19	19	19
fs_541_4	19	19	19	19	19	19
fs_680_1	6	6	6	6	6	6
fs_680_2	6	6	6	6	6	6
fs_680_3	6	6	6	6	6	6
fs_760_1	22	22	22	22	22	22
fs_760_2	22	22	22	22	22	22
fs_760_3	22	22	22	22	22	22
gr_30_30	30	30	30	30	30	30
gre_512	36	36	36	36	36	36
gre_1107	90	90 (93.4)	90 (91)	90 (91.2)	90 (90.2)	90 (92)
hypercube_10_1024.txt	252	252	252	252	312(354.7)	252
hypercube_9_512.txt	126	126	126	126	157(169.4)	126
jagmesh1	26	26	26	26	26	26
jagmesh2	31	31	31	31	31	31
jagmesh3	33	33	33	33	33	33
jagmesh7	14	14 (14.4)	14 (14.3)	14 (14.1)	14 (16.3)	14 (14.5)
jpwh_991	63	63 (67.6)	63 (64)	63 (64.2)	63 (64.8)	63 (63.8)
lms_511	31	31 (31.4)	31 (31.3)	31	31 (31.8)	31 (31.5)
lshp1009	31	31	31	31	31	31
mcf	89	89 (89.1)	104(128.8)	106(126.7)	89 (89.2)	106(143.9)
nnc666	18	18	18	18	18	18 (19)
nos2	3	3 (4.2)	3 (3.4)	3(3.8)	3(3.6)	3(5)
nos3	40	40	40	40	40	40 (44)
nos6	15	15	15	15	15	15
nos7	65	65	65	65	65 (66.4)	65
orsirr_2	50	50 (50.7)	50 (55.4)	50 (54.8)	51	50 (53.2)
saylr3	30	30 (30.7)	30 (30.3)	30 (30.3)	34(58.3)	30 (30.6)
sherman1	30	30 (30.7)	30 (30.3)	30 (30.3)	34(58.3)	30 (30.4)
sherman4	22	22	22 (23.4)	22 (22.8)	31	22 (22.8)
shl_----0	82	80 (80.6)	82(82.3)	82	82	82
shl_---200	90	88 (88.5)	89(90.5)	89(90.8)	90(91)	90(90.8)
steam2	60	60	60	60	60	60
west0655	108	108	108 (109.5)	108 (108.8)	108 (109.2)	108 (110.2)

Table 5: Results for all algorithms on classic_large benchmark