

## Lecture 2: Linear Algebra

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Mathematics for Machine Learning

<https://yung-web.github.io/home/courses/mathml.html>

KAIST EE

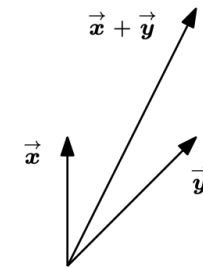
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- Scalars:  $a, b, c, \alpha, \beta, \gamma$
- Vectors:  $\mathbf{x}, \mathbf{y}, \mathbf{z}$
- Matrices:  $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$
- Sets:  $\mathcal{A}, \mathcal{B}, \mathcal{C}$
- (Ordered) tuple:  $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$
- Matrix of column vectors:  $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$  or  $\mathbf{B} = (\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3)$
- Set of vectors:  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$
- $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{N}, \mathbb{R}^n$ , etc

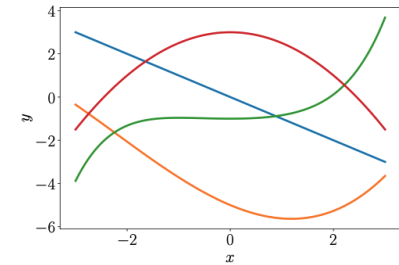
- (1) Systems of Linear Equations
- (2) Matrices
- (3) Solving Systems of Linear Equations
- (4) Vector Spaces
- (5) Linear Independence
- (6) Basis and Rank
- (7) Linear Mappings
- (8) Affine Spaces

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- Algebra: a set of objects and a set of rules or operations to manipulate those objects
- Linear algebra
  - Object: vectors  $\mathbf{v}$
  - Operations: their additions ( $\mathbf{v} + \mathbf{w}$ ) and scalar multiplication ( $k\mathbf{v}$ )
- Examples
  - Geometric vectors
    - High school physics
  - Polynomials
  - Audio signals
  - Elements of  $\mathbb{R}^n$



(a) Geometric vectors.



(b) Polynomials.

- For unknown variables  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

- Three cases of solutions

- No solution

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 3 \\ x_1 - x_2 + 2x_3 & = & 2 \\ 2x_1 & + & 3x_3 = 1 \end{array}$$

- Unique solution

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 3 \\ x_1 - x_2 + 2x_3 & = & 2 \\ & x_2 + 3x_3 & = 1 \end{array}$$

- Infinitely many solutions

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 3 \\ x_1 - x_2 + 2x_3 & = & 2 \\ 2x_1 & + & 3x_3 = 5 \end{array}$$

- Question.** Under what conditions, one of the above three cases occur?

- A collection of linear equations

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

- Matrix representations:

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \cdots + \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \iff \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}}$$

- Understanding  $\mathbf{A}$  is the key to answering various questions about this linear system  $\mathbf{Ax} = \mathbf{b}$ .

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- For two matrices  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{B} \in \mathbb{R}^{m \times n}$ ,

$$\mathbf{A} + \mathbf{B} := \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

- For two matrices  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{B} \in \mathbb{R}^{n \times k}$ , the elements  $c_{ij}$  of the product  $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times k}$  is:

$$c_{ij} = \sum_{l=1}^n a_{il} b_{lj}, \quad i = 1, \dots, m, \quad j = 1, \dots, k.$$

- Example.**  $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$ , compute  $\mathbf{AB}$  and  $\mathbf{BA}$ .

- A square matrix<sup>1</sup>  $I_n$  with  $I_{ii} = 1$  and  $I_{ij}=0$  for  $i \neq j$ , where  $n$  is the number of rows and columns. For example,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **Associativity**: For  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times p}$ ,  $\mathbf{C} \in \mathbb{R}^{p \times q}$ ,  $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- **Distributivity**: For  $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$ , and  $\mathbf{C}, \mathbf{D} \in \mathbb{R}^{n \times p}$ ,  
(i)  $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$  and (ii)  $\mathbf{A}(\mathbf{C} + \mathbf{D}) = \mathbf{AC} + \mathbf{AD}$
- **Multiplication with the identity matrix**: For  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $I_m \mathbf{A} = \mathbf{A} I_n = \mathbf{A}$

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<sup>1</sup># of rows = # of cols

# Inverse and Transpose

- For a square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B}$  is the **inverse** of  $\mathbf{A}$ , denoted by  $\mathbf{A}^{-1}$ , if

$$\mathbf{AB} = \mathbf{I}_n = \mathbf{BA}.$$

- Called **regular/invertible/nonsingular**, if it exists.
- If it exists, it is unique.
- How to compute? For  $2 \times 2$  matrix,

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

- For a matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times m}$  with  $b_{ij} = a_{ji}$  is the **transpose** of  $\mathbf{A}$ , which we denote by  $\mathbf{A}^T$ .

- Example.** For  $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$ ,

$$\mathbf{A}^T = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

- If  $\mathbf{A} = \mathbf{A}^T$ ,  $\mathbf{A}$  is called **symmetric**.

- $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$
- $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$
- $(\mathbf{A}^{\top})^{\top} = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})^{\top} = \mathbf{A}^{\top} + \mathbf{B}^{\top}$
- $(\mathbf{A}\mathbf{B})^{\top} = \mathbf{B}^{\top}\mathbf{A}^{\top}$
- If  $\mathbf{A}$  is invertible, so is  $\mathbf{A}^{\top}$ .

- Multiplication by a scalar  $\lambda \in \mathbb{R}$  to  $\mathbf{A} \in \mathbb{R}^{m \times n}$

- **Example.** For  $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$ ,  $3 \times \mathbf{A} = \begin{pmatrix} 0 & 6 \\ 3 & -3 \\ 0 & 3 \end{pmatrix}$

- **Associativity**

- $(\lambda\psi)\mathbf{C} = \lambda(\psi\mathbf{C})$
- $\lambda(\mathbf{BC}) = (\lambda\mathbf{B})\mathbf{C} = \mathbf{B}(\lambda\mathbf{C}) = (\mathbf{BC})\lambda$
- $(\lambda\mathbf{C})^\top = \mathbf{C}^\top\lambda^\top = \mathbf{C}^\top\lambda = \lambda\mathbf{C}^\top$

- **Distributivity**

- $(\lambda + \psi)\mathbf{C} = \lambda\mathbf{C} + \psi\mathbf{C}$
- $\lambda(\mathbf{B} + \mathbf{C}) = \lambda\mathbf{B} + \lambda\mathbf{C}$

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## Example

$$\begin{aligned} -3x \quad \quad + 2z &= -1 \\ x - 2y + 2z &= -5/3 \\ -x - 4y + 6z &= -13/3 \end{aligned}$$

- $\rho_i$ :  $i$ -th equation
- Express the equation as its **augmented matrix**.

$$\begin{pmatrix} -3 & 0 & 2 & | & -1 \\ 1 & -2 & 2 & | & -5/3 \\ -1 & -4 & 6 & | & -13/3 \end{pmatrix} \xrightarrow[\substack{(1/3)\rho_1 + \rho_2 \\ -(1/3)\rho_1 + \rho_3}]{-2\rho_2 + \rho_3} \begin{pmatrix} -3 & 0 & 2 & | & -1 \\ 0 & -2 & 8/3 & | & -2 \\ 0 & -4 & 16/3 & | & -4 \end{pmatrix} \xrightarrow{-2\rho_2 + \rho_3} \begin{pmatrix} -3 & 0 & 2 & | & -1 \\ 0 & -2 & 8/3 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The two nonzero rows give  $-3x + 2z = -1$  and  $-2y + (8/3)z = -2$ .

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<sup>1</sup>Examples from this slide to the next several slides come from Jim Hefferson's Linear Algebra book.

- Parametrizing  $-3x + 2z = -1$  and  $-2y + (8/3)z = -2$  gives:

$$x = (1/3) + (2/3)z$$

$$y = 1 + (4/3)z$$

$$z = z$$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 4/3 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \right\}$$

This helps us understand the set of solutions, e.g., each value of  $z$  gives a different solution.

$z$		0	1	2	$-1/2$
solution	$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 7/3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 5/3 \\ 11/3 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1/3 \\ -1/2 \end{pmatrix}$



- The system 
$$\begin{array}{rcrcrcrcrcrcl} x & + & 2y & - & z & & & & & = & 2 \\ 2x & - & y & - & 2z & + & w & & & = & 5 \end{array}$$
 reduces in this way.

$$\left( \begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 2 & -1 & -2 & 1 & 5 \end{array} \right) \xrightarrow{-2\rho_1 + \rho_2} \left( \begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & -5 & 0 & 1 & 1 \end{array} \right)$$

- It has solutions of this form.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \quad \text{for } z, w \in \mathbb{R}$$

- Note that taking  $z = w = 0$  shows that the first vector is a **particular solution** of the system.

# General = Particular + Homogeneous

- General approach
  1. Find a particular solution to  $\mathbf{Ax} = \mathbf{b}$
  2. Find all solutions to the homogeneous equation  $\mathbf{Ax} = 0$ 
    - ▶ 0 is a trivial solution
  3. Combine the solutions from steps 1. and 2. to the general solution
- Questions: A formal algorithm that performs the above?
  - Gauss-Jordan method: convert into a “beautiful” form (formally reduced row-echelon form)
  - Elementary transformations: (i) row swapping (ii) multiply by a constant (iii) row addition
- Such a form allows an algorithmic way of solving linear equations

## Example: Unique Solution

- Start as usual by getting echelon form.

$$\begin{array}{rcl}
 x + y - z = 2 & & x + y - z = 2 \\
 2x - y = -1 & \xrightarrow{-2\rho_1 + \rho_2} & -3y + 2z = -5 \\
 x - 2y + 2z = -1 & \xrightarrow{-1\rho_1 + \rho_3} & -3y + 3z = -3
 \end{array}
 \xrightarrow{-1\rho_2 + \rho_3}
 \begin{array}{rcl}
 x + y - z = 2 & & x + y - z = 2 \\
 -3y + 2z = -5 & & -3y + 2z = -5 \\
 z = 2 & & z = 2
 \end{array}$$

- Make all the leading entries one.

$$\begin{array}{rcl}
 x + y - z = 2 & & x + y - z = 2 \\
 y - (2/3)z = 5/3 & \xrightarrow{(-1/3)\rho_2} & y - (2/3)z = 5/3 \\
 z = 2 & & z = 2
 \end{array}$$

- Finish by using the leading entries to eliminate upwards, until we can read off the solution.

$$\begin{array}{rcl}
 x + y - z = 2 & & x + y = 4 \\
 y - (2/3)z = 5/3 & \xrightarrow{\rho_3 + \rho_1} & y = 3 \\
 z = 2 & \xrightarrow{(2/3)\rho_3 + \rho_2} & z = 2
 \end{array}
 \xrightarrow{-\rho_2 + \rho_1}
 \begin{array}{rcl}
 x & & x = 1 \\
 y & & y = 3 \\
 z & & z = 2
 \end{array}$$

## Example: Infinite Number of Solutions

$$\begin{aligned}x - y - 2w &= 2 \\x + y + 3z + w &= 1 \\-y + z - w &= 0\end{aligned}$$

- Start by getting echelon form and turn the leading entries to 1's.

$$\begin{aligned}&\xrightarrow{-1\rho_1+\rho_2} \left( \begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & -1 \\ 0 & -1 & 1 & -1 & 0 \end{array} \right) \\&\xrightarrow{(1/2)\rho_2+\rho_3} \left( \begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & -1 \\ 0 & 0 & 5/2 & 1/2 & -1/2 \end{array} \right) \\&\xrightarrow[(2/5)\rho_3]{(1/2)\rho_2} \left( \begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 3/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array} \right)\end{aligned}$$

- Eliminate upwards.

$$\begin{aligned}&\xrightarrow{-(3/2)\rho_3+\rho_2} \left( \begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array} \right) \\&\xrightarrow{\rho_2+\rho_1} \left( \begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array} \right)\end{aligned}$$

- The parameterized solution set is:

$$\left\{ \begin{pmatrix} 9/5 \\ -1/5 \\ -1/5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4/5 \\ -6/5 \\ -1/5 \\ 1 \end{pmatrix} w \mid w \in \mathbb{R} \right\}$$

		<i>number of solutions of the homogeneous system</i>	
		<i>one</i>	<i>infinitely many</i>
<i>particular solution exists?</i>	<i>yes</i>	<b>unique</b> solution	<b>infinitely many</b> solutions
	<i>no</i>	<b>no</b> solutions	<b>no</b> solutions

## 1. Pseudo-inverse

$$\mathbf{Ax} = \mathbf{b} \iff \mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b} \iff \mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

- $(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ : *Moore-Penrose pseudo-inverse*
- many computations: matrix product, inverse, etc

## 2. Gaussian elimination

- intuitive and constructive way
- cubic complexity (in terms of # of simultaneous equations)

## 3. Iterative methods

- practical ways to solve indirectly
- (a) stationary iterative methods: Richardson method, Jacobi method, Gaus-Seidel method, successive over-relaxation method
- (b) Krylov subspace methods: conjugate gradients, generalized minimal residual, biconjugate gradients

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- A set  $\mathcal{G}$  and an operation  $\otimes : \mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$ .  $G := (\mathcal{G}, \otimes)$  is called a **group**, if:
  1. **Closure.**  $\forall x, y \in \mathcal{G}, x \otimes y \in \mathcal{G}$
  2. **Associativity.**  $\forall x, y, z \in \mathcal{G}, (x \otimes y) \otimes z = x \otimes (y \otimes z)$
  3. **Neutral element.**  $\exists e \in \mathcal{G}, \forall x \in \mathcal{G}, x \otimes e = x$  and  $e \otimes x = x$
  4. **Inverse element.**  $\forall x \in \mathcal{G}, \exists y \in \mathcal{G}, x \otimes y = e$  and  $y \otimes x = e$ . We often use  $x^{-1} = y$ .
- $G = (\mathcal{G}, \otimes)$  is an **Abelian group**, if the following is additionally met:
  - **Communicativity.**  $\forall x, y \in \mathcal{G}, x \otimes y = y \otimes x$



- $(\mathbb{Z}, +)$  is an Abelian group
- $(\mathbb{N} \cup \{0\}, +)$  is not a group (because inverses are missing)
- $(\mathbb{Z}, \cdot)$  is not a group
- $(\mathbb{R}, \cdot)$  is not a group (because of no inverse for 0)
- $(\mathbb{R}^n, +)$ ,  $(\mathbb{Z}^n, +)$  are Abelian, if  $+$  is defined componentwise
- $(\mathbb{R}^{m \times n}, +)$  is Abelian (with componentwise  $+$ )
- $(\mathbb{R}^{n \times n}, \cdot)$ 
  - Closure and associativity follow directly
  - Neutral element:  $\mathbf{I}_n$
  - The inverse  $\mathbf{A}^{-1}$  may exist or not. So, generally, it is not a group. However, the set of invertible matrices in  $\mathbb{R}^{n \times n}$  with matrix multiplication is a group, called **general linear group**.

**Definition.** A real-valued vector space  $V = (\mathcal{V}, +, \cdot)$  is a set  $\mathcal{V}$  with two operations

- (a)  $+: \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$  (vector addition)
- (b)  $\cdot: \mathbb{R} \times \mathcal{V} \mapsto \mathcal{V}$  (scalar multiplication),

where

1.  $(\mathcal{V}, +)$  is an Abelian group
2. **Distributivity.**
  - $\forall \lambda \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathcal{V}, \lambda \cdot (\mathbf{x} + \mathbf{y}) = \lambda \cdot \mathbf{x} + \lambda \mathbf{y}$
  - $\forall \lambda, \psi \in \mathbb{R}, \mathbf{x} \in \mathcal{V}, (\lambda + \psi) \cdot \mathbf{x} = \lambda \cdot \mathbf{x} + \psi \cdot \mathbf{x}$
3. **Associativity.**  $\forall \lambda, \psi \in \mathbb{R}, \mathbf{x} \in \mathcal{V}, \lambda \cdot (\psi \cdot \mathbf{x}) = (\lambda\psi) \cdot \mathbf{x}$
4. **Neutral element.**  $\forall \mathbf{x} \in \mathcal{V}, 1 \cdot \mathbf{x} = \mathbf{x}$

## Example

- $\mathcal{V} = \mathbb{R}^n$  with
  - Vector addition:  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$
  - Scalar multiplication:  $\lambda \mathbf{x} = (\lambda x_1, \dots, \lambda x_n)$
- $\mathcal{V} = \mathbb{R}^{m \times n}$  with
  - Vector addition:  $\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$
  - Scalar multiplication:  $\lambda \mathbf{A} = \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$

**Definition.** Consider a vector space  $V = (\mathcal{V}, +, \cdot)$  and  $\mathcal{U} \subset \mathcal{V}$ . Then,  $U = (\mathcal{U}, +, \cdot)$  is called **vector subspace** (simply linear subspace or subspace) of  $V$  if  $U$  is a vector space with two operations '+' and '·' restricted to  $\mathcal{U} \times \mathcal{U}$  and  $\mathbb{R} \times \mathcal{U}$ .

## Examples

- For every vector space  $V$ ,  $V$  and  $\{0\}$  are the trivial subspaces.
- The solution set of  $\mathbf{Ax} = 0$  is the subspace of  $\mathbb{R}^n$ .
- The solution of  $\mathbf{Ax} = \mathbf{b}$  ( $\mathbf{b} \neq 0$ ) is not a subspace of  $\mathbb{R}^n$ .
- The intersection of arbitrarily many subspaces is a subspace itself.

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- **Definition.** For a vector space  $V$  and vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ , every  $\mathbf{v} \in V$  of the form  $\mathbf{v} = \lambda_1 \mathbf{x}_1 + \dots + \lambda_k \mathbf{x}_k$  with  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  is a **linear combination** of the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ .
- **Definition.** If there is a non-trivial linear combination such that  $0 = \sum_{i=1}^k \lambda_i \mathbf{x}_i$  with at least one  $\lambda_i \neq 0$ , the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are **linearly dependent**. If only the trivial solution exists, i.e.,  $\lambda_1 = \dots = \lambda_k = 0$ ,  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are **linearly independent**.
- **Meaning.** A set of linearly independent vectors consists of vectors that have no redundancy.
- **Useful fact.** The vectors  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  are linearly dependent, iff (at least) one of them is a linear combination of the others.
  - $x - 2y = 2$  and  $2x - 4y = 4$  are linearly dependent.

# Checking Linear Independence

- Gauss elimination to get the row echelon form
- All column vectors are linearly independent iff all columns are pivot columns (why?).
- Example.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix} \rightsquigarrow \dots \rightsquigarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- Every column is a pivot column. Thus,  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ ,  $\mathbf{x}_3$  are linearly independent.

# Linear Combinations of Linearly Independent Vectors

- Vector space  $V$  with  $k$  linearly independent vectors  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$
- $m$  linear combinations  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ . (Q) Are they linearly independent?

$$\begin{aligned}\mathbf{x}_1 &= \lambda_{11}\mathbf{b}_1 + \lambda_{21}\mathbf{b}_2 + \cdots + \lambda_{k1}\mathbf{b}_k \\ &\vdots \\ \mathbf{x}_m &= \lambda_{1m}\mathbf{b}_1 + \lambda_{2m}\mathbf{b}_2 + \cdots + \lambda_{km}\mathbf{b}_k\end{aligned}$$

$$\mathbf{x}_j = \underbrace{(\mathbf{b}_1, \dots, \mathbf{b}_k)}_{\mathbf{B}} \overbrace{\begin{pmatrix} \lambda_{1j} \\ \vdots \\ \lambda_{kj} \end{pmatrix}}^{\lambda_j}, \quad \mathbf{x}_j = \mathbf{B}\lambda_j$$

- $\sum_{j=1}^m \psi_j \mathbf{x}_j = \sum_{j=1}^m \psi_j \mathbf{B}\lambda_j = \mathbf{B} \sum_{j=1}^m \psi_j \lambda_j$
- $\{\mathbf{x}\}$  linearly independent  $\iff \{\lambda\}$  linearly independent



$$\begin{aligned} \mathbf{x}_1 &= \mathbf{b}_1 - 2\mathbf{b}_2 + \mathbf{b}_3 - \mathbf{b}_4 \\ \mathbf{x}_2 &= -4\mathbf{b}_1 - 2\mathbf{b}_2 + 4\mathbf{b}_4 \\ \mathbf{x}_3 &= 2\mathbf{b}_1 + 3\mathbf{b}_2 - \mathbf{b}_3 - 3\mathbf{b}_4 \\ \mathbf{x}_4 &= 17\mathbf{b}_1 - 10\mathbf{b}_2 + 11\mathbf{b}_3 + \mathbf{b}_4 \end{aligned}$$

$$\mathbf{A} = (\lambda_1 \ \lambda_2 \ \lambda_3 \ \lambda_4) = \begin{pmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & -4 & -3 & 1 \end{pmatrix} \rightsquigarrow \dots \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The last column is not a pivot column. Thus,  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$  are linearly dependent.

- (1) Systems of Linear Equations
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- (5) Linear Independence
- (6) **Basis and Rank**
- (7) Linear Mappings
- (8) Affine Spaces

- **Definition.** A vector space  $V = (\mathcal{V}, +, \cdot)$  and a set of vectors  $\mathcal{A} = \{\mathbf{x}_1, \dots, \mathbf{x}_k\} \subset \mathcal{V}$ .
  - If every  $v \in \mathcal{V}$  can be expressed as a linear combination of  $\mathbf{x}_1, \dots, \mathbf{x}_k$ ,  $\mathcal{A}$  is called a **generating set** of  $V$ .
  - The set of all linear combinations of  $\mathcal{A}$  is called the **span** of  $\mathcal{A}$ .
  - If  $\mathcal{A}$  spans the vector space  $V$ , we use  $V = \text{span}[\mathcal{A}]$  or  $V = \text{span}[\mathbf{x}_1, \dots, \mathbf{x}_k]$
- **Definition.** The minimal generating set  $\mathcal{B}$  of  $V$  is called **basis** of  $V$ . We call each element of  $\mathcal{B}$  **basis vector**. The number of basis vectors is called **dimension** of  $V$ .
- Properties
  - $\mathcal{B}$  is a maximally<sup>2</sup> linearly independent set of vectors in  $V$ .
  - Every vector  $x \in V$  is a linear combination of  $\mathcal{B}$ , which is unique.

---

<sup>2</sup>Adding any other vector to this set will make it linearly dependent.

- Different bases  $\mathbb{R}^3$

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{B}_3 = \left\{ \begin{pmatrix} 0.5 \\ 0.8 \\ 0.4 \end{pmatrix}, \begin{pmatrix} 1.8 \\ 0.3 \\ 0.3 \end{pmatrix}, \begin{pmatrix} -2.2 \\ -1.3 \\ 3.5 \end{pmatrix} \right\}$$

- Linearly independent, but not maximal. Thus, not a basis.

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -4 \end{pmatrix} \right\}$$

- Want to find a basis of a subspace  $U = \text{span}[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m]$ 
  1. Construct a matrix  $\mathbf{A} = (\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_m)$
  2. Find the row-echelon form of  $\mathbf{A}$ .
  3. Collect the pivot columns.
- Logic: Collect  $\mathbf{x}_i$  so that we have only trivial solution. Pivot columns tell us which set of vectors is linearly independent.
- See example 2.17 (pp. 35)

- **Definition.** The **rank** of  $\mathbf{A} \in \mathbb{R}^{m \times n}$  denoted by  $\text{rk}(\mathbf{A})$  is # of linearly independent columns
  - Same as the number of linearly independent rows

- $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix} \rightsquigarrow \dots \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

Thus,  $\text{rk}(\mathbf{A}) = 2$ .

- $\text{rk}(\mathbf{A}) = \text{rk}(\mathbf{A}^T)$

- The **columns** (resp. **rows**) of  $\mathbf{A}$  span a subspace  $U$  (resp.  $W$ ) with  $\dim(U) = \text{rk}(\mathbf{A})$  (resp.  $\dim(W) = \text{rk}(\mathbf{A})$ ), and a basis of  $U$  (resp.  $W$ ) can be found by Gauss elimination of  $\mathbf{A}$  (resp.  $\mathbf{A}^T$ ).
- For all  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\text{rk}(\mathbf{A}) = n$ , iff  $\mathbf{A}$  is regular (invertible).
- The linear system  $\mathbf{A}\mathbf{x} = \mathbf{b}$  is solvable, iff  $\text{rk}(\mathbf{A}) = \text{rk}(\mathbf{A}|\mathbf{b})$ .
- For  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , the subspace of solutions for  $\mathbf{A}\mathbf{x} = 0$  possesses dimension  $n - \text{rk}(\mathbf{A})$ .
- $\mathbf{A} \in \mathbb{R}^{m \times n}$  has **full rank** if its rank equals the largest possible rank for a matrix of the same dimensions. The rank of the full-rank matrix  $\mathbf{A}$  is  $\min(\# \text{ of cols}, \# \text{ of rows})$ .

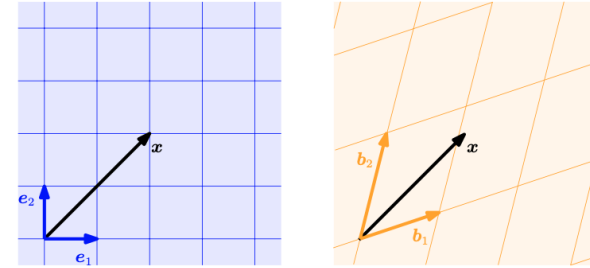
- (1) Systems of Linear Equations
- (2) Matrices
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- (6) Basis and Rank
- (7) **Linear Mappings**
- (8) Affine Spaces



- Interest: A mapping that preserves the structure of the vector space
- **Definition.** For vector spaces  $V, W$ , a mapping  $\Phi : V \mapsto W$  is called a **linear mapping** (or homomorphism/linear transformation), if, for all  $\mathbf{x}, \mathbf{y} \in V$  and all  $\lambda \in \mathbb{R}$ ,
  - $\Phi(\mathbf{x} + \mathbf{y}) = \Phi(\mathbf{x}) + \Phi(\mathbf{y})$
  - $\Phi(\lambda \mathbf{x}) = \lambda \Phi(\mathbf{x})$
- **Definition.** A mapping  $\Phi : \mathcal{V} \mapsto \mathcal{W}$  is called
  - **Injective** (단사), if  $\forall \mathbf{x}, \mathbf{y} \in \mathcal{V}, \Phi(\mathbf{x}) = \Phi(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$
  - **Surjective** (전사), if  $\Phi(\mathcal{V}) = \mathcal{W}$
  - **Bijjective** (전단사), if it is injective and surjective.

- For bijective mapping, there exists an inverse mapping  $\Phi^{-1}$ .
- **Isomorphism** if  $\Psi$  is linear and bijective.
- **Theorem.** Vector spaces  $V$  and  $W$  are isomorphic, iff  $\dim(V) = \dim(W)$ .
  - Vector spaces of the same dimension are kind of the same thing.
- Other properties
  - For two linear mappings  $\Phi$  and  $\Psi$ ,  $\Phi \circ \Psi$  is also a linear mapping.
  - If  $\Phi$  is an isomorphism, so is  $\Phi^{-1}$ .
  - For two linear mappings  $\Phi$  and  $\Psi$ ,  $\Phi + \Psi$  and  $\lambda\Psi$  for  $\lambda \in \mathbb{R}$  are linear.

- A basis defines a coordinate system.



- Consider an ordered basis  $B = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$  of vector space  $V$ . Then, for any  $\mathbf{x} \in V$ , there exists a unique linear combination

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n.$$

- We call  $\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$  the coordinate of  $\mathbf{x}$  with respect to  $B = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$ .
- Basis change  $\implies$  Coordinate change

- Consider a vector space  $V$  and two coordinate systems defined by  $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$  and  $B' = (\mathbf{b}'_1, \dots, \mathbf{b}'_n)$ .
- **Question.** For  $(x_1, \dots, x_n)_B \rightarrow (y_1, \dots, y_n)_{B'}$ , what is  $(y_1, \dots, y_n)_{B'}$ ?
- **Theorem.** 
$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (\mathbf{b}'_1 \ \dots \ \mathbf{b}'_n)^{-1} (\mathbf{b}_1 \ \dots \ \mathbf{b}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$
- Regard  $\mathbf{A}_\Phi = (\mathbf{b}'_1 \ \dots \ \mathbf{b}'_n)^{-1} (\mathbf{b}_1 \ \dots \ \mathbf{b}_n)$  as a linear map

## Example

- $B = ((1, 0), (0, 1))$  and  $B' = ((2, 1), (1, 2))$

- $(4, 2)_B \rightarrow (x, y)_{B'}$ ?

- Using 
$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (\mathbf{b}'_1 \ \dots \ \mathbf{b}'_n)^{-1} (\mathbf{b}_1 \ \dots \ \mathbf{b}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

- Two vector spaces
  - $V$  with basis  $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$  and  $W$  with basis  $C = (\mathbf{c}_1, \dots, \mathbf{c}_m)$
- What is the coordinate in  $C$ -system for each basis  $\mathbf{b}_j$ ? For  $j = 1, \dots, n$ ,

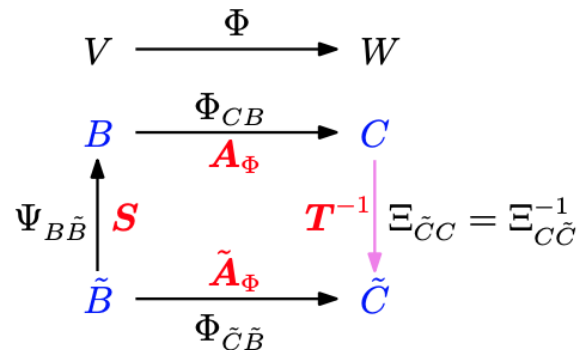
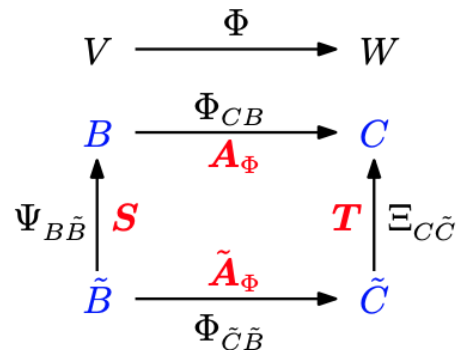
$$\mathbf{b}_j = \alpha_{1j}\mathbf{c}_1 + \dots + \alpha_{mj}\mathbf{c}_m \iff \mathbf{b}_j = (\mathbf{c}_1 \ \dots \ \mathbf{c}_m) \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}$$

$$\implies (\mathbf{b}_1 \ \dots \ \mathbf{b}_n) = (\mathbf{c}_1 \ \dots \ \mathbf{c}_m) \overbrace{\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}}^{\mathbf{A}_\Phi}$$

- $\hat{x} = \mathbf{A}_\Phi \hat{y}$ , where  $\hat{x}$  is the vector w.r.t  $B$  and  $\hat{y}$  is the vector w.r.t.  $C$

# Basis Change: General Case

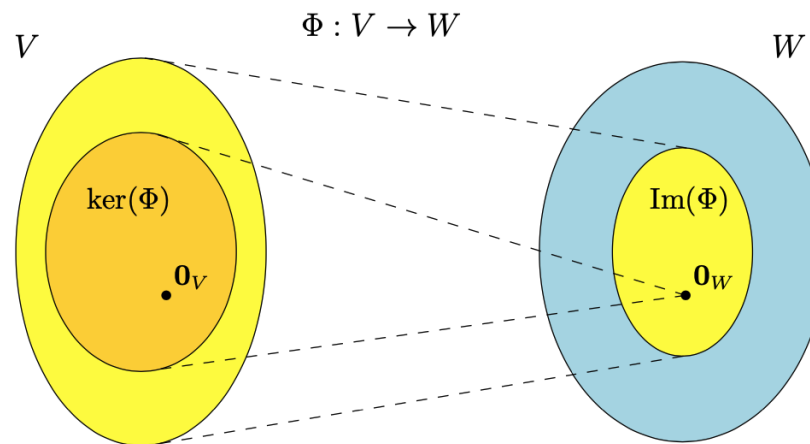
- For linear mapping  $\Phi : V \mapsto W$ , consider bases  $B, B'$  of  $V$  and  $C, C'$  of  $W$   
 $B = (\mathbf{b}_1 \cdots \mathbf{b}_n), B' = (\mathbf{b}'_1 \cdots \mathbf{b}'_n) \quad C = (\mathbf{c}_1 \cdots \mathbf{c}_m), C' = (\mathbf{c}'_1 \cdots \mathbf{c}'_m).$
- (inter) transformation matrices  $\mathbf{A}_\Phi$  from  $B$  to  $C$  and  $\mathbf{A}'_\Phi$  from  $B'$  to  $C'$
- (intra) transformation matrices  $S$  from  $B'$  to  $B$  and  $T$  from  $C'$  to  $C$
- Theorem.**  $\mathbf{A}'_\Phi = T^{-1} \mathbf{A}_\Phi S$



- Consider a linear mapping  $\Phi : V \mapsto W$ . The **kernel** (or **null space**) is the set of vectors in  $V$  that maps to  $0 \in W$  (i.e., neutral element).

**Definition.**  $\ker(\Phi) := \Phi^{-1}(0_W) = \{\mathbf{v} \in V : \Phi(\mathbf{v}) = 0_W\}$

- Image/range:** set of vectors  $w \in W$  that can be reached by  $\Phi$  from any vector in  $V$
- $V$ : **domain**,  $W$ : **codomain**





- $0_V \in \ker(\Phi)$  (because  $\Phi(0_V) = 0_W$ )
- Both  $\text{Im}(\Phi)$  and  $\ker(\Phi)$  are subspaces of  $W$  and  $V$ , respectively.
- $\Phi$  is one-to-one (injective)  $\iff \ker(\Phi) = \{0\}$  (i.e., only 0 is mapped to 0)
- Since  $\Phi$  is a linear mapping, there exists  $\mathbf{A} \in \mathbb{R}^{m \times n}$  such that  $\Phi : \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ . Then,  $\text{Im}(\Phi) = \text{column space of } \mathbf{A}$  which is the span of column vectors of  $\mathbf{A}$ .
- $\text{rk}(\mathbf{A}) = \dim(\text{Im}(\Phi))$
- $\ker(\Phi)$  is the solution set of the homogeneous system of linear equations  $\mathbf{A}\mathbf{x} = 0$

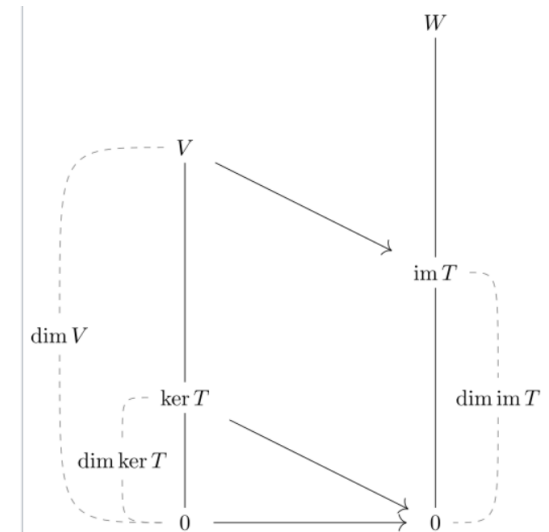
# Rank-Nullity Theorem

## Theorem.

$$\dim(\ker(\Phi)) + \dim(\text{Im}(\Phi)) = \dim(V)$$

- If  $\dim(\text{Im}(\Phi)) < \dim(V)$ , the kernel contains more than just 0.
- If  $\dim(\text{Im}(\Phi)) < \dim(V)$ ,  $\mathbf{A}_\Phi \mathbf{x} = 0$  has infinitely many solutions.
- If  $\dim(V) = \dim(W)$  (e.g.,  $V = W = \mathbb{R}^n$ ), the followings are equivalent:  $\Phi$  is
  - (1) injective, (2) surjective, (3) bijective,
  - In this case,  $\Phi$  defines  $\mathbf{y} = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is regular.
- **Simplified version.** For  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,

$$\text{rk}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$$



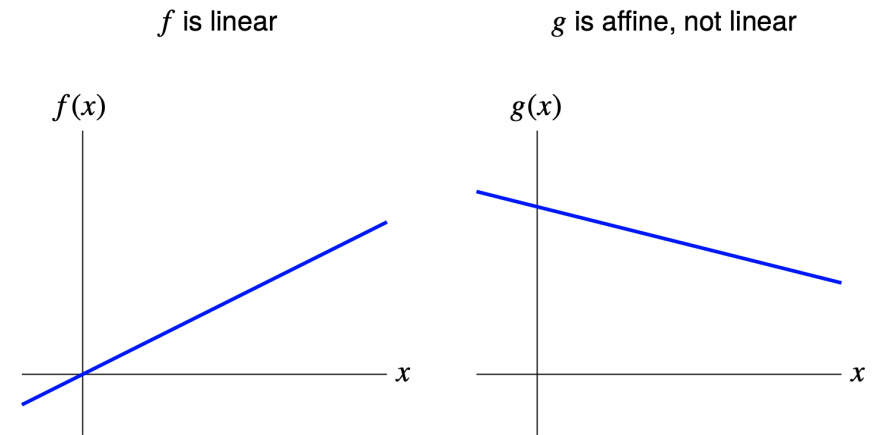
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<sup>2</sup>Nullity: the dimension of null space (kernel)

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# Linear vs. Affine Function

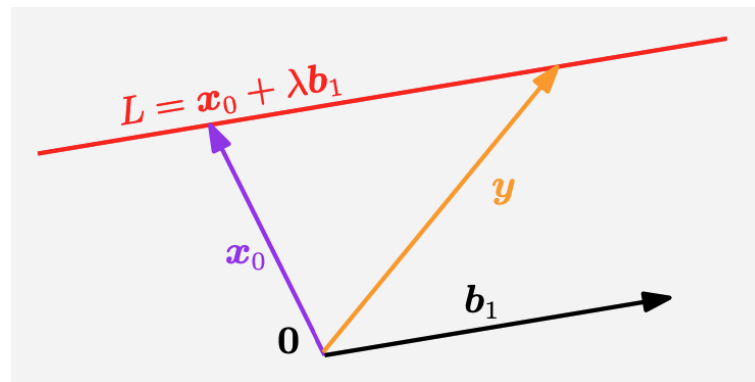
- **linear function**:  $f(x) = ax$
- **affine function**:  $f(x) = ax + b$
- sometimes (ignorant) people refer to affine functions as linear



- Spaces that are offset from the origin. Not a vector space.
- **Definition.** Consider a vector space  $V$ ,  $\mathbf{x}_0 \in V$  and a subspace  $U \subset V$ . Then, the subset  $L = \mathbf{x}_0 + U := \{\mathbf{x}_0 + \mathbf{u} : \mathbf{u} \in U\}$  is called **affine subspace** or **linear manifold** of  $V$ .
- $U$  is called **direction** or **direction space**, and  $\mathbf{x}_0$  is **support** point.
- An affine subspace is not a vector subspace of  $V$  for  $\mathbf{x}_0 \notin U$ .
- **Parametric equation.** A  $k$ -dimensional affine space  $L = \mathbf{x}_0 + U$ . If  $(\mathbf{b}_1, \dots, \mathbf{b}_k)$  is an ordered basis of  $U$ , any element  $\mathbf{x} \in L$  can be uniquely described as
$$\mathbf{x} = \mathbf{x}_0 + \lambda_1 \mathbf{b}_1 + \dots + \lambda_k \mathbf{b}_k, \quad \lambda_1, \dots, \lambda_k \in \mathbb{R}$$

## Example

- In  $\mathbb{R}^2$ , one-dimensional affine subspace: **line**.  $\mathbf{y} = \mathbf{x}_0 + \lambda \mathbf{b}_1$ .  $U = \text{span}[\mathbf{b}_1]$
- In  $\mathbb{R}^3$ , two-dimensional affine subspace: **plane**.  $\mathbf{y} = \mathbf{x}_0 + \lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2$ .  $U = \text{span}[\mathbf{b}_1, \mathbf{b}_2]$
- In  $\mathbb{R}^n$ ,  $(n - 1)$ -dimensional affine subspace: **hyperplane**.  $\mathbf{y} = \mathbf{x}_0 + \sum_{k=1}^{n-1} \lambda_k \mathbf{b}_k$ .  
 $U = \text{span}[\mathbf{b}_1, \dots, \mathbf{b}_n]$



- For a linear mapping  $\Phi : V \mapsto W$  and a vector  $\mathbf{a} \in W$ , the mapping  $\phi : V \mapsto W$  with  $\phi(\mathbf{x}) = \mathbf{a} + \Phi(\mathbf{x})$  is an **affine mapping** from  $V$  to  $W$ . The vector  $\mathbf{a}$  is called the **translation vector**.

Questions?

1)