Lecture 2: Linear Algebra

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Roadmap

(1) Systems of Linear Equations

(2) Matrices

(3) Solving Systems of Linear Equations

(4) Vector Spaces

(5) Linear Independence

(6) Basis and Rank

(7) Linear Mappings

(8) Affine Spaces

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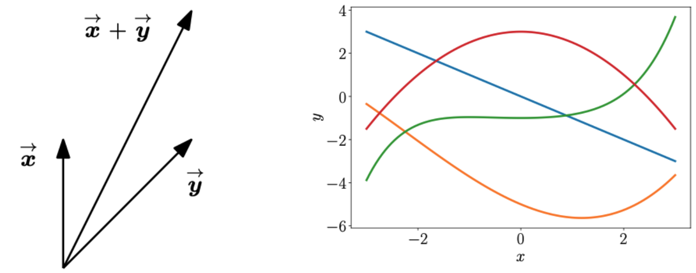
Basic Notations

- Scalars: $a, b, c, \alpha, \beta, \gamma$
- Vectors: $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$
- Matrices: $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$
- Sets: $\mathcal{A}, \mathcal{B}, \mathcal{C}$
- (Ordered) tuple: $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$
- Matrix of column vectors: $\boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3]$ or $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{pmatrix}$
- Set of vectors: $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}^n$, etc

Linear Algebra

- Algebra: a set of objects and a set of rules or operations to manipulate those objects
- Linear algebra
  - Object: vectors $\boldsymbol{v}$
  - Operations: their additions $(\boldsymbol{v} + \boldsymbol{w})$ and scalar multiplication $(k\boldsymbol{v})$
- Examples
  - Geometric vectors
    - High school physics
  - Polynomials
  - Audio signals
  - Elements of $\mathbb{R}^n$



(a) Geometric vectors. (b) Polynomials.

System of Linear Equations

- For unknown variables $(x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

- Three cases of solutions

- No solution

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 \phantom{{}+x_2{}} + 3x_3 &= 1 \end{aligned}$$

- Unique solution

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ x_2 + 3x_3 &= 1 \end{aligned}$$

- Infinitely many solutions

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 \phantom{{}+x_2{}} + 3x_3 &= 5 \end{aligned}$$

- Question: Under what conditions, one of the above three cases occur.

Matrix Representation

- A collection of linear equations

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
&\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- Matrix representations:

$$
\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \cdots + \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \iff \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{\boldsymbol{b}}
$$

- Understanding $\boldsymbol{A}$ is the key to answering various questions about this linear system $\boldsymbol{Ax} = \boldsymbol{b}$.

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Matrix: Addition and Multiplication

- For two matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{m \times n}$,

$$\boldsymbol{A} + \boldsymbol{B} := \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

- For two matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times k}$, the elements c_{ij} of the product $\boldsymbol{C} = \boldsymbol{AB} \in \mathbb{R}^{m \times k}$ is:

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, k.$$

- Example. $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$, compute $\boldsymbol{AB}$ and $\boldsymbol{BA}$.

Identity Matrix and Matrix Properties

- A square matrix[1] $\boldsymbol{I}_n$ with $I_{ii} = 1$ and $I_{ij=0}$ for $i \neq j$, where n is the number of rows and columns. For example,

$$\boldsymbol{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Associativity: For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n \times p}$, $\boldsymbol{C} \in \mathbb{R}^{p \times q}$, $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- Distributivity: For $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$, and $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{n \times p}$,
$$\text{(i) } (\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC} \text{ and (ii) } \boldsymbol{A}(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{AC} + \boldsymbol{AD}$$
- Multiplication with the identity matrix: For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{A}$

[1] # of rows = # of cols

Inverse and Transpose

- For a square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\boldsymbol{B}$ is the inverse of A, denoted by $\boldsymbol{A}^{-1}$, if

$$\boldsymbol{A}\boldsymbol{B} = \boldsymbol{I}_n = \boldsymbol{B}\boldsymbol{A}.$$

- Called regular/invertible/nonsingular, if it exists.
- If it exists, it is unique.
- (Q). How to compute?
- For 2×2 matrix,

$$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

- For a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, the $\boldsymbol{B} \in \mathbb{R}^{n \times m}$ with $b_{ij} = a_{ji}$ is the transpose of $\boldsymbol{A}$, which we denote by $\boldsymbol{A}^\mathsf{T}$.
- Example. For $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$,

$$\boldsymbol{A}^\mathsf{T} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

- If $\boldsymbol{A} = \boldsymbol{A}^\mathsf{T}$, $\boldsymbol{A}$ is called symmetric.

Inverse and Transpose: More Properties

- $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A}$
- $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- $(\boldsymbol{A}+\boldsymbol{B})^{-1} \neq \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$
- $(\boldsymbol{A}^{\mathsf{T}})^{\mathsf{T}} = \boldsymbol{A}$
- $(\boldsymbol{A}+\boldsymbol{B})^{\mathsf{T}} = \boldsymbol{A}^{\mathsf{T}} + \boldsymbol{B}^{\mathsf{T}}$
- $(\boldsymbol{A}\boldsymbol{B})^{\mathsf{T}} = \boldsymbol{B}^{\mathsf{T}}\boldsymbol{A}^{\mathsf{T}}$
- If $\boldsymbol{A}$ is invertible, so is $\boldsymbol{A}^{\mathsf{T}}$.

Scalar Multiplication

- Multiplication by a scalar $\lambda \in \mathbb{R}$ to $\boldsymbol{A} \in \mathbb{R}^{m \times n}$
- Example. For $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$, $3 \times \boldsymbol{A} = \begin{pmatrix} 0 & 6 \\ 3 & -3 \\ 0 & 3 \end{pmatrix}$
- Associativity
  - $(\lambda\psi)\boldsymbol{C} = \lambda(\psi\boldsymbol{C})$
  - $\lambda(\boldsymbol{BC}) = (\lambda\boldsymbol{B})\boldsymbol{C} = \boldsymbol{B}(\lambda\boldsymbol{C}) = (\boldsymbol{BC})\lambda$
  - $(\lambda C)^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{T}}\lambda^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{T}}\lambda = \lambda\boldsymbol{C}^{\mathsf{T}}$
- Distributivity
  - $(\lambda + \psi)\boldsymbol{C} = \lambda\boldsymbol{C} + \psi\boldsymbol{C}$
  - $\lambda(\boldsymbol{B} + \boldsymbol{C}) = \lambda\boldsymbol{B} + \lambda\boldsymbol{C}$

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Example

$$\begin{aligned} -3x \qquad\quad + 2z &= \quad -1 \\ x - 2y + 2z &= \; -5/3 \\ -x - 4y + 6z &= -13/3 \end{aligned}$$

- ρ_i: i-th equation
- Express the equation as its augmented matrix.

$$\left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 1 & -2 & 2 & -5/3 \\ -1 & -4 & 6 & -13/3 \end{array}\right) \xrightarrow[-(1/3)\rho_1+\rho_3]{(1/3)\rho_1+\rho_2} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & -4 & 16/3 & -4 \end{array}\right)$$

$$\xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The two nonzero rows give $-3x + 2z = -1$ and $-2y + (8/3)z = -2$.

- Parametrizing $-3x + 2z = -1$ and $-2y + (8/3)z = -2$ gives:

$$x = (1/3) + (2/3)z$$
$$y = 1 + (4/3)z$$
$$z = z$$

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 4/3 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R}\}$$

This helps us understand the set of solutions, e.g., each value of z gives a different solution.

$$\begin{array}{r|cccc} z & 0 & 1 & 2 & -1/2 \\ \hline \text{solution } \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 7/3 \\ 1 \end{pmatrix} & \begin{pmatrix} 5/3 \\ 11/3 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 1/3 \\ -1/2 \end{pmatrix} \end{array}$$

Form of solution sets

- The system $\begin{array}{rcl} x+2y-z\phantom{{}+w} &=& 2 \\ 2x-y-2z+w &=& 5 \end{array}$ reduces in this way.

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 2 & -1 & -2 & 1 & 5 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & -5 & 0 & 1 & 1 \end{array}\right)$$

- It has solutions of this form.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \qquad \text{for } z, w \in \Re$$

- Note that taking $z = w = 0$ shows that the first vector is a particular solution of the system.

General = Particular + Homogeneous

- General approach
  1. Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$
  2. Find all solutions to the homogeneous equation $\boldsymbol{Ax} = 0$
     - 0 is a trivial solution
  3. Combine the solutions from steps 1. and 2. to the general solution
- Questions: A formal algorithm that performs the above?
  - Gauss-Jordan method: convert into a "beautiful" form (formally reduced row-echelon form)
  - Elementary transformations: (i) row swapping (ii) multiply by a constant (iii) row addition
- Such a form allows an algorithmic way of solving linear equations

Example: Unique Solution

- Start as usual by getting echelon form.

$$\begin{array}{rcl} x + y - z &=& 2 \\ 2x - y \phantom{{}+2z} &=& -1 \\ x - 2y + 2z &=& -1 \end{array} \xrightarrow[-1\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcl} x + y - z &=& 2 \\ -3y + 2z &=& -5 \\ -3y + 3z &=& -3 \end{array} \xrightarrow{-1\rho_2+\rho_3} \begin{array}{rcl} x + y - z &=& 2 \\ -3y + 2z &=& -5 \\ z &=& 2 \end{array}$$

- Make all the leading entries one.

$$\xrightarrow{(-1/3)\rho_2} \begin{array}{rcl} x + y - z &=& 2 \\ y - (2/3)z &=& 5/3 \\ z &=& 2 \end{array}$$

- Finish by using the leading entries to eliminate upwards, until we can read off the solution.

$$\begin{array}{rcl} x + y - z &=& 2 \\ y - (2/3)z &=& 5/3 \\ z &=& 2 \end{array} \xrightarrow[(2/3)\rho_3+\rho_2]{\rho_3+\rho_1} \begin{array}{rcl} x + y &=& 4 \\ y &=& 3 \\ z &=& 2 \end{array} \xrightarrow{-\rho_2+\rho_1} \begin{array}{rcl} x &=& 1 \\ y &=& 3 \\ z &=& 2 \end{array}$$

Example: Infinite Number of Solutions

$$\begin{aligned} x - y \qquad - 2w &= 2 \\ x + y + 3z + w &= 1 \\ -y + z - w &= 0 \end{aligned}$$

- Start by getting echelon form and turn the leading entries to 1's.

$$\xrightarrow{-1\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & -1 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & -1 \\ 0 & 0 & 5/2 & 1/2 & -1/2 \end{array}\right)$$

$$\underset{(2/5)\rho_3}{\xrightarrow{(1/2)\rho_2}} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 3/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

- Eliminate upwards.

$$\xrightarrow{-(3/2)\rho_3+\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

$$\xrightarrow{\rho_2+\rho_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

- The parameterized solution set is:

$$\left\{ \begin{pmatrix} 9/5 \\ -1/5 \\ -1/5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4/5 \\ -6/5 \\ -1/5 \\ 1 \end{pmatrix} w \;\middle|\; w \in \mathbb{R} \right\}$$

Cases of Solution Sets

particular solution exists?		*number of solutions of the homogeneous system*: *one*	*number of solutions of the homogeneous system*: *infinitely many*
	yes	unique solution	infinitely many solutions
	no	no solutions	no solutions

Algorithms for Solving System of Linear Equations

1. Pseudo-inverse

   $$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Longleftrightarrow \boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^\top \boldsymbol{b} \Longleftrightarrow \boldsymbol{x} = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{b}$$

   - $(\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top$: *Moore-Penrose pseudo-inverse*
   - many computations: matrix product, inverse, etc
2. Gaussian elimination
   - intuitive and constructive way
   - cubic complexity (in terms of $\#$ of simultaneous equations)
3. Iterative methods
   - practical ways to solve indirectly

   (a) stationary iterative methods: Richardson method, Jacobi method, Gaus-Seidel method, successive over-relaxation method

   (b) Krylov subspace methods: conjugate gradients, generalized minimal residual, biconjugate gradients

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Group

- A set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$. $G := (\mathcal{G}, \otimes)$ is called a group, if:
    1. Closure. $\forall x, y \in \mathcal{G}$, $x \otimes y \in \mathcal{G}$
    2. Associativity. $\forall x, y, z \in \mathcal{G}$, $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
    3. Neutral element. $\exists e \in \mathcal{G}$, $\forall x \in \mathcal{G}$, $x \otimes e = x$ and $e \otimes x = x$
    4. Inverse element. $\forall x \in \mathcal{G}$, $\exists y \in \mathcal{G}$, $x \otimes y = e$ and $y \otimes x = e$. We often use $x^{-1} = y$.

- $G = (\mathcal{G}, \otimes)$ is an Abelian group, if the following is additionally met:
    - Communicativity. $\forall x, y \in \mathcal{G}$, $x \otimes y = y \otimes x$

Examples

- $(\mathbb{Z}, +)$ is an Abelian group
- $(\mathbb{N} \cup \{0\}, +)$ is not a group (because inverses are missing)
- $(\mathbb{Z}, \cdot)$ is not a group
- $(\mathbb{R}, \cdot)$ is not a group (because of no inverse for 0)
- $(\mathbb{R}^n, +)$, $(\mathbb{Z}^n, +)$ are Abelian, if $+$ is defined componentwise
- $(\mathbb{R}^{m \times n}, +)$ is Abelian (with componentwise $+$)
- $(\mathbb{R}^{n \times n}, \cdot)$
  - Closure and associativity follow directly
  - Neutral element: $\boldsymbol{I}_n$
  - The inverse $\boldsymbol{A}^{-1}$ may exist or not. So, generally, it is not a group. However, the set of invertible matrices in $\mathbb{R}^{n \times n}$ with matrix multiplication is a group, called general linear group.

Vector Spaces

Definition. A real-valued vector space $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

$$+ : \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V} \quad \text{(vector addition)}$$
$$\cdot : \mathbb{R} \times \mathcal{V} \mapsto \mathcal{V} \quad \text{(scalar multiplication)},$$

where

1. $(\mathcal{V}, +)$ is an Abelian group
2. Distributivity.
   - $\forall \lambda \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V},\ \lambda \cdot (\boldsymbol{x} + \boldsymbol{y}) = \lambda \cdot \boldsymbol{x} + \lambda \boldsymbol{y}$
   - $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V},\ (\lambda + \psi) \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x} + \psi \cdot \boldsymbol{x}$
3. Associativity. $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \boldsymbol{\mathcal{V}},\ \lambda \cdot (\psi \cdot \boldsymbol{x}) = (\lambda \psi) \cdot \boldsymbol{x}$
4. Neutral element. $\forall \boldsymbol{x} \in \mathcal{V},\ 1 \cdot \boldsymbol{x} = \boldsymbol{x}$

Example

- $\mathcal{V} = \mathbb{R}^n$ with
  - Vector addition: $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \dots, x_n + y_n)$
  - Scalar multiplication: $\lambda \boldsymbol{x} = (\lambda x_1, \dots, \lambda x_n)$
- $\mathcal{V} = \mathbb{R}^{m \times n}$ with
  - Vector addition: $\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$
  - Scalar multiplication: $\lambda \boldsymbol{A} = \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}$

Vector Subspaces

Definition. Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and $\mathcal{U} \subset \mathcal{V}$. Then, $U = (\mathcal{U}, +, \cdot)$ is called vector subspace (simply linear subspace or subspace) of V if U is a vector space with two operations '$+$' and '$\cdot$' restricted to $\mathcal{U} \times \mathcal{U}$ and $\mathbb{R} \times \mathcal{U}$.

Examples

- For every vector space V, V and $\{0\}$ are the trivial subspaces.
- The solution set of $\boldsymbol{Ax} = 0$ is the subspace of $\mathbb{R}^n$.
- The solution of $\boldsymbol{Ax} = \boldsymbol{b}$ ($\boldsymbol{b} \neq 0$) is not a subspace of $\mathbb{R}^n$.
- The intersection of arbitrarily many subspaces is a subspace itself.

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Linear Independence

- Definition. For a vector space V and vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in V$, every $\boldsymbol{v} \in V$ of the form $\boldsymbol{v} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k$ with $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ is a linear combination of the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in V$.
- Definition. If there is a non-trivial linear combination such that $0 = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ with at least one $\lambda_i \neq 0$, the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ are linearly dependent. If only the trivial solution exists, i.e., $\lambda_1 = \ldots = \lambda_k = 0$, $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ are linearly independent.
- Meaning. A set of linearly independent vectors consists of vectors that have no redundancy.
- Useful fact. The vectors $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$ are linearly dependent, iff (at least) one of them is a linear combination of the others.
  - $x - 2y = 2$ and $2x - 4y = 4$ are linearly dependent.

Checking Linear Independence

- Gauss elimination to get the row echelon form
- All column vectors are linearly independent iff all columns are pivot columns (why?).
- Example.

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix} \quad \rightsquigarrow \cdots \rightsquigarrow \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- Every column is a pivot column. Thus, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are linearly independent.

Linear Combinations of Linearly Independent Vectors

- Vector space V with k linearly independent vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k$
- m linear combinations $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$. (Q) Are they linearly independent?

$$\begin{aligned} \boldsymbol{x}_1 &= \lambda_{11}\boldsymbol{b}_1 + \lambda_{21}\boldsymbol{b}_2 + \cdots + \lambda_{k1}\boldsymbol{b}_k \\ &\vdots \\ \boldsymbol{x}_m &= \lambda_{1m}\boldsymbol{b}_1 + \lambda_{2m}\boldsymbol{b}_2 + \cdots + \lambda_{km}\boldsymbol{b}_k \end{aligned}$$

$$\boldsymbol{x}_j = \overbrace{\begin{pmatrix}\boldsymbol{b}_1, & \cdots, & \boldsymbol{b}_k\end{pmatrix}}^{\boldsymbol{B}} \overbrace{\begin{pmatrix}\lambda_{1j} \\ \vdots \\ \lambda_{kj}\end{pmatrix}}^{\boldsymbol{\lambda}_j}, \quad \boldsymbol{x}_j = \boldsymbol{B}\boldsymbol{\lambda}_j$$

- $\sum_{j=1}^m \psi_j \boldsymbol{x}_j = \sum_{j=1}^m \psi_j \boldsymbol{B}\boldsymbol{\lambda}_j = \boldsymbol{B}\sum_{j=1}^m \psi_j \boldsymbol{\lambda}_j$
- $\{\boldsymbol{x}\}$ linearly independent $\Longleftrightarrow$ $\{\boldsymbol{\lambda}\}$ linearly independent

Example

$$\begin{aligned}
\boldsymbol{x}_1 &= \boldsymbol{b}_1 - 2\boldsymbol{b}_2 + \boldsymbol{b}_3 - \boldsymbol{b}_4 \\
\boldsymbol{x}_2 &= -4\boldsymbol{b}_1 - 2\boldsymbol{b}_2 + 4\boldsymbol{b}_4 \\
\boldsymbol{x}_3 &= 2\boldsymbol{b}_1 + 3\boldsymbol{b}_2 - \boldsymbol{b}_3 - 3\boldsymbol{b}_4 \\
\boldsymbol{x}_4 &= 17\boldsymbol{b}_1 - 10\boldsymbol{b}_2 + 11\boldsymbol{b}_3 + \boldsymbol{b}_4
\end{aligned}$$

$$\boldsymbol{A} = (\boldsymbol{\lambda}_1 \ \boldsymbol{\lambda}_2 \ \boldsymbol{\lambda}_3 \ \boldsymbol{\lambda}_4) = \begin{pmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & -4 & -3 & 1 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The last column is not a pivot column. Thus, $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_3$ are linearly dependent.

Roadmap

(1) Systems of Linear Equations

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(3) Solving Systems of Linear Equations

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(6) Basis and Rank

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Generating Set and Basis

- Definition. A vector space $V = (\mathcal{V}, +, \cdot)$ and a set of vectors $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathcal{V}$.
  - If every $v \in \mathcal{V}$ can be expressed as a linear combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$, $\mathcal{A}$ is called a generating set of V.
  - The set of all linear combinations of $\mathcal{A}$ is called the span of $\mathcal{A}$.
  - If $\mathcal{A}$ spans the vector space V, we use $V = \text{span}[\mathcal{A}]$ or $V = \text{span}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k]$
- Definition. The minimal generating set $\mathcal{B}$ of V is called basis of V. We call each element of $\mathcal{B}$ basis vector. The number of basis vectors is called dimension of V.
- Properties
  - $\mathcal{B}$ is a maximally[2] linearly independent set of vectors in V.
  - Every vector $x \in V$ is a linear combination of $\mathcal{B}$, which is unique.

[2] Adding any other vector to this set will make it linearly dependent.

Examples

- Different bases $\mathbb{R}^3$

$$\mathcal{B}_1 = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}, \mathcal{B}_2 = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \},$$

$$\mathcal{B}_3 = \{ \begin{pmatrix} 0.5 \\ 0.8 \\ 0.4 \end{pmatrix}, \begin{pmatrix} 1.8 \\ 0.3 \\ 0.3 \end{pmatrix}, \begin{pmatrix} -2.2 \\ -1.3 \\ 3.5 \end{pmatrix} \}$$

- Linearly independent, but not maximal. Thus, not a basis.

$$\mathcal{A} = \{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -4 \end{pmatrix} \}$$

Determining a Basis

- Want to find a basis of a subspace $U = \text{span}[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m]$
  1. Construct a matrix $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_m \end{pmatrix}$
  2. Find the row-echelon form of $\boldsymbol{A}$.
  3. Collect the pivot columns.
- Logic: Collect $\boldsymbol{x}_i$ so that we have only trivial solution. Pivot columns tell us which set of vectors is linearly independent.
- See example 2.17 (pp. 35)

Rank (1)

- Definition. The rank of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ denoted by $\text{rk}(\boldsymbol{A})$ is $\#$ of linearly independent columns
  - Same as the number of linearly independent rows

- $$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

  Thus, $\text{rk}(\boldsymbol{A}) = 2$.

- $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}^{\mathsf{T}})$

Rank (2)

- The columns (resp. rows) of $\boldsymbol{A}$ span a subspace U (resp. W) with $\dim(U) = \text{rk}(\boldsymbol{A})$ (resp. $\dim(W) = \text{rk}(\boldsymbol{A})$), and a basis of U (resp. W) can be found by Gauss elimination of $\boldsymbol{A}$ (resp. $\boldsymbol{A}^\mathsf{T}$).
- For all $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\text{rk}(\boldsymbol{A}) = n$, iff $\boldsymbol{A}$ is regular (invertible).
- The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is solvable, iff $\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}|\boldsymbol{b})$.
- For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, the subspace of solutions for $\boldsymbol{Ax} = 0$ possesses dimension $n - \text{rk}(\boldsymbol{A})$.
- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible rank for a matrix of the same dimensions. The rank of the full-rank matrix $\boldsymbol{A}$ is min(# of cols, # of rows).

Roadmap

(1) Systems of Linear Equations

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Linear Mapping (1)

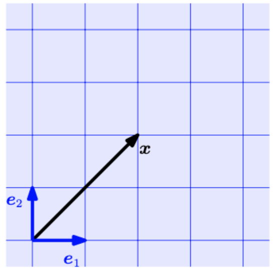
- Interest: A mapping that preserves the structure of the vector space
- Definition. For vector spaces V, W, a mapping $\Phi : V \mapsto W$ is called a linear mapping (or homomorphism/linear transformation), if, for all $\boldsymbol{x}, \boldsymbol{y} \in V$ and all $\lambda \in \mathbb{R}$,
  - $\Phi(\boldsymbol{x} + \boldsymbol{y}) = \Phi(\boldsymbol{x}) + \Phi(\boldsymbol{y})$
  - $\Phi(\lambda \boldsymbol{x}) = \lambda \Phi(\boldsymbol{x})$
- Definition. A mapping $\Phi : \mathcal{V} \mapsto \mathcal{W}$ is called
  - Injective (단사), if $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, $\Phi(\boldsymbol{x}) = \Phi(\boldsymbol{y}) \implies \boldsymbol{x} = \boldsymbol{y}$
  - Surjective (전사), if $\Phi(\mathcal{V}) = \mathcal{W}$
  - Bijective (전단사), if it is injenctive and surjective.

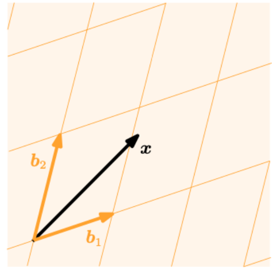
Linear Mapping (2)

- For bjective mapping, there exists an inverse mapping Φ^{-1}.
- Isomorphism if Ψ is linear and bijective.
- Theorem. Vector spaces V and W are isomorphic, iff $\dim(V) = \dim(W)$.
  - Vector spaces of the same dimension are kind of the same thing.
- Other properties
  - For two linear mappings Φ and Ψ, $\Phi \circ \Psi$ is also a linear mapping.
  - If Φ is an isomorphism, so is Φ^{-1}.
  - For two linear mappings Φ and Ψ, $\Phi + \Psi$ and $\lambda\Psi$ for $\lambda \in \mathbb{R}$ are linear.

Coordinates

- A basis defines a coordinate system.





- Consider an ordered basis $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n)$ of vector space V. Then, for any $\boldsymbol{x} \in V$, there exists a unique linear combination

$$\boldsymbol{x} = \alpha_1 \boldsymbol{b}_1 + \ldots + \alpha_n \boldsymbol{b}_n.$$

- We call $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ the coordinate of $\boldsymbol{x}$ with respect to $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n)$.

- Consider a vector space V and two coordinate systems defined by $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ and $B' = (\boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_n)$.
- Question. For $(x_1, \ldots, x_n)_B \to (y_1, \ldots, y_n)_{B'}$, what is $(y_1, \ldots, y_n)_{B'}$?
- Theorem. $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}'_1 & \ldots & \boldsymbol{b}'_n \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{b}_1 & \ldots & \boldsymbol{b}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
- Regard $\boldsymbol{A}_\Phi = \begin{pmatrix} \boldsymbol{b}'_1 & \ldots & \boldsymbol{b}'_n \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{b}_1 & \ldots & \boldsymbol{b}_n \end{pmatrix}$ as a linear map

Example

- $B = ((1,0),(0,1)$ and $B' = ((2,1),(1,2))$
- $(4,2)_B \to (x,y)_{B'}$?
- Using $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}'_1 & \dots & \boldsymbol{b}'_n \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{b}_1 & \dots & \boldsymbol{b}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Transformation Matrix

- Two vector spaces
  - V with basis $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ and W with basis $C = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$
- What is the coordinate in C-system for each basis $\boldsymbol{b}_j$? For $j = 1, \ldots, n$,

$$\boldsymbol{b}_j = \alpha_{1j}\boldsymbol{c}_1 + \cdots + \alpha_{mj}\boldsymbol{c}_m \Longleftrightarrow \boldsymbol{b}_j = \begin{pmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{pmatrix} \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{pmatrix} \overbrace{\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}}^{\boldsymbol{A}_\Phi}$$

- $\hat{x} = \boldsymbol{A}_\Phi \hat{y}$, where $\hat{x}$ is the vector w.r.t B and $\hat{y}$ is the vector w.r.t. C

KAIST EE

- For linear mapping $\Phi : V \mapsto W$, consider bases B, B' of V and C, C' of W

$$B = \begin{pmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{pmatrix}, \; B' = \begin{pmatrix} \boldsymbol{b}'_1 & \cdots & \boldsymbol{b}'_n \end{pmatrix} \quad C = \begin{pmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{pmatrix}, \; C' = \begin{pmatrix} \boldsymbol{c}'_1 & \cdots & \boldsymbol{c}'_m \end{pmatrix}.$$

- (inter) transformation matrices $\boldsymbol{A}_\Phi$ from B to C and $\boldsymbol{A}'_\Phi$ from B' to C'
- (intra) transformation matrices S from B' to B and T from C' to C
- Theorem. $\boldsymbol{A}'_\Phi = T^{-1}\boldsymbol{A}_\Phi S$

$$\begin{tikzcd}
V \arrow[r, "\Phi"] & W \\
B \arrow[r, "\Phi_{CB}", "\boldsymbol{A}_\Phi"'] & C \\
\tilde{B} \arrow[u, "\Psi_{B\tilde{B}}", "\boldsymbol{S}"'] \arrow[r, "\tilde{\boldsymbol{A}}_\Phi", "\Phi_{\tilde{C}\tilde{B}}"'] & \tilde{C} \arrow[u, "\boldsymbol{T}", "\Xi_{C\tilde{C}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\Phi"] & W \\
B \arrow[r, "\Phi_{CB}", "\boldsymbol{A}_\Phi"'] & C \arrow[d, "\boldsymbol{T}^{-1}"', "\Xi_{\tilde{C}C} = \Xi^{-1}_{C\tilde{C}}"] \\
\tilde{B} \arrow[u, "\Psi_{B\tilde{B}}", "\boldsymbol{S}"'] \arrow[r, "\tilde{\boldsymbol{A}}_\Phi", "\Phi_{\tilde{C}\tilde{B}}"'] & \tilde{C}
\end{tikzcd}$$

Image and Kernel

- Consider a linear mapping $\Phi : V \mapsto W$. The kernel (or null space) is the set of vectors in V that maps to $0 \in W$ (i.e., neutral element).

  Definition. $\ker(\Phi) := \Phi^{-1}(0_W) = \{\boldsymbol{v} \in V : \Phi(\boldsymbol{v}) = 0_W\}$

- Image/range: set of vectors $w \in W$ that can be reached by Φ from any vector in V
- V: domain, W: codomain

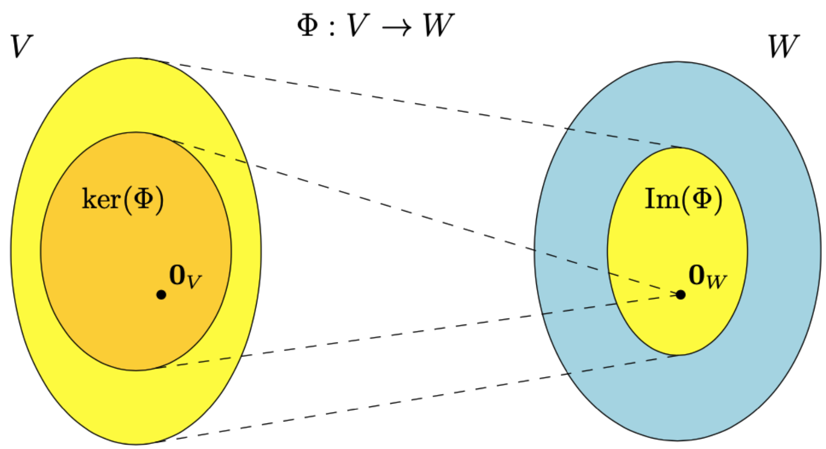


Image and Kernel: Properties

- $0_V \in \ker(\Phi)$ (because $\Phi(0_V) = 0_W$)
- Both $\mathrm{Im}(\Phi)$ and $\ker(\Phi)$ are subspaces of W and V, respectively.
- Φ is one-to-one (injective) $\iff \ker(\Phi) = \{0\}$ (i.e., only 0 is mapped to 0)
- Since Φ is a linear mapping, there exists $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ such that $\Phi : \boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$. Then, $\mathrm{Im}(\Phi) =$ column space of $\boldsymbol{A}$ which is the span of column vectors of $\boldsymbol{A}$.
- $\mathrm{rk}(\boldsymbol{A}) = \dim(\mathrm{Im}(\Phi))$
- $\ker(\Phi)$ is the solution set of the homogeneous system of linear equations $\boldsymbol{A}\boldsymbol{x} = 0$

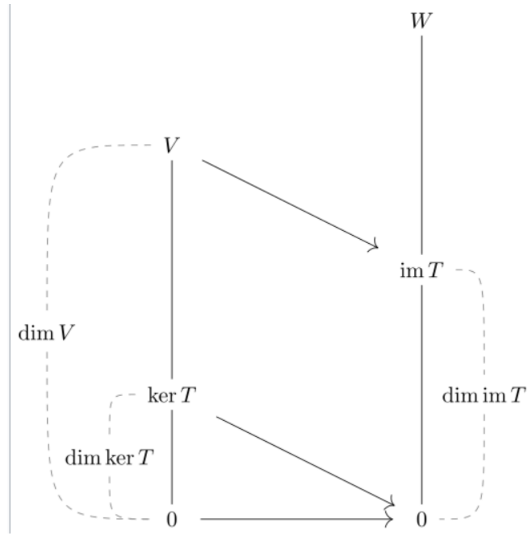
Rank-Nullity Theorem

Theorem.

$$\dim(\ker(\Phi)) + \dim(\mathrm{Im}(\Phi)) = \dim(V)$$

- If $\dim(\mathrm{Im}(\Phi)) < \dim(V)$, the kernel contains more than just 0.
- If $\dim(\mathrm{Im}(\Phi)) < \dim(V)$, $\boldsymbol{A}_\Phi \boldsymbol{x} = 0$ has infinitely many solutions.
- If $\dim(V) = \dim(W)$ (e.g., $V = W = \mathbb{R}^n$), the followings are equivalent: Φ is
  - (1) injective, (2) surjective, (3) bijective,
  - In this case, Φ defines $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A}$ is regular.
- Simplified version. For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$,

$$\mathrm{rk}(\boldsymbol{A}) + \mathrm{nullity}(\boldsymbol{A}) = n$$



[2] Nullity: the dimension of null space (kernel)

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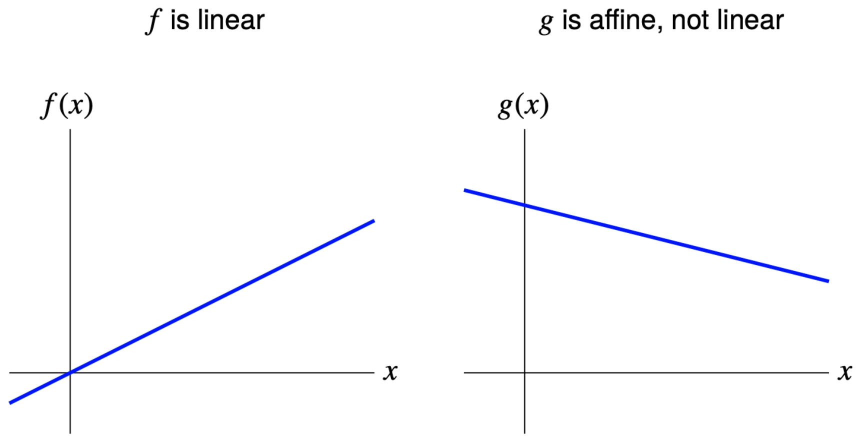
Linear vs. Affine Function

- linear function: $f(x) = ax$
- affine function: $f(x) = ax + b$
- sometimes (ignorant) people refer to affine functions as linear



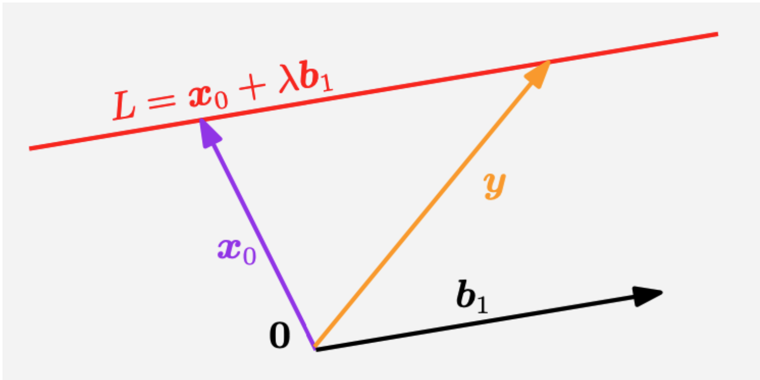
Affine Subspace

- Spaces that are offset from the origin. Not a vector space.
- Definition. Consider a vector space V, $\boldsymbol{x}_0 \in V$ and a subspace $U \subset V$. Then, the subset $L = \boldsymbol{x}_0 + U := \{\boldsymbol{x}_0 + \boldsymbol{u} : \boldsymbol{u} \in U\}$ is called affine subspace or linear manifold of V.
- U is called direction or direction space, and $\boldsymbol{x}_0$ is support. point.
- An affine subspace is not a vector subspace of V for $\boldsymbol{x}_0 \notin U$.
- **Parametric equation.** A k-dimensional affine space $L = \boldsymbol{x}_0 + U$. If $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k)$ is an ordered basis of U, any element $\boldsymbol{x} \in L$ can be uniquely described as

$$\boldsymbol{x} = \boldsymbol{x}_0 + \lambda_1 \boldsymbol{b}_1 + \cdots + \lambda_k \boldsymbol{b}_k, \quad \lambda_1, \ldots, \lambda_k \in \mathbb{R}$$

Example

- In $\mathbb{R}^2$, one-dimensional affine subspace: line. $\boldsymbol{y} = \boldsymbol{x}_0 + \lambda \boldsymbol{b}_1$. $U = \text{span}[\boldsymbol{b}_1]$
- In $\mathbb{R}^3$, two-dimensional affine subspace: plane. $\boldsymbol{y} = \boldsymbol{x}_0 + \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2$. $U = \text{span}[\boldsymbol{b}_1, \boldsymbol{b}_2]$
- In $\mathbb{R}^n$, $(n-1)$-dimensional affine subspace: hyperplane. $\boldsymbol{y} = \boldsymbol{x}_0 + \sum_{k=1}^{n-1} \lambda_i \boldsymbol{b}_i$. $U = \text{span}[\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n]$



- For a linear mapping $\Phi : V \mapsto W$ and a vector $\boldsymbol{a} \in W$, the mapping $\phi : V \mapsto W$ with $\phi(\boldsymbol{x}) = \boldsymbol{a} + \Phi(\boldsymbol{x})$ is an affine mapping from V to W. The vector $\boldsymbol{a}$ is called the translation vector.

Questions?

Review Questions

1)