Lecture 7: Optimization

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Mathematics for Machine Learning
https://yung-web.github.io/home/courses/mathml.html
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(1) Optimization Using Gradient Descent

(2) Constrained Optimization and Lagrange Multipliers

(3) Convex Sets and Functions

(4) Convex Optimization

(5) Convex Conjugate

- Training machine learning models = finding a good set of parameters

- A good set of parameters = Solution (or close to solution) to some optimization problem

- Directions: Unconstrained optimization, Constrained optimization, Convex optimization

- High-school math: A necessary condition for the optimal point: $f'(x) = 0$ (stationary point)
  - Gradient will play an important role

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- Goal
$$\min f(\boldsymbol{x}), \quad f(\boldsymbol{x}) : \mathbb{R}^n \mapsto \mathbb{R}, \quad f \in C^1$$

- Graident-type algorithms
$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \gamma_k \boldsymbol{d}_k, \quad k = 0, 1, 2, \dots$$

- Lemma. Any direction $\boldsymbol{d} \in \mathbb{R}^{n \times 1}$ that satisfies $\nabla f(\boldsymbol{x}) \cdot \boldsymbol{d} < 0$ is a descent direction of f at $\boldsymbol{x}$. That is, if we let $\boldsymbol{x}_\alpha = \boldsymbol{x} + \alpha \boldsymbol{d}$, $\exists \bar{\alpha} > 0$, such that for all $\alpha \in (0, \bar{\alpha}]$, $f(\boldsymbol{x}_\alpha) < f(\boldsymbol{x})$.

- Steepest gradient descent[1]. $\boldsymbol{d}_k = -\nabla f(\boldsymbol{x}_k)^\mathsf{T}$.

- Finding a local optimum $f(\boldsymbol{x}_\star)$, if the step-size γ_k is suitably chosen.

- Question. How do we choose $\boldsymbol{d}_k$ for a constrained optimization?

[1] In some cases, just gradient descent often means this steepest gradient descent.

- A quadratic function $f : \mathbb{R}^2 \mapsto \mathbb{R}$.
$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^\mathsf{T} \begin{pmatrix} 2 & 1 \\ 1 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix}^\mathsf{T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
whose gradient is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^\mathsf{T} \begin{pmatrix} 2 & 1 \\ 1 & 20 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix}^\mathsf{T}$

- $\boldsymbol{x}_0 = (-3 \ -1)^\mathsf{T}$

- constant step size $\alpha = 0.085$

- Zigzag pattern
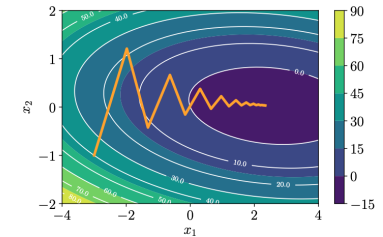
- Goal: $\min L(\boldsymbol{\theta})$ for n training data

- Based on the amount of training data used for each iteration
  - Batch gradient descent (the entire n)
  - Mini-batch gradient descent($k < n$ data)
  - Stochastic gradient descent (one sampled data)

- Based on the adaptive method of update
  - Momentum, NAG, Adagrad, RMSprop, Adam, etc

- https://ruder.io/optimizing-gradient-descent/

- Assume $L(\boldsymbol{\theta}) = \sum_{i=1}^{n} L_n(\boldsymbol{\theta})$ (which happens in many cases in machine learning, e.g., negative log-likelihood in regression)

- Gradient update
$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k - \gamma_k \nabla L(\boldsymbol{\theta}_k)^\mathsf{T} = \boldsymbol{\theta}_k - \gamma_k \sum_{n=1}^{N} \nabla L_n(\boldsymbol{\theta}_k)^\mathsf{T}$$

  - Batch gradient: $\sum_{n=1}^{N} \nabla L_n(\boldsymbol{\theta}_k)^\mathsf{T}$
  - Mini-batch gradient: $\sum_{n \in \mathcal{K}} \nabla L_n(\boldsymbol{\theta}_k)^\mathsf{T}$ for a suitable choice of $\mathcal{K}, |\mathcal{K}| < n$
  - Stochastic gradient: $\nabla L_n(\boldsymbol{\theta}_i)^\mathsf{T}$ for some (randomly chosen) i. Noisy approximation to the real gradient.

- Tradeoff: computation burden vs. exactness

- Step size.
  - Too small: slow update, Too big: overshoot, zig-zag, often fail to converge

- Adaptive update: smooth out the erratic behavior and dampens oscillations

- Gradient descent with momentum

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \gamma_i \nabla f(\boldsymbol{x}_k)^{\mathsf{T}} + \alpha \Delta \boldsymbol{x}_k, \quad \alpha \in [0,1]$$

$$\Delta \boldsymbol{x}_k = \boldsymbol{x}_k - \boldsymbol{x}_{k-1}$$

  - Memory term: $\alpha \Delta \boldsymbol{x}_k$, where α is the degree of how much we remember the past
  - Next update = a linear combination of current and previous updates

- An optimization problem in standard form:

  minimize $f(\boldsymbol{x})$

  subject to $g_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \ldots, m$ (Inequality constraints)

  $\qquad\qquad h_j(\boldsymbol{x}) = 0, \quad j = 1, 2, \ldots, p$ (Equality constraints)

- Variables: $\boldsymbol{x} \in \mathbb{R}^n$. Assume nonempty feasible set

- Optimal value: p^*. Optimizer: $\boldsymbol{x}^*$

- Duality Mentality
  - Bound or solve an optimization problem via a different optimization problem!
  - We'll develop the basic Lagrange duality theory for a general optimization problem, then specialize for convex optimization

- Idea: augment the objective with a weighted sum of constraints
  - Lagrangian:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x}) + \sum_{i=1}^{p} \nu_i h_i(\boldsymbol{x})$$

  - Lagrange multipliers (dual variables): $\boldsymbol{\lambda} = (\lambda_i : i = 1, \cdots, m) \succeq 0, \ \boldsymbol{\nu} = (\nu_1, \cdots, \nu_p)$

  - Lagrange dual function:

$$\mathcal{D}(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \inf_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu})$$

- The dual function $\mathcal{D}(\boldsymbol{\lambda}, \boldsymbol{\nu})$ is the lower bound on the optimal value p^*.

- Theorem. $\mathcal{D}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \leq p^*, \ \ \forall \boldsymbol{\lambda} \succeq 0, \ \boldsymbol{\nu}$

- Proof. Consider feasible $\tilde{\boldsymbol{x}}$. Then,

$$\mathcal{L}(\tilde{\boldsymbol{x}}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f(\tilde{\boldsymbol{x}}) + \sum_{i=1}^{m} \lambda_i g_i(\tilde{\boldsymbol{x}}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{\boldsymbol{x}}) \leq f(\tilde{\boldsymbol{x}})$$

since $f_i(\tilde{\boldsymbol{x}}) \leq 0$ and $\lambda_i \geq 0$.
Hence, $\mathcal{D}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \leq \mathcal{L}(\tilde{\boldsymbol{x}}, \boldsymbol{\lambda}, \boldsymbol{\nu}) \leq f(\tilde{\boldsymbol{x}})$ for all feasible $\tilde{\boldsymbol{x}}$. Therefore, $\mathcal{D}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \leq p^*$.

- Lower bound from Lagrange dual function depends on $(\boldsymbol{\lambda}, \boldsymbol{\nu})$.

- Question. What's the best lower bound?

  Langrangian dual problem   maximize $\mathcal{D}(\boldsymbol{\lambda}, \boldsymbol{\nu})$
       subject to $\boldsymbol{\lambda} \succeq 0$

- Dual variables: $(\boldsymbol{\lambda}, \boldsymbol{\nu})$

- Always a convex optimization, because $\mathcal{D}(\boldsymbol{\lambda}, \boldsymbol{\nu})$ is always concave over $\boldsymbol{\lambda}, \boldsymbol{\nu}$.
  - Infimum over $\boldsymbol{x}$ of a family of affine functions in $(\boldsymbol{\lambda}, \boldsymbol{\nu})$ (we will see this later)

- Denote the optimal value of Lagrange dual problem by d^*.

- What's the relationship between d^* and p^*?

Weak Duality

$d^* \leq p^*$

- Weak duality always hold (even if the primal problem is not convex):

- Optimal duality gap: $p^* - d^*$

- Efficient generation of the lower bounds through the dual problem

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- Convex optimization problem

  minimize $f(\boldsymbol{x})$

  subject to $\boldsymbol{x} \in \mathcal{X}$,

  where $f(\boldsymbol{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function, and $\mathcal{X}$ is a convex set.

- The watershed between easily solvable problem and intractable ones is not 'linearity', but 'convexity'

- Let's overview the background of convex functions, convex sets, and their basic properties.

- Set $\mathcal{C}$ is a convex set if the line segment between any two points in $\mathcal{C}$ lies in $\mathcal{C}$, i.e., if for any $x_1, x_2 \in \mathcal{C}$ and any $\theta \in [0, 1]$, we have $\theta x_1 + (1 - \theta)x_2 \in \mathcal{C}$

- Convex hull of $\mathcal{C}$ is the set of all convex combinations of points in $\mathcal{C}$:

$$\left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in \mathcal{C}, \theta_i \geq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} \theta_i = 1 \right\}$$

  - What is k? For all k? For some k?

- Generalize to infinite sums and integrals:

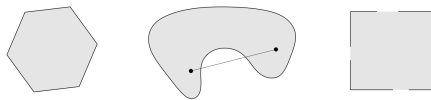$$\sum_{i=1}^{\infty} \theta_i x_i \in \mathcal{C}, \quad \int_{\mathcal{C}} p(x)x dx \in \mathcal{C},$$

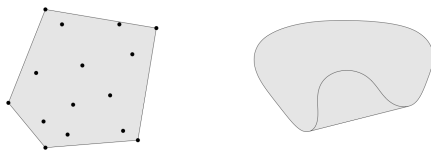where $\sum_{i=1}^{\infty} \theta_i = 1$ and $p(x)$ is a pdf of some random variable.
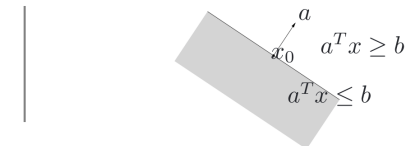
- Convex and Non-convex sets

- Convex hulls

- Hyperplane in $\mathbb{R}^n$ is a set: $\{x \mid a^\mathsf{T} x = b\}$ where $a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R}$
  In other words, $\{x \mid a^\mathsf{T}(x - x_0) = 0\}$, where x_0 is any point in the hyperplane, i.e., $a^\mathsf{T} x_0 = b$.

- Divides $\mathbb{R}^n$ into two halfspaces:
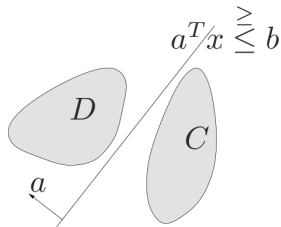 $\{x|a^\mathsf{T} x \leq b\}$ and $\{x|a^\mathsf{T} x > b\}$

- Polyhedron is the solution set of a finite number of linear equalities and inequalities (intersection of finite number of halfspaces and hyperplanes)
  $\mathcal{P} = \{x \mid a_j^\mathsf{T} x \leq b_j, j = 1, \ldots, m, c_j^\mathsf{T} x = d_j, j = 1, \ldots, p\} = \{x \mid Ax \leq b, Cx = d\}$
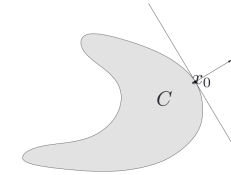
- Polytope: a bounded polyhedron

$$a^\mathsf{T} x \overset{\geq}{\underset{\leq}{}} b$$

- $\mathcal{C}$ and $\mathcal{D}$: non-intersecting convex sets, i.e., $\mathcal{C} \bigcap \mathcal{D} = \phi$.
- Then, there exist $a \neq 0$ and b such that $a^\mathsf{T} x \leq b$ for all $x \in \mathcal{C}$ and $a^\mathsf{T} x \geq b$ for all $x \in \mathcal{D}$.

- Given a set $\mathcal{C} \in \mathbb{R}^n$ and a point x_0 on its boundary, if $a \neq 0$ satisfies $a^\mathsf{T} x \leq a^\mathsf{T} x_0$ for all $x \in \mathcal{C}$, then $\{x | a^\mathsf{T} x = a^\mathsf{T} x_0\}$ is called a supporting hyperplane to $\mathcal{C}$ at x_0
- For any nonempty convex set $\mathcal{C}$ and any x_0 on boundary of $\mathcal{C}$, there exists a supporting hyperplane to $\mathcal{C}$ at x_0
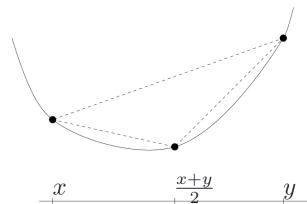- What happens if $\mathcal{C}$ is non-convex?

- $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function if dom f is a convex set and for all $x, y \in$ dom f and $\theta \in [0, 1]$, we have
$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$
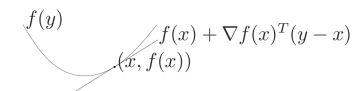- f is strictly convex if the strict inequality in the above holds for all $x \neq y$ and $0 < \theta < 1$.
  - f is concave if $-f$ is convex
  - Affine functions are convex and concave
  - Jensen's inequality. For a rv X, $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$.

- First-order condition. For differentiable functions, f is convex iff
$$f(y) - f(x) \geq \nabla f(x)^\mathsf{T}(y - x), \quad \forall x, y \in \text{dom } f, \text{ and dom } f \text{ is convex}$$

- Example. $f(y) = y^2$.
- $f(y) \geq \tilde{f}_x(y)$ where $\tilde{f}_x(y)$ is the first order Taylor expansion of $f(y)$ at x.
- Local information (first order Taylor approximation) about a convex function provides global information (global underestimator).
- If $\nabla f(x) = 0$, then $f(y) \geq f(x)$, $\forall y$. Thus, x is a global minimizer of f

- Second-order condition. For twice differentiable functions, f is convex iff
$$\nabla^2 f(x) \succeq 0$$
for all $x \in \text{dom } f$ (upward slope) and $\text{dom } f$ is convex
- Example: $f(x) = x^2$.
- Meaning: The graph of the function have positive (upward) curvature at x.

- e^{ax} is convex on $\mathbb{R}$, for any $a \in \mathbb{R}$
- x^a is convex on $\mathbb{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$
- $|x|^p$ is convex on $\mathbb{R}$ for $p \geq 1$
- $\log x$ is concave on $\mathbb{R}_{++}$
- $x \log x$ is strictly convex on $\mathbb{R}_{++}$
- Every norm on $\mathbb{R}^n$ is convex
- $f(x) = \max\{x_1, \ldots, x_n\}$ is convex on $\mathbb{R}^n$
- $f(x) = \log \sum_{i=1}^n e^{x_i}$ is convex on $\mathbb{R}^n$
- $f(x) = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$ is concave on $\mathbb{R}_{++}^n$

- $f = \sum_{i=1}^n w_i f_i$ convex if f_i are all convex and $w_i \geq 0$
- $g(x) = f(Ax + b)$ is convex iff $f(x)$ is convex
- $f(x) = \max\{f_1(x), f_2(x)\}$ convex if f_i convex, e.g., sum of r largest components is convex
- $f(x) = h(g(x))$, where $h : \mathbb{R}^k \mapsto \mathbb{R}$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^k$.

  If $k = 1$: $f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$. So, f is convex if h is convex and nondecreasing and g is convex, or if h is convex and nonincreasing and g is concave ...
- $g(x) = \inf_{y \in \mathcal{C}} f(x, y)$ is convex if f is convex in (x, y) and $\mathcal{C}$ is convex

- If $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then
$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$
is convex. Similarly, if $f(x, y)$ is concave in x for each $y \in \mathcal{A}$, then
$$g(x) = \inf_{y \in \mathcal{A}} f(x, y)$$
is concave.
- Example. distance to farthest point in a set $\mathcal{C}$: $f(x) = \sup_{y \in \mathcal{C}} ||x - y||$ is convex.
- Example. Lagrange dual function
$$\mathcal{D}(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \inf_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu})$$
is concave.

- A standard convex optimization problem with variables $\boldsymbol{x}$:

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) \\
\text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \ldots, m \\
& a_i^{\mathsf{T}} \boldsymbol{x} = b_i, \quad i = 1, 2, \ldots, p
\end{aligned}$$

where $f, f_1, \ldots, f_m$ are convex functions.

- Minimize convex objective function (or maximize concave objective function)

- Upper bound inequality constraints on convex functions ($\Rightarrow$ Constraint set is convex)

- Equality constraints must be affine (Only affine functions leads to a convex set for the equality constraints)

Properties for Optimality

Strong Duality

- Local optimality implies global optimality.
  - Given x is locally optimal for a convex optimization problem, i.e., $\boldsymbol{x}$ is feasible and for some $R > 0$,

  $$f(\boldsymbol{x}) = \inf\{f(\boldsymbol{z}) \mid \boldsymbol{z} \text{ is feasible}, \|\boldsymbol{z} - \boldsymbol{x}\|_2 \leq R\}$$

  - Theorem. if $\boldsymbol{x}$ is locally optimal in convex program, then globally optimal.
- Optimal condition for differentiable f
  - $\boldsymbol{x}$ is optimal for a convex optimization problem iff $\boldsymbol{x}$ is feasible and for all feasible $\boldsymbol{y}$:

  $$\nabla f(\boldsymbol{x})^{\mathsf{T}}(\boldsymbol{y} - \boldsymbol{x}) \geq 0$$

  - $-\nabla f(\boldsymbol{x})$ defines a supporting hyperplane to the feasible set ($\{\boldsymbol{y} \mid -\nabla f(\boldsymbol{x})^{\mathsf{T}} \boldsymbol{y} \leq -\nabla f(\boldsymbol{x})^{\mathsf{T}} \boldsymbol{x}\}$).
  - (Note) Unconstrained convex optimization: condition reduces to: $\nabla f(\boldsymbol{x}) = 0$

- Strong duality (zero optimal duality gap):

  $$d^* = p^*$$

- If strong duality holds, solving dual is 'equivalent' to solving primal. But strong duality does not always hold

- Convexity and constraint qualifications $\implies$ Strong duality

- A simple constraint qualification: Slater's condition (there exists strictly feasible primal variables $f_i(\boldsymbol{x}) < 0$ for non-affine f_i) (see Section 5.3.2 of Boyd's book).

- Another reason why convex optimization is 'easy'

Complementary Slackness

- Assume strong duality holds. Then,

$$f(\boldsymbol{x}^*) = \mathcal{D}(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*) = \inf_{\boldsymbol{x}} \left(f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i^* f_i(\boldsymbol{x}) + \sum_{i=1}^{p} \nu_i^* h_i(\boldsymbol{x}) \right)$$

$$\leq f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i^* f_i(\boldsymbol{x}^*) + \sum_{i=1}^{p} \nu_i^* h_i(\boldsymbol{x}^*) \leq f(\boldsymbol{x}^*)$$

- Thus, the two inequalities must hold with equality, implying: $\lambda_i^* f_i(\boldsymbol{x}^*) = 0, \ \forall i$
- Complementary slackness condition:

$$\lambda_i^* > 0 \implies f_i(\boldsymbol{x}^*) = 0$$
$$f_i(\boldsymbol{x}^*) < 0 \implies \lambda_i^* = 0$$

- i-th optimal Lagrange multiplier is zero unless the ith constraint is active at the optimum.

KKT Condition

- Since $\boldsymbol{x}^*$ minimizes $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ over $\boldsymbol{x}$,

$$\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\boldsymbol{x}^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(\boldsymbol{x}^*) = 0$$

Karush-Kuhn-Tucker optimality condition
$$f_i(\boldsymbol{x}^*) \leq 0, \ \ h_i(\boldsymbol{x}^*) = 0, \ \ \lambda_i^* \succeq 0$$
$$\lambda_i^* f_i(\boldsymbol{x}^*) = 0$$
$$\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\boldsymbol{x}^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(\boldsymbol{x}^*) = 0$$

- Any optimization with strong duality, KKT condition is necessary for primal-dual optimality
- Convex optimization with Slater's condition, KKT is also sufficient for primal-dual optimality.

Useful Tips

- Minimization problem (min-min-max rule)
  - Problem: min $f(x)$ s.t. $f_i(x) \leq 0$, $g_i(x) = 0$, x
  - $f(x)$: convex, $f_i(x)$: convex, $g_i(x)$: affine
  - $L(x, \lambda, \mu) = f(x) + \sum_i \lambda_i f_i(x) + \sum_i \mu_i g_i(x)$
  - $\inf_x L(x, \lambda, \mu) = \mathcal{D}(\lambda, \mu)$
  - $\max_{\lambda \geq 0} \mathcal{D}(\lambda, \mu)$
- Maximization problem (max-max-min rule)
  - Problem: max $f(x)$ s.t. $f_i(x) \geq 0$, $g_i(x) = 0$, x
  - $f(x)$: concave, $f_i(x)$: concave, $g_i(x)$: affine
  - $L(x, \lambda, \mu) = f(x) + \sum_i \lambda_i f_i(x) + \sum_i \mu_i g_i(x)$
  - $\sup_x L(x, \lambda, \mu) = \mathcal{D}(\lambda, \mu)$
  - $\min_{\lambda \geq 0} \mathcal{D}(\lambda, \mu)$

Linear Programming

- Primal problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \quad \boldsymbol{c}^\mathsf{T} \boldsymbol{x}$$
$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b},$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times d}$ and $\boldsymbol{b} \in \mathbb{R}^m$.

- Dual problem

$$\max_{\boldsymbol{\lambda} \in \mathbb{R}^m} \quad -\boldsymbol{b}^\mathsf{T} \boldsymbol{\lambda}$$
$$\text{subject to} \quad \boldsymbol{c} + \boldsymbol{A}^\mathsf{T} \boldsymbol{\lambda} = 0, \ \boldsymbol{\lambda} \succeq 0,$$

where $\boldsymbol{\lambda} \in \mathbb{R}^m$.

- The Lagrangian: $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = (\boldsymbol{c} + \boldsymbol{A}^\mathsf{T} \boldsymbol{\lambda})^\mathsf{T} \boldsymbol{x} - \boldsymbol{\lambda}^\mathsf{T} \boldsymbol{b}$, whose derivative w.r.t. $\boldsymbol{x}$ becomes zero, when $\boldsymbol{c} + \boldsymbol{A}^\mathsf{T} \boldsymbol{\lambda} = 0$.
- The dual function: $\mathcal{D}(\boldsymbol{\lambda}) = -\boldsymbol{\lambda}^\mathsf{T} \boldsymbol{b}$

- Primal problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} \quad \tfrac{1}{2}\boldsymbol{x}^\mathsf{T}\boldsymbol{Q}\boldsymbol{x} + c^\mathsf{T}\boldsymbol{x}$$
$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b},$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times d}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^d$, the square matrix $\boldsymbol{Q}$ is symmetric, positive definite.

- Dual problem

$$\max_{\boldsymbol{\lambda}\in\mathbb{R}^m} \quad \left(-\tfrac{1}{2}(\boldsymbol{c}+\boldsymbol{A}^\mathsf{T}\boldsymbol{\lambda})^\mathsf{T}\boldsymbol{A}\boldsymbol{Q}^{-1}(\boldsymbol{c}+\boldsymbol{A}^\mathsf{T}\boldsymbol{\lambda}) - \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \right)$$
$$\text{subject to} \quad \boldsymbol{\lambda} \succeq 0,$$

where $\boldsymbol{\lambda} \in \mathbb{R}^m$.

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(2) Constrained Optimization and Lagrange Multipliers
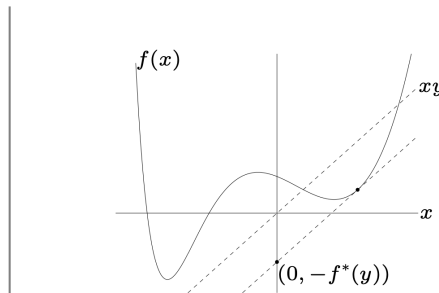
(3) Convex Sets and Functions

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(5) Convex Conjugate

- Given $f : \mathbb{R}^n \mapsto \mathbb{R}$, the conjugate function $f^* : \mathbb{R}^n \mapsto \mathbb{R}$ defined as:

$$f^*(\boldsymbol{y}) = \sup_{\boldsymbol{x}\in\text{dom } f} (\boldsymbol{y}^\mathsf{T}\boldsymbol{x} - f(\boldsymbol{x}))$$

with domain consisting of $\boldsymbol{y} \in \mathbb{R}^n$ for which the supremum is finite

- Assume $\mathbb{R}^1$.

- For a given slope of y, $yx - f(x)$ is the vertical distance between the line yx and $f(x)$.

- Thus, $f^*(y)$ is the maximum distance

- Given $f : \mathbb{R}^n \mapsto \mathbb{R}$, the conjugate function $f^* : \mathbb{R}^n \mapsto \mathbb{R}$ defined as:

$$f^*(\boldsymbol{y}) = \sup_{\boldsymbol{x}\in\text{dom } f} (\boldsymbol{y}^\mathsf{T}\boldsymbol{x} - f(\boldsymbol{x}))$$

with domain consisting of $\boldsymbol{y} \in \mathbb{R}^n$ for which the supremum is finite

- $f^*(\boldsymbol{y})$: always convex (the pointwise supremum of a family of affine functions of $\boldsymbol{y}$)

- $f^* = f$ if f is convex and closed

- Fenchel's inequality: $f(\boldsymbol{x}) + f^*(\boldsymbol{y}) \geq \boldsymbol{x}^\mathsf{T}\boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y}$ (by definition)
  - Example. $f(x) = |x|^2/2$. Then, $f^*(y) = |y|^2/2$. Thus, F-inequality tells us:

$$\frac{1}{2}(|x|^2 + |y|^2) \geq xy$$

- $f(x) = ax + b$, $f^*(a) = -b$
- $f(x) = -\log x$, $f^*(y) = -\log(-y) - 1$ for $y < 0$
- $f(x) = e^x$, $f^*(y) = y \log y - y$
- $f(x) = x \log x$, $f^*(y) = e^{y-1}$
- $f(x) = \frac{1}{2} x^T Q x$, $f^*(y) = \frac{1}{2} y^T Q^{-1} y$ (Q is positive definite)
- $f(x) = \log \sum_{i=1}^n e^{x_i}$,

$$f^*(y) = \begin{cases} \sum_{i=1}^n y_i \log y_i & \text{if } y \succeq 0 \text{ and } \sum_{i=1}^n y_i = 1, \\ \infty & \text{otherwise} \end{cases}$$

- They are closely related. Consider the following problem:

  minimize $\quad f(x)$
  subject to $\quad Ax \preceq b$,
  $\qquad\qquad Cx = d$

- Using the conjugate of f, we can write the dual function as:

$$\mathcal{D}(\lambda, \nu) = \inf_x \left(f(x) + \lambda^T (Ax - b) + \nu^T (Cx - d) \right)$$
$$= -b^T \lambda - d^T \nu + \inf_x \left(f(x) + (A^T \lambda + C^T \nu)^T x \right)$$
$$= -b^T \lambda - d^T \nu - f^* \left(-A^T \lambda - C^T \nu \right)$$

Questions?

1)