Full Specification of the Object Language

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1 Syntax

Γ	$::= \cdot \mid \Gamma, x : A$	Typing contexts
$A, B, C,$		
a, b, c, P	$::=$	Terms
	$\mid x \mid \mathcal{U}_i$	*variables, type universes*
	$\mid \Pi x {:} A.\, B \mid \lambda x.\, b \mid b\ a$	*function types, abstractions, applications*
	$\mid \Sigma x {:} A.\, B \mid (a, b) \mid \pi_1\, x \mid \pi_2\, x$	*pair types, pairs, projections*
	$\mid \mathbf{Nat} \mid \mathbf{zero} \mid \mathbf{succ}\ a \mid \mathbf{ind}\ P\ a\ b\ (\lambda x\ y.\ c)$	*naturals, zero, successor, induction*

2 Typing

$\boxed{\vdash \Gamma}$ *(Context well-formedness)*

Wf-Nil
$$\frac{}{\vdash \cdot}$$

Wf-Cons
$$\frac{\vdash \Gamma \qquad \Gamma \vdash A : \mathcal{U}_i}{\vdash \Gamma, x : A}$$

$\boxed{\Gamma \vdash a : A}$ *(Typing)*

Wt-Var
$$\frac{\vdash \Gamma \qquad x : A \in \Gamma}{\Gamma \vdash x : A}$$

Wt-Pi
$$\frac{\Gamma \vdash A : \mathcal{U}_i \qquad \Gamma, x : A \vdash B : \mathcal{U}_i}{\Gamma \vdash \Pi x {:} A.\, B : \mathcal{U}_i}$$

Wt-Abs
$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.\, b : \Pi x {:} A.\, B}$$

Wt-App
$$\frac{\Gamma \vdash b : \Pi x {:} A.\, B \qquad \Gamma \vdash a : A}{\Gamma \vdash b\ a : B[a/x]}$$

Wt-Univ
$$\frac{\vdash \Gamma}{\Gamma \vdash \mathcal{U}_i : \mathcal{U}_{i+1}}$$

Wt-Sig
$$\frac{\Gamma \vdash A : \mathcal{U}_i \qquad \Gamma, x : A \vdash B : \mathcal{U}_i}{\Gamma \vdash \Sigma x {:} A.\, B : \mathcal{U}_i}$$

Wt-Pair
$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B[a/x] \qquad \Gamma \vdash \Sigma x {:} A.\, B : \mathcal{U}_i}{\Gamma \vdash (a, b) : \Sigma x {:} A.\, B}$$

Wt-Proj1
$$\frac{\Gamma \vdash a : \Sigma x {:} A.\, B}{\Gamma \vdash \pi_1\, a : A}$$

Wt-Proj2
$$\frac{\Gamma \vdash a : \Sigma x {:} A.\, B}{\Gamma \vdash \pi_2\, a : B[\pi_1\, a/x]}$$

$$\frac{\vdash \Gamma}{\Gamma \vdash \mathbf{Nat} : \mathcal{U}_i}\ \textsc{Wt-Nat} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \mathbf{zero} : \mathbf{Nat}}\ \textsc{Wt-Zero} \qquad \frac{\Gamma \vdash a : \mathbf{Nat}}{\Gamma \vdash \mathbf{succ}\, a : \mathbf{Nat}}\ \textsc{Wt-Succ}$$

$$\frac{\begin{array}{c}\Gamma, x : \mathbf{Nat} \vdash P : \mathcal{U}_i \\ \Gamma \vdash a : \mathbf{Nat} \\ \Gamma \vdash b : P[\mathbf{zero}/x] \\ \Gamma, x : \mathbf{Nat}, y : P \vdash b_1 : P[\mathbf{succ}\, x/z]\end{array}}{\Gamma \vdash \mathbf{ind}\, P\ a\ b\ (\lambda x\, y.\, c) : A[a/z]}\ \textsc{Wt-Ind}$$

$\boxed{\Gamma \vdash a = b : A}$ *(Equality)*

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a = a : A}\ \textsc{E-Refl} \qquad \frac{\Gamma \vdash b = a : A}{\Gamma \vdash a = b : A}\ \textsc{E-Sym} \qquad \frac{\begin{array}{c}\Gamma \vdash a = b : A \\ \Gamma \vdash b = c : A\end{array}}{\Gamma \vdash a = c : A}\ \textsc{E-Trans}$$

$$\frac{\begin{array}{c}\Gamma \vdash A_0 = A_1 : \mathcal{U}_i \\ \Gamma, x : A_0 \vdash B_0 = B_1 : \mathcal{U}_i\end{array}}{\Gamma \vdash \Pi x : A_0.\, B_0 = \Pi x : A_1.\, B_1 : \mathcal{U}_i}\ \textsc{E-Pi} \qquad \frac{\begin{array}{c}x \notin \Gamma \\ \Gamma, x : A \vdash a\, x = b\, x : B\end{array}}{\Gamma \vdash a = b : \Pi x : A.\, B}\ \textsc{E-AbsExt}$$

$$\frac{\begin{array}{c}\Gamma \vdash b_0 = b_1 : \Pi x : A.\, B \\ \Gamma \vdash a_0 = a_1 : A\end{array}}{\Gamma \vdash b_0\, a_0 = b_1\, a_1 : B[a_0/x]}\ \textsc{E-App} \qquad \frac{\begin{array}{c}\Gamma \vdash A_0 = A_1 : \mathcal{U}_i \\ \Gamma, x : A_0 \vdash B_0 = B_1 : \mathcal{U}_i\end{array}}{\Gamma \vdash \Sigma x : A_0.\, B_0 = \Sigma x : A_1.\, B_1 : \mathcal{U}_i}\ \textsc{E-Sig}$$

$$\frac{\begin{array}{c}\Gamma \vdash \pi_1\, a = \pi_1\, b : A \\ \Gamma \vdash \pi_2\, a = \pi_2\, b : B[\pi_1\, a/x]\end{array}}{\Gamma \vdash a = b : \Sigma x : A.\, B}\ \textsc{E-PairExt} \qquad \frac{\Gamma \vdash a = b : \Sigma x : A.\, B}{\Gamma \vdash \pi_1\, a = \pi_1\, b : A}\ \textsc{E-Proj1}$$

$$\frac{\Gamma \vdash a = b : \Sigma x : A.\, B}{\Gamma \vdash \pi_2\, a = \pi_2\, b : B[\pi_1\, a/x]}\ \textsc{E-Proj2} \qquad \frac{\Gamma \vdash a = b : \mathbf{Nat}}{\Gamma \vdash \mathbf{succ}\, a = \mathbf{succ}\, b : \mathbf{Nat}}\ \textsc{E-Suc}$$

$$\frac{\begin{array}{c}\Gamma, x : \mathbf{Nat} \vdash P_0 = P_1 : \mathcal{U}_i \\ \Gamma \vdash a_0 = a_1 : \mathbf{Nat} \\ \Gamma \vdash b_0 = b_1 : P_0[\mathbf{zero}/x] \\ \Gamma, x : \mathbf{Nat}, y : P_0 \vdash c_0 = c_1 : P_0[\mathbf{succ}\, x/x]\end{array}}{\Gamma \vdash \mathbf{ind}\, P_0\ a_0\ b_0\ (\lambda x\, y.\, c_0) = \mathbf{ind}\, P_1\ a_1\ b_1\ (\lambda x\, y.\, c_1) : P_0[a_0/x]}\ \textsc{E-Ind}$$

$$\frac{\Gamma \vdash a = b : A \qquad \Gamma \vdash A \leq B}{\Gamma \vdash a = b : B}\ \textsc{E-Conv}$$

$$\frac{\Gamma \vdash b : A \qquad \Gamma, x : A \vdash a : B}{\Gamma \vdash (\lambda x.\, a)\, b = a[b/x] : B[b/x]}\ \textsc{E-AppAbs}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B[a/x]}{\Gamma \vdash \pi_1\, (a, b) = a : A}\ \textsc{E-ProjPair1}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B[a/x]}{\Gamma \vdash \pi_2\, (a, b) = b : B[a/x]}\ \textsc{E-ProjPair2}$$

$$\frac{\Gamma \vdash b : P[\mathbf{zero}/x] \qquad \Gamma, x : \mathbf{Nat}, y : P \vdash c : P[\mathbf{succ}\, x/x]}{\Gamma \vdash \mathbf{ind}\, P\, \mathbf{zero}\, b\, (\lambda x\, y.\, c) = b : P[\mathbf{zero}/x]}\ \textsc{E-IndZero}$$

$$\frac{\Gamma \vdash a : \mathbf{Nat} \qquad \Gamma \vdash b : P[\mathbf{zero}/x] \qquad \Gamma, x : \mathbf{Nat}, y : P \vdash c : P[\mathbf{succ}\, x/x]}{\Gamma \vdash \mathbf{ind}\, P\, (\mathbf{succ}\, a)\, b\, (\lambda x\, y.\, c) = c[a/x, \mathbf{ind}\, P\; a\; b\, (\lambda x\, y.\, c)/y] : P[\mathbf{succ}\, a/x]}\ \textsc{E-IndSuc}$$

$\boxed{\Gamma \vdash A \leq B}$ *(Subtyping)*

$$\frac{\Gamma \vdash A \leq B \qquad \Gamma \vdash B \leq C}{\Gamma \vdash A \leq C}\ \textsc{L-Trans}$$

$$\frac{\Gamma \vdash A = B : \mathcal{U}_i}{\Gamma \vdash A \leq B}\ \textsc{L-Eq}$$

$$\frac{\vdash \Gamma \qquad i \leq j}{\Gamma \vdash \mathcal{U}_i \leq \mathcal{U}_j}\ \textsc{L-Univ}$$

$$\frac{\Gamma \vdash A_1 \leq A_0 \qquad \Gamma, x : A_0 \vdash B_0 \leq B_1}{\Gamma \vdash \Pi x \!:\! A_0.\, B_0 \leq \Pi x \!:\! A_1.\, B_1}\ \textsc{L-Pi}$$

$$\frac{\Gamma \vdash A_0 \leq A_1 \qquad \Gamma, x : A_1 \vdash B_0 \leq B_1}{\Gamma \vdash \Sigma x \!:\! A_0.\, B_0 \leq \Sigma x \!:\! A_1.\, B_1}\ \textsc{L-Sig}$$

$$\frac{\Gamma \vdash \Pi x \!:\! A_0.\, B_0 \leq \Pi x \!:\! A_1.\, B_1}{\Gamma \vdash A_1 \leq A_0}\ \textsc{L-PiProj1}$$

$$\frac{\Gamma \vdash \Pi x \!:\! A_0.\, B_0 \leq \Pi x \!:\! A_1.\, B_1 \qquad \Gamma \vdash a_0 = a_1 : A_1}{\Gamma \vdash B_0[a_0/x] \leq B_1[a_1/x]}\ \textsc{L-PiProj2}$$

$$\frac{\Gamma \vdash \Sigma x \!:\! A_0.\, B_0 \leq \Sigma x \!:\! A_1.\, B_1}{\Gamma \vdash A_0 \leq A_1}\ \textsc{L-SigProj1}$$

$$\frac{\Gamma \vdash \Sigma x \!:\! A_0.\, B_0 \leq \Sigma x \!:\! A_1.\, B_1 \qquad \Gamma \vdash a_0 = a_1 : A_0}{\Gamma \vdash B_0[a_0/x] \leq B_1[a_1/x]}\ \textsc{L-SigProj2}$$

3 Reductions

Full β, η, and $\beta\eta$-reductions ($\rightsquigarrow_\beta, \rightsquigarrow_\eta, \rightsquigarrow_{\beta\eta}$) are the congruence closures of the respective primitive reductions and we omit their definitions because they are standard.

$\boxed{a \rhd_\beta b}$ *(Primitive β-Reduction)*

$$\textsc{PB-AppAbs}\quad \frac{}{(\lambda x.\, a)\; b \rhd_\beta a[b/x]}$$

$$\textsc{PB-ProjPair1}\quad \frac{}{\pi_1\,(a, b) \rhd_\beta a}$$

$$\textsc{PB-ProjPair2}\quad \frac{}{\pi_2\,(a, b) \rhd_\beta b}$$

$$\textsc{PB-IndZero}\quad \frac{}{\textbf{ind}\, P\, \textbf{zero}\, b\, (\lambda x\, y.\, c) \rhd_\beta b}$$

$$\textsc{PB-IndSuc}\quad \frac{}{\textbf{ind}\, P\, (\textbf{succ}\, a)\, b\, (\lambda x\, y.\, c) \rhd_\beta c[a/x, \textbf{ind}\; P\; a\; b\; (\lambda x\, y.\, c)/y]}$$

$\boxed{a \rhd_\eta b}$ *(Primitive η-Reduction)*

$$\textsc{PE-AppEta}\quad \frac{x \notin \textbf{freevar}(a)}{\lambda x.\, a\, x \rhd_\eta a}$$

$$\textsc{PE-PairEta}\quad \frac{}{(\pi_1\; a, \pi_2\; a) \rhd_\eta a}$$

$\boxed{a \Rightarrow_\eta b}$ *(Parallel η-Reduction)*

$$\textsc{P-AppEta}\quad \frac{a_0 \Rightarrow_\eta a_1 \qquad x \notin \textbf{freevar}(a_0)}{(\lambda x.\, a_0\, x) \Rightarrow_\eta a_1}$$

$$\textsc{P-PairEta}\quad \frac{a_0 \Rightarrow_\eta a_1}{(\pi_1\; a_0, \pi_2\; a_0) \Rightarrow_\eta a_1}$$

$$\textsc{P-AbsCong}\quad \frac{a_0 \Rightarrow_\eta a_1}{\lambda x.\, a_0 \Rightarrow_\eta \lambda x.\, a_1}$$

$$\textsc{P-AppCong}\quad \frac{a_0 \Rightarrow_\eta a_1 \qquad b_0 \Rightarrow_\eta b_1}{a_0\; b_0 \Rightarrow_\eta a_1\; b_1}$$

$$\textsc{P-PairCong}\quad \frac{a_0 \Rightarrow_\eta a_1 \qquad b_0 \Rightarrow_\eta b_1}{(a_0, b_0) \Rightarrow_\eta (a_1, b_1)}$$

$$\textsc{P-ProjCong1}\quad \frac{a_0 \Rightarrow_\eta a_1}{\pi_1\; a_0 \Rightarrow_\eta \pi_1\; a_1}$$

$$\textsc{P-ProjCong2}\quad \frac{a_0 \Rightarrow_\eta a_1}{\pi_2\; a_0 \Rightarrow_\eta \pi_2\; a_1}$$

$$\textsc{P-Var}\quad \frac{}{x \Rightarrow_\eta x}$$

$$\textsc{P-Univ}\quad \frac{}{\mathcal{U}_i \Rightarrow_\eta \mathcal{U}_i}$$

$$\textsc{P-PiCong}\quad \frac{A_0 \Rightarrow_\eta A_1 \qquad B_0 \Rightarrow_\eta B_1}{\Pi x\!:\!A_0.\, B_0 \Rightarrow_\eta \Pi x\!:\!A_1.\, B_1}$$

$$\textsc{P-SigCong}\quad \frac{A_0 \Rightarrow_\eta A_1 \qquad B_0 \Rightarrow_\eta B_1}{\Sigma x\!:\!A_0.\, B_0 \Rightarrow_\eta \Sigma x\!:\!A_1.\, B_1}$$

$$\textsc{P-Nat}\quad \frac{}{\textbf{Nat} \Rightarrow_\eta \textbf{Nat}}$$

$$\textsc{P-IndCong}\quad \frac{P_0 \Rightarrow_\eta P_1 \qquad a_0 \Rightarrow_\eta a_1 \qquad b_0 \Rightarrow_\eta b_1 \qquad c_0 \Rightarrow_\eta c_1}{\textbf{ind}\; P_0\; a_0\; b_0\; (\lambda x\, y.\, c_0) \Rightarrow_\eta \textbf{ind}\; P_1\; a_1\; b_1\; (\lambda x\, y.\, c_1)}$$

$$\textsc{P-ZeroCong}\quad \frac{}{\textbf{zero} \Rightarrow_\eta \textbf{zero}}$$

P-SucCong
$$\frac{a_0 \Rightarrow_\eta a_1}{\mathbf{succ}\ a_0 \Rightarrow_\eta \mathbf{succ}\ a_1}$$

$\boxed{a \Rightarrow_r b}$ *(Restrictive η-Reduction (not eliminated))*

EP-AppEta
$$\frac{a_0 \notin \mathbf{canf} \quad a_0 \Rightarrow_{\bar{r}} a_1 \quad x \notin \mathbf{freevar}(a_0)}{(\lambda x.\, a_0\ x) \Rightarrow_r a_1}$$

EP-PairEta
$$\frac{a_0 \notin \mathbf{canf} \quad a_0 \Rightarrow_{\bar{r}} a_1}{(\pi_1\ a_0, \pi_2\ a_0) \Rightarrow_r a_1}$$

EP-Embed
$$\frac{a \Rightarrow_{\bar{r}} b}{a \Rightarrow_r b}$$

$\boxed{a \Rightarrow_{\bar{r}} b}$ *(Restrictive η-Reduction (eliminated))*

NEP-AbsCong
$$\frac{a_0 \Rightarrow_r a_1}{\lambda x.\, a_0 \Rightarrow_{\bar{r}} \lambda x.\, a_1}$$

NEP-AppCong
$$\frac{a_0 \Rightarrow_{\bar{r}} a_1 \qquad b_0 \Rightarrow_r b_1}{a_0\ b_0 \Rightarrow_{\bar{r}} a_1\ b_1}$$

NEP-PairCong
$$\frac{a_0 \Rightarrow_r a_1 \qquad b_0 \Rightarrow_r b_1}{(a_0, b_0) \Rightarrow_{\bar{r}} (a_1, b_1)}$$

NEP-ProjCong1
$$\frac{a_0 \Rightarrow_{\bar{r}} a_1}{\pi_1\ a_0 \Rightarrow_{\bar{r}} \pi_1\ a_1}$$

NEP-ProjCong2
$$\frac{a_0 \Rightarrow_{\bar{r}} a_1}{\pi_2\ a_0 \Rightarrow_{\bar{r}} \pi_2\ a_1}$$

NEP-Var
$$\frac{}{x \Rightarrow_{\bar{r}} x}$$

NEP-Univ
$$\frac{}{\mathcal{U}_i \Rightarrow_{\bar{r}} \mathcal{U}_i}$$

NEP-PiCong
$$\frac{A_0 \Rightarrow_r A_1 \quad B_0 \Rightarrow_r B_1}{\Pi x : A_0.\, B_0 \Rightarrow_{\bar{r}} \Pi x : A_1.\, B_1}$$

NEP-SigCong
$$\frac{A_0 \Rightarrow_r A_1 \quad B_0 \Rightarrow_r B_1}{\Sigma x : A_0.\, B_0 \Rightarrow_{\bar{r}} \Sigma x : A_1.\, B_1}$$

NEP-Nat
$$\frac{}{\mathbf{Nat} \Rightarrow_{\bar{r}} \mathbf{Nat}}$$

NEP-IndCong
$$\frac{P_0 \Rightarrow_r P_1 \quad a_0 \Rightarrow_{\bar{r}} a_1 \quad b_0 \Rightarrow_r b_1 \qquad c_0 \Rightarrow_r c_1}{\mathbf{ind}\ P_0\ a_0\ b_0\ (\lambda x\ y.\, c_0) \Rightarrow_{\bar{r}} \mathbf{ind}\ P_1\ a_1\ b_1\ (\lambda x\ y.\, c_1)}$$

NEP-ZeroCong
$$\frac{}{\mathbf{zero} \Rightarrow_{\bar{r}} \mathbf{zero}}$$

NEP-SucCong
$$\frac{a_0 \Rightarrow_r a_1}{\mathbf{succ}\ a_0 \Rightarrow_{\bar{r}} \mathbf{succ}\ a_1}$$

4 Strong Normalization

$\boxed{a \in \mathbf{SN}}$ *(Strong Normal Forms)*

N-Abs
$$\frac{b \in \mathbf{SN}}{\lambda x.\, b \in \mathbf{SN}}$$

N-Pair
$$\frac{a \in \mathbf{SN} \qquad b \in \mathbf{SN}}{(a, b) \in \mathbf{SN}}$$

N-Pi
$$\frac{A \in \mathbf{SN} \qquad B \in \mathbf{SN}}{\Pi x : A.\, B \in \mathbf{SN}}$$

$$\frac{A \in \mathbf{SN} \qquad B \in \mathbf{SN}}{\Sigma x : A.\, B \in \mathbf{SN}}\ \textsc{N-Sig} \qquad \frac{}{\mathcal{U}_i \in \mathbf{SN}}\ \textsc{N-Univ} \qquad \frac{a \in \mathbf{SNe}}{a \in \mathbf{SN}}\ \textsc{N-SNe} \qquad \frac{}{\mathbf{zero} \in \mathbf{SN}}\ \textsc{N-Zero}$$

$$\frac{a \in \mathbf{SN}}{\mathbf{succ}\ a \in \mathbf{SN}}\ \textsc{N-Succ} \qquad \frac{}{\mathbf{Nat} \in \mathbf{SN}}\ \textsc{N-Nat} \qquad \frac{a \leadsto_{\mathbf{SN}} b \qquad b \in \mathbf{SN}}{a \in \mathbf{SN}}\ \textsc{N-Exp}$$

$\boxed{a \in \mathbf{SNe}}$ *(Strong Neutral Forms)*

$$\frac{}{x \in \mathbf{SNe}}\ \textsc{N-Var} \qquad \frac{a \in \mathbf{SNe} \qquad b \in \mathbf{SN}}{a\ b \in \mathbf{SNe}}\ \textsc{N-App} \qquad \frac{a \in \mathbf{SNe}}{\pi_1\ a \in \mathbf{SNe}}\ \textsc{N-Proj1} \qquad \frac{a \in \mathbf{SNe}}{\pi_2\ a \in \mathbf{SNe}}\ \textsc{N-Proj2}$$

$$\frac{\begin{array}{cc} P \in \mathbf{SN} & a \in \mathbf{SNe} \\ b \in \mathbf{SN} & c \in \mathbf{SN} \end{array}}{\mathbf{ind}\ P\ a\ b\ (\lambda x\ y.\ c) \in \mathbf{SNe}}\ \textsc{N-Ind}$$

$\boxed{a \leadsto_{\mathbf{SN}} b}$ *(Strong Weak Head Reduction)*

$$\frac{b \in \mathbf{SN}}{(\lambda x.\ a)\ b \leadsto_{\mathbf{SN}} a[b/x]}\ \textsc{N-AppAbs} \qquad \frac{\begin{array}{c} b \in \mathbf{SN} \\ a_0 \leadsto_{\mathbf{SN}} a_1 \end{array}}{a_0\ b \leadsto_{\mathbf{SN}} a_1\ b}\ \textsc{N-AppCong} \qquad \frac{b \in \mathbf{SN}}{\pi_1\ (a, b) \leadsto_{\mathbf{SN}} a}\ \textsc{N-ProjPair1}$$

$$\frac{a \in \mathbf{SN}}{\pi_2\ (a, b) \leadsto_{\mathbf{SN}} b}\ \textsc{N-ProjPair2} \qquad \frac{a \leadsto_{\mathbf{SN}} b}{\pi_1\ a \leadsto_{\mathbf{SN}} \pi_1\ b}\ \textsc{N-ProjCong1} \qquad \frac{a \leadsto_{\mathbf{SN}} b}{\pi_2\ a \leadsto_{\mathbf{SN}} \pi_2\ b}\ \textsc{N-ProjCong2}$$

$$\frac{\begin{array}{c} P \in \mathbf{SN} \\ b \in \mathbf{SN} \qquad c \in \mathbf{SN} \end{array}}{\mathbf{ind}\ P\ \mathbf{zero}\ b\ (\lambda x\ y.\ c) \leadsto_{\mathbf{SN}} b}\ \textsc{N-IndZero}$$

$$\frac{\begin{array}{cc} P \in \mathbf{SN} & a \in \mathbf{SN} \\ b \in \mathbf{SN} & c \in \mathbf{SN} \end{array}}{\mathbf{ind}\ P\ (\mathbf{succ}\ a)\ b\ (\lambda x\ y.\ c) \leadsto_{\mathbf{SN}} c[a/x, \mathbf{ind}\ P\ a\ b\ (\lambda x\ y.\ c)/y]}\ \textsc{N-IndSuc}$$

$$\frac{\begin{array}{c} P \in \mathbf{SN} \\ b \in \mathbf{SN} \qquad c \in \mathbf{SN} \\ a_0 \leadsto_{\mathbf{SN}} a_1 \end{array}}{\mathbf{ind}\ P\ a_0\ b\ (\lambda x\ y.\ c) \leadsto_{\mathbf{SN}} \mathbf{ind}\ P\ a_1\ b\ (\lambda x\ y.\ c)}\ \textsc{N-IndCong}$$

5 Coquand's Algorithm

$\boxed{a \leftrightarrow b}$ *(Algorithmic equality)*

$$\textsc{CE-Red}\quad \frac{a \leadsto_h^* f_0 \qquad b \leadsto_h^* f_1 \qquad f_0 \sim f_1}{a \leftrightarrow b}$$

$\boxed{f_0 \sim f_1}$ *(Algorithmic equality for head normal forms)*

$$\textsc{CE-Zero}\quad \frac{}{\mathbf{zero} \sim \mathbf{zero}} \qquad \textsc{CE-Suc}\quad \frac{a \leftrightarrow b}{\mathbf{succ}\ a \sim \mathbf{succ}\ b} \qquad \textsc{CE-Var}\quad \frac{}{x \sim x} \qquad \textsc{CE-Abs}\quad \frac{a \leftrightarrow b}{\lambda x.\, a \sim \lambda x.\, b}$$

$$\textsc{CE-AbsNeu}\quad \frac{a \leftrightarrow e\ x \qquad x \notin \mathbf{freevar}(e)}{\lambda x.\, a \sim e} \qquad \textsc{CE-NeuAbs}\quad \frac{e\ x \leftrightarrow b \qquad x \notin \mathbf{freevar}(e)}{e \sim \lambda x.\, b} \qquad \textsc{CE-Pair}\quad \frac{a_0 \leftrightarrow a_1 \qquad b_0 \leftrightarrow b_1}{(a_0, b_0) \sim (a_1, b_1)}$$

$$\textsc{CE-PairNeu}\quad \frac{a_0 \leftrightarrow \pi_1\ e \qquad b_0 \leftrightarrow \pi_2\ e}{(a_0, b_0) \sim e} \qquad \textsc{CE-NeuPair}\quad \frac{\pi_1\ e \leftrightarrow a_0 \qquad \pi_2\ e \leftrightarrow b_0}{e \sim (a_0, b_0)} \qquad \textsc{CE-Univ}\quad \frac{}{\mathcal{U}_i \sim \mathcal{U}_i} \qquad \textsc{CE-Nat}\quad \frac{}{\mathbf{Nat} \sim \mathbf{Nat}}$$

$$\textsc{CE-Pi}\quad \frac{A_0 \leftrightarrow A_1 \qquad B_0 \leftrightarrow B_1}{\Pi x : A_0.\, B_0 \sim \Pi x : A_1.\, B_1} \qquad \textsc{CE-Sig}\quad \frac{A_0 \leftrightarrow A_1 \qquad B_0 \leftrightarrow B_1}{\Sigma x : A_0.\, B_0 \sim \Sigma x : A_1.\, B_1}$$

$$\textsc{CE-App}\quad \frac{e_0 \sim e_1 \qquad a_0 \leftrightarrow a_1}{e_0\ a_0 \sim e_1\ a_1} \qquad \textsc{CE-Fst}\quad \frac{e_0 \sim e_1}{\pi_1\ e_0 \sim \pi_1\ e_1} \qquad \textsc{CE-Snd}\quad \frac{e_0 \sim e_1}{\pi_2\ e_0 \sim \pi_2\ e_1}$$

$$\textsc{CE-Ind}\quad \frac{P_0 \leftrightarrow P_1 \qquad e_0 \sim e_1 \qquad b_0 \leftrightarrow b_1 \qquad c_0 \leftrightarrow c_1}{\mathbf{ind}\ P_0\ e_0\ b_0\ (\lambda x\ y.\, c_0) \sim \mathbf{ind}\ P_1\ e_1\ b_1\ (\lambda x\ y.\, c_1)}$$

$\boxed{A \ll B}$ *(Algorithmic subtyping)*

$$\textsc{CLE-Red}\quad \frac{A \leadsto_h^* f_0 \qquad B \leadsto_h^* f_1 \qquad f_0 \lesssim f_1}{A \ll B}$$

$\boxed{f_0 \lesssim f_1}$ *(Algorithmic subtyping for head normal forms)*

$$\text{CLE-UNIV}\ \frac{i \le j}{\mathcal{U}_i \lesssim \mathcal{U}_j} \qquad \text{CLE-NAT}\ \frac{}{\mathbf{Nat} \lesssim \mathbf{Nat}} \qquad \text{CLE-PI}\ \frac{A_1 \ll A_0 \qquad B_0 \ll B_1}{\Pi x : A_0.\, B_0 \lesssim \Pi x : A_1.\, B_1}$$

$$\text{CLE-SIG}\ \frac{A_0 \ll A_1 \qquad B_0 \ll B_1}{\Sigma x : A_0.\, B_0 \lesssim \Sigma x : A_1.\, B_1} \qquad \text{CLE-NEUNEU}\ \frac{e_0 \sim e_1}{e_0 \lesssim e_1}$$

5.1 Domains

The diamond operator ($\diamond$) in rules SA-Conf and A-Conf is meant for the catch-all cases where none of the other cases match (i.e. the input terms have conflicting weak-head normal forms that can never be convertible).

$\boxed{(a, b) \in \mathcal{A}^*}$ *(Domain for Algorithmic Equality)*

$$\text{A-NFNF}\ \frac{(f_0, f_1) \in \mathcal{A}}{(f_0, f_1) \in \mathcal{A}^*} \qquad \text{A-REDL}\ \frac{a \rightsquigarrow_h a_0 \qquad (a_0, b) \in \mathcal{A}^*}{(a, b) \in \mathcal{A}^*} \qquad \text{A-REDR}\ \frac{b \rightsquigarrow_h b_0 \qquad (f, b_0) \in \mathcal{A}^*}{(f, b) \in \mathcal{A}^*}$$

$\boxed{(a, b) \in \mathcal{A}}$ *(Domain for Algorithmic Equality (normal forms))*

$$\text{A-UNIV}\ \frac{}{(\mathcal{U}_i, \mathcal{U}_j) \in \mathcal{A}} \qquad \text{A-ABSABS}\ \frac{}{(\lambda x.\, a, \lambda x.\, b) \in \mathcal{A}} \qquad \text{A-ABSNEU}\ \frac{(a, e\, x) \in \mathcal{A} \qquad x \notin \mathbf{freevar}(b)}{(\lambda x.\, a, e) \in \mathcal{A}} \qquad \text{A-NEUABS}\ \frac{(e\, x, a) \in \mathcal{A}^*}{(e, \lambda x.\, a) \in \mathcal{A}}$$

$$\text{A-PAIRPAIR}\ \frac{(a_0, a_1) \in \mathcal{A}^* \qquad (b_0, b_1) \in \mathcal{A}^*}{((a_0, b_0), (a_1, b_1)) \in \mathcal{A}} \qquad \text{A-PAIRNEU}\ \frac{(a_0, \pi_1\, e) \in \mathcal{A}^* \qquad (a_1, \pi_2\, e) \in \mathcal{A}^*}{((a_0, a_1), e) \in \mathcal{A}} \qquad \text{A-NEUPAIR}\ \frac{(\pi_1\, e, a_0) \in \mathcal{A}^* \qquad (\pi_2\, e, a_1) \in \mathcal{A}^*}{(e, (a_0, a_1)) \in \mathcal{A}}$$

$$\text{A-ZERO}\ \frac{}{(\mathbf{zero}, \mathbf{zero}) \in \mathcal{A}} \qquad \text{A-SUCC}\ \frac{(a, b) \in \mathcal{A}^*}{(\mathbf{succ}\, a, \mathbf{succ}\, b) \in \mathcal{A}} \qquad \text{A-UNIV}\ \frac{}{(\mathcal{U}_i, \mathcal{U}_j) \in \mathcal{A}}$$

$$\text{A-PI}\ \frac{(A_0, A_1) \in \mathcal{A}^* \qquad (B_0, B_1) \in \mathcal{A}^*}{(\Pi x : A_0.\, B_0, \Pi x : A_1.\, B_1) \in \mathcal{A}} \qquad \text{A-SIG}\ \frac{(A_0, A_1) \in \mathcal{A}^* \qquad (B_0, B_1) \in \mathcal{A}^*}{(\Sigma x : A_0.\, B_0, \Sigma x : A_1.\, B_1) \in \mathcal{A}}$$

$$\frac{}{(\mathbf{Nat}, \mathbf{Nat}) \in \mathcal{A}}\ \textsc{A-Nat}$$

$$\frac{}{(x, y) \in \mathcal{A}}\ \textsc{A-Var}$$

$$\frac{(e_0, e_1) \in \mathcal{A} \quad (a_0, a_1) \in \mathcal{A}^*}{(e_0\ a_0, e_1\ a_1) \in \mathcal{A}}\ \textsc{A-AppCong}$$

$$\frac{(e_0, e_1) \in \mathcal{A}}{(\pi_1\ e_0, \pi_1\ e_1) \in \mathcal{A}}\ \textsc{A-ProjCong1}$$

$$\frac{(e_0, e_1) \in \mathcal{A}}{(\pi_2\ e_0, \pi_2\ e_1) \in \mathcal{A}}\ \textsc{A-ProjCong2}$$

$$\frac{(P_0, P_1) \in \mathcal{A}^* \quad (e_0, e_1) \in \mathcal{A} \quad (b_0, b_1) \in \mathcal{A}^* \quad (c_0, c_1) \in \mathcal{A}^*}{(\mathbf{ind}\ P_0\ e_0\ b_0\ (\lambda x\ y.\ c_0), \mathbf{ind}\ P_1\ e_1\ b_1\ (\lambda x\ y.\ c_1)) \in \mathcal{A}}\ \textsc{A-IndCong}$$

$$\frac{h_0 \diamond h_1}{(h_0, h_1) \in \mathcal{A}}\ \textsc{A-Conf}$$

$\boxed{(A, B) \in \mathcal{S}^*}$ *(Domain for Algorithmic Subtyping)*

$$\frac{(f_0, f_1) \in \mathcal{S}}{(f_0, f_1) \in \mathcal{S}^*}\ \textsc{SA-NfNf}$$

$$\frac{a \leadsto_h a_0 \quad (a_0, b) \in \mathcal{S}^*}{(a, b) \in \mathcal{S}^*}\ \textsc{SA-RedL}$$

$$\frac{b \leadsto_h b_0 \quad (f, b_0) \in \mathcal{S}^*}{(f, b) \in \mathcal{S}^*}\ \textsc{SA-RedR}$$

$\boxed{(f_0, f_1) \in \mathcal{S}}$ *(Domain for Algorithmic Subtyping (normal forms))*

$$\frac{}{(\mathcal{U}_i, \mathcal{U}_j) \in \mathcal{S}}\ \textsc{SA-Univ}$$

$$\frac{(A_1, A_0) \in \mathcal{S}^* \quad (B_0, B_1) \in \mathcal{S}^*}{(\Pi x : A_0.\ B_0, \Pi x : A_1.\ B_1) \in \mathcal{S}}\ \textsc{SA-Pi}$$

$$\frac{(A_0, A_1) \in \mathcal{S}^* \quad (B_0, B_1) \in \mathcal{S}^*}{(\Sigma x : A_0.\ B_0, \Sigma x : A_1.\ B_1) \in \mathcal{S}}\ \textsc{SA-Sig}$$

$$\frac{}{(\mathbf{Nat}, \mathbf{Nat}) \in \mathcal{S}}\ \textsc{SA-Nat}$$

$$\frac{(e_0, e_1) \in \mathcal{A}}{(A, B) \in \mathcal{S}}\ \textsc{SA-NeuNeu}$$

$$\frac{h_0 \diamond_S h_1}{(h_0, h_1) \in \mathcal{S}}\ \textsc{SA-Conf}$$