Q1

$$f(x) = \begin{cases} cx^2, & \text{for } 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of constant c and sketch the p.d.f

Because

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{1}$$

Subtitute $f(x)$ into that, we get

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{1}^{2} cx^2 dx \\ &= F(2) - F(1) = 1 \end{aligned}$$

where

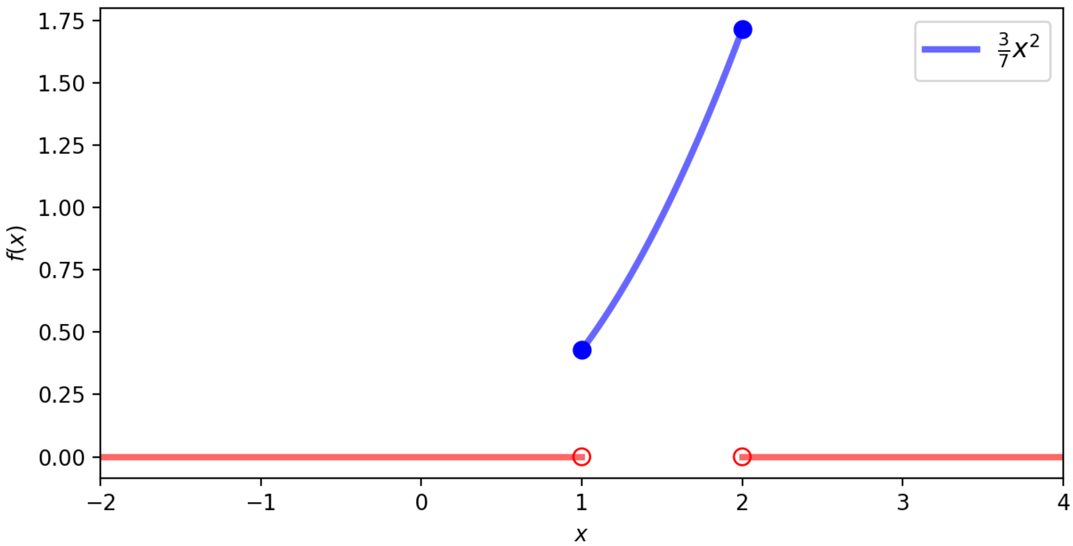
$$F(x) = \frac{1}{3}cx^3 \tag{2}$$

Therefore,

$$\frac{1}{3}c(2^3 - 1^3) = \frac{7}{3}c = 1 \tag{3}$$

Finally, we get

$$c = \frac{3}{7} \tag{4}$$



(b) Find the value of $Pr(X > \frac{3}{2})$

This question is to ask

$$\int_{\frac{3}{2}}^{\infty} f(x)dx \tag{5}$$

And we can simplify that as

$$\int_{\frac{3}{2}}^{\infty} f(x)dx = \int_{\frac{3}{2}}^{2} f(x)dx$$
$$= F(2) - F(\frac{3}{2})$$
$$= \frac{1}{7}[2^3 - (\frac{3}{2})^3] = \frac{37}{56} = 0.661$$

where we used

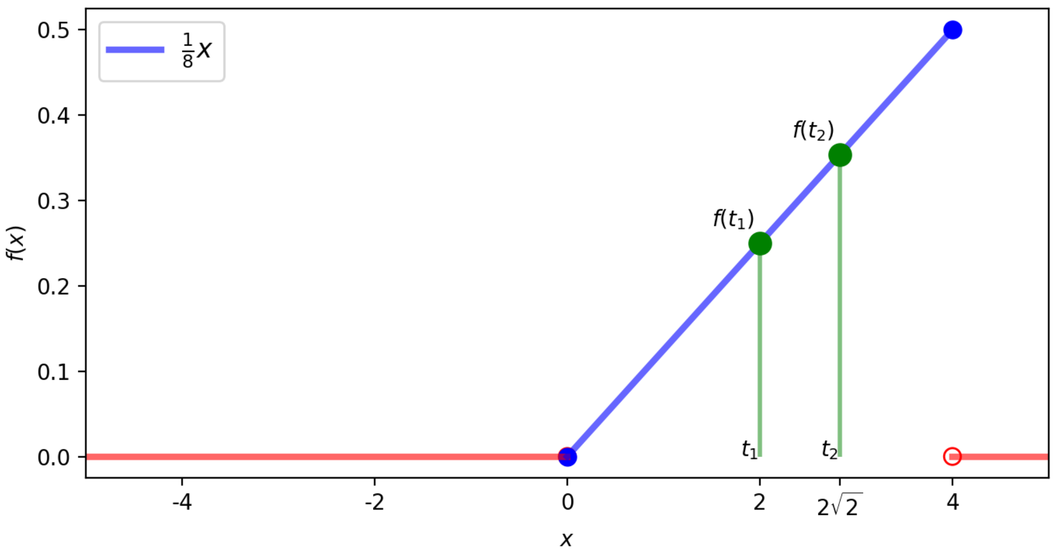
$$F(x) = \frac{1}{3}cx^3 = \frac{1}{7}x^3 \tag{6}$$

Finally, we get

$$Pr(X > \frac{3}{2}) = 0.661 \tag{7}$$

Q2

$$f(x) = \begin{cases} \frac{1}{8}x, & \text{for } 0 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$



As you can see in the plot, we can use the formula of triangle's area to calculate probability.

$$\text{Area} = Pr(X \leq t) = \frac{t \times f(t)}{2} = \frac{t^2}{16} \tag{8}$$

(a) Find the value of t such that $Pr(X \leq t) = \frac{1}{4}$

$$Pr(X \geq t) = \frac{t^2}{16} = \frac{1}{4} \tag{9}$$

and

$$t^2 = 4 \tag{10}$$

so

$$t = 2 \tag{11}$$

(b) Find the value of t such that $Pr(X \geq t) = \frac{1}{2}$

Here, let us change the target by

$$Pr(X \leq t) = 1 - Pr(X \geq t) = 1 - \frac{1}{2} = \frac{1}{2} \tag{12}$$

Therefore, as the general formula I mentioned above, we just try to find

$$\frac{t^2}{16} = \frac{1}{2} \tag{13}$$

so

$$t = 2\sqrt{2} \tag{14}$$

Q3

Show that there does not exist any number c such that the following function $f(x)$ would be a p.d.f.:

$$f(x) = \begin{cases} \frac{c}{x} & \text{for } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

If $f(x)$ is a p.d.f, it should satisfy:

$$\int_{-\infty}^{\infty} f(x) = 1 \tag{15}$$

In this question, the integral becomes

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) &\approx \int_0^1 \frac{c}{x} \\ &= c\int_0^1 \frac{1}{x} \\ &= c[\ln x]_0^1 \\ &= c[\ln 1 - \ln 0] = \infty \end{aligned}$$

As you can see, the integral diverges. Therefore, there does not exist any number c such that $f(x)$ would be a p.d.f.

Q4

Bigger mammals tend to carry their young longer before birth. The length of horse pregnancies from conception to birth varies according to a roughly Normal distribution with mean 336 days and standard deviation 3 days. Use the 68-95-99.7 rule to answer the following questions.

(a)Almost all (99.7%) horse pregnancies fall within what range of lengths?

According to

$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997 \tag{16}$$

The left side of the range is

$$336 - 3 \times 3 = 327 \tag{17}$$

The right side of the range is

$$336 + 3 \times 3 = 345 \tag{18}$$

The range of lengths is between 327 days and 345 days.

(b) What percent of horse pregnancies are longer than 339 days?

First, we notice that

$$339 = 336 + 3 = \mu + \sigma \tag{19}$$

and we also know

$$P(\mu - \sigma < X < \mu + \sigma) \approx 0.68 \tag{20}$$

Furthermore, because the normal distribution is symmetry, we get

$$P(X > \mu + \sigma) \approx 0.16 \tag{21}$$

Therefore, 16% of horse pregnancies are longer than 339 days.

Q5

Use Z table to find the proportion of observations from standard Normal distribution that falls in each of the following regions. In each case, sketch a standard Normal curve and shade of the area representing the region.

(a) $z < 1.85$

By Z-table, The proportion of observations is 0.9678

(b) $z > -0.66$

Because the Normal distribution is symmetry, the area for $P(z > -0.66)$ would be the same as $P(z < 0.66)$

Therefore, by Z-table, the proportion of observations is 0.7454

(c) $z > 1.85$

We can use the result of (a)

$$P(z > 1.85) = 1 - P(z < 1.85) = 1 - 0.9678 = 0.0322 \tag{22}$$

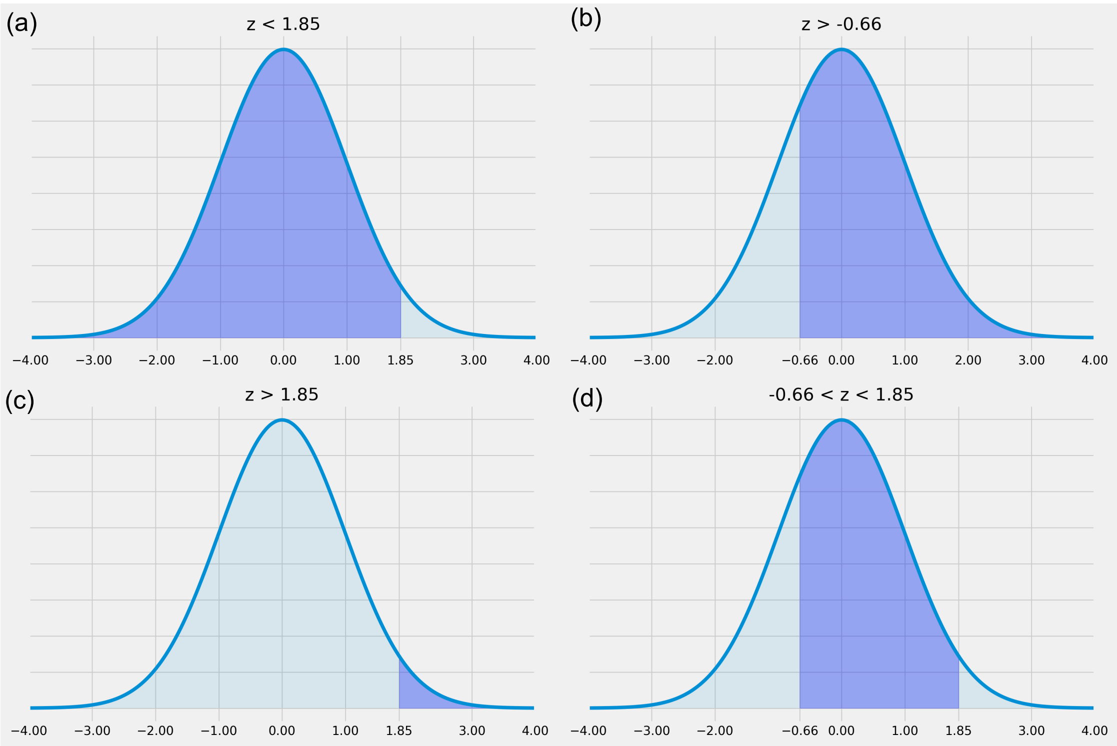
Therefore, the proportion of observations is 0.0322

(d) $-0.66

We can use the result of (b) and (c)

$$P(-0.66 < z < 1.85) = P(z > -0.66) - P(z > 1.85) = 0.7454 - 0.0322 = 0.7132 \tag{23}$$

Therefore, the proportion of observations is 0.7132



Q6

Find the z value that satisfies each of the following conditions (report the value of z that comes closest to satisfying the condition). In each case, sketch a standard Normal curve with your value of z marked on the axis

(a) 20% of the observations fall below z.

First, I try to find z' which 80% of the observations fall below z'. That is

$$P(z < z') = 0.8 \tag{24}$$

From the z-table, we know

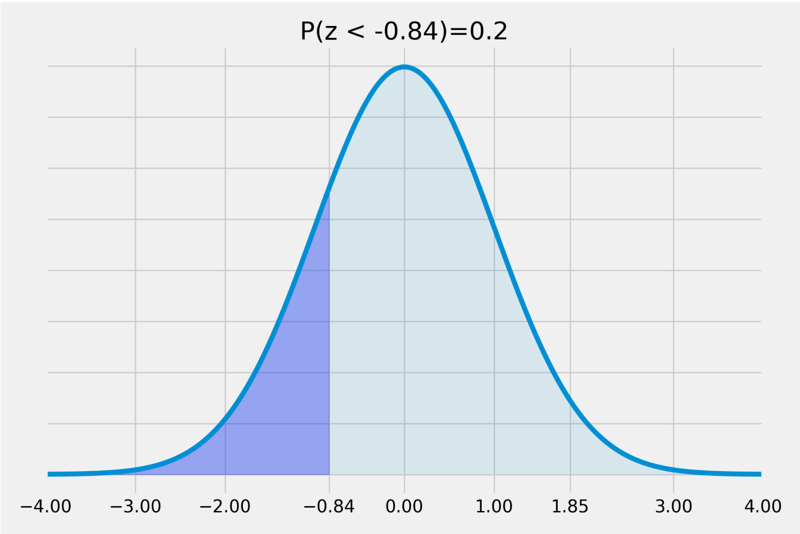
$$P(z < 0.84) = 0.7995 \tag{25}$$

Therefore, it is reasonable to say

$$z' \approx 0.84 \tag{26}$$

Because of the symmetry, the z value in this question is

$$z = -0.84 \tag{27}$$



(b) 30% of the observations fall below z.

First, I try to find z' which 70% of the observations fall below z'. That is

$$P(z < z') = 0.7 \tag{28}$$

From the z-table, we know

$$P(z < 0.52) = 0.6985 \tag{29}$$
$$P(z < 0.53) = 0.7019 \tag{30}$$

Therfore, it is reasonable to say

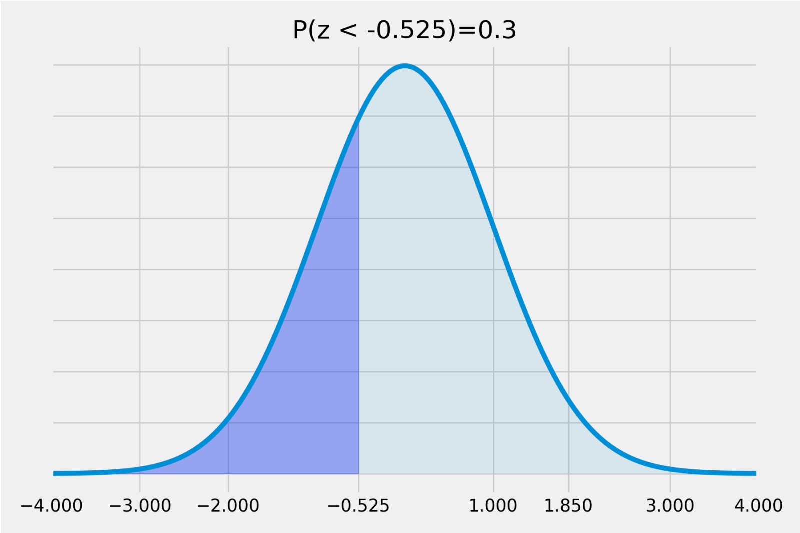
$$P(z < 0.525) \approx 0.7 \tag{31}$$

so

$$z' = 0.525 \tag{32}$$

Finally, by the symmetrical property of the Normal distribution, the z value in this question is

$$z = -0.525 \tag{33}$$



(c) 30% of the observations fall above z.

From (b), we have already known

$$z = 0.525 \tag{34}$$