Lecture 7 Continuous Random Variable, LLT & CLT

10/28/2020

- Today's Main Topics
  - Practices on normal random variables
  - From Probability to Statistics : RandomSampling
  - Law of Large Number
  - The Central Limit Theorem

補充 (不考)

- Uniform distribution and Random Number Generation
- Turing and the Central Limit Theorem
- Gaussian Copula and 金融風暴

- Lecture 6: Normal distribution

$X \sim N(\mu,\sigma^2)$, where $E(X)=\mu$, $Var(X)=\sigma^2$

Standardization: $Z=\dfrac{X-\mu}{\sigma} \sim N(0,1)$

Important skill: Use Z~N(0,1) table to compute the probability of $X \sim N(\mu,\sigma^2)$

Two types of questions (Exercises in Lecture 6**)**

- 給切點，求機率 (Given cut-off points, find the probability)
- 給機率，求切點 (Given a probability, find the cutoff point)

Exercise 1: Given cut-off points, find the probability

YouTube 解說 https://youtu.be/n_3_PX093Sg

$X \sim N(\mu=37,\sigma=2)$ Find K such that $\Pr(|X-37|<0.5)=K$

Sol: $\Pr(|X-37|<0.5)=\Pr(\dfrac{|X-\mu|}{\sigma}<\dfrac{0.5}{2})=\Pr(|Z|<0.25)$

$\Pr(|X-37|<0.5)=\Pr(\dfrac{|X-37|}{\sigma}<\dfrac{0.5}{2})=\Pr(|Z|<0.25)$

$\Pr(|Z|<0.25)=\Pr(-0.25<Z<0.25)=\Pr(Z<0.25)-\Pr(Z<-0.25)$

$\Pr(Z<0.25)=0.5987$, $\Pr(Z<-0.25)=\Pr(Z>0.25)=1-0.5987=0.4013$

0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879

$\Pr(|Z|<0.25)=\Pr(-0.25<Z<0.25)=\Pr(Z<0.25)-\Pr(Z<-0.25)=0.1974$

Exercise 2: Given a probability, find the cutoff point.

https://youtu.be/ofY0G1lu-0o

$X \sim N(\mu = 37, \sigma = 2)$ Find x such that $\Pr(|X - 37| < x) = 0.5$

$$\Pr(|X - 37| < x) = \Pr(\frac{|X - 37|}{\sigma} < \frac{x}{2}) = \Pr(|Z| < \frac{x}{2}) = 0.5$$

$$\Pr(|Z| < \frac{x}{2}) = \Pr(Z < \frac{x}{2}) - \Pr(Z < -\frac{x}{2}) = 0.5$$

Let $\Pr(Z < \frac{x}{2}) = p$ → $\Pr(Z < \frac{x}{2}) - \Pr(Z < -\frac{x}{2}) = p - (1 - p) = 2p - 1 = 0.5$

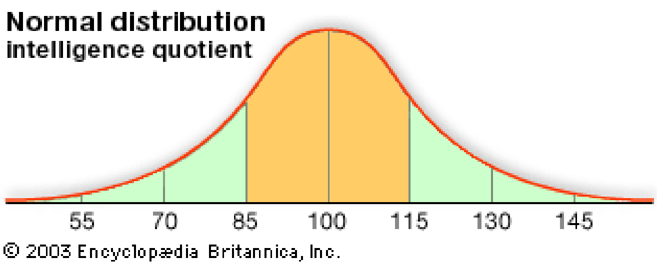
$p = 0.75$

0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852

From the table, $\Pr(Z < 0.67) = 0.7486$ and $\Pr(Z < 0.68) = 0.7517$

Thus, $\Pr(Z < 0.675) \approx 0.75$, $\frac{x}{2} \approx 0.675$, Ans: $x \approx 1.35$.

Note: $\Pr(|Z| < \frac{1.35}{2}) = \Pr(Z < 0.675) - \Pr(Z < -0.675) = 0.75 - 0.25 = 0.5$



Ex: $X \sim$ IQ score

Intelligence quotients (IQs) measured on the Stanford Revision of the Binet-Simon Intelligence Scale are normally distributed with a mean of 100 and a standard deviation of 15.

(a) Determine the percentage of people who have IQs between 115 and 140.

$$\Pr(115 < X < 140) = \Pr(\frac{115 - 100}{15} < \frac{X - \mu}{\sigma} < \frac{140 - 100}{15})$$

$$= \Pr(1 < Z < 2) = \Pr(Z < 2) - \Pr(Z < 1)$$

$$= 0.9772\text{-}0.8413 = 0.1359$$

(b) Determine the percentage of people who have IQs above 135.

$$\Pr(X > 135) = \Pr(\frac{X-\mu}{\sigma} > \frac{135-100}{15}) = \Pr(Z > 2.33)$$

$$= 1 - \Pr(Z < 2.33) = 1 - 0.9901 = 0.0009$$

(c) Determine the Q1 and Q3 IQ scores.

$$\Pr(X < Q_1) = \Pr(\frac{X-\mu}{\sigma} < \frac{Q_1 - 100}{15}) = \Pr(Z < \frac{Q_1 - 100}{15}) = 0.25$$

−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776

$$\frac{Q_1 - 100}{15} \approx -0.675 \rightarrow Q_1 \approx 100 - 10.125 = 89.88$$

$$\Pr(X < Q_3) = \Pr(\frac{X-\mu}{\sigma} < \frac{Q_3 - 100}{15}) = \Pr(Z < \frac{Q_3 - 100}{15}) = 0.75$$

0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852

$$\frac{Q_3 - 100}{15} \approx 0.675 \rightarrow Q_3 \approx 100 + 10.125 = 110.13$$

Log-normal distribution (will NOT be on the exam)

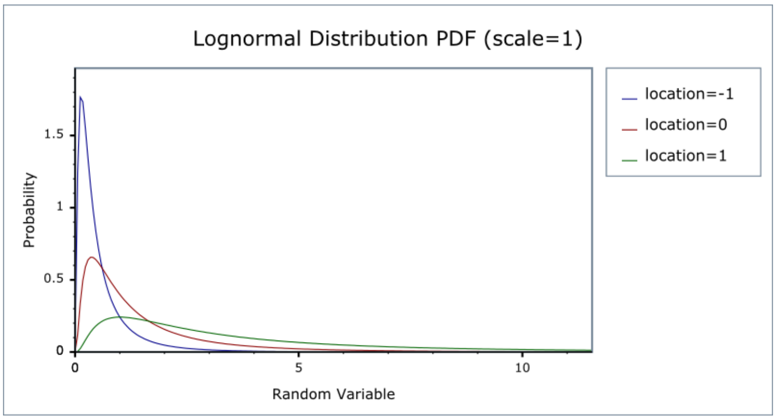
Consider $X > 0$ such that $\log(X) = Y \sim N(\mu, Var(Y) = \sigma^2)$

$\Pr(a < X < b)$

$= \Pr(\log(a) < \log(X) < \log(b))$ (since log(.) is monotone increasing)

$$= \Pr(\frac{\log(a)-\mu}{\sigma} < \frac{Y-\mu}{\sigma} < \frac{\log(b)-\mu}{\sigma})$$

$$= \Pr(\frac{\log(a)-\mu}{\sigma} < Z < \frac{\log(b)-\mu}{\sigma})$$



Normal distribution = Gaussian Distribution

http://old.nationalcurvebank.org/gaussdist/gaussdist.htm

The Gaussian Distribution, also called the Frequency Curve, Bell Curve, or Normal Distribution, is one of the most widely studied topics in all mathematics. Two of the most common variations of the equations are . . .

As a probability function:	*As the so-called Standard Normal Distribution:*
$P(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$	For mean $\mu = 0$ and variance $\sigma^2 = 1$, $P(x)\,dx = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,dx$

The Deutsche Mark note with Gauss' picture and his hallmark distribution curve has been replaced in circulation by the Euro.

Application of U(0,1) in generating continuous random variables

https://youtu.be/7DkfHf4y1Lg

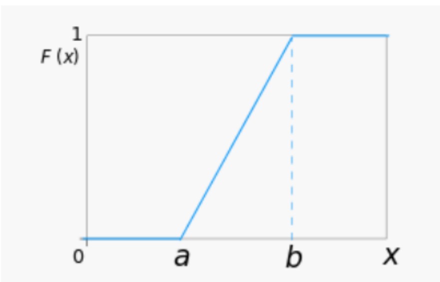
Suppose X *is a continuous random variable with the cumulative distribution function* $F(x)$ *such that* $F(x) = \Pr(X \le x)$. *Then* $U = F(X) \sim U(0,1)$

Proof: Let $U = F(X)$.

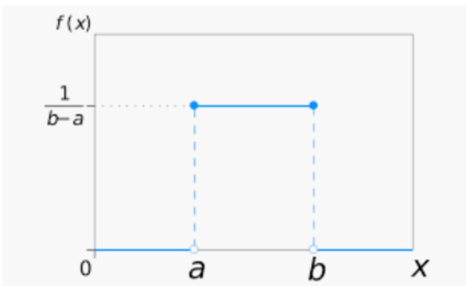
Since $F(.)$ is an increasing function, $F^{-1}(.)$ is also increasing. Hence

$$\begin{aligned}\Pr(U \le u) = \Pr(F(X) \le u) &= \Pr(X \le F^{-1}(u)) \\ &= F\{F^{-1}(u)\} \quad \text{(by definition } \Pr(X \le x) = F(x)\text{)} \\ &= u\end{aligned}$$

$\Pr(U \le u) = u$ implies that $U \sim U(0,1)$



Uniform distribution (continuous) - Wikipedia
en.wikipedia.org



Uniform distribution (continuous) - Wikipedia
en.wikipedia.org

See Lecture 6 for the idea of random number generation.

From Probability to Statistics

Random sample (隨機樣本):

- a good representative of the unknown population of interest,

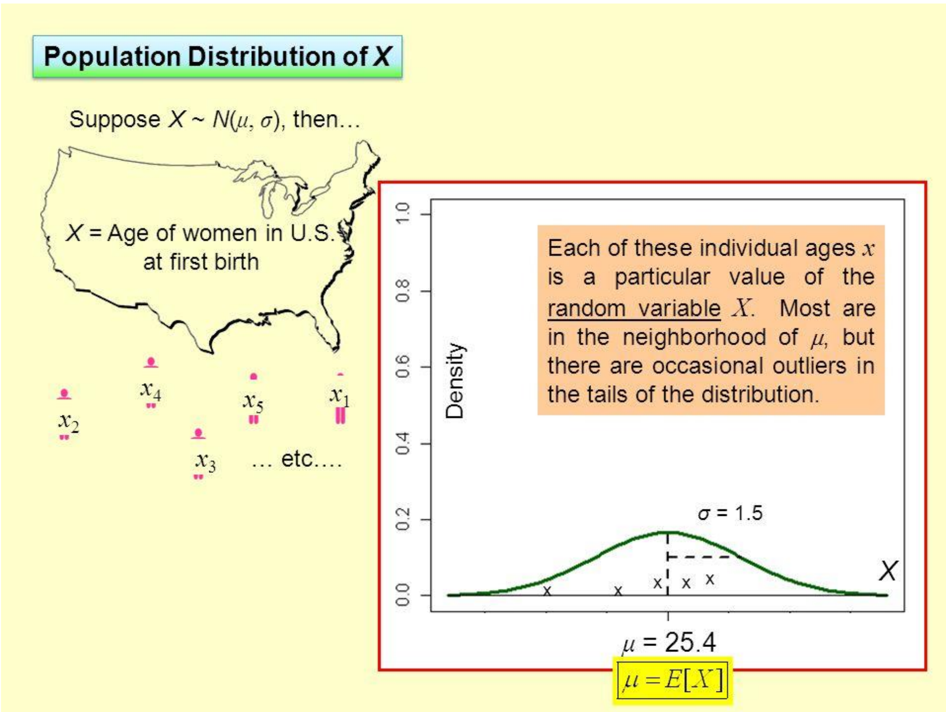
- Notation: $(X_1,\ldots,X_n)$ are *iid* distributed with $E(X_i)=\mu$, $Var(X_i)=\sigma^2$

Remarks:

- "iid": abbreviation of "identically and independently distribution"
  - identically distributed → X_i has the same distribution for all $i=1,\ldots,n$
  - independently distributed → X_i and X_j are distributed for $i \neq j$

Framework of statistical inferences:

- From the population, we define a random variable of interest and study some important parameters of the distribution.



- The purpose of sampling:
  - the population is too large
  - the outcome is unknown (use polls to predict the result of elections)

- Statistical inference:
  - Estimation
    - Point estimation
    - Interval estimation
  - Hypothesis testing

The second-half of the semester:

- Focus on $E(X_i) = \mu$
  - μ unknown (of interest), but σ^2 known
  - μ unknown (of interest), but σ^2 also unknown
- Paired sample or two samples: Focus on $\mu_1 - \mu_2$

Remark: "Probability" vs. "Statistics"

- In probability, the values of parameters such as p, μ and σ^2 are given. The main objective is to calculate $\Pr(a < X < b)$
- In statistics, the parameter values are unknown. The main objective is to make statistical inference of these parameters based on the random sample $(X_1, ..., X_n)$.

Big data: are we making a big mistake?

https://rss.onlinelibrary.wiley.com/doi/pdf/10.1111/j.1740-9713.2014.00778.x

- "Google Flu Trends": published in *Nature*

- Impacts:

Since every single data point can be captured, *making old statistical sampling techniques obsolete.* There is no need to know what causes what, because statistical correlation tells us what we need to know Scientific or statistical models aren't needed because, to quote "The End of Theory", a provocative essay published in Wired in 2008, "with enough data, the numbers speak for themselves".

- Author's claim: But a theory-free analysis of mere correlations is inevitably fragile. If you have no idea what is behind a correlation, you have no idea what might cause that correlation to break down.

Statisticians have spent the past 200 years figuring out what traps lie in wait when we try to understand the world through data. We must not pretend that the traps have all been made safe

Topic: Properties of the sample mean

Q: Why study $\overline{X}$?

A: We often use $\overline{X}$ to estimate μ given that $(X_1,\dots,X_n) \sim$ SRS (simple random sample)

Properties: $X_i \sim$ any distribution (i.e. discrete, continuous or mixed)

a. $\overline{X}$ is a random variable and has its distribution

b. $E(\overline{X}) = E(\frac{X_1+\dots+X_n}{n}) = \frac{1}{n}[E(X_1)+\dots+E(X_n)] = \frac{n\cdot\mu}{n} = \mu$

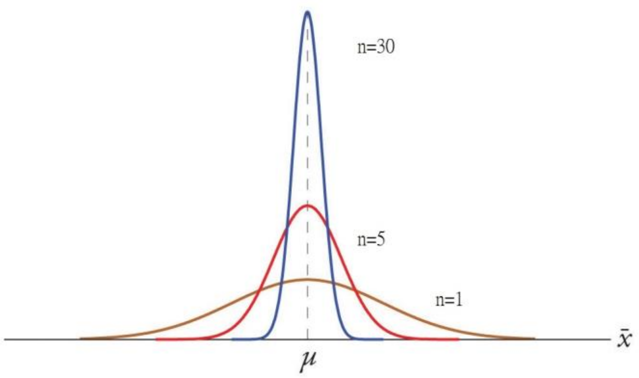
c. $Var(\overline{X}) = Var(\frac{X_1+\dots+X_n}{n}) = \frac{1}{n^2}[Var(X_1)+\dots+Var(X_n)]$

$$= \frac{1}{n^2}[\sigma^2+\dots+\sigma^2] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

Remarks:

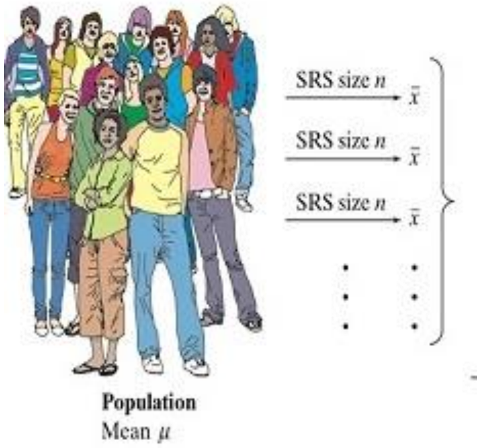
- $Var(\overline{X}) < Var(X_i)$ for $n \geq 2$

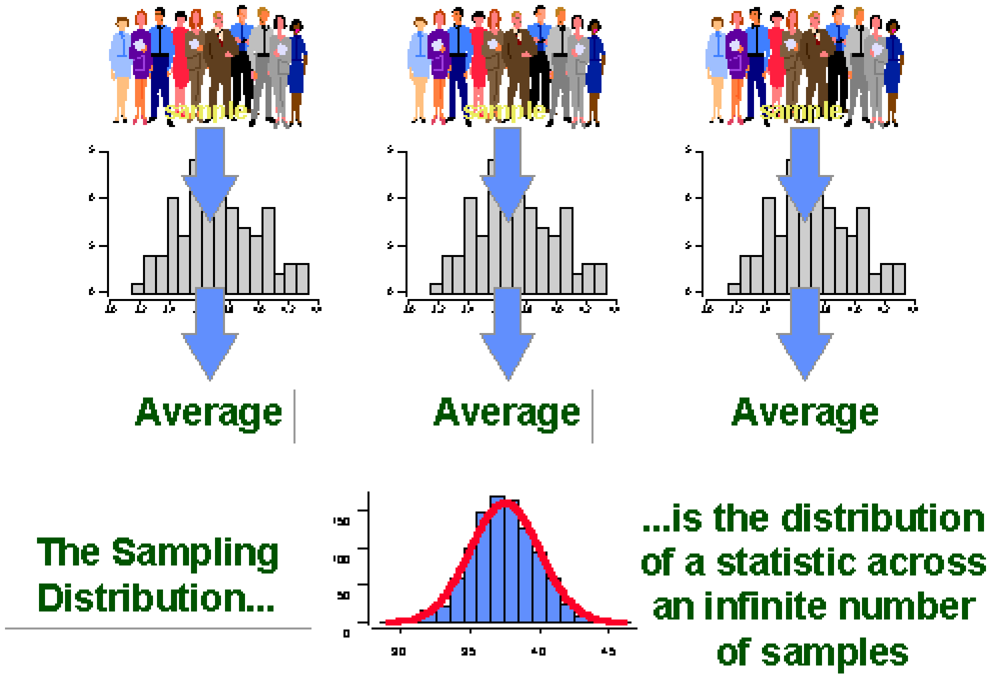
- $Var(\overline{X})$ decreases as n increases.

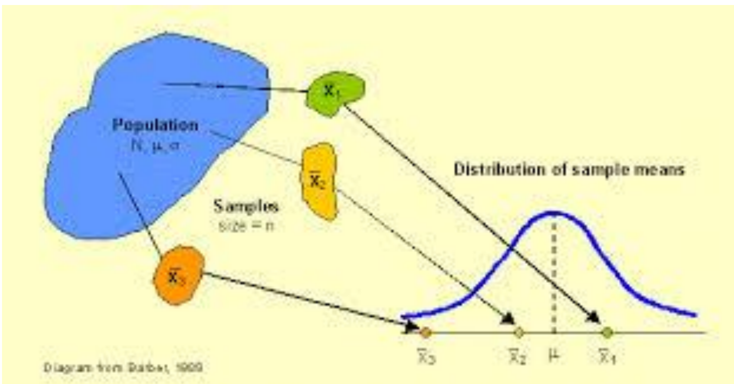


The distribution of $\overline{X}$:

- From the population, select a random sample of size n ($X_1,\dots,X_n$) and then compute its average.
- Repeat the sampling procedure r times and get r averages: denoted as $\overline{x}_n^{(1)},\dots,\overline{x}_n^{(R)}$.
- When $r \to \infty$, the distribution of $\overline{x}_n^{(1)},\dots,\overline{x}_n^{(R)}$ is the “sampling distribution” of $\overline{X}_n = (X_1+\dots+X_n)/n$


SRS size n
SRS size n
SRS size n
Population
Mean μ


Average
Average
Average
The Sampling Distribution...
...is the distribution of a statistic across an infinite number of samples


Population
Samples
Distribution of sample means

- **Sample mean under the normal population:**

When $X_i \overset{iid}{\sim} N(\mu, \sigma^2) \rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ for <u>all</u> n.

Equivalently $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \underset{\text{exactly}}{\sim} N(0,1)$

Example: Given a test that is normally distributed with a mean of 100 and a <u>standard deviation</u> of 12, find:

(a) the probability that <u>a single score</u> drawn at random will be greater than 110;

(b) the probability that <u>a sample of 25 scores</u> will have a mean greater than 105;

(d) the probability that the mean of <u>a sample of 16 scores</u> will be either less than 95 or greater than 105.

$X_i \overset{iid}{\sim} N(\mu = 100, \sigma^2 = 12^2)$

(a) $\Pr(X_i > 110) = \Pr(\frac{X-\mu}{\sigma} > \frac{110-\mu}{\sigma}) = \Pr(Z > \frac{110-100}{12}) = \Pr(Z > 0.83)$

$\Pr(Z > 0.83) = 1 - \Pr(Z \le 0.83) =$ 1- 0.9769 = 0.2033

0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

(b) $\Pr(\bar{X}_{n=25} > 105) = \Pr(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > \frac{105-100}{12/\sqrt{25}}) = \Pr(Z > \frac{105-100}{12/\sqrt{25}}) = \Pr(Z > 2.083)$

$\Pr(Z > 2.08) = 1 - \Pr(Z < 2.08) = 1 - 0.9812 = 0.0188$

1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

(c) $\Pr(\bar{X}_{n=16} < 95) + \Pr(\bar{X}_{n=16} > 105)$

$= \Pr(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{95-100}{12/\sqrt{16}}) + \Pr(Z > \frac{105-100}{12/\sqrt{16}})$

$= \Pr(Z < \frac{-5}{12/4}) + \Pr(Z > \frac{5}{12/\sqrt{16}})$

$= \Pr(Z < -1.67) + \Pr(Z > 1.67) = 2 \times 0.0475 = 0.095$

−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

Large-sample Properties of the Sample Mean

***** *The Law of Large Number (L.L.N., 大數法則):*

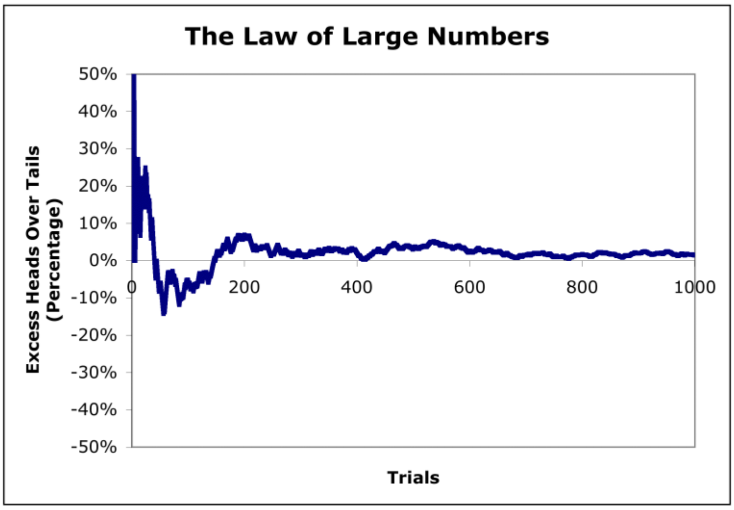
$$\bar{X} \to \mu \quad \text{as n large}$$

You can see that when $n \to \infty$, $Var(\bar{X}) = \dfrac{\sigma^2}{n} \to 0$,

that is $\bar{X}$ approaches to a fixed value (i.e. $E(X_i) = \mu$).

X-axis: n

Y-axis: $\left[\sum_{i=1}^{n} I(Y_i=1)/n\right] - 0.5 \quad (Y_i = 1 \Leftrightarrow \text{head occurs});$



請參照 R 補充講義

Example 1: X_i takes values of 1,2,3,4,5,6 with prob = 1/6

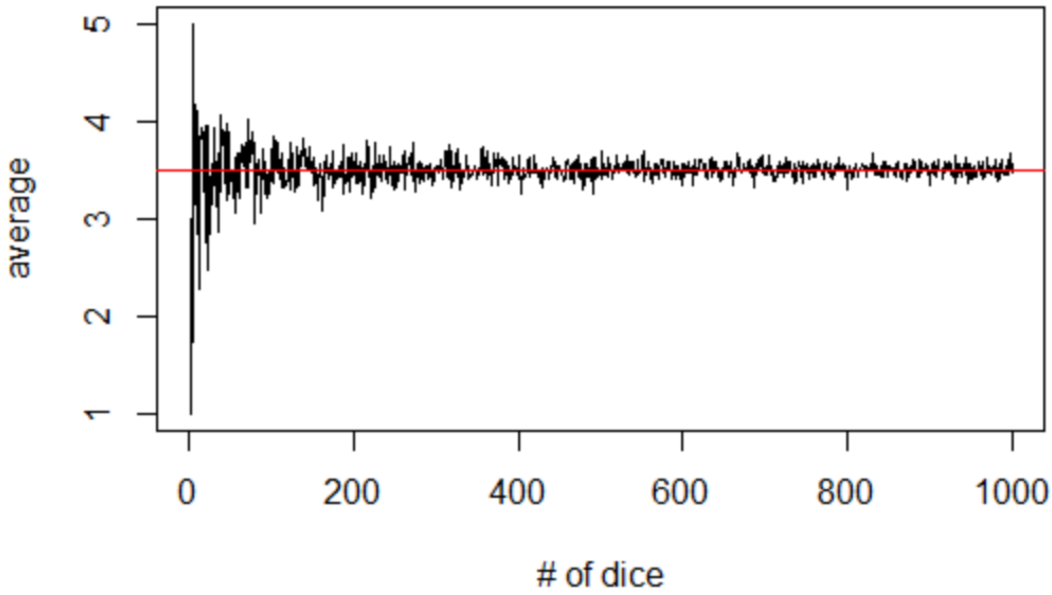
```
> die <- 1:6
> die
[1] 1 2 3 4 5 6
```

Sample 10 observations $X_1,\ldots,X_{10}$ from this distribution with replacement

```
> sample(die, size = 10, replace = TRUE)
 [1] 1 6 2 2 2 5 6 4 4 4
```

Write a function of $\bar{X}$ with n being an argument

```
> roll <- function(n) {  mean(sample(die, size = n, replace = TRUE))}
> plot(sapply(1:1000, roll), type = "l", xlab = "# of dice", ylab = "av
erage")
> abline(h = 3.5, col = "red")
```



Question: what is the limit of $\left(\sum_{i=1}^{n} X_i^2\right)/n$? → $E(X_i^2)$

$$\left(\sum_{i=1}^{n} g(X_i)\right)/n \;\rightarrow\; E[g(X_i)]$$

Note: When n gets larger, the sample mean of a random variable will converge to its expected value.

Law of Large Number in Simulations

Let $f(x)$ be the density of X

Objective: find $E(X)=\int xf(x)dx$ or $E[g(X)]=\int g(x)f(x)dx$

→ require integration

Approximation by LLN

Generate $X_i \sim^{iid}$ with density $f(x)$ for $i=1,\ldots,n$

- Use $\sum_{i=1}^{n} X_i/n$ to estimate $E(X)=\int xf(x)dx$
- Use $\sum_{i=1}^{n} g(X_i)/n$ to estimate $E[g(X)]=\int g(x)f(x)dx$
- The approximation is more accurate if n is very large.

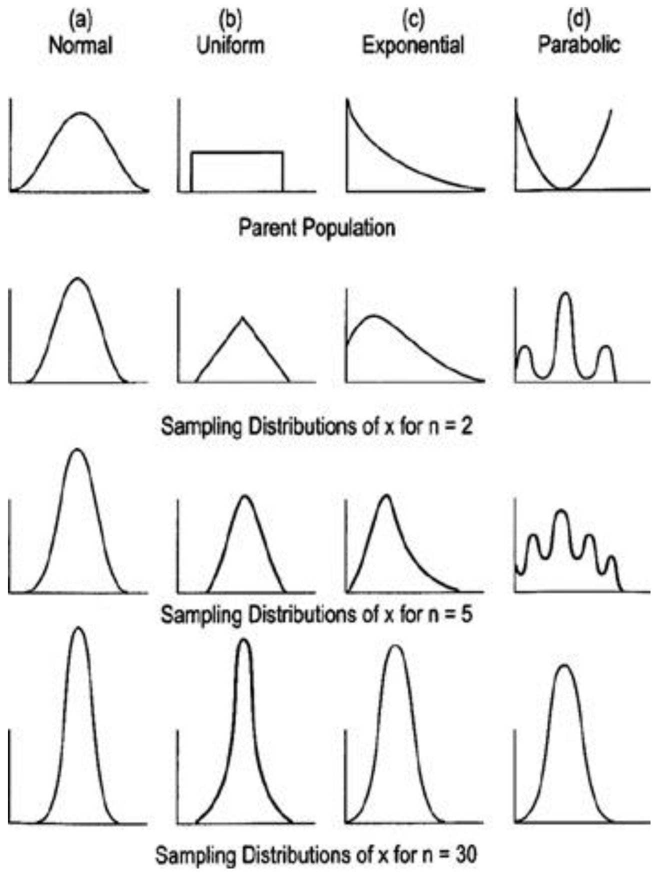
Central Limit Theorem **(中央極限定理)**

Random sample: $(X_1,...,X_n)$ where $X_i \sim$ <u>any</u> distribution with $E(X_i)=\mu$ and $E(X_i)=\sigma^2$. Then

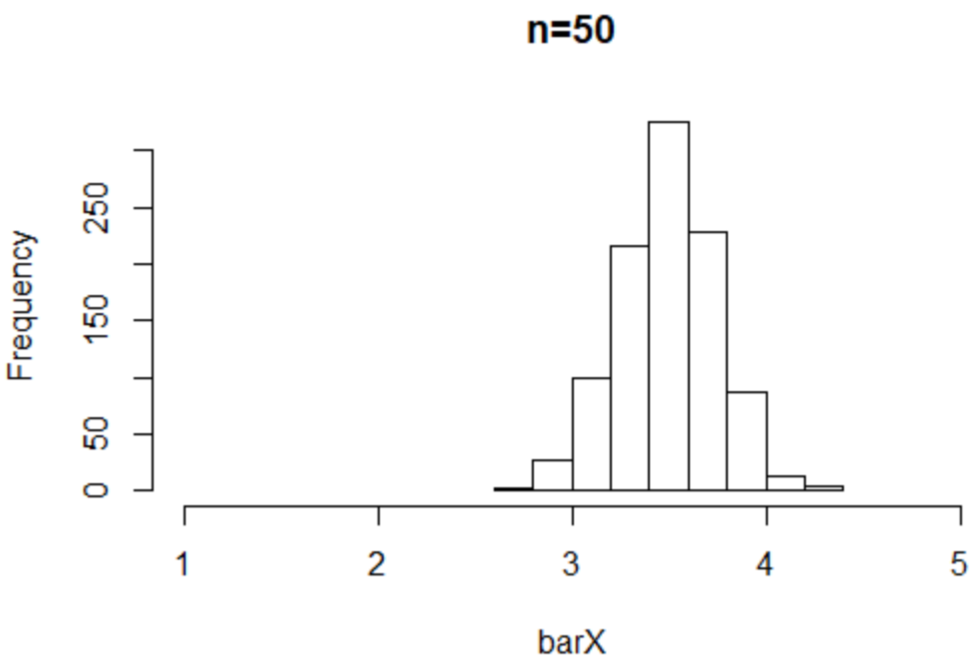
$$\bar{X}_n \underset{n\to\infty}{\sim} N(\mu, Var(\bar{X}_n)=\sigma^2/n) \Leftrightarrow \frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \underset{n\to\infty}{\sim} N(0,1)$$

Plot: The distribution of $\bar{X}_n$ with $n=1,2,5,30$ (上→下) from four different distributions

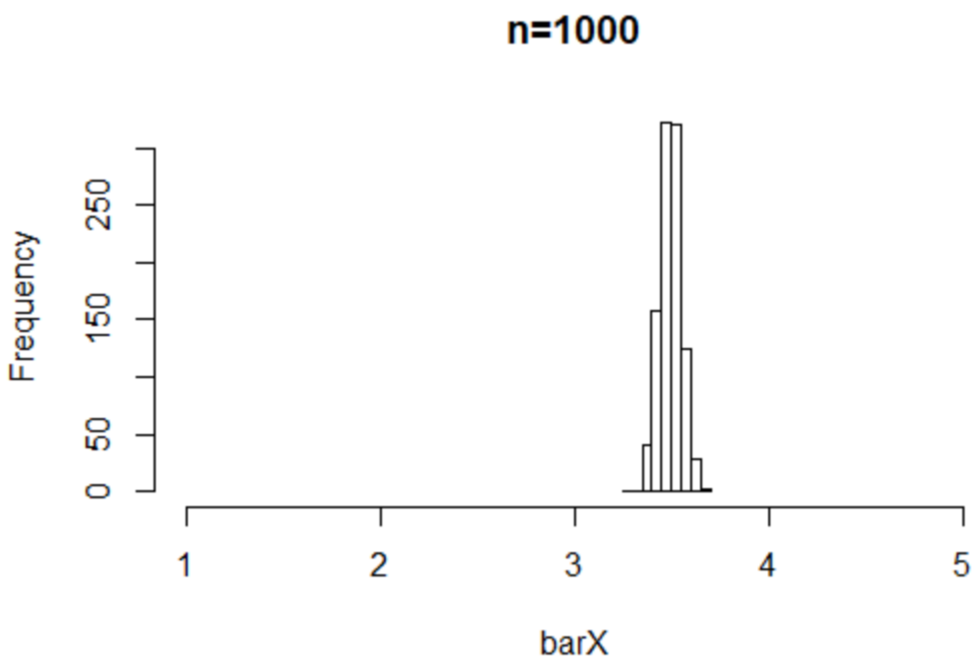
(n=1 → a: Bell curve; b: Uniform,c:skewed,d: Parabola: Open upward).



```
> for (i in 1:1000){barX[i]=mean(sample(1:6,50,replace=T))}
> hist(barX,freq=NULL,main="n=50",xlim=c(1,5))
```



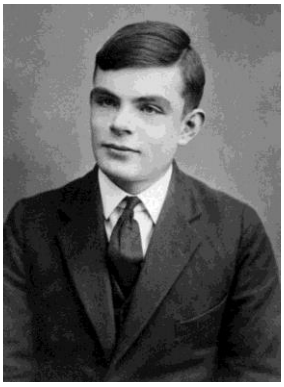

```
> for (i in 1:1000){barX[i]=mean(sample(1:6,1000,replace=T))}
> hist(barX,freq=NULL,main="n=1000",xlim=c(1,5))
```

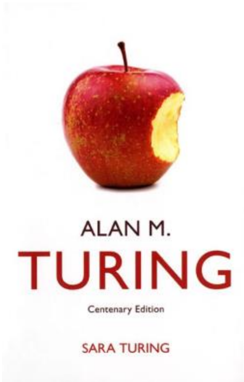


Story: Alan Turing (圖靈) and CLT

Alan Turing (1912–1954) was an English mathematician, computer scientist, logician, cryptanalyst, philosopher, and theoretical biologist. Turing was highly influential in the development of theoretical computer science, providing a formalization of the concepts of algorithm and computation with the Turing machine, which can be considered a model of a general purpose computer.

Turing is widely considered to be <u>the father of theoretical computer science and artificial intelligence</u>. Despite these accomplishments, he was never fully recognized in his home country during his lifetime due to his homosexuality, which was then a crime in the UK.





During the Second World War, Turing worked for the Government Code and Cypher School (GC&CS) at Bletchley Park, Britain's codebreaking centre that produced Ultra intelligence. … Turing played a pivotal role in cracking intercepted coded messages that enabled the Allies to defeat the Nazis in many crucial engagements, including the Battle of the Atlantic, and in so doing helped win the war. … It has been estimated that this work shortened the war in Europe by more than two years and saved over fourteen million lives.

After the war, Turing worked at the National Physical Laboratory, where he designed the ACE, among the first designs for a stored-program computer. In 1948 Turing joined Max Newman's Computing Machine Laboratory at the Victoria University of Manchester, where he helped develop the Manchester computers and became interested in mathematical biology.

Turing was prosecuted in 1952 for homosexual acts …. He accepted chemical castration treatment, with DES, as an alternative to prison. Turing died in 1954, 16 days before his 42nd birthday, from cyanide poisoning. An inquest determined his death as suicide, but it has been noted that the known evidence is also consistent with accidental poisoning. **In 2009, following an Internet campaign, British Prime Minister Gordon Brown made an official public apology on behalf of the British government for "the appalling way he was treated." Queen Elizabeth II granted him a posthumous pardon in 2013**.

Alan Turing and the Central Limit Theorem

Alan Turing and the Central Limit Theorem

S. L. Zabell

Mathematical Monthly (1995)

In 1934 Turing, while still an undergraduate, rediscovered a version of Lindeberg's 1922 Central Limit theorem and … (unpublished).

Turing went up to Cambridge as an undergraduate in the Fall Term of 1931, having gained a scholarship to King's College.

Two years later, during the course of his studies, Turing attended a series of lectures on the Methodology of Science, … one topic Eddington discussed was the tendency of experimental measurements subject to errors of observation to often have an approximately normal or Gaussian distribution. …

Turing set out to derive a rigorous mathematical proof of what is today termed the central limit theorem for independent (but not necessarily identically distributed) random variables. This revision, entitled "On the Gaussian Error Function," was completed and submitted in November, 1934. On the strength of this paper Turing was elected a Fellow of King's four months later (March 16, 1935) at the age of 22; his nomination supported by the group theorist Philip Hall and the economists John Maynard Keynes and Alfred Cecil Pigou. Later that year the paper was awarded the prestigious Smith's prize by the University. Turing never published his paper. Its major result had been anticipated, although, as will be seen, it contains other results that were both interesting and novel at the time.

But in the interim Turing's mathematical interests had taken a very different turn. **During the spring of 1935, awaiting the outcome of his application for a Fellowship at King's, Turing attended a course of lectures by the topologist M. H. A. Newman on the Foundations of Mathematics.** During the International Congress of Mathematicians in 1928, David Hilbert had posed three questions: is mathematics complete (that is, can every statement in the language of number theory be either proved or disproved?), is it consistent, and is it decidable? (This last is the Entscheidungsproblem, or the "decision problem"; does there exist an algorithm for deciding whether or not a specific mathematical assertion does or does not have a proof.)

Bivariate Normal Distribution https://www.youtube.com/watch?v=2xHeg6n89tU

6.4 Bivariate Normal Distribution

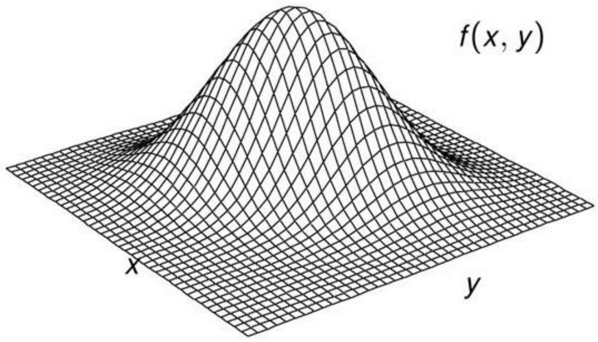
The continuous random variables X and Y with joint pdf

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

on $\mathcal{A} = \{-\infty < x < \infty, -\infty < y < \infty\}$ with parameter space

$\Omega = \{(\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho) | -\infty < \mu_X < \infty, -\infty < \mu_Y < \infty, \sigma_X > 0, \sigma_Y > 0, -1 < \rho < 1$

are *bivariate normal random variables* with parameters μ_X, μ_Y, σ_X, σ_Y, and ρ



Bivariate Normal Distribution $\begin{bmatrix} X \\ Y \end{bmatrix}$

- characterized by the mean vector $\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$ and the 2×2 covariance matrix

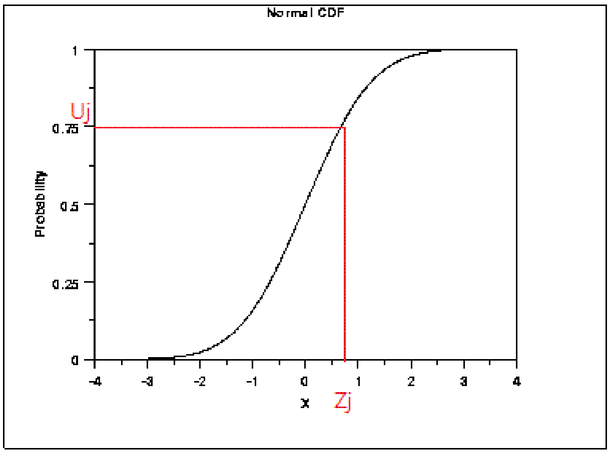
$\begin{bmatrix} \sigma_X^2 & Cov(X,Y) \\ Cov(X,Y) & \sigma_Y^2 \end{bmatrix}$ where $Cov(X,Y) = \rho_{X,Y}\sigma_X\sigma_Y$

Multivariate Normal (Gaussian) Distribution $\begin{bmatrix} X_1 \\ \dots \\ X_p \end{bmatrix} \sim N_p(\begin{bmatrix} \mu_1 \\ \dots \\ \mu_p \end{bmatrix}, \Sigma)$

→ 描述 p 個變數的聯合行為（個別的 X_i 均為 Normal)

- characterized by the $p\times 1$ mean vector and the $p\times p$ covariance matrix Σ with the (i, j) component being $Cov(X_i, X_j) = \rho_{X_i,X_j}\sigma_{X_i}\sigma_{X_j}$

Applications in Finance: $\begin{bmatrix} X_1 \\ \ldots \\ X_p \end{bmatrix}$



= 投資的組合 ~ usually not be normal

Gaussian Copula

$$F_i(X_i) = U_i \Rightarrow \Phi^{-1}(U_i) = Z_i$$

(先將原來的 X_i 轉到 U(0,1), 再透過 U_i 轉到 Z_i ~N(0,1))

再建立相關模型 $\begin{bmatrix} Z_1 \\ \ldots \\ Z_p \end{bmatrix} \sim N_p(\begin{bmatrix} 0 \\ \ldots \\ 0 \end{bmatrix}, \Sigma)$ Σ: 對角線 = 1, 其他 = (Z_i, Z_j) 相關係數

2007年–2008年全球金融危機（英語：Financial crisis of 2007–2008），又稱**2008年世界金融危機**、**次貸危機**、**信用危機**、**2008年華爾街金融危機**、**2008年金融崩潰**，在2008年又出現了**金融海嘯**及**華爾街海嘯**等名稱，是一場在2007年8月9日開始浮現的金融危機。自次級房屋信貸危機爆發後，投資者開始對抵押證券的價值失去信心，引發流動性危機。即使多國中央銀行多次向金融市場注入巨額資金，也無法阻止這場金融危機的爆發。直到2008年9月[1]，這場金融危機開始失控，並導致多間相當大型的金融機構倒閉或被政府接管，引發經濟衰退。

导致华尔街灾难的数学模型Gaussian copula function --- 源自 ...
https://sites.google.com › site › zgzqit › securitiesit › daozhi... ▾ 轉為繁體網頁
2009年4月25日 - 导致华尔街灾难的数学模型Gaussian copula function --- 源自中国人 ... 当因金融系统基础动摇而爆发的危机吞噬了数万亿美元，使全球银行体系 ...

外電：重傷華爾街的中國人數學模型| 大紀元
www.epochtimes.com › 首頁 › 新聞 › 北美新聞 ▾
2009年3月8日 - 【大紀元3月8日訊】（大紀元記者吳英編譯）自金融危機以來，媒體、學術界及 ... 模型計算，稱之為「高斯－聯結相依函數」（Gaussian copula function）。

次贷危机回忆录：怪数学公式or 怪市场太贪婪？ | 每日经济新闻
www.nbd.com.cn › articles ▾ 轉為繁體網頁
2017年7月27日 - 十年前，引爆全球金融危机的那场美国次贷危机，就像恶魔的影子一样， ... 违约相关性模型进行描述的公式：高斯联结相依函数(Gaussian Copula)。