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1. a) Solve $Xw = y$,

$$w = (USV^T)^{-1}y = VS^{-1}U^{-1}y$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{r}\begin{bmatrix}r & 0\\ 0 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}r}\begin{bmatrix}r+1\\ r-1\end{bmatrix}.$$

$$\|w\|_2^2 = \frac{1}{2r^2}(r^2+2r+1+r^2-2r+1) = 1+\frac{1}{r^2}$$

if $r = 0.1$, $X = \frac{1}{r} = 10$, $\|w\|_2^2 = \left(\frac{10}{\sqrt{2}}\begin{bmatrix}11/10\\ -9/10\end{bmatrix}\right)^2 = 101$

if $r = 10^{-8}$, similar as above, $\|w\|_2^2 = 10^{16}+1$.

b).

$$w = \frac{1}{2\sqrt{2}r}\begin{bmatrix}r+1 & r-1 & r-1 & r+1\\ r-1 & r+1 & r+1 & r-1\end{bmatrix}\begin{bmatrix}1+\varepsilon\\0\\0\\1\end{bmatrix} = w_0 + w_\varepsilon.$$

$$\to \quad w_0 = \frac{1}{\sqrt{2}r}\begin{bmatrix}r+1\\ r-1\end{bmatrix}, \quad w_\varepsilon = \frac{\varepsilon}{2\sqrt{2}r}\begin{bmatrix}r+1\\ r-1\end{bmatrix},$$

$$\|w_\varepsilon\|_2^2 = \frac{\varepsilon^2}{4}\left(1+\frac{1}{r^2}\right),$$

if $r = 0.1$, $\|w_\varepsilon\|_2^2 = \frac{10^{-4}}{4}(1+100) = 0.0025$

if $r = 10^{-8}$, $\|w_\varepsilon\|_2^2 = \frac{10^{-4}}{4}(1+10^{16}) = 2.5\times 10^{[illegible]}$

Norm of perturbation increase as condition number increase

c) From a) & b),

easy to get $w = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2+\varepsilon \\ 2+\varepsilon \end{bmatrix} = w_0 + w_\varepsilon$

$w_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $w_\varepsilon = \frac{\varepsilon}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So, $W = W_0 + W_\varepsilon = \frac{201}{200\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ when $\gamma = 0.1$ & $\gamma = 10^{-8}$,

$\|w_\varepsilon\|_2^2 = 2.5 \times 10^{-5}$.

and $\|w_\varepsilon\|_2^2$ remains consistent even if we ~~change~~ increase the conditioned number.

Compare with b), $\|w_\varepsilon\|_2^2 \downarrow$ when $\gamma = 0.1$ & $\gamma = 10^{-8}$.