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40. if n is odd, we find

f	I	τ	σ	$\tau\circ\sigma$
mon(f)	z_1^{2n}	$z_1^2 z_2^{n-1}$	z_2^n	z_2^n.

then $P_G(z_1, z_2, \cdots, z_{2n}) = \dfrac{z_1^{2n} + z_1^2 z_2^{n-1} + 2z_2^n}{4}$

# of 2-sided n-ominoes is $\dfrac{p^{2n} + p^{n+1} + 2p^n}{4}$

if n is even, we find

f	I	τ	σ	$\tau\circ\sigma$
mon(f)	z_1^{2n}	z_2^n	z_2^n	z_2^n

then $P_G(z_1, z_2, \cdots, z_{2n}) = \dfrac{z_1^{2n} + 3z_2^n}{4}$

# of 2-sided n-ominoes $= \dfrac{p^{2n} + 3p^n}{4}$

41. let cycle index of $G = D_6$ be $\sum_{f\in G} \text{mon}(f) \cdot |G|^{-1}$.

then,

f	I	ρ	ρ^2	ρ^3	ρ^4	ρ^5	τ	$\rho\circ\tau$	$\rho^2\circ\tau$	$\rho^3\circ\tau$	$\rho^4\circ\tau$	$\rho^5\circ\tau$
mon(f)	z_1^6	z_6	z_3^2	z_2^3	z_3^2	z_6	z_2^3	$z_1^2 z_2^2$	z_2^3	$z_1^2 z_2^2$	z_2^3	$z_1^2 z_2^2$

so, cycle index $= \dfrac{z_1^6 + 3z_1^2 z_2^2 + 4z_2^3 + 2z_3^2 + 2z_6}{12}$

43. We set the edges of square as 1, 2, 3, 4 (ie, $X = \{1,2,3,4\}$). G is the edge-symmetry group, with dihedral group D_4 order 8, $G = \{\rho^i\}_{i=0}^{3} \cup \{\rho^i \circ \tau\}_{i=0}^{3}$

$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

By definition, cycle index of G is $\dfrac{\sum_{f\in G} \text{mon}(f)}{|G|}$

We get

f	I	ρ	ρ^2	ρ^3	τ	$\rho\circ\tau$	$\rho^2\circ\tau$	$\rho^3\circ\tau$
mon(f).	z_1^4	z_4	z_2^2	z_4.	z_2^2	$z_1^2 z_2$	z_2^2	$z_1^2 z_2$

Then, the cycle index $= \dfrac{z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4}{8}$.

44. By exercise 43, we get (r refers to red, b refers to blue).

$$P_G(r+b, r^2+b^2, r^3+b^3, r^4+b^4) = \frac{(r+b)^4+2(r+b)^2(r^2+b^2)+3(r^2+b^2)^2+2(r^4+b^4)}{8}$$

Then, $P_G(k,k,k,k)$ = # nonequivalent colorings with k colors $= \frac{k^4+2k^3+3k^2+2k}{8}$

46. As usual, label beads $1,2,\dots,n$, $X=\{1,2,\dots,n\}$. And G is the symmetry group, with dihedral group D_n of order $2n$. So $G=\{\rho^i\}_{i=0}^{n-1}\cup\{\rho^i\circ\tau\}_{i=0}^{n-1}$ when ρ is clockwise rotation and τ is reflection about line of symmetry

By definition, cycle index of $G = \frac{\sum_{f\in G} mon(f)}{|G|}$

Since the monomial for ρ^i is z_n, $\rho^i\circ\tau$ is $z_1z_2^{(n-1)/2}$,

$$P_G(z_1,z_2,\dots,z_n) = \frac{z_1^n+(n-1)z_n+nz_1z_2^{(n-1)/2}}{2n}.$$

# of necklaces $= \frac{k^n+(n-1)k+nk^{(n+1)/2}}{2n}$.

47. We define $G=\{\rho^i\}_{i=0}^{3}$ with ρ a clockwise rotation.

By definition, $P_G(z_1,z_2,\dots,z_9)=\frac{\sum_{f\in G} mon(f)}{|G|}$

Since

f	I	ρ	ρ^2	ρ^3
mon(f)	z_1^9	$z_1z_4^2$	$z_1z_2^4$	$z_1z_4^2$

Then, $P_G(z_1,z_2,\dots,z_9) = \frac{z_1^9+2z_1z_4^2+z_1z_2^4}{4}$

We get the generating func. of $P_G(r+b, r^2+b^2,\dots,r^9+b^9) = \frac{(r+b)^9+2(r+b)(r^4+b^4)^2+(r+b)(r^2+b^2)^4}{4}$

So, # of colorings $= \frac{2^9+2\cdot 2^3+2^5}{2} = 140$.

48. We define G be the symmetry group, with $G=\{\rho^i\}_{i=0}^{3}\cup\{\rho^i\circ\tau\}_{i=0}^{3}$ (i.e., ρ is the clockwise 90 degree rotation, τ is the upside down window condition)

By definition, $P_G(z_1,z_2,\dots,z_9)=\frac{\sum_{f\in G} mon(f)}{|G|}$

Since

f	I	ρ	ρ^2	ρ^3	τ	$\rho\circ\tau$	$\rho^2\circ\tau$	$\rho^3\circ\tau$
mon(f)	z_1^9	$z_1z_4^2$	$z_1z_2^4$	$z_1z_4^2$	$z_1^3z_2^3$	$z_1^3z_2^3$	$z_1^3z_2^3$	$z_1^3z_2^3$

then $P_G(z_1, z_2, \dots z_9) = \dfrac{z_1^9 + 2z_1z_4^2 + z_1z_2^4 + 4z_1^3z_2^3}{8}$

For color r & b, $P_G(r+b, r^2+b^2, \dots r^9+b^9) = \dfrac{(r+b)^9 + 2(r+b)(r^4+b^4)^2 + (r+b)(r^2+b^2)^4 + 4(r+b)^3(r^2+b^2)^3}{8}$

# of stained glass windows $= \dfrac{2^9 + 2\cdot 2^3 + 2^5 + 4\cdot 2^6}{8} = 102$

49. We now get a checkerboard of size 4×4, so we get.

f	I	ρ	ρ^2	ρ^3	τ	$\rho\circ\tau$	$\rho^2\circ\tau$	$\rho^3\circ\tau$
mon(f)	z_1^{16}	z_4^4	z_2^8	z_4^4	z_2^8	$z_1^4z_2^6$	z_2^8	$z_1^4z_2^6$

So, $P_G(z_1, z_2, \dots z_{16}) = \dfrac{z_1^{16} + 2z_4^4 + 3z_2^8 + 2z_1^4z_2^6}{8}$

For colors r & b, $P_G(r+b, r^2+b^2, \dots, r^{16}+b^{16}) = \dfrac{(r+b)^{16} + 2(r^4+b^4)^4 + 3(r^2+b^2)^8 + 2(r+b)^4(r^2+b^2)^6}{8}$

# of stained glass window $= \dfrac{2^{16} + 2\cdot 2^4 + 3\cdot 2^8 + 2\cdot 2^{10}}{8}$

50. Similar to exercise 46, P_G be the cycle index

$$P_G(z_1, z_2, \dots z_p) = \frac{z_1^p + (p-1)z_p + p z_1 z_2^{(p-1)/2}}{2p}$$

For color r & b, $P_G(r+b, r^2+b^2, \dots r^p+b^p) = \dfrac{(r+b)^p + (p-1)(r^p+b^p) + p(r+b)(r^2+b^2)^{\frac{(p-1)}{2}}}{2p}$

51. $G = D_n$ be the corner-symmetry group of regular n-gon, with $G = \{\rho^i\}_{i=0}^{n-1} \cup \{\rho^i\circ\tau\}_{i=0}^{n-1}$ (ρ is clockwise rotation, τ is reflection about the line of symmetry passing through corners 1 and p.

By definition, $P_G(z_1, z_2, \dots z_n) = \dfrac{\sum_{f\in G} \text{mon}(f)}{|G|}$.

Since monomial ρ^i is z_n (i is odd), z_p^2 (i is even), z_2^p ($i=p$),
monomial $\rho^i\circ\tau$ is z_2^p (i is odd), $z_1^2z_2^{p-1}$ (i is even)

So, $P_G(z_1, z_2, \dots z_n) = \dfrac{z_1^n + (p-1)z_n + (p-1)z_p^2 + (p+1)z_2^p + p z_1^2 z_2^{p-1}}{4p}$

(replace $n = 2p$)

52. Similar as exercise 51, we define P_G as the cycle index

By definition, $P_G(z_1, z_2, \cdots z_r) = \dfrac{\sum_{f \in G} \operatorname{mon}(f)}{|G|}$.

And at this time, we set $z_i = r^i + b^i$,

So generating func. $= \dfrac{(r+b)^{2p} + (p-1)(r^p+b^p)^2 + (p-1)(r^{2p}+b^{2p}) + (p+1)(r^2+b^2)^p + p(r+b)^2(r^2+b^2)^{p-1}}{4p}$

53. We define G be the symmetry group with $G = \{\rho^i\}_{i=0}^{2}$ (ρ is 120° rotation)

By definition, $P_G(z_1, z_2, \cdots z_{10}) = \dfrac{\sum_{f \in G} \operatorname{mon}(f)}{|G|}$

Since

f	I	ρ	ρ^2
$\operatorname{mon}(f)$	z_1^{10}	$z_1 z_3^3$	$z_1 z_3^3$

, $P_G(z_1, z_2 \cdots z_{10}) = \dfrac{z_1^{10} + 2z_1 z_3^3}{3}$

For color r & b, $P_G(r+b, r^2+b^2, \cdots, r^{10}+b^{10}) = \dfrac{(r+b)^{10} + 2(r+b)(r^3+b^3)^3}{3}$.

if we are also allowed to turn over the array,

$G = \{\rho^i\}_{i=0}^{2} \cup \{\rho^i \circ \tau\}_{i=0}^{2}$ (τ is reflection about a vertical line through the $\triangle$ containing center and one corner).

Since

f	I	ρ	ρ^2	τ	$\rho \circ \tau$	$\rho^2 \circ \tau$
$\operatorname{mon}(f)$	z_1^{10}	$z_1 z_3^3$	$z_1 z_3^3$	$z_1^2 z_2^4$	$z_1^2 z_2^4$	$z_1^2 z_2^4$

,

$P_G(z_1, z_2, \cdots, z_{10}) = \dfrac{z_1^{10} + 2z_1 z_3^3 + 3z_1^2 z_2^4}{6}$

For color r & b, generating func $= P_G(r+b, r^2+b^2, \cdots, r^{10}+b^{10})$

$= \dfrac{(r+b)^{10} + 2(r+b)(r^3+b^3)^3 + 3(r+b)^2(r^2+b^2)^4}{6}$.