period_10_starter_script-2

February 25, 2021

0.1 CS/ECE/ME532 Period 10 Activity

Estimated Time:

P1: 25 mins

P2: 25 mins

0.1.1 Preambles

[145]:
```
import numpy as np # numpy
from scipy.io import loadmat # load & save data
from scipy.io import savemat
import matplotlib.pyplot as plt # plot
np.set_printoptions(formatter={'float': lambda x: "{0:0.2f}".format(x)})
```

0.1.2 Q1. K-means

Let $\boldsymbol{A} = \begin{bmatrix} 3 & 3 & 3 & -1 & -1 & -1 \\ 1 & 1 & 1 & -3 & -3 & -3 \\ 1 & 1 & 1 & -3 & -3 & -3 \\ 3 & 3 & 3 & -1 & -1 & -1 \end{bmatrix}$. Use the provided script to help you complete the problem.

[146]:
```
A = np.
 ↪array([[3,3,3,-1,-1,-1],[1,1,1,-3,-3,-3],[1,1,1,-3,-3,-3],[3,3,3,-1,-1,-1]],␣
 ↪float)
rows, cols = A.shape
print('A = \n', A)
```

```
A =
 [[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
```

[147]:
```
# numpy iterates over the 0th dimension first (over the rows)

for each_entry in A:
    print(each_entry) # This prints iterates the "rows" of A
```

```
for each_entry in A.transpose():
    print(each_entry) # This prints iterates the "columns" of A
```

```
[3.00 3.00 3.00 -1.00 -1.00 -1.00]
[1.00 1.00 1.00 -3.00 -3.00 -3.00]
[1.00 1.00 1.00 -3.00 -3.00 -3.00]
[3.00 3.00 3.00 -1.00 -1.00 -1.00]
[3.00 1.00 1.00 3.00]
[3.00 1.00 1.00 3.00]
[3.00 1.00 1.00 3.00]
[-1.00 -3.00 -3.00 -1.00]
[-1.00 -3.00 -3.00 -1.00]
[-1.00 -3.00 -3.00 -1.00]
```

a) Understand the following implementation of the k-means algorithm and fill in the blank to define the distance function.

[148]:
```
def dist(x, y):
    """
    this function takes in two 1-d numpy as input an outputs
    Euclidean the distance between them
    """
    return (x-y).transpose()@(x-y) ## Fill in the blank: Recall the 'distance'␣
↪function used in the kMeans algorithm

def kMeans(X, K, maxIters = 20):
    """
    this implementation of k-means takes as input (i) a matrix X
    (with the data points as columns) (ii) an integer K representing the number
    of clusters, and returns (i) a matrix with the K columns representing
    the cluster centers and (ii) a list C of the assigned cluster centers
    """
    X_transpose = X.transpose()
    centroids = X_transpose[np.random.choice(X.shape[0], K)]
    for i in range(maxIters):
        # Cluster Assignment step
        C = np.array([np.argmin([dist(x_i, y_k) for y_k in centroids]) for x_i␣
↪in X_transpose])
        # Update centroids step
        for k in range(K):
            if (C == k).any():
                centroids[k] = X_transpose[C == k].mean(axis = 0)
            else: # if there are no data points assigned to this certain␣
↪centroid
                centroids[k] = X_transpose[np.random.choice(len(X))]
    return centroids.transpose() , C
```

b) Use the K-means algorithm to represent the columns of A with a single cluster.

[149]:
```
# k-means with 1 cluster
centroids, C = kMeans(A,1) ## Fill in the blank: call the "kMeans" algorithm␣
 ↪with proper input arguments
print('A = \n', A)
print('centroids = \n', centroids)
print('centroid assignment = \n', C)
```

```
A =
 [[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
centroids =
 [[1.00]
 [-1.00]
 [-1.00]
 [1.00]]
centroid assignment =
 [0 0 0 0 0 0]
```

c) Construct a matrix $\hat{A}_{r=1}$ whose i-th column is the centroid corresponding to the i-th column of A. Note that this can be viewed as a rank-1 approximation to A. Compare the rank-1 approximation to the original matrix and explain the nature of the approximation in terms of the properties of the K-means algorithm.

[150]:
```
# Construct rank-1 approximation using cluster
centroids_transposed = centroids.transpose() # transpose "centroids" to iterate␣
 ↪over columns
A_hat_1 = (np.array([centroids for i in range(len(C))])).transpose() # Fill in␣
 ↪the blank: pick the columns of centroids indexed by C and then transpose it␣
 ↪again
print('Rank-1 Approximation, \n A_hat_1 = \n', A_hat_1)
```

```
Rank-1 Approximation,
 A_hat_1 =
 [[[1.00 1.00 1.00 1.00 1.00 1.00]
  [-1.00 -1.00 -1.00 -1.00 -1.00 -1.00]
  [-1.00 -1.00 -1.00 -1.00 -1.00 -1.00]
  [1.00 1.00 1.00 1.00 1.00 1.00]]]
```

d) Repeat b) and c) with $K = 2$. Compare the rank-2 approximation to the original matrix and explain the nature of the approximation in terms of the properties of the K-means algorithm.

[151]:
```
# k-means with 2 cluster
centroids, C = kMeans(A,2) ## Fill in the blank: call the "kMeans" method with␣
 ↪proper input arguments
```

```
print('A = \n', A)
print('centroids = \n', centroids)
print('centroid assignment = \n', C)
centroids_transposed = centroids.transpose() # transpose "centroids" to iterate␣
↪over columns
A_hat_2 = [centroids_transposed[0] for i in range(len(C))  if C[i]==0 ]+␣
↪[centroids_transposed[1] for i in range(len(C)) if C[i]==1]
A_hat_2 = (np.array(A_hat_2)).transpose()

  # Fill in the blank: pick the columns of centroids indexed by C and then␣
↪transpose it again
print('Rank-2 Approximation \n', A_hat_2)
```

```
A =
 [[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
centroids =
 [[-1.00 3.00]
 [-3.00 1.00]
 [-3.00 1.00]
 [-1.00 3.00]]
centroid assignment =
 [1 1 1 0 0 0]
Rank-2 Approximation
 [[-1.00 -1.00 -1.00 3.00 3.00 3.00]
 [-3.00 -3.00 -3.00 1.00 1.00 1.00]
 [-3.00 -3.00 -3.00 1.00 1.00 1.00]
 [-1.00 -1.00 -1.00 3.00 3.00 3.00]]
```

[152]:
```
# Write code to compare A_hat_1 and A_hat_2 to the original matrix A
A_hat_1_res = A-A_hat_1
A_hat_2_res = A-A_hat_2

print(np.linalg.norm(A_hat_1_res))
print(np.linalg.norm(A_hat_2_res))

# Thus the error when using A_hat_2 is much smaller than using A_hat_1,
```

```
9.797958971132712
19.595917942265423
```

0.1.3 Q2. SVD

Again let $\boldsymbol{A} = \begin{bmatrix} 3 & 3 & 3 & -1 & -1 & -1 \\ 1 & 1 & 1 & -3 & -3 & -3 \\ 1 & 1 & 1 & -3 & -3 & -3 \\ 3 & 3 & 3 & -1 & -1 & -1 \end{bmatrix}$. Now consider the singular value decomposition (SVD) $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$

a) If the full SVD is computed, find the dimensions of $\boldsymbol{U}, \boldsymbol{S}$, and $\boldsymbol{V}$.

U is 4 by 4

S is 4 by 6

V is 6 by 6

b) Find the dimensions of $\boldsymbol{U}, \boldsymbol{S}$, and $\boldsymbol{V}$ in the economy or skinny SVD of $\boldsymbol{A}$.

since for A is N by M, N = 4, M = 6, we know M > N

We could have:

U is still 4 by 4

S can be reduced to 4 by 4

V can be reduced to 4 by 6

c) The Python and NumPy command `U, s, VT = np.linalg.svd(A, full_matrices=True)` computes the singular value decomposition, $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$ where $\boldsymbol{U}$ and $\boldsymbol{V}$ are matrices with orthonormal columns comprising the left and right singular vectors and $\boldsymbol{S}$ is a diagonal matrix of singular values.

i. Compute the SVD of $\boldsymbol{A}$. Make sure $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$ holds.

ii. Find $\boldsymbol{U}^T\boldsymbol{U}$ and $\boldsymbol{V}^T\boldsymbol{V}$. Are the columns of $\boldsymbol{U}$ and $\boldsymbol{V}$ orthonormal? Why? *Hint:* compute $\boldsymbol{U}^T\boldsymbol{U}$.

iii. Find $\boldsymbol{U}\boldsymbol{U}^T$ and $\boldsymbol{V}\boldsymbol{V}^T$. Are the rows of $\boldsymbol{U}$ and $\boldsymbol{V}$ orthonormal? Why?

iv. Find the left and right singular vectors associated with the largest singular value.

v. What is the rank of $\boldsymbol{A}$?

[153]:
```
# i)
U, s, VT = np.linalg.svd(A, full_matrices=True)
S_matrix = np.zeros_like(U) ## Fill in the blank: Size of S should be equal to␣
 ↪size of U
np.fill_diagonal(S_matrix, s) ## Fill in the diagonal entries of S_matrix with s
S_matrix =  np.hstack((S_matrix,np.zeros((len(S_matrix),2)) ))
print(U@S_matrix@VT)
print(A)
print(f"U=\n{U}")
print(f"S=\n{S_matrix}")
print(f"VT=\n{VT}")
```

```
[[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
[[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
U=
[[-0.50 -0.50 -0.71 0.04]
 [-0.50 0.50 -0.04 -0.71]
 [-0.50 0.50 0.04 0.71]
 [-0.50 -0.50 0.71 -0.04]]
S=
[[9.80 0.00 0.00 0.00 0.00 0.00]
 [0.00 4.90 0.00 0.00 0.00 0.00]
 [0.00 0.00 0.00 0.00 0.00 0.00]
 [0.00 0.00 0.00 0.00 0.00 0.00]]
VT=
[[-0.41 -0.41 -0.41 0.41 0.41 0.41]
 [-0.41 -0.41 -0.41 -0.41 -0.41 -0.41]
 [0.79 -0.58 -0.21 0.00 0.00 0.00]
 [-0.22 -0.57 0.79 -0.00 0.00 0.00]
 [-0.00 0.00 -0.00 -0.58 0.79 -0.21]
 [-0.00 0.00 -0.00 -0.58 -0.21 0.79]]
```

[154]:
```
# ii)
print('UTU: \n', U.T@U) # i. Printing U^T*U
print('VTV: \n', VT@VT.T) # i. Printing V^T*V
```

```
UTU:
 [[1.00 0.00 0.00 -0.00]
 [0.00 1.00 -0.00 0.00]
 [0.00 -0.00 1.00 -0.00]
 [-0.00 0.00 -0.00 1.00]]
VTV:
 [[1.00 0.00 -0.00 0.00 -0.00 -0.00]
 [0.00 1.00 -0.00 -0.00 0.00 0.00]
 [-0.00 -0.00 1.00 -0.00 -0.00 -0.00]
 [0.00 -0.00 -0.00 1.00 -0.00 -0.00]
 [-0.00 0.00 -0.00 -0.00 1.00 0.00]
 [-0.00 0.00 -0.00 -0.00 0.00 1.00]]
```

since for both UTU, and VTV, the columns are also of unit norm and are orthogonal to each other, their columns are orthonormal

[155]:
```
# iii)
print('UUT: \n', U@U.T) # i. Printing U*U^T
```

```
print('VVT: \n', VT.T@VT) # i. Printing V*V^T
```

```
UUT:
 [[1.00 -0.00 0.00 -0.00]
 [-0.00 1.00 -0.00 -0.00]
 [0.00 -0.00 1.00 -0.00]
 [-0.00 -0.00 -0.00 1.00]]
VVT:
 [[1.00 0.00 0.00 -0.00 -0.00 -0.00]
 [0.00 1.00 -0.00 -0.00 -0.00 -0.00]
 [0.00 -0.00 1.00 -0.00 -0.00 -0.00]
 [-0.00 -0.00 -0.00 1.00 -0.00 -0.00]
 [-0.00 -0.00 -0.00 -0.00 1.00 -0.00]
 [-0.00 -0.00 -0.00 -0.00 -0.00 1.00]]
```

yes, because for both matrices, the rows are of unit norm and are orthogonal to each other

[156]:
```
# iv)
print('First left singular vector: \n', U[:,[0]])
print('Largest singular value:', s[0])
print('First right singular vector: \n', VT[0])

# v)
print(np.sum(np.abs(s)>1e-6))
print(f'rank A = {np.linalg.matrix_rank(A)}' )
```

```
First left singular vector:
 [[-0.50]
 [-0.50]
 [-0.50]
 [-0.50]]
Largest singular value: 9.797958971132713
First right singular vector:
 [-0.41 -0.41 -0.41 0.41 0.41 0.41]
2
rank A = 2
```

d) The Python and NumPy command `U, s, VT = np.linalg.svd(A, full_matrices=False)` computes the economy or skinny singular value decomposition, $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$ where $\boldsymbol{U}$ and $\boldsymbol{V}$ are matrices with orthonormal columns comprising the left and right singular vectors and $\boldsymbol{S}$ is a square diagonal matrix of singular values.

i. Compute the SVD of $\boldsymbol{A}$. Make sure $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$ holds.

ii. Find $\boldsymbol{U}^T\boldsymbol{U}$ and $\boldsymbol{V}^T\boldsymbol{V}$. Are the columns of $\boldsymbol{U}$ and $\boldsymbol{V}$ orthonormal? Why? *Hint:* compute $\boldsymbol{U}^T\boldsymbol{U}$.

iii. Find $\boldsymbol{U}\boldsymbol{U}^T$ and $\boldsymbol{V}\boldsymbol{V}^T$. Are the rows of $\boldsymbol{U}$ and $\boldsymbol{V}$ orthonormal? Why?

[157]:
```
# i)
U, s, VT = np.linalg.svd(A, full_matrices=False)
S_matrix = np.diag(s)
print(U@S_matrix@VT)
print(A)
```

```
[[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
[[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
```

[158]:
```
# ii)
print('UTU: \n', U.T@U) # i. Printing U^T*U
print('VTV: \n', VT@VT.T) # i. Printing V^T*V
```

```
UTU:
 [[1.00 0.00 0.00 -0.00]
 [0.00 1.00 -0.00 0.00]
 [0.00 -0.00 1.00 -0.00]
 [-0.00 0.00 -0.00 1.00]]
VTV:
 [[1.00 0.00 -0.00 0.00]
 [0.00 1.00 -0.00 -0.00]
 [-0.00 -0.00 1.00 -0.00]
 [0.00 -0.00 -0.00 1.00]]
```

since for both UTU, and VTV, the columns are also of unit norm and are orthogonal to each other, their columns are orthonormal

[159]:
```
# iii)
print('UUT: \n', U@U.T) # i. Printing U*U^T
print('VVT: \n', VT.T@VT) # i. Printing V*V^T
```

```
UUT:
 [[1.00 -0.00 0.00 -0.00]
 [-0.00 1.00 -0.00 -0.00]
 [0.00 -0.00 1.00 -0.00]
 [-0.00 -0.00 -0.00 1.00]]
VVT:
 [[1.00 0.00 0.00 -0.00 -0.00 -0.00]
 [0.00 1.00 -0.00 -0.00 -0.00 -0.00]
 [0.00 -0.00 1.00 -0.00 -0.00 -0.00]
 [-0.00 -0.00 -0.00 0.33 0.33 0.33]
 [-0.00 -0.00 -0.00 0.33 0.33 0.33]
 [-0.00 -0.00 -0.00 0.33 0.33 0.33]]
```

yes, because for both matrices, the rows are of unit norm and are orthogonal to each other

e) Compare the singular vectors and singular values of the economy and full SVD. How do they differ?

[160]:
```
# FOR ECONOMY SVD
print('First left singular vector: \n', U[:,[0]])
print('Largest singular value:', s[0])
print('First right singular vector: \n', VT[0])
```

```
First left singular vector:
 [[-0.50]
 [-0.50]
 [-0.50]
 [-0.50]]
Largest singular value: 9.797958971132713
First right singular vector:
 [-0.41 -0.41 -0.41 0.41 0.41 0.41]
```

Thus they are the same

f) Identify an orthonormal basis for the space spanned by the columns of $\boldsymbol{A}$.

[161]:
```
U
```

[161]:
```
array([[-0.50, -0.50, -0.71, 0.04],
       [-0.50, 0.50, -0.04, -0.71],
       [-0.50, 0.50, 0.04, 0.71],
       [-0.50, -0.50, 0.71, -0.04]])
```

g) Identify an orthonormal basis for the space spanned by the rows of $\boldsymbol{A}$.

[162]:
```
VT.T
```

[162]:
```
array([[-0.41, -0.41, 0.79, -0.22],
       [-0.41, -0.41, -0.58, -0.57],
       [-0.41, -0.41, -0.21, 0.79],
       [0.41, -0.41, 0.00, -0.00],
       [0.41, -0.41, 0.00, 0.00],
       [0.41, -0.41, 0.00, 0.00]])
```

h) Define the rank-r approximation to $\boldsymbol{A}$ as $\boldsymbol{A}_r = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$ where σ_i is the ith singular value with left singular vector $\boldsymbol{u}_i$ and right singular vector $\boldsymbol{v}_i$.

i. Find the rank-1 approximation $\boldsymbol{A}_1$. How does $\boldsymbol{A}_1$ compare to $\boldsymbol{A}$?

[163]:
```
A_rank1=s[0]*(np.outer(U[:,[0]],VT[0]))
print(f'rank 1 approximation: \n{A_rank1}')
print(A-A_rank1)
```

```
rank 1 approximation:
[[2.00 2.00 2.00 -2.00 -2.00 -2.00]
 [2.00 2.00 2.00 -2.00 -2.00 -2.00]
 [2.00 2.00 2.00 -2.00 -2.00 -2.00]
 [2.00 2.00 2.00 -2.00 -2.00 -2.00]]
[[1.00 1.00 1.00 1.00 1.00 1.00]
 [-1.00 -1.00 -1.00 -1.00 -1.00 -1.00]
 [-1.00 -1.00 -1.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 1.00 1.00 1.00]]
```

For all entries, the approximation is 1 off in absolute value

ii. Find the rank-2 approximation 2 . How does 2 compare to ?

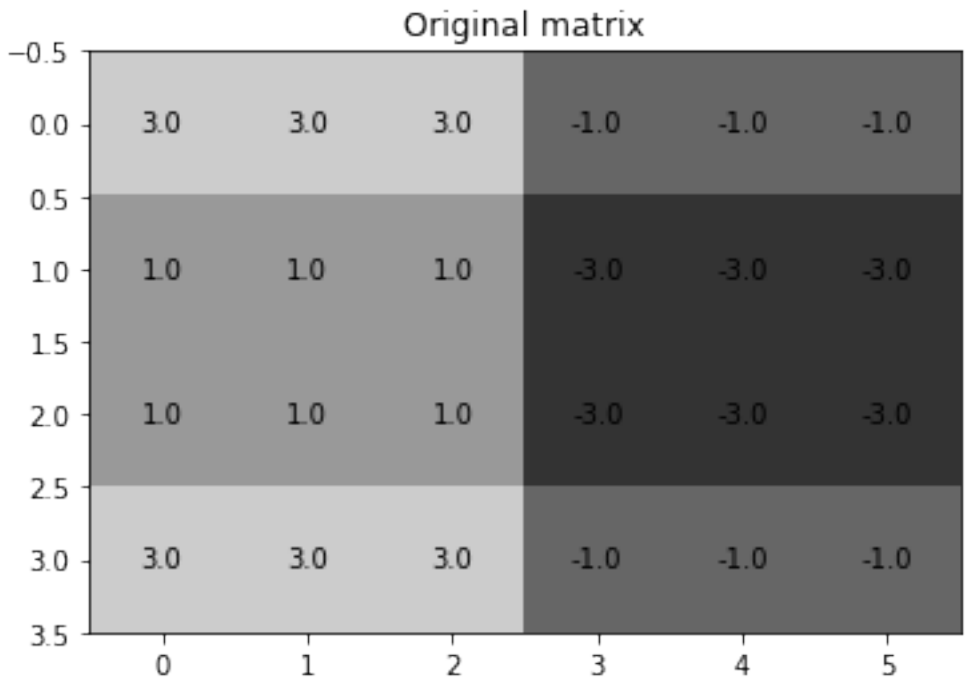
[164]:
```
A_rank2=s[0]*(np.outer(U[:,[0]],VT[0]))+s[1]*(np.outer(U[:,[1]],VT[1]))
print(f'rank 1 approximation: \n{A_rank2}')
print(A-A_rank2)
```

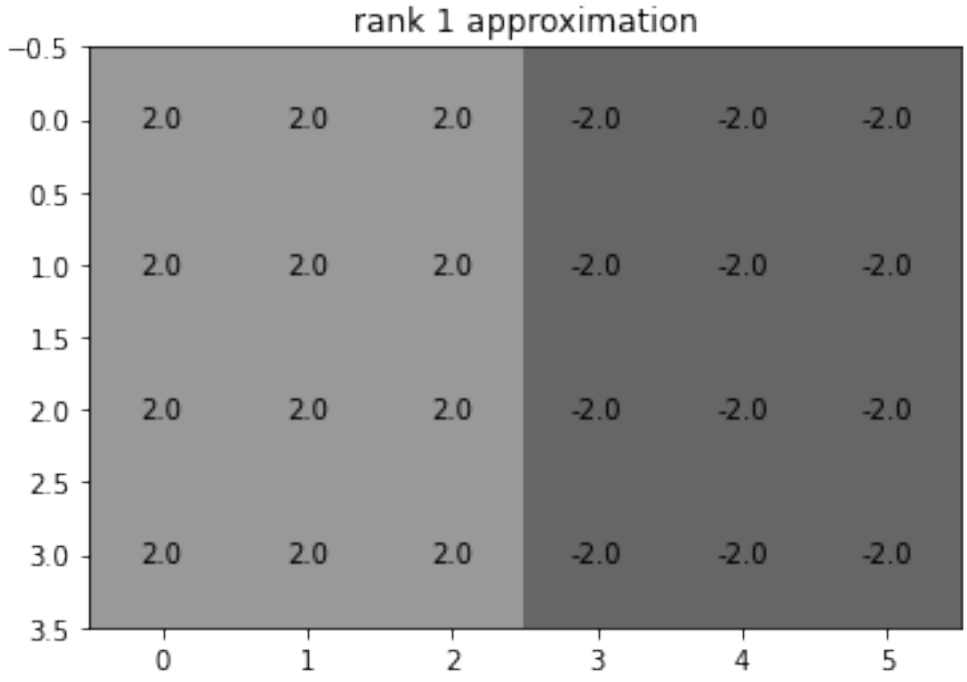
```
rank 1 approximation:
[[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
[[-0.00 -0.00 -0.00 0.00 0.00 0.00]
 [-0.00 0.00 0.00 -0.00 -0.00 -0.00]
 [-0.00 0.00 0.00 0.00 0.00 0.00]
 [-0.00 -0.00 -0.00 0.00 0.00 0.00]]
```

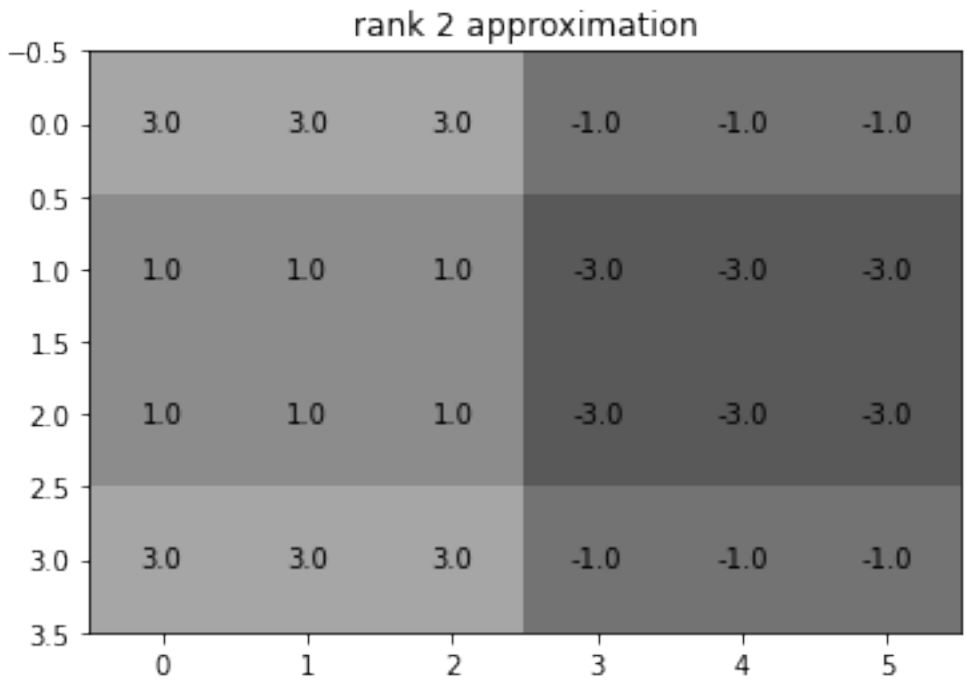
For all entries, the approximation is very close to the truth in absolute value

[165]:
```
## display the original matrix using a heatmap
plt.figure(num=None)
for (j,i),label in np.ndenumerate(A):
    plt.text(i,j,np.round(label,1),ha='center',va='center')
plt.imshow(A, vmin=-5, vmax=5, interpolation='none', cmap='gray')
plt.title('Original matrix' )

## display the rank-r approximations using a heatmap
for r in range(1,3):
    ## Fill in the blank: choose the first r columnns of U, first r singular␣
 ↪values, etc...
    A_rank_r_approx = U[:,:r]@S_matrix[:r,:]@VT[:,:]
    plt.figure(num=None)
    for (j,i),label in np.ndenumerate(A_rank_r_approx):
        plt.text(i,j,np.round(label,1),ha='center',va='center')
    plt.imshow(A_rank_r_approx, vmin=-10, vmax=10, interpolation='none',␣
 ↪cmap='gray')
    plt.title('rank ' + str(r) + ' approximation'  )
```


Original matrix
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
3.0 3.0 3.0 -1.0 -1.0 -1.0
1.0 1.0 1.0 -3.0 -3.0 -3.0
1.0 1.0 1.0 -3.0 -3.0 -3.0
3.0 3.0 3.0 -1.0 -1.0 -1.0
0
1
2
3
4
5


rank 1 approximation
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
2.0 2.0 2.0 -2.0 -2.0 -2.0
2.0 2.0 2.0 -2.0 -2.0 -2.0
2.0 2.0 2.0 -2.0 -2.0 -2.0
2.0 2.0 2.0 -2.0 -2.0 -2.0
0
1
2
3
4
5



i) The economy SVD is based on the dimension of the matrices and does not consider the rank of the matrix. What is the smallest economy SVD (minimum dimension of the square matrix $\boldsymbol{S}$) possible for the matrix $\boldsymbol{A}$? Find $\boldsymbol{U}, \boldsymbol{S}$, and $\boldsymbol{V}$ for this minimal economy SVD.

[166]:
```
print(f'rank A = {np.linalg.matrix_rank(A)}' )
U =U[:,[0,1]]
S_matrix = S_matrix[:2,:2]
VT = VT[:2,:]

print(f'U = \n {U}')
print(f'S = \n {S_matrix}')
print(f'V = \n {VT.T}')

print(U@S_matrix@VT)
print('S is 2 by 2' )
```

```
rank A = 2
U =
 [[-0.50 -0.50]
 [-0.50 0.50]
 [-0.50 0.50]
 [-0.50 -0.50]]
S =
 [[9.80 0.00]
 [0.00 4.90]]
```

```
V =
 [[-0.41 -0.41]
 [-0.41 -0.41]
 [-0.41 -0.41]
 [0.41 -0.41]
 [0.41 -0.41]
 [0.41 -0.41]]
[[3.00 3.00 3.00 -1.00 -1.00 -1.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [1.00 1.00 1.00 -3.00 -3.00 -3.00]
 [3.00 3.00 3.00 -1.00 -1.00 -1.00]]
S is 2 by 2
```