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a) $W^T = \beta_p a_i^p + \beta_{p-1} a_i^{p-1} + \cdots + \beta a_i + \beta_0 , \quad \beta_0 \in \mathbb{R}$

b) $= [a_i^p \; a_i^{p-1} \cdots a_i \; 1][\beta_p \; \beta_{p-1} \cdots \beta_0]^T \approx b_i$

$Ax = d.$

$$A = \begin{bmatrix} a_1^p & \cdots & a_1 & 1 \\ a_2^p & \cdots & a_2 & 1 \\ \vdots & & \vdots & \vdots \\ a_m^p & \cdots & a_m & 1 \end{bmatrix} \quad d = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad x = \begin{bmatrix} \beta_p \\ \beta_{p-1} \\ \vdots \\ \beta_0 \end{bmatrix}$$

c)

$p = 1, \quad b = 0.214a - 0.119$

$p = 2, \quad b = 0.705a^2 - 0.533a + 0.009$

$p = 3, \quad b = 0.796a^3 - 0.536$

for the plots, see figures below.

2. a) $\hat{w} = (T^T T)^{-1} T x = I_{r \times r} T x = T^T x.$

b). $\begin{bmatrix} | & & | \\ t_1 & \cdots & t_r \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ w_1 & \cdots & w_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_p \\ | & | & & | \end{bmatrix}$

$W = [T^T x_1 \quad T^T x_2 \cdots T^T x_p] = T^T x.$

3. a)

$$\tilde{x} = \begin{bmatrix} 0.447 & -0.365 & -0.632 & -0.516 & 0 & 0 & 0 & 1.13 \times 10^{-14} \\ 0.447 & 0.548 & 0.316 & -0.387 & 0 & 0 & 0 & 0.5 \\ 0.447 & -0.365 & 2.3 \times 10^{-16} & 0.645 & 0 & 0 & 0 & 0.5 \\ 0.447 & 0.548 & 0.316 & 0.387 & 0 & 0 & 0 & -0.5 \\ 0.447 & -0.365 & -0.632 & -0.129 & 0 & 0 & 0 & -0.5 \end{bmatrix}.$$

Yes, the first vector equal to t_1.

b).

$$\hat{x}_j = \text{proj}_{t_1} x_j = (t_1 t_1^T) x_j = \begin{bmatrix} 1/5 & \cdots & 1/5 \\ & \vdots & \\ 1/5 & \cdots & 1/5 \end{bmatrix} x_j.$$

$$W = t_1^T X = [13.42 \quad 12.97, \quad 8.05 \quad 15.21 \quad 17.44 \quad 9.84 \quad 5.81]$$

$$\text{So, } t_1 W = \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}}_{5 \times 7 \text{ matrix}}.$$

$$X - t_1 W = \begin{bmatrix} -2 & 1.2 & -1.6 & 1.2 & -0.8 & -0.4 & -0.6 \\ 3 & -2.8 & 1.4 & -0.8 & 2.2 & 0.6 & 2.4 \\ -2 & 2.2 & -0.6 & 0.2 & -1.8 & -0.4 & -1.6 \\ 3 & -3.8 & 2.4 & -1.8 & 1.2 & 0.6 & 1.4 \\ -2 & 3.2 & -1.6 & 1.2 & -0.8 & -0.4 & -1.6 \end{bmatrix}.$$

c)

$$T = [t_1 \ t_2] = \begin{bmatrix} 0.447 & -0.365 \\ 0.447 & 0.548 \\ 0.447 & -0.365 \\ 0.447 & 0.548 \\ 0.447 & -0.365 \end{bmatrix}$$

$$W = T^T X = \begin{bmatrix} 13.42 & 12.97 & 8.05 & 15.21 & 17.44 & 9.84 & 5.81 \\ 5.48 & -6.02 & 3.47 & -2.37 & 3.1 & 1.1 & 3.47 \end{bmatrix}.$$

$$X - TW = \begin{bmatrix} 0 & -1 & -1/3 & 1/3 & 1/3 & 0 & 2/3 \\ 0 & 1/2 & -1/2 & 1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 2/3 & -2/3 & -2/3 & 0 & -1/3 \\ 0 & -1/2 & 1/2 & -1/2 & -1/2 & 0 & -1/2 \\ 0 & 1 & -1/3 & 1/3 & 1/3 & 0 & -1/3 \end{bmatrix}$$

t_2 negative value for sci-fi, positive value for romance.

d)

$$T = \begin{bmatrix} 0.447 & -0.768 & -0.632 \\ 0.447 & 0.548 & 0.316 \\ 0.447 & -0.765 & 2.8\times10^{-16} \\ -0.447 & 0.548 & -0.316 \\ 0.447 & -0.765 & -0.632 \end{bmatrix}$$

$$W = T^T x = \begin{bmatrix} 13.42 & 12.97 & 8.05 & 13.21 & 13.44 & 8.84 & 5.81 \\ 5.48 & -6.02 & 3.47 & -2.77 & 3.1 & 1.1 & 3.47 \\ 0 & 1.58 & -0.32 & 0.32 & 0.32 & 0 & -0.32 \end{bmatrix}$$

$$x - TW = \begin{bmatrix} 0 & 0 & -0.53 & 0.53 & 0.53 & 0 & 0.47 \\ 0 & 0 & -0.4 & 0.4 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0.67 & -0.67 & -0.67 & 0 & -0.33 \\ 0 & 0 & -0.4 & -0.4 & -0.4 & 0 & -0.6 \\ 0 & 0 & -0.13 & 0.13 & 0.13 & 0 & -0.13 \end{bmatrix}$$

e)

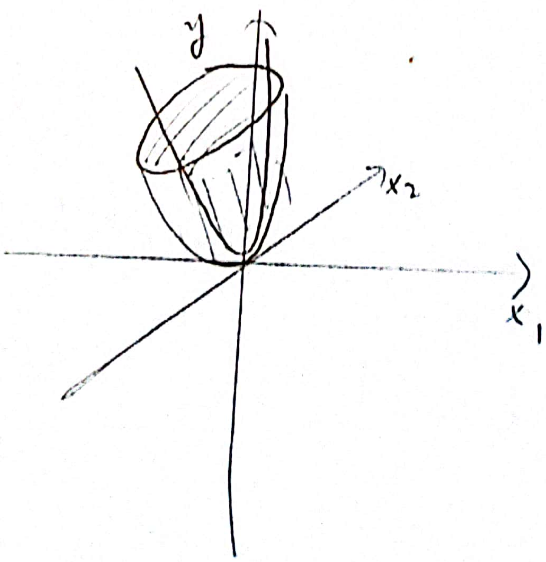
Values won't change, Projecting to the same orthonormal Basis.

4. a)

$$v^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} v = [v_1 \; v_2] \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1^2 + 2v_2^2 > 0$$

So, $Q > 0$

b).



$$y = x_1^2 + 2x_2^2$$

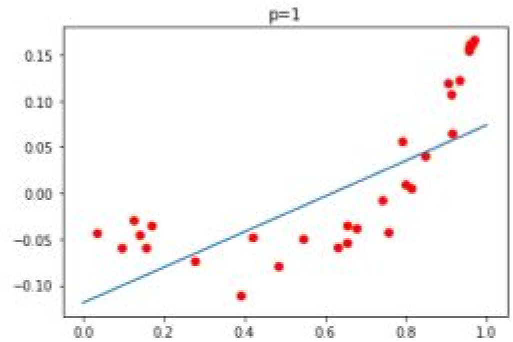
5. Given $P > \emptyset$, $Q > \emptyset$, then $v^T P v > \phi$, $v^T Q v > \phi$.

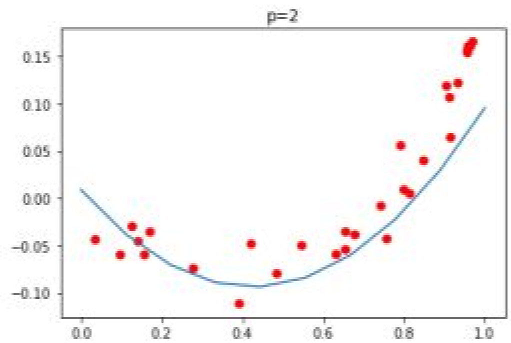
$$(v^T Q v) \cdot (v^T P v) \cdot (v^T Q v) > \phi$$

$$v^T Q v v^T (v v^T) P Q v > \phi$$

Since $v v^T > \phi$, we can easily omit all $v v^T$ from the left hand side

Then $v^T Q P Q v > \phi$, $QPQ > \phi$.


p=1
0.15
0.10
0.05
0.00
−0.05
−0.10
0.0
0.2
0.4
0.6
0.8
1.0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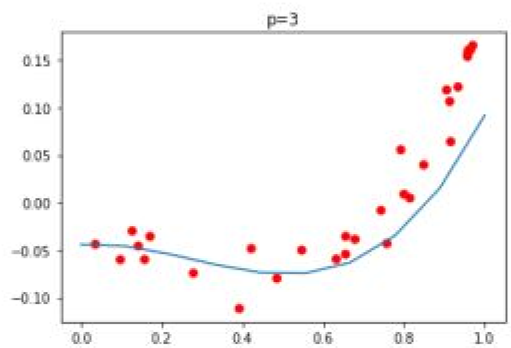
0.00
−0.05
−0.10
0.0
0.2
0.4
0.6
0.8
1.0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0.05
0.00
−0.05
−0.10
0.0
0.2
0.4
0.6
0.8
1.0