CS 577-2: Introduction to Algorithms Fall 2019

Practice Design Problems for Final Exam

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The following are some old exam problems that you can use as design practice. Don't forget to do the additional problems on each of the assignments!

1. You are given an $n \times n$ table with entries in $\{0, 1\}$. Design an algorithm to determine the largest size of a contiguous square in the table that consists solely of 1s. Your algorithm should run in time $O(n^2)$.

2. Alice and Barbara are playing the following game with a chocolate bar consisting of $n \times m$ squares, one of which is white while the others are dark. Alice starts with the original bar, breaks it into two parts, and passes the part with the white square to Barbara. The split does not need to be even but has to run along square boundaries, either horizontally or vertically. Barbara then does the same with the bar she gets from Alice, passing back to Alice. The game continues until a player is given the white square only; that player wins the game.

   Design an algorithm that, given n, m, and the position of the white square, determines whether Alice is guaranteed to win when playing optimally. Your algorithm should run in time polynomial in n and m.

3. You are given a sequence $p_1, p_2, \ldots, p_n$ of integers, and an integer k. Consider open intervals (x, y) with end points $x, y \in [n] \doteq \{1, 2, \ldots, n\}$ and $x \leq y$, and define the value of the interval (x, y) as $\mathrm{val}(x, y) \doteq p_y - p_x$. Let M_k denote the maximum of $\sum_{i=1}^{k} \mathrm{val}(x_i, y_i)$ over all choices of k pairwise disjoint intervals (x_i, y_i) of the above type.

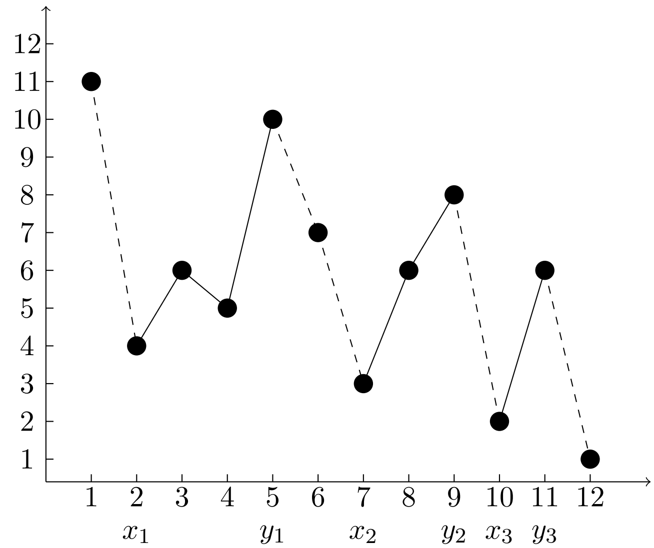
   The figure below illustrates an instance consisting of the sequence of 12 integers 11, 4, 6, 5, 10, 7, 3, 6, 8, 2, 6, 1, and $k = 3$. The indices $i \in [12]$ are on the x-axis and the values p_i on the y-axis. For this instance $M_3 = 15$ and is realized by the three disjoint intervals $(x_1, y_1) = (2, 5)$, $(x_2, y_2) = (7, 9)$, and $(x_3, y_3) = (10, 11)$.

   You are given a blackbox that takes as input $b, f \in [n]$ with $b \leq f$ and immediately returns $x, y \in [n]$ with $b \leq x \leq y \leq f$ such that $\mathrm{val}(x, y)$ equals the maximum value of any single interval in $[b, f]$.

   (a) Design an algorithm to compute M_k given $p_1, p_2, \ldots, p_n$ and k. Your algorithm can make use of the blackbox and should run in time time $O(kn^2)$ and space $O(n)$ (assuming every call to the blackbox takes no time and no space).

   (b) Design an algorithm to find a choice of k pairwise disjoint intervals (x_i, y_i) that realizes M_k given $p_1, p_2, \ldots, p_n$ and k. Your algorithm can make use of part (a) and of the blackbox, and should run in time $O(kn^2)$ and space $O(kn)$ (assuming every call to the blackbox takes no time and no space). Analyze the run time and space usage; you do not need to argue correctness.

   Extra credit: Reduce the space complexity to $O(n)$ while maintaining a running time of $O(kn^2)$.



4. In 1-in-3-SAT, we are given a CNF formula in which each clause contains exactly three distinct literals, and we want to find an assignment that satisfies exactly one literal in every clause, or report that none exists.

   Show how to polynomial-time reduce search to decision for this problem.

5. You are given a graph G with non-negative integer edge weights, vertices s and t, and an integer ℓ, and would like to find a (not necessarily simple) path from s to t of length exactly ℓ, or report that none exists.

   Show that the problem is NP-hard.

6. Show that the following search problem is NP-hard: Given a graph G, construct a spanning tree of G with at most 577 leaves, or report that none exists.

List of NP-complete problems that you can use

- 3-Sat: Given a Boolean formula in 3-CNF form, does there exist an assignment that makes the formula true?
- Independent Set: Given a graph $G = (V, E)$ and an integer k, does there exist $I \subseteq V$ with $|I| \geq k$ such that there is no edge between any of the vertices in I?
- Vertex Cover: Given a graph G and an integer k, does there exists $C \subseteq V$ with $|C| \leq k$ such that for every $e = (u, v) \in E$, at least one of u or v is in C? size at most k?
- 3-Coloring: Given a graph $G = (V, E)$, does there exist a mapping $c : V \to \{R, G, B\}$ such that for every $e = (u, v) \in E$, $c(u) \neq c(v)$?
- Hamiltonian Cycle/Path: Given a (di)graph G, does it have a (directed) cycle/path that visits every vertex once?
- Subset Sum: Given nonnegative integers $a_1, a_2, \ldots, a_n$ and t, does there exist $I \subseteq [n]$ such that $\sum_{i \in I} a_i = t$?