CS 577: Introduction to Algorithms Fall 2019

Homework 2

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This homework covers the divide and conquer paradigm. In order to get feedback on problem 3, you need to submit it via Canvas by **11:59pm on 9/24**. Please refer to the homework guidelines on Canvas for detailed instructions.

Warm-up problems

1. (a) Design an algorithm to compute the binary representation of 10^n in $O(n^{\log_2 3})$ time.

   (b) Design an algorithm that converts a given n-digit decimal number to binary in $O(n^{\log_2 3})$ time.

   Hint: Use the integer multiplication algorithm from class as a blackbox.

2. Given an array $A[1, \ldots, n]$ of integers and a positive integer k, we want to rearrange A such that the subarrays $A_1 \doteq A[1, \ldots, k], A_2 \doteq A[k+1, \ldots, 2k], \ldots$, satisfy the following property: For every $i < j$, every element of A_i is less than or equal to every element of A_j.

   Design an $O(n \log(n/k))$ algorithm for this problem. You can assume that all elements in the array are distinct.

   Hint: First solve the case where $n = 2k$.

Feedback problem

3. You are given a sequence of n real numbers $a_1, a_2, \ldots, a_n$ and a corresponding sequence of weights $w_1, w_2, \ldots, w_n$. The weights are nonnegative reals that add up to 1, i.e., $\sum_{i=1}^{n} w_i = 1$. The weighted median of the sequence is the number a_k such that

$$\sum_{a_i < a_k} w_i < \frac{1}{2} \qquad \text{and} \qquad \sum_{a_i \leq a_k} w_i \geq \frac{1}{2}.$$

   For example, the weighted median of the following instance is 2.5:

i	1	2	3	4	5	6	7
a_i	40	-5	4	0	2.5	6	-2
w_i	.25	.1	.05	.18	.15	.2	.07

   Design an $O(n)$ algorithm that finds the weighted median using elementary operations. An addition or a multiplication of two real numbers counts as one elementary operation.

Additional problems

4. You are given an $n \times n$ grid, and a procedure $V(i, j)$ that assigns an integer value to each position (i, j) in the grid, where i and j are integers such that $1 \leq i, j \leq n$. Your goal is to find a local minimum (or sink) in the grid (i.e., integers i^*, j^* with $1 \leq i^*, j^* \leq n$ such that for all neighbors (i, j) of (i^*, j^*) in the grid, $V(i^*, j^*) \leq V(i, j)$). The neighbors of (i, j) are $(i-1, j)$, $(i+1, j)$, $(i, j+1)$, $(i, j-1)$; the elements along the diagonals do not count as neighbors.

   Design an algorithm that makes $O(n)$ calls to V. Note that the grid has n^2 nodes.

5. You are given a topographical map that provides the maximum altitude along the direct road between any two neighboring cities, and two cities s and t. Design a linear-time algorithm that finds a route from s to t that minimizes the maximum altitude. All roads can be traversed in both directions.

Challenge problem

The following is one of the nicest introductory algorithm problems I know. Give it a try!

6. You are given n coins, at least one of which is bad. All the good coins weigh the same, and all the bad coins weigh the same. The bad coins are lighter than the good coins.

   Design an algorithm that makes $O((\log n)^2)$ weighings on a balance to find the exact number of bad coins. Each weighing tells you whether the total weight of the coins on the left side of the balance is smaller than, equal to, or larger than the total weight of the coins on the right side.

Programming problem

7. SPOJ problem Aggressive Cows (problem code AGGRCOW).