Dynamic causal inference notes

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Chapter 1

Overview of project

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1.2 Description of circular firefly task

The task has several phases. We will only explicitly model two of them: the **observation phase** and the **action phase**. These correspond to the causal inference and navigation segments of the task, respectively.

Observation phase. This phase lasts for an amount of time t_{obs}. The subject begins at position $x_0 = 0$ and moves at a constant (possibly zero) angular speed w. The firefly begins at f_0, ends at $f_a := f_0 + vt_{obs}$, and moves at a (possibly zero) velocity v, i.e.

ff pos= $$f(t) = f_0 + vt \qquad t \in [0, t_{obs}] \ . \tag{1.1}$$

From the point of view of the (possibly rotating) observer, the distance $d(t) := f(t) - x(t)$ between the firefly and the observer is

$$d(t) = f_0 + (v - w)t = f_a - (v - w)(t_{obs} - t) \qquad t \in [0, t_{obs}] \ . \tag{1.2}$$

We assume that the subject continuously receives noisy observations of the relative firefly position and their self-motion during the observation phase. In particular, discretize the time interval $[0, t_{obs}]$ into N time bins, so we have $\Delta t := t_{obs}/N$ and $t_i := i \ \Delta t$. The subject receives N self-motion observations $\{w_1, ..., w_N\}$ and firefly location observations $\{d_1, ..., d_N\}$. Assume these observations are noisy, and in particular that

$$\begin{aligned} d_i &\sim \mathcal{N}(d(t_i), \sigma_f^2/\Delta t) \\ w_i &\sim \mathcal{N}(w, \sigma_w^2/\Delta t) \end{aligned} \tag{1.3}$$

where σ_f is the distance observation noise, and σ_w is the self-motion observation noise.

1.3 Overview of theory

Our model assumes that well-trained subjects perform each component of the task—causal inference and navigation—'optimally', but possibly with wrong assumptions. This idea has occasionally been called 'rational', as opposed to 'optimal', decision-making.

In particular, we assume subjects construct a belief about firefly motion and self-motion in accordance with Bayes' rule. For convenience, we assume they make a binary decision about the underlying causal structure (i.e., the firefly either moves or does not) and use their estimates conditioned on that causal structure to determine a steering plan for the navigation part of the task. See Ch. for more details.

We assume subjects navigate 'optimally' in the sense that their steering trajectory minimizes a certain objective. However, we allow this objective to include factors other than reaching the target using the least amount of effort. Hence, trajectories may appear suboptimal from the experimenter's point of view, since the objective includes other terms. The specific objective we assume is

$$-V := \int_0^T \left\{ \frac{\alpha}{2} u^2 + \frac{\kappa^3}{2} \dot{u}^2 - \rho u \dot{f} + \frac{1}{2\beta}(x - f)^2 \right\} e^{-t/\delta} \, dt + \frac{(x_T - f_T)^2}{2} \; . \tag{1.4}$$

See Ch. 4 for more details, including an explanation for each of the terms.

1.4 Summary of useful results

Chapter 2

Bayesian inference for each model

2.1 No object motion, no self-motion

There is only one latent variable, so $z = f_0$. The observations are relative positions $\{f_t\}$.

Generative model. Assume an observation model and prior

$$p(f_t|z) = \mathcal{N}(f_0, \frac{\sigma_f^2}{\Delta t}) \qquad p(z) = \mathcal{N}(f_0; 0, \xi_f^2)\ . \tag{2.1}$$

We will summarize observations as $y := \langle f_t \rangle$. Note,

$$\mathbb{E}[y|z] = f_0 = 1z \qquad \Sigma_{obs}^{-1} = \frac{t_{obs}}{\sigma_f^2}\ . \tag{2.2}$$

Uniform priors. Assume an infinite-width prior. The posterior inverse covariance is

$$\Sigma^{-1} = \frac{t_{obs}}{\sigma_f^2}\ . \tag{2.3}$$

The posterior covariance and mean are

$$\Sigma = \frac{\sigma_f^2}{t_{obs}} \qquad \mu = y = \langle f_t \rangle\ . \tag{2.4}$$

The mean is an unbiased estimator, since $[\mu] = f_0$.

Posterior. The posterior inverse covariance and covariance are

$$\Sigma^{-1} = \frac{t_{obs}}{\sigma_f^2}\left[1 + \frac{1}{\tilde{\xi}_f^2}\right] \qquad \Sigma = \frac{\sigma_f^2}{t_{obs}}\frac{1}{1+\frac{1}{\tilde{\xi}_f^2}}\ . \tag{2.5}$$

The posterior mean is

$$\mu = \frac{1}{1+\frac{1}{\tilde{\xi}_f^2}}\langle f_t \rangle = \frac{\frac{t_{obs}}{\sigma_f^2}}{\frac{t_{obs}}{\sigma_f^2}+\frac{1}{\xi_f^2}}\langle f_t \rangle\ . \tag{2.6}$$

Distribution of estimates. Consider the set of variables $\hat{\boldsymbol{z}} := (f_0, \hat{v})^T$, where $\hat{v}$ is the posterior mean velocity for the model that *does* assume object motion. We want to compute the mean and covariance of $\hat{\boldsymbol{z}}$.

Redefine $\boldsymbol{y} := (\langle f_t \rangle, \hat{v})^T$. Allowing for a nonzero latent velocity, note that

$$\mathbb{E}[\boldsymbol{y}|\boldsymbol{z}] = \begin{pmatrix} f_0 + v\langle t \rangle \\ v \end{pmatrix} = \begin{pmatrix} 1 & \langle t \rangle \\ 0 & 1 \end{pmatrix} \boldsymbol{z} \ . \tag{2.7}$$

By definition, we have

$$\begin{aligned} \mu_f &= \frac{1}{1 + \frac{1}{\tilde{\xi}_f^2}} \langle f_t \rangle \\ \mu_v &= \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \left[\frac{\langle t \rangle}{\tilde{\xi}_f^2} \langle f_t \rangle + C_t \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \hat{v} \right] \ . \end{aligned} \tag{2.8}$$

In matrix form, we have $\boldsymbol{\mu} = \boldsymbol{B}\boldsymbol{y}$, where

$$\boldsymbol{B} := \begin{pmatrix} \frac{1}{1+\frac{1}{\tilde{\xi}_f^2}} & 0 \\ \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \frac{\langle t \rangle}{\tilde{\xi}_f^2} & \frac{1}{|\boldsymbol{\Sigma}^{-1}|} C_t \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \end{pmatrix} \tag{2.9}$$

The average mean is $[\boldsymbol{\mu}] = \boldsymbol{B}\boldsymbol{M}\boldsymbol{z}$. Explicitly:

$$\begin{aligned} [\mu_f] &= \frac{1}{1 + \frac{1}{\tilde{\xi}_f^2}} f_0 + \frac{\langle t \rangle}{1 + \frac{1}{\tilde{\xi}_f^2}} v \\ [\mu_v] &= \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \left[\frac{\langle t \rangle}{\tilde{\xi}_f^2} f_0 + \left(C_t + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right) v \right] \ . \end{aligned} \tag{2.10}$$

The average covariance is $[\boldsymbol{\Sigma}] = \boldsymbol{B}\boldsymbol{\Sigma}_{obs}\boldsymbol{B}^T$. Explicitly:

$$[\boldsymbol{\Sigma}] = \frac{\sigma_f^2}{t_{obs}} \begin{pmatrix} \frac{1}{(1+\frac{1}{\tilde{\xi}_f^2})^2} & \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \frac{\frac{1}{\tilde{\xi}_f^2}}{1+\frac{1}{\tilde{\xi}_f^2}} \langle t \rangle \\ \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \frac{\frac{1}{\tilde{\xi}_f^2}}{1+\frac{1}{\tilde{\xi}_f^2}} \langle t \rangle & \frac{1}{|\boldsymbol{\Sigma}^{-1}|^2} \left[\frac{\langle t \rangle^2}{\tilde{\xi}_f^4} + C_t \left(1 + \frac{1}{\tilde{\xi}_f^2} \right)^2 \right] \end{pmatrix} \tag{2.11}$$

The determinant of this matrix (ignoring the prefactor) is

$$|[\boldsymbol{\Sigma}]| = \frac{1}{|\boldsymbol{\Sigma}^{-1}|^2} C_t \ . \tag{2.12}$$

The inverse is

$$[\boldsymbol{\Sigma}]^{-1} = \frac{t_{obs}}{\sigma_f^2} \frac{1}{C_t} \begin{pmatrix} \frac{\langle t \rangle^2}{\tilde{\xi}_f^4} + C_t \left(1 + \frac{1}{\tilde{\xi}_f^2} \right)^2 & -|\boldsymbol{\Sigma}^{-1}| \frac{\frac{1}{\tilde{\xi}_f^2}}{1+\frac{1}{\tilde{\xi}_f^2}} \langle t \rangle \\ -|\boldsymbol{\Sigma}^{-1}| \frac{\frac{1}{\tilde{\xi}_f^2}}{1+\frac{1}{\tilde{\xi}_f^2}} \langle t \rangle & |\boldsymbol{\Sigma}^{-1}|^2 \frac{1}{(1+\frac{1}{\tilde{\xi}_f^2})^2} \end{pmatrix} \ . \tag{2.13}$$

2.2 Object motion, no self-motion

In 1D, the set of latents is $\boldsymbol{z} := (f_0, v)^T$. The observations are relative positions $\{f_t\}$.

Generative model. Assume an observation model and prior

$$p(f_t|\boldsymbol{z}) = \mathcal{N}(f_0 + vt, \frac{\sigma_f^2}{\Delta t}) \qquad\qquad p(\boldsymbol{z}) = \mathcal{N}(f_0; 0, \xi_f^2)\, \mathcal{N}(v; 0, \xi_v^2)\ . \tag{2.14}$$

It is useful to introduce the sufficient statistics $\langle f_t \rangle$ and $\hat{v} = \frac{\mathrm{Cov}(t, f_t)}{\mathrm{Cov}(t)}$. These statistics $\boldsymbol{y} := (\langle f_t \rangle, \hat{v})^T$ satisfy

$$\mathbb{E}[\boldsymbol{y}|\boldsymbol{z}] = \begin{pmatrix} f_0 + v\langle t \rangle \\ v \end{pmatrix} = \begin{pmatrix} 1 & \langle t \rangle \\ 0 & 1 \end{pmatrix} \boldsymbol{z} = \boldsymbol{M}\boldsymbol{z} \qquad\qquad \boldsymbol{\Sigma}_{obs}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & 0 \\ 0 & C(t) \end{pmatrix}\ . \tag{2.15}$$

Uniform priors. Assume infinite-width priors. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\boldsymbol{\Sigma}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & \langle t \rangle \\ \langle t \rangle & \langle t^2 \rangle \end{pmatrix} \qquad\qquad |\boldsymbol{\Sigma}^{-1}| = \langle t^2 \rangle - \langle t \rangle^2 = C(t)\ . \tag{2.16}$$

The posterior covariance and mean are

$$\boldsymbol{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \langle t^2 \rangle & -\langle t \rangle \\ -\langle t \rangle & 1 \end{pmatrix} \qquad\qquad \boldsymbol{\mu} = \begin{pmatrix} \langle f_t \rangle - \langle t \rangle \hat{v} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} 1 & -\langle t \rangle \\ 0 & 1 \end{pmatrix} \boldsymbol{y}\ . \tag{2.17}$$

The mean is an unbiased estimator of $\boldsymbol{z}$, since $[\boldsymbol{\mu}] = \boldsymbol{z}$.

Posterior. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\begin{aligned} \boldsymbol{\Sigma}^{-1} &= \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 + \frac{1}{\tilde{\xi}_f^2} & \langle t \rangle \\ \langle t \rangle & \langle t^2 \rangle + \frac{1}{\tilde{\xi}_v^2} \end{pmatrix} \\ |\boldsymbol{\Sigma}^{-1}| &= \left(1 + \frac{1}{\tilde{\xi}_f^2}\right) \left(\langle t^2 \rangle + \frac{1}{\tilde{\xi}_v^2}\right) - \langle t \rangle^2 = \left(1 + \frac{1}{\tilde{\xi}_f^2}\right) \left(C(t) + \frac{1}{\tilde{\xi}_v^2}\right) + \frac{\langle t \rangle^2}{\tilde{\xi}_f^2}\ . \end{aligned} \tag{2.18}$$

The posterior covariance is

$$\boldsymbol{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \langle t^2 \rangle + \frac{1}{\tilde{\xi}_v^2} & -\langle t \rangle \\ -\langle t \rangle & 1 + \frac{1}{\tilde{\xi}_f^2} \end{pmatrix}\ . \tag{2.19}$$

The posterior mean is

$$\boldsymbol{\mu} = \frac{C_t}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \left[1 + \frac{1}{C_t\ \tilde{\xi}_v^2}\right] \langle f_t \rangle - \langle t \rangle \hat{v} \\ \frac{\langle t \rangle}{C_t\ \tilde{\xi}_f^2} \langle f_t \rangle + \left(1 + \frac{1}{\tilde{\xi}_f^2}\right) \hat{v} \end{pmatrix} = \frac{C_t}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \left[1 + \frac{1}{C_t\ \tilde{\xi}_v^2}\right] & -\langle t \rangle \\ \frac{\langle t \rangle}{C_t\ \tilde{\xi}_f^2} & 1 + \frac{1}{\tilde{\xi}_f^2} \end{pmatrix} \boldsymbol{y}\ .$$

Distribution of estimates. The posterior mean estimates are

$$\begin{aligned}\mu_f &= \frac{1}{|\mathbf{\Sigma}^{-1}|}\left[\left(C_t + \frac{1}{\tilde{\xi}_v^2}\right)\langle f_t\rangle - C_t\langle t\rangle\hat{v}\right] \\ \mu_v &= \frac{1}{|\mathbf{\Sigma}^{-1}|}\left[\frac{\langle t\rangle}{\tilde{\xi}_f^2}\langle f_t\rangle + C_t\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)\hat{v}\right] \ .\end{aligned} \tag{2.20}$$

The average estimates are

$$\begin{aligned}[\mu_f] &= \frac{1}{|\mathbf{\Sigma}^{-1}|}\left[\left(C_t + \frac{1}{\tilde{\xi}_v^2}\right)f_0 + \frac{\langle t\rangle}{\tilde{\xi}_v^2}v\right] \\ [\mu_v] &= \frac{1}{|\mathbf{\Sigma}^{-1}|}\left[\frac{\langle t\rangle}{\tilde{\xi}_f^2}f_0 + \left(C_t + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right)v\right] \ .\end{aligned} \tag{2.21}$$

The average covariance is

$$[\mathbf{\Sigma}] = \frac{\sigma_f^2}{t_{obs}}\frac{1}{|\mathbf{\Sigma}^{-1}|^2}\begin{pmatrix} \left(C_t + \frac{1}{\tilde{\xi}_v^2}\right)^2 + C_t\langle t\rangle^2 & \langle t\rangle\left(\frac{1}{\tilde{\xi}_f^2\tilde{\xi}_v^2} - C_t\right) \\ \langle t\rangle\left(\frac{1}{\tilde{\xi}_f^2\tilde{\xi}_v^2} - C_t\right) & \frac{\langle t\rangle^2}{\tilde{\xi}_f^4} + C_t\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2 \ . \end{pmatrix} \tag{2.22}$$

Ignoring the prefactors, the determinant of this matrix is (after some algebra)

$$\det[\mathbf{\Sigma}] = C_t|\mathbf{\Sigma}^{-1}|^2 \ . \tag{2.23}$$

Hence, the inverse average covariance is

$$[\mathbf{\Sigma}]^{-1} = \frac{t_{obs}}{\sigma_f^2}\frac{1}{C_t}\begin{pmatrix} \frac{\langle t\rangle^2}{\tilde{\xi}_f^4} + C_t\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2 & -\langle t\rangle\left(\frac{1}{\tilde{\xi}_f^2\tilde{\xi}_v^2} - C_t\right) \\ -\langle t\rangle\left(\frac{1}{\tilde{\xi}_f^2\tilde{\xi}_v^2} - C_t\right) & \left(C_t + \frac{1}{\tilde{\xi}_v^2}\right)^2 + C_t\langle t\rangle^2 \end{pmatrix} \ . \tag{2.24}$$

2.3 No object motion, self-motion

In 1D, the set of latents is $\boldsymbol{z} := (f_0, w)^T$. The observations are relative positions and vestibular signals $\{f_t, w_t\}$.

Generative model. Assume an observation model and prior

$$p(f_t, w_t|\boldsymbol{z}) = \mathcal{N}(f_0 - wt, \frac{\sigma_f^2}{\Delta t})\ \mathcal{N}(w, \frac{\sigma_w^2}{\Delta t}) \qquad\qquad p(\boldsymbol{z}) = \mathcal{N}(f_0; 0, \xi_f^2)\ \mathcal{N}(w; 0, \xi_w^2)\ .$$

The summary statistics $\boldsymbol{y} := (\langle f_t\rangle, \langle w_t\rangle, \hat{v}_{rel})^T$ satisfy

$$\mathbb{E}[\boldsymbol{y}|\boldsymbol{z}] = \begin{pmatrix} f_0 - w\langle t\rangle \\ w \\ -w \end{pmatrix} = \begin{pmatrix} 1 & -\langle t\rangle \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \boldsymbol{z} \qquad\qquad \boldsymbol{\Sigma}_{obs}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sigma_f^2}{\sigma_w^2} & 0 \\ 0 & 0 & C_t \end{pmatrix}\ . \tag{2.25}$$

Uniform priors. Assume infinite-width priors. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\boldsymbol{\Sigma}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & -\langle t\rangle \\ -\langle t\rangle & \langle t^2\rangle + \frac{\sigma_f^2}{\sigma_w^2} \end{pmatrix} \qquad\qquad |\boldsymbol{\Sigma}^{-1}| = C_t + \frac{\sigma_f^2}{\sigma_w^2}\ .$$

The posterior covariance is

$$\boldsymbol{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \langle t^2\rangle + \frac{\sigma_f^2}{\sigma_w^2} & \langle t\rangle \\ \langle t\rangle & 1 \end{pmatrix}\ . \tag{2.26}$$

The posterior mean is

$$\boldsymbol{\mu} = \begin{pmatrix} \langle f_t\rangle + \frac{\frac{\sigma_f^2}{\sigma_w^2}}{C_t + \frac{\sigma_f^2}{\sigma_w^2}} \langle t\rangle\langle w_t\rangle - \frac{C_t}{C_t + \frac{\sigma_f^2}{\sigma_w^2}} \langle t\rangle \hat{v}_{rel} \\ \frac{\frac{\sigma_f^2}{\sigma_w^2}}{C_t + \frac{\sigma_f^2}{\sigma_w^2}} \langle w_t\rangle - \frac{C_t}{C_t + \frac{\sigma_f^2}{\sigma_w^2}} \hat{v}_{rel} \end{pmatrix} = \frac{1}{C_t + \frac{\sigma_f^2}{\sigma_w^2}} \begin{pmatrix} C_t + \frac{\sigma_f^2}{\sigma_w^2} & \frac{\sigma_f^2}{\sigma_w^2}\langle t\rangle & -C_t\langle t\rangle \\ 0 & \frac{\sigma_f^2}{\sigma_w^2} & -C_t \end{pmatrix} \boldsymbol{y}\ . \tag{2.27}$$

As before, $\boldsymbol{\mu}$ is unbiased since $[\boldsymbol{\mu}] = \boldsymbol{z}$.

Posterior. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\boldsymbol{\Sigma}^{-1} = \frac{t_{obs}}{\sigma_f^2} \left[\begin{pmatrix} \frac{1}{\tilde{\xi}_f^2} & 0 \\ 0 & \frac{1}{\tilde{\xi}_w^2} \end{pmatrix} + \begin{pmatrix} 1 & -\langle t\rangle \\ -\langle t\rangle & \langle t^2\rangle + \frac{\sigma_f^2}{\sigma_w^2} \end{pmatrix} \right] = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 + \frac{1}{\tilde{\xi}_f^2} & -\langle t\rangle \\ -\langle t\rangle & \langle t^2\rangle + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \end{pmatrix}$$

$$|\boldsymbol{\Sigma}^{-1}| = \left(C_t + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2} \right) + \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right)\ .$$

The posterior covariance is

$$\mathbf{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\mathbf{\Sigma}^{-1}|} \begin{pmatrix} \langle t^2 \rangle + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} & \langle t \rangle \\ \langle t \rangle & 1 + \frac{1}{\tilde{\xi}_f^2} \end{pmatrix} .$$

The posterior mean is

$$\boldsymbol{\mu} = \frac{1}{|\mathbf{\Sigma}^{-1}|} \begin{pmatrix} C_t + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} & \frac{\sigma_f^2}{\sigma_w^2} \langle t \rangle & -C_t \langle t \rangle \\ -\frac{1}{\tilde{\xi}_f^2} \langle t \rangle & \frac{\sigma_f^2}{\sigma_w^2}(1 + \frac{1}{\tilde{\xi}_f^2}) & -C_t(1 + \frac{1}{\tilde{\xi}_f^2}) \end{pmatrix} \boldsymbol{y} . \tag{2.28}$$

More explicitly,

$$\begin{aligned} \mu_f &= \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ \left(C_t + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \langle f_t \rangle + \frac{\sigma_f^2}{\sigma_w^2} \langle t \rangle \langle w_t \rangle - C_t \langle t \rangle \hat{v}_{rel} \right\} \\ \mu_w &= \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ -\frac{1}{\tilde{\xi}_f^2} \langle t \rangle \langle f_t \rangle + \frac{\sigma_f^2}{\sigma_w^2} \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \langle w_t \rangle - C_t \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \hat{v}_{rel} \right\} \end{aligned} \tag{2.29}$$

Distribution of estimates. Consider the collection $\hat{\boldsymbol{z}} = (\mu_f, \mu_w, \mu_v^*)^T$, where μ_v^* is from the model where it is assumed to be nonzero.

The estimates themselves are

$$\begin{aligned} \mu_f &= \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ \left(C_t + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \langle f_t \rangle + \frac{\sigma_f^2}{\sigma_w^2} \langle t \rangle \langle w_t \rangle - C_t \langle t \rangle \hat{v}_{rel} \right\} \\ \mu_w &= \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ -\frac{1}{\tilde{\xi}_f^2} \langle t \rangle \langle f_t \rangle + \frac{\sigma_f^2}{\sigma_w^2} \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \langle w_t \rangle - C_t \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \hat{v}_{rel} \right\} \end{aligned} \tag{2.30}$$

$$\mu_v = \frac{1}{|\mathbf{\Sigma}_*^{-1}|} \left\{ \frac{\langle t \rangle}{\tilde{\xi}_f^2} \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \langle f_t \rangle + \frac{\sigma_f^2}{\sigma_w^2} \left[\mathrm{C}(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right] \langle w_t \rangle + C(t) \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \hat{v}_{rel} \right\}$$

where we have used an asterisk to distinguish the determinant associated with the bigger model.

The average estimates are

$$\begin{aligned} [\mu_f] &= \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ \left(C_t + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) f_0 - \frac{1}{\tilde{\xi}_w^2} \langle t \rangle w + \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \langle t \rangle v \right\} \\ [\mu_w] &= \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ -\frac{1}{\tilde{\xi}_f^2} \langle t \rangle f_0 + \left[C_t + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} + \frac{\sigma_f^2}{\sigma_w^2} \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \right] w - \left[C_t + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right] v \right\} . \end{aligned} \tag{2.31}$$

$$[\mu_v] = \frac{1}{|\mathbf{\Sigma}_*^{-1}|} \left\{ \frac{\langle t \rangle}{\tilde{\xi}_f^2} \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) f_0 - \frac{1}{\tilde{\xi}_w^2} \left[C_t + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right] w + \left(C_t + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) v \right\}$$

The average covariance is

$$[\boldsymbol{\Sigma}] = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\boldsymbol{\Sigma}^{-1}|^2} \begin{pmatrix} \left(C_t + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right)^2 + \left(C_t + \frac{\sigma_f^2}{\sigma_w^2}\right)\langle t\rangle^2 & .... & \\ ... & \frac{1}{\tilde{\xi}_f^4}\langle t\rangle^2 + \left(C_t + \frac{\sigma_f^2}{\sigma_w^2}\right)\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2 & \\ ... & ... & \frac{|\boldsymbol{\Sigma}^{-1}|^2}{|\boldsymbol{\Sigma}_*^{-1}|^2}\left\{ \frac{\langle t\rangle^2}{\tilde{\xi}_f^4}\left(\frac{\sigma_f^2}{\sigma_w^2} + \right.\right. \end{pmatrix} \tag{2.32}$$

MAYBE JUST DO THIS IN THE OBVIOUS WAY TO AVOID HAVING TO WRITE OUT ALL THE GARBAGE...JUST WRITE IN TERMS OF THE COEFFICIENTS OF THE ORIGINAL ESTIMATES

2.4 Full 1D model

In 1D, the set of latents is $\boldsymbol{z} := (f_0, w, v)^T$. The observations are relative positions and vestibular signals $\{f_t, w_t\}$.

Generative model. Assume an observation model and prior

$$p(f_t, w_t|\boldsymbol{z}) = \mathcal{N}(f_0+(v-w)t, \frac{\sigma_f^2}{\Delta t})\, \mathcal{N}(w, \frac{\sigma_w^2}{\Delta t}) \qquad p(\boldsymbol{z}) = \mathcal{N}(f_0; 0, \xi_f^2)\, \mathcal{N}(w; 0, \xi_w^2)\, \mathcal{N}(v; 0, \xi_v^2)\, .$$

It is useful to introduce the sufficient statistics $\langle f_t \rangle$, $\langle w_t \rangle$, and $\hat{v}_{rel} = \frac{\mathrm{Cov}(t,f_t)}{\mathrm{Cov}(t)}$. These statistics $\boldsymbol{y} := (\langle f_t \rangle, \langle w_t \rangle, \hat{v}_{rel})^T$ satisfy

$$\mathbb{E}[\boldsymbol{y}|\boldsymbol{z}] = \begin{pmatrix} f_0 + (v-w)\langle t \rangle \\ w \\ v-w \end{pmatrix} = \begin{pmatrix} 1 & -\langle t \rangle & \langle t \rangle \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \boldsymbol{z} \qquad \boldsymbol{\Sigma}_{obs}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sigma_f^2}{\sigma_w^2} & 0 \\ 0 & 0 & C(t) \end{pmatrix}\, . \tag{2.33}$$

Uniform priors. Assume infinite-width priors. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\boldsymbol{\Sigma}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & -\langle t \rangle & \langle t \rangle \\ -\langle t \rangle & \langle t^2 \rangle + \frac{\sigma_f^2}{\sigma_w^2} & -\langle t^2 \rangle \\ \langle t \rangle & -\langle t^2 \rangle & \langle t^2 \rangle \end{pmatrix} \qquad |\boldsymbol{\Sigma}^{-1}| = C_t \frac{\sigma_f^2}{\sigma_w^2}\, .$$

The posterior covariance is

$$\boldsymbol{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \langle t^2 \rangle \frac{\sigma_f^2}{\sigma_w^2} & 0 & -\langle t \rangle \frac{\sigma_f^2}{\sigma_w^2} \\ 0 & C_t & C_t \\ -\langle t \rangle \frac{\sigma_f^2}{\sigma_w^2} & C_t & C_t + \frac{\sigma_f^2}{\sigma_w^2} \end{pmatrix} = \frac{\sigma_f^2}{t_{obs}} \begin{pmatrix} \frac{\langle t^2 \rangle}{C_t} & 0 & -\frac{\langle t \rangle}{C_t} \\ 0 & \frac{\sigma_w^2}{\sigma_f^2} & \frac{\sigma_w^2}{\sigma_f^2} \\ -\frac{\langle t \rangle}{C_t} & \frac{\sigma_w^2}{\sigma_f^2} & \frac{\sigma_w^2}{\sigma_f^2} + \frac{1}{C_t} \end{pmatrix}\, . \tag{2.34}$$

The posterior mean is

$$\boldsymbol{\mu} = \begin{pmatrix} \langle f_t \rangle - \langle t \rangle \hat{v}_{rel} \\ \langle w_t \rangle \\ \langle w_t \rangle + \hat{v}_{rel} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\langle t \rangle \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \boldsymbol{y}\, . \tag{2.35}$$

As before, $\boldsymbol{\mu}$ is unbiased since $[\boldsymbol{\mu}] = \boldsymbol{z}$.

Posterior. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\boldsymbol{\Sigma}^{-1} = \frac{t_{obs}}{\sigma_f^2} \left[\begin{pmatrix} \frac{1}{\tilde{\xi}_f^2} & 0 & 0 \\ 0 & \frac{1}{\tilde{\xi}_w^2} & 0 \\ 0 & 0 & \frac{1}{\tilde{\xi}_v^2} \end{pmatrix} + \begin{pmatrix} 1 & -\langle t \rangle & \langle t \rangle \\ -\langle t \rangle & \langle t^2 \rangle + \frac{\sigma_f^2}{\sigma_w^2} & -\langle t^2 \rangle \\ \langle t \rangle & -\langle t^2 \rangle & \langle t^2 \rangle \end{pmatrix} \right] = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 + \frac{1}{\tilde{\xi}_f^2} & -\langle t \rangle & \langle t \rangle \\ -\langle t \rangle & \langle t^2 \rangle + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} & -\langle t^2 \rangle \\ \langle t \rangle & -\langle t^2 \rangle & \langle t^2 \rangle + \frac{1}{\tilde{\xi}_v^2} \end{pmatrix}$$

$$|\boldsymbol{\Sigma}^{-1}| = \left(C(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_v^2} + \frac{1}{\tilde{\xi}_w^2} \right) + \frac{1}{\tilde{\xi}_v^2} \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right)\, .$$

The posterior covariance is

$$\mathbf{\Sigma} = \frac{\sigma_f^2}{t_{obs}}\frac{1}{|\mathbf{\Sigma}^{-1}|}\begin{pmatrix} (\frac{1}{\tilde{\xi}_v^2}+\langle t^2\rangle)(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2})+\frac{\langle t^2\rangle}{\tilde{\xi}_v^2} & \frac{\langle t\rangle}{\tilde{\xi}_v^2} & -\langle t\rangle\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right) \\ \frac{\langle t\rangle}{\tilde{\xi}_v^2} & \frac{1}{\tilde{\xi}_v^2}\left[1+\frac{1}{\tilde{\xi}_f^2}\right]+\mathrm{C}(t)+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2} & \mathrm{C}(t)+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2} \\ -\langle t\rangle\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right) & \mathrm{C}(t)+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2} & \mathrm{C}(t)+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}+(1+\frac{1}{\tilde{\xi}_f^2})\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right) \end{pmatrix}$$

The posterior mean is

$$\mu_f = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{\left[\left(\frac{1}{\tilde{\xi}_v^2}+C(t)\right)\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)+\frac{C(t)}{\tilde{\xi}_v^2}\right]\langle f_t\rangle+\frac{\langle t\rangle}{\tilde{\xi}_v^2}\frac{\sigma_f^2}{\sigma_w^2}\langle w_t\rangle-\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}+\frac{1}{\tilde{\xi}_v^2}\right)\langle t\rangle C(t)\hat{v}_{rel}\right\}$$

$$\mu_w = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{-\frac{\langle t\rangle}{\tilde{\xi}_f^2\tilde{\xi}_v^2}\langle f_t\rangle+\frac{\sigma_f^2}{\sigma_w^2}\left[\frac{1}{\tilde{\xi}_v^2}\left(1+\frac{1}{\tilde{\xi}_f^2}\right)+\mathrm{C}(t)+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right]\langle w_t\rangle-\frac{C(t)}{\tilde{\xi}_v^2}\left[1+\frac{1}{\tilde{\xi}_f^2}\right]\hat{v}_{rel}\right\}$$

$$\mu_v = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{\frac{\langle t\rangle}{\tilde{\xi}_f^2}\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)\langle f_t\rangle+\frac{\sigma_f^2}{\sigma_w^2}\left[\mathrm{C}(t)+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right]\langle w_t\rangle+C(t)\left(1+\frac{1}{\tilde{\xi}_f^2}\right)\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)\hat{v}_{rel}\right\}$$

As a sanity check, note that the above expressions reduce to the uniform prior ones when the prior widths are taken to infinity.

Chapter 3

Bayesian dynamic causal inference

3.1 Requirements for Bayesian causal inference

3.2 Model hierarchies and sufficient statistics

In the abstract, the relevant generative model is

$$\boldsymbol{z} \rightarrow \{\boldsymbol{x}_t\} \tag{3.1}$$

where each $\boldsymbol{x}_t$ observation is independent, but not identically distributed. Ignoring additive constants, the minus log-likelihood is

$$\begin{aligned} -\log p(\{\boldsymbol{x}_t\}|\boldsymbol{z}) &= \frac{t_{obs}}{\sigma_f^2} \sum_{\{t\}} \frac{\Delta t}{2t_{obs}} \left[f_0 + (v-w)t - f_t\right]^2 + \frac{t_{obs}}{\sigma_w^2}\frac{\Delta t}{2t_{obs}}(w - w_t)^2 + \text{const.} \\ &= \frac{t_{obs}}{2\sigma_f^2}\langle [f_0 + (v-w)t - f_t]^2\rangle + \frac{t_{obs}}{2\sigma_w^2}\langle (w-w_t)^2\rangle + \text{const.} \end{aligned} \tag{3.2}$$

where $\langle\cdot\rangle$ denotes time-averages. Ultimately, we are interested in the observation-associated posterior $p_{obs}(\boldsymbol{z}|\{\boldsymbol{x}_t\})$, which is normal, and hence is characterized by a mean and covariance matrix. We can write

$$p_{obs}(\boldsymbol{z}|\{\boldsymbol{x}_t\}) = \mathcal{N}(\boldsymbol{z}; \boldsymbol{z}_{obs}, \boldsymbol{\Sigma}_{obs}) \ . \tag{3.3}$$

We can compute the mean and covariance by taking derivatives of the (minus) log-likelihood above.

Here, we make an important observation. Taking two derivatives of the above with respect to any of the latent variables (f_0, w, or v) produces expressions that do not depend on observations. Taking a single derivative of the above produces

$$-\frac{\partial}{\partial \boldsymbol{z}} \log p(\{\boldsymbol{x}_t\}|\boldsymbol{z}) = \boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{z} - \boldsymbol{z}_{obs}) \tag{3.4}$$

where

$$\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{z}_{obs} = \frac{t_{obs}}{\sigma_f^2}\begin{pmatrix} \langle f_t \rangle \\ -\langle t f_t \rangle + \frac{\sigma_f^2}{\sigma_w^2}\langle w_t \rangle \\ \langle t f_t \rangle \ . \end{pmatrix} \tag{3.5}$$

Here, $\boldsymbol{z}_{obs} := (f_{obs}, w_{obs}, v_{obs})^T$ denotes the mean and $\boldsymbol{\Sigma}_{obs}$ denotes the covariance. In order to determine the mean, we only need to compute the covariance, invert it, and multiply it against the above vector.

The important observation is that this implies the mean only depends on certain combinations of observations, rather than on observations in a more arbitrary fashion. Specifically, the posterior depends on three: the time-averaged distance-to-object observations $\langle f_t \rangle$, the time-averaged self-motion-observations $\langle w_t \rangle$, and a time-weighted average of distance observations $\langle t f_t \rangle$. Hence, we can write $p_{obs}(\boldsymbol{z}|\{\boldsymbol{x}_t\}) = p_{obs}(\boldsymbol{z}|\boldsymbol{y})$, where we use $\boldsymbol{y}$ to denote the vector of sufficient statistics.

Note that

$$\begin{aligned} [\langle f_t \rangle] &= \langle [f_t] \rangle = \langle f(t) \rangle = f_0 + (v - w)\langle t \rangle \\ [\langle w_t \rangle] &= \langle [w_t] \rangle = \langle w \rangle = w \end{aligned} \tag{3.6}$$

where $[\cdot]$ denotes an average over sample realizations. We also have

$$\begin{aligned} \mathrm{var}(\langle f_t \rangle) &= \sum_{t,t'} \frac{(\Delta t)^2}{t_{obs}^2}\mathrm{Cov}(f_t, f_{t'}) = \sum_{t,t'} \frac{(\Delta t)^2}{t_{obs}^2}\frac{\sigma_f^2}{\Delta t}\delta_{tt'} = \frac{\sigma_f^2}{t_{obs}} \\ \mathrm{var}(\langle w_t \rangle) &= \sum_{t,t'} \frac{(\Delta t)^2}{t_{obs}^2}\mathrm{Cov}(w_t, w_{t'}) = \sum_{t,t'} \frac{(\Delta t)^2}{t_{obs}^2}\frac{\sigma_w^2}{\Delta t}\delta_{tt'} = \frac{\sigma_w^2}{t_{obs}} \\ \mathrm{Cov}(\langle f_t \rangle, \langle w_t \rangle) &= \sum_{t,t'} \frac{(\Delta t)^2}{t_{obs}^2}\mathrm{Cov}(f_t, w_{t'}) = 0 \ . \end{aligned} \tag{3.7}$$

Life will be easiest if each sufficient statistic is statistically independent and interpretable; for these reasons, we will use something slightly different than $\langle t f_t \rangle$, which does not have an interpretable sample average and is not independent of $\langle f_t \rangle$. Consider

$$\hat{v}_{rel} := \frac{\mathrm{Cov}(t, f_t)}{\mathrm{Cov}(t,t)} = \frac{1}{\mathrm{Cov}(t,t)}\langle (t - t_{obs}/2) f_t \rangle \ . \tag{3.8}$$

The mean and variance of this quantity are

$$\begin{aligned} [\hat{v}_{rel}] &= \frac{\langle (t - t_{obs}/2)[f_t] \rangle}{C(t)} = \frac{\langle (t - t_{obs}/2) f(t) \rangle}{C(t)} = \frac{\langle (t - t_{obs}/2) t \rangle (v - w)}{C(t)} = v - w \\ \mathrm{var}(\hat{v}_{rel}) &= \frac{1}{C(t)^2}\sum_{t,t'} \frac{(\Delta t)^2}{t_{obs}^2}(t - \langle t \rangle)(t' - \langle t \rangle)\mathrm{Cov}(f_t, f_{t'}) = \frac{\sigma_f^2}{C(t) t_{obs}} \ . \end{aligned} \tag{3.9}$$

That is, $\hat{v}_{rel}$ estimates the relative velocity $v - w$. Importantly, we also have

$$\mathrm{Cov}(\langle f_t \rangle, \hat{v}_{rel}) = \frac{1}{C(t)} \frac{(\Delta t)^2}{t_{obs}^2} \sum_{t,t'} (t - \langle t \rangle) \mathrm{Cov}(f_t, f_{t'}) = \frac{\sigma_f^2}{C(t) t_{obs}} \sum_t \frac{\Delta t}{t_{obs}} (t - \langle t \rangle) = 0 \ . \quad (3.10)$$

Hence, we have three interpretable and statistically independent sufficient statistics. In summary,

$$-\log p(\boldsymbol{y}|\boldsymbol{z}) = \frac{t_{obs}}{2\sigma_f^2} \left[\langle f_t \rangle - f_0 - (v-w)t\right]^2 + \frac{t_{obs}}{2\sigma_w^2} \left[\langle w_t \rangle - w\right]^2 + \frac{C(t) t_{obs}}{2\sigma_f^2} \left[\hat{v}_{rel} - (v-w)\right]^2 + \mathrm{const.}$$

Worth pointing out is that these sufficient statistics are the same for both models of interest (moving and stationary firefly).

3.3 Model reduction, in brief

Model definition. Assume that the observation model and priors are

$$p(\{\boldsymbol{x}_t\}|\boldsymbol{z}) = \mathcal{N}(\boldsymbol{M}\boldsymbol{z}, \boldsymbol{\Sigma}_{obs}) \qquad p(\boldsymbol{z}) = \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_0) \ . \quad (3.11)$$

Posterior mean and covariance:

$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_0^{-1} + \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M} \qquad \boldsymbol{\mu} = \boldsymbol{\Sigma} \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \{\boldsymbol{x}_t\} \quad (3.12)$$

Model comparison setup. Model likelihood:

$$p(\{\boldsymbol{x}_t\}) = \int p(\{\boldsymbol{x}_t\}|\boldsymbol{z}) p(\boldsymbol{z}) \ d\boldsymbol{z} = \mathcal{N}(\{\boldsymbol{x}_t\}; \boldsymbol{0}, \boldsymbol{\Sigma}_{obs} + \boldsymbol{M}\boldsymbol{\Sigma}_0\boldsymbol{M}^T) = \mathcal{N}(\{\boldsymbol{x}_t\}; \boldsymbol{0}, \boldsymbol{\Sigma}_{data}) \quad (3.13)$$

Inverse of data covariance:

$$\boldsymbol{\Sigma}_{data}^{-1} = \boldsymbol{\Sigma}_{obs}^{-1} - \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M} (\boldsymbol{\Sigma}_0^{-1} + \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M})^{-1} \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} = \boldsymbol{\Sigma}_{obs}^{-1} - \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M} \boldsymbol{\Sigma} \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \quad (3.14)$$

The log marginal likelihood can be written as

$$\log p(\boldsymbol{y}) = -\frac{1}{2} \{\boldsymbol{x}_t\}^T \boldsymbol{\Sigma}_{obs}^{-1} \{\boldsymbol{x}_t\} + \frac{1}{2} \{\boldsymbol{x}_t\}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M} \boldsymbol{\Sigma} \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \{\boldsymbol{x}_t\} - \frac{1}{2} \log[(2\pi)^D \det \boldsymbol{\Sigma}_{data}] \ .$$

Consider the difference of two such log-likelihoods, $\Delta := \log p(\boldsymbol{y}|M_B) - \log p(\boldsymbol{y}|M_A)$, where the only thing that differs is $\boldsymbol{\Sigma}_0$:

$$\Delta = \frac{1}{2} \{\boldsymbol{x}_t\}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M} (\boldsymbol{\Sigma}_B - \boldsymbol{\Sigma}_A) \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \{\boldsymbol{x}_t\} - \frac{1}{2} \log \left[\frac{\det \boldsymbol{\Sigma}_{data,B}}{\det \boldsymbol{\Sigma}_{data,A}} \right] \ . \quad (3.15)$$

Simplifying the first term. Assume the difference between the inverse covariances is rank 1. Then

$$\boldsymbol{\Sigma}_B - \boldsymbol{\Sigma}_A = \frac{\det \boldsymbol{\Sigma}_B^{-1}}{\det \boldsymbol{\Sigma}_A^{-1}} \boldsymbol{\Sigma}_B (\boldsymbol{\Sigma}_A^{-1} - \boldsymbol{\Sigma}_B^{-1}) \boldsymbol{\Sigma}_B \quad (3.16)$$

and

$$\Delta = \frac{1}{2}\frac{\det \boldsymbol{\Sigma}_B^{-1}}{\det \boldsymbol{\Sigma}_A^{-1}}\boldsymbol{\mu}^T(\boldsymbol{\Sigma}_{0A}^{-1} - \boldsymbol{\Sigma}_{0B}^{-1})\boldsymbol{\mu} - \frac{1}{2}\log\left[\frac{\det \boldsymbol{\Sigma}_{data,B}}{\det \boldsymbol{\Sigma}_{data,A}}\right] \tag{3.17}$$

What is the determinant of $\boldsymbol{\Sigma}_A$, assuming this is the submodel? We are taking one of the prior widths to zero, meaning one entry of $\boldsymbol{\Sigma}_0^{-1}$ (and hence an entry on the diagonal of $\boldsymbol{\Sigma}^{-1}$) is being taken to infinity. Using the Laplace (cofactor) expansion, in the width to zero limit,

$$\det \boldsymbol{\Sigma}^{-1} \to \frac{1}{\xi_v^2}\det \boldsymbol{\Sigma}^{-1} \tag{3.18}$$

where we abuse notation somewhat to refer to the matrix with one row and column (corresponding to the missing latent) removed. The prior parameter that's taken to infinity cancels with the corresponding parameter in the numerator, leaving us with

$$\Delta = \frac{1}{2}\frac{\det \boldsymbol{\Sigma}_B^{-1}}{\det \boldsymbol{\Sigma}_A^{-1}}\mu_v^2 - \frac{1}{2}\log\left[\frac{\det \boldsymbol{\Sigma}_{data,B}}{\det \boldsymbol{\Sigma}_{data,A}}\right] \tag{3.19}$$

Units here check out; be careful about including the prefactor (t_{obs}/σ_f^2) in the determinant, which appears a different number of times in the numerator and denominator.

Simplifying second term. Using Sylvester's theorem,

$$\begin{aligned}\det\big(\boldsymbol{\Sigma}_{obs} + \boldsymbol{M}\boldsymbol{\Sigma}_0\boldsymbol{M}^T\big) &= \det(\boldsymbol{\Sigma}_{obs})\det\Big(\boldsymbol{I} + \boldsymbol{\Sigma}_0^{1/2}\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\boldsymbol{\Sigma}_0^{1/2}\Big)\\ &= \det(\boldsymbol{\Sigma}_{obs})\det(\boldsymbol{\Sigma}_0)\det\big(\boldsymbol{\Sigma}_0^{-1} + \boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\big)\\ &= \det(\boldsymbol{\Sigma}_{obs})\det(\boldsymbol{\Sigma}_0)\det\big(\boldsymbol{\Sigma}^{-1}\big)\end{aligned} \tag{3.20}$$

One has to be a little careful about $\boldsymbol{\Sigma}_0$ in the prior width to zero limit; pre- and post-multiplying by it removes the relevant row and column. The determinant that is pulled out is not the full-size one, but the corresponding matrix with the offending row and column removed. Then

$$\frac{\det \boldsymbol{\Sigma}_{data,B}}{\det \boldsymbol{\Sigma}_{data,A}} = \frac{\det(\boldsymbol{\Sigma}_{0B})\det\big(\boldsymbol{\Sigma}_B^{-1}\big)}{\det(\boldsymbol{\Sigma}_{0A})\det\big(\boldsymbol{\Sigma}_A^{-1}\big)} = \xi_v^2\frac{\det\big(\boldsymbol{\Sigma}_B^{-1}\big)}{\det\big(\boldsymbol{\Sigma}_A^{-1}\big)} \tag{3.21}$$

Decision threshold. Combining the previous results and adding a prior model preference, we have

$$\Delta = \frac{1}{2}\frac{\det \boldsymbol{\Sigma}_B^{-1}}{\det \boldsymbol{\Sigma}_A^{-1}}\mu_v^2 - \frac{1}{2}\log\left[\xi_v^2\frac{\det\big(\boldsymbol{\Sigma}_B^{-1}\big)}{\det\big(\boldsymbol{\Sigma}_A^{-1}\big)}\right] + \Delta_0\ . \tag{3.22}$$

Define $c := \det \boldsymbol{\Sigma}_B^{-1}/\det \boldsymbol{\Sigma}_A^{-1}$. The more complex model is favored when

$$|\mu_v| \geq \sqrt{\frac{[\ \log(\xi_v^2 c) - 2\Delta_0\ ]_+}{c}}\ . \tag{3.23}$$

Decision confidence. The agent's decision confidence, i.e., their subjective belief about whether they made the correct model choice, is

$$p(\text{correct choice}) = \frac{\exp(|\Delta|)}{1+\exp(|\Delta|)} \ . \tag{3.24}$$

Distribution of decisions. The experimenter does not have access to the specific noisy observations subjects use to make decisions. Hence, we cannot predict individual decisions, but we can predict distributions of decisions. The distribution of the posterior mean $\boldsymbol{\mu}$ given $\boldsymbol{z}$ is normal with

$$\begin{aligned} [\boldsymbol{\mu}] &= \boldsymbol{\Sigma M}^T \boldsymbol{\Sigma}_{obs}^{-1} [\{\boldsymbol{x}_t\}] = \boldsymbol{\Sigma M}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M z} = (\boldsymbol{I} - \boldsymbol{\Sigma}\boldsymbol{\Sigma}_0^{-1}) \boldsymbol{z} \\ \mathrm{Cov}(\boldsymbol{\mu}) &= \boldsymbol{\Sigma M}^T \boldsymbol{\Sigma}_{obs}^{-1} \mathrm{Cov}(\{\boldsymbol{x}_t\}) \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M \Sigma} = \boldsymbol{\Sigma M}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M \Sigma} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\boldsymbol{\Sigma}_0^{-1} \boldsymbol{\Sigma} \ . \end{aligned} \tag{3.25}$$

Denote the mean and variance by $[\mu_v]$ and σ_v^2. The probability that μ_v exceeds some threshold $q > 0$ is

$$P(|\mu_v| \geq q) = \int_q^\infty \mathcal{N}(\mu_v; [\mu_v], \sigma_v^2) \ d\mu_v + \int_{-\infty}^{-q} \mathcal{N}(\mu_v; [\mu_v], \sigma_v^2) \ d\mu_v \ . \tag{3.26}$$

We can perform a change of variables to express these integrals in terms of error functions. We find

$$p(\text{report move}) = P(|\mu_v| \geq q) = 1 - \frac{1}{2} \left[\mathrm{erf}\left(\frac{q - [\mu_v]}{\sqrt{2} \ \sigma_v} \right) + \mathrm{erf}\left(\frac{q + [\mu_v]}{\sqrt{2} \ \sigma_v} \right) \right] \ . \tag{3.27}$$

As a sanity check, note that this probability goes to zero as $q \to \infty$. (Note that we are not allowed to take q negative, since we assumed it was positive when we derived this.)

3.4 Decision times

One can compute observation-averaged decision times in order to predict reaction times. This can be formalized as the time at which $\mathbb{E}[\Delta_t]$ is equal to some (generally collapsing) bound. The time-dependence of the accumulated evidence comes both from the determinant ratio, and the posterior mean and variance of the relevant quantity (which appears when we take the average).

More precisely,

$$\mathbb{E}[\Delta_t] = \frac{1}{2} \frac{\det \boldsymbol{\Sigma}_B^{-1}}{\det \boldsymbol{\Sigma}_A^{-1}} \left\{ [\mu_v] + \sigma_v^2 \right\} - \frac{1}{2} \log \left[\xi_v^2 \frac{\det\left(\boldsymbol{\Sigma}_B^{-1}\right)}{\det\left(\boldsymbol{\Sigma}_A^{-1}\right)} \right] + \Delta_0 \ . \tag{3.28}$$

3.5 Deciding whether or not there is self-motion

Consider the problem of determining whether there is self-motion or not given vestibular signals. This is a 'toy' causal inference problem that allows us to test our complex math. There is only one latent and one type of observation.

Generative model. Assume an observation model and prior

$$p(w_t|w) = \mathcal{N}(w, \frac{\sigma_w^2}{\Delta t}) \qquad\qquad p(w) = \mathcal{N}(0, \xi_w^2) \ . \tag{3.29}$$

It is useful to express observations in terms of the sufficient statistic $\langle w_t \rangle$, the time-average of all self-motion observations. The corresponding posterior mean and covariance are

$$\Sigma^{-1} = \frac{1}{\xi_w^2} + \frac{t_{obs}}{\sigma_w^2} \qquad\qquad \mu_w = \frac{\frac{t_{obs}}{\sigma_w^2}}{\frac{1}{\xi_w^2} + \frac{t_{obs}}{\sigma_w^2}} \langle w_t \rangle \ . \tag{3.30}$$

For the model that assumes there is no self-motion, the posterior and prior are both $\delta(w)$.

Model comparison. The difference of log-likelihoods (positive favors self-motion) is

$$\Delta = \frac{1}{2}\left[\frac{1}{\xi_w^2} + \frac{t_{obs}}{\sigma_w^2}\right]\mu_w^2 - \frac{1}{2}\log\left[1 + \xi_w^2 \frac{t_{obs}}{\sigma_w^2}\right] + \Delta_0 \tag{3.31}$$

where we note that the determinant of the stationary model is 1 (i.e., the determinant is 'vacuous'). Hence, the self-motion model is favored when

$$|\mu_w| \geq \sqrt{\frac{[\ \log\left(1 + \xi_w^2 \frac{t_{obs}}{\sigma_w^2}\right) - 2\Delta_0\ ]_+}{\frac{1}{\xi_w^2} + \frac{t_{obs}}{\sigma_w^2}}} \ . \tag{3.32}$$

Distribution of decisions. The distribution of μ_w has

$$[\mu_w] = \frac{\frac{t_{obs}}{\sigma_w^2}}{\frac{1}{\xi_w^2} + \frac{t_{obs}}{\sigma_w^2}}\ w \qquad\qquad \sigma^2 = \frac{\frac{t_{obs}}{\sigma_w^2}}{\left(\frac{1}{\xi_w^2} + \frac{t_{obs}}{\sigma_w^2}\right)^2} \tag{3.33}$$

and the probability of reporting self-motion is

$$\begin{aligned} &1 - \frac{1}{2}\left[\text{erf}\left(\frac{q - [\mu_w]}{\sqrt{2}\ \sigma}\right) + \text{erf}\left(\frac{q + [\mu_w]}{\sqrt{2}\ \sigma}\right)\right] \\ &= 1 - \frac{1}{2}\text{erf}\left(\sqrt{\frac{1}{2}\left(1 + \frac{\sigma_w^2}{t_{obs}}\frac{1}{\xi_w^2}\right)[\ \log\left(1 + \xi_w^2\frac{t_{obs}}{\sigma_w^2}\right) - 2\Delta_0\ ]_+} - \sqrt{\frac{t_{obs}}{2\sigma_w^2}}w\right) \\ &\quad - \frac{1}{2}\text{erf}\left(\sqrt{\frac{1}{2}\left(1 + \frac{\sigma_w^2}{t_{obs}}\frac{1}{\xi_w^2}\right)[\ \log\left(1 + \xi_w^2\frac{t_{obs}}{\sigma_w^2}\right) - 2\Delta_0\ ]_+} + \sqrt{\frac{t_{obs}}{2\sigma_w^2}}w\right) \end{aligned}$$

Note that the error function is determined by only 2 parameter combinations (q/σ and c_w/σ), but there are 3 parameters to infer ($\sigma_w^2, \xi_w^2, \Delta_0$). Hence, not all parameters are identifiable. Would have to vary t_{obs} to identify more.

3.6 Deciding whether or not an object moves

In 1D, the set of latents is $\boldsymbol{z} := (f_0, v)^T$. The observations are relative positions $\{f_t\}$.

Generative model. Assume an observation model and prior

$$p(f_t|\boldsymbol{z}) = \mathcal{N}(f_0 + vt, \frac{\sigma_f^2}{\Delta t}) \qquad\qquad p(\boldsymbol{z}) = \mathcal{N}(f_0; 0, \xi_f^2)\, \mathcal{N}(v; 0, \xi_v^2)\ . \tag{3.34}$$

It is useful to introduce the sufficient statistics $\langle f_t \rangle$ and $\hat{v} = \frac{\mathrm{Cov}(t, f_t)}{\mathrm{Cov}(t)}$. These statistics $\boldsymbol{y} := (\langle f_t \rangle, \hat{v})^T$ satisfy

$$\mathbb{E}[\boldsymbol{y}|\boldsymbol{z}] = \begin{pmatrix} f_0 + v\langle t \rangle \\ v \end{pmatrix} = \begin{pmatrix} 1 & \langle t \rangle \\ 0 & 1 \end{pmatrix} \boldsymbol{z} = \boldsymbol{M}\boldsymbol{z} \qquad\qquad \boldsymbol{\Sigma}_{obs}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & 0 \\ 0 & C(t) \end{pmatrix}\ . \tag{3.35}$$

Uniform priors. Assume infinite-width priors. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\boldsymbol{\Sigma}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & \langle t \rangle \\ \langle t \rangle & \langle t^2 \rangle \end{pmatrix} \qquad\qquad |\boldsymbol{\Sigma}^{-1}| = \langle t^2 \rangle - \langle t \rangle^2 = C(t)\ . \tag{3.36}$$

The posterior covariance and mean are

$$\boldsymbol{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \langle t^2 \rangle & -\langle t \rangle \\ -\langle t \rangle & 1 \end{pmatrix} \qquad\qquad \boldsymbol{\mu} = \begin{pmatrix} \langle f_t \rangle - \langle t \rangle \hat{v} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} 1 & -\langle t \rangle \\ 0 & 1 \end{pmatrix} \boldsymbol{y}\ . \tag{3.37}$$

The mean is an unbiased estimator of $\boldsymbol{z}$, since $[\boldsymbol{\mu}] = \boldsymbol{z}$.

Posterior. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\begin{aligned} \boldsymbol{\Sigma}^{-1} &= \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 + \frac{1}{\tilde{\xi}_f^2} & \langle t \rangle \\ \langle t \rangle & \langle t^2 \rangle + \frac{1}{\tilde{\xi}_v^2} \end{pmatrix} \\ |\boldsymbol{\Sigma}^{-1}| &= \left(1 + \frac{1}{\tilde{\xi}_f^2}\right) \left(\langle t^2 \rangle + \frac{1}{\tilde{\xi}_v^2}\right) - \langle t \rangle^2 = \left(1 + \frac{1}{\tilde{\xi}_f^2}\right) \left(C(t) + \frac{1}{\tilde{\xi}_v^2}\right) + \frac{\langle t \rangle^2}{\tilde{\xi}_f^2}\ . \end{aligned} \tag{3.38}$$

The posterior covariance is

$$\boldsymbol{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \langle t^2 \rangle + \frac{1}{\tilde{\xi}_v^2} & -\langle t \rangle \\ -\langle t \rangle & 1 + \frac{1}{\tilde{\xi}_f^2} \end{pmatrix}\ . \tag{3.39}$$

The posterior mean is

$$\boldsymbol{\mu} = \frac{C_t}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \left[1 + \frac{1}{C_t\ \tilde{\xi}_v^2}\right] \langle f_t \rangle - \langle t \rangle \hat{v} \\ \frac{\langle t \rangle}{C_t\ \tilde{\xi}_f^2} \langle f_t \rangle + \left(1 + \frac{1}{\tilde{\xi}_f^2}\right) \hat{v} \end{pmatrix} = \frac{C_t}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \left[1 + \frac{1}{C_t\ \tilde{\xi}_v^2}\right] & -\langle t \rangle \\ \frac{\langle t \rangle}{C_t\ \tilde{\xi}_f^2} & 1 + \frac{1}{\tilde{\xi}_f^2} \end{pmatrix} \boldsymbol{y}\ .$$

The determinant of the inverse covariance of the reduced model can be found by taking the $\xi_v \to 0$ limit and removing the overall ξ_v-dependent factor. We have

$$|\mathbf{\Sigma}_{red}^{-1}| = \lim_{\xi_v \to 0} \xi_v^2 |\mathbf{\Sigma}^{-1}| = 1 + \frac{1}{\tilde{\xi}_f^2} \ . \tag{3.40}$$

Model comparison. The difference of log-likelihoods (positive favors motion) is

$$\begin{aligned} \Delta &= \frac{1}{2}\frac{|\mathbf{\Sigma}^{-1}|}{|\mathbf{\Sigma}_{red}^{-1}|}\mu_v^2 - \frac{1}{2}\log\left[\xi_v^2 \frac{|\mathbf{\Sigma}^{-1}|}{|\mathbf{\Sigma}_{red}^{-1}|}\right] + \Delta_0 \\ &= \frac{1}{2}\frac{t_{obs}}{\sigma_f^2}\left[C(t) + \frac{1}{\tilde{\xi}_v^2} + \frac{\langle t^2\rangle}{1+\tilde{\xi}_f^2}\right]\mu_v^2 - \frac{1}{2}\log\left[1 + \tilde{\xi}_v^2\left(C(t) + \frac{\langle t^2\rangle}{1+\tilde{\xi}_f^2}\right)\right] + \Delta_0 \ . \end{aligned} \tag{3.41}$$

The motion model is favored when

$$|\mu_v| \geq \sqrt{\frac{[\ \log\left(1 + \tilde{\xi}_v^2(C(t) + \frac{\langle t^2\rangle}{1+\tilde{\xi}_f^2})\ \right) - 2\Delta_0\ ]_+}{\frac{t_{obs}}{\sigma_f^2}\left[C(t) + \frac{1}{\tilde{\xi}_v^2} + \frac{\langle t^2\rangle}{1+\tilde{\xi}_f^2}\right]}} \ . \tag{3.42}$$

Distribution of decisions. The distribution of μ_v has

$$\begin{aligned} [\mu_v] &= \frac{1}{|\mathbf{\Sigma}^{-1}|}\left[\frac{\langle t\rangle}{\tilde{\xi}_f^2} f_0 + \left(C(t) + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right) v\right] \\ \sigma^2 &= \frac{1}{|\mathbf{\Sigma}^{-1}|^2}\left[\left(\frac{\langle t\rangle}{\tilde{\xi}_f^2}\right)^2 \frac{\sigma_f^2}{t_{obs}} + C(t)^2\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2 \frac{\sigma_f^2}{t_{obs}}\frac{1}{C(t)}\right] \\ &= \frac{1}{|\mathbf{\Sigma}^{-1}|^2}\frac{\sigma_f^2}{t_{obs}}\left[\left(\frac{\langle t\rangle}{\tilde{\xi}_f^2}\right)^2 + C(t)\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2\right] \\ \sigma &= \frac{1}{|\mathbf{\Sigma}^{-1}|}\sqrt{\frac{\sigma_f^2}{t_{obs}}}\sqrt{\left(\frac{\langle t\rangle}{\tilde{\xi}_f^2}\right)^2 + C(t)\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2} \ . \end{aligned} \tag{3.43}$$

The probability of reporting motion is determined by the quantities

$$\begin{aligned} \frac{q}{\sigma} &= \frac{\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)\left(C(t) + \frac{1}{\tilde{\xi}_v^2}\right) + \frac{\langle t\rangle^2}{\tilde{\xi}_f^2}}{\sqrt{\left[\left(\frac{\langle t\rangle}{\tilde{\xi}_f^2}\right)^2 + C(t)\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2\right]\left[C(t) + \frac{1}{\tilde{\xi}_v^2} + \frac{\langle t^2\rangle}{1+\tilde{\xi}_f^2}\right]}}\sqrt{[\ \log\left(1 + \tilde{\xi}_v^2(C(t) + \frac{\langle t^2\rangle}{1+\tilde{\xi}_f^2})\ \right) - 2\Delta_0\ ]_+} \\ \frac{[\mu_v]}{\sigma} &= \sqrt{\frac{\frac{t_{obs}}{\sigma_f^2}}{\left(\frac{\langle t\rangle}{\tilde{\xi}_f^2}\right)^2 + C(t)\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2}}\left[\frac{\langle t\rangle}{\tilde{\xi}_f^2} f_0 + \left(C(t) + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right) v\right] \ . \end{aligned} \tag{3.44}$$

The probability itself is

$$p(\text{report move}) = 1 - \frac{1}{2}\left[\operatorname{erf}\left(\frac{q - [\mu_v]}{\sqrt{2}\ \sigma}\right) + \operatorname{erf}\left(\frac{q + [\mu_v]}{\sqrt{2}\ \sigma}\right)\right] . \tag{3.45}$$

Note: there are 4 parameters to infer (σ_f^2, ξ_f^2, ξ_w^2, Δ_0) and 3 parameter combinations that shape the above probability (q/σ, c_f/σ, c_v/σ). Hence, there is degeneracy. Also kind of unclear which parameter combinations are most 'important'. Fisher information? Empirical tests?

3.7 Full 1D model

In 1D, the set of latents is $\boldsymbol{z} := (f_0, w, v)^T$. The observations are relative positions and vestibular signals $\{f_t, w_t\}$.

Generative model. Assume an observation model and prior

$$p(f_t, w_t|\boldsymbol{z}) = \mathcal{N}(f_0+(v-w)t, \frac{\sigma_f^2}{\Delta t})\,\mathcal{N}(w, \frac{\sigma_w^2}{\Delta t}) \qquad p(\boldsymbol{z}) = \mathcal{N}(f_0; 0, \xi_f^2)\,\mathcal{N}(w; 0, \xi_w^2)\,\mathcal{N}(v; 0, \xi_v^2)\,.$$

It is useful to introduce the sufficient statistics $\langle f_t\rangle$, $\langle w_t\rangle$, and $\hat{v}_{rel} = \frac{\mathrm{Cov}(t,f_t)}{\mathrm{Cov}(t)}$. These statistics $\boldsymbol{y} := (\langle f_t\rangle, \langle w_t\rangle, \hat{v}_{rel})^T$ satisfy

$$\mathbb{E}[\boldsymbol{y}|\boldsymbol{z}] = \begin{pmatrix} f_0 + (v-w)\langle t\rangle \\ w \\ v-w \end{pmatrix} = \begin{pmatrix} 1 & -\langle t\rangle & \langle t\rangle \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \boldsymbol{z} \qquad \boldsymbol{\Sigma}_{obs}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sigma_f^2}{\sigma_w^2} & 0 \\ 0 & 0 & C(t) \end{pmatrix}\,. \tag{3.46}$$

Uniform priors. Assume infinite-width priors. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\boldsymbol{\Sigma}^{-1} = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 & -\langle t\rangle & \langle t\rangle \\ -\langle t\rangle & \langle t^2\rangle + \frac{\sigma_f^2}{\sigma_w^2} & -\langle t^2\rangle \\ \langle t\rangle & -\langle t^2\rangle & \langle t^2\rangle \end{pmatrix} \qquad |\boldsymbol{\Sigma}^{-1}| = C_t \frac{\sigma_f^2}{\sigma_w^2}\,.$$

The posterior covariance is

$$\boldsymbol{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\boldsymbol{\Sigma}^{-1}|} \begin{pmatrix} \langle t^2\rangle \frac{\sigma_f^2}{\sigma_w^2} & 0 & -\langle t\rangle \frac{\sigma_f^2}{\sigma_w^2} \\ 0 & C_t & C_t \\ -\langle t\rangle \frac{\sigma_f^2}{\sigma_w^2} & C_t & C_t + \frac{\sigma_f^2}{\sigma_w^2} \end{pmatrix} = \frac{\sigma_f^2}{t_{obs}} \begin{pmatrix} \frac{\langle t^2\rangle}{C_t} & 0 & -\frac{\langle t\rangle}{C_t} \\ 0 & \frac{\sigma_w^2}{\sigma_f^2} & \frac{\sigma_w^2}{\sigma_f^2} \\ -\frac{\langle t\rangle}{C_t} & \frac{\sigma_w^2}{\sigma_f^2} & \frac{\sigma_w^2}{\sigma_f^2} + \frac{1}{C_t} \end{pmatrix}\,. \tag{3.47}$$

The posterior mean is

$$\boldsymbol{\mu} = \begin{pmatrix} \langle f_t\rangle - \langle t\rangle \hat{v}_{rel} \\ \langle w_t\rangle \\ \langle w_t\rangle + \hat{v}_{rel} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\langle t\rangle \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \boldsymbol{y}\,. \tag{3.48}$$

As before, $\boldsymbol{\mu}$ is unbiased since $[\boldsymbol{\mu}] = \boldsymbol{z}$.

Posterior. The posterior inverse covariance and its (prefactor-adjusted) determinant are

$$\boldsymbol{\Sigma}^{-1} = \frac{t_{obs}}{\sigma_f^2} \left[\begin{pmatrix} \frac{1}{\tilde{\xi}_f^2} & 0 & 0 \\ 0 & \frac{1}{\tilde{\xi}_w^2} & 0 \\ 0 & 0 & \frac{1}{\tilde{\xi}_v^2} \end{pmatrix} + \begin{pmatrix} 1 & -\langle t\rangle & \langle t\rangle \\ -\langle t\rangle & \langle t^2\rangle + \frac{\sigma_f^2}{\sigma_w^2} & -\langle t^2\rangle \\ \langle t\rangle & -\langle t^2\rangle & \langle t^2\rangle \end{pmatrix} \right] = \frac{t_{obs}}{\sigma_f^2} \begin{pmatrix} 1 + \frac{1}{\tilde{\xi}_f^2} & -\langle t\rangle & \langle t\rangle \\ -\langle t\rangle & \langle t^2\rangle + \frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} & -\langle t^2\rangle \\ \langle t\rangle & -\langle t^2\rangle & \langle t^2\rangle + \frac{1}{\tilde{\xi}_v^2} \end{pmatrix}$$

$$|\boldsymbol{\Sigma}^{-1}| = \left(C(t) + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_v^2} + \frac{1}{\tilde{\xi}_w^2} \right) + \frac{1}{\tilde{\xi}_v^2} \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right)\,.$$

The posterior covariance is

$$\mathbf{\Sigma} = \frac{\sigma_f^2}{t_{obs}} \frac{1}{|\mathbf{\Sigma}^{-1}|} \begin{pmatrix} (\frac{1}{\tilde{\xi}_v^2} + \langle t^2 \rangle)(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}) + \frac{\langle t^2 \rangle}{\tilde{\xi}_v^2} & \frac{\langle t \rangle}{\tilde{\xi}_v^2} & -\langle t \rangle \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \\ \frac{\langle t \rangle}{\tilde{\xi}_v^2} & \frac{1}{\tilde{\xi}_v^2} \left[1 + \frac{1}{\tilde{\xi}_f^2} \right] + \mathrm{C}(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} & \mathrm{C}(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \\ -\langle t \rangle \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) & \mathrm{C}(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} & \mathrm{C}(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} + (1 + \frac{1}{\tilde{\xi}_f^2}) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \end{pmatrix}$$

The posterior mean is

$$\mu_f = \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ \left[\left(\frac{1}{\tilde{\xi}_v^2} + C(t) \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) + \frac{C(t)}{\tilde{\xi}_v^2} \right] \langle f_t \rangle + \frac{\langle t \rangle}{\tilde{\xi}_v^2} \frac{\sigma_f^2}{\sigma_w^2} \langle w_t \rangle - \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} + \frac{1}{\tilde{\xi}_v^2} \right) \langle t \rangle C(t) \hat{v}_{rel} \right\}$$

$$\mu_w = \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ -\frac{\langle t \rangle}{\tilde{\xi}_f^2 \tilde{\xi}_v^2} \langle f_t \rangle + \frac{\sigma_f^2}{\sigma_w^2} \left[\frac{1}{\tilde{\xi}_v^2} \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) + \mathrm{C}(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right] \langle w_t \rangle - \frac{C(t)}{\tilde{\xi}_v^2} \left[1 + \frac{1}{\tilde{\xi}_f^2} \right] \hat{v}_{rel} \right\}$$

$$\mu_v = \frac{1}{|\mathbf{\Sigma}^{-1}|} \left\{ \frac{\langle t \rangle}{\tilde{\xi}_f^2} \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \langle f_t \rangle + \frac{\sigma_f^2}{\sigma_w^2} \left[\mathrm{C}(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right] \langle w_t \rangle + C(t) \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \hat{v}_{rel} \right\}$$

As a sanity check, note that the above expressions reduce to the uniform prior ones when the prior widths are taken to infinity.

The determinant of the inverse covariance of the reduced model is

$$|\mathbf{\Sigma}_{red}^{-1}| = \lim_{\xi_v \to 0} \xi_v^2 |\mathbf{\Sigma}^{-1}| = \left(C(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right) + \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) . \tag{3.49}$$

The determinant ratio is

$$\begin{aligned} \frac{|\mathbf{\Sigma}^{-1}|}{|\mathbf{\Sigma}_{red}^{-1}|} = & \frac{\left(C(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right)}{\left(C(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right) + \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right)} \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_v^2} + \frac{1}{\tilde{\xi}_w^2} \right) \\ & + \frac{\left(1 + \frac{1}{\tilde{\xi}_f^2} \right)}{\left(C(t) + \frac{\langle t^2 \rangle}{\tilde{\xi}_f^2} \right) + \left(1 + \frac{1}{\tilde{\xi}_f^2} \right) \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right)} \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2} \right) \frac{1}{\tilde{\xi}_v^2} . \end{aligned} \tag{3.50}$$

Model comparison. The difference of log-likelihoods (positive favors motion) is

$$\Delta = \frac{1}{2} \frac{t_{obs}}{\sigma_f^2} \frac{|\mathbf{\Sigma}^{-1}|}{|\mathbf{\Sigma}_{red}^{-1}|} \mu_v^2 - \frac{1}{2} \log \left[\tilde{\xi}_v^2 \frac{|\mathbf{\Sigma}^{-1}|}{|\mathbf{\Sigma}_{red}^{-1}|} \right] + \Delta_0 . \tag{3.51}$$

The motion model is favored when

$$|\mu_v| \geq \sqrt{\frac{\left[\log \left(\tilde{\xi}_v^2 \frac{|\mathbf{\Sigma}^{-1}|}{|\mathbf{\Sigma}_{red}^{-1}|} \right) - 2\Delta_0 \right]_+}{\frac{t_{obs}}{\sigma_f^2} \frac{|\mathbf{\Sigma}^{-1}|}{|\mathbf{\Sigma}_{red}^{-1}|}}} . \tag{3.52}$$

Distribution of decisions. Note that

$$[\mu_f] = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{\left[\left(\frac{1}{\tilde{\xi}_v^2}+C(t)\right)\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)+\frac{C(t)}{\tilde{\xi}_v^2}\right]f_0 - \frac{\langle t\rangle}{\tilde{\xi}_v^2\tilde{\xi}_w^2}w + \frac{\langle t\rangle}{\tilde{\xi}_v^2}\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)v\right\}$$

$$[\mu_w] = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{-\frac{\langle t\rangle}{\tilde{\xi}_f^2\tilde{\xi}_v^2}f_0 + \left[\frac{\sigma_f^2}{\sigma_w^2}\frac{1}{\tilde{\xi}_v^2}\left(1+\frac{1}{\tilde{\xi}_f^2}\right)+\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_v^2}\right)\left(C_t+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right)\right]w - \frac{1}{\tilde{\xi}_v^2}\left[C_t+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right]v\right\}$$

$$[\mu_v] = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{\frac{\langle t\rangle}{\tilde{\xi}_f^2}\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)f_0 - \frac{1}{\tilde{\xi}_w^2}\left[C_t+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right]w + \left(C_t+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right)\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)v\right\}$$

Posterior mean velocity given observations:

$$\mu_v = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{\frac{\langle t\rangle}{\tilde{\xi}_f^2}\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)\langle f_t\rangle + \frac{\sigma_f^2}{\sigma_w^2}\left[\mathrm{C}(t)+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right]\langle w_t\rangle + C(t)\left(1+\frac{1}{\tilde{\xi}_f^2}\right)\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)\hat{v}_{rel}\right\}$$

We can find the corresponding velocity variance:

$$\sigma_v^2 = \frac{\sigma_f^2}{t_{obs}}\frac{1}{|\mathbf{\Sigma}^{-1}|^2}\left\{\frac{\langle t\rangle^2}{\tilde{\xi}_f^4}\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)^2 + \frac{\sigma_f^2}{\sigma_w^2}\left[\mathrm{C}(t)+\frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right]^2 + C(t)\left(1+\frac{1}{\tilde{\xi}_f^2}\right)^2\left(\frac{\sigma_f^2}{\sigma_w^2}+\frac{1}{\tilde{\xi}_w^2}\right)^2\right\} \tag{3.53}$$

$$p(\text{report stat}) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{q-[\mu_v]}{\sqrt{2}\,\sigma}\right)+\mathrm{erf}\left(\frac{q+[\mu_v]}{\sqrt{2}\,\sigma}\right)\right] . \tag{3.54}$$

Derivative of this prob with respect to v:

$$\frac{d}{dv}p(\text{report stat}) = -\frac{c_v}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(q-[\mu_v])^2}{2\sigma^2}\right\}+\frac{c_v}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(q+[\mu_v])^2}{2\sigma^2}\right\} = 0 \tag{3.55}$$

The optimum happens where

$$q[\mu_v] = -q[\mu_v] \implies [\mu_v] = 0 \text{ if } q > 0 \tag{3.56}$$

This happens where

$$v = -\frac{(c_f f_0 + c_w w)}{c_v} . \tag{3.57}$$

More explicitly,

$$v = -\frac{(c_f f_0 + c_w w)}{c_v} . \tag{3.58}$$

$$p(\text{report stat}) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{q-(c_f f_0 + c_w w + c_v v)}{\sqrt{2}\,\sigma}\right)+\mathrm{erf}\left(\frac{q+(c_f f_0 + c_w w + c_v v)}{\sqrt{2}\,\sigma}\right)\right] . \tag{3.59}$$

Distribution of decisions, no self-motion. Note that

$$[\mu_f] = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{\left[\left(\frac{1}{\tilde{\xi}_v^2} + C(t)\right)\left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right) + \frac{C(t)}{\tilde{\xi}_v^2}\right] f_0 - \frac{\langle t\rangle}{\tilde{\xi}_v^2\tilde{\xi}_w^2} w + \frac{\langle t\rangle}{\tilde{\xi}_v^2}\left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right) v\right\}$$

$$[\mu_w] = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{-\frac{\langle t\rangle}{\tilde{\xi}_f^2\tilde{\xi}_v^2} f_0 + \left[\frac{\sigma_f^2}{\sigma_w^2}\frac{1}{\tilde{\xi}_v^2}\left(1 + \frac{1}{\tilde{\xi}_f^2}\right) + \left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_v^2}\right)\left(C_t + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right)\right] w - \frac{1}{\tilde{\xi}_v^2}\left[C_t + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right] v\right\}$$

$$[\mu_v] = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{\frac{\langle t\rangle}{\tilde{\xi}_f^2}\left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right) f_0 - \frac{1}{\tilde{\xi}_w^2}\left[C_t + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right] w + \left(C_t + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right)\left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right) v\right\}$$

Posterior mean velocity given observations:

$$\mu_v = \frac{1}{|\mathbf{\Sigma}^{-1}|}\left\{\frac{\langle t\rangle}{\tilde{\xi}_f^2}\left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right)\langle f_t\rangle + \frac{\sigma_f^2}{\sigma_w^2}\left[\mathrm{C}(t) + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right]\langle w_t\rangle + C(t)\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)\left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right)\hat{v}_{rel}\right\}$$

We can find the corresponding velocity variance:

$$\sigma_v^2 = \frac{\sigma_f^2}{t_{obs}}\frac{1}{|\mathbf{\Sigma}^{-1}|^2}\left\{\frac{\langle t\rangle^2}{\tilde{\xi}_f^4}\left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right)^2 + \frac{\sigma_f^2}{\sigma_w^2}\left[\mathrm{C}(t) + \frac{\langle t^2\rangle}{\tilde{\xi}_f^2}\right]^2 + C(t)\left(1 + \frac{1}{\tilde{\xi}_f^2}\right)^2\left(\frac{\sigma_f^2}{\sigma_w^2} + \frac{1}{\tilde{\xi}_w^2}\right)^2\right\} \tag{3.60}$$

Chapter 4

Normative target pursuit strategies

4.1 General setup

Assume a linear control model $\dot{\boldsymbol{x}} = \boldsymbol{u}$ and a quadratic cost function. In particular, assume the target $\boldsymbol{f}_t$ has a motion model, and the agent cares about minimizing a set of costs (or maximizing a set of rewards) which involves:

- $\boldsymbol{x} - \boldsymbol{f}$, the agent's position relative to the target
- $\boldsymbol{u}$, the agent's velocity
- $\dot{\boldsymbol{u}}$, the agent's acceleration
- $\boldsymbol{x}$

We will include all possible quadratic terms, as long as each term involves a controllable component (e.g., no reason to include $\dot{f}^2$ since it cannot be influenced directly). Assume an overall value function

$$-V := \int_0^T \left\{ \frac{1}{2}\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u} + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{f})^T \boldsymbol{B}^{-1}(\boldsymbol{x}-\boldsymbol{f}) - \rho\boldsymbol{u}\cdot\dot{\boldsymbol{f}} + \frac{1}{2}\dot{\boldsymbol{u}}^T\boldsymbol{K}\dot{\boldsymbol{u}} \right\} e^{-t/\delta}\, dt + \frac{\|\boldsymbol{x}_T - \boldsymbol{f}_T\|_2^2}{2} \,. \tag{4.1}$$

For simplicity, we assume all directions are the same, i.e. that matrices like $\boldsymbol{A}$ are proportional to the identity matrix. Then the value function simplifies to

$$-V := \int_0^T \left\{ \frac{\alpha}{2}\|\boldsymbol{u}\|_2^2 + \frac{1}{2\beta}\|\boldsymbol{x}-\boldsymbol{f}\|_2^2 - \rho\boldsymbol{u}\cdot\dot{\boldsymbol{f}} + \frac{\kappa^3}{2}\|\dot{\boldsymbol{u}}\|_2^2 \right\} e^{-t/\delta}\, dt + \frac{\|\boldsymbol{x}_T - \boldsymbol{f}_T\|_2^2}{2} \,. \tag{4.2}$$

Importantly, the objective can be decomposed as a sum of sub-objectives, each of which only depends on information happening in one dimension. Hence, the D-dimensional control problem can be viewed as D one-dimensional control problems. (This would not be true if there were cross-terms, e.g., terms which depend on $u_x u_y$.)

The analogous one-dimensional objective is

$$-V := \int_0^T \left\{ \frac{\alpha}{2} u^2 + \frac{\kappa^3}{2} \dot{u}^2 - \rho u \dot{f} + \frac{1}{2\beta}(x-f)^2 \right\} e^{-t/\delta} \, dt + \frac{(x_T - f_T)^2}{2} \ . \tag{4.3}$$

With the acceleration term included, one must specify the initial position and velocity of the agent. Without it, one must only specify the agent's initial position.

To solve this problem, one writes down a Lagrangian

$$L = p(\dot{x} - u) + \left\{ \frac{\alpha}{2} u^2 + \frac{\kappa^3}{2} \dot{u}^2 - \rho u \dot{f} + \frac{1}{2\beta}(x-f)^2 \right\} e^{-t/\delta} \tag{4.4}$$

and computes the corresponding Euler-Lagrange equations. The relevant derivatives are

$$\begin{aligned} \frac{\partial L}{\partial u} &= -p + \left\{ \alpha u - \rho \dot{f} \right\} e^{-t/\delta} \\ \frac{\partial L}{\partial x} &= \frac{x - f}{\beta} e^{-t/\delta} \\ \frac{\partial L}{\partial \dot{x}} &= p \\ \frac{\partial L}{\partial \dot{u}} &= \kappa^3 \dot{u} e^{-t/\delta} \ . \end{aligned} \tag{4.5}$$

The Euler-Lagrange equations are

$$\begin{aligned} \kappa^3(\ddot{u} - \dot{u}/\delta) e^{-t/\delta} &= -p + (\alpha u - \rho \dot{f}) e^{-t/\delta} \\ \dot{p} &= \frac{(x-f)}{\beta} e^{-t/\delta} \ . \end{aligned} \tag{4.6}$$

We want to eliminate the Lagrange multiplier p, so that our problem is only in terms of 'physical' navigation variables. We can take a derivative of the above equation and rewrite it as

$$\kappa^3(\dddot{u} - \ddot{u}/\delta) = -\frac{(x-f)}{\beta} - \frac{1}{\delta}\left[\alpha u - \rho \dot{f} - \kappa^3(\ddot{u} - \dot{u}/\delta) \right] + \alpha \dot{u} - \rho \ddot{f} \ . \tag{4.7}$$

We can reorganize this equation in terms of derivatives of x. We find an inhomogeneous linear ODE:

$$\kappa^3 x^{(4)} - 2\frac{\kappa^3}{\delta} x^{(3)} + \left(\frac{\kappa^3}{\delta^2} - \alpha \right) \ddot{x} + \frac{\alpha}{\delta} \dot{x} + \frac{1}{\beta} x = \frac{1}{\beta} f + \frac{\rho}{\delta} \dot{f} - \rho \ddot{f} \ . \tag{4.8}$$

The corresponding characteristic polynomial is

$$\kappa^3 r^4 - 2\frac{\kappa^3}{\delta} r^3 + \left(\frac{\kappa^3}{\delta^2} - \alpha \right) r^2 + \frac{\alpha}{\delta} r + \frac{1}{\beta} = 0 \ . \tag{4.9}$$

Another way to write it is

$$\kappa^3 (r - 1/\delta)^2 r^2 - \alpha(r^2 - \frac{1}{\delta} r - \frac{1}{\alpha\beta}) = 0 \ . \tag{4.10}$$

The second polynomial can be written as

$$\left(r-\frac{1}{2\delta}+q\right)\left(r-\frac{1}{2\delta}-q\right) \tag{4.11}$$

where $q \geq 0$. For convenience, define $\tilde{r} := r - \frac{1}{2\delta}$. We have

$$\kappa^3(\tilde{r}-\frac{1}{2\delta})^2(\tilde{r}+\frac{1}{2\delta})^2-\alpha(\tilde{r}-q)(\tilde{r}+q)=\kappa^3(\tilde{r}^2-\frac{1}{4\delta^2})^2-\alpha(\tilde{r}^2-q^2)=0\ . \tag{4.12}$$

Surprisingly, we can solve for these roots exactly. Define $R := \tilde{r}^2 - 1/(4\delta^2)$. Then

$$\kappa^3 R^2-\alpha R-\alpha\left(\frac{1}{4\delta^2}-q^2\right)=0\ . \tag{4.13}$$

Since $q^2 = \frac{1}{4\delta^2} + \frac{1}{\alpha\beta}$, we have

$$\kappa^3 R^2-\alpha R+\frac{1}{\beta}=0\ . \tag{4.14}$$

Now,

$$R=\frac{\alpha}{2\kappa^3}\pm\frac{1}{2\kappa^3}\sqrt{\alpha^2-4\frac{\kappa^3}{\beta}}\ . \tag{4.15}$$

Then

$$\tilde{r}^2=\frac{1}{4\delta^2}+\frac{\alpha}{2\kappa^3}\pm\frac{1}{2\kappa^3}\sqrt{\alpha^2-4\frac{\kappa^3}{\beta}}\ , \tag{4.16}$$

Use s_1 and s_2 to denote signs. We have

$$r(s_1,s_2)=\frac{1}{2\delta}+s_1\sqrt{\frac{1}{4\delta^2}+\frac{\alpha}{2\kappa^3}+s_2\frac{1}{2\kappa^3}\sqrt{\alpha^2-4\frac{\kappa^3}{\beta}}}\ . \tag{4.17}$$

The fact that the roots are hierarchical is kind of interesting. In general, there are four distinct roots $r_1, ..., r_4$. Depending on the sign of either determinant, the roots may be complex.

Some limiting cases.

- $\kappa \to 0$, in order for the roots to not diverge to infinity we must take s_2 negative. Using a first-order expansion of the inner square root in κ^3, we find

$$r(s_1,-1)=\frac{1}{2\delta}+s_1\sqrt{\frac{1}{4\delta^2}+\frac{1}{\alpha\beta}} \tag{4.18}$$

- $\gamma \to 0$, in order for non-divergence we need $s_1 = -1$.

- x

The general solution for x_t is

$$x_t = x_0 + \sum_{i=1}^{4} c_i e^{r_i t} + x_P(t) \tag{4.19}$$

where x_P is a particular solution dependent on f_t and its derivatives.

Note: what can we assume about f_t? Doesn't need to be constant velocity or acceleration. Really f_t and its derivatives should just be expressible as a system closed under differentiation. Polynomials are one example and linear combinations of exponentials (e.g., Fourier series) are another example. Have a slight problem when f_t's components overlap with x_t components, so will ignore that case.

There are four c_i to determine in general. Two are determined by initial conditions (initial x and u). One is determined by a self-consistency condition (the ODE implies a relationship between initial acceleration and jerk). One is determined by minimizing the objective. All except the objective minimization condition are somewhat easy to handle.

The ODE is inhomogeneous and takes the form

$$\hat{L}_1 x_t = \hat{L}_2 f_t \ . \tag{4.20}$$

There are two canonical approaches to determining the solution to this ODE. One is the method of undetermined coefficients, while the other is the method of Green's functions.

Either way, assume that

$$f_t = \boldsymbol{q}_y^T \boldsymbol{\phi}_t \tag{4.21}$$

where $\boldsymbol{\phi}_t$ is a vector of time-dependent functions and $\boldsymbol{q}_y$ is a vector of coefficients. For example, this can be a neural network that learns a model of target motion. We are interested in computing $\tilde{x}_t$, a particular solution of the ODE that satisfies the inhomogeneous form. We will be agnostic about boundary conditions for now.

4.1.1 Method of undetermined coefficients

Assume that the $\boldsymbol{\phi}_t$ comprise a (generally finite) basis for a function space, with each function independent of the others, and that

$$\dot{\boldsymbol{\phi}}_t = \boldsymbol{Q} \boldsymbol{\phi}_t \ . \tag{4.22}$$

This allows us to write derivatives of f_t in terms of $\boldsymbol{\phi}_t$ and $\boldsymbol{Q}$. For example, $\dot{f}_t = \boldsymbol{q}_y^T \boldsymbol{Q} \boldsymbol{\phi}_t$. Consider the ansatz

$$\tilde{x}_t := \boldsymbol{q}^T \boldsymbol{\phi}_t \ . \tag{4.23}$$

Our task is to determine the coefficients $\boldsymbol{q}$. By substitution, the ODE

$$\sum_n a_n x^{(n)} = \sum_n b_n f^{(n)} \tag{4.24}$$

becomes

$$\left[\sum_n a_n \boldsymbol{q}^T \boldsymbol{Q}^n\right] \boldsymbol{\phi}_t = \left[\sum_n b_n \boldsymbol{q}_y^T \boldsymbol{Q}^n\right] \boldsymbol{\phi}_t \ . \tag{4.25}$$

By independence, the expression is zero only when all of its coefficients are zero, so

$$\left[\sum_n a_n (\boldsymbol{Q}^T)^n\right] \boldsymbol{q} = \left[\sum_n b_n (\boldsymbol{Q}^T)^n\right] \boldsymbol{q}_y \ . \tag{4.26}$$

Hence,

$$\boldsymbol{q} = \left[\sum_n a_n (\boldsymbol{Q}^T)^n\right]^{-1} \left[\sum_n b_n (\boldsymbol{Q}^T)^n\right] \boldsymbol{q}_y \ . \tag{4.27}$$

The boundary conditions satisfied by $\tilde{x}_t$ depend on the properties of the basis functions used.

4.1.2 Method of Green's functions

Consider a function $G(t,t')$ that satisfies $\hat{L}G(t,t') = \delta(t-t')$. Assuming the characteristic polynomial has distinct roots, and enforcing the boundary conditions $G(0,t') = \dot{G}(0,t') = \cdots = G^{(n)}(0,t') = 0$ for our later convenience, we can Laplace transform to find

$$a_N (s-r_1)\cdots(s-r_N)\tilde{G}(s,t') = e^{-st'} \implies \tilde{G}(s,t') = \frac{1}{a_N}\frac{e^{-st'}}{(s-r_1)\cdots(s-r_N)} \ . \tag{4.28}$$

Using a partial fraction decomposition,

$$\frac{e^{-st'}}{(s-r_1)\cdots(s-r_N)} = \sum_i \left[\prod_{j\neq i} \frac{1}{(r_i - r_j)}\right] \frac{e^{-st'}}{(s-r_i)} \ . \tag{4.29}$$

Taking an inverse Laplace transform,

$$G(t,t') = \frac{1}{a_N} \sum_i \left[\prod_{j\neq i} \frac{1}{(r_i - r_j)}\right] e^{r_i(t-t')}\Theta(t-t') \tag{4.30}$$

where $\Theta(t-t')$ is the Heaviside step function. Note that, because the sum of the inverse root-differences is zero when the exponential factor is equal to one, $G(t,t) = 0$ as well.

Our particular solution is

$$\tilde{x}_t := \int_0^t G(t,t') f_{t'} \ dt' \tag{4.31}$$

and satisfies $\tilde{x}_0 = 0$ and $\dot{\tilde{x}}_0 = 0$. One nice feature of this approach is that it does not matter what properties $f_{t'}$ has; one can even use a learned approximator like a neural network.

Either way, the overall solution is

$$x_t = \boldsymbol{c}^T \boldsymbol{k}_t + \tilde{x}_t \tag{4.32}$$

where $k_{ti} := e^{r_i t}$ and $\boldsymbol{c}$ are the to-be-determined coefficients.

4.1.3 Determining coefficients

One constraint is that V is minimized—subject to all the other constraints. This problem can conveniently be formulated as an optimization involving Lagrange multipliers. The initial condition constraints are:

$$x_0 = \boldsymbol{c}^T \boldsymbol{k}_0 = \boldsymbol{c}^T \mathbf{1} \qquad \dot{x}_0 = \boldsymbol{c}^T \boldsymbol{R} \mathbf{1} \qquad \ddot{x}_0 = \boldsymbol{c}^T \boldsymbol{R}^2 \mathbf{1} \; . \tag{4.33}$$

Unfortunately, with the acceleration penalty added, one must specify the initial velocity and acceleration in addition to the initial position. (Continue and come back to ensure this is true.)

4.2 Steering cost only

The one-dimensional steering problem is

$$V = -\int_0^T \frac{\alpha}{2} u^2 \; dt - \frac{(x_T - f_T)^2}{2} \tag{4.34}$$

where α is the time scale of accruing movement costs. The Lagrangian is

$$L = -\frac{\alpha}{2} u^2 - p(\dot{x} - u) \tag{4.35}$$

where $p(t)$ is a Lagrange multiplier that implements our control constraint. The corresponding Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial L}{\partial u} = 0 \quad &\Longrightarrow \quad p = \alpha u \\ \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad &\Longrightarrow \quad p = \text{const.} \end{aligned} \tag{4.36}$$

We can combine these equations to find that $u(t)$ satisfies

$$u = \text{const.} \tag{4.37}$$

The corresponding trajectory is

$$x(t) = x_0 + ut \; . \tag{4.38}$$

The objective function we're interested in is the (negative) value function V:

$$J = -V = \int_0^T \frac{\alpha}{2} u^2 \; dt + \frac{(x_T - f_T)^2}{2} = \frac{\alpha}{2} u^2 T + \frac{(x_T - f_T)^2}{2} \; . \tag{4.39}$$

The derivative of this with respect to u is

$$\frac{\partial J}{\partial u} = \alpha u T + (x_T - f_T) \frac{\partial}{\partial u}(x_T - f_T) = \alpha u T + (x_0 + uT - f_T) T = 0 \; . \tag{4.40}$$

Rearranging, we find

$$u = -\frac{(x_0 - f_T)}{\alpha + T} \ . \tag{4.41}$$

Our final answer for x and u are

$$\begin{aligned} u(t) &= -\frac{(x_0 - f_T)}{\alpha + T} \\ x(t) &= x_0 - \frac{(x_0 - f_T)}{\alpha + T} t \ . \end{aligned} \tag{4.42}$$

The endpoint is

$$x_T = \frac{\alpha}{\alpha + T} x_0 + \frac{T}{\alpha + T} f_T \ . \tag{4.43}$$

The value function is

$$\begin{aligned} V(x_0) &= -\frac{\alpha}{2} u^2 T - \frac{(x_T - f_T)^2}{2} \\ &= -\frac{\alpha}{2} \frac{(x_0 - f_T)^2}{(\alpha + T)^2} T - \frac{\alpha^2 (x_0 - f_T)^2}{2(\alpha + T)^2} \\ &= -\frac{\alpha}{2} \frac{(x_0 - f_T)^2}{(\alpha + T)^2} [T + \alpha] \\ &= -\frac{\alpha}{\alpha + T} \frac{(x_0 - f_T)^2}{2} \ . \end{aligned} \tag{4.44}$$

Maximized when $T \to \infty$: $V = 0$ if the target is stationary. Take as long as possible to reach target to avoid accruing movement costs!

If the target is *not* stationary, there is a nonzero and non-infinite solution for T. If the target is moving in the subject's direction, the optimal strategy is to wait for it to reach you. If the target is moving away from you...

$$\begin{aligned} V(x_0) &= -\frac{\alpha}{\alpha + T} \frac{(x_0 - f_0 - vT)^2}{2} \\ \frac{\partial V}{\partial T} &= -\frac{\alpha}{2} \left\{ -\frac{(x_0 - f_T)^2}{(\alpha + T)^2} - 2v \frac{(x_0 - f_T)}{(\alpha + T)} \right\} \ . \end{aligned} \tag{4.45}$$

Zero when

$$\begin{aligned} (x_0 - f_0 - vT) + 2v(\alpha + T) = 0 \\ \implies \quad T = \frac{f_0 - x_0}{v} - 2\alpha \ . \end{aligned} \tag{4.46}$$

This means that

$$\begin{aligned} u &= 2v \\ V(x_0) &= -2\alpha \left| \frac{f_0 - x_0}{v} - \alpha \right| v^2 \ . \end{aligned} \tag{4.47}$$

Final distance to target:

$$x_T - f_T = x_0 - f_0 + vT = -2\alpha v \ . \tag{4.48}$$

Note that

$$u(x_t - f_t) = u(x_0 + ut - f_0 - vt) = u[x_0 + ut - f_T + v(T - t)] = u[x_0 - f_T + (u - v)t + vT] = -\frac{(x_0 - f_T)^2}{\alpha + T} + u\,[(u - v) \tag{4.49}$$

(cosine of) Angle between line of sight and animal velocity:

$$\frac{u_x(x_t - f_t) + u_y(y_t - g_t)}{\sqrt{u_x^2 + u_y^2}\sqrt{f_t^2 + g_t^2}} = \frac{-(x_0 - f_T)(x_t - f_t) - (y_0 - g_T)(y_t - g_t)}{d(x_0, f_T)\sqrt{(f_0 + v_x t)^2 + (g_0 + v_y t)^2}} \tag{4.50}$$

4.2.1 Commentary on limiting behavior

Note that the (fixed T) control solution is

$$u(x_0) = \frac{f_T - x_0}{\alpha + T} \ . \tag{4.51}$$

In the $\alpha \to 0$ limit (technically $\alpha/T \ll 1$), we recover the strategy that moves at a constant speed to the predicted final location of the target:

$$u(x_0) = \frac{f_T - x_0}{T} \ . \tag{4.52}$$

4.3 Steering cost and discounting

The one-dimensional steering problem is

$$V = -\int_0^T \frac{\alpha}{2} u^2 e^{-t/\delta}\ dt - \frac{(x_T - f_T)^2}{2} e^{-T/\delta} \tag{4.53}$$

where α is the time scale of accruing movement costs, and δ is the time scale of temporal discounting. The Lagrangian is

$$L = -\frac{\alpha}{2} u^2 e^{-t/\delta} - p(\dot{x} - u) \tag{4.54}$$

where $p(t)$ is a Lagrange multiplier that implements our control constraint. The corresponding Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial L}{\partial u} = 0 \quad &\implies \qquad p = \alpha u e^{-t/\delta} \\ \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad &\implies \qquad p = \text{const.} \end{aligned} \tag{4.55}$$

We can combine these equations to find that $u(t)$ satisfies

$$u = c e^{t/\delta}\ . \tag{4.56}$$

The corresponding trajectory is

$$x(t) = x_0 + c\delta \left(e^{t/\delta} - 1\right)\ . \tag{4.57}$$

The objective function we're interested in is the (negative) value function V:

$$\begin{aligned} J = -V &= \int_0^T \frac{\alpha}{2} u^2 e^{-t/\delta}\ dt + \frac{(x_T - f_T)^2}{2} e^{-T/\delta} \\ &= \frac{\alpha}{2} c^2 \delta \left(e^{T/\delta} - 1\right) + \frac{\left[x_0 + c\delta\left(e^{T/\delta} - 1\right) - f_T\right]^2}{2} e^{-T/\delta}\ . \end{aligned} \tag{4.58}$$

The derivative of this with respect to c is

$$\begin{aligned} \frac{\partial J}{\partial c} &= \alpha c \delta \left(e^{T/\delta} - 1\right) + (x_T - f_T) \frac{\partial}{\partial c}(x_T - f_T) \\ &= \alpha c \delta \left(e^{T/\delta} - 1\right) + \left[x_0 + c\delta\left(e^{T/\delta} - 1\right) - f_T\right] \delta \left(e^{T/\delta} - 1\right) e^{-T/\delta} = 0\ . \end{aligned} \tag{4.59}$$

Rearranging, we find

$$c = -\frac{(x_0 - f_T) e^{-T/\delta}}{\alpha + \delta\left(1 - e^{-T/\delta}\right)}\ . \tag{4.60}$$

Our final answer for x and u are

$$
\begin{aligned}
u(t) &= -\frac{(x_0 - f_T)e^{-T/\delta}}{\alpha + \delta\left(1 - e^{-T/\delta}\right)} e^{t/\delta} \\
x(t) &= x_0 - \frac{(x_0 - f_T)e^{-T/\delta}}{\alpha + \delta\left(1 - e^{-T/\delta}\right)} \delta\left(e^{t/\delta} - 1\right) \ .
\end{aligned}
\tag{4.61}
$$

The endpoint is

$$
x_T = \frac{\alpha}{\alpha + \delta\left(1 - e^{-T/\delta}\right)} x_0 + \frac{\delta\left(1 - e^{-T/\delta}\right)}{\alpha + \delta\left(1 - e^{-T/\delta}\right)} f_T \ . \tag{4.62}
$$

The value function is

$$
\begin{aligned}
V &= -\frac{\alpha}{2} c^2 \delta\left(e^{T/\delta} - 1\right) - \frac{\left(x_T - f_T\right)^2}{2} e^{-T/\delta} \\
&= -\frac{\alpha}{2} \frac{(x_0 - f_T)^2 e^{-2T/\delta}}{\left[\alpha + \delta\left(1 - e^{-T/\delta}\right)\right]^2} \delta\left(e^{T/\delta} - 1\right) - \frac{\alpha^2}{\left[\alpha + \delta\left(1 - e^{-T/\delta}\right)\right]^2} \frac{\left(x_0 - f_T\right)^2}{2} e^{-T/\delta} \\
&= -\frac{\alpha e^{-T/\delta}}{\left[\alpha + \delta\left(1 - e^{-T/\delta}\right)\right]^2} \frac{\left(x_0 - f_T\right)^2}{2} \left\{\delta(1 - e^{-T/\delta}) + \alpha\right\} \\
&= -\frac{\alpha e^{-T/\delta}}{\alpha + \delta\left(1 - e^{-T/\delta}\right)} \frac{\left(x_0 - f_T\right)^2}{2} \ .
\end{aligned}
\tag{4.63}
$$

Maximized when $T \to \infty$: $V = 0$. Take as long as possible to reach target to avoid accruing movement costs!

4.3.1 Commentary on limiting behavior

The control solution is

$$
u_t = \frac{(f_T - x_0)e^{-T/\delta}}{\alpha + \delta(1 - e^{-T/\delta})} \ e^{t/\delta} \ . \tag{4.64}
$$

In the $\alpha \to 0$ limit, the agent moves directly to the predicted target location, but with exponentially increasing control (most control is done at end):

$$
\begin{aligned}
u_t^* &= \frac{(f_T - x_0)e^{-T/\delta}}{\delta(1 - e^{-T/\delta})} \ e^{t/\delta} \\
x_T &= f_T \ .
\end{aligned}
\tag{4.65}
$$

4.4 Steering cost and proximity reward

Assume the specific firefly motion model

$$\begin{aligned} f(t) &= vt + \frac{\ddot{f}}{2}t^2 \\ \dot{f}(t) &= v + \ddot{f}t \ . \end{aligned} \tag{4.66}$$

The one-dimensional steering problem is

$$V = \int_0^T -\left[\frac{\alpha}{2}u^2 + \frac{(x-f)^2}{2\beta}\right] \, dt - \frac{(x_T - f_T)^2}{2} \tag{4.67}$$

where α is the time scale of accruing movement costs, and β is the time scale of accruing penalties for being too far away from the target.

4.4.1 Solving for the shape of control trajectories

The Lagrangian is

$$L = -\frac{\alpha}{2}u^2 - \frac{(x-f)^2}{2\beta} - p(\dot{x} - u) \tag{4.68}$$

where $p(t)$ is a Lagrange multiplier that implements our control constraint. The corresponding Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial L}{\partial u} = 0 \quad &\Longrightarrow \quad p = \alpha u \\ \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad &\Longrightarrow \quad \dot{p} = \frac{x-f}{\beta} \ . \end{aligned} \tag{4.69}$$

We can combine these equations to find that $u(t)$ satisfies

$$\dot{u} = \frac{x-f}{\tau^2} \tag{4.70}$$

where $\tau := \sqrt{\alpha\beta}$. Differentiate to obtain

$$\ddot{u} - \frac{u}{\tau^2} = -\frac{\dot{f}}{\tau^2} \tag{4.71}$$

where $\dot{f} = v + \ddot{f}t$. The roots of the characteristic equation are $\pm\tau$, and a satisfactory particular solution is just $\dot{f}$. We have

$$u_t = c_+ e^{t/\tau} + c_- e^{-t/\tau} + \dot{f} \ . \tag{4.72}$$

The constants c_+ and c_- are determined by the boundary conditions, and the fact that we want to maximize the value function. One constraint follows from the ODE satisfied by u:

$$\dot{u}_0 = \frac{x_a - f_a}{\tau^2} = \frac{(c_+ - c_-)}{\tau} + \ddot{f} \ . \tag{4.73}$$

Equivalently,

$$c_+ - c_- = \frac{\Delta x_a}{\tau} - \ddot{f}\tau \implies c_- = c_+ + \ddot{f}\tau - \frac{\Delta x_a}{\tau} \ . \tag{4.74}$$

Our solutions for $u(t)$ and $x(t)$ currently look like

$$\begin{aligned} u(t) &= c\left(e^{t/\tau} + e^{-t/\tau}\right) + \left(\ddot{f}\tau - \frac{\Delta x_a}{\tau}\right) e^{-t/\tau} + \dot{f} \\ x(t) - f(t) &= c\tau\left(e^{t/\tau} - e^{-t/\tau}\right) + \left(\Delta x_a - \ddot{f}\tau^2\right) e^{-t/\tau} + \ddot{f}\tau^2 \ . \end{aligned} \tag{4.75}$$

4.4.2 Minimizing objective function with respect to c

The objective function we're interested in is the (negative) value function $J := -V$. The derivative of J with respect to c is

$$\frac{\partial J}{\partial c} = \int_0^T \alpha u \frac{\partial u}{\partial c} + \frac{(x-f)}{\beta}\frac{\partial (x-f)}{\partial c} \ dt + (x_T - f_T)\frac{\partial (x_T - f_T)}{\partial c} \ . \tag{4.76}$$

For convenience, write u and x as

$$\begin{aligned} u(t) &= c u_c + u_r \\ x(t) - f(t) &= c x_c + x_r \end{aligned} \tag{4.77}$$

where

$$\begin{aligned} u_c &= \left(e^{t/\tau} + e^{-t/\tau}\right) \\ x_c &= \tau\left(e^{t/\tau} - e^{-t/\tau}\right) \\ u_r &= \left(\ddot{f}\tau - \frac{\Delta x_a}{\tau}\right) e^{-t/\tau} + \dot{f} \\ x_r &= \left(\Delta x_a - \ddot{f}\tau^2\right) e^{-t/\tau} + \ddot{f}\tau^2 \ . \end{aligned} \tag{4.78}$$

Then we have

$$\frac{\partial J}{\partial c} = \int_0^T \alpha(c u_c + u_r) u_c + \frac{(c x_c + x_r) x_c}{\beta} \ dt + [c x_c(T) + x_r(T)] x_c(T) \ . \tag{4.79}$$

Setting this equal to zero, we find

$$\left[\int_0^T \left(\alpha u_c^2 + \frac{x_c^2}{\beta}\right) \ dt + x_c(T)^2\right] c + \left[\int_0^T \left(\alpha u_r u_c + \frac{x_r x_c}{\beta}\right) \ dt + x_r(T) x_c(T)\right] = 0 \tag{4.80}$$

or equivalently

$$c = \frac{-\int_0^T \left(\alpha u_r u_c + \frac{x_r x_c}{\beta}\right) \ dt - x_r(T) x_c(T)}{\int_0^T \left(\alpha u_c^2 + \frac{x_c^2}{\beta}\right) \ dt + x_c(T)^2} \ . \tag{4.81}$$

In principle, we now know the full solution of our control problem—we only need to evaluate the integrals that appear in the expression above, either analytically or numerically.

4.4.3 Obtaining explicit form for constant c denominator

Denote the denominator of c by D. We have

$$
\begin{aligned}
D &:= \int_0^T \left(\alpha u_c^2 + \frac{x_c^2}{\beta}\right) e^{-t/\delta}\ dt + x_c(T)^2 e^{-T/\delta} \\
&= \int_0^T \alpha \left(e^{t/\tau} + e^{-t/\tau}\right)^2 + \frac{\tau^2}{\beta}\left(e^{t/\tau} - e^{-t/\tau}\right)^2\ dt + \alpha\beta\left(e^{T/\tau} - e^{-T/\tau}\right)^2 \\
&= \int_0^T 2\alpha\left[e^{2t/\tau} + e^{-2t/\tau}\right]\ dt + \alpha\beta\left(e^{T/\tau} - e^{-T/\tau}\right)^2 \\
&= \alpha\tau\left[e^{2T/\tau} - e^{-2T/\tau}\right] + \alpha\beta\left(e^{T/\tau} - e^{-T/\tau}\right)^2 \\
&= \tau^2\left(e^{T/\tau} - e^{-T/\tau}\right)\left[\left(1 + \frac{\alpha}{\tau}\right)e^{T/\tau} - \left(1 - \frac{\alpha}{\tau}\right)e^{-T/\tau}\right] \ .
\end{aligned}
$$

4.4.4 Obtaining explicit form for constant c numerator

Denote the numerator of c by $-N$. We have

$$
-N := \int_0^T \left(\alpha u_r u_c + \frac{x_r x_c}{\beta}\right)\ dt + x_r(T)x_c(T)\ . \tag{4.82}
$$

The following expressions will be somewhat more complicated, and it is helpful to group them in a particular way. Write

$$
N = N_x \Delta x_a + N_v v + N_a \ddot{f} \tag{4.83}
$$

since each term in the above integral is proportional to either Δx_a, v, or $\ddot{f}$. Let's take these one at a time. First, $-N_x$ is (using that $\alpha/\tau = \tau/\beta$)

$$
\begin{aligned}
-N_x &:= \int_0^T \left[-\frac{\alpha}{\tau}e^{-t/\tau}\left(e^{t/\tau} + e^{-t/\tau}\right) + \frac{\tau}{\beta}e^{-t/\tau}\left(e^{t/\tau} - e^{-t/\tau}\right)\right]\ dt + \tau e^{-T/\tau}\left(e^{T/\tau} - e^{-T/\tau}\right) \\
&= -2\frac{\alpha}{\tau}\int_0^T e^{-2t/\tau}\ dt + \tau e^{-T/\tau}\left(e^{T/\tau} - e^{-T/\tau}\right) \\
&= \alpha\left(e^{-2T/\tau} - 1\right) + \tau e^{-T/\tau}\left(e^{T/\tau} - e^{-T/\tau}\right) \\
&= -\alpha e^{-T/\tau}\left(e^{T/\tau} - e^{-T/\tau}\right) + \tau e^{-T/\tau}\left(e^{T/\tau} - e^{-T/\tau}\right) \\
&= \tau\left(e^{T/\tau} - e^{-T/\tau}\right)\left(1 - \frac{\alpha}{\tau}\right)e^{-T/\tau}\ .
\end{aligned} \tag{4.84}
$$

Next, $-N_v$ is

$$
\begin{aligned}
-N_v &:= \int_0^T \alpha\left(e^{t/\tau} + e^{-t/\tau}\right)\ dt \\
&= \alpha\tau\left(e^{T/\tau} - e^{-T/\tau}\right)\ .
\end{aligned} \tag{4.85}
$$

Finally, $-N_a$ is

$$
\begin{aligned}
&:= \int_0^T \left[\alpha(\tau e^{-t/\tau}+t)\left(e^{t/\tau}+e^{-t/\tau}\right)+\frac{\tau^3}{\beta}\left(1-e^{-t/\tau}\right)\left(e^{t/\tau}-e^{-t/\tau}\right)\right]\, dt + \alpha\beta\tau\left(1-e^{-T/\tau}\right)\left(e^{T/\tau}-e^{-T/\tau}\right) \\
&= \int_0^T \left[2\alpha\tau e^{-2t/\tau}+\alpha t\left(e^{t/\tau}+e^{-t/\tau}\right)+\alpha\tau\left(e^{t/\tau}-e^{-t/\tau}\right)\right]\, dt + \alpha\beta\tau\left(1-e^{-T/\tau}\right)\left(e^{T/\tau}-e^{-T/\tau}\right) \\
&= -\alpha\tau^2\left(e^{-2T/\tau}-1\right)+\alpha\tau^2\left(e^{T/\tau}+e^{-T/\tau}-2\right)+2\alpha\int_0^T t\cosh(t/\tau)dt+\alpha\beta\tau\left(1-e^{-T/\tau}\right)\left(e^{T/\tau}-e^{-T/\tau}\right) \ .
\end{aligned}
\tag{4.86}
$$

We need the fact that

$$
2\int_0^T t\cosh(t/\tau)dt = -\tau^2\left(e^{T/\tau}+e^{-T/\tau}-2\right)+\tau T\left(e^{T/\tau}-e^{-T/\tau}\right) \tag{4.87}
$$

Now, (the integral part of) $-N_a$ is

$$
\begin{aligned}
&= -\alpha\tau^2\left(e^{-2T/\tau}-1\right)+\alpha\tau^2\left(e^{T/\tau}+e^{-T/\tau}-2\right)-\alpha\tau^2\left(e^{T/\tau}+e^{-T/\tau}-2\right)+\alpha\tau T\left(e^{T/\tau}-e^{-T/\tau}\right) \\
&= \alpha\tau^2 e^{-T/\tau}\left(e^{T/\tau}-e^{-T/\tau}\right)+\alpha\tau^2\frac{T}{\tau}\left(e^{T/\tau}-e^{-T/\tau}\right) \\
&= \alpha\tau^2\left(e^{T/\tau}-e^{-T/\tau}\right)\left(e^{-T/\tau}+\frac{T}{\tau}\right) \ .
\end{aligned}
\tag{4.88}
$$

All together, $-N_a$ is

$$
\begin{aligned}
&= \alpha\tau^2\left(e^{T/\tau}-e^{-T/\tau}\right)\left(e^{-T/\tau}+\frac{T}{\tau}\right)+\alpha\beta\tau\left(1-e^{-T/\tau}\right)\left(e^{T/\tau}-e^{-T/\tau}\right) \\
&= \alpha\tau^2\left(e^{T/\tau}-e^{-T/\tau}\right)\left[e^{-T/\tau}+\frac{T}{\tau}+\frac{\beta}{\tau}\left(1-e^{-T/\tau}\right)\right] \\
&= \alpha\tau^2\left(e^{T/\tau}-e^{-T/\tau}\right)\left[\left(1-\frac{\beta}{\tau}\right)e^{-T/\tau}+\frac{\beta+T}{\tau}\right] \ .
\end{aligned}
\tag{4.89}
$$

Collecting our results, we have found that

$$
\begin{aligned}
N_x &= -\tau\left(e^{T/\tau}-e^{-T/\tau}\right)\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau} \\
N_v &= -\alpha\tau\left(e^{T/\tau}-e^{-T/\tau}\right) \\
N_a &= -\alpha\tau^2\left(e^{T/\tau}-e^{-T/\tau}\right)\left[\left(1-\frac{\beta}{\tau}\right)e^{-T/\tau}+\frac{\beta+T}{\tau}\right] \ .
\end{aligned}
\tag{4.90}
$$

4.4.5 Explicit form for constant

The quantity that actually appears in $x(t)$ is $N\tau/D$. Note,

$$\begin{aligned}
\frac{N_x\tau}{D} &= -\frac{\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau}-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}} \\
\frac{N_v\tau}{D} &= -\frac{\alpha}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau}-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}} \\
\frac{N_a\tau}{D} &= \frac{\tau^2\left[\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}-\left(1+\frac{\alpha}{\tau^2}T\right)\right]}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau}-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}} \ .
\end{aligned} \tag{4.91}$$

Then

$$\frac{N\tau}{D} = \frac{-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}\Delta x_a - \alpha v + \tau^2\left[\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}-\left(1+\frac{\alpha}{\tau^2}T\right)\right]\ddot{f}}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau}-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}} \tag{4.92}$$

4.4.6 Final expression for control and trajectories

$$\begin{aligned}
u(t) &= \frac{N}{D}\left(e^{t/\tau}+e^{-t/\tau}\right) + \left(\ddot{f}\tau - \frac{\Delta x_a}{\tau}\right)e^{-t/\tau} + \dot{f} \\
x(t) - f(t) &= \frac{N\tau}{D}\left(e^{t/\tau}-e^{-t/\tau}\right) + \left(\Delta x_a - \ddot{f}\tau^2\right)e^{-t/\tau} + \ddot{f}\tau^2 \ .
\end{aligned} \tag{4.93}$$

4.4.7 Endpoint

$$\begin{aligned}
x_T - f_T &= \frac{N\tau}{D}\left(e^{T/\tau}-e^{-T/\tau}\right) + \left(\Delta x_a - \ddot{f}\tau^2\right)e^{-T/\tau} + \ddot{f}\tau^2 \\
&= R_x\Delta x_a + R_v v + R_a\ddot{f} \ .
\end{aligned} \tag{4.94}$$

The x coefficient is

$$\begin{aligned}
R_x &= \frac{N_x\tau}{D}\left(e^{T/\tau}-e^{-T/\tau}\right) + e^{-T/\tau} \\
&= \frac{-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau}-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}}\left(e^{T/\tau}-e^{-T/\tau}\right) + e^{-T/\tau} \\
&= \frac{e^{-T/\tau}}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau}-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}}\left\{-\left(1-\frac{\alpha}{\tau}\right)\left(e^{T/\tau}-e^{-T/\tau}\right) + \left(1+\frac{\alpha}{\tau}\right)e^{T/\tau} - \left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}\right\} \\
&= \frac{\frac{2\alpha}{\tau}}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau}-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}} \ .
\end{aligned} \tag{4.95}$$

The v coefficient is

$$\begin{aligned}
R_v &= \frac{N_v\tau}{D}\left(e^{T/\tau}-e^{-T/\tau}\right) \\
&= -\frac{\alpha\left(e^{T/\tau}-e^{-T/\tau}\right)}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau}-\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}} \ .
\end{aligned} \tag{4.96}$$

The $\ddot{f}$ coefficient is

$$
\begin{aligned}
R_a &= \frac{N_a \tau}{D}\left(e^{T/\tau} - e^{-T/\tau}\right) + \tau^2\left(1 - e^{-T/\tau}\right) \\
&= \frac{\tau^2\left[\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau} - \left(1+\frac{\alpha}{\tau^2}T\right)\right]}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau} - \left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}}\left(e^{T/\tau} - e^{-T/\tau}\right) + \tau^2\left(1 - e^{-T/\tau}\right) \qquad (4.97) \\
&= \frac{\tau^2}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau} - \left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}} Q
\end{aligned}
$$

where

$$
\begin{aligned}
Q = &\left[\left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau} - \left(1+\frac{\alpha}{\tau^2}T\right)\right]\left(e^{T/\tau} - e^{-T/\tau}\right) + \left(1 - e^{-T/\tau}\right)\left[\left(e^{T/\tau} - e^{-T/\tau}\right) + \frac{\alpha}{\tau}\left(e^{T/\tau} + e^{-T/\tau}\right)\right] \\
= &\frac{\alpha}{\tau}\left\{-\left(e^{-T/\tau} + \frac{T}{\tau}\right)\left(e^{T/\tau} - e^{-T/\tau}\right) + \left(1 - e^{-T/\tau}\right)\left(e^{T/\tau} + e^{-T/\tau}\right)\right\} \\
= &\frac{\alpha}{\tau}\left\{-2 - \frac{T}{\tau}\left(e^{T/\tau} - e^{-T/\tau}\right) + e^{T/\tau} + e^{-T/\tau}\right\} \\
= &\frac{\alpha}{\tau}\left\{-2 + \left(1 - \frac{T}{\tau}\right)e^{T/\tau} + \left(1 + \frac{T}{\tau}\right)e^{-T/\tau}\right\} \ . \qquad (4.98)
\end{aligned}
$$

Finally, the $\ddot{f}$ coefficient is

$$
\begin{aligned}
R_a &= \frac{\alpha\tau}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau} - \left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}}\left[-2 + e^{T/\tau} + e^{-T/\tau} - \frac{T}{\tau}\left(e^{T/\tau} - e^{-T/\tau}\right)\right] \\
&= \frac{\alpha\tau}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau} - \left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}}\left(e^{T/(2\tau)} - e^{-T/(2\tau)}\right)\left[e^{T/(2\tau)} - e^{-T/(2\tau)} - \frac{T}{\tau}\left(e^{T/(2\tau)} + e^{-T/(2\tau)}\right)\right] \\
&= \frac{\alpha\tau}{\left(1+\frac{\alpha}{\tau}\right)e^{T/\tau} - \left(1-\frac{\alpha}{\tau}\right)e^{-T/\tau}}\left(e^{T/\tau} - e^{-T/\tau}\right)\left[\tanh\left(\frac{T}{2\tau}\right) - \frac{T}{\tau}\right] \ . \qquad (4.99)
\end{aligned}
$$

Since $\tanh x < x$, R_a is always negative. In fact, assuming Δx_0 is negative, v is positive, and $\ddot{f}$ is positive, *all* of the coefficients give negative contributions to the endpoint.

4.5 Continuous reward + discounting

The one-dimensional steering problem is

$$V = \int_0^T - \left[\frac{\alpha}{2} u^2 + \frac{(x-f)^2}{2\beta} \right] e^{-t/\delta} \, dt - \frac{(x_T - f_T)^2}{2} e^{-T/\delta} \tag{4.100}$$

where α is the time scale of accruing movement costs, β is the time scale of proximity costs, and δ is the time scale of temporal discounting.

4.5.1 Deriving the ODE that describes control trajectories

To solve for the shape of trajectories, we must analyze the Lagrangian that corresponds to the integral part of this value function:

$$L = - \left[\frac{\alpha}{2} u^2 + \frac{(x-f)^2}{2\beta} \right] e^{-t/\delta} - p(\dot{x} - u) \tag{4.101}$$

where $p(t)$ is a Lagrange multiplier that implements our control constraint. The corresponding Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial L}{\partial u} = 0 \quad &\Longrightarrow \quad p = \alpha u e^{-t/\delta} \\ \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad &\Longrightarrow \quad \dot{p} = \frac{(x-f)}{\beta} e^{-t/\delta} \ . \end{aligned} \tag{4.102}$$

We can combine these equations to find that $u(t)$ satisfies

$$\dot{u} - u/\delta = \frac{(x-f)}{\alpha\beta} \ . \tag{4.103}$$

To get something completely in terms of u, we can differentiate:

$$\ddot{u} - \frac{\dot{u}}{\delta} - \frac{u}{\alpha\beta} = -\frac{\dot{f}}{\alpha\beta} \ . \tag{4.104}$$

4.5.2 Solving for the shape of control trajectories

First, let's compute the solution to the homogeneous ODE. The characteristic equation for the roots is

$$r^2 - \frac{r}{\delta} - \frac{1}{\alpha\beta} = 0 \ , \tag{4.105}$$

and has solutions

$$r_\pm = \frac{1}{2\delta} \pm \frac{1}{2}\sqrt{\frac{1}{\delta^2} + \frac{4}{\alpha\beta}} \ . \tag{4.106}$$

Hence, the solution to the control ODE is

$$u(t) = c_+ e^{r_+ t} + c_- e^{r_- t} + u_p(t) \tag{4.107}$$

where $u_p(t)$ is a particular solution. A general and sophisticated approach is to use the method of Green's functions to write down an expression for $u_p(t)$; here, however, a linear ansatz for $u_p(t)$ suffices. By substitution, we find that

$$u_p(t) = -\frac{\alpha\beta}{\delta}\ddot{f} + \dot{f}(t) = -\frac{\alpha\beta}{\delta}\ddot{f} + v + \ddot{f}t \ . \tag{4.108}$$

Then we have

$$\begin{aligned} u(t) &= c_+ e^{r_+ t} + c_- e^{r_- t} + \dot{f}(t) - \frac{\alpha\beta}{\delta}\ddot{f} \\ x(t) - f(t) &= \Delta x_a + c_+ \frac{(e^{r_+ t} - 1)}{r_+} + c_- \frac{(e^{r_- t} - 1)}{r_-} - \frac{\alpha\beta}{\delta}\ddot{f}t \ . \end{aligned} \tag{4.109}$$

4.5.3 Enforcing self-consistency initial condition

The ODE for u implies

$$\dot{u}_0 - \frac{u_0}{\delta} = \frac{\Delta x_0}{\alpha\beta} \ . \tag{4.110}$$

where $\Delta x_0 := x_0 - f_0$. Using our expression for $u(t)$ (recall that both G and its derivative are zero at $t = 0$), this becomes

$$\left(r_+ c_+ + r_- c_- + \ddot{f}\right) - \frac{1}{\delta}\left(c_+ + c_- + v - \frac{\alpha\beta}{\delta}\ddot{f}\right) = \frac{\Delta x_0}{\alpha\beta} \ . \tag{4.111}$$

Using the facts that $r_+ - 1/\delta = -r_-$ and $r_- - 1/\delta = -r_+$, this condition becomes

$$r_- c_+ + r_+ c_- = -\frac{\Delta x_0}{\alpha\beta} - \frac{v}{\delta} + \left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f} \ . \tag{4.112}$$

Equivalently,

$$\begin{aligned} c_- &= -\frac{1}{r_+}\frac{\Delta x_0}{\alpha\beta} - \frac{1}{r_+}\frac{v}{\delta} + \frac{1}{r_+}\left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f} - \frac{r_-}{r_+}c_+ \\ &= r_-\left[\Delta x_0 + \frac{\alpha\beta}{\delta}v - \alpha\beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right] - \frac{r_-}{r_+}c_+ \ . \end{aligned} \tag{4.113}$$

Using simplified notation $c := c_+$, our solutions for u and x become

$$\begin{aligned} u(t) &= c\left(e^{r_+ t} - \frac{r_-}{r_+}e^{r_- t}\right) + r_-\left[\Delta x_0 + \frac{\alpha\beta}{\delta}v - \alpha\beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right]e^{r_- t} + \dot{f}(t) - \frac{\alpha\beta}{\delta}\ddot{f} \\ \Delta x_t &= \frac{c}{r_+}\left(e^{r_+ t} - e^{r_- t}\right) + \left[\Delta x_0 + \frac{\alpha\beta}{\delta}v - \alpha\beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right]\ e^{r_- t} + \left[-\frac{\alpha\beta}{\delta}v + \alpha\beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right] - \frac{\alpha\beta}{\delta}\ddot{f}t \ . \end{aligned} \tag{4.114}$$

4.5.4 Minimizing objective function with respect to c

The objective function we're interested in is the (negative) value function V:

$$J = -V = \int_0^T \left[\frac{\alpha}{2} u^2 + \frac{(x-f)^2}{2\beta} \right] e^{-t/\delta} \, dt + \frac{(x_T - f_T)^2}{2} e^{-T/\delta} \ . \tag{4.115}$$

The derivative of this with respect to c is

$$\frac{\partial J}{\partial c} = \int_0^T \left[\alpha u \frac{\partial u}{\partial c} + \frac{(x-f)}{\beta} \frac{\partial (x-f)}{\partial c} \right] e^{-t/\delta} \, dt + (x_T - f_T) \frac{\partial (x_T - f_T)}{\partial c} e^{-T/\delta} \ . \tag{4.116}$$

We will need the facts that

$$\begin{aligned} \frac{\partial u}{\partial c} &= e^{r_+ t} - \frac{r_-}{r_+} e^{r_- t} \\ \frac{\partial (x-f)}{\partial c} &= \frac{1}{r_+} \left(e^{r_+ t} - e^{r_- t} \right) \ . \end{aligned} \tag{4.117}$$

Notice that neither of these quantities depends on c. To make things as easy as possible, write u and x as

$$\begin{aligned} u(t) &= c u_c + u_r \\ x(t) - f(t) &= c x_c + x_r \ . \end{aligned} \tag{4.118}$$

Then we have

$$\frac{\partial J}{\partial c} = \int_0^T \left[\alpha (c u_c + u_r) u_c + \frac{(c x_c + x_r) x_c}{\beta} \right] e^{-t/\delta} \, dt + [c x_c(T) + x_r(T)] x_c(T) e^{-T/\delta} \ . \tag{4.119}$$

Setting this equal to zero, we find

$$\left[\int_0^T \left(\alpha u_c^2 + \frac{x_c^2}{\beta} \right) e^{-t/\delta} \, dt + x_c(T)^2 e^{-T/\delta} \right] c + \left[\int_0^T \left(\alpha u_r u_c + \frac{x_r x_c}{\beta} \right) e^{-t/\delta} \, dt + x_r(T) x_c(T) e^{-T/\delta} \right] = 0 \tag{4.120}$$

or equivalently

$$c = \frac{ - \int_0^T \left(\alpha u_r u_c + \frac{x_r x_c}{\beta} \right) e^{-t/\delta} \, dt - x_r(T) x_c(T) e^{-T/\delta} }{ \int_0^T \left(\alpha u_c^2 + \frac{x_c^2}{\beta} \right) e^{-t/\delta} \, dt + x_c(T)^2 e^{-T/\delta} } \ . \tag{4.121}$$

In principle, we now know the full solution of our control problem—we only need to evaluate the integrals that appear in the expression above, either analytically or numerically.

4.5.5 Obtaining explicit form for constant c denominator

Let's start with the denominator, which we will denote by D. We have

$$\begin{aligned}
D &:= \int_0^T \left(\alpha u_c^2 + \frac{x_c^2}{\beta}\right) e^{-t/\delta}\ dt + x_c(T)^2 e^{-T/\delta} \\
&= \int_0^T \left[\alpha \left(e^{r_+ t} - \frac{r_-}{r_+} e^{r_- t}\right)^2 + \frac{\left(e^{r_+ t} - e^{r_- t}\right)^2}{\beta r_+^2}\right] e^{-t/\delta}\ dt + \frac{1}{r_+^2}\left(e^{r_+ T} - e^{r_- T}\right)^2 e^{-T/\delta} \\
&= \left[\alpha + \frac{1}{\beta r_+^2}\right] \frac{\left[e^{(2r_+ - 1/\delta)T} - 1\right]}{2r_+ - 1/\delta} + \frac{1}{r_+^2}\left[\alpha r_-^2 + \frac{1}{\beta}\right] \frac{\left[e^{(2r_- - 1/\delta)T} - 1\right]}{2r_- - 1/\delta} + \frac{1}{r_+^2}\left(e^{r_+ T} - e^{r_- T}\right)^2 e^{-T/\delta}\ .
\end{aligned}$$

If we define the quantity

$$\omega = r_+ - r_- = 2r_+ - \frac{1}{\delta} = -2r_- + \frac{1}{\delta} = \sqrt{\frac{1}{\delta^2} + \frac{4}{\alpha\beta}}\ , \tag{4.122}$$

we can write D as

$$D = \frac{1}{r_+^2}\left[\alpha r_+^2 + \frac{1}{\beta}\right] \frac{\left[e^{\omega T} - 1\right]}{\omega} - \frac{1}{r_+^2}\left[\alpha r_-^2 + \frac{1}{\beta}\right] \frac{\left[e^{-\omega T} - 1\right]}{\omega} + \frac{1}{r_+^2}\left(e^{r_+ T} - e^{r_- T}\right)^2 e^{-T/\delta}\ .$$

This can be further simplified. We will use a few facts:

$$\begin{aligned}
r_+^2 &= \frac{1}{2\delta^2} + \frac{1}{\alpha\beta} + \frac{\omega}{2\delta} \\
r_-^2 &= \frac{1}{2\delta^2} + \frac{1}{\alpha\beta} - \frac{\omega}{2\delta} \\
\alpha r_+^2 + \frac{1}{\beta} &= \frac{\alpha}{2}\left[\omega^2 + \frac{\omega}{\delta}\right] = \alpha\omega r_+ \\
\alpha r_-^2 + \frac{1}{\beta} &= \frac{\alpha}{2}\left[\omega^2 - \frac{\omega}{\delta}\right] = -\alpha\omega r_- \\
\left(e^{r_+ T} - e^{r_- T}\right)^2 e^{-T/\delta} &= \left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right)^2 \\
e^{\omega T} - 1 &= e^{\frac{\omega}{2}T}\left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right) \\
e^{-\omega T} - 1 &= -e^{-\frac{\omega}{2}T}\left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right)\ .
\end{aligned}$$

Putting these together, D is

$$\begin{aligned}
\frac{D r_+^2}{e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}} &= \alpha r_+ e^{\frac{\omega}{2}T} - \alpha r_- e^{-\frac{\omega}{2}T} + \left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right) \\
&= \left(1 + \alpha r_+\right) e^{\frac{\omega}{2}T} - \left(1 + \alpha r_-\right) e^{-\frac{\omega}{2}T}\ .
\end{aligned}$$

4.5.6 Obtaining explicit form for constant c numerator

Denote the numerator of c by $-N$. We have

$$
\begin{aligned}
-N &:= \int_0^T \left(\alpha u_r u_c + \frac{x_r x_c}{\beta}\right) e^{-t/\delta}\, dt + x_r(T) x_c(T) e^{-T/\delta} \\
&= \int_0^T \left[\alpha u_r \left(e^{r_+ t} - \frac{r_-}{r_+} e^{r_- t}\right) + \frac{x_r}{\beta r_+}\left(e^{r_+ t} - e^{r_- t}\right)\right] e^{-t/\delta}\, dt + \frac{x_r(T)}{r_+}\left(e^{r_+ T} - e^{r_- T}\right) e^{-T/\delta}\ .
\end{aligned}
$$

Recall that

$$
\begin{aligned}
u_r &= r_- \left[\Delta x_0 + \frac{\alpha\beta}{\delta} v - \alpha\beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right] e^{r_- t} + \dot{f}(t) - \frac{\alpha\beta}{\delta}\ddot{f} \\
x_r &= \left[\Delta x_0 + \frac{\alpha\beta}{\delta} v - \alpha\beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right]\ e^{r_- t} + \left[-\frac{\alpha\beta}{\delta} v + \alpha\beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right] - \frac{\alpha\beta}{\delta}\ddot{f} t\ .
\end{aligned}
\tag{4.123}
$$

The following expressions will be somewhat more complicated, and it is helpful to group them in a particular way. Write

$$N = N_x \Delta x_a + N_v v + N_a \ddot{f} \tag{4.124}$$

since each term in the above integral is proportional to either Δx_a, v, or $\ddot{f}$. Let's take these one at a time.

$-N_x$

The integral terms are

$$
\begin{aligned}
&\int_0^T \left[\alpha r_- e^{r_- t}\left(e^{r_+ t} - \frac{r_-}{r_+} e^{r_- t}\right) + \frac{1}{\beta r_+} e^{r_- t}\left(e^{r_+ t} - e^{r_- t}\right)\right] e^{-t/\delta}\, dt \\
&= \alpha\omega \frac{r_-}{r_+} \int_0^T e^{-\omega t}\, dt \\
&= -\alpha \frac{r_-}{r_+}\left(e^{-\omega T} - 1\right) \\
&= \frac{\alpha r_-}{r_+}\left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right) e^{-\frac{\omega}{2}T}\ .
\end{aligned}
$$

All together, $-N_x$ is

$$
\begin{aligned}
-N_x &= \frac{\alpha r_-}{r_+}\left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right) e^{-\frac{\omega}{2}T} + \frac{1}{r_+}\left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right) e^{-\frac{\omega}{2}T} \\
&= \left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right) e^{-\frac{\omega}{2}T} \frac{(1 + \alpha r_-)}{r_+}\ .
\end{aligned}
$$

$-N_v$

The integral terms are

$$\begin{aligned}
&\int_0^T \left[\alpha\left(r_-\frac{\alpha\beta}{\delta}e^{r_-t}+1\right)\left(e^{r_+t}-\frac{r_-}{r_+}e^{r_-t}\right)+\frac{1}{\beta r_+}\frac{\alpha\beta}{\delta}\left(e^{r_-t}-1\right)\left(e^{r_+t}-e^{r_-t}\right)\right]e^{-t/\delta}\,dt \\
&= \int_0^T \left[\alpha\left(r_-\frac{\alpha\beta}{\delta}e^{r_-t}+1\right)\left(e^{r_+t}-\frac{r_-}{r_+}e^{r_-t}\right)-\alpha r_-\frac{\alpha\beta}{\delta}\left(e^{r_-t}-1\right)\left(e^{r_+t}-e^{r_-t}\right)\right]e^{-t/\delta}\,dt \\
&= \int_0^T \alpha\frac{r_-}{r_+}\frac{\alpha\beta}{\delta}\omega e^{-\omega t}+\alpha\left(1+r_-\frac{\alpha\beta}{\delta}\right)e^{(r_+-1/\delta)t}-\alpha\frac{r_-}{r_+}\left(1+r_+\frac{\alpha\beta}{\delta}\right)e^{(r_--1/\delta)t}\,dt \\
&= \int_0^T \alpha\frac{r_-}{r_+}\frac{\alpha\beta}{\delta}\omega e^{-\omega t}+\alpha\frac{(r_+-1/\delta)}{r_+}e^{(r_+-1/\delta)t}-\alpha\frac{r_-}{r_+}\frac{(r_--1/\delta)}{r_-}e^{(r_--1/\delta)t}\,dt \\
&= \alpha\frac{r_-}{r_+}\frac{\alpha\beta}{\delta}\left(e^{-\omega T}-1\right)+\frac{\alpha}{r_+}\left(e^{(r_+-1/\delta)T}-1\right)-\frac{\alpha}{r_+}\left(e^{(r_--1/\delta)T}-1\right) \\
&= \frac{\alpha}{r_+}\left\{r_-\frac{\alpha\beta}{\delta}\left(e^{-\omega T}-1\right)+\left(e^{(r_+-1/\delta)T}-e^{(r_--1/\delta)T}\right)\right\}\ .
\end{aligned}$$

All together, $-N_v$ is

$$\begin{aligned}
-N_v &= \frac{\alpha}{r_+}\left\{r_-\frac{\alpha\beta}{\delta}\left(e^{-\omega T}-1\right)+\left(e^{(r_+-1/\delta)T}-e^{(r_--1/\delta)T}\right)+\frac{\beta}{\delta}\left(e^{r_-T}-1\right)\left(e^{r_+T}-e^{r_-T}\right)e^{-T/\delta}\right\} \\
&= \frac{\alpha}{r_+}\left\{\frac{\beta}{\delta}\left(1-\alpha r_-\right)\left(1-e^{-\omega T}\right)+\left(1-\frac{\beta}{\delta}\right)\left(e^{(r_+-1/\delta)T}-e^{(r_--1/\delta)T}\right)\right\} \\
&= \frac{\alpha}{r_+}\left(1-e^{-\omega T}\right)\left\{\frac{\beta}{\delta}\left(1-\alpha r_-\right)+\left(1-\frac{\beta}{\delta}\right)e^{-r_-T}\right\}\ .
\end{aligned}$$

$-N_a$

The integral terms are

$$\begin{aligned}
&\int_0^T \alpha\left(-\alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)r_-e^{r_-t}+t-\frac{\alpha\beta}{\delta}\right)\left(e^{r_+t}-\frac{r_-}{r_+}e^{r_-t}\right)e^{-t/\delta} \\
&+\frac{1}{\beta r_+}\left[-\alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)\left(e^{r_-t}-1\right)-\frac{\alpha\beta}{\delta}t\right]\left(e^{r_+t}-e^{r_-t}\right)e^{-t/\delta}\,dt\ .
\end{aligned}$$

Using the fact that $r_+r_-=-1/(\alpha\beta)$, we can rewrite this as

$$\begin{aligned}
&\int_0^T \alpha\left(-\alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)r_-e^{r_-t}+t-\frac{\alpha\beta}{\delta}\right)\left(e^{r_+t}-\frac{r_-}{r_+}e^{r_-t}\right)e^{-t/\delta} \\
&+\alpha r_-\left[\alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)e^{r_-t}-\alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)+\frac{\alpha\beta}{\delta}t\right]\left(e^{r_+t}-e^{r_-t}\right)e^{-t/\delta}\,dt\ .
\end{aligned}$$

Expanding a bit, we have

$$\int_0^T \alpha \left(-\alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) r_- e^{r_- t} + t - \frac{\alpha\beta}{\delta}\right) e^{r_+ t} e^{-t/\delta}$$
$$- \frac{r_-}{r_+}\alpha \left(-\alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) r_- e^{r_- t} + t - \frac{\alpha\beta}{\delta}\right) e^{r_- t} e^{-t/\delta}$$
$$+ \alpha r_- \left[\alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) e^{r_- t} - \alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) + \frac{\alpha\beta}{\delta} t\right] e^{r_+ t} e^{-t/\delta} \; dt$$
$$- \alpha r_- \left[\alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) e^{r_- t} - \alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) + \frac{\alpha\beta}{\delta} t\right] e^{r_- t} e^{-t/\delta} \; dt \; .$$

Canceling some terms, we have

$$\int_0^T \alpha \left(t - \frac{\alpha\beta}{\delta}\right) e^{r_+ t} e^{-t/\delta}$$
$$- \frac{r_-}{r_+}\alpha \left(-\alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) r_- e^{r_- t} + t - \frac{\alpha\beta}{\delta}\right) e^{r_- t} e^{-t/\delta}$$
$$+ \alpha r_- \left[-\alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) + \frac{\alpha\beta}{\delta} t\right] e^{r_+ t} e^{-t/\delta} \; dt$$
$$- \alpha r_- \left[\alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) e^{r_- t} - \alpha\beta \left(1 + \frac{\alpha\beta}{\delta^2}\right) + \frac{\alpha\beta}{\delta} t\right] e^{r_- t} e^{-t/\delta} \; dt \; .$$

Collecting terms,

$$\int_0^T \alpha \left(1 + \frac{\alpha\beta}{\delta} r_-\right) \left(t - \frac{\alpha\beta}{\delta}\right) e^{r_+ t} e^{-t/\delta}$$
$$- \alpha^2 \beta r_- e^{r_+ t} e^{-t/\delta} + \alpha^2 \beta r_- e^{r_- t} e^{-t/\delta}$$
$$- \alpha r_- \left(\frac{1}{r_+} + \frac{\alpha\beta}{\delta}\right) \left(t - \frac{\alpha\beta}{\delta}\right) e^{r_- t} e^{-t/\delta} \; dt$$
$$- \alpha^2 \beta r_- \left(1 + \frac{\alpha\beta}{\delta^2}\right) \left(1 - \frac{r_-}{r_+}\right) e^{2 r_- t} e^{-t/\delta} \; dt \; .$$

Simplifying,

$$\int_0^T \frac{\alpha}{r_+} \left(r_+ - \frac{1}{\delta}\right) \left(t - \frac{\alpha\beta}{\delta}\right) e^{r_+ t} e^{-t/\delta}$$
$$- \alpha^2 \beta r_- e^{r_+ t} e^{-t/\delta} + \alpha^2 \beta r_- e^{r_- t} e^{-t/\delta}$$
$$- \frac{\alpha}{r_+} \left(r_- - \frac{1}{\delta}\right) \left(t - \frac{\alpha\beta}{\delta}\right) e^{r_- t} e^{-t/\delta} \; dt$$
$$- \alpha^2 \beta \frac{r_-}{r_+} \omega \left(1 + \frac{\alpha\beta}{\delta^2}\right) e^{-\omega t} \; dt \; .$$

More simplifying...

$$
\begin{aligned}
&\int_0^T \frac{\alpha}{r_+}\left[\left(r_+ - \frac{1}{\delta}\right) t\ e^{r_+ t} e^{-t/\delta} - \left(r_- - \frac{1}{\delta}\right) t\ e^{r_- t} e^{-t/\delta}\right] \\
&- \alpha^2\beta\left[\frac{r_-}{r_+}\left(r_+ - \frac{1}{\delta}\right) e^{r_+ t} e^{-t/\delta} - \left(r_- - \frac{1}{\delta}\right) e^{r_- t} e^{-t/\delta}\right] \\
&- \alpha^2\beta\frac{r_-}{r_+}\omega\left(1 + \frac{\alpha\beta}{\delta^2}\right) e^{-\omega t}\ dt\ .
\end{aligned}
$$

Evaluating these integrals, we have

$$
\begin{aligned}
&\frac{\alpha}{r_+}\left[\frac{1 + e^{(r_+ - 1/\delta)T}\left((r_+ - 1/\delta)T - 1\right)}{\left(r_+ - \frac{1}{\delta}\right)} - \frac{1 + e^{(r_- - 1/\delta)T}\left((r_- - 1/\delta)T - 1\right)}{\left(r_- - \frac{1}{\delta}\right)}\right] \\
&- \alpha^2\beta\left[\frac{r_-}{r_+}\left(e^{(r_+ - 1/\delta)T} - 1\right) - \left(e^{(r_- - 1/\delta)T} - 1\right)\right] \\
&+ \alpha^2\beta\frac{r_-}{r_+}\left(1 + \frac{\alpha\beta}{\delta^2}\right)\left(e^{-\omega T} - 1\right)\ .
\end{aligned}
$$

Simplifying the result,

$$
\begin{aligned}
&\frac{\alpha^2\beta}{\delta r_+}\left(1 - e^{(r_+ - 1/\delta)T}\right) - \frac{\alpha^2\beta}{\delta r_+}\left(1 - e^{(r_- - 1/\delta)T}\right) + \frac{\alpha}{r_+}T\left[e^{(r_+ - 1/\delta)T} - e^{(r_- - 1/\delta)T}\right] \\
&+ \alpha^2\beta\frac{r_-}{r_+}\left(1 + \frac{\alpha\beta}{\delta^2}\right)\left(e^{-\omega T} - 1\right)\ .
\end{aligned}
$$

Simplifying more,

$$
\begin{aligned}
&-\frac{\alpha^2\beta}{\delta r_+}\left(e^{(r_+ - 1/\delta)T} - e^{(r_- - 1/\delta)T}\right) + \frac{\alpha}{r_+}T\left[e^{(r_+ - 1/\delta)T} - e^{(r_- - 1/\delta)T}\right] \\
&+ \alpha^2\beta\frac{r_-}{r_+}\left(1 + \frac{\alpha\beta}{\delta^2}\right)\left(e^{-\omega T} - 1\right)\ .
\end{aligned}
$$

Simplifying even more...

$$
\frac{\alpha}{r_+}\left(e^{(r_+ - 1/\delta)T} - e^{(r_- - 1/\delta)T}\right)\left\{-\frac{\alpha\beta}{\delta} + T - \alpha\beta r_-\left(1 + \frac{\alpha\beta}{\delta^2}\right) e^{r_- T}\right\}\ .
$$

These were only the integral terms. Adding back in the other terms, we have

$$
\begin{aligned}
&\frac{\alpha}{r_+}\left(e^{(r_+ - 1/\delta)T} - e^{(r_- - 1/\delta)T}\right)\left\{-\frac{\alpha\beta}{\delta} + T - \alpha\beta r_-\left(1 + \frac{\alpha\beta}{\delta^2}\right) e^{r_- T} - \beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\left(e^{r_- T} - 1\right) - \frac{\beta}{\delta}T\right\} \\
&\frac{\alpha}{r_+}\left(e^{(r_+ - 1/\delta)T} - e^{(r_- - 1/\delta)T}\right)\left\{\beta\left(1 + \frac{\alpha\beta}{\delta^2} - \frac{\alpha}{\delta}\right) + \left(1 - \frac{\beta}{\delta}\right)T - \beta\left(1 + \frac{\alpha\beta}{\delta^2}\right)\left(1 + \alpha r_-\right) e^{r_- T}\right\}\ .
\end{aligned}
$$

4.5.7 Explicit form for constant

The quantity that actually appears in $x(t)$ is $\frac{N}{Dr_+}$. Note,

$$\begin{aligned}
\frac{N_x}{Dr_+} &= -\frac{(1+\alpha r_-)\,e^{-\frac{\omega}{2}T}}{(1+\alpha r_+)\,e^{\frac{\omega}{2}T} - (1+\alpha r_-)\,e^{-\frac{\omega}{2}T}} \\
\frac{N_v}{Dr_+} &= -\alpha\frac{\left[\frac{\beta}{\delta}(1-\alpha r_-) + \left(1-\frac{\beta}{\delta}\right)e^{-r_-T}\right]e^{-\frac{\omega}{2}T}}{(1+\alpha r_+)\,e^{\frac{\omega}{2}T} - (1+\alpha r_-)\,e^{-\frac{\omega}{2}T}} \\
\frac{N_a}{Dr_+} &= -\alpha e^{-T/(2\delta)}\frac{\left\{\beta\left(1+\frac{\alpha\beta}{\delta^2}-\frac{\alpha}{\delta}\right) + \left(1-\frac{\beta}{\delta}\right)T - \beta\left(1+\frac{\alpha\beta}{\delta^2}\right)(1+\alpha r_-)\,e^{r_-T}\right\}}{(1+\alpha r_+)\,e^{\frac{\omega}{2}T} - (1+\alpha r_-)\,e^{-\frac{\omega}{2}T}}\ .
\end{aligned} \tag{4.125}$$

4.5.8 Final expression for control and trajectories

$$\begin{aligned}
u(t) &= \frac{N}{D}\left(e^{r_+t} - \frac{r_-}{r_+}e^{r_-t}\right) + r_-\left[\Delta x_0 + \frac{\alpha\beta}{\delta}v - \alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right]e^{r_-t} + \dot{f}(t) - \frac{\alpha\beta}{\delta}\ddot{f} \\
\Delta x_t &= \frac{N}{Dr_+}\left(e^{r_+t} - e^{r_-t}\right) + \left[\Delta x_0 + \frac{\alpha\beta}{\delta}v - \alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right]\ e^{r_-t} + \left[-\frac{\alpha\beta}{\delta}v + \alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right] - \frac{\alpha\beta}{\delta}\ddot{f}t\ .
\end{aligned} \tag{4.126}$$

where

$$r_\pm = \frac{1}{2\delta} \pm \frac{1}{2}\sqrt{\frac{1}{\delta^2} + \frac{4}{\alpha\beta}}\ . \tag{4.127}$$

$$\omega = r_+ - r_- = 2r_+ - \frac{1}{\delta} = -2r_- + \frac{1}{\delta} = \sqrt{\frac{1}{\delta^2} + \frac{4}{\alpha\beta}}\ , \tag{4.128}$$

4.5.9 Endpoint

$$\begin{aligned}
\Delta x_T &= \frac{N\left(e^{r_+T} - e^{r_-T}\right)}{Dr_+} + \left[\Delta x_0 + \frac{\alpha\beta}{\delta}v - \alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right]\ e^{r_-T} + \left[-\frac{\alpha\beta}{\delta}v + \alpha\beta\left(1+\frac{\alpha\beta}{\delta^2}\right)\ddot{f}\right] - \frac{\alpha\beta}{\delta}\ddot{f}T \\
&= R_x\Delta x_a + R_v v + R_a\ddot{f}\ .
\end{aligned} \tag{4.129}$$

The x coefficient is

$$\begin{aligned}
R_x &= \frac{N_x\left(e^{r_+T} - e^{r_-T}\right)}{Dr_+} + e^{-r_-T} \\
&= -\frac{(1+\alpha r_-)\,e^{-\frac{\omega}{2}T}}{(1+\alpha r_+)\,e^{\frac{\omega}{2}T} - (1+\alpha r_-)\,e^{-\frac{\omega}{2}T}}\left(e^{r_+T} - e^{r_-T}\right) + e^{-r_-T} \\
&= -\frac{\left\{(1+\alpha r_-)\,e^{-\frac{\omega}{2}T}\left(e^{r_+T} - e^{r_-T}\right) - \left[(1+\alpha r_+)\,e^{\frac{\omega}{2}T} - (1+\alpha r_-)\,e^{-\frac{\omega}{2}T}\right]e^{-r_-T}\right\}}{(1+\alpha r_+)\,e^{\frac{\omega}{2}T} - (1+\alpha r_-)\,e^{-\frac{\omega}{2}T}} \\
&= -\frac{\left\{\left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right)\left[(1+\alpha r_-)\,e^{r_-T} - e^{-r_-T}\right] - \alpha\left[r_+e^{\frac{\omega}{2}T} - r_-e^{-\frac{\omega}{2}T}\right]e^{-r_-T}\right\}}{(1+\alpha r_+)\,e^{\frac{\omega}{2}T} - (1+\alpha r_-)\,e^{-\frac{\omega}{2}T}}\ .
\end{aligned} \tag{4.130}$$

This is always nonnegative. The v coefficient is

$$
\begin{aligned}
R_v &= \frac{N_v \left(e^{r_+ T} - e^{r_- T}\right)}{D r_+} + \frac{\alpha\beta}{\delta}\left(e^{-r_- T} - 1\right) \\
&= -\frac{\left(1+\alpha r_-\right) e^{-\frac{\omega}{2}T}}{\left(1+\alpha r_+\right) e^{\frac{\omega}{2}T} - \left(1+\alpha r_-\right) e^{-\frac{\omega}{2}T}}\left(e^{r_+ T} - e^{r_- T}\right) + \frac{\alpha\beta}{\delta}\left(e^{-r_- T} - 1\right) \\
&= -\frac{\left\{\left(1+\alpha r_-\right) e^{-\frac{\omega}{2}T}\left(e^{r_+ T} - e^{r_- T}\right) - \frac{\alpha\beta}{\delta}\left[\left(1+\alpha r_+\right) e^{\frac{\omega}{2}T} - \left(1+\alpha r_-\right) e^{-\frac{\omega}{2}T}\right]\left(e^{-r_- T} - 1\right)\right\}}{\left(1+\alpha r_+\right) e^{\frac{\omega}{2}T} - \left(1+\alpha r_-\right) e^{-\frac{\omega}{2}T}} \\
&= -\frac{\left\{\left(e^{\frac{\omega}{2}T} - e^{-\frac{\omega}{2}T}\right)\left[\left(1+\alpha r_-\right) e^{r_- T} - \frac{\alpha\beta}{\delta}\left(e^{-r_- T} - 1\right)\right] - \frac{\alpha^2\beta}{\delta}\left[r_+ e^{\frac{\omega}{2}T} - r_- e^{-\frac{\omega}{2}T}\right]\left(e^{-r_- T} - 1\right)\right\}}{\left(1+\alpha r_+\right) e^{\frac{\omega}{2}T} - \left(1+\alpha r_-\right) e^{-\frac{\omega}{2}T}} \, .
\end{aligned}
\tag{4.131}
$$

4.6 Velocity and acceleration penalty

The objective is

$$-V = \int_0^T \frac{\alpha}{2}u^2 + \frac{\kappa^3}{2}\dot{u}^2 \; dt + \frac{(x_T - f_T)^2}{2} \; . \tag{4.132}$$

The Lagrangian is

$$L = p(\dot{x} - u) + \frac{\alpha}{2}u^2 + \frac{\kappa^3}{2}\dot{u}^2 \; . \tag{4.133}$$

The relevant derivatives are

$$\begin{aligned} \frac{\partial L}{\partial u} &= \alpha u - p \\ \frac{\partial L}{\partial \dot{u}} &= \kappa^3 \dot{u} \\ \frac{\partial L}{\partial \dot{x}} &= p \; . \end{aligned} \tag{4.134}$$

The Euler-Lagrange equations are

$$\begin{aligned} \kappa^3 \ddot{u} &= \alpha u - p \\ p &= \text{const.} \end{aligned} \tag{4.135}$$

This becomes

$$\ddot{u} - \frac{\alpha}{\kappa^3} u = C \tag{4.136}$$

where C is an undetermined constant. This is a linear first-order ODE whose solution is

$$u_t = c_+ e^{rt} + c_- e^{-rt} + C' \tag{4.137}$$

where $r := \sqrt{\alpha/\kappa^3}$ and C' is a different (still arbitrary) constant. Note that there are three constants we must determine. One comes from minimizing the objective, and another comes from enforcing the initial condition on x; we need one more, which has to come from enforcing an initial condition on u.

Note,

$$u_0 = c_+ + c_- + C' \implies C' = u_0 - c_+ - c_- \; . \tag{4.138}$$

Then

$$\begin{aligned} u_t &= c_+(e^{rt} - 1) + c_-(e^{-rt} - 1) + u_0 \\ \dot{u}_t &= c_+ r e^{rt} - c_- r e^{-rt} \; . \end{aligned} \tag{4.139}$$

Integrating, we have

$$\begin{aligned} x_t &= x_0 + \frac{1}{r}\left[c_+(e^{rt} - 1) - c_-(e^{-rt} - 1)\right] + C't \\ &= \frac{c_+}{r}(e^{rt} - 1 - rt) - \frac{c_-}{r}(e^{-rt} - 1 + rt) + x_0 + u_0 t \; . \end{aligned} \tag{4.140}$$

The original objective is optimized when

$$-\frac{\partial V}{\partial \theta} = \int \alpha u \frac{\partial u}{\partial \theta} + \kappa^3 \dot{u} \frac{\partial \dot{u}}{\partial \theta} \, dt + (x_T - f_T) \frac{\partial x_T}{\partial \theta} = 0 \; . \tag{4.141}$$

In particular,

$$\begin{aligned} -\frac{\partial V}{\partial c_+} &= \int \alpha \left[c_+ u_+ + c_- u_- + u_0\right] u_+ + \kappa^3 \left[c_+ \dot{u}_+ + c_- \dot{u}_-\right] \dot{u}_+ \, dt + \left[c_+ x_+ + c_- x_- + x_0 + u_0 T - f_T\right] x_+ \\ -\frac{\partial V}{\partial c_-} &= \int \alpha \left[c_+ u_+ + c_- u_- + u_0\right] u_- + \kappa^3 \left[c_+ \dot{u}_+ + c_- \dot{u}_-\right] \dot{u}_- \, dt + \left[c_+ x_+ + c_- x_- + x_0 + u_0 T - f_T\right] x_- \end{aligned} \tag{4.142}$$

Rearranging, this implies

$$\begin{aligned} &\left[\int \alpha u_+^2 + \kappa^3 \dot{u}_+^2 \, dt + x_+(T)^2\right] c_+ + \left[\int \alpha u_+ u_- + \kappa^3 \dot{u}_+ \dot{u}_- \, dt + x_+(T) x_-(T)\right] c_- \\ = &\int -\alpha u_0 u_+ dt - (x_0 + u_0 T - f_T) x_+(T) \end{aligned} \tag{4.143}$$

$$\begin{aligned} &\left[\int \alpha u_+ u_- + \kappa^3 \dot{u}_+ \dot{u}_- \, dt + x_+(T) x_-(T)\right] c_+ + \left[\int \alpha u_-^2 + \kappa^3 \dot{u}_-^2 \, dt + x_-(T)^2\right] c_- \\ = &\int -\alpha u_0 u_- dt - (x_0 + u_0 T - f_T) x_-(T) \end{aligned} \tag{4.144}$$

In matrix form, we have

$$\begin{pmatrix} M_{++} & M_{+-} \\ M_{+-} & M_{--} \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} d_+ \\ d_- \end{pmatrix} \; . \tag{4.145}$$

Computing relevant quantities:

$$\begin{aligned} d_+ &= -\alpha u_0 \left[\frac{1}{r}(e^{rT} - 1) - T\right] - (x_0 + u_0 T - f_T) x_+(T) \\ d_- &= -\alpha u_0 \left[-\frac{1}{r}(e^{-rT} - 1) - T\right] - (x_0 + u_0 T - f_T) x_-(T) \\ M_{++} &= \alpha \left[\frac{1}{2r}(e^{2rT} - 1) - \frac{2}{r}(e^{rT} - 1) + T\right] + \kappa^3 \frac{r}{2} \left[e^{2rT} - 1\right] + \frac{1}{r^2} \left[e^{rT} - 1 - rT\right]^2 \\ &= \frac{\alpha}{r} \left[(e^{2rT} - 1) - 2(e^{rT} - 1) + rT\right] + \frac{1}{r^2} \left[e^{rT} - 1 - rT\right]^2 \\ &= \frac{\alpha}{r} \left[(e^{rT} - 1)^2 + rT\right] + \frac{1}{r^2} \left[e^{rT} - 1 - rT\right]^2 \\ M_{--} &= -\frac{\alpha}{r} \left[(e^{-rT} - 1)^2 - rT\right] + \frac{1}{r^2} \left[e^{-rT} - 1 + rT\right]^2 \\ M_{+-} &= \alpha \left[2T - \frac{1}{r}(e^{rT} - e^{-rT})\right] - r^2 \kappa^3 T - \frac{1}{r^2} \left[e^{rT} - 1 - rT\right] \left[e^{-rT} - 1 + rT\right] \\ &= \alpha \left[T - \frac{1}{r}(e^{rT} - e^{-rT})\right] - \frac{1}{r^2} \left[2 - e^{rT} - e^{-rT} + rT(e^{rT} - e^{-rT}) - r^2 T^2\right] \end{aligned} \tag{4.146}$$

Chapter 5

Online Bayesian causal inference

5.1 Problem formulation

5.2 Special approach for exponential families

5.3 Useful mathematical facts

One property of the determinant is that

$$\det(A + \epsilon X) \approx \det(A) + \det(A)\mathrm{tr}(A^{-1}X)\epsilon \tag{5.1}$$

for small ϵ. Taking the logarithm,

$$\log\det(A + \epsilon X) \approx \log\det A + \log\left[1 + \mathrm{tr}(A^{-1}X)\epsilon\right] \approx \log\det A + \mathrm{tr}(A^{-1}X)\epsilon \ . \tag{5.2}$$

Trace trick: https://stats.stackexchange.com/questions/544311/why-does-hutchinsons-trace-estimator-reduce-computation-complexity; https://www.nowozin.net/sebastian/blog/thoughts-on-trace-estimation-in-deep-learning.html

$$mmmm \tag{5.3}$$

5.4 Filtering notes

Suppose there is a latent variable $\boldsymbol{z}_t$ which has stochastic dynamics

$$\begin{aligned}\dot{\boldsymbol{z}}_t &= \boldsymbol{b} - \boldsymbol{A}\boldsymbol{z}_t + \boldsymbol{g}\ \boldsymbol{\eta}_t \\ p(\boldsymbol{z}_t|\boldsymbol{z}_{t-1}) &= \mathcal{N}(\boldsymbol{z}_t; \boldsymbol{z}_{t-1} + (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{z}_t)\Delta t, \boldsymbol{\Sigma}_{noise}\Delta t)\ .\end{aligned} \tag{5.4}$$

where $\boldsymbol{\Sigma}_{noise} := \boldsymbol{g}\boldsymbol{g}^T$. Suppose $\boldsymbol{z}_t$ is observed according to

$$p(\boldsymbol{x}_t|\boldsymbol{z}_t) = \mathcal{N}(\boldsymbol{x}_t; \boldsymbol{M}\boldsymbol{z}_t, \boldsymbol{\Sigma}_{obs}/\Delta t) \tag{5.5}$$

and that observations at different moments in time are independent. Suppose we're interested in computing the parameters of $p(\boldsymbol{z}_t|\boldsymbol{X}_t) = \mathcal{N}(\boldsymbol{z}_t; \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$, where $\boldsymbol{X}_t := \{\boldsymbol{x}_1, ..., \boldsymbol{x}_t\}$. Note,

$$p(\boldsymbol{z}_t|\boldsymbol{X}_t) = \frac{p(\boldsymbol{x}_t|\boldsymbol{z}_t)p(\boldsymbol{z}_t|\boldsymbol{X}_{t-1})}{p(\boldsymbol{x}_t|\boldsymbol{X}_{t-1})} = \frac{p(\boldsymbol{x}_t|\boldsymbol{z}_t)}{p(\boldsymbol{x}_t|\boldsymbol{X}_{t-1})} \int p(\boldsymbol{z}_t|\boldsymbol{z}_{t-1})p(\boldsymbol{z}_{t-1}|\boldsymbol{X}_{t-1})\ d\boldsymbol{z}_{t-1}\ . \tag{5.6}$$

The integral can be computed exactly. We have

$$\begin{aligned}&\int \frac{d\boldsymbol{p}}{(2\pi)^D} \exp\left\{-i\boldsymbol{p}^T[\boldsymbol{z}_t - \boldsymbol{z}_{t-1} - (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{z}_{t-1})\Delta t] - \frac{1}{2}\boldsymbol{p}^T\boldsymbol{\Sigma}_{noise}\boldsymbol{p}\Delta t\right\} \mathcal{N}(\boldsymbol{z}_{t-1}; \boldsymbol{\mu}_{t-1}, \boldsymbol{\Sigma}_{t-1})\ d\boldsymbol{z}_{t-1} \\ &= \int \frac{d\boldsymbol{p}}{(2\pi)^D}\ e^{-i\boldsymbol{p}^T[\boldsymbol{z}_t - \boldsymbol{b}\Delta t] - \frac{1}{2}\boldsymbol{p}^T\boldsymbol{\Sigma}_{noise}\boldsymbol{p}\Delta t + i\boldsymbol{p}^T[\boldsymbol{I} - \boldsymbol{A}\Delta t]\boldsymbol{\mu}_{t-1} - \frac{1}{2}\boldsymbol{p}^T[\boldsymbol{I} - \boldsymbol{A}\Delta t]\boldsymbol{\Sigma}_{t-1}\left[\boldsymbol{I} - \boldsymbol{A}^T\Delta t\right]^T\boldsymbol{p}} \\ &= \mathcal{N}(\boldsymbol{z}_t; \boldsymbol{\mu}_{t-1} + (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{\mu}_{t-1})\Delta t, \boldsymbol{\Sigma}_{t-1} + \Delta t\left[\boldsymbol{\Sigma}_{noise} - \boldsymbol{A}\boldsymbol{\Sigma}_{t-1} - \boldsymbol{\Sigma}_{t-1}\boldsymbol{A}^T\right])\end{aligned}$$

to first order in Δt. Define the quantities

$$\boldsymbol{\mu}_t' := \boldsymbol{\mu}_{t-1} + (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{\mu}_{t-1})\Delta t \qquad \boldsymbol{\Sigma}_t' := \boldsymbol{\Sigma}_{t-1} + \Delta t\left[\boldsymbol{\Sigma}_{noise} - \boldsymbol{A}\boldsymbol{\Sigma}_{t-1} - \boldsymbol{\Sigma}_{t-1}\boldsymbol{A}^T\right]\ . \tag{5.7}$$

By Bayesian cue combination, $\boldsymbol{\Sigma}_t^{-1} = (\boldsymbol{\Sigma}_t')^{-1} + \boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\Delta t$. Inverting this using Woodbury's identity, to first order in Δt we have

$$\begin{aligned}\boldsymbol{\Sigma}_t &= \boldsymbol{\Sigma}_t' - \boldsymbol{\Sigma}_t'\boldsymbol{M}^T\left[\boldsymbol{\Sigma}_{obs} + \boldsymbol{M}\boldsymbol{\Sigma}_t'\boldsymbol{M}^T\Delta t\right]^{-1}\boldsymbol{M}\boldsymbol{\Sigma}_t'\Delta t \\ &\approx \boldsymbol{\Sigma}_t' - \boldsymbol{\Sigma}_{t-1}\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\boldsymbol{\Sigma}_{t-1}\Delta t \\ &= \boldsymbol{\Sigma}_{t-1} + \Delta t\left[\boldsymbol{\Sigma}_{noise} - \boldsymbol{A}\boldsymbol{\Sigma}_{t-1} - \boldsymbol{\Sigma}_{t-1}\boldsymbol{A}^T\right] - \boldsymbol{\Sigma}_{t-1}\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\boldsymbol{\Sigma}_{t-1}\Delta t\ .\end{aligned} \tag{5.8}$$

Also by Bayesian cue combination,

$$\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t = \boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\Delta t\ \boldsymbol{x}_t + (\boldsymbol{\Sigma}_t')^{-1}\boldsymbol{\mu}_t'\ . \tag{5.9}$$

Since $\boldsymbol{\Sigma}_t(\boldsymbol{\Sigma}_t')^{-1} = \boldsymbol{I} - \boldsymbol{\Sigma}_t\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\Delta t \approx \boldsymbol{I} - \boldsymbol{\Sigma}_{t-1}\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\Delta t$, we have

$$\begin{aligned}\boldsymbol{\mu}_t &= \boldsymbol{\Sigma}_{t-1}\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{x}_t\Delta t + \left[\boldsymbol{I} - \boldsymbol{\Sigma}_{t-1}\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\Delta t\right][\boldsymbol{\mu}_{t-1} + (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{\mu}_{t-1})\Delta t] \\ &\approx \boldsymbol{\mu}_{t-1} + \Delta t\left[\ \boldsymbol{b} - \boldsymbol{A}\boldsymbol{\mu}_{t-1} + \boldsymbol{\Sigma}_{t-1}\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_{t-1})\ \right]\ .\end{aligned} \tag{5.10}$$

In the continuous-time limit, the whole system of updates is given by

$$\begin{aligned}\dot{\boldsymbol{\mu}}_t &= \boldsymbol{b} - \boldsymbol{A}\boldsymbol{\mu}_t + \boldsymbol{\Sigma}_t \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t) \\ \dot{\boldsymbol{\Sigma}}_t &= \boldsymbol{\Sigma}_{noise} - (\boldsymbol{A}\boldsymbol{\Sigma}_t + \boldsymbol{\Sigma}_t \boldsymbol{A}^T) - \boldsymbol{\Sigma}_t \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M} \boldsymbol{\Sigma}_t\end{aligned} \tag{5.11}$$

As we see, this is a weighted combination of the target expected from prediction and the target from observation. The current covariance plays an interesting role. When it is large, we update substantially according to new evidence; when it is small, we only update slightly.

Let's consider some special cases.

- When observation noise is negligible or zero, $\boldsymbol{\mu}_t$ is tied to the current observation and $\boldsymbol{\Sigma}_t$ is rapidly driven to zero.
- When observation noise is infinite, we recover the dynamics of the mean and covariance of the underlying distribution.

A specific case for which explicit computations are possible has a one-dimensional process ($M = 1$) with no dynamics ($A = 0$). The equations for the mean and variance are

$$\begin{aligned}\dot{\mu}_t &= \frac{v_t}{\sigma_{obs}^2}(x_t - \mu_t) \\ \dot{v}_t &= -\frac{v_t^2}{\sigma_{obs}^2}\ .\end{aligned} \tag{5.12}$$

The explicit solution to this system of ODEs is straightforward to compute, and is

$$\begin{aligned}v_t &= \frac{1}{\frac{1}{v_0} + \frac{t}{\sigma_{obs}^2}} \\ \mu_t &= v_t \left(\frac{\mu_0}{v_0} + \frac{t}{\sigma_{obs}^2} \langle x_t \rangle \right) = \frac{\frac{\mu_0}{v_0} + \frac{t}{\sigma_{obs}^2}\langle x_t \rangle}{\frac{1}{v_0} + \frac{t}{\sigma_{obs}^2}}\end{aligned} \tag{5.13}$$

where the angular brackets denote time-averaging. This is the usual, familiar solution.

5.4.1 Log-likelihood updates

We can also derive an ODE that describes data likelihood $p(\boldsymbol{X}_t)$ updates. Note that

$$\begin{aligned}p(\boldsymbol{X}_t) &= p(\boldsymbol{X}_{t-1}) \int p(\boldsymbol{x}_t|\boldsymbol{z}_t) p(\boldsymbol{z}_t|\boldsymbol{z}_{t-1}) p(\boldsymbol{z}_{t-1}|\boldsymbol{X}_{t-1})\ d\boldsymbol{z}_t d\boldsymbol{z}_{t-1} \\ &= p(\boldsymbol{X}_{t-1}) \int p(\boldsymbol{x}_t|\boldsymbol{z}_t) \mathcal{N}(\boldsymbol{z}_t; \boldsymbol{\mu}_t', \boldsymbol{\Sigma}_t')\ d\boldsymbol{z}_t\ .\end{aligned} \tag{5.14}$$

Also,

$$\begin{aligned}&\frac{\Delta t}{2}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{z}_t)^T \boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{z}_t) + \frac{1}{2}(\boldsymbol{z}_t - \boldsymbol{\mu}_t')^T (\boldsymbol{\Sigma}_t')^{-1}(\boldsymbol{z}_t - \boldsymbol{\mu}_t') \\ =\ &\frac{\Delta t}{2}\boldsymbol{x}_t^T \boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{x}_t + \frac{1}{2}(\boldsymbol{\mu}_t')^T(\boldsymbol{\Sigma}_t')^{-1}(\boldsymbol{\mu}_t') + \frac{1}{2}\boldsymbol{z}^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{z}_t - \boldsymbol{z}^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t \\ =\ &\frac{\Delta t}{2}\boldsymbol{x}_t^T \boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{x}_t + \frac{1}{2}(\boldsymbol{\mu}_t')^T(\boldsymbol{\Sigma}_t')^{-1}(\boldsymbol{\mu}_t') - \frac{1}{2}\boldsymbol{\mu}_t^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t + \frac{1}{2}(\boldsymbol{z} - \boldsymbol{\mu}_t)^T \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{z}_t - \boldsymbol{\mu}_t)\ .\end{aligned} \tag{5.15}$$

Additionally, up to order Δt,

$$
\begin{aligned}
&(\boldsymbol{\mu}'_t)^T(\boldsymbol{\Sigma}'_t)^{-1}(\boldsymbol{\mu}'_t) \\
&= (\boldsymbol{\mu}_t - \boldsymbol{\Sigma}_t\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)\Delta t)^T \left(\boldsymbol{\Sigma}_t^{-1} - \boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\Delta t\right)(\boldsymbol{\mu}_t - \boldsymbol{\Sigma}_t\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)\Delta t) \\
&\approx \boldsymbol{\mu}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t + \left\{-\boldsymbol{\mu}_t^T\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\boldsymbol{\mu}_t - 2\boldsymbol{\mu}_t^T\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)\right\}\Delta t \\
&= \boldsymbol{\mu}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t + \left\{\boldsymbol{\mu}_t^T\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\boldsymbol{\mu}_t - \boldsymbol{\mu}_t^T\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{x}_t\right\}\Delta t \ ,
\end{aligned}
$$

which allows us to write

$$
\frac{\Delta t}{2}\boldsymbol{x}_t^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{x}_t + \frac{1}{2}(\boldsymbol{\mu}'_t)^T(\boldsymbol{\Sigma}'_t)^{-1}(\boldsymbol{\mu}'_t) - \frac{1}{2}\boldsymbol{\mu}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t \approx \frac{\Delta t}{2}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t) \ .
$$

Combining these results,

$$
\begin{aligned}
\frac{p(\boldsymbol{X}_t)}{p(\boldsymbol{X}_{t-1})} &= \exp\left\{-\frac{\Delta t}{2}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)\right\}\sqrt{\frac{\det[\boldsymbol{\Sigma}_t(\boldsymbol{\Sigma}'_t)^{-1}]}{(2\pi)^D\det\frac{\boldsymbol{\Sigma}_{obs}}{\Delta t}}}\int p(\boldsymbol{z}_t|\boldsymbol{X}_t)\ d\boldsymbol{z}_t \\
&= \exp\left\{-\frac{\Delta t}{2}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)\right\}\sqrt{\frac{\det\left[\boldsymbol{I} - \boldsymbol{\Sigma}_t\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\Delta t\right]}{(2\pi)^D\det\frac{\boldsymbol{\Sigma}_{obs}}{\Delta t}}} \ .
\end{aligned}
$$

Here, we make two comments. First, we can expand the determinant to first order in Δt:

$$
\det\left[\boldsymbol{I} - \boldsymbol{\Sigma}_t\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\Delta t\right] \approx 1 - \mathrm{tr}(\boldsymbol{\Sigma}_t\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M})\Delta t \ . \tag{5.16}
$$

The square root of this determinant is approximately $1 - \frac{1}{2}\mathrm{tr}(\boldsymbol{\Sigma}_t\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M})\Delta t$.

The next comment is subtle but important. Consider the special case that we are comparing two models which only differ in $\boldsymbol{M}$. For two such models, the $(2\pi)^D\det\frac{\boldsymbol{\Sigma}_{obs}}{\Delta t}$ term in the denominator is the same, and so disappears when we take the difference of their log-likelihoods. Hence, it is reasonable to ignore it. Taking the logarithm of our final result,

$$
\log p(\boldsymbol{X}_t) = \log p(\boldsymbol{X}_{t-1}) - \frac{\Delta t}{2}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t) - \frac{1}{2}\mathrm{tr}(\boldsymbol{\Sigma}_t\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M})\Delta t
$$

where we have abused notation by dropping the aforementioned term. In continuous time,

$$
\frac{d}{dt}\log p(\boldsymbol{X}_t) = -\frac{1}{2}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)^T\boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t) - \frac{1}{2}\mathrm{tr}(\boldsymbol{M}^T\boldsymbol{\Sigma}_{obs}^{-1}\boldsymbol{M}\boldsymbol{\Sigma}_t) \ . \tag{5.17}
$$

5.4.2 Full model equations

For the causal inference model of interest, the defining dynamics equations are

$$
\begin{aligned}
\dot{f}_t &= -w_t + v_t \\
\dot{w}_t &= 0 \\
\dot{v}_t &= 0 \ .
\end{aligned} \tag{5.18}
$$

Equivalently, the corresponding $\boldsymbol{A}$ matrix is (partly due to a weird sign thing; might decide to switch sign of $\boldsymbol{A}$ above)

$$\boldsymbol{A} := \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{5.19}$$

The observation model is defined by

$$\boldsymbol{M} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{\Sigma}_{obs} := \begin{pmatrix} \sigma_f^2 & 0 \\ 0 & \sigma_w^2 \end{pmatrix} \ . \tag{5.20}$$

The relevant ODEs are

$$\begin{aligned} \dot{\boldsymbol{\mu}}_t &= -\boldsymbol{A}\boldsymbol{\mu}_t + \boldsymbol{\Sigma}_t \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t) \\ \dot{\boldsymbol{\Sigma}}_t &= -(\boldsymbol{A}\boldsymbol{\Sigma}_t + \boldsymbol{\Sigma}_t \boldsymbol{A}^T) - \boldsymbol{\Sigma}_t \boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M} \boldsymbol{\Sigma}_t \\ \frac{d}{dt} \log p(\boldsymbol{X}_t) &= -\frac{1}{2}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t)^T \boldsymbol{\Sigma}_{obs}^{-1}(\boldsymbol{x}_t - \boldsymbol{M}\boldsymbol{\mu}_t) - \frac{1}{2}\mathrm{tr}(\boldsymbol{M}^T \boldsymbol{\Sigma}_{obs}^{-1} \boldsymbol{M} \boldsymbol{\Sigma}_t) \ . \end{aligned} \tag{5.21}$$

5.5 Simplifications for rank-1 model comparison

Chapter 6

Population codes for causal inference

6.1 Why PPCs?

6.2 Natural parameter estimates have correct distribution

6.3 Hierarchical PPCs

6.4 Simple CI algorithm using PPCs

6.5 Full CI algorithm using PPCs

Chapter 7

Fitting

7.1 Steering models

Need to be mindful of the fact that the f_0 relevant for inference not the same as the f_a (sometimes I use f_0) relevant for control. The relationship between the two is that

$$f_a = f_0 + (v - w)t_{obs} + vt_{wait} + \frac{a}{2}t_{wait}^2 \ . \tag{7.1}$$

Here, t_{wait} denotes the empirical amount of time post-firefly-disappearance until the subject begins exerting control. T denotes the amount of time between the beginning of control and the end of the trial (different from trial to trial!).

7.1.1 Control cost only

Control cost only:

$$u_t^* = \frac{f_a + vT + \frac{a}{2}T^2 - x_t}{\alpha + T - t} \ . \tag{7.2}$$

In terms of f_0:

$$u_t^* = -\frac{1}{\alpha + T - t}x_t + \frac{1}{\alpha + T - t}f_0 - \frac{t_{obs}}{\alpha + T - t}w + \frac{t_{obs} + t_{wait} + T}{\alpha + T - t}v + \frac{1}{2}\frac{t_{wait}^2 + T^2}{\alpha + T - t}a \ . \tag{7.3}$$

In shorthand:

$$u_t^* = w_{tx}x_t + w_{ta}a + w_{tv}v + \boldsymbol{w}_{to} \cdot \boldsymbol{z}_o \ . \tag{7.4}$$

7.1.2 Control and proximity cost

$$xxx \tag{7.5}$$

$$\frac{N\tau}{D} = \frac{-\left(1 - \frac{\alpha}{\tau}\right)e^{-T/\tau}\Delta x_a - \alpha v + \tau^2\left[\left(1 - \frac{\alpha}{\tau}\right)e^{-T/\tau} - \left(1 + \frac{\alpha}{\tau^2}T\right)\right]\ddot{f}}{\left(1 + \frac{\alpha}{\tau}\right)e^{T/\tau} - \left(1 - \frac{\alpha}{\tau}\right)e^{-T/\tau}} \tag{7.6}$$

$$u_0 = 2\frac{N}{D} + \ddot{f}\tau - \frac{\Delta x_a}{\tau} + \dot{f} \tag{7.7}$$

7.2 Preliminaries

Before we can compute the objective, we need to determine a certain distribution. Consider the distribution

$$p(\hat{v}, \hat{\boldsymbol{z}}_s|\boldsymbol{z}) := \int p(\hat{v}, \hat{\boldsymbol{z}}_s|\boldsymbol{y})p(\boldsymbol{y}|\boldsymbol{z})\ d\boldsymbol{y} \tag{7.8}$$

where $\hat{\boldsymbol{z}}_s$ refers to the posterior mean under the stationary model. This distribution is analogous to the distribution of posterior means under the moving model, and has the same number of variables.

Because it is a convolution of normal distributions, it is normal, so we only need to compute its mean and covariance. Assume that

$$\hat{v} = \boldsymbol{w} \cdot \boldsymbol{y} \qquad\qquad \hat{\boldsymbol{z}}_s = \boldsymbol{W}_{stat}\boldsymbol{y}\ . \tag{7.9}$$

Then

$$\mathbb{E}[\hat{v}|\boldsymbol{z}] = \boldsymbol{w}^T\boldsymbol{M}\boldsymbol{z} \qquad\qquad \hat{\boldsymbol{z}}_s = \boldsymbol{W}_{stat}\boldsymbol{M}\boldsymbol{z}\ . \tag{7.10}$$

Assuming the covariance of $\boldsymbol{y}$ is $\boldsymbol{\Sigma}_{obs}$, we also have

$$\begin{aligned} \Sigma_{vv} &= \boldsymbol{w}^T\boldsymbol{\Sigma}_{obs}\boldsymbol{w} \\ \boldsymbol{\Sigma}_{oo} &= \boldsymbol{W}_{stat}\boldsymbol{\Sigma}_{obs}\boldsymbol{W}_{stat}^T \\ \boldsymbol{\Sigma}_{ov} &= \boldsymbol{W}_{stat}\boldsymbol{\Sigma}_{obs}\boldsymbol{w} \end{aligned} \tag{7.11}$$

7.2.1 Explicit result for no self-motion

7.3 Explicit result for self-motion model

7.4 Deriving the objective

The overall likelihood of a particular steering trajectory $\{u_t\}$ given a task condition $\boldsymbol{z}$ is

$$L(\{u_t\}|\boldsymbol{z}) = L_{move} + L_{stat} \tag{7.12}$$

where

$$\begin{aligned} L_{move} &= \int p(\{u_t\}|\hat{\boldsymbol{z}}_m, \hat{v}, \hat{a})p(\hat{a}|a)p_{C=1}(\hat{v})p(\hat{v}, \hat{\boldsymbol{z}}_m|\boldsymbol{y})p(\boldsymbol{y}|\boldsymbol{z})\ d\hat{a}d\hat{\boldsymbol{z}}_m d\hat{v}d\boldsymbol{y} \\ L_{stat} &= \int p(\{u_t\}|\hat{\boldsymbol{z}}_s, 0, 0)p_{C=0}(\hat{v})p(\hat{v}, \hat{\boldsymbol{z}}_s|\boldsymbol{y})p(\boldsymbol{y}|\boldsymbol{z})\ d\hat{\boldsymbol{z}}_s d\hat{v}d\boldsymbol{y}\ . \end{aligned} \tag{7.13}$$

Here, $\hat{v}$ denotes the posterior mean under the moving model, $\hat{\boldsymbol{z}}_m$ denotes the posterior mean of the other variables under the moving model (2 variables if there is self-motion, and 1 variable otherwise), and $\hat{\boldsymbol{z}}_s$ denotes the posterior mean of the other variables under the stationary model. Not including the observation summary statistics $\boldsymbol{y}$, we must integrate over either 3 or 2 variables to compute each integral.

Assume that there is (isotropic, state-independent) steering noise, and that control is normally distributed about the optimal control. In all models we consider, the optimal control is linear in the relevant variables—here, $\hat{v}$ and $\hat{\boldsymbol{z}}_o$. In symbols:

$$p(u_t|\hat{\boldsymbol{z}}, x_t) = \mathcal{N}(u_t; w_{tx}x_t + w_{tv}\hat{v} + \boldsymbol{w}_{to}\hat{\boldsymbol{z}}_o, g^2/\Delta t) \ . \tag{7.14}$$

Assume that the full collection of estimates $\hat{\boldsymbol{z}}$ (which refers to either $\hat{\boldsymbol{z}}_m$ and $\hat{v}$, or $\hat{\boldsymbol{z}}_s$ and $\hat{v}$, depending on which integral we are calculating) is normally distributed. That is, assume

$$p(\hat{\boldsymbol{z}}|\boldsymbol{z}) = \mathcal{N}(\hat{\boldsymbol{z}}; \boldsymbol{M}\boldsymbol{z}, \boldsymbol{\Sigma}) \ . \tag{7.15}$$

Assume we can break down the estimate distribution, with

$$\begin{aligned} &\frac{1}{2}(\hat{\boldsymbol{z}} - \boldsymbol{M}\boldsymbol{z})^T\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{z}} - \boldsymbol{M}\boldsymbol{z}) \\ &= \frac{1}{2}(\hat{v} - \boldsymbol{M}_v\boldsymbol{z})^2\Sigma_{vv}^{-1} + \frac{1}{2}(\hat{\boldsymbol{z}}_o - \boldsymbol{M}_o\boldsymbol{z})^T\boldsymbol{\Sigma}_{oo}^{-1}(\hat{\boldsymbol{z}}_o - \boldsymbol{M}_o\boldsymbol{z}) + (\hat{v} - \boldsymbol{M}_v\boldsymbol{z})\boldsymbol{\Sigma}_{vo}^{-1}(\hat{\boldsymbol{z}}_o - \boldsymbol{M}_o\boldsymbol{z}) \end{aligned} \tag{7.16}$$

Similarly,

$$\begin{aligned} &\frac{\Delta t}{2g^2}(u_t - w_{tx}x_t - w_{tv}\hat{v} - \boldsymbol{w}_{to}\hat{\boldsymbol{z}}_o)^2 \\ &= \frac{\Delta t}{2g^2}(u_t - w_{tx}x_t)^2 + \frac{\Delta t}{2g^2}w_{tv}^2\hat{v}^2 + \frac{\Delta t}{2g^2}\hat{\boldsymbol{z}}_o^T\boldsymbol{w}_{to}\boldsymbol{w}_{to}^T\hat{\boldsymbol{z}}_o - \frac{\Delta t}{g^2}(u_t - w_{tx}x_t)\left[w_{tv}\hat{v} + \boldsymbol{w}_{to}\hat{\boldsymbol{z}}_o\right] + \frac{\Delta t}{g^2}w_{tv}\hat{v}\boldsymbol{w}_{to}\hat{\boldsymbol{z}}_o \end{aligned} \tag{7.17}$$

In order to evaluate the likelihood, we need to compute a few integrals. Consider integrating with respect to $\hat{\boldsymbol{z}}_o$. The relevant terms are

$$\begin{aligned} I_1 = \int d\hat{\boldsymbol{z}}_o \ \exp\Bigg\{ &-\frac{1}{2}\hat{\boldsymbol{z}}_o^T\left[\boldsymbol{\Sigma}_{oo}^{-1} + \sum_t \frac{\boldsymbol{w}_{to}\boldsymbol{w}_{to}^T \ \Delta t}{g^2}\right]\hat{\boldsymbol{z}}_o \\ &+\hat{\boldsymbol{z}}_o \cdot \left[\boldsymbol{\Sigma}_{oo}^{-1}\boldsymbol{M}_o\boldsymbol{z} + \boldsymbol{\Sigma}_{vo}^{-1}(\boldsymbol{M}_v\boldsymbol{z} - \hat{v}) + \sum_t \frac{\boldsymbol{w}_{to}(u_t - w_{tx}x_t - w_{tv}\hat{v}) \ \Delta t}{g^2}\right]\Bigg\} \end{aligned} \tag{7.18}$$

This is just a (multivariate) Gaussian integral. Define

$$\begin{aligned} \boldsymbol{A} &:= \boldsymbol{\Sigma}_{oo}^{-1} + \sum_t \frac{\boldsymbol{w}_{to}\boldsymbol{w}_{to}^T \ \Delta t}{g^2} \\ \boldsymbol{J} &:= \boldsymbol{\Sigma}_{oo}^{-1}\boldsymbol{M}_o\boldsymbol{z} + \boldsymbol{\Sigma}_{vo}^{-1}(\boldsymbol{M}_v\boldsymbol{z} - \hat{v}) + \sum_t \frac{\boldsymbol{w}_{to}(u_t - w_{tx}x_t - w_{tv}\hat{v}) \ \Delta t}{g^2} \end{aligned} \tag{7.19}$$

The answer is

$$I_1 = \sqrt{\frac{(2\pi)^D}{\det \boldsymbol{A}}} \exp\left\{\frac{1}{2}\boldsymbol{J}^T\boldsymbol{A}^{-1}\boldsymbol{J}\right\} \tag{7.20}$$

where $D = 2$ (for model with self-motion) or $D = 1$ (for model without self-motion).

For convenience, write $\boldsymbol{J} = -\boldsymbol{J}_v\hat{v} + \boldsymbol{J}_r$ where

$$\begin{aligned}
\boldsymbol{J}_v &:= \boldsymbol{\Sigma}_{vo}^{-1} + \sum_t w_{tv}\frac{\boldsymbol{w}_{to}\Delta t}{g^2} \\
\boldsymbol{J}_r &:= \boldsymbol{\Sigma}_{oo}^{-1}\boldsymbol{M}_o\boldsymbol{z} + \boldsymbol{\Sigma}_{vo}^{-1}\boldsymbol{M}_v\boldsymbol{z} + \sum_t \frac{\boldsymbol{w}_{to}(u_t - w_{tx}x_t)\ \Delta t}{g^2}
\end{aligned} \tag{7.21}$$

We can now write I_1 as

$$I_1 = \sqrt{\frac{(2\pi)^D}{\det \boldsymbol{A}}} \exp\left\{\frac{1}{2}\hat{v}^2\boldsymbol{J}_v^T\boldsymbol{A}^{-1}\boldsymbol{J}_v - \boldsymbol{J}_r^T\boldsymbol{A}^{-1}\boldsymbol{J}_v\hat{v} + \frac{1}{2}\boldsymbol{J}_r^T\boldsymbol{A}^{-1}\boldsymbol{J}_r\right\} \tag{7.22}$$

The final integral we have to do (ignoring constant prefactors) is

$$\begin{aligned}
I_2 = &\int d\hat{v}\ \exp\left\{-\frac{1}{2}\hat{v}^2\left[\Sigma_{vv}^{-1} + \sum_t \frac{w_{tv}^2\Delta t}{g^2} - \boldsymbol{J}_v^T\boldsymbol{A}^{-1}\boldsymbol{J}_v\right]\right. \\
&\left. + \left[\Sigma_{vv}^{-1}\boldsymbol{M}_v\boldsymbol{z} + \boldsymbol{\Sigma}_{vo}^{-1}\boldsymbol{M}_o\boldsymbol{z} + \sum_t \frac{(u_t - w_{tx}x_t)w_{tv}\Delta t}{g^2} - \boldsymbol{J}_r^T\boldsymbol{A}^{-1}\boldsymbol{J}_v\right]\hat{v}\right\}
\end{aligned} \tag{7.23}$$

This is an almost Gaussian integral. Note that

$$\begin{aligned}
\int_q^\infty e^{-\frac{a}{2}x^2+Jx}dx &= \sqrt{\frac{\pi}{2a}}e^{\frac{J^2}{2a}}\left[1 + \operatorname{erf}\left(\frac{J-aq}{\sqrt{2a}}\right)\right] \\
\int_{-\infty}^{-q} e^{-\frac{a}{2}x^2+Jx}dx &= \sqrt{\frac{\pi}{2a}}e^{\frac{J^2}{2a}}\left[1 - \operatorname{erf}\left(\frac{J+aq}{\sqrt{2a}}\right)\right] \\
\int_q^\infty e^{-\frac{a}{2}x^2+Jx}dx + \int_{-\infty}^{-q} e^{-\frac{a}{2}x^2+Jx}dx &= \sqrt{\frac{2\pi}{a}}e^{\frac{J^2}{2a}}\left[1 + \frac{1}{2}\operatorname{erf}\left(\frac{J-aq}{\sqrt{2a}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{J+aq}{\sqrt{2a}}\right)\right] \\
\int_{-q}^{q} e^{-\frac{a}{2}x^2+Jx}dx &= \sqrt{\frac{2\pi}{a}}e^{\frac{J^2}{2a}}\left[-\frac{1}{2}\operatorname{erf}\left(\frac{J-aq}{\sqrt{2a}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{J+aq}{\sqrt{2a}}\right)\right]
\end{aligned} \tag{7.24}$$

The overall prefactor is

$$C = \left(\sqrt{\frac{\Delta t}{2\pi g^2}}\right)^T \sqrt{\frac{1}{\det \boldsymbol{A}\ \ \det \boldsymbol{\Sigma}}} \exp\left\{-\frac{1}{2}\boldsymbol{z}^T\boldsymbol{M}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{z} - \sum_t \frac{(u_t - w_{tx}x_t)^2\Delta t}{2g^2} + \frac{1}{2}\boldsymbol{J}_r^T\boldsymbol{A}^{-1}\boldsymbol{J}_r\right\} \tag{7.25}$$

Our final answer is

$$
\begin{aligned}
L_{move} &= \left(\sqrt{\frac{\Delta t}{2\pi g^2}}\right)^T \sqrt{\frac{1}{k_m\ \det \boldsymbol{A}_m\ \det \boldsymbol{\Sigma}_m}} e^{-Z_m} \left[1+\frac{1}{2}\mathrm{erf}\left(\frac{J_m-k_m q}{\sqrt{2k_m}}\right)-\frac{1}{2}\mathrm{erf}\left(\frac{J_m+k_m q}{\sqrt{2k_m}}\right)\right] \\
L_{stat} &= \left(\sqrt{\frac{\Delta t}{2\pi g^2}}\right)^T \sqrt{\frac{1}{k_s\ \det \boldsymbol{A}_s\ \det \boldsymbol{\Sigma}_s}} e^{-Z_s} \left[-\frac{1}{2}\mathrm{erf}\left(\frac{J_s-k_s q}{\sqrt{2k_s}}\right)+\frac{1}{2}\mathrm{erf}\left(\frac{J_s+k_s q}{\sqrt{2k_s}}\right)\right] .
\end{aligned}
\tag{7.26}
$$

where

$$
\begin{aligned}
Z &:= \frac{1}{2}\boldsymbol{z}^T \boldsymbol{M}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{M}\boldsymbol{z} + \sum_t \frac{(u_t - w_{tx}x_t)^2 \Delta t}{2g^2} - \frac{1}{2}\boldsymbol{J}_r^T \boldsymbol{A}^{-1} \boldsymbol{J}_r - \frac{J^2}{2k} \\
k &:= \Sigma_{vv}^{-1} + \sum_t \frac{w_{tv}^2 \Delta t}{g^2} - \boldsymbol{J}_v^T \boldsymbol{A}^{-1} \boldsymbol{J}_v \\
J &:= \Sigma_{vv}^{-1} \boldsymbol{M}_v \boldsymbol{z} + \boldsymbol{\Sigma}_{vo}^{-1} \boldsymbol{M}_o \boldsymbol{z} + \sum_t \frac{(u_t - w_{tx}x_t) w_{tv} \Delta t}{g^2} - \boldsymbol{J}_r^T \boldsymbol{A}^{-1} \boldsymbol{J}_v \\
\boldsymbol{J}_v &:= \boldsymbol{\Sigma}_{vo}^{-1} + \sum_t w_{tv} \frac{\boldsymbol{w}_{to} \Delta t}{g^2} \\
\boldsymbol{J}_r &:= \boldsymbol{\Sigma}_{oo}^{-1} \boldsymbol{M}_o \boldsymbol{z} + \boldsymbol{\Sigma}_{vo}^{-1} \boldsymbol{M}_v \boldsymbol{z} + \sum_t \frac{\boldsymbol{w}_{to}(u_t - w_{tx}x_t)\ \Delta t}{g^2} \\
\boldsymbol{A} &:= \boldsymbol{\Sigma}_{oo}^{-1} + \sum_t \frac{\boldsymbol{w}_{to} \boldsymbol{w}_{to}^T\ \Delta t}{g^2}
\end{aligned}
\tag{7.27}
$$

The minus log-likelihood is

$$
-\log L = \frac{T}{2}\log\bigl(2\pi g^2/\Delta t\bigr) + \log(L'_{move} + L'_{stat}) \tag{7.28}
$$

7.5 Explicit quantities for each control model

7.5.1 Control cost only

$$
u_t^* = -\frac{1}{\alpha+T-t}x_t + \frac{1}{\alpha+T-t}f_0 - \frac{t_{obs}}{\alpha+T-t}w + \frac{t_{obs}+t_{wait}+T}{\alpha+T-t}v + \frac{1}{2}\frac{t_{wait}^2+T^2}{\alpha+T-t}a\ . \tag{7.29}
$$

In shorthand:

$$
u_t^* = w_{tx}x_t + w_{ta}a + w_{tv}v + \boldsymbol{w}_{to} \cdot \boldsymbol{z}_o\ . \tag{7.30}
$$

Relevant quantities:

$$\begin{aligned}
\boldsymbol{w}_{to} &= \frac{1}{\alpha+T-t}\begin{pmatrix}1 & -t_{obs} & t_{obs}+t_{wait}+T & \frac{1}{2}(t_{wait}^2+T^2)\end{pmatrix}^T = \frac{1}{\alpha+T-t}\boldsymbol{w}_o \\
\sum_t \frac{\boldsymbol{w}_{to}\boldsymbol{w}_{to}^T\,\Delta t}{g^2} &= \frac{1}{g^2}\frac{T}{\alpha(\alpha+T)}\boldsymbol{w}_o\boldsymbol{w}_o^T \\
\det \boldsymbol{A} &= \det \boldsymbol{\Sigma}_{oo}^{-1}\left[1+\frac{1}{g^2}\frac{T}{\alpha(\alpha+T)}\boldsymbol{w}_o^T\boldsymbol{\Sigma}_{oo}\boldsymbol{w}_o\right] \\
\boldsymbol{A}^{-1} &= \boldsymbol{\Sigma}_{oo} - \frac{1}{g^2}\frac{T}{\alpha(\alpha+T)}\frac{\boldsymbol{\Sigma}_{oo}\boldsymbol{w}_o\boldsymbol{w}_o^T\boldsymbol{\Sigma}_{oo}}{1+\frac{1}{g^2}\frac{T}{\alpha(\alpha+T)}\boldsymbol{w}_o^T\boldsymbol{\Sigma}_{oo}\boldsymbol{w}_o}
\end{aligned} \tag{7.31}$$

7.6 Explicit objective, absence of self-motion

7.7 Problem, redux

$$p(\{u_t\}|\boldsymbol{z}) = \sum_C \int p(\{u_t\}|\hat{\boldsymbol{z}})p(\hat{\boldsymbol{z}}|C, \hat{\boldsymbol{z}}_{move}, \hat{\boldsymbol{z}}_{stat})p(C|\hat{\boldsymbol{z}}_{move})p(\hat{\boldsymbol{z}}_{move}, \hat{\boldsymbol{z}}_{stat}|\boldsymbol{z})\ d\hat{\boldsymbol{z}}_{move}d\hat{\boldsymbol{z}}_{stat}d\hat{\boldsymbol{z}}$$
$$= \int p(\{u_t\}|\hat{\boldsymbol{z}}_{move})p_{C=1}(\hat{\boldsymbol{z}}_{move})p(\hat{\boldsymbol{z}}_{move}|\boldsymbol{z})\ d\hat{\boldsymbol{z}}_{move} + \int p(\{u_t\}|\hat{\boldsymbol{z}}_{stat})p_{C=0}(\hat{\boldsymbol{z}}_{move})p(\hat{\boldsymbol{z}}_{move}, \hat{\boldsymbol{z}}_{stat}|\boldsymbol{z})\ d\hat{\boldsymbol{z}}_{move}d\hat{\boldsymbol{z}}_{stat} \tag{7.32}$$

Let's be more concrete. Assume

$$\begin{aligned} p(u_t|\hat{\boldsymbol{z}}, x_t) &= \mathcal{N}(u_t; \boldsymbol{U}_t\hat{\boldsymbol{z}}, g^2\Delta t) \\ p_{C=1}(\hat{\boldsymbol{z}}_{move}) &= \Theta(\hat{v}_{move} - q) + \Theta(-q - \hat{v}_{move}) \\ p(\hat{\boldsymbol{z}}_{move}|\boldsymbol{z}) &= \mathcal{N}(\boldsymbol{M}_{move}\boldsymbol{z}, \boldsymbol{\Sigma}_{move}) \\ p(\hat{\boldsymbol{z}}_{move}, \hat{\boldsymbol{z}}_{stat}|\boldsymbol{z}) &= \mathcal{N}(\boldsymbol{M}\boldsymbol{z}, \boldsymbol{\Sigma}) \end{aligned} \tag{7.33}$$

7.8 Objective for steering only

Assume steering noise:

$$p(x_{t+1}|x_t, u_t) = \delta(x_{t+1} - x_t - u_t\Delta t) \tag{7.34}$$

Assume subjects pick the optimal control given their starting state and firefly belief:

$$p(u_t|\boldsymbol{z}, x_t) = \mathcal{N}(u_t; u_*(\boldsymbol{z}, x_t), g^2\Delta t) \tag{7.35}$$

Then the likelihood is:

$$p(\{u_t\}|\boldsymbol{z}) = p(u_{T-1}|\boldsymbol{z}, x_{T-1}) \cdots p(u_0|\boldsymbol{z}, x_0) \tag{7.36}$$

7.9 Problem and general concerns

Decision-making model

$$C \xrightarrow{p(\boldsymbol{z}|C)} \boldsymbol{z} \xrightarrow{p(\{\boldsymbol{x}_t\}|\boldsymbol{z})} \{\boldsymbol{x}_t\} \xrightarrow{p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \hat{C}|\{\boldsymbol{x}_t\})} \boldsymbol{\mu}, \boldsymbol{\Sigma}, \hat{C} \to \{u_t\} \tag{7.37}$$

Probability (control, belief, observation):

$$p(\{u_t\}|\boldsymbol{z}) = \int p(\{u_t\}|\boldsymbol{\mu}, \hat{C})\ p(\boldsymbol{\mu}, \hat{C}|\{\boldsymbol{x}_t\})\ p(\{\boldsymbol{x}_t\}|\boldsymbol{z})\ d\boldsymbol{\mu}d\{\boldsymbol{x}_t\} \tag{7.38}$$

Control.

$$p(\{u_t\}|\boldsymbol{\mu}, \hat{C}) = \prod_{t=0}^{T-1} p(u_t|x_t, t, \boldsymbol{\mu}, \hat{C}) = \prod_{t=0}^{T-1} \mathcal{N}(\ u_t; u_t^*(x_t, t, \boldsymbol{\mu}, \hat{C}), \sigma_u^2/\Delta t\ ) \tag{7.39}$$

Belief.

$$p(\hat{C}|\{\boldsymbol{x}_t\}) = \begin{cases} \text{object moving} & |\mu_v| \geq b \\ \text{object stationary} & |\mu_v| < b \end{cases} \qquad \boldsymbol{\mu} = \boldsymbol{A}_{\hat{C}}\{\boldsymbol{x}_t\} \tag{7.40}$$

Observation.

$$p(\{\boldsymbol{x}_t\}|\boldsymbol{z}) = \prod_{t=1}^{T} \mathcal{N}(\boldsymbol{x}_t; \boldsymbol{o}(t, \boldsymbol{z}), \boldsymbol{\Sigma}_{obs}) \tag{7.41}$$

7.10 Fitting moving-stationary judgments

7.11 Fitting steering endpoints

7.12 Fitting control trajectories

Chapter 8

Numerical concerns

8.1 Data preprocessing

Given: sequence of inputs $\{u_t\}$, sequence of positions $\{x_t\}$, wait time t_{wait}, vector of times $\{t\}$, trial-specific dt.

Assume each vector is the same length, so that each quantity can be in a large (num trials, num time pts) array.

To ensure each sequence is the same length, we do not use the raw data, but a spline interpolation of it with a similar framerate. This also corrects for trials where the trial time is recorded to be slightly different.

Trials with and without self-motion are NOT different data sets fit with different models, since the same underlying parameters are shared. The true objective is a sum of the objectives for both types of data.

Appendix A

Useful mathematical facts

A.1 Useful mathematical preliminaries

We will find a few identities useful. One is the Woodbury identity, which states that

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1} \; . \tag{A.1}$$

Another is the fact that, if A and B are invertible matrices and $A - B$ is rank 1, then

$$A^{-1} - B^{-1} = -\frac{B^{-1}(A-B)B^{-1}}{1 + \operatorname{tr}(B^{-1}(A-B))} \; . \tag{A.2}$$

This identity can be rewritten using the fact that

$$1 + \operatorname{tr}(M) = \det(I + M) \tag{A.3}$$

for a rank 1 matrix M. In particular,

$$1 + \operatorname{tr}(B^{-1}(A-B)) = \det\bigl(B^{-1}A\bigr) = \det(A)/\det(B) \; . \tag{A.4}$$

Finally,

$$A^{-1} - B^{-1} = -B^{-1}(A-B)B^{-1}\,\frac{\det(B)}{\det(A)} \; . \tag{A.5}$$

Consequence of Sylvester's theorem:, if $\boldsymbol{X}$ invertible,

$$\det(\boldsymbol{X} + \boldsymbol{A}\boldsymbol{B}) = \det(\boldsymbol{X})\det\bigl(\boldsymbol{I} + \boldsymbol{B}\boldsymbol{X}^{-1}\boldsymbol{A}\bigr) \tag{A.6}$$

A.2 Exponential families