This paper designs a tax-efficient portfolio rebalancing strategy, given the historical returns of the portfolio and three equally-weighted investments. Since the timing of rebalancing is fixed on a specific date, the main concern is how to optimally execute trade to minimize the capital gain so as to achieve tax-efficiency.

1 Security-level tax-efficient strategy

1.1 Assumptions

I make the following assumptions on the calculation of capital gain tax:

Assumption 1: If the investor makes multiple purchases of a single security, she can calculate the capital gain on a transaction level. Namely, she can tell the IRS which particular security she is selling and the capital gain/loss will be calculated based on the holding duration and the price change of that particular security.

Assumption 2: The investor doesn't account for the potential change in tax rates in the future. This assumption will be relaxed later.

1.2 Security-level strategy: an example

Consider a simple example. Suppose the investor made two purchases of stock j on two previous dates (T_1, T_2) with costs (p_1^j, p_2^j) and number of shares $(\bar{s}_1^j, \bar{s}_2^j)$. We further assume that the two purchases are subject to two different tax rates (τ_1, τ_2) on net capital gains since they are held for different duration, with $\tau_1 < \tau_2$. Now, for rebalancing needs the investor wants to sell $\Delta x^j > 0$ dollars on stock j. What would be the optimal strategy? Specifically, how to allocate the liquidation of stock j on the existing position $(\bar{s}_1^j, \bar{s}_2^j)$?

Consider a case where the price of stock j has experienced a run-up from T_1 to T_2 and a slight down-turn from T_2 to the current date t: $p_1^j < p_t^j < p_2^j$. If the investor sells $\Delta s_2^j > 0$ shares of stock j purchased on date T_2, the trade triggers a capital loss and the investor doesn't have to pay tax. More importantly, the capital loss grants the investor a tax reduction of the amount

$$\tau_2 \Delta p_2^j \Delta s_2^j < 0$$

where Δp_2^j is the difference between current price p_t^j and p_2^j. Similarly, if the investor sells Δs_1^j, the trade triggers a capital gain and the capital gain tax is

$$\tau_1 \Delta p_1^j \Delta s_1^j > 0$$

In sum, the trade $(\Delta s_1^j, \Delta s_2^j)$ incurs a tax on net capital gains

$$T^j \equiv \sum_{i=1,2} \tau_i \Delta p_i^j \Delta s_i^j$$

Depending on the relative size of capital gains and losses, T^j can be positive or negative. Our object is to minimize this tax on net capital gains while achieving the desired risk exposure. A negative T^j can be used to reduce tax on other capital gains and sources of income (up to a limit) for the current fiscal year. Formally, given the value of sale Δx^j, the rebalancing problem is

$$\min_{(\Delta s_1^j, \Delta s_2^j)} \sum_{i=1,2} \tau_i \Delta p_i^j \Delta s_i^j$$

$$\text{s.t.} \quad p_t^j \sum_{i=1,2} \Delta s_i^j = \Delta x^j$$

$$\Delta s_i^j \leq \bar{s}_i^j \quad i = 1,2$$

The optimization problem is essentially a constrained linear programming problem. Since $\tau_1 \Delta p_1^j > 0$ and $\tau_2 \Delta p_2^j < 0$, the optimal solution has the following structure:

$$\Delta s_2^{j,*} = \min\left\{\frac{\Delta x^j}{p_t^j}, \bar{s}_2^j\right\}$$

$$\Delta s_1^{j,*} = \max\left\{\frac{\Delta x^j}{p_t^j} - s_2^{j,*}, \bar{s}_2^j\right\} = \max\left\{\frac{\Delta x^j}{p_t^j} - \bar{s}_2^j, 0\right\}$$

and the resulting tax on net capital gains is

$$T^{j,*} = \tau_1 \Delta p_1^j \max\left\{\frac{\Delta x^j}{p_t^j} - \bar{s}_2^j, 0\right\} + \tau_2 \Delta p_2^j \min\left\{\frac{\Delta x^j}{p_t^j}, \bar{s}_2^j\right\}$$

Intuitively, for the tax-efficiency purpose, the investor should liquidate the stock that has the highest marginal tax-adjusted loss first. In comparison, a first-in first-out liquidation strategy has *a much higher current tax loss plus the loss on future returns from tax savings.*

$$T^{j,FIFO} = \tau_1 \Delta p_1^j \min\left\{\frac{\Delta x^j}{p_t^j}, \bar{s}_1^j\right\} + \tau_2 \Delta p_2^j \max\left\{\frac{\Delta x^j}{p_t^j} - \bar{s}_1^j, 0\right\}$$

Note that if the rebalancing strategy requires us to *buy* Δx^j dollars of security j, instead of directly purchasing $\Delta x^j / p_t^j$ shares of security j, the investor should first sell all the losing positions and buy back those positions. In this case, the tax on net capital gains is

$$T^{j,*} = \bar{s}_2^j \tau_2 \Delta p_2^j < 0$$

1.3 Security-level strategy: a general case

Consider a general case where the investor has N distinct purchases of security j in her portfolio. As before we use $\bar{\boldsymbol{s}}^j \equiv \{\bar{s}_i^j\}_{i=1,..,N}$ to denote the existing positions, $\bar{\boldsymbol{p}}^j \equiv \{p_i^j\}_{i=1,...,N}$ for the costs, and $\{\tau_i^j\}_{i=1,...,N}$ for the applicable tax rates on capital gains. The objective function is still the tax on net capital gains as a function of $\{\Delta s_i^j; \bar{s}_i^j, p_i^j, \tau_i^j\}_{i=1,...,N}$ and $(\Delta x^j, p_t^j)$, except that we interpret $\Delta^+ s^j \geq 0$ ($\Delta^- s_i^j \geq 0$) as a buy (sell) order and Δx^j as the amount of investment in security j.

Conditional on $\Delta x^j < 0$, meaning that the investor sells Δx^j of security j, the minimization problem is

$$\min_{\{\Delta^- s_i^j\}_i} \sum_i \tau_i^j \Delta p_i^j \Delta^- s_i^j$$

$$\text{s.t.} \quad p_t^j \sum_i \Delta^- s_i^j = |\Delta x^j|$$

$$\Delta^- s_i^j \leq \bar{s}_i^j \quad i = 1, 2, ..., N$$

The optimal solution is to maximize the stock liquidation with the largest marginal tax-adjusted loss. First, sort all the marginal tax-adjusted losses $\{\tau_i^j \Delta p_i^j\}_{i=1,...,N}$ in an ascending order and define $\tau^j_{(i)}\Delta p^j_{(i)}$ as the i-th item. Call this *the i-th largest marginal tax-adjusted loss.* Then the optimal strategy is derived recursively:

$$\Delta s^{j,*}_{(1)} = \min\left\{\frac{|\Delta x^j|}{p^j_t}, \bar{s}^j_{(1)}\right\}$$

$$\Delta s^{j,*}_{(i)} = \min\left\{\frac{|\Delta x^j|}{p^j_t} - \sum_{h<i} \bar{s}^{j,*}_{(h)}, \bar{s}^j_{(i)}\right\} \quad i = 2, ..., N$$

Note that once $\Delta^- s^{j,*}_{(i)} = \bar{s}^j_{(i)}$, $\Delta^- s^{j,*}_{(k)} = 0$ for $k > i$, and there is no buy order.

Conditional on $\Delta x^j > 0$, the investor purchases Δx^j security j. She should first sell all the losing stocks and buy backs the sold positions plus the new position.

$$\Delta^- s^{j,*} = \sum_i \max\left\{\frac{-\Delta p^j_i}{|\Delta p^j_i|}, 0\right\} \bar{s}^j_i$$

$$\Delta^+ s^{j,*} = \sum_i \max\left\{\frac{-\Delta p^j_i}{|\Delta p^j_i|}, 0\right\} \bar{s}^j_i + \frac{\Delta x^j}{p^j_t}$$

$$T^{j,*} = \sum_i \max\left\{\frac{-\Delta p^j_i}{|\Delta p^j_i|}, 0\right\} \tau^j_i \bar{s}^j_i \Delta p^j_i$$

Note that I ignore the impact of a change in the current cost basis on the future tax-efficiency. By replacing the losing stocks with stocks at the current market price, we can achieve the maximum amount of current tax reduction but also reduce the average cost, which generates larger future capital gains and the associated tax. So dynamically, the investor has to trade off between current and future tax reduction. However, the dynamic optimization requires us to make assumptions on the timing of future rebalancing, which, in my opinion, is beyond the scope of this assignment. So I choose to leave this trade-off unintended.

The optimization gives us the tax on net capital gains as a function of the amount rebalancing:

$$T^{j,*} = F(\Delta x^j; \bar{\boldsymbol{s}}^j, \bar{\boldsymbol{p}}^j) = \begin{cases} \sum_i \max\left\{\frac{-\Delta p^j_i}{|\Delta p^j_i|}, 0\right\} \tau^j_i \Delta p^j_i \bar{s}^j_i & \text{, if } \Delta x^j > 0 \\ \sum_i \tau^j_i \Delta p^j_{(i)} \Delta s^{j,*}_{(i)} & \text{, if } \Delta x^j < 0 \end{cases} \tag{1.1}$$

For $\Delta x^j > 0$, the net capital gain is constant which only depends on $(\bar{\boldsymbol{s}}^j, \bar{\boldsymbol{p}}^j)$, while for negative Δx^j, the net capital gain is increasing in the absolute value $|\Delta x^j|$.

1.4 Discussion

Once the investor decided to liquidate a particular security in her portfolio for a certain amount, the security-level tax-efficient trading strategy becomes independent of future states and we end up solving a deterministic problem. The reason is that under my assumptions, *looking forward*, every share of security j is the same[1], while *looking backward*, they have different marginal tax-adjusted

[1] However, if we relax the second assumption, so the investor's investment horizon is long-enough that the short-term capital gain/loss becomes long-term, the strategy should be forward-looking. If we view the tax rate as a function of holding period, then the future rebalancing scheme, especially the timing, matters for the current rebalancing strategy. I will discuss this later.

losses. So we can break the rebalancing problem into two seprate tasks: we first determine the security-level strategy given the amount of rebalancing for each security, then we determine the portfolio weights taking into account the optimal security-level tax reduction. Based on the analysis above, the optimal strategy is to liquidate the portion that has the most tax reduction and the tax benefit is increasing in the amount of liquidation. But too much liquidation may cause a big deviation of the resulting portfolio weights from the target weights. We will model this trade-off at the portfolio level.

2 Portfolio-level tax-efficient rebalancing strategy

In the previous section I discuss the optimal strategy to sell a single security in order to achieve tax-efficiency, given the amount of liquidation. At the portfolio level, we need to further determine the optimal portfolio allocation given the optimal single security strategy. Namely, we want to know how much to sell the losing stock. Generally, selling more losing stocks generates a larger tax deduction for the whole portfolio. However, this might make the portfolio less diversified, especially if the losing stock is currently below the target weight. At the portfolio level, the optimal strategy is based on the trade-off between higher tax reduction and larger tracking errors.

Assumption 3: The only concern for investment-efficiency is the tracking error of the portfolio to the target index, defined as the absolute values of the difference between portfolio weights and target weights. Let $\boldsymbol{w}_t$ be the current weight of the portfolio and $\bar{\boldsymbol{w}}$ as the target weights, which is an equal-weight scheme in this assignment, the tracking error is a vector $(|w_t^j - \bar{w}^j|)_{j \le N}$.

2.1 A portfolio optimization problem

The portfolio-level tax-efficient rebalancing problem can be stated as the following portfolio optimization problem. Let the portfolio position and current prices be the state variables, denoted by $\boldsymbol{s}_t \equiv (\bar{\boldsymbol{s}}_t^j)_{j \le N}$ and $\boldsymbol{p}_t \equiv (p_t^j)_{j \le N}$, where N is the number of stocks in the portfolio.

I still keep the assumption that the tax rate is time-invariant. Given the optimal security-level strategy $\Delta \boldsymbol{s}_t^{j,*} \equiv (\Delta s_{i,t}^{j,*})_{i \le N_i^j}$, the investor tries to minimize the tax on net capital gains plus the size of the tracking error by choosing $\Delta \boldsymbol{x}_t \equiv (\Delta x^j)_{j \le N}$:

$$\min_{\Delta \boldsymbol{x}_t} \sum_j T^{j,*} + \lambda \sum_j |w_t^j - \bar{w}^j|^2$$

subject to

$$\sum_{j=1}^{N} \Delta x^j = 0$$

$$w^j = \frac{p_t^j \left[\Delta^+ s_t^{j,*} + \sum_i (\bar{s}_{i,t}^j - \Delta^- s_{i,t}^{j,*})\right]}{\sum_j p_t^j \left[\Delta^+ s_t^{j,*} + \sum_i (\bar{s}_{i,t}^j - \Delta^- s_{i,t}^{j,*})\right]} \qquad j = 1, 2, ..., N$$

$$T^{j,*} = F(\Delta x^j; \bar{\boldsymbol{s}}^j, \bar{\boldsymbol{p}}^j) = \begin{cases} \sum_i \max\left\{\dfrac{-\Delta p_i^j}{|\Delta p_i^j|}, 0\right\} \tau_i^j \Delta p_i^j \bar{s}_i^j & \text{, if } \Delta x^j > 0 \\ \sum_i \tau_i^j \Delta p_{(i)}^j \Delta s_{(i)}^{j,*} & \text{, if } \Delta x^j < 0 \end{cases}$$