1.

$$p(\lambda|y) = \frac{p(y|\lambda)\cdot p(\lambda)}{p(y)}$$

$$p(y|\lambda) = \frac{(\lambda t)^r \cdot e^{-\lambda t}}{r!}$$

$$p(y,\lambda) = p(y|\lambda)\cdot p(\lambda) = \frac{(\lambda t)^r \cdot e^{-\lambda t}}{r!} p(\lambda)$$

$$p(y) = \underbrace{p(y,\lambda_1)}_{p(y|\lambda_1)\cdot p(\lambda_1)} + \underbrace{p(y,\lambda_2)}_{p(y|\lambda_2)\cdot p(\lambda_2)} + \underbrace{p(y,\lambda_3)}_{p(y|\lambda_3)\cdot p(\lambda_3)}$$

For t:

In this question, we can regard the 380 services were done by one person, So the t should be 5, 2 = 10.

For P:

$$\begin{cases} P_{30} = \frac{1-P}{2}, \ P_{40} = P, \ P_{50} = \frac{1-P}{2} \\ P_{40} = 2P_{30} = 2P_{50} \end{cases} \rightarrow \begin{cases} P_{40} = 0.5 \\ P_{30} = P_{50} = 0.25 \end{cases}$$

$$p(y|\lambda_1)\cdot p(\lambda_1) = \frac{(10\cdot 30)^{380}\cdot e^{-10\cdot 30}}{380!}\cdot 0.25 = p(y,\lambda_1)$$

$$p(y|\lambda_2)\cdot p(\lambda_2) = \frac{(10\cdot 40)^{380}\cdot e^{-10\cdot 40}}{380!}\cdot 0.5 = p(y,\lambda_2)$$

$$p(y|\lambda_3)\cdot p(\lambda_3) = \frac{(10\cdot 50)^{380}\cdot e^{-10\cdot 50}}{380!}\cdot 0.25 = p(y,\lambda_3)$$

$$p(\lambda|y) = \frac{p(\lambda,y)}{p(y,\lambda_1)+p(y,\lambda_2)+p(y,\lambda_3)}$$

$$= \frac{2\cdot\lambda^{380}\cdot e^{-10\lambda}\cdot p(\lambda)}{0.5\cdot 30^{380}\cdot e^{-10\cdot 30} + 1\cdot 40^{380}\cdot e^{-10\cdot 40} + 50^{380}\cdot e^{-10\cdot 50}\cdot 0.5}$$

We can validate the results by using R.

2.(a)

$$f(y|a,b)=\frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}=e^{\ln\left(\frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}\right)}=e^{\ln(y^{a-1}\cdot(1-y)^{b-1})-\ln(B(a,b))}$$

$$=e^{(a-1)\ln y+(b-1)\ln(1-y)-\ln(B(a,b))}$$

According to the formula $f(x|\theta)=e^{\sum_{j=1}^{k}A_j(\theta)B_j(x)+C(x)+D(\theta)}$

So $A_1(a,b)=a$, $B_1(y)=\ln y$,

$A_2(a,b)=b$, $B_2(y)=\ln(1-y)$,

$C=-\ln y-\ln(1-y)$.

$D=-\ln(B(a,b))$

This way, the prior distribution should be:

$f(a,b|\alpha_1,\alpha_2,\alpha_3)$

$=A_1(a,b)\cdot\alpha_1+A_2(a,b)\cdot\alpha_2+\alpha_3 D(a,b)+k(\alpha_1,\alpha_2,\alpha_3)$

$=a\cdot\alpha_1+b\cdot\alpha_2+\alpha_3-\ln(B(a,b))+k(\alpha_1,\alpha_2,\alpha_3)$

$\propto\alpha_1\cdot a+\alpha_2\cdot b+\alpha_3\cdot-\ln(B(a,b))$

$\propto e^{a\cdot\alpha_1+\alpha_2\cdot b+\alpha_3\cdot-\ln(B(a,b))}$

$\propto\frac{e^{\alpha_1 a+\alpha_2\cdot b}}{B(a,b)^{\alpha_3}}$

Let's assume $\gamma=\alpha_1$, $\delta=\alpha_2$, $\theta=\alpha_3$.

The final result should be

$f(a,b|\gamma,\delta,\theta)\propto\frac{e^{\gamma a+\delta\alpha_2}}{B(a,b)^{\theta}}$.

2.(b)

posterior $\propto$ prior $\times$ likelihood

$$f''(a,b|\gamma,\delta,\theta) \propto \prod_{i=1}^{n} f(y_i|a,b) \cdot f'(a,b|\gamma,\delta,\theta)$$

$$\propto e^{\sum_{j=1}^{2} A_j(a,b)\left(\sum_{i=1}^{n} B_j(y_i)+\alpha_j\right)+(n+\alpha_3)\cdot D(a,b)}$$

$$\propto e^{A_1(a,b)\left(\sum_{i=1}^{n} B_1(y_i)+\alpha_1\right)+A_2(a,b)\left(\sum_{i=1}^{n} B_2(y_i)+\alpha_2\right)+(n+\alpha_3)D(a,b)}$$

$$\propto e^{a\left(\sum_{i=1}^{n} \ln(y_i)+\gamma\right)+b\left(\sum_{i=1}^{n} \ln(1-y_i)+\delta\right)+(n+\theta)\cdot -\ln(B(a,b))}$$

$$\propto \frac{e^{a\cdot\gamma^* + b\cdot\delta^*}}{B(a,b)^{\theta^*}}$$

$$\gamma^* = \sum_{i=1}^{n} \ln(y_i)+\gamma$$

$$\delta^* = \sum_{i=1}^{n} \ln(1-y_i)+\delta$$

$$\theta^* = \theta + n.$$