

Meeting 5:

Decision-making – the building blocks

Example

Assume you are about to buy a car and you have a budget of SEK 260 000.

You choose between a Skoda (SEK 260 000) and a Renault (SEK 150 000).

Both cars are of model 2020, and have about the same mileage. You are indifferent between them when it comes to exterior and interior looks, comfort, manoeuvring, equipment etc.



The relevant difference between them is that the Skoda is an electric car and the Renault is petrol-driven.

With today's prices of petrol and electricity the Skoda will cost you SEK 4 and the Renault will cost you SEK 9 per 10 km mixed driving.

You estimate to drive a total of 50 000 km with the car.

- Which car would you choose based on today's "fuel" prices?
- How would future prices changes affect your choice?

What characterises a decision problem?

You have a set of alternative *acts* (*actions, decisions*) to choose from.

An act can be beneficial for you depending on the *state* (*of the world, of nature*)

This depends on the *outcome* (*consequence*) of the combination of act and state.

You (normally) *know*

- which the different acts are
- which the different states are
- your relative preferences of the different consequences

You *don't know*

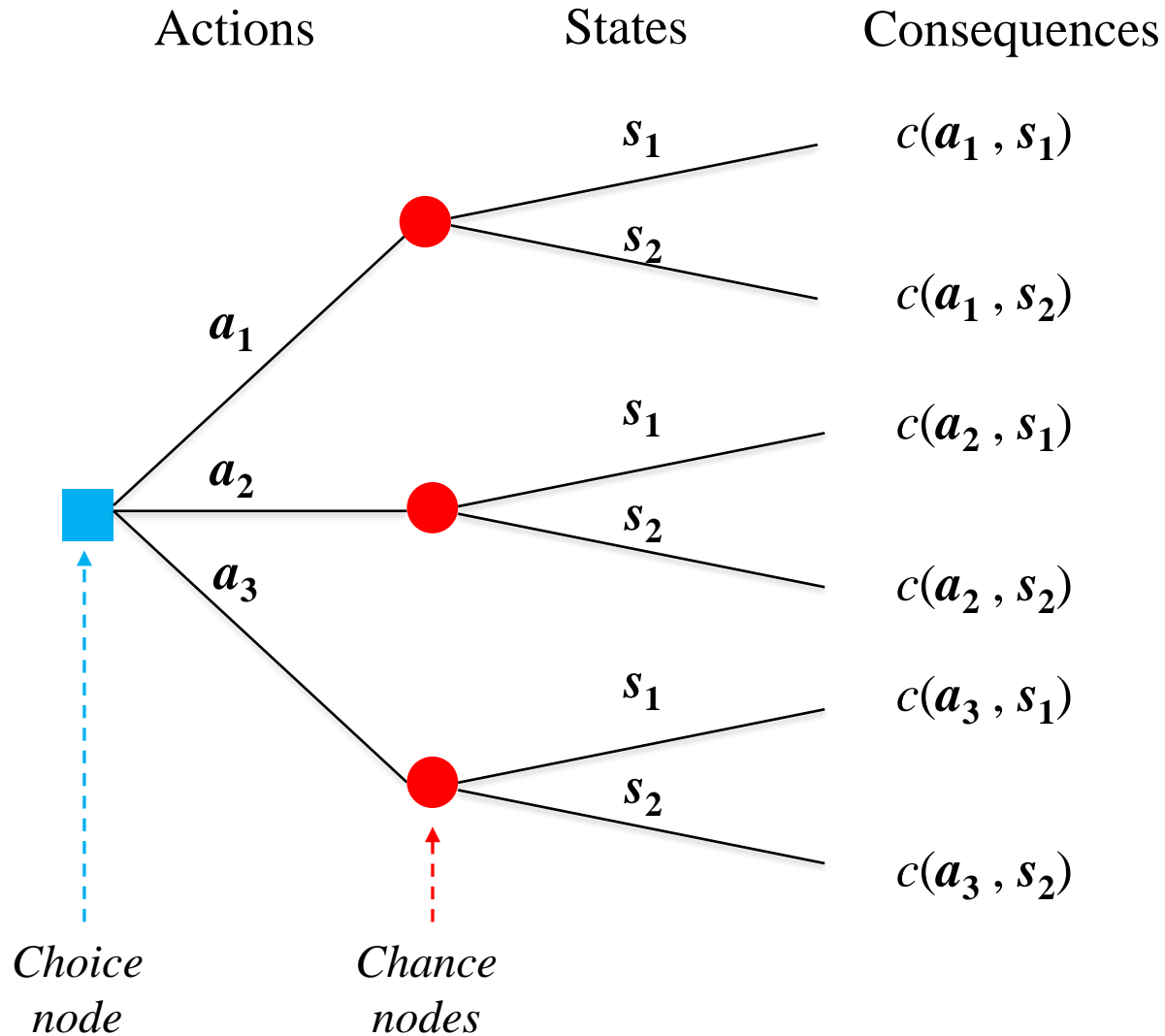
- the *true* state

You *may*

- be able to assign the *probabilities* of the different states
- have access to *data* to assist in your choice

Visualisation of a decision problem

Decision tree



Decision matrix

Actions	States	
	s_1	s_2
a_1	$c(a_1, s_1)$	$c(a_1, s_2)$
a_2	$c(a_2, s_1)$	$c(a_2, s_2)$
a_3	$c(a_3, s_1)$	$c(a_3, s_2)$

Example

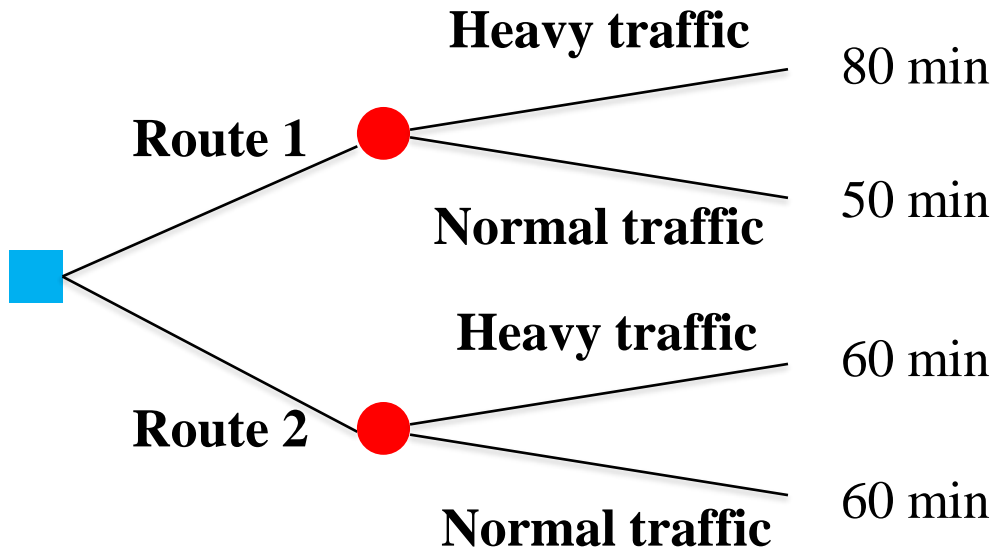
You are about to travel from city A to city B by car.

You can choose between taking route 1 (a_1) or route 2 (a_2).

If the traffic is heavy (s_1), route 1 will take you approximately 80 minutes, while route 2 will take you approximately 60 minutes.

If the traffic is normal (s_2), route 1 will take you approximately 50 minutes while route 2 will take you approximately 60 minutes.

Your aim is to get to city B as quickly as possible.



Actions	States	
	Heavy traffic	Normal traffic
Route 1	80 min	50 min
Route 2	60 min	60 min

The classical description (*not in the course book(s)*)

True state of world: /nature	θ	Unknown. The Bayesian description of this uncertainty is in terms of a random variable $\tilde{\theta}$ with prior density $p(\theta)$
Data:	\mathbf{x}	Observation of $\tilde{\mathbf{x}}$, whose pdf (or pmf) depends on θ (data is thus assumed to be available)
Decision procedure:	δ	(also referred to as a decision rule)
<u>Action (act):</u>	$\delta(\mathbf{x}), a$	The decision rule becomes an action when applied to given data \mathbf{x}
Loss function:	$L(\delta(\mathbf{x}), \theta)$	measures the <i>loss</i> from taking action $\delta(\mathbf{x})$ when θ holds
Payoff function	$R(\delta(\mathbf{x}), \theta)$	measures the <i>payoff</i> from taking action $\delta(\mathbf{x})$ when θ holds
<u>Risk function:</u>	$D(\delta, \theta) = \int_{\mathbf{x}} L(\delta(\mathbf{x}), \theta) f(\mathbf{x} \theta) d\mathbf{x} = E_{\tilde{\mathbf{x}}}(L(\delta(\tilde{\mathbf{x}}), \theta))$	
<u>Chance function</u>	$C(\delta, \theta) = \int_{\mathbf{x}} R(\delta(\mathbf{x}), \theta) f(\mathbf{x} \theta) d\mathbf{x} = E_{\tilde{\mathbf{x}}}(R(\delta(\tilde{\mathbf{x}}), \theta))$	

Expected loss/payoff with respect to variation in \mathbf{x}

Functions of the state of world/nature and the *decision rule* (and not the action)

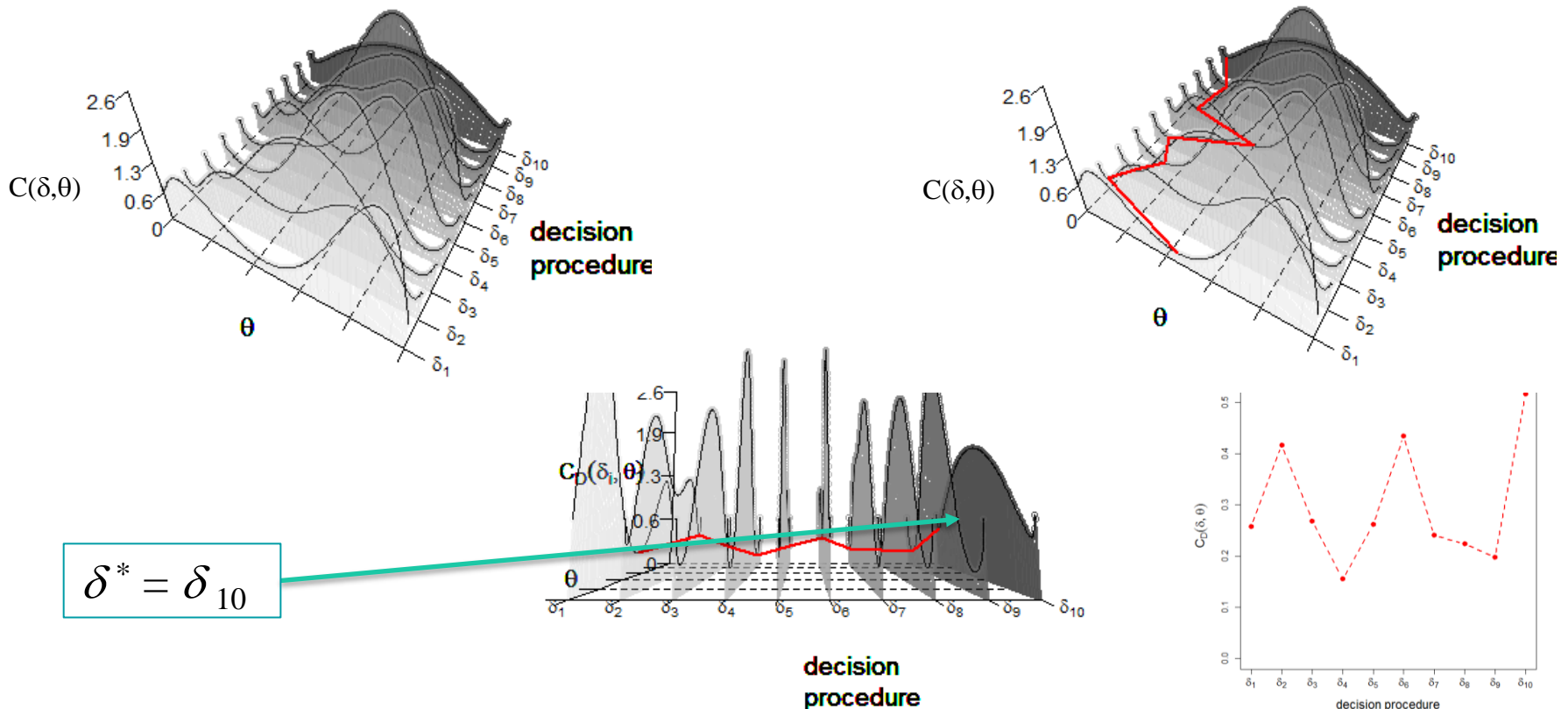
Decision-making under ignorance [non-probabilistic criteria]

no assigned probabilities of the states

A decision rule δ^* is a *maximin* rule if

$$C(\delta^*, (\theta)) = \max_{\delta} \left\{ \min_{\theta} C(\delta, \theta) \right\}$$

i.e. θ is for each decision rule chosen to be the “worst” state \Rightarrow lowest chance, and among these lowest chances the decision rule that gives the highest one of them is chosen.

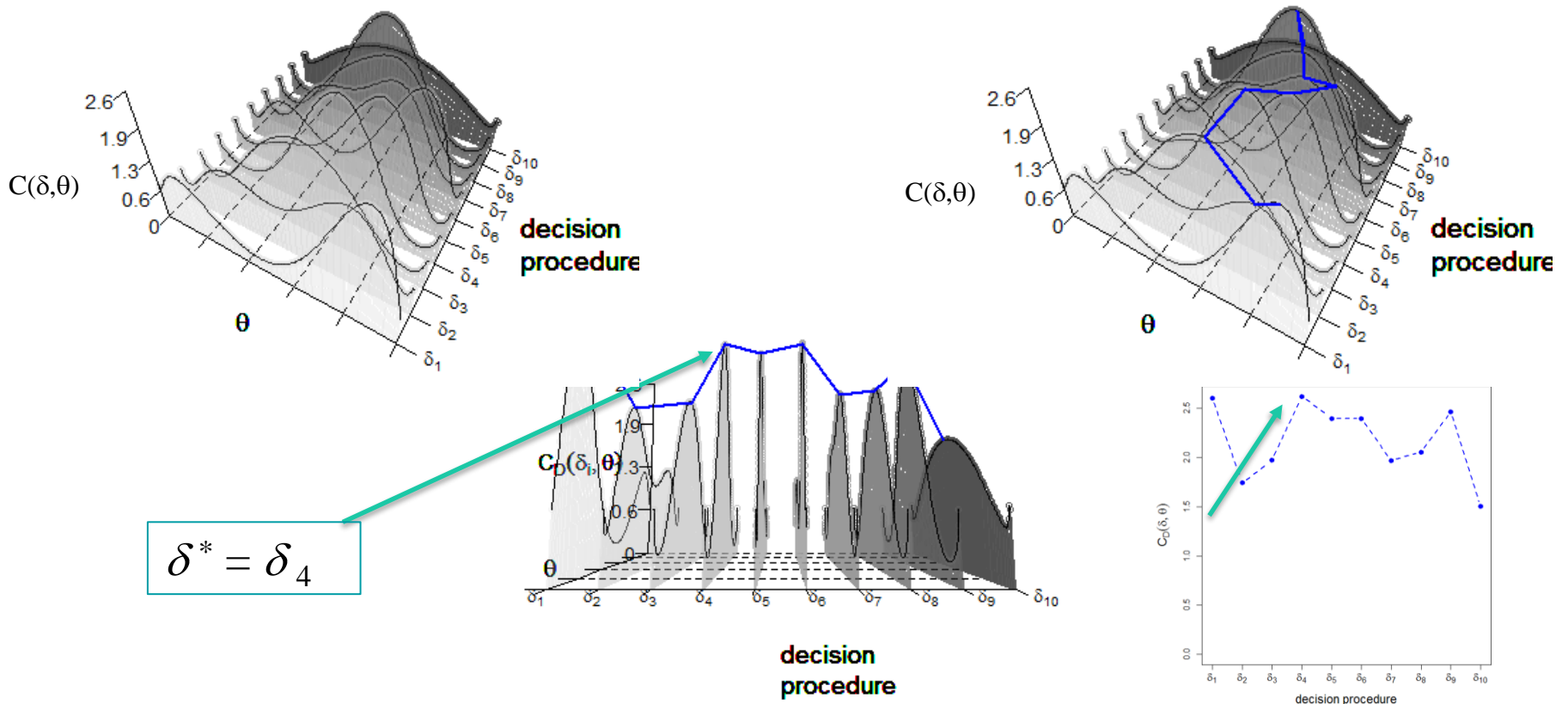


A procedure δ^* is a *maximax* rule if

$$C(\delta^*, (\theta)) = \max_{\delta} \left\{ \max_{\theta} C(\delta, \theta) \right\}$$

i.e. θ is for each decision rule chosen to be the “best” state \Rightarrow highest chance, and among these highest chances the decision rule that gives the highest one of them is chosen.

The maximax rule is a typical *optimistic* rule

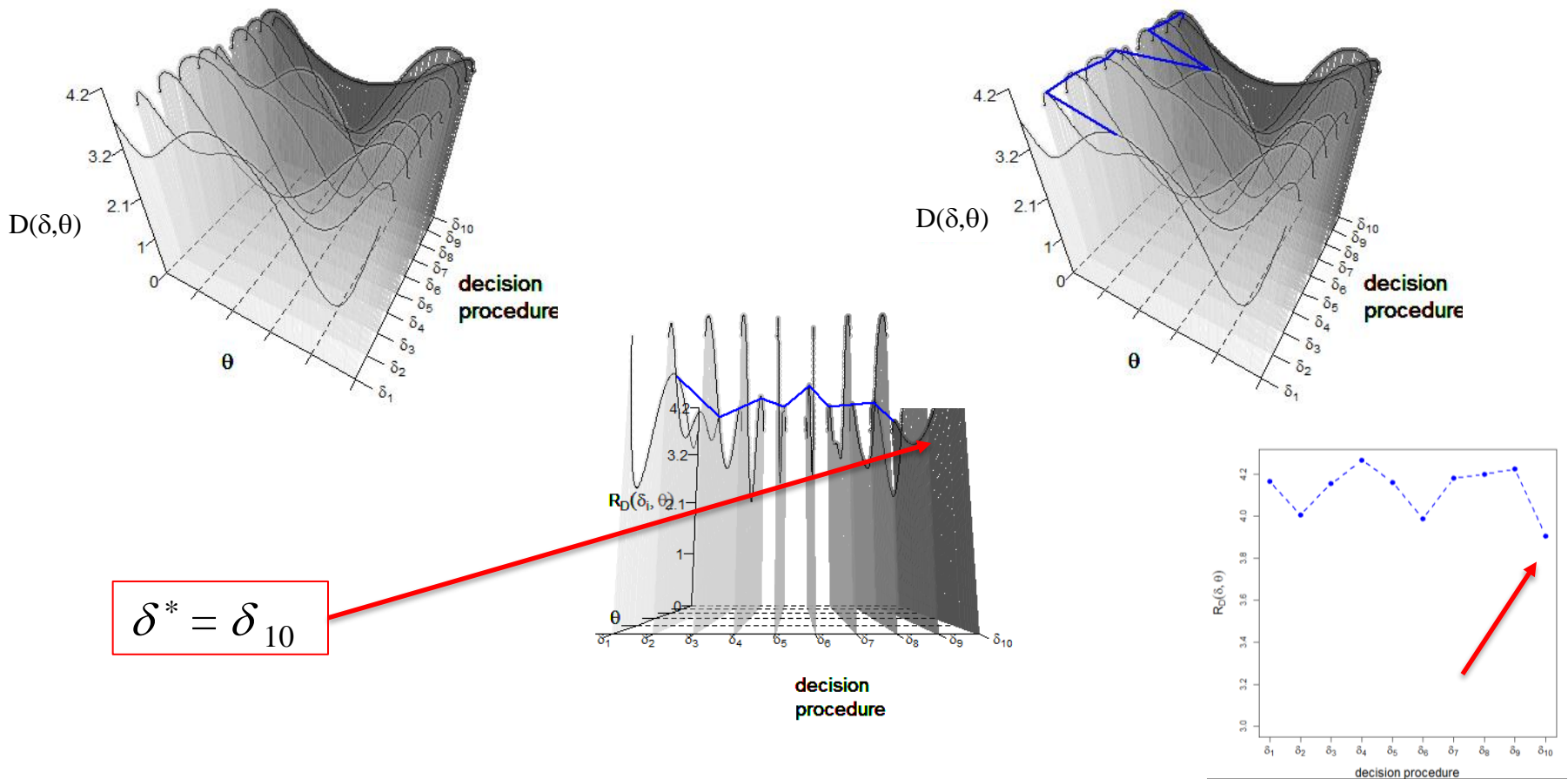


A procedure δ^* is a *minimax* rule if

$$D(\delta^*, (\theta)) = \min_{\delta} \left\{ \max_{\theta} D(\delta, \theta) \right\}$$

i.e. θ is for each decision rule chosen to be the “worst” state \Rightarrow highest risk, and among these highest risks the decision rule that gives the lowest one is chosen.

The minimax rule is a typical *pessimistic* rule



Maximin $C(\delta^*, (\boldsymbol{\theta})) = \max_{\delta} \left\{ \min_{\boldsymbol{\theta}} C(\delta, \boldsymbol{\theta}) \right\}$

Maximax $C(\delta^*, (\boldsymbol{\theta})) = \max_{\delta} \left\{ \max_{\boldsymbol{\theta}} C(\delta, \boldsymbol{\theta}) \right\}$

Minimax ~~$R(\delta^*, (\boldsymbol{\theta})) = \min_{\delta} \left\{ \max_{\boldsymbol{\theta}} R(\delta, \boldsymbol{\theta}) \right\}$~~

Another decision rule under ignorance:

Lexical maximin (or minimax)

If the criteria maximin is used and it happens that two rules δ_1 and δ_2 both are solutions to $\max_{\delta} \left\{ \min_{\boldsymbol{\theta}} C(\delta, \boldsymbol{\theta}) \right\}$

- try $\max_{\delta_1, \delta_2} \left\{ \text{next-to-min}_{\boldsymbol{\theta}} C(\delta, \boldsymbol{\theta}) \right\}$

-

...and if that cannot separate them

- try $\max_{\delta_1, \delta_2} \left\{ \text{next-to-next-to-min}_{\boldsymbol{\theta}} C(\delta, \boldsymbol{\theta}) \right\}$

- etc.

Example

Suppose you are about to make a decision on whether you should buy or rent a new laptop to have for two years = 24 months.



→ $\delta_1 = \text{“Buy the laptop”}$ $\delta_2 = \text{“Rent the laptop”}$

Now, assume θ is the mean time until the laptop breaks down for the first time.

Let θ assume three possible values: 6, 12 and 24 months.

The cost of the laptop is \$500 if you buy it and \$30 per month if you rent it.

If the laptop breaks down after 12 months (length of warranty) you'll have to replace it for the same cost as you bought it if you bought it. If you rented it you will get a new laptop for no cost provided you proceed with your contract.

Let X be the time in months until the laptop breaks down and assume this variable is exponentially distributed with mean θ .

→ A cost function (negative payoff function) for an ownership of maximum 24 months may be defined as

$$L(\delta_1(X), \theta) = 500 + 500 \cdot \mathbf{1}_{\{X < 12\}} \quad \text{and}$$

$$L(\delta_2(X), \theta) = 30 \cdot 24 = 720$$

$$\text{where } \mathbf{1}_{\{y\}} = \begin{cases} 0 & y < 0 \\ 1 & y \geq 0 \end{cases}$$

As we will see later, this is not a proper loss function, but we use it like this here for purpose of illustration.

Then
$$D(\delta_1, \theta) = E(500 + 500 \cdot \mathbf{1}_{\{X-12\}}) = 500 + 500 \int_{12}^{\infty} 1 \cdot \frac{1}{\theta} e^{-\frac{x}{\theta}} dx =$$
$$= 500 \cdot (1 + e^{-12/\theta})$$

$$D(\delta_2, \theta) = 720$$

Now compare the risks for the three possible values of θ .

θ	$D(\delta_1, \theta)$	$D(\delta_2, \theta)$
6	$500 \cdot (1 + e^{-12/6}) = 568$	720
12	$500 \cdot (1 + e^{-12/12}) = 684$	720
24	$500 \cdot (1 + e^{-12/24}) = 803$	720

Clearly the risk for the first rule increases with θ while the risk for the second is constant. In searching for the minimax rule we therefore focus on the largest possible value of θ and there δ_2 has the smallest risk.

Minimax decision rule = δ_2 (rent the laptop)

For the maximin and maximax rules, let $C(\delta_i, \theta) = K - D(\delta_i, \theta)$

\Rightarrow Maximin decision rule = δ_2 and Maximax decision rule = δ_1

Details about the relation will come

Note! In this description the rules are defined with expected payoffs and losses with respect to data (Chance and Risk functions). All rules may however be applied directly to payoff or loss tables for different states of the world/nature.

Example Route 1 and 2 revisited

The consequences are not expressed in terms of payoff or losses.

But the aim is to reach city B as quickly as possible.

Acts	States	
	Heavy traffic	Normal traffic
Route 1	80 min	50 min
Route 2	60 min	60 min

\Rightarrow Preference order is $50 \text{ min} \succ 60 \text{ min} \succ 80 \text{ min}$

“ \succ ” means “preferred to”

Maximin and Minimax:

Route 1: consequence with lowest preference is 80 min

Route 2: consequence with lowest preference is 60 min.

$60 \text{ min} \succ 80 \text{ min} \Rightarrow$ The maximin (and minimax) rule is to choose Route 2.

Maximax:

Route 1: consequence with highest preference is 50 min.

Route 2: consequence with highest preference is 60 min.

$50 \text{ min} \succ 60 \text{ min} \Rightarrow$ The maximax rule is to choose Route 1.

Decision-making under risk (*Bayes decision rule(s)*) [probabilistic criteria]

To make formulas a bit simpler we use the notation

$$\begin{array}{ll} p(\boldsymbol{\theta}) = f'(\boldsymbol{\theta}) & \text{prior probability density (or mass) function for } \tilde{\boldsymbol{\theta}} \\ f(\mathbf{x}|\boldsymbol{\theta}) & \text{likelihood} \\ q(\boldsymbol{\theta}|\mathbf{x}) = f''(\boldsymbol{\theta}|\mathbf{x}) & \text{posterior probability density (or mass) function for } \tilde{\boldsymbol{\theta}} \end{array}$$

Bayes risk of procedure δ :
$$B(\delta) = \int_{\boldsymbol{\theta} \in \Theta} D(\delta, \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

i.e. the risk function averaged over the prior distribution.

Note! The integral is a sum when $p(\boldsymbol{\theta})$ is a pmf.

A *Bayes rule* is a procedure that minimizes the Bayes risk

$$\delta_B = \arg \min_{\delta} \left\{ \int_{\boldsymbol{\theta} \in \Theta} D(\delta, \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \right\}$$

Note! This is about the decision rule, not a specific action

However,

$$\begin{aligned}\int_{\boldsymbol{\theta} \in \Theta} D(\delta, \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} &= \int_{\boldsymbol{\theta} \in \Theta} \left(\int_x L(\delta(\mathbf{x}), \boldsymbol{\theta}) f(\mathbf{x}|\boldsymbol{\theta}) d\mathbf{x} \right) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &= \int_x \int_{\boldsymbol{\theta} \in \Theta} L(\delta(\mathbf{x}), \boldsymbol{\theta}) f(\mathbf{x}|\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} d\mathbf{x} = \int_x \int_{\boldsymbol{\theta} \in \Theta} L(\delta(\mathbf{x}), \boldsymbol{\theta}) q(\boldsymbol{\theta}|\mathbf{x}) h(\mathbf{x}) d\boldsymbol{\theta}\end{aligned}$$

where $h(\mathbf{x})$ is the marginal likelihood

$$q(\boldsymbol{\theta}|\mathbf{x}) = \frac{f(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{h(\mathbf{x})}$$

$$\Rightarrow \int_{\boldsymbol{\theta} \in \Theta} D(\delta, \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} = \int_x h(\mathbf{x}) \int_{\boldsymbol{\theta} \in \Theta} L(\delta(\mathbf{x}), \boldsymbol{\theta}) q(\boldsymbol{\theta}|\mathbf{x}) d\boldsymbol{\theta} d\mathbf{x} = \int_x h(\mathbf{x}) E_{\tilde{\boldsymbol{\theta}}|\mathbf{x}} \left(L(\delta(\mathbf{x}), \tilde{\boldsymbol{\theta}}) \right) d\mathbf{x}$$

$$\Rightarrow \arg \min_{\delta} \left\{ \int_{\boldsymbol{\theta} \in \Theta} D(\delta, \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \right\} = \arg \min_{\delta} \left\{ E_{\tilde{\boldsymbol{\theta}}|\mathbf{x}} \left(L(\delta(\mathbf{x}), \tilde{\boldsymbol{\theta}}) \right) \right\} \quad \text{for any given value of } \mathbf{x}$$

Hence, a procedure that minimises the posterior expected loss is a Bayes decision rule (a probabilistic criterion).

Example: Rent or buy laptop cont.

Assume the three possible values of θ (6, 12 and 24) have the prior probabilities 0.2, 0.3 and 0.5 respectively.

$$L(\delta_1(X), \theta) = 500 + 500 \cdot \mathbf{1}_{\{X > 12\}}$$

$$L(\delta_2(X), \theta) = 30 \cdot 24 = 720$$

$$f(x/\theta) = \theta^{-1} e^{-x/\theta}$$

Using the Bayes risks directly

$$p(\theta) = \begin{cases} 0.2 & \theta = 6 \\ 0.3 & \theta = 12 \\ 0.5 & \theta = 24 \\ 0 & \text{otherwise} \end{cases} \quad (\text{pmf})$$

$$B(\delta_1) = \left\langle \theta \text{ is discrete} \right\rangle = \sum_{\theta \in \Theta} D(\delta_1, \theta) p(\theta) = \sum_{\theta \in \Theta} 500(1 - e^{-12/\theta}) p(\theta)$$

$$= 500(1 - e^{-12/6}) \cdot 0.2 + 500(1 - e^{-12/12}) \cdot 0.3 + 500(1 - e^{-12/24}) \cdot 0.5 = 280$$

$$B(\delta_2) = \sum_{\theta \in \Theta} D(\delta_2, \theta) p(\theta) = \langle R_D(\delta_2, \theta) \equiv 720 \rangle = 720 \cdot 0.2 + 720 \cdot 0.3 + 720 \cdot 0.5 = 720$$

Thus, the minimal Bayes risk is with procedure δ_1 and therefore δ_1 is the Bayes decision rule (among δ_1 and δ_2).

Working with payoffs instead of losses

How do we define payoff and how do we define loss?

The **payoff** represents the *net change* in your total “wealth” as a function of your action and the actual state of the world/nature.

The payoff can then be seen as the *consequence* of your action with the actual state of the world/nature.

The term “wealth” is not necessarily to be interpreted in monetary units, but very often it should be possible – but perhaps difficult – to translate non-monetary consequences to cash equivalents

Defining payoff as the net change means that all costs involved are taken into consideration. Hence the payoff may be negative.

Example

Suppose you are about to sell apples on an open market a Saturday in October.



You are choosing between selling one of two kinds of apples.

For the first kind – a lower quality apple - you can buy 100 kg apples for SEK 1800, and you deem a reasonable highest selling price to be SEK 30 per kg.

For the second kind – a higher quality apple – you can buy 80 kg for SEK 2000, and you deem a reasonable highest selling price to be 40 per kg.

Lower quality: 100 kg cost SEK 1800, selling price SEK 30 per kg
Higher quality: 80 kg cost SEK 2000, selling price SEK 40 per kg



Assume the total demand for your apples that day is 50 kg (no matter what kind of apple).

Your payoff with the action to sell apples of the lower quality will be $\text{SEK } 50 \times 30 - 1800 = -300$.

Your payoff with the action to sell apples of the higher quality will be $\text{SEK } 50 \times 40 - 2000 = 0$.

If the total demand is 100 kg? If your action is to sell apples of the lower quality, your payoff will be $\text{SEK } 100 \times 30 - 1800 = 1200$, and if your action is to sell apples of the higher quality, your payoff will be $\text{SEK } 80 \times 40 - 2000 = 1200$.

The **loss** function is to be interpreted in terms of *opportunity* loss, i.e. for each state of the world it is the difference between the maximal payoff that can be obtained with that certain state and the payoff of the particular action – referred to as *regret* in the course book by Peterson.

This means,

$$\begin{aligned}\delta_B &= \arg \min_{\delta} \left\{ E_{\tilde{\theta}|x} \left(L(\delta(x), \tilde{\theta}) \right) \right\} = \underline{\arg \min_{\delta}} \left\{ E_{\tilde{\theta}|x} \left(\underbrace{\max_d \{ R(d(x), \tilde{\theta}) \}}_{\text{not depending on } \delta} - R(x(\delta(x), \tilde{\theta})) \right) \right\} \\ &= \arg \min_{\delta} \left\{ -E_{\tilde{\theta}|x} \left(R(\delta(x), \tilde{\theta}) \right) \right\} = \arg \max_{\delta} \left\{ E_{\tilde{\theta}|x} \left(R(\delta(x), \tilde{\theta}) \right) \right\}\end{aligned}$$

Hence, a procedure that maximises the posterior expected payoff is (also) a Bayes procedure.

Important, though: Payoff is not synonymous with *utility*

Note that implicit in the derivation above is:

$$E_{\tilde{\theta}|x} \left(L(\delta(x), \tilde{\theta}) \right) = E_{\tilde{\theta}|x} \left(\max_d \{ R(d(x), \tilde{\theta}) \} \right) - E_{\tilde{\theta}|x} \left(R(\delta(x), \tilde{\theta}) \right) = T - E_{\tilde{\theta}|x} \left(R(\delta(x), \tilde{\theta}) \right)$$

where T is a quantity that does not depend on the action $\delta(x)$

The “complete” theoretical description hence takes its standpoint from decision procedures.

This covers both decision-making under ignorance (non-Bayesian) and decision-making under risk (Bayesian).

However, most accounts (including the course book(s)) of this course would focus on particular actions.

Thus we can say

Under ignorance (no probabilities and no data)...

- The maximin action is the action that has highest minimum payoff
- The minimax action is the action that has the lowest maximum loss
- The maximax action is the action that has the highest maximum payoff

Under risk (probabilities of states assigned, data may be available) ...

- A Bayes action is an action that maximises the expected payoff (prior or posterior) – equivalent to an action that minimises the expected loss (prior or posterior)

The relation between (opportunity) loss (L) and payoff (R) for an action a^* is thus

$$L(a^*, \boldsymbol{\theta}) = \max_a \{R(a, \boldsymbol{\theta})\} - R(a^*, \boldsymbol{\theta})$$

This implies that the loss can never be negative , while the payoff can be positive, negative or zero.

Applied to a payoff table with m rows and n column, where each row represents an action and each column a state of the world:

$$L_{ij} = \max_k \{R_{kj}\} - R_{ij} \quad i = 1, \dots, m, \quad j = 1, \dots, n$$

	θ_1	θ_2	...
a_1	L_{11}, R_{11}	L_{12}, R_{12}	
a_2	L_{21}, R_{21}	L_{22}, R_{22}	
...			

where L_{ij} and R_{ij} stand for the loss and the payoff respectively when action i is chosen with state j of the world.

(In)Admissibility of actions

An action a_1 is *inadmissible* if there exists another action a_2 such that

$$R(a_2, \boldsymbol{\theta}) \geq R(a_1, \boldsymbol{\theta}) \text{ or } L(a_2, \boldsymbol{\theta}) \leq L(a_1, \boldsymbol{\theta})$$

for all states of the world/nature $\boldsymbol{\theta}$ and

$$R(a_2, \boldsymbol{\theta}) > R(a_1, \boldsymbol{\theta}) \text{ or } L(a_2, \boldsymbol{\theta}) < L(a_1, \boldsymbol{\theta})$$

(strict inequality) for at least one state $\boldsymbol{\theta}$

The action a_2 is then said to *dominate* action a_1 .

If an action is inadmissible, it should not be considered.

If an action is not dominated by any other action, it is *admissible*.

Example, apples cont.



We can form a payoff table as

	Demand is 50 kg	Demand is 100 kg
a_1 = Sell lower quality apples	$R_{11} = -300$	$R_{12} = 1200$
a_2 = Sell higher quality apples	$R_{21} = 0$	$R_{22} = 1200$

Since the decision to sell higher quality apples (a_2) would give payoffs that do not fall short of the payoffs given by selling lower quality apples (a_1), and the payoff with action a_2 when the demand is 50 kg is higher than the payoff with action a_1 , the decision to sell lower quality apples is inadmissible.

provided the only states of the world (possible demands) considered are 50 kg and 100 kg

Example

Given the loss table below ...

States:		<i>I</i>	<i>II</i>	<i>III</i>
<i>Actions</i>	<i>1</i>	0	3	6
	<i>2</i>	1	1	0
	<i>3</i>	4	0	1

...complete the Payoff table below

	<i>I</i>	<i>II</i>	<i>III</i>
<i>1</i>	12	7	9
<i>2</i>			
<i>3</i>			

The relationship between Loss and Payoff for action i , state j , here L_{ij} and R_{ij} is

$$L_{ij} = \max_k R_{kj} - R_{ij} \Leftrightarrow R_{ij} = \max_k R_{kj} - L_{ij}$$

Loss table

States:		<i>I</i>	<i>II</i>	<i>III</i>
<i>Actions</i>	<i>1</i>	0	3	6
	<i>2</i>	1	1	0
	<i>3</i>	4	0	1

Payoff table

	<i>I</i>	<i>II</i>	<i>III</i>
<i>1</i>	12	7	9
<i>2</i>			
<i>3</i>			

$$L_{ij} = \max_k R_{kj} - R_{ij} \Leftrightarrow$$

$$R_{ij} = \max_k R_{kj} - L_{ij}$$

$$\Rightarrow \max_k R_{k1} - R_{11} = L_{11} = 0 \Rightarrow \max_k R_{k1} = R_{11} = 12$$

$$\max_k R_{k2} - R_{12} = L_{12} = 3 \Rightarrow \max_k R_{k2} = 3 + R_{12} = 10$$

$$\max_k R_{k3} - R_{13} = L_{13} = 6 \Rightarrow \max_k R_{k3} = 6 + R_{13} = 15$$

\Rightarrow The payoff table is

	<i>I</i>	<i>II</i>	<i>III</i>
<i>1</i>	12	7	9
<i>2</i>	$12 - 1 = 11$	$10 - 1 = 9$	$15 - 0 = 15$
<i>3</i>	$12 - 4 = 8$	$10 - 0 = 10$	$15 - 1 = 14$



	<i>I</i>	<i>II</i>	<i>III</i>
<i>1</i>	12	7	9
<i>2</i>	11	9	15
<i>3</i>	8	10	14

Using the loss table

	<i>I</i>	<i>II</i>	<i>III</i>
<i>1</i>	0	3	6
<i>2</i>	1	1	0
<i>3</i>	4	0	1

we see that no action gives a higher or equal loss compared to any of the other actions for all three states of the world. Hence, there is no inadmissible action.

Using the payoff table

	<i>I</i>	<i>II</i>	<i>III</i>
<i>1</i>	12	7	9
<i>2</i>	11	9	15
<i>3</i>	8	10	14

we see that no action gives a lower or equal payoff compared to any of the other actions for all three states of the world .