SEMINAR EXERCISES IN PROBABILITY THEORY 732A63

Hector Rodriguez-Deniz

6 October 2023

6 Convergence

Exercise 6.0.1 (self-study)

Take a look at the Examples 1.1 to 1.4 in pages 148-149 in the book, and then try to solve them on your own. It does not matter if you have seen the solution/steps beforehand, it will be good practice towards more elaborated exercises.

Exercise 6.1

Assume that $Y, X_1, X_2, \ldots$ are independent random variables with $Y \sim U(0,1)$ and $X_n \sim U(\frac{1}{2n}, 1)$ for $n = 1, 2, \ldots$.

(a) Does $X_n \overset{a.s.}{\to} Y$ as $n \to \infty$?

(b) Does $X_n \overset{p}{\to} Y$ as $n \to \infty$?

(c) Does $X_n \overset{d}{\to} Y$ as $n \to \infty$?

Exercise 6.2 (6.1 in Gut's book)

Let $X_1, X_2, \ldots$ be $U(0,1)$-distributed independent random variables. Show that

(a) $\max_{1 \le k \le n} X_k \overset{p}{\to} 1$ as $n \to \infty$

(b) $\min_{1 \le k \le n} X_k \overset{p}{\to} 0$ as $n \to \infty$

Exercise 6.3 (6.6 in Gut's book)

Suppose that $X_1, X_2, \ldots$ are independent, $Pa(1,2)$-distributed random variables, and set $Y_n = \min\{X_1, X_2, \ldots, X_n\}$.

(a) Show that $Y_n \overset{p}{\to} 1$ as $n \to \infty$. It thus follows that $Y_n \approx 1$ with a probability close to 1 when n is large. One might therefore suspect that there exists a limit theorem to the effect that $Y_n - 1$, suitably rescaled, converges in distribution as $n \to \infty$ (note that $Y_n > 1$ always).

(b) Show that $n(Y_n - 1)$ converges in distribution as $n \to \infty$, and determine the limit distribution.

Exercise 6.4 (6.15 in Gut's book)

Let X and Y be random variables such that

$$Y|X = x \sim N(0, x) \text{ with } X \sim Po(\lambda).$$

(a) Find the characteristic function of Y

(b) Show that $Y/\sqrt{\lambda} \xrightarrow{d} N(0,1)$ as $\lambda \to \infty$

Exercise 6.5* (6.19 in Gut's book)

Suppose that the random variables $N_n, X_1, X_2, \ldots$ are independent, that $N_n \sim Ge(p_n),\ 0 < p_n < 1$, and that $X_1, X_2, \ldots$ are equidistributed with finite mean μ. Show that if $p_n \to 0$ as $n \to \infty$ then $p_n(X_1 + X_2 + \cdots + X_{N_n})$ converges in distribution as $n \to \infty$, and determine the limit distribution.

Exercise 6.6* (6.25 in Gut's book)

Let $X_n \sim \Gamma(n, 1)$, and set

$$Y_n = \frac{X_n - n}{\sqrt{X_n}}.$$

Show that $Y_n \xrightarrow{d} N(0,1)$ as $n \to \infty$.

Self-study exercises are excluded from the bonus-point deal. Exercises marked with * are a bit more challenging.

May Gauss be with you!