Probability Theory

Lecture 4: Order Statistics

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Overview Lecture 4

- Order statistics
- Distribution of extremes
- Marginal distribution of order statistics
- Joint distribution of order statistics

ORDER STATISTICS

- $X_1, X_2, ..., X_n$ i.i.d. (in general same distribution not necessary)
- **Extremes**:
  - $\min(X_1, X_2, ..., X_n)$
  - $\max(X_1, X_2, ..., X_n)$.
- **Range**: $R = \max(X_1, X_2, ..., X_n) - \min(X_1, X_2, ..., X_n)$
- Applications in extreme value theory, derivatives pricing, etc

DEF The *k*th **order variable**

$$X_{(k)} = \text{the } k\text{th smallest of } X_1, X_2, ..., X_n$$

- $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$

DEF The order statistic: $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$

- $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are not necessarily independent

Distribution of the maximum

Th Distribution of $X_{(n)}$:

$$F_{X_{(n)}}(x) = \boxed{P\left(X_1 \leq x, X_2 \leq x, ..., X_n \leq x\right)}$$
$$= \prod_{i=1}^{n} P(X_i \leq x) = \boxed{[F(x)]^n}$$

- Density of $X_{(n)}$:

$$f_{X_{(n)}}(x) = n\left[F(x)\right]^{n-1} f(x)$$

- Example: $X_1, ..., X_n \sim L(a)$

$$F(x) = \begin{cases} \frac{1}{2}\exp\left(\frac{x}{a}\right) & \text{if } x < 0 \\ 1 - \frac{1}{2}\exp\left(-\frac{x}{a}\right) & \text{if } x \geq 0 \end{cases}$$

$$F_{X_{(n)}}(x) = [F(x)]^n = \begin{cases} \frac{1}{2^n}\exp\left(\frac{nx}{a}\right) & \text{if } x < 0 \\ \left[1 - \frac{1}{2}\exp\left(-\frac{x}{a}\right)\right]^n & \text{if } x \geq 0 \end{cases}$$

Distribution of the minimum

TH Distribution of $X_{(1)}$

$$F_{X_{(1)}}(x) = 1 - P\left(X_{(1)} > x\right)$$
$$= 1 - P\left(X_1 > x, X_2 > x, ..., X_n > x\right)$$
$$= 1 - \prod_{i=1}^{n} P(X_i > x) = 1 - \left[1 - F(x)\right]^n .$$

- Density of $X_{(n)}$

$$f_{X_{(1)}}(x) = n\left[1 - F(x)\right]^{n-1} f(x)$$

- Example: $X_1, ..., X_n \sim Exp(a)$

$$F(x) = 1 - e^{-x/a}$$

$$f_{X_{(1)}}(x) = n\left[e^{-x/a}\right]^{n-1} 1/ae^{-x/a} = n/ae^{-n/ax}$$

so $X_{(1)} \sim Exp(\frac{a}{n})$

Marginal distribution of $X_{(k)}$

Th The distribution of the kth order variable $X_{(k)}$ from a random sample with cdf $F(x)$:

$$F_{X_{(k)}}(x) = F_{\beta(k,n+1-k)}\left[F(x)\right]$$

where $F_{\beta(k,n+1-k)}(\cdot)$ is the cdf of $Beta(k, n+1-k)$ distribution.

- Example: $X \sim U(0,1) \quad \Rightarrow \quad X_{(k)} \sim \beta(k, n+1-k)$

Marginal distribution of $X_{(k)}$ - example

- Example: Let the individual jumps of n athletes in a long jump tournament be independently $U(a, b)$ distributed. Three jumps per athlete. What is the probability that the recorded score of the silver medalist is longer than c meters?
  Solution: First, calculate the distribution of Y_i = longest jump out of three jumps for the ith athlete, for $i = 1, ..., n$:

$$F_{Y_i}(y) = [F(y)]^3 = \left(\frac{y-a}{b-a}\right)^3$$

  Then derive distribution of $Y_{(n-1)}$

$$F_{Y_{(n-1)}}(y) = F_{\beta(n-1,2)}\left(\left(\frac{y-a}{b-a}\right)^3\right)$$

Joint distribution of extremes and range

TH Joint density of $X_{(1)}$ and $X_{(n)}$

$$f_{X_{(1)},X_{(n)}}(x,y) = \begin{cases} n(n-1)\left(F(y)-F(x)\right)^{n-2} f(y)f(x) & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

- From $f_{X_{(1)},X_{(n)}}(x,y)$ we can derive the distribution of the range $R_n = X_{(n)} - X_{(1)}$ using Transformation Theorem.

TH Density of **range** $R_n = X_{(n)} - X_{(1)}$:

$$f_{R_n}(r) = \begin{cases} n(n-1)\int_{-\infty}^{\infty}\left(F(u+r)-F(u)\right)^{n-2} f(u+r)f(u)du & \text{if } r > 0 \\ 0 & \text{otherw.} \end{cases}$$

Joint distributions of order statistics

Th Joint density of $X_{(1)}, \ldots, X_{(n)}$:

$$f_{X_{(1)},\ldots,X_{(n)}}(y_1, \ldots, y_n) = \begin{cases} n! \prod_{k=1}^{n} f(y_k) & \text{if } y_1 < y_2 < \cdots < y_n \\ 0 & \text{otherwise} \end{cases}$$

- Marginal density of any order variable can be derived by integrating $f_{X_{(1)},\ldots,X_{(n)}}(y_1, \ldots, y_n)$
- $X_1, X_2 \sim Exp(1)$, independent

$$\begin{aligned} f_{X_{(1)},X_{(2)}}(y_1, y_2) &= 2e^{-y_1} e^{-y_2},\ y_1 < y_2 \\ f_{X_{(1)}}(y_1) &= \int_{y_1}^{\infty} 2e^{-y_1} e^{-y_2} dy_2 = 2e^{-2y_1} \end{aligned}$$

Thank you for your attention!