SEMINAR EXERCISES IN PROBABILITY THEORY 732A63

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5 Multivariate Normal Distribution

Exercise 5.0.1 (self-study)

Your task here is to download *The Matrix Cookbook* [1] and take a look at it. This is a document I always come back to when I have to work with Gaussians. You can get it from

`https://www2.imm.dtu.dk/pubdb/edoc/imm3274.pdf`

Chapter 8 is fully dedicated to the multivariate Gaussian distribution and its properties. Get familiar with it, you will probably find results, like the conditioning formulas in subsection 8.1.3, that are useful for the exercises.

Exercise 5.1 (5.4 in Gut's book)

The random vector $(X, Y)'$ has a two-dimensional normal distribution with $Var(X) = Var(Y)$. Show that $X + Y$ and $X - Y$ are independent random variables.

Exercise 5.2 (5.12 in Gut's book)

Let X_1 and X_2 be independent, $N(0, 1)$-distributed random variables. Set $Y_1 = X_1 - 3X_2 + 2$ and $Y_2 = 2X_1 - X_2 - 1$. Determine the distribution of

(a) $\mathbf{Y}$

(b) $Y_1 | Y_2 = y$

Exercise 5.3 (5.18 in Gut's book)

The random vector $\mathbf{X}$ has a three-dimensional normal distribution with expectation $\mathbf{0}$ and covariance matrix $\mathbf{\Lambda}$ given by

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ -1 & 0 & 7 \end{pmatrix}.$$

Find the distribution of X_3 given that $X_1 = 1$.

Exercise 5.4 (5.20 in Gut's book)

The random vector $\mathbf{X}$ has a three-dimensional normal distribution with mean vector $\mu = \mathbf{0}$ and covariance matrix

$$\mathbf{\Lambda} = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the distribution of $X_1 + X_3$ given that

(a) $X_2 = 0$

(b) $X_2 = 2$

Exercise 5.5* (5.33 in Gut's book)

Let X and Y be random variables such that

$$Y|X = x \sim N(x, \tau^2) \text{ with } X \sim N(\mu, \sigma^2).$$

(a) Compute $E(Y)$, $Var(Y)$, and $Cov(X, Y)$

(b) Determine the distribution of the vector $(X, Y)'$

(c) Determine the (posterior) distribution of $X|Y = y$

Exercise 5.6* (5.34 in Gut's book)

Let X and Y be jointly normal with means 0, variances 1, and correlation coefficient ρ. Compute the moment generating function of $X \cdot Y$ for

(a) $\rho = 0$

(b) general ρ

Self-study exercises are excluded from the bonus-point deal. Exercises marked with * are a bit more challenging.

May Gauss be with you!

References

[1] K. B. Petersen and M. S. Pedersen. The matrix cookbook, nov 2012. Version 20121115.