Exam 25.10 solution

(1)

Ex. 1 a) $g_Y(t) = E t^Y = E t^{X_1 + \dots + X_n} = \prod_{i=1}^{n} \underbrace{E t^{X_i}}_{= g_{X_i}(t)}$

$= \left(E t^{X_1}\right)^n = \left(g_{X_1}(t)\right)^n.$

b) $g_Y(t) = \prod_{i=1}^{3} g_{X_i}(t) = \prod_{i=1}^{3} e^{\lambda_i (t-1)} = e^{(\lambda_1+\lambda_2+\lambda_3)(t-1)}$

$= e^{\tilde{\lambda}(t-1)}, \quad \tilde{\lambda} = \lambda_1+\lambda_2+\lambda_3$

$\Rightarrow Y \sim P_0(\lambda_1+\lambda_2+\lambda_3).$

c) $g_Y(t) = g_{2X_1}(t) \cdot g_{X_2}(t) = g_{X_1}(t^2) \cdot g_{X_1}(t)$

$= (t^2 p + 1-p)^k \cdot (tp + 1-p)^k = \left[(t^2 p + 1-p)(tp+1-p)\right]^k.$

Ex. 2 a) $F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(X_1 \le x, \dots, X_n \le x)$

$= \prod_{i=1}^{n} P(X_i \le x) = \left(P(X_1 \le x)\right)^n = \left(F_{X_1}(x)\right)^n.$

$f_{X_{(n)}}(x) = n \cdot F_{X_1}(x)^{n-1} \cdot f_{X_1}(x).$

b) $F_{X_{(1)}}(x) = 1 - (1 - F_{X_1}(x))^n$

$F_{X_1}(x) = 1 - e^{-x/a}, \quad x > 0$ (otherwise 0)

$F_{X_{(1)}}(x) = 1 - e^{-xn/a}, \quad x > 0$ (otherwise 0)

$\Rightarrow X_{(1)} \sim Exp(a/n).$

Alternatively

$f_{X_{(1)}}(x) = n(1 - F_{X_1}(x))^{n-1} \cdot f_{X_1}(x)$ ($x > 0$, otherwise 0).

$= n e^{-x(n-1)/a} \cdot \frac{1}{a} e^{-x/a}$

$= \frac{n}{a} \cancel{e^{x/a}} e^{-xn/a}$

$\Rightarrow X_{(1)} \sim Exp(a/n).$

c) $f_{X_{(1)}, X_{(2)}, X_{(3)}}(x_1, x_2, x_3) = n! \prod_{i=1}^{3} f_{X_i}(x_i)$, $x_1 < x_2 < x_3$ (otherwise 0)

$= 3! \cdot \frac{1}{a} e^{-x_1/a} \cdot \frac{1}{a} e^{-x_2/a} \cdot \frac{1}{a} e^{-x_3/a}$

$= 6 \cdot \frac{1}{a^3} \cdot e^{-(x_1+x_2+x_3)/a}$, $x_1 < x_2 < x_3$ (otherwise 0).

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Ex. 3 a) $Y \sim N(\mu_Y, \Sigma_Y)$ with $\mu_Y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $\Sigma_Y = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$.

$$\Psi_Y(t) = \exp(\underbrace{t^T \mu_Y}_{t_1 + t_2} + \underbrace{\tfrac{1}{2} t^T \Sigma_Y t}_{\frac{1}{2} \cdot (4t_1 - 2t_2, \, -2t_1 + 3t_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}})$$

$$= \frac{1}{2}(4t_1^2 - 2t_1 t_2 - 2t_1 t_2 + 3t_2^2)$$

$$= \frac{1}{2}(4t_1^2 - 4t_1 t_2 + 3t_2^2)$$

$$= \exp(t_1 + t_2 + \tfrac{1}{2}(4t_1^2 - 4t_1 t_2 + 3t_2^2))$$

b) $Z = (\underbrace{X_1 + 2X_2}_{Z_1}, \underbrace{X_2 - X_3}_{Z_2})^T \sim N(\mu_Z, \Sigma_Z)$

$$Z = \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}}_{A} \cdot X$$

$$\Rightarrow \mu_Z = A\mu = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \Sigma_Z = A \cdot \Sigma A^T = \underbrace{\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & -2 & 1 \\ -2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & 4 & 1 \\ -3 & 3 & -1 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -1 \end{pmatrix}}_{\begin{pmatrix} 8 & 3 \\ 3 & 4 \end{pmatrix}}$$

$Z_1 = Y$

$Z_2 = X$

$$Z_1 / Z_2 = 1 \sim N(\tilde{\mu}, \tilde{\sigma}^2)$$

$$\tilde{\mu} = \underset{\underset{3}{\shortparallel}}{\mu_{Z_1}} + \rho \frac{\sigma_{Z_1}}{\sigma_{Z_2}} (1 - \underset{\underset{1}{\shortparallel}}{\mu_{Z_2}}) = 3$$

$$\tilde{\sigma}^2 = \sigma_{Z_1}^2 (1 - \rho^2) = 8 \cdot \left(1 - \frac{9}{32}\right) = \frac{23}{4}$$

$$\rho = \frac{cov(Z_1, Z_2)}{\sqrt{\underset{\sigma^2_{Z_1}}{Var(Z_1)} \cdot \underset{\sigma^2_{Z_2}}{Var(Z_2)}}} = \frac{3}{\sqrt{8 \cdot 4}}$$

$$\Rightarrow Z_1 / Z_2 = 1 \sim N\left(3, \frac{23}{4}\right).$$

Ex. 4 To show: $P(|X_n - \overset{\overset{1}{\shortparallel}}{X}| > \varepsilon) \to 0$ as $n \to \infty$

$E(X_n) = 1 \quad Var(X_n) = \frac{1}{n} \quad$ (from $F(n, \frac{1}{n})$)

$$P(|X_n - \overset{\overset{1}{\shortparallel}}{X}| > \varepsilon) \leq P(|X_n - \overset{\overset{1}{\shortparallel}}{X}| \geq \varepsilon)$$

$$= P(|X_n - \underset{E(X_n)}{\underset{\shortparallel}{1}}| \geq \varepsilon) \underset{\text{chebyshev's inequality}}{\leq} \frac{Var(X_n)}{\varepsilon^2} = \frac{1}{\varepsilon^2 n} \to 0, \; n \to \infty$$