SEMINAR EXERCISES IN PROBABILITY THEORY 732A63

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4 Order Statistics

Exercise 4.0.1 (self-study)

Get familiar with the distribution functions for the minimum and maximum, $F_{X_{(1)}}(x)$ and $F_{X_{(n)}}(x)$, and do yourself the Example 1.1 in pages 101-102 in Gut's book.

Exercise 4.1 (4.1 in Gut's book)

Let X, Y and Z have the joint density

$$f_{X,Y,Z}(x,y,z) = \begin{cases} e^{-(x+y+z)} & , x,y,z > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Compute $P(X < Y < Z)$, and $P(X = Y < Z)$.

Exercise 4.2 (4.5 in Gut's book)

Let $X_1, X_2, \ldots, X_n$ be independent, continous random variables with common distribution function $F(x)$, and consider the order statistic $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$. Compute $E(F(X_{(n)}) - F(X_{(1)}))$.

Exercise 4.3 (4.6 in Gut's book)

Let X_1, X_2, X_3 and X_4 be independent, $U(0,1)$-distributed random variables. Compute

(a) $P(X_{(3)} + X_{(4)} \leq 1)$.

(b) $P(X_3 + X_4 \leq 1)$.

Exercise 4.4 (4.24 in Gut's book)

Let $X_1, X_2, \ldots, X_n$ be independent, $Exp(a)$-distributed random variables. Determine the distribution of $\sum_{k=1}^{n} X_{(k)}$.

Exercise 4.5* (4.16 in Gut's book)

Let X_1 and X_2 be independent, $Exp(a)$-distributed random variables.

(a) Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent, and determine their distributions.

(b) Compute $E(X_{(2)}|X_{(1)} = y)$, and $E(X_{(1)}|X_{(2)} = x)$.

Exercise 4.6* (4.18 in Gut's book)

Suppose that $X \sim U(0,1)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order variables corresponding to a sample of n independent observations of X, and set

$$V_i = \frac{X_{(i)}}{X_{(i+1)}},\ i = 1, 2, \ldots, n-1, \text{ and } V_n = X_{(n)}.$$

Show that

(a) $V_1, V_2, \ldots, V_n$ are independent.

(b) $V_i^i \sim U(0,1)$ for $i = 1, 2, \ldots, n$.

Self-study exercises are excluded from the bonus-point deal. Exercises marked with * are a bit more challenging.

May Gauss be with you!