LINKÖPINGS UNIVERSITET
IDA, STIMA

Examination - Probability Theory
2022-10-24

Course: 732A63 Probability Theory

Date and time: 2022/10/24, 14-18

Examinator: Maryna Prus

Allowed aids: Pocket calculator.
Table with common formulas and moment generating functions (distributed with the exam).
Table of integrals (distributed with the exam).
Table with distributions from Appendix B in the course book (distributed with the exam).
An A4 paper with your own hand-written notes (front- and backside).

Grades: Final grade is based on sum of points from written exam (max 20) and bonus points from seminars (max 0.5 to reach A, max 1 to reach B).

A= from 18 points,	D= [12 - 14) points,
B= [16 - 18) points,	E= [10 - 12) points,
C= [14 - 16) points,	F= [0 - 10) points.

Instructions: Write clear and concise answers to the questions.

Problem 1 (4 p)

(a) (2 p) Suppose that X is a continuously distributed random variable with density function f_X. For $Y = aX + b$, $a, b \in \mathbb{R}$, $a \neq 0$, prove that

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

(b) (2 p) Suppose that $X \sim LN(\mu, \sigma^2)$ and $Y = \ln X$. Show that $Y \sim N(\mu, \sigma^2)$.

Problem 2 (7 p)

Suppose that $X_1, X_2, \ldots, X_n$ are non-negative, integer-valued random variables.

(a) (2 p) Suppose that $X_1, X_2, \ldots, X_n$ are also independent and identically distributed with probability generating function $g_{X_1}(t)$. Derive the formula for the probability generating function $g_Y(t)$ for $Y = X_1 + X_2 + \cdots + X_n$.

(b) (2 p) Suppose that n and m are non-negative integers. Show that

$$g_{nX_1+m}(t) = t^m g_{X_1}(t^n).$$

(c) (3 p) Suppose that $X_1 \sim Fs(\theta)$, $0 < \theta < 1$ (First success distribution). Derive the formula for the probability generating function of X_1.

Problem 3 (5 p)

Suppose that $X = (X_1, X_2, X_3)^\top$ and $X \sim N(\mu, \Sigma)$ with $\mu = (1, 1, 0)^\top$ and

$$\Sigma = \begin{pmatrix} 4 & -2 & 1 \\ -2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) (2 p) Determine the moment generating function $\psi_Y(t)$, where $t = (t_1, t_2)^\top$, of $Y = (Y_1, Y_2)^\top$ with $Y_1 = X_1$ and $Y_2 = X_2$.
Note that the moment generating function should be written explicitly as a function of t_1 and t_2.

(b) (3 p) Determine the conditional distribution of $X_1 + 2X_2$ given $X_2 - X_3 = 1$.

Problem 4 (4 p)

Let X_1, X_2, ... be independent $U(1, 2)$-distributed random variables.

(a) (3 p) Show for $Y_n = max\,\{X_1, \ldots, X_n\}$ that $Y_n \xrightarrow{p} 2$.

(b) (1 p) Would it be correct to conclude that $Y_n \xrightarrow{d} 2$ using the statement in part a)? Would it be correct to conclude that $Y_n \xrightarrow{a.s.} 2$?