LINKÖPINGS UNIVERSITET
IDA, STIMA

Examination - Probability Theory
2021-10-25

Course: 732A63 Probability Theory

Date and time: 2021/10/25, 08-12

Examinator: Maryna Prus

Allowed aids: Pocket calculator.
Table with common formulas and moment generating functions (distributed with the exam).
Table of integrals (distributed with the exam).
Table with distributions from Appendix B in the course book (distributed with the exam).
An A4 paper with your own hand-written notes (front- and backside).

Grades: Final grade is based on sum of points from written exam (max 20) and bonus points from seminars (max 1.5 to reach A, max 2 to reach B).

A= from 19 points,	D= [12 - 14) points,
B= [17 - 19) points,	E= [10 - 12) points,
C= [14 - 17) points,	F= [0 - 10) points.

Instructions: Write clear and concise answers to the questions.

Problem 1 (6 p)

Suppose that $X_1, X_2, \ldots, X_n$ are independent, non-negative, integer-valued random variables.

(a) (2 p) Suppose that $X_1, X_2, \ldots, X_n$ are also identically distributed with probability generating function $g_{X_1}(t)$. Derive the formula for the probability generating function $g_Y(t)$ for $Y = X_1 + X_2 + \cdots + X_n$.

(b) (2 p) Suppose that $X_i \sim Po(\lambda_i)$, for $i = 1, \ldots, n$. Determine the distribution of $Y = X_1 + X_2 + X_3$ using probability generating functions.

(c) (2 p) Suppose that $X_i \sim Bin(k, p)$, for $i = 1, \ldots, n$. Determine the probability generating function of $Y = 2X_1 + X_2$.

Problem 2 (6 p)

Suppose that X_1, X_2, ..., X_n are independent, identically distributed, continuous random variables with (cumulative) distribution function F_X and the density function f_X for each X_i, $i = 1, \ldots, n$.

(a) (2 p) Derive the formulas for the (cumulative) distribution function and the density function of $X_{(n)} = max\,\{X_1, \ldots, X_n\}$.

(b) (2 p) Suppose that $X_i \sim Exp(a)$, for $i = 1, \ldots, n$. Determine the distribution of $X_{(1)} = min\,\{X_1, \ldots, X_n\}$.

(c) (2 p) Suppose that $X_i \sim Exp(a)$, for $i = 1, \ldots, n$ and $n = 3$. Determine the joint density function of the order variables $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$ $(X_{(1)} \leq X_{(2)} \leq X_{(3)})$.

Problem 3 (5 p)

Suppose that $X = (X_1, X_2, X_3)^\top$ and $X \sim N(\mu, \Sigma)$ with $\mu = (1, 1, 0)^\top$ and

$$\Sigma = \begin{pmatrix} 4 & -2 & 1 \\ -2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) (2 p) Determine the moment generating function $\psi_Y(t)$, where $t = (t_1, t_2)^\top$, of $Y = (Y_1, Y_2)^\top$ with $Y_1 = X_1$ and $Y_2 = X_2$.
Note that the moment generating function should be written explicitly as a function of t_1 and t_2.

(b) (3 p) Determine the conditional distribution of $X_1 + 2X_2$ given $X_2 - X_3 = 1$.

Problem 4 (3 p)

Suppose that X_1, X_2, ... are independent random variables with $X_n \sim \Gamma(n, \frac{1}{n})$. Show that $X_n \xrightarrow{p} 1$ as $n \to \infty$.

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	EX	$\operatorname{Var} X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n = 1, 2, \ldots,\ 0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$	$\dfrac{p}{1 - qe^{it}}$
First success $\mathrm{Fs}(p)$, $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	$\dfrac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n = 1, 2, 3, \ldots,$ $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\dfrac{q}{p}$	$n\dfrac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m > 0$	$p(k) = e^{-m}\dfrac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N,n,p)$, $n = 0, 1, \ldots, N,$ $N = 1, 2, \ldots,$ $p = 0, \frac{1}{N}, \frac{2}{N}, \ldots, 1$	$p(k) = \dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\ k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\dfrac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	EX	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular $U(a,b)$	$f(x) = \dfrac{1}{b-a},\ a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb} - e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x) = 1,\ 0 < x < 1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x) = \frac{1}{2},\ \lvert x\rvert < 1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular $\mathrm{Tri}(a,b)$	$f(x) = \dfrac{2}{b-a}\left(1 - \dfrac{2}{b-a}\left\lvert x - \dfrac{a+b}{2}\right\rvert\right)$ $a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2} - e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x) = 1 - \lvert x\rvert,\ \lvert x\rvert < 1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a)$, $a > 0$	$f(x) = \dfrac{1}{a}e^{-x/a},\ x > 0$	a	a^2	$\dfrac{1}{1 - ait}$
Gamma $\Gamma(p,a)$, $a > 0$, $p > 0$	$f(x) = \dfrac{1}{\Gamma(p)} x^{p-1} \dfrac{1}{a^p} e^{-x/a},\ x > 0$	pa	pa^2	$\dfrac{1}{(1 - ait)^p}$
Chi-square $\chi^2(n)$, $n = 1, 2, 3, \ldots$	$f(x) = \dfrac{1}{\Gamma(\frac{n}{2})} x^{\frac{1}{2}n-1} \left(\frac{1}{2}\right)^{n/2} e^{-x/2},\ x > 0$	n	$2n$	$\dfrac{1}{(1 - 2it)^{n/2}}$
Laplace $L(a)$, $a > 0$	$f(x) = \dfrac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty < x < \infty$	0	$2a^2$	$\dfrac{1}{1 + a^2t^2}$
Beta $\beta(r,s)$, $r, s > 0$	$f(x) = \dfrac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1},$ $0 < x < 1$	$\dfrac{r}{r+s}$	$\dfrac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	EX	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha}$, $x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)$ $-\Gamma(\beta+1)^2\big)$	$*$
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha}$, $x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	$*$
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$, $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2}$, $-\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2}$, $x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	$*$
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}}$, $-\infty<x<\infty$	0	$\dfrac{n}{n-2}$, $n>2$	$*$
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}}$, $x>0$	$\frac{n}{n-2}$, $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\left(\frac{n}{n-2}\right)^2$, $n>4$	$*$

Continuous Distributions (continued)

Distribution, notation	Density	EX	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy $C(m,a)$	$f(x)=\frac{1}{\pi}\cdot\frac{a}{a^2+(x-m)^2}$, $-\infty<x<\infty$	$\nexists$	$\nexists$	$e^{imt-a\lvert t\rvert}$
$C(0,1)$	$f(x)=\dfrac{1}{\pi}\cdot\dfrac{1}{1+x^2}$, $-\infty<x<\infty$	$\nexists$	$\nexists$	$e^{-\lvert t\rvert}$
Pareto $\mathrm{Pa}\,(k,\alpha)$, $k>0$, $\alpha>0$	$f(x)=\dfrac{\alpha k^{\alpha}}{x^{\alpha+1}}$, $x>k$	$\dfrac{\alpha k}{\alpha-1}$, $\alpha>1$	$\dfrac{\alpha k^2}{(\alpha-2)(\alpha-1)^2}$, $\alpha>2$,	$*$

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2016-10-18
Probability Theory, 6 hp
732A63

Table with common formulas and moment generating functions

Some common mathematical results

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$
$\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$ if $r \neq 1$
$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if $|r| < 1$
$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Moment generating functions for some common distributions

Distribution	Abbreviation	Moment generating function
Bernoulli	$Be(p)$	$q + pe^t$
Binomial	$Bin(n,p)$	$[q + pe^t]^n$
Poisson	$Po(m)$	$e^{m(e^t-1)}$
Uniform	$U(a,b)$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Exponential	$Exp(a)$	$\frac{1}{1-at}$ for $t < 1/a$
Gamma	$\Gamma(p,a)$	$\frac{1}{(1-at)^p}$ for $t < 1/a$
Laplace	$L(a)$	$\frac{1}{1-a^2t^2}$ for $\|t\| < 1/a$
Normal	$N(\mu,\sigma^2)$	$e^{t\mu+\sigma^2t^2/2}$

Some statistical results

$Y|X = x \sim N\left[\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1-\rho^2)\right]$ if Y and X are jointly normal.

$\vec{Y}|\vec{X} = \vec{x} \sim N\left[\vec{\mu}_y + \mathbf{\Sigma}_{yx}\mathbf{\Sigma}_{xx}^{-1}(\vec{x} - \vec{\mu}_x), \mathbf{\Sigma}_{yy} - \mathbf{\Sigma}_{yx}\mathbf{\Sigma}_{xx}^{-1}\mathbf{\Sigma}_{xy}\right]$ if Y and X are jointly normal.

Table of Integrals*

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \tag{1}$$

$$\int \frac{1}{x}dx = \ln|x| \tag{2}$$

$$\int udv = uv - \int vdu \tag{3}$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| \tag{4}$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a} \tag{5}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1 \tag{6}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \tag{7}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1}x \tag{8}$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \tag{9}$$

$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2| \tag{10}$$

$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a} \tag{11}$$

$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2| \tag{12}$$

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \tag{13}$$

$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x},\ a \neq b \tag{14}$$

$$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x| \tag{15}$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \tag{16}$$

Integrals with Roots

$$\int \sqrt{x-a}dx = \frac{2}{3}(x-a)^{3/2} \tag{17}$$

$$\int \frac{1}{\sqrt{x\pm a}}dx = 2\sqrt{x\pm a} \tag{18}$$

$$\int \frac{1}{\sqrt{a-x}}dx = -2\sqrt{a-x} \tag{19}$$

$$\int x\sqrt{x-a}dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2} \tag{20}$$

$$\int \sqrt{ax+b}dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b} \tag{21}$$

$$\int (ax+b)^{3/2}dx = \frac{2}{5a}(ax+b)^{5/2} \tag{22}$$

$$\int \frac{x}{\sqrt{x\pm a}}dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a} \tag{23}$$

$$\int \sqrt{\frac{x}{a-x}}dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a} \tag{24}$$

$$\int \sqrt{\frac{x}{a+x}}dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x}+\sqrt{x+a}\right] \tag{25}$$

$$\int x\sqrt{ax+b}dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b} \tag{26}$$

$$\int \sqrt{x(ax+b)}dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right|\right] \tag{27}$$

$$\int \sqrt{x^3(ax+b)}dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right| \tag{28}$$

$$\int \sqrt{x^2\pm a^2}dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{29}$$

$$\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} \tag{30}$$

$$\int x\sqrt{x^2\pm a^2}dx = \frac{1}{3}\left(x^2\pm a^2\right)^{3/2} \tag{31}$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{32}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} \tag{33}$$

$$\int \frac{x}{\sqrt{x^2\pm a^2}}dx = \sqrt{x^2\pm a^2} \tag{34}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2} \tag{35}$$

$$\int \frac{x^2}{\sqrt{x^2\pm a^2}}dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \mp \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{36}$$

$$\int \sqrt{ax^2+bx+c}dx = \frac{b+2ax}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx^+c)}\right| \tag{37}$$

$$\int x\sqrt{ax^2+bx+c} = \frac{1}{48a^{5/2}}\left(2\sqrt{a}\sqrt{ax^2+bx+c} \times \left(-3b^2+2abx+8a(c+ax^2)\right) + 3(b^3-4abc)\ln\left|b+2ax+2\sqrt{a}\sqrt{ax^2+bx+c}\right|\right) \tag{38}$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{39}$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{40}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \tag{41}$$

Integrals with Logarithms

$$\int \ln ax dx = x\ln ax - x \tag{42}$$

$$\int \frac{\ln ax}{x}dx = \frac{1}{2}(\ln ax)^2 \tag{43}$$

$$\int \ln(ax+b)dx = \left(x+\frac{b}{a}\right)\ln(ax+b) - x, a \neq 0 \tag{44}$$

$$\int \ln(x^2+a^2)\,\mathrm{d}x = x\ln(x^2+a^2) + 2a\tan^{-1}\frac{x}{a} - 2x \tag{45}$$

$$\int \ln(x^2-a^2)\,\mathrm{d}x = x\ln(x^2-a^2) + a\ln\frac{x+a}{x-a} - 2x \tag{46}$$

$$\int \ln\left(ax^2+bx+c\right)dx = \frac{1}{a}\sqrt{4ac-b^2}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a}+x\right)\ln\left(ax^2+bx+c\right) \tag{47}$$

$$\int x\ln(ax+b)dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2-\frac{b^2}{a^2}\right)\ln(ax+b) \tag{48}$$

$$\int x\ln\left(a^2-b^2x^2\right)dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2-\frac{a^2}{b^2}\right)\ln\left(a^2-b^2x^2\right) \tag{49}$$

Integrals with Exponentials

$$\int e^{ax}dx = \frac{1}{a}e^{ax} \tag{50}$$

$$\int \sqrt{x}e^{ax}dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\mathrm{erf}\left(i\sqrt{ax}\right), \quad \text{where } \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{51}$$

$$\int xe^x dx = (x-1)e^x \tag{52}$$

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \tag{53}$$

$$\int x^2e^x dx = \left(x^2-2x+2\right)e^x \tag{54}$$

$$\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} \tag{55}$$

$$\int x^3e^x dx = \left(x^3-3x^2+6x-6\right)e^x \tag{56}$$

$$\int x^ne^{ax}\,\mathrm{d}x = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x \tag{57}$$

$$\int x^ne^{ax}\,\mathrm{d}x = \frac{(-1)^n}{a^{n+1}}\Gamma[1+n,-ax], \quad \text{where } \Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,\mathrm{d}t \tag{58}$$

$$\int e^{ax^2}\,\mathrm{d}x = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(ix\sqrt{a}\right) \tag{59}$$

$$\int e^{-ax^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(x\sqrt{a}\right) \tag{60}$$

$$\int xe^{-ax^2}\,\mathrm{d}x = -\frac{1}{2a}e^{-ax^2} \tag{61}$$

$$\int x^2e^{-ax^2}\,\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\mathrm{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2} \tag{62}$$

Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a}\cos ax \tag{63}$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \tag{64}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \;\; {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \tag{65}$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} \tag{66}$$

$$\int \cos ax dx = \frac{1}{a}\sin ax \tag{67}$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \tag{68}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \tag{69}$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} \tag{70}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \tag{71}$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \tag{72}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x \tag{73}$$

$$\int \cos^2 ax \sin bx dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} \tag{74}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a}\cos^3 ax \tag{75}$$

$$\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \tag{76}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \tag{77}$$

$$\int \tan ax dx = -\frac{1}{a}\ln\cos ax \tag{78}$$

$$\int \tan^2 ax dx = -x + \frac{1}{a}\tan ax \tag{79}$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \tag{80}$$

$$\int \tan^3 ax dx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax \tag{81}$$

$$\int \sec x dx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan\frac{x}{2}\right) \tag{82}$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax \tag{83}$$

$$\int \sec^3 x \, \mathrm{d}x = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| \tag{84}$$

$$\int \sec x \tan x dx = \sec x \tag{85}$$

$$\int \sec^2 x \tan x dx = \frac{1}{2}\sec^2 x \tag{86}$$

$$\int \sec^n x \tan x dx = \frac{1}{n}\sec^n x, n \neq 0 \tag{87}$$

$$\int \csc x dx = \ln\left|\tan\frac{x}{2}\right| = \ln|\csc x - \cot x| + C \tag{88}$$

$$\int \csc^2 ax dx = -\frac{1}{a}\cot ax \tag{89}$$

$$\int \csc^3 x dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\ln|\csc x - \cot x| \tag{90}$$

$$\int \csc^n x \cot x dx = -\frac{1}{n}\csc^n x, n \neq 0 \tag{91}$$

$$\int \sec x \csc x dx = \ln|\tan x| \tag{92}$$

Products of Trigonometric Functions and Monomials

$$\int x\cos x dx = \cos x + x\sin x \tag{93}$$

$$\int x\cos ax dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax \tag{94}$$

$$\int x^2\cos x dx = 2x\cos x + \left(x^2-2\right)\sin x \tag{95}$$

$$\int x^2\cos ax dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax \tag{96}$$

$$\int x^n cos x dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1,-ix) + (-1)^n\Gamma(n+1,ix)\right] \tag{97}$$

$$\int x^n cos ax dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1,-iax) - \Gamma(n+1,ixa)\right] \tag{98}$$

$$\int x\sin x dx = -x\cos x + \sin x \tag{99}$$

$$\int x\sin ax dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2} \tag{100}$$

$$\int x^2\sin x dx = \left(2-x^2\right)\cos x + 2x\sin x \tag{101}$$

$$\int x^2\sin ax dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2} \tag{102}$$

$$\int x^n\sin x dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1,-ix) - (-1)^n\Gamma(n+1,-ix)\right] \tag{103}$$

Products of Trigonometric Functions and Exponentials

$$\int e^x\sin x dx = \frac{1}{2}e^x(\sin x - \cos x) \tag{104}$$

$$\int e^{bx}\sin ax dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax) \tag{105}$$

$$\int e^x\cos x dx = \frac{1}{2}e^x(\sin x + \cos x) \tag{106}$$

$$\int e^{bx}\cos ax dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax) \tag{107}$$

$$\int xe^x\sin x dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x) \tag{108}$$

$$\int xe^x\cos x dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x) \tag{109}$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a}\sinh ax \tag{110}$$

$$\int e^{ax}\cosh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2} & a = b \end{cases} \tag{111}$$

$$\int \sinh ax dx = \frac{1}{a}\cosh ax \tag{112}$$

$$\int e^{ax}\sinh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} & a = b \end{cases} \tag{113}$$

$$\int e^{ax}\tanh bx dx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)}{}_2F_1\left[1+\dfrac{a}{2b}, 1, 2+\dfrac{a}{2b}, -e^{2bx}\right] - \dfrac{1}{a}e^{ax}{}_2F_1\left[\dfrac{a}{2b}, 1, 1E, -e^{2bx}\right] & a \neq b \\ \dfrac{e^{ax} - 2\tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \tag{114}$$

$$\int \tanh ax \, dx = \frac{1}{a}\ln\cosh ax \tag{115}$$

$$\int \cos ax\cosh bx dx = \frac{1}{a^2+b^2}\left[a\sin ax\cosh bx + b\cos ax\sinh bx\right] \tag{116}$$

$$\int \cos ax\sinh bx dx = \frac{1}{a^2+b^2}\left[b\cos ax\cosh bx + a\sin ax\sinh bx\right] \tag{117}$$

$$\int \sin ax\cosh bx dx = \frac{1}{a^2+b^2}\left[-a\cos ax\cosh bx + b\sin ax\sinh bx\right] \tag{118}$$

$$\int \sin ax\sinh bx dx = \frac{1}{a^2+b^2}\left[b\cosh bx\sin ax - a\cos ax\sinh bx\right] \tag{119}$$

$$\int \sinh ax\cosh ax dx = \frac{1}{4a}\left[-2ax + \sinh 2ax\right] \tag{120}$$

$$\int \sinh ax\cosh bx dx = \frac{1}{b^2-a^2}\left[b\cosh bx\sinh ax - a\cosh ax\sinh bx\right] \tag{121}$$