

SEMINAR EXERCISES IN PROBABILITY THEORY

732A63

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3 Transforms

Exercise 3.0.1 (self-study)

Prove Theorem 3.2 in page 63 in Gut's book.

Exercise 3.1

Suppose X has probability function $p_X(k) = (1 - \theta)\theta^{k-1}$, $k = 1, 2, \dots$, for $0 \leq \theta \leq 1$ and that Y has density function $f_Y(y) = 2y$, $0 \leq y \leq 1$.

(a) Derive the probability generating function of X .

(b) Derive the moment generating function of Y .

(c) Let $S_X = Y_1 + Y_2 + \dots + Y_X$ be the sum of X i.i.d. random variables with the same distribution as Y in (b), and assume that Y_1, Y_2, \dots, Y_X are all independent of X , where X is distributed as in (a). Compute the characteristic function of S_X .

Exercise 3.2 (3.1 in Gut's book)

The non-negative, integer-valued random variable X has generating function $g_X(t) = \log(1/(1 - qt))$. Determine $P(X = k)$ for $k = 0, 1, 2, \dots$, $E(X)$, and $Var(X)$.

Exercise 3.3 (3.6 in Gut's book)

Show, by using moment-generating functions, that if $X \sim L(1)$, then $X \stackrel{d}{=} Y_1 - Y_2$, where Y_1 and Y_2 are independent, exponentially-distributed random variables.

Exercise 3.4 (3.34 in Gut's book)

Suppose that X is a non-negative, integer-valued random variable, and let n and m be non-negative integers. Show that

$$g_{nX+m}(t) = t^m \cdot g_X(t^n).$$

Exercise 3.5* (3.5 in Gut's book)

Let $Y \sim \beta(n, m)$, where n, m are integers.

- (a) Derive the moment generating function of $-\log Y$.
- (b) Show that $-\log Y$ has the same distribution as $\sum_{k=1}^m X_k$, where X_1, X_2, \dots are independent, exponentially-distributed random variables.

Remark. The formula $\Gamma(r + s)/\Gamma(r) = (r + s - 1) \cdots (r + 1)r$, which holds when s is an integer, might be useful.

Exercise 3.6* (3.26 in Gut's book)

The number of cars passing a road crossing during an hour is $Po(b)$ -distributed. The number of passengers in each car is $Po(p)$ -distributed. Find the generating function of the total number of passengers, Y , passing the road crossing during one hour, and compute $E(Y)$ and $Var(Y)$.

Self-study exercises are excluded from the bonus-point deal. Exercises marked with * are a bit more challenging.

May Gauss be with you!