Exam 24.10.2022

Solution

Pr. 1.

(2) a) $Y = aX + b \rightarrow X = \frac{Y-b}{a} = h(y)$

$$f_Y(y) = f_X(h(y)) \cdot \left| \frac{\partial h(y)}{\partial y} \right|$$

$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}.$$

(2) b) $Y = \ln X \rightarrow X = \exp(Y) = h(y)$

$$f_Y(y) = f_X(h(y)) \left| \frac{\partial h(y)}{\partial y} \right|$$

$$= \frac{1}{\sigma \exp(y) \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}(y-\mu)^2/\sigma^2\right) \cdot |\underbrace{\exp(y)}_{>0}|$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y-\mu)^2/\sigma^2\right).$$

Pr. 2

(2) a)

$$g_Y(t) = Et^Y = Et^{X_1+\dots+X_n} \overset{\text{ind.}}{=} \prod_{i=1}^{n} \underbrace{Et^{X_i}}_{g_{X_1}(t)} = \left(g_{X_1}(t)\right)^n.$$

(2) b) $g_{nX_1+m}(t) = Et^{nX_1+m} = t^m E t^{nX_1} = t^m E(t^n)^{X_1}$

$$= t^m g_{X_1}(t^n).$$

(3) c) $g_{X_1}(t) = Et^{X_1} = \sum_{k=1}^{\infty} t^k P(X_1 = k) = \sum_{k=1}^{\infty} t^k \cdot p(1-p)^{k-1}$

$$= p \cdot t \sum_{k=1}^{\infty} (t(1-p))^{k-1} = pt \sum_{k=0}^{\infty} (t(1-p))^k$$

$$\sum_{k=0}^{\infty} (t(1-p))^k = \frac{1}{1-t(1-p)}$$

$$= p \cdot t \cdot \frac{1}{1-t(1-p)}.$$

② a) $Y \sim N(\mu_Y, \Sigma_Y)$, where $\mu_Y = (1,1)^T$

$$\Sigma_Y = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}.$$

$$\varphi_Y(t) = \exp\left(t^T\mu_Y + \frac{1}{2} t^T \Sigma_Y t\right)$$

$$t^T\mu_Y = t_1 + t_2$$

$$t^T\Sigma_Y t = \underbrace{(t_1, t_2)\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}}_{(4t_1 - 2t_2,\ -2t_1 + 3t_2)} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$= 4t_1^2 - 2t_1t_2 - 2t_1t_2 + 3t_2^2$$

$$= 4t_1^2 - 4t_1t_2 + 3t_2^2$$

$$\Rightarrow \varphi_Y(t) = \exp\left(t_1 + t_2 + \frac{1}{2}\left(4t_1^2 - 4t_1t_2 + 3t_2^2\right)\right)$$

③ b) $Z = (Z_1, Z_2)^T = (X_1 + 2X_2, X_2 - X_3)^T$

$$= \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}}_{A} \cdot X$$

$$\Rightarrow \mu_Z = A \cdot (1,1,0)^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \underline{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}.$$

$$\Sigma_Z = \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 & -2 & 1 \\ -2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & 4 & 1 \\ -3 & 3 & -1 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 3 \\ 3 & 4 \end{pmatrix}.$$

$$Z_1 \mid Z_2 = 1 \sim N(\overset{=③}{\tilde{\mu}}, \overset{=23/4}{\tilde{\sigma}^2}), \qquad \rho = \frac{\operatorname{cov}(Z_1, Z_2)}{\sqrt{\operatorname{Var} Z_1 \operatorname{Var} Z_2}} = \frac{3}{4\sqrt{2}} = \frac{3}{\sqrt{32}}$$

$$\tilde{\mu} = \overset{=3}{\mu_{Z_1}} + \rho \frac{\sigma_{Z_1}}{\sigma_{Z_2}} \underbrace{(1 - \overset{=1}{\mu_{Z_2}})}_{=0} = 3$$

$$\tilde{\sigma}^2 = \sigma_{Z_1}^2(1 - \rho^2) = 8\left(1 - \frac{9}{32}\right) = 8 \cdot \frac{23}{32} = \frac{23}{4}$$

Pr. 4.

3 a) $P(|\underbrace{Y_n}_{\in(1,2)} - 2| > \varepsilon) \to 0$

$$P(|Y_n - 2| > \varepsilon) = P(Y_n - 2 < -\varepsilon)$$
$$= P(Y_n < 2-\varepsilon)$$
$$= P(\max(X_1,..,X_n) < 2-\varepsilon)$$
$$\overset{\text{cont. distr.}}{=} F_{\max(X_1,..,X_n)}(2-\varepsilon)$$
$$= \left(F_{X_1}(2-\varepsilon)\right)^n$$
$$= \left(\frac{2-\varepsilon-1}{1}\right)^n = (1-\varepsilon)^n \xrightarrow[n\to\infty]{} 0$$

if $\varepsilon \leq 1$.

If $\varepsilon > 1$, $P(|Y_n - 2| > \varepsilon) < P(|Y_n-2| > 1)$
$= 0$ as $Y_n \in (1,2)$.

$\Rightarrow$ $Y_n \xrightarrow{P} 2$.

1 b) $Y_n \xrightarrow{d} 2$ – yes

$Y_n \xrightarrow{a.s.}$ – no.