

Examination - Probability Theory
2021-12-6

Course: 732A63 Probability Theory

Date and time: 2021/12/6, 08-12

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Allowed aids: Pocket calculator.

Table with common formulas and moment generating functions (distributed with the exam).

Table of integrals (distributed with the exam).

Table with distributions from Appendix B in the course book (distributed with the exam).

An A4 paper with your own hand-written notes (front- and backside).

Grades: Final grade is based on sum of points from written exam (max 20) and bonus points from seminars (max 1.5 to reach A, max 2 to reach B).

A= from 19 points,

D= [12 - 14) points,

B= [17 - 19) points,

E= [10 - 12) points,

C= [14 - 17) points,

F= [0 - 10) points.

Instructions: Write clear and concise answers to the questions.

Problem 1 (4 p)

- (a) (2 p) Suppose that X is a continuously distributed random variable with density function f_X . For $Y = aX + b$, $a, b \in \mathbb{R}$, $a \neq 0$, prove that

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

- (b) (2 p) Suppose that $X \sim LN(\mu, \sigma^2)$ and $Y = \ln X$. Show that $Y \sim N(\mu, \sigma^2)$.

Problem 2 (6 p)

Suppose that X_1, X_2, \dots, X_n are independent, non-negative, integer-valued random variables.

- (a) (2 p) Suppose that X_1, X_2, \dots, X_n are also identically distributed with probability generating function $g_{X_1}(t)$. Derive the formula for the probability generating function $g_Y(t)$ for $Y = X_1 + X_2 + \dots + X_n$.
- (b) (2 p) Suppose that $X_i \sim Po(\lambda_i)$, for $i = 1, \dots, n$. Determine the distribution of $Y = X_1 + X_2 + X_3$ using probability generating functions.
- (c) (2 p) Suppose that $X_1 \sim Bin(k, p)$, $X_2 \sim Be(p)$. Determine the distribution of $Y = X_1 + X_2$ using probability generating functions.

Problem 3 (5 p)

Suppose that X_1, X_2, \dots, X_n are independent, identically distributed, continuous random variables with (cumulative) distribution function F_X and the density function f_X for each $X_i, i = 1, \dots, n$.

- (a) (2 p) Derive the formulas for the (cumulative) distribution function and the density function of $X_{(1)} = \min \{X_1, \dots, X_n\}$.
- (b) (1.5 p) Suppose that $X_i \sim \text{Exp}(a)$, for $i = 1, \dots, n$. Determine the distribution of $X_{(1)} = \min \{X_1, \dots, X_n\}$.
- (c) (1.5 p) Suppose that $X_i \sim \text{Exp}(a)$, for $i = 1, \dots, n$ and $n = 3$. Determine the (cumulative) distribution function of $X_{(3)} = \max \{X_1, X_2, X_3\}$.

Problem 4 (5 p)

Suppose that $X = (X_1, X_2, X_3)^\top$ and $X \sim N(\mu, \Sigma)$ with $\mu = (0, 1, 1)^\top$ and

$$\Sigma = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

- (a) (2 p) Determine the moment generating function $\psi_Y(t)$, where $t = (t_1, t_2)^\top$, of $Y = (Y_1, Y_2)^\top$ with $Y_1 = X_1$ and $Y_2 = X_3$.
Note that the moment generating function should be written explicitly as a function of t_1 and t_2 .
- (b) (3 p) Determine the conditional distribution of $X_1 + X_2$ given $X_2 - X_3 = x$.

Discrete Distributions

Following is a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	EX	$\text{Var } X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\text{Be}(p)$, $0 \leq p \leq 1$	$p(0) = q$, $p(1) = p$; $q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\text{Bin}(n, p)$, $n = 1, 2, \dots$, $0 \leq p \leq 1$	$p(k) = \binom{n}{k} p^k q^{n-k}$, $k = 0, 1, \dots, n$; $q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\text{Ge}(p)$, $0 \leq p \leq 1$	$p(k) = pq^k$, $k = 0, 1, 2, \dots$; $q = 1 - p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1 - qe^{it}}$
First success $\text{Fs}(p)$, $0 \leq p \leq 1$	$p(k) = pq^{k-1}$, $k = 1, 2, \dots$; $q = 1 - p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\text{NBin}(n, p)$, $n = 1, 2, 3, \dots$, $0 \leq p \leq 1$	$p(k) = \binom{n+k-1}{k} p^n q^k$, $k = 0, 1, 2, \dots$; $q = 1 - p$	$n \frac{q}{p}$	$n \frac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson $\text{Po}(m)$, $m > 0$	$p(k) = e^{-m} \frac{m^k}{k!}$, $k = 0, 1, 2, \dots$	m	m	$e^{m(e^{it} - 1)}$
Hypergeometric $H(N, n, p)$, $n = 0, 1, \dots, N$, $N = 1, 2, \dots$, $p = 0, \frac{1}{N}, \frac{2}{N}, \dots, 1$	$p(k) = \frac{\binom{Np}{k} \binom{Nq}{n-k}}{\binom{N}{n}}$, $k = 0, 1, \dots, Np$; $q = 1 - p$; $n - k = 0, \dots, Nq$	np	$npq \frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	EX	$\text{Var } X$	$\varphi_X(t)$
Uniform/Rectangular $U(a, b)$	$f(x) = \frac{1}{b-a}$, $a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb} - e^{ita}}{it(b-a)}$
$U(0, 1)$	$f(x) = 1$, $0 < x < 1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it} - 1}{it}$
$U(-1, 1)$	$f(x) = \frac{1}{2}$, $ x < 1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular $\text{Tri}(a, b)$	$f(x) = \frac{2}{b-a} \left(1 - \frac{2}{b-a} \left x - \frac{a+b}{2} \right \right)$ $a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2} - e^{ita/2}}{\frac{1}{2}it(b-a)} \right)^2$
$\text{Tri}(-1, 1)$	$f(x) = 1 - x $, $ x < 1$	0	$\frac{1}{6}$	$\left(\frac{\sin \frac{t}{2}}{\frac{t}{2}} \right)^2$
Exponential $\text{Exp}(a)$, $a > 0$	$f(x) = \frac{1}{a} e^{-x/a}$, $x > 0$	a	a^2	$\frac{1}{1 - ait}$
Gamma $\Gamma(p, a)$, $a > 0$, $p > 0$	$f(x) = \frac{1}{\Gamma(p)} x^{p-1} \frac{1}{a^p} e^{-x/a}$, $x > 0$	pa	pa^2	$\frac{1}{(1 - ait)^p}$
Chi-square $\chi^2(n)$, $n = 1, 2, 3, \dots$	$f(x) = \frac{1}{\Gamma(\frac{n}{2})} x^{\frac{1}{2}n-1} \left(\frac{1}{2} \right)^{n/2} e^{-x/2}$, $x > 0$	n	$2n$	$\frac{1}{(1 - 2it)^{n/2}}$
Laplace $L(a)$, $a > 0$	$f(x) = \frac{1}{2a} e^{- x /a}$, $-\infty < x < \infty$	0	$2a^2$	$\frac{1}{1 + a^2 t^2}$
Beta $\beta(r, s)$, $r, s > 0$	$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1} (1-x)^{s-1}$, $0 < x < 1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	EX	$\text{Var } X$	$\varphi_X(t)$
Weibull $W(\alpha, \beta), \alpha, \beta > 0$	$f(x) = \frac{1}{\alpha\beta} x^{(1/\beta)-1} e^{-x^{1/\beta}/\alpha}, x > 0$	$\alpha^\beta \Gamma(\beta + 1)$	$\alpha^{2\beta} (\Gamma(2\beta + 1) - \Gamma(\beta + 1)^2)$	*
Rayleigh $\text{Ra}(\alpha), \alpha > 0$	$f(x) = \frac{2}{\alpha} x e^{-x^2/\alpha}, x > 0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1 - \frac{1}{4}\pi)$	*
Normal $N(\mu, \sigma^2),$ $-\infty < \mu < \infty, \sigma > 0$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty < x < \infty$	μ	σ^2	$e^{i\mu t - \frac{1}{2}t^2\sigma^2}$
$N(0, 1)$	$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < x < \infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu, \sigma^2),$ $-\infty < \mu < \infty, \sigma > 0$	$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{1}{2}(\log x - \mu)^2/\sigma^2}, x > 0$	$e^{\mu + \frac{1}{2}\sigma^2}$	$e^{2\mu}(e^{2\sigma^2} - e^{\sigma^2})$	*
(Student's) t $t(n), n = 1, 2, \dots$	$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n} \Gamma(\frac{n}{2})} \cdot d \frac{1}{(1 + \frac{x^2}{n})(n+1)/2},$ $-\infty < x < \infty$	0	$\frac{n}{n-2}, n > 2$	*
(Fisher's) F $F(m, n), m, n = 1, 2, \dots$	$f(x) = \frac{\Gamma(\frac{m+n}{2})\Gamma(\frac{m}{2})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})^{n/2}} \cdot \frac{x^{m/2-1}}{(1 + \frac{mx}{n})^{(m+n)/2}},$ $x > 0$	$\frac{n}{n-2},$ $n > 2$	$\frac{n^2(m+2)}{m(n-2)(n-4)} - \left(\frac{n}{n-2}\right)^2,$ $n > 4$	*

Continuous Distributions (continued)

Distribution, notation	Density	EX	$\text{Var } X$	$\varphi_X(t)$
Cauchy $C(m, a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2 + (x-m)^2}, -\infty < x < \infty$	$\bar{\exists}$	$\bar{\exists}$	$e^{imt - a t }$
$C(0, 1)$	$f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + x^2}, -\infty < x < \infty$	$\bar{\exists}$	$\bar{\exists}$	$e^{- t }$
Pareto $\text{Pa}(k, \alpha), k > 0, \alpha > 0$	$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}}, x > k$	$\frac{\alpha k}{\alpha - 1}, \alpha > 1$	$\frac{\alpha k^2}{(\alpha - 2)(\alpha - 1)^2}, \alpha > 2,$	*

Table of Integrals*

Basic Forms

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \quad (1)$$

$$\int \frac{1}{x} dx = \ln |x| \quad (2)$$

$$\int u dv = uv - \int v du \quad (3)$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| \quad (4)$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} \quad (5)$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1 \quad (6)$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \quad (7)$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x \quad (8)$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad (9)$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln |a^2+x^2| \quad (10)$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a \tan^{-1} \frac{x}{a} \quad (11)$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2} x^2 - \frac{1}{2} a^2 \ln |a^2+x^2| \quad (12)$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (13)$$

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, a \neq b \quad (14)$$

$$\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln |a+x| \quad (15)$$

$$\int \frac{x}{ax^2+bx+c} dx = \frac{1}{2a} \ln |ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (16)$$

Integrals with Roots

$$\int \sqrt{x-a} dx = \frac{2}{3} (x-a)^{3/2} \quad (17)$$

$$\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a} \quad (18)$$

$$\int \frac{1}{\sqrt{a-x}} dx = -2\sqrt{a-x} \quad (19)$$

$$\int x\sqrt{x-a} dx = \frac{2}{3} a(x-a)^{3/2} + \frac{2}{5} (x-a)^{5/2} \quad (20)$$

$$\int \sqrt{ax+b} dx = \left(\frac{2b}{3a} + \frac{2x}{3} \right) \sqrt{ax+b} \quad (21)$$

$$\int (ax+b)^{3/2} dx = \frac{2}{5a} (ax+b)^{5/2} \quad (22)$$

$$\int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3} (x \mp 2a) \sqrt{x \pm a} \quad (23)$$

$$\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x(a-x)} - a \tan^{-1} \frac{\sqrt{x(a-x)}}{x-a} \quad (24)$$

$$\int \sqrt{\frac{x}{a+x}} dx = \sqrt{x(a+x)} - a \ln [\sqrt{x} + \sqrt{x+a}] \quad (25)$$

$$\int x\sqrt{ax+b} dx = \frac{2}{15a^2} (-2b^2 + abx + 3a^2 x^2) \sqrt{ax+b} \quad (26)$$

$$\int \sqrt{x(ax+b)} dx = \frac{1}{4a^{3/2}} \left[(2ax+b) \sqrt{ax(ax+b)} - b^2 \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \right] \quad (27)$$

$$\int \sqrt{x^3(ax+b)} dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2 x} + \frac{x}{3} \right] \sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}} \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \quad (28)$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} x \sqrt{x^2 \pm a^2} \pm \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (29)$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} \quad (30)$$

$$\int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2} \quad (31)$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (32)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \quad (33)$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} \quad (34)$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} \quad (35)$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2} x \sqrt{x^2 \pm a^2} \mp \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (36)$$

$$\int \sqrt{ax^2+bx+c} dx = \frac{b+2ax}{4a} \sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}} \ln \left| 2ax+b+2\sqrt{a(ax^2+bx+c)} \right| \quad (37)$$

$$\int x\sqrt{ax^2+bx+c} dx = \frac{1}{48a^{5/2}} \left(2\sqrt{a}\sqrt{ax^2+bx+c} \times (-3b^2+2abx+8a(c+ax^2)) + 3(b^3-4abc) \ln \left| b+2ax+2\sqrt{a}\sqrt{ax^2+bx+c} \right| \right) \quad (38)$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax+b+2\sqrt{a(ax^2+bx+c)} \right| \quad (39)$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}} dx = \frac{1}{a} \sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}} \ln \left| 2ax+b+2\sqrt{a(ax^2+bx+c)} \right| \quad (40)$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \quad (41)$$

Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x \quad (42)$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2} (\ln ax)^2 \quad (43)$$

$$\int \ln(ax+b) dx = \left(x + \frac{b}{a} \right) \ln(ax+b) - x, a \neq 0 \quad (44)$$

$$\int \ln(x^2+a^2) dx = x \ln(x^2+a^2) + 2a \tan^{-1} \frac{x}{a} - 2x \quad (45)$$

$$\int \ln(x^2-a^2) dx = x \ln(x^2-a^2) + a \ln \frac{x+a}{x-a} - 2x \quad (46)$$

$$\int \ln(ax^2+bx+c) dx = \frac{1}{a} \sqrt{4ac-b^2} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a} + x \right) \ln(ax^2+bx+c) \quad (47)$$

$$\int x \ln(ax+b) dx = \frac{bx}{2a} - \frac{1}{4} x^2 + \frac{1}{2} \left(x^2 - \frac{b^2}{a^2} \right) \ln(ax+b) \quad (48)$$

$$\int x \ln(a^2-b^2x^2) dx = -\frac{1}{2} x^2 + \frac{1}{2} \left(x^2 - \frac{a^2}{b^2} \right) \ln(a^2-b^2x^2) \quad (49)$$

Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a} e^{ax} \quad (50)$$

$$\int \sqrt{x} e^{ax} dx = \frac{1}{a} \sqrt{x} e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}(i\sqrt{ax}), \text{ where } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (51)$$

$$\int x e^x dx = (x-1) e^x \quad (52)$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2} \right) e^{ax} \quad (53)$$

$$\int x^2 e^x dx = (x^2 - 2x + 2) e^x \quad (54)$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) e^{ax} \quad (55)$$

$$\int x^3 e^x dx = (x^3 - 3x^2 + 6x - 6) e^x \quad (56)$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx \quad (57)$$

$$\int x^n e^{ax} dx = \frac{(-1)^n}{a^{n+1}} \Gamma[1+n, -ax], \text{ where } \Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt \quad (58)$$

$$\int e^{ax^2} dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(i\sqrt{a}x) \quad (59)$$

$$\int e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(x\sqrt{a}) \quad (60)$$

$$\int x e^{-ax^2} dx = -\frac{1}{2a} e^{-ax^2} \quad (61)$$

$$\int x^2 e^{-ax^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{a^3}} \operatorname{erf}(x\sqrt{a}) - \frac{x}{2a} e^{-ax^2} \quad (62)$$

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Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a} \cos ax \quad (63)$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \quad (64)$$

$$\int \sin^n ax dx = -\frac{1}{a} \cos ax {}_2F_1 \left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax \right] \quad (65)$$

$$\int \sin^3 ax dx = -\frac{3 \cos ax}{4a} + \frac{\cos 3ax}{12a} \quad (66)$$

$$\int \cos ax dx = \frac{1}{a} \sin ax \quad (67)$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \quad (68)$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)} \cos^{1+p} ax \times {}_2F_1 \left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax \right] \quad (69)$$

$$\int \cos^3 ax dx = \frac{3 \sin ax}{4a} + \frac{\sin 3ax}{12a} \quad (70)$$

$$\int \cos ax \sin bxdx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \quad (71)$$

$$\int \sin^2 ax \cos bxdx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \quad (72)$$

$$\int \sin^2 x \cos x dx = \frac{1}{3} \sin^3 x \quad (73)$$

$$\int \cos^2 ax \sin bxdx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} \quad (74)$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a} \cos^3 ax \quad (75)$$

$$\int \sin^2 ax \cos^2 bxdx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \quad (76)$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \quad (77)$$

$$\int \tan ax dx = -\frac{1}{a} \ln \cos ax \quad (78)$$

$$\int \tan^2 ax dx = -x + \frac{1}{a} \tan ax \quad (79)$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1 \left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax \right) \quad (80)$$

$$\int \tan^3 ax dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax \quad (81)$$

$$\int \sec x dx = \ln |\sec x + \tan x| = 2 \tanh^{-1} \left(\tan \frac{x}{2} \right) \quad (82)$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax \quad (83)$$

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| \quad (84)$$

$$\int \sec x \tan x dx = \sec x \quad (85)$$

$$\int \sec^2 x \tan x dx = \frac{1}{2} \sec^2 x \quad (86)$$

$$\int \sec^n x \tan x dx = \frac{1}{n} \sec^n x, n \neq 0 \quad (87)$$

$$\int \csc x dx = \ln \left| \tan \frac{x}{2} \right| = \ln |\csc x - \cot x| + C \quad (88)$$

$$\int \csc^2 ax dx = -\frac{1}{a} \cot ax \quad (89)$$

$$\int \csc^3 x dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln |\csc x - \cot x| \quad (90)$$

$$\int \csc^n x \cot x dx = -\frac{1}{n} \csc^n x, n \neq 0 \quad (91)$$

$$\int \sec x \csc x dx = \ln |\tan x| \quad (92)$$

Products of Trigonometric Functions and Monomials

$$\int x \cos x dx = \cos x + x \sin x \quad (93)$$

$$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax \quad (94)$$

$$\int x^2 \cos x dx = 2x \cos x + (x^2 - 2) \sin x \quad (95)$$

$$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax \quad (96)$$

$$\int x^n \cos x dx = -\frac{1}{2} (i)^{n+1} [\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)] \quad (97)$$

$$\int x^n \cos ax dx = \frac{1}{2} (ia)^{1-n} [(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, iax)] \quad (98)$$

$$\int x \sin x dx = -x \cos x + \sin x \quad (99)$$

$$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} \quad (100)$$

$$\int x^2 \sin x dx = (2 - x^2) \cos x + 2x \sin x \quad (101)$$

$$\int x^2 \sin ax dx = \frac{2 - a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} \quad (102)$$

$$\int x^n \sin x dx = -\frac{1}{2} (i)^n [\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)] \quad (103)$$

Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) \quad (104)$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) \quad (105)$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) \quad (106)$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) \quad (107)$$

$$\int x e^x \sin x dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) \quad (108)$$

$$\int x e^x \cos x dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) \quad (109)$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a} \sinh ax \quad (110)$$

$$\int e^{ax} \cosh bxdx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [a \cosh bx - b \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} + \frac{x}{2} & a = b \end{cases} \quad (111)$$

$$\int \sinh ax dx = \frac{1}{a} \cosh ax \quad (112)$$

$$\int e^{ax} \sinh bxdx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [-b \cosh bx + a \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} - \frac{x}{2} & a = b \end{cases} \quad (113)$$

$$\int e^{ax} \tanh bxdx = \begin{cases} \frac{e^{(a+2b)x}}{(a+2b)^2} {}_2F_1 \left[1 + \frac{a}{2b}, 1, 2 + \frac{a}{2b}, -e^{2bx} \right] \\ \quad - \frac{1}{a} e^{ax} {}_2F_1 \left[\frac{a}{2b}, 1, 1E, -e^{2bx} \right] & a \neq b \\ \frac{e^{ax} - 2 \tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \quad (114)$$

$$\int \tanh ax dx = \frac{1}{a} \ln \cosh ax \quad (115)$$

$$\int \cos ax \cosh bxdx = \frac{1}{a^2 + b^2} [a \sin ax \cosh bx + b \cos ax \sinh bx] \quad (116)$$

$$\int \cos ax \sinh bxdx = \frac{1}{a^2 + b^2} [b \cos ax \cosh bx + a \sin ax \sinh bx] \quad (117)$$

$$\int \sin ax \cosh bxdx = \frac{1}{a^2 + b^2} [-a \cos ax \cosh bx + b \sin ax \sinh bx] \quad (118)$$

$$\int \sin ax \sinh bxdx = \frac{1}{a^2 + b^2} [b \cosh bx \sin ax - a \cos ax \sinh bx] \quad (119)$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a} [-2ax + \sinh 2ax] \quad (120)$$

$$\int \sinh ax \cosh bxdx = \frac{1}{b^2 - a^2} [b \cosh bx \sinh ax - a \cosh ax \sinh bx] \quad (121)$$

Table with common formulas and moment generating functions

Some common mathematical results

$$\begin{aligned}
 e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
 e^x &= \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \\
 \sum_{k=0}^{n-1} ar^k &= a \frac{1-r^n}{1-r} \quad \text{if } r \neq 1 \\
 \sum_{k=0}^{\infty} ar^k &= \frac{a}{1-r} \quad \text{if } |r| < 1 \\
 (x+y)^n &= \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}
 \end{aligned}$$

Moment generating functions for some common distributions

Distribution	Abbreviation	Moment generating function
Bernoulli	$Be(p)$	$q + pe^t$
Binomial	$Bin(n, p)$	$[q + pe^t]^n$
Poisson	$Po(m)$	$e^{m(e^t - 1)}$
Uniform	$U(a, b)$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$
Exponential	$Exp(a)$	$\frac{1}{1-at}$ for $t < 1/a$
Gamma	$\Gamma(p, a)$	$\frac{1}{(1-at)^p}$ for $t < 1/a$
Laplace	$L(a)$	$\frac{1}{1-a^2 t^2}$ for $ t < 1/a$
Normal	$N(\mu, \sigma^2)$	$e^{t\mu + \sigma^2 t^2 / 2}$

Some statistical results

$Y|X = x \sim N\left[\mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2)\right]$ if Y and X are jointly normal.

$\vec{Y}|\vec{X} = \vec{x} \sim N\left[\vec{\mu}_y + \Sigma_{yx}\Sigma_{xx}^{-1}(\vec{x} - \vec{\mu}_x), \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}\right]$ if Y and X are jointly normal.