COSETS

- Let $(G, *)$ be a group and H be any subgroup of G.

Let $a \in G$ be any element, then the set

$H*a = \{h*a\}/\forall h \in H\}$ is called a right coset of H in G.

and

$a*H = \{a*h/\forall h \in H\}$ is called a left coset of H in G.

Note :

1) $H*a$ and $a*H$ are subsets of G.
2) If $(G, *)$ is an abelian group then $H*a = a*H$ in G.

e.g. 1) Let $(\mathbf{Z}, +)$ is a group and

$H = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$ is a subgroup of $G = \mathbf{Z}$

$\therefore$ For $1 \in \mathbf{Z}$, $H + 1 = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$

$3 \in \mathbf{Z}$, $H + 3 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$

$5 \in \mathbf{Z}$, $H + 5 = \{\ldots, -5, 0, 5, 10, \ldots\} = H$

are right cosets of H in G.

$$2 + H = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$$

and $1 + H = \{\ldots, -9, -4, 1, 6, \ldots\}$

are left cosets of H in G.

Theorem 1 : Any two right cosets of a group are either identical or disjoint.

Proof : Let Ha and Hb be any two right cosets of H in G, where $b \in G$

Claim : Prove that $Ha \cap Hb = \phi$ or $Ha = Hb$

Suppose Ha and Hb are not disjoint i.e. $Ha \cap Hb \neq \phi$

$\therefore \ \exists x \in Ha \cap Hb \Rightarrow x \in Ha$ and $x \in Hb$

$$\Rightarrow x = h_1 a \text{ and } x = h_2 b \ ; \ h_1, h_2 \in H$$

$$\Rightarrow x = h_1 a = h_2 b$$

$$\Rightarrow a = h_1^{-1} h_2 b$$

$$\Rightarrow Ha = H(h_1^{-1} h_2) b = (H h_1^{-1} h_2) b$$

$$\Rightarrow Ha = Hb \qquad (\because H = H h_1^{-1} h_2)$$

i.e. If two right cosets are not disjoint then they are identical.

Hence either $Ha \cap Hb = \phi$ or $Ha = Hb$

Theorem 2 : If H is any subgroup of a group G then G is equal to the union of all right cosets of H in G i.e. $G = H \cup Ha \cup Hb \cup Hc$ where $a, b, c \ldots \in G$.

Proof : Let $x \in G$ and $e \in H$ then $ex \in Hx$

$\therefore$ $x \in Hx$, Thus for any $x \in G$, x belongs to any one of the right coset of H.

$$\therefore \quad G \subseteq H \cup Ha \cup Hb \cup \cdots \cup Hx \cdots \qquad \ldots (5.8.1)$$

where a, b, c, ... x ..., $\in G$

Let $y \in H \cup Ha \cup Hb \cup \cdots$

$\therefore \exists$ some $d \in G$ such that $y \in Hd$

As $H \subseteq G$ and $d \in G$, therefore $y \in Hd \Rightarrow y \in G$

$$\therefore H \cup Ha \cup Hb \cup \cdots \subseteq G \qquad \ldots (5.8.2)$$

From (5.8.1) and (5.8.2), we get $G = H \cup Ha \cup Hb \cup \cdots$

Theorem 3 : Show that the set of the inverse of the elements of a right coset is a left coset or more precisely show that $(Ha)^{-1} = a^{-1}H$.

Proof : We have $Ha = \{ha \,/\, h \in H \text{ and } a \in G\}$

Consider : $(Ha)^{-1} = \{(ha)^{-1} \,/\, h \in H, a \in G\}$

$$= \{a^{-1}h^{-1} \,/\, h \in H,\ a \in G\}$$

$$= \{a^{-1}h^{-1} \,/\, h^{-1} \in H,\ a^{-1} \in G\}$$

$$= \{\, a^{-1}h_1 \,/\, h^{-1} = h_1 \in H,\ a^{-1} \in G \,\}$$

$= a^{-1}H$ Hence the proof

Lagranges Theorem: The order of each subgroup of a finite group is divisor of order of group.

Proof : Let G be a group of finite order n. Let H be a subgroup of G and let O (H) = m. Suppose h_1, h_2 ... h_m are the m members of H.

Let $a \in G$. Then Ha is a right coset of H in G and we have

$$Ha = \{h_1 a, h_2 a, \ldots\ldots, h_m a\}$$

Ha has m distinct members, since $h_i a = h_j a \Rightarrow h_i = h_j$.

Therefore each right coset of H in G has m distinct members. Any two distinct right cosets of H in G are disjoint i.e. they have no element in common. Since G is a finite group, the number of distinct right cosets of H in G will be finite, say equal to K. The union of these K distinct right cosets of H in G is equal to G.

Thus, if Ha_1, Ha_2, ... Ha_k are the K distinct right cosets of H in G, then

$$G = Ha_1 \cup Ha_2 \cup \ldots\ldots\ldots\ldots \cup Ha_k$$

$\Rightarrow$ The number of elements in G = The number of elements in Ha_1 + The number of elements in Ha_2 + ... + The number of elements in Ha_k

... [$\because$ two distinct right cosets are mutually disjoint]

$$\Rightarrow \quad O(G) = Km \Rightarrow n = Km$$

$$\Rightarrow \quad K = \frac{n}{m} \Rightarrow \text{m is a divisor of n}$$

$\Rightarrow$ O (H) is a divisor of O (G).