RING :-

A ring $(R, +, \cdot)$ is a set R together with two binary opr $(+)$ and $(\times)$ defined on R for the following axioms.

(i) $(R, +)$ is an abelian grp.

(ii) $(R, \cdot)$ is semigrp i.e

$\rightarrow$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall$ $a, b, c \in R$.
(Associative)

$\rightarrow$ $(a \cdot b) \in R$. $\forall$ $a, b \in R$.

(iii) The operation $(\cdot)$ is distributive over the opr $(+)$

(a) $a \cdot (b+c) = a \cdot b + a \cdot c$. $\forall$ $a, b, c \in R$ (Left distributive)

(b) $(b+c) \cdot a = (b \cdot a) + (c \cdot a)$ (Right distributive)

NOTE :-

The set R consisting of single element 0 with two binary opr defined by $0+0$, $0 \cdot 0 = 0$ is a ZERO | NULL RING.

Eg. The set of all matrices of the form $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ where a, b are real nos with matrix addition and matrix multiplication is a ring.

$$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

9) Show that a set of integers w.r.t + and ... is a ring i.e $(I, +, \cdot)$

Sol^n (i) $(I, +)$ is abelian

(ii) $(I, \cdot)$ is semi grp.

(iii) $\cdot$ distributive over $+$.

(a) ABELIAN GROUP:- Let $x, y \in I$ then $(x+y = y+x)$

Closure:- if $x, y \in I$

$\Rightarrow (x+y) \in I$

i.e $4+6 \in I$

$\therefore (I, +)$ is a closure

Associative - if $x, y, z \in I$.

$(x+y)+z = x+(y+z)$.

$\therefore (I, +)$ is associative.

Identity for $a \in I$

$a+e = a$.

$e = 0$

Inverse: $\forall a \in G$,

$a+a^{-1} = e$.

$a+a^{-1} = 0$

$a = -a^{-1}$.

Commutative :- $\forall\ x, y \in I$

$x+y = y+x$

$\therefore$ Comm. holds.

$\therefore (I, +)$ is Abelian group.

$(I, \cdot)$ is SEMIGRP :-

Closure : $x \cdot y \in I$

$\forall\ x, y \in I$

Associative :- $x, y \in I$

$x(yz) = (xy)z$.

DISTRIBUTIVE :-

$a.(b+c) = ab + ac$

$-2(-6+9) = -2\times-6 + -2\times 9$

$-2\times 3 = 12+(-18)$

$-6 = -6$. ✓

$\forall\ a, b, c \in I$.

$\therefore$ distributive law holds.

Hence $(I, +, \cdot)$ is a Ring.

A ring R is called COMMUTATIVE if $(a.b = b.a)$ $\forall\ a, b \in G$ Means if the semigrp $(R, \cdot)$ is commutative.

Q) Set $R = \{0, 1, 2, 3, 4, 5\}$ is a commutative ring

Sol^n Semigroup $(R, \times_6)$ $e = 0$

closure ($\checkmark$), (associative) ($\checkmark$)

distributive :- $\times_6$ over $+_6$.

$$a \times_6 (b +_6 c) = a \times_6 (b + c)$$

$$= (a \times_6 b) +_6 (a \times_6 c)$$

$\therefore (R +_6, \times_6)$ is a ring.

Now $\times_6$ is commutative :-

$$a \times_6 b = b \times_6 a$$

$$(a \times b) \bmod 6 = (b \times a) \bmod 6.$$

$\therefore$ R is a commutative ring.

PROPERTIES OF RING :-

If $(R, +, \cdot)$ is a ring the $\forall a, b, c \in R$ the following are satisfied where 0 is additive identity & $(-a)$ denotes additive inverse of $a \in R$

① $a \cdot 0 = 0 = 0 \cdot a$

② $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

③ $(-a) \cdot (-b) = a \cdot b$

④ $a \cdot (b - c) = a \cdot b - a \cdot c$

$(b - c) \cdot a = (b \cdot a) - (c \cdot a)$

(1) RING WITH UNITY :-

In a ring R, there exist an element 1 such that :- $(1 \in R)$

$$1 \cdot a = a \cdot 1 \quad \forall a \in R \text{ then.}$$

Then R is called ring with unity i.e $(1 \in R)$ is unit element of ring.

(2) RING WITHOUT UNITY :-

A ring R which doesn't contain the multiplicative identity is called ring without unity.

(3) Boolean Ring :-

A ring whose element is idempotent i.e $a^2 = a \cdot a = a \quad \forall a \in R$ is called boolean ring.

(4) Order of Ring :-

The no of elements in a finite ring R is called order of Ring. $(O(R))$

(5) INVERTIBLE RING :-

Let $(R, +, \cdot)$ be ring with unity then an element $a \in R$ is said to be invertible if $\exists\ a^{-1} \in R$ called the inverse such that

$$\boxed{a \cdot a^{-1} = a^{-1} \cdot a = 1}$$

ZERO DIVISOR OF RING :—

A non zero element a $(a \neq 0)$ is called zero divisor or divisor of zero if there exist an an element $(b \neq 0)$ $(b \in R)$ of Ring R such that either

$$\boxed{ab = 0 \quad \text{or} \quad ba = 0}$$

Ex Let M is a ring of all 2×2 matrices with their elements as integers, the addition and multiplication of matrices being the two ring compositions. Then M is ring with zero divisions.

Soln)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0 \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0$$

$\in R$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(zero element (Null Ring) of R)

Here 'a' is called as divisor of Ring i.e. M is a ring with 0 divisors.

If $b \cdot a = 0$ then b is a zero divisor of R

RING WITHOUT ZERO DIVISOR :—

A ring R is called Ring without zero divisor if the product of no two non zero element is 0. if $ab = 0$ then either $(a = 0$ or $b = 0)$

INTEGRAL DOMAIN:-

A ring R is called Integral Domain if it follows :-

(i) It is commutative ring
(ii) It has a unit element
(iii) It is a ring without zero divisors.

Eg. Ring of integers.
" " Real nos.
" " Rational nos.

FIELD:-

A ring is called field if it is

(i) commutative ring
(ii) It has a unit element
(iii) each non-zero element having a multiplicative inverse.

Eg. Ring of Real nos
" " Rational nos.

Ring of integers is not a FIELD.

Thus in a field if $ab = 0 \Rightarrow$

$$a = 0 \text{ or } b = 0.$$

$\therefore$ F is field without zero divisors.

Q) Prove that ring R is commutative iff :-
$(a+b)^2 = a^2 + 2ab + b^2 \quad \forall a, b \in R$.

Sol) Let ring R is commutative then
$ab = ba$.

Taking LHS :-

$(a+b)^2 = (a+b)(a+b)$ (Distributive)
$= a^2 + ab + ba + b^2$ (comm law)
$= a^2 + ab + ab + b^2$
$= a^2 + 2ab + b^2$

Also $(a+b)^2 = a^2 + 2ab + b^2$ then show
R is commutative

$(a+b)^2 = a^2 + 2ab + b^2$

$\cancel{a^2} + ab + ba + \cancel{b^2} = \cancel{a^2} + 2ab + \cancel{b^2}$

$ab + ba = 2ab$

$ba = ab \quad \forall a, b \in R.$

$\therefore$ R is commutative ring.