NORMAL SUBGROUPS :-

→ A subgroup H of a group G is said to be normal subgrp of G. if for every $x \in G$, and for every $h \in H$,

$$\boxed{x h x^{-1} \in H}$$

→ A subgrp H of G is called normal if all the left cosets are equal to right cosets.

→ Every subgroup of Abelian grp is a normal subgroup.

→ If $aH = Ha \ \forall a \in G$

Theorem: A subgroup H of a group G is normal iff $xHx^{-1} = H \ \forall x \in G$.

PROOF: Let $xHx^{-1} = H \ \forall x \in G$.

$\Rightarrow xHx^{-1} \subseteq H \quad \forall x \in G$.

$\Rightarrow H$ is normal subgroup of G.

QUOTIENT GROUP / FACTOR GROUP:-

Let H be a normal subgroup of G. The set of all cosets of H in G is known as quotient or Factor group

$$\boxed{G/H = \{Ha : a \in G\}}$$

Theorem: The product of two right (left) cosets in G is also right (left) coset in G.

Sol^n Proof: Let G is group and H is normal subgp. of G.

~~Then~~, $G/H = \{Ha : a \in G\}$ be quotient group.

Let $Ha, Hb \in G/H. \Rightarrow$

$$(Ha)(Hb) = H(aH)b$$

$$= H(Ha)b$$

$$(\because Ha = aH)$$

$$\Rightarrow (HH)(ab)$$

$$\Rightarrow H(ab) \ \forall a \in G, b \in G.$$

$\Rightarrow$ right coset of H in G generated by ab.

<u>Theorem</u> There is one to one correspondence b/w the elements of subgroup H and those of any coset of H in G.

<u>Soln</u> We define mapping $f: H \to aH$ by

$$f(h) = ah. \quad \forall h \in H.$$

Let $h_1, h_2 \in H$ such that

$$f(h_1) = f(h_2)$$

$$ah_1 = ah_2$$

$$\Rightarrow h_1 = h_2. \quad \text{(Left cancellation)}$$

$\therefore f$ is one-one.

Every element of aH of form ah for some $h \in H$. $\therefore f$ is onto.

Theorem

If G is finite group, H is normal subgroup of G then $O(G/H) = \dfrac{O(G)}{O(H)}$

Proof We have;

$O(G/H)$ = no. of distinct right cosets of H in G.

$= \dfrac{\text{no of element in } G}{\text{no. of elements in } H}$

$= \dfrac{O(G)}{O(H)}.$

PERMUTATION GROUP:— (SYMMETRIC GROUP)

Let S be finite set consisting of n elements then the ~~set of~~ ~~all~~ one-one mapping of S onto itself is called permutation if:-

$$(f : S \to S)$$

① f is one to one.

② f is onto.

No of distinct elements in finite set S is called degree of permutation. It is denoted by:-

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ f(a_1) & f(a_2) & f(a_3) & \cdots & f(a_n) \end{pmatrix}$$

$S = \{a_1, a_2, a_3 \; ---- \; a_n\}$.

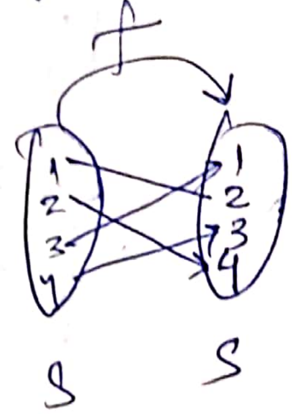
eg: $S = \{1, 2, 3, 4\}$ and mapping $f : S \to S$.

is given as.

$f(1) = 2 \quad f(2) = 4 \quad f(3) = 1$

$f(4) = 3$.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$



EQUALITY OF PERMUTATIONS :-

If f and g are 2 permutations on sets. then $f = g$ iff

$$\boxed{f(x) = g(x)} \quad \forall \; x \in S.$$

eq.
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$g = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

$\therefore \; (f = g)$

$f(1) = 2$
$f(2) = 3$
$f(3) = 1$
$f(4) = 4$

$g(1) = 2$
$g(2) = 3$
$g(3) = 1$
$g(4) = 4$.

Identity Permutation :-

If each element of permutation is mapped by itself then it is called identity Per. (I)

eg. $I = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$ is an identity permutation

$S = \{a, b, c \ldots\}$.

$\boxed{I(a) = a} \ \forall a \in S$.

Total no of Permutations :-

If there are n elements in set S then total no of permutations is $n!$

Inverse of Permutation :-

If f is a permutation on 'S'

$S = \{a_1, a_2 \ldots a_n\}$ such that

$f = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}$

then there exist a permutation called inverse denoted as f^{-1} such that.

$\boxed{f \circ f^{-1} = I}$

$\begin{pmatrix} f(a_1) = y. \\ f^{-1}(y) = a_1 \end{pmatrix}$

$f^{-1} = \begin{pmatrix} b_1 & b_2 & \ldots & b_n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$

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$\boxed{I(a) = a}\ \forall a \in S$.

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PRODUCT OF PERMUTATIONS (COMPOSITIONS OF PERMUTATIONS) :-

The product of two permutations f and g of same degree is denoted by fog or fg meaning perform f and then g :-

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{pmatrix} \text{ and } g = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

then $fog = fg = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}$

- Now fog is also permutation on $S = \{a_1, a_2 \cdots a_n\}$

TOTAL NO. OF PERMUTATIONS

Q) # $A = \{1,2,3\}$ then $S_3 = \{P_0, P_1, P_2, P_3, P_4, P_5\}$

$S_n \rightarrow$ set of all permutations of degree n, then set S_n be the set consisting of all permutations of degree n, then the S_n will have $n!$ distinct permutations of degree n. This set S_n is called symmetric set of permutations of degree n.

$P_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

PERMUTATION GROUP:-

Let A be a set of degree n. Let P_n be the set of all permutations of degree n on A. Then $(P, *)$ is a group called as Permutation Group and the opr $*$ is the composition of permutations.

CLOSURE PROP:-

Let f and g be any two permutations in P_n. where,

$$f = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix} \quad g = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

then

$$fg = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

(Hence P_n is closed for the composition as product of two permutation)

Since $a_1 a_2 \cdots a_n$ are also arrangements of n element of ~~A~~ set S $\therefore fg$ is a permutation of degree n $fg \in P_n$ $\forall f, g \in P_n$.

2) ASSOCIATIVITY :-

$$h = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ d_1 & d_2 & \cdots & d_n \end{pmatrix}$$

$$(fg)h) = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ d_1 & d_2 & \cdots & d_n \end{pmatrix} = fg(h).$$

(3) EXISTENCE OF IDENTITY :-

$$\boxed{fI = If = f} \qquad I = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

(4) EXISTENCE OF INVERSE :-

Let f be the permutation of degree 'n' then the permutation f^{-1} is also permutation of degree 'n'.

Now, $\boxed{ff^{-1} = f^{-1}f = I}$

$\therefore (P_n, *)$ is a group of order $n!$ with respect to composition of permutations.

CYCLIC PERMUTATION :-

Let 'f' be a permutation of degree 'n' on set S having 'n' distinct elements. Suppose it is possible to arrange 'm' elements of set 'S' in a row in such a way that the f-image of each element of set 'S' in a row is the element following it, the f-image of last element is the first element and the remaining $(n-m)$ elements of the set 'S' are left unchanged by 'f'. Then 'f' is called a cyclic permutation.

eg. $f = (1, 3, 4, 5) = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 \end{pmatrix}$

$(1\ 2\ 3\ 4) = (2\ 3\ 4\ 1) = (3\ 4\ 1\ 2) = (4\ 1\ 2\ 3)$.

→ A circular permutation may be denoted by more than one rowed symbols.

⇒ Length of cycle means the no of elements permuted by the cycle.

⇒ DISJOINT CYCLES are those which have no common elements. Every permutation of a finite set can be expressed as a cycle or as a product of disjoint cycles.

eq $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{pmatrix}$

this is written as

$\left.\begin{matrix} (1\,2), & (3, 4, 6), & (5) \\ \downarrow \text{Length} & \downarrow & \downarrow \\ 2 & 3 & 1 \end{matrix}\right\}$ 3 cycles.

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