Module - 1

Partial Differential Equation

A differential equation which involves more than one independent variable is called partial differential equation.

Eg $$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = c$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \; ; \; x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$$

Order & degree of partial differential eq^n

Order of a partial differential equation is the order of highest ordered derivative (differential coefficient) present in the eq^n.

Date: Page:

Methods of forming partial differential equation

A partial differential equation is formed by 2 methods

1. by elimination arbitary constants
2. by eliminating arbitary functions

$p = \frac{\partial z}{\partial x}$ f_x

$q = \frac{\partial z}{\partial y}$ f_y

one variable

$rt - s^2 > 0$

$r = \frac{\partial^2 z}{\partial x^2}$ f_{xx}

$s = \frac{\partial^2 z}{\partial x \partial y}$ f_{xy}

$t = \frac{\partial^2 z}{\partial y^2}$ f_{yy}

two variable

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Date: ______ Page: ______

Q $x^2+y^2+(z-c)^2=a^2$ — (1)

here arbitary constant are $\boxed{a, c}$

partial differentiation w.r.t (x)

$$\frac{\partial}{\partial x}[x^2+y^2+(z-c)^2]=\frac{\partial}{\partial x}[a^2]$$

(y → in

$$2x+2(z-c)\frac{\partial z}{\partial x}=0$$

Treating z as a function of x, y

$$\boxed{x+(z-c)p=0} \quad \text{— (2)}$$

differentiating partially w.r.t (y)

$$2y+2(z-c)\frac{\partial z}{\partial y}=0$$

$$\boxed{y+(z-c)q=0} \quad \text{— (3)}$$

Let us ~~elimination~~ eliminate c from (2

$$(z-c)=-\frac{x}{p}$$

$$y-\frac{x}{p}q=0 \Rightarrow \boxed{yp-xq=0}$$

$$y\frac{\partial z}{} - x\frac{\partial z}{}$$

DATE
PAGE

Q $z = x + y + f(xy)$

$p = \frac{\partial z}{\partial x} = 1 + y f'(xy)$ $\Rightarrow p - 1 = y f'(xy)$ — (1)

$q = \frac{\partial z}{\partial y} = 1 + x f'(xy)$ $\Rightarrow q - 1 = x f'(xy)$ — (2)

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$$\frac{p-1}{q-1} = \frac{y f'(xy)}{x f'(xy)} \Rightarrow \boxed{\frac{p-1}{q-1} = \frac{y}{x}}$$

Linear partial differential equation of the 1st order.

A differential equation involving first order partial derivatives p and q only is called a partial differential equation of the 1st order

If p & q both occur in the first degree only and are not multiplied together than it is called linear partial differential equation of the 1st order.

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DATE
PAGE

Lagrange's Linear Equation

The partial differential equation of the form

$$\boxed{Pp + Qq = R}$$

$p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$

where P, Q, R are the functions of x, y, z is the standard form of linear partial differential equation of the 1st order and is called lagrange's equation

Working Rule

Step 1: Write down the auxillary equations

is
$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

↳ lagranges auxillary equation

Step 2: Solve the auxillary equations

Let the two solⁿ $u = C_1$ & $v = C_2$

DATE
PAGE

Step 3 Required Sol^n $f(u, v) = 0$

or

$v = f(u)$ or $u = f(v)$

Ques 1 $yzp - xzq = xy$

Sol^n $\boxed{Pp + Qq = R}$

$P = yz$ $Q = -xz$ $R = xy$

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$
(1) (2) (3)

$\Rightarrow \frac{dx}{zy} = \frac{dy}{-xz} = \frac{dz}{xy}$

$\frac{dx}{\cancel{z}y} = -\frac{dy}{x\cancel{z}} \Rightarrow x\,dx = -y\,dy \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + C = 0$

Integrating

$x^2 + y^2 = C_1$; $u = x^2 + y^2$

$u = C_1$

$\frac{dy}{-xz} = \frac{dz}{xy} \Rightarrow \frac{dy}{-z} = \frac{dz}{y}$

$-y\,dy = z\,dz$

Integrating

$y^2 + z^2 = C_2$

$v = C_2$

$\Rightarrow f(u, v) = 0$

$f(x^2 + y^2, y^2 + z^2) = 0$

$x^2 + y^2 = f(y^2 + z^2)$

Imp

DATE

PAGE

Q $y^2 p - xyq = x(z-2y)$

$$P_p + Q_q = R$$

$P = y^2$ $\quad Q = -xy$ $\quad R = x(z-2y)$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

(1) → $\frac{dx}{y^2} = -\frac{dy}{xy}$

$$\frac{dx}{y} = -\frac{dy}{x}$$

$$x\,dx = -y\,dy$$

$$x\,dx + y\,dy = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + c = 0$$

$$x^2 + y^2 + 2C = 0$$

$$x^2 + y^2 = -2C$$

$$\boxed{x^2 + y^2 = C_1}$$

$$\boxed{u = x^2 + y^2}$$

Now, 2nd eqn

(2) → $\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$$\frac{dy}{-y} = \frac{dz}{(z-2y)}$$

$$(z-2y)\,dy = -y\,dz$$

$$z\,dy - 2y\,dy = -y\,dz$$

$$\underline{z\,dy + y\,dz} = 2y\,dy$$

$d(xy)$
$x\,dy + y\,dx$]

$$\int d(yz) = \int 2y\,dy$$

$$yz + c_2 = y^2$$

DATE
PAGE

$$\boxed{C_2 = y^2 - yz}$$

$$v = y^2 - yz$$

$$\boxed{\phi\,(x^2+y^2,\ y^2-yz) = 0}$$ Ans

Q3 $\frac{y^2 z}{x} p + xzq = y^2$

$$Pp + Qq = R$$

$$y^2 z\, p + x^2 z\, q = y^2 x$$

$P = y^2 z \qquad Q = x^2 z \qquad R = y^2 x$

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

1st eqn

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$$

$$x^2\,dx = y^2\,dy$$

$$x^2\,dx - y^2\,dy = 0$$

$$\frac{x^3}{3} - \frac{y^3}{3} + c_1 = 0$$

$$x^3 - y^3 + c_1 = 0$$

$$\boxed{x^3 - y^3 = c_1}$$

$$\boxed{u = (x^3 - y^3)}$$

2nd eqn

$$\frac{dx}{y^2 z} = \frac{dz}{y^2 x}$$

$$x\,dx = z\,dz$$

$$x\,dx - z\,dz = 0$$

$$\boxed{x^2 - z^2 = C_2}$$

$$\boxed{v = x^2 - z^2}$$

$$\boxed{\phi\,(x^3-y^3,\ x^2-z^2) = 0}$$ Ans

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Q4 $x^2p + y^2q = (x+y)z$

$\boxed{Pp + Qq = R}$ equation (partial diff.)

$\Rightarrow P = x^2 \qquad Q = y^2 \qquad R = (x+y)z$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$\cancel{y^2\,dx} = \cancel{x^2\,dy} \qquad \int x^{-2}\,dx = \int y^{-2}\,dy$

$\cancel{y^2x - x^2y} = C \qquad \boxed{\frac{1}{x} - \frac{1}{y} = C_1 = u}$

$\cancel{xy(y-x)}$

$$\frac{dx-dy}{x^2-y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx-dy}{(x+y)(x-y)} = \frac{dz}{(x+y)z}$$

$$\frac{dx-dy}{x-y} = \frac{dz}{z}$$

$$\int \frac{d(x-y)}{x-y} = \int \frac{dz}{z}$$

Point to remember

$\log(x-y) - \log z = \log C_2$

$\log\left(\frac{x-y}{z}\right) = \log C_2$

$\frac{x-y}{z} = C_2$

$\log(x-y) = \log z + \log C_2 \Rightarrow \frac{x-y}{z} = C_2 = V$

DATE PAGE

$$\boxed{\phi\left(\frac{1}{x}-\frac{1}{y},\ \frac{x-y}{z}\right)=0}$$ Ans.

Q5 $pz - qz = z^2 + (x+y)^2$

$P = z \qquad Q = -z \qquad R = z^2 + (x+y)^2$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$
① ② ③

1) 1st eq^n

$$\frac{dx}{z} = \frac{dy}{-z}$$

$dx = -dy$

$\int dx + \int dy = 0$

$\boxed{x+y=c_1}$

$\boxed{u = x+y = c_1}$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + C}$$

2nd eq^n

$$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$$

$$\int dx = \int \frac{z\,dz}{z^2+c_1^2}$$

$$\boxed{x = \frac{1}{2}\log(z^2+c_1^2) + c_2}$$

$\cancel{z^2+c_1^2 = t}$

$\cancel{2z\,dz = dt}$

$$\boxed{v = x - \frac{1}{2}\log(z^2+c_1^2)}$$

$c_1 = x+y$

$$\boxed{\phi\left((x+y),\ x-\frac{1}{2}\log\left(z^2+(x+y)^2\right)\right)=0}$$ Ans

DATE

PAGE

Q 6 $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

$P = x^2 - yz$

$Q = y^2 - zx$

$R = z^2 - xy$

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx}$$

$$\frac{dx - dy}{x^2 - yz - (y^2 - zx)} =$$

$$\frac{dx - dy}{x^2 - yz - y^2 + zx}$$

$$\frac{dx - dy}{(x - y)(x + y) + z(y - x)}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

(1) (2) (3)

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$$\log(x-y) = \log(y-z) + \log C_1$$

$$\boxed{u = C_1 = \frac{x-y}{y-z}}$$

$$\frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x}$$

$$\boxed{v = C_2 = \frac{y-z}{z-x}}$$

$$\boxed{\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0}$$ Ans

DATE
PAGE

Methods of Multiplier [when seperation is not possible]

Let the auxillary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Let l, m, n may be constants or functions of x, y, z than we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = K$$

l, m, n are choosen in a such way $lP + mQ + nR = 0$

$$l\,dx + m\,dy + n\,dz = 0$$

Ques $(mz - ny)p + (nx - lz)q = ly - mx$

$P = mz - ny$ $\quad Q = nx - lz$ $\quad R = ly - mx$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Taking x, y, z as a multiplier

$$\frac{x\,dx + y\,dy + z\,dz}{xmz - nxy + ynx - ylz + zly - zmx} = K$$

DATE PAGE

$$\frac{x\,dx + y\,dy + z\,dz}{0} = K$$

$$x\,dx + y\,dy + z\,dz = K$$

$$\boxed{x^2 + y^2 + z^2 = C_1}$$

Taking l, m, n as a multipliers

$$\frac{l\,dx + m\,dy + n\,dz}{lmz - nly + mnx - lmz + lyn - mnx}$$

$$\frac{l\,dx + m\,dy + n\,dz}{0} = K$$

$$l\,dx + m\,dy + n\,dz = 0$$

$$\boxed{lx + my + nz = 0} = C_2$$

$$\phi\,(x^2 + y^2 + z^2,\ lx + my + nz) = 0$$

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Q $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

$$\cancel{\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)}}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as a multiplier

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{x^2(y-z)}{x^2} + \frac{y^2}{y^2}(z-x) + \frac{z^2}{z^2}(x-y)} = K$$

$$\int \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = C_1$$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as a multiplier

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\underbrace{x(y-z) + y(z-x) + z(x-y)}_{\to 0}} =$$

DATE ☐☐ ☐☐ ☐☐
PAGE ☐☐

$$\int \frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\boxed{c_2 = xyz}$$

$$\boxed{\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right) = 0}\ \text{Ans.}$$

Q 2 $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

$y^2+z-x^2-z+x^2-y^2$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as a multiplier

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2} = K \Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\int \frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$$

$$\log x + \log y + \log z = \log c$$

$$\log xyz = \log c \Rightarrow \boxed{xyz = c_1}$$

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DATE PAGE

Taking $x, y, -1$ as a multiplier

$$x^2y^2+x^2z - y^2x^2 - y^2z - zx^2 + y^2z$$

$\therefore$

$$\int x\,dx + y\,dy - dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} - z = C$$

$$x^2 + y^2 - 2z = 2C$$

$$\boxed{v = x^2 + y^2 - 2z = C_2}$$

Ques 3 $p + 3q = 5z + \tan(y - 3x)$

$P = 1 \qquad Q = 3 \qquad R = 5z + \tan(y-3x)$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

1st eq^n

$$3\,dx = dy$$

$$3\,dx - dy = 0$$

$$\boxed{3x - y = C_1}$$

$$u = \boxed{C_1 = y - 3x}$$

2nd eq^n

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y-3x)}$$

$$\int dx = \int \frac{dz}{5z + \tan C_1}$$

$$x = \frac{1}{5}\log(5z + \tan C_1) + C_2$$

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DATE
PAGE

$$v = x - \frac{1}{5}\log\left(5z + \tan(y-3x)\right)$$

Ans :-

$$\phi\left((y-3x),\ x - \frac{1}{5}\log\left(5z + \tan(y-3x)\right)\right)$$

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Q 4 $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$

$Pp + Qq = R$

☆ $P = y^2 + z^2 - x^2$ $\quad Q = -2xy \quad R = -2xz$

☆ $$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

☆

$$+\frac{dy}{2xy} = +\frac{dz}{2xz}$$

$$\frac{1}{y}\,dy = \frac{1}{z}\,dz$$

$$\int \frac{1}{y}\,dy - \int \frac{1}{z}\,dz = 0$$

$$\log y - \log z = \log C$$

$$\log\left(\frac{y}{z}\right) = \log C$$

$$u = \boxed{\frac{y}{z} = C_1}$$

$y^2x + z^2x - x^3 - 2x^3y^2 - 2xz^2$
$-xy^2 - xz^2 - x^3$
$-x(y^2+z^2+x^2)$

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$$\frac{x\,dx + y\,dy + z\,dz}{Px + Qy + Rz}$$

DATE PAGE

Taking x, y, z as multiplier

$$\frac{x\,dx + y\,dy + z\,dz}{+\cancel{x}(x^2+y^2+z^2)} = \frac{dz}{+2\cancel{x}z}$$

$$\int \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = \int \frac{dz}{z}$$

$$\log(x^2+y^2+z^2) = \log z + \log c$$

$$\boxed{\frac{x^2+y^2+z^2}{z} = v}$$

$$\phi(u, v) = 0$$

Q5 $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \left(\frac{x-y}{xy}\right)$

Ans xyz multiply both side

$$xyz\left(\frac{y-z}{yz}\right)p + xyz\left(\frac{z-x}{zx}\right)q = xyz\left(\frac{x-y}{xy}\right)$$

$$x(y-z)p + y(z-x)q = z(x-y)$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

DATE

PAGE

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y} = K$$

$$\int \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = \int 0$$

$$\log x + \log y + \log z = \log c$$

$$\log xyz = \log c$$

$$\boxed{xyz = c_1 = u}$$ Ans 1

Taking 1, 1, 1 as a multipliers

$$\frac{dx + dy + dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{yx} + \cancel{zx} - \cancel{yz}} = K$$

$$dx + dy + dz$$

$$\boxed{x + y + z = c_2 = v}$$ Ans 2

$$\boxed{\phi(u, v) = 0}$$

DATE 05 08 19
PAGE

Non Linear partial differential

The partial differential eq^n which involves 1st order partial ~~det~~ derivative p & q with degree higher than one and the product ~~are~~ of p and q is called non linear partial differential eq^n of 1st order.

The comple sol^n of such a eq^n involves only two arbitary constants

① $(y+zx)p - (x+yz)q = x^2-y^2$

$P = y+zx \qquad Q = -(x+yz) \qquad R = x^2-y^2$

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

$\cancel{zy + z^2x = x - yz}$

$y\cancel{x} + zx^2 \cancel{-} xy \cancel{-} y^2 z \cancel{+ x^2 - y^2}$

$y\cancel{x} + z\cancel{x}^2 - x\cancel{y} - y^2 z \cancel{+} z\cancel{x^2} + \cancel{y^2} z$

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DATE

PAGE

Taking $x, y, -z$ as multipliers

$$\int x\,dx + y\,dy - z\,dz = 0$$

$$\boxed{x^2 + y^2 - z^2 = c_1 = u}$$

Taking $y \quad x \quad 1$

$$\int y\,dx + x\,dy + \int dz = 0$$

$$\boxed{\cancel{v = yx + xy + z = c_2}}$$

$$v = d(xy) + dz = 0$$

$$v = \int d(xy) + \int dz$$

$$\boxed{v = xy + z}$$

$$\boxed{\phi(x^2 + y^2 - z^2,\ xy + z) = 0}$$ Ans

Q2 $(x^2-y^2-yz)p + (x^2-y^2-zx)q = z(x-y)$

$P = x^2-y^2-yz \quad Q = x^2-y^2-zx \quad R = z(x-y)$

$$\frac{dx}{(x^2-y^2-yz)} = \frac{dy}{x^2-y^2-zx} = \frac{dz}{z(x-y)}$$

R/W

$$\cancel{x^2}-\cancel{y^2}-yz-\cancel{x^2}+\cancel{y^2}+zx$$
$$\rightarrow z(x-y) - z(x-y)$$

$$\frac{dx-dy}{x^2-y^2-yz-x^2+y^2+zx} = \frac{dz}{}$$

Taking 1, -1, -1 as multipliers

$$\int dx - \int dy - \int dz = 0$$

$$u = \boxed{x-y-z = 0}$$

$$\rightarrow \boxed{\frac{x\,dx - y\,dy}{xP - yQ} = \frac{dz}{R}}$$

$$x^3 - y^2x - yx\cancel{z} - x^2y + y^3 + zx\cancel{y}$$

$$x^3 - xy^2 - yx^2 + y^3$$

$$\frac{x\,dx - y\,dy}{(x-y)\;(x^2-y^2)} = \frac{dz}{z(x-y)}$$

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$$\int \frac{x\,dx - y\,dy}{x^2-y^2} = \int \frac{dz}{z}$$

$$\frac{1}{2}\log(x^2-y^2) = \log z + \log C_2$$

$$\log(x^2-y^2) = 2\log z + 2\log C_2$$

$$\log(x^2-y^2) = \log z^2 + \log C_2^2$$

$$\log(x^2-y^2) - \log z^2 = \log C_2^2$$

$$\boxed{\frac{x^2-y^2}{z^2} = C = v}$$

$$\phi\left(x-y-z,\ \frac{x^2-y^2}{z^2}\right) = 0$$

5 Aug

DATE
PAGE

Non Linear

form :
1) $pq = 1$
2) $p + q = 1$
3) $p - q = 1$
4) $\sqrt{p} \pm \sqrt{q} = 1$

Type 1

Equation of the type $f(p, q) = 0$ —— (1)
such that equation containing p & q only

Method Let the required sol^n

$$\boxed{z = ax + by + c} \quad \text{—— (2)}$$

where: $p = \frac{\partial z}{\partial x} = a$

$q \ \frac{\partial z}{\partial y} = b$

On putting these value in eq^n (1)

$$f(p, q) = 0 \Rightarrow f(a, b) = 0$$

from this find the value of b in terms of a and subsitute the value of b in equation (2)

That will be the required sol^n

DATE
PAGE

Ques 1: 2 Marks

Solve: $p^2+q^2=1$

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=1$$

$p^2+q^2=1$ — (1)

Assume required solⁿ is $\boxed{z=ax+by+c}$ — (2)

We know that $\boxed{p=\frac{\partial z}{\partial x}=a}$; $\boxed{q=\frac{\partial z}{\partial y}=b}$

Subsituting p & q in (1)

$a^2+b^2=1$; $\boxed{b=\sqrt{1-a^2}}$

Putting the value of b in eqⁿ (2)

$$\boxed{z=ax+\left(\sqrt{1-a^2}\right)y+c}$$

Q2 $pq=1$ — (1)

$z=ax+by+c$ — (2)

$p=\frac{\partial z}{\partial x}=a$; $q=\frac{\partial z}{\partial y}=b$

$ab=1$; $\boxed{b=\frac{1}{a}}$

$$\boxed{z=ax+\frac{y}{a}+c}$$

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DATE
PAGE

Q3 $\sqrt{p} + \sqrt{q} = 1$

$z = ax + by + C$

$p = \frac{\partial z}{\partial x} = a \qquad q = \frac{\partial z}{\partial y} = b$

$\sqrt{a} + \sqrt{b} = 1 \Rightarrow \boxed{\sqrt{b} = 1 - \sqrt{a}}$

$b = (1 - \sqrt{a})^2$

$$\boxed{z = ax + (1 - \sqrt{a})^2 y + C}$$

Q4 $p^2 - q^2 = 1$

$z = ax + by + c$

$a = \frac{\partial z}{\partial x} = p \qquad b = \frac{\partial z}{\partial y} = q$

$a^2 - b^2 = 1$

$a^2 - 1 = b^2 \Rightarrow \boxed{\sqrt{a^2-1} = b}$

$$\boxed{z = ax + (\sqrt{a^2-1})y + c}$$

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Q 5 $\qquad x^2p^2 + y^2q^2 = z^2 \qquad \Big| \; p+q=1$

$\cancel{x^2\left(p^2 + y^2\right.}$

$$\frac{x^2}{z^2}p^2 + \frac{y^2}{z^2}q^2 = 1$$

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\left(\frac{\frac{\partial z}{z}}{\frac{\partial x}{x}}\right)^2 + \left(\frac{\frac{\partial z}{z}}{\frac{\partial y}{y}}\right)^2 = 1 \quad \text{—— } A$$

Let $\frac{\partial z}{z} = \partial Z$, $\frac{\partial x}{x} = \partial X$, $\frac{\partial y}{y} = \partial Y$

$\log z = Z \qquad \log x = X \qquad \log y = Y$

Eqn A $\qquad \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$

$f(P, Q) = 0 \qquad P^2 + Q^2 = 1$

$$Z = aX + bY + c$$

$P = \frac{\partial Z}{\partial X} = a \qquad Q = \frac{\partial Z}{\partial Y} = b$

$$a^2 + b^2 = 1 \; ; \; b = \sqrt{1-a^2}$$

$$Z = aX + \sqrt{1-a^2}\,Y + c$$

$$\boxed{\log z = a\log x + \sqrt{1-a^2}\log y + c}$$

DATE PAGE

Type-2 : CLAIRAUT TYPE

$$z = px + qy + f(p, q)$$

Solⁿ is : $z = ax + by + f(a, b)$

Ques 1 $z = px + qy + p^2 + q^2$

Solⁿ $z = ax + by + a^2 + b^2$

Ques 2 $z = px + qy + 2\sqrt{pq}$

Solⁿ $z = ax + by + 2\sqrt{ab}$

Imp

CHARPITS METHOD

This is the general method for finding the complete solⁿ of non-linear partial differential equation of the 1st order

Let the given equation be

$$f(x, y, z, p, q) = 0 \quad \text{—} ①$$

if we can find another relation

$$F(x, y, z, p, q) = 0 \quad \text{—} ②$$

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DATE
PAGE

involving x, y, z, p & q than we can solve 1 and 2 for p and q and subsititute in :-

$$\boxed{dz = p\,dx + q\,dy} \quad - (3)$$

Sol^n of equation 3rd is the complete solution.

The auxillary equation are :-

$$\Rightarrow \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

$$\Rightarrow \frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0}$$

form $\Rightarrow$ $p+q+z+x+y+K=0$

DATE
PAGE

Find the complete integral of :-

$$q = 3p^2 \quad -①$$

Solⁿ By charpit method

$$f(x,y,z,p,q) = 3p^2 - q = 0$$

$f_x = 0$ $\quad f_p = 6p$

$f_y = 0$

$f_z = 0$ $\quad f_q = -1$

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-6p^2+q^2} = \frac{dx}{-6p} = \frac{dy}{1}$$

$$\frac{dp}{0} = K$$

$$\int dp = \int 0$$

$$\boxed{p = a}$$

Put the value in eqⁿ $-①$

$$q = 3a^2$$

$$dz = p\,dx + q\,dy$$

$$\int dz = a\,dx + 3a^2\,dy$$

$$z = ax + 3a^2y + b$$

DATE
PAGE

Q find the complete integral of :-

$$zpq = p+q$$

By charpit method

$$f(x,y,z,p,q) = zpq - p - q = 0$$

$f_x = 0$ $\quad f_p = zq - 1$

$f_y = 0$ $\quad f_q = zp - 1$

$f_z = pq$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0}$$

(1) (2)

$$\boxed{\frac{dp}{p^2q} = \frac{dq}{q^2p}} = \frac{dz}{-2zpq + p + q} = \frac{dx}{1-2z} = \frac{dy}{1-pz} = \frac{df}{0}$$

from 1 & 2

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\log p = \log q + \log a$$

$$\frac{p}{q} = a$$

$$\boxed{p = aq}$$

$$\boxed{zpq = p+q} \quad -(3)$$

putting p in (3)

$$zaq^2 = aq + q$$

$$zaq = a+1$$

$$\boxed{z = \frac{a+1}{aq}} \rightarrow q = \frac{a+1}{az}$$

$$p = \frac{a(a+1)}{az} \Rightarrow \frac{a+1}{z} = p$$

DATE

PAGE

Putting in eq^n

$$dz = p\,dx + q\,dy$$

$$dz = \left(\frac{1+a}{z}\right)dx + \left(\frac{1+a}{az}\right)dy$$

$$z\,dz = (1+a)\left[dx + \frac{1}{a}dy\right]$$

Integrating both Sid

$$\frac{z^2}{2} = (1+a)\left[x + \frac{y}{a}\right) + b$$ Ans.

$\hookrightarrow$ constant

Q3 $\quad 2zx - px^2 - 2qxy + pq = 0$

$$f(x, y, z, p, q) = 2zx - px^2 - 2qxy + pq$$

$$f_x = 2z - 2px - 2qy$$

$$f_y = -2qx$$

$$f_z = 2x$$

$$f_p = -x^2 + q$$

$$f_q = -2xy + p$$

DATE

PAGE

$$\frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{dz}{-px^2+2xyq}$$

$\int dq = \int 0$

$q = a$

$p(q-x^2) - 2qxy + 2zx = 0$

$p = \frac{2x(z-ay)}{x^2-a}$

$p = \frac{2qxy - 2zx}{q - x^2}$

$$dz = \frac{2x(z-ay)}{x^2-a}\,dx + a\,dy$$

$$\frac{dz - a\,dy}{z-ay} = \frac{2x}{x^2-a}\,dx$$

$$\frac{d(z-ay)}{(z-ay)} = \frac{2x}{x^2-a}\,dx$$

$\log(z-ay) = \log(x^2-a) + \log b$

$$\frac{z-ay}{x^2-a} = b$$

DATE

PAGE

Q4 $z^2(p^2z^2+q^2)=1 \Rightarrow z^2(p^2z^2+q^2)-1=0$

$f(x,y,z,p,q) = p^2z^4+q^2z^2$

$f_x=0 \qquad f_y=0 \qquad f_z = 4p^2z^3+2q^2z$

$f_p = 2pz^4 \qquad f_q = 2qz^2$

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0}$$

$$\frac{dq}{4p^2z^3q+2q^3z} = \frac{dz}{-2p^2z^4-2q^2z^2} = \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2} = \frac{df}{0}$$

$$\frac{dp}{p(4p^2z^3+2q^2z)} = \frac{dq}{q(4p^2z^3+2q^2z)}$$

DATE

PAGE

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log a$$

$$\log p = \log(qa) \Rightarrow \boxed{p = qa}$$

$$z^2(p^2z^2+q^2)-1=0$$

$$p^2z^4+q^2z^2-1=0$$

$$q^2a^2z^4+q^2z^2-1=0$$

$$q^2(a^2z^4+z^2)=1$$

$$q^2 = \frac{1}{z^2(a^2z^2+1)} \Rightarrow \boxed{q = \frac{1}{z\sqrt{1+a^2z^2}}}$$

$$q = \frac{p}{a}$$

$$z^2(p^2z^2+q^2)-1=0$$

$$z^4p^2+q^2z^2-1=0$$

$$z^4p^2+z^2\frac{p^2}{a^2} \cancel{-1} = \cancel{0}\ 1$$

$$p^2\left(z^4+\frac{z^2}{a^2}\right)=1$$

$$p^2(z^4a^2+z^2)=a^2 \Rightarrow \boxed{p^2 = \frac{a^2}{z^4a^2+z^2}}$$

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DATE
PAGE

$$p = \frac{a}{\sqrt{z^2(z^2a^2+1)}} \Rightarrow \boxed{p = \frac{a}{z\sqrt{a^2z^2+1}}}$$

Putting in eq^n

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{z\sqrt{a^2z^2+1}}\,dx + \frac{1}{z\sqrt{a^2z^2+1}}\,dy$$

$$dz = \frac{a\,dx + dy}{z\sqrt{a^2z^2+1}}$$

$$z\sqrt{a^2z^2+1}\;dz = a\,dx + dy$$

On integrating

$$\int a\,dx + dy = \int z\sqrt{a^2z^2+1}\;dz$$

Let $\underline{a^2z^2+1 = t^2}$

$$2a^2z\,dz = 2t\,dt$$

Solve rest at home

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Q5 $q = (z+px)^2$

$q = z^2 + p^2x^2 + 2zpx$

$0 = z^2 + p^2x^2 + 2zpx - q$

$f_x = 2p^2x + 2zp$

$f_y = 0$

$f_z = 2z + 2px$

$f_p = 2px^2 + 2zx$

$f_q = -1$

$f(x, y, z, p, q) = z^2 + p^2x^2 + 2zpx - q$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0}$$

$$\Rightarrow \frac{dp}{2p^2x + 2zp + p(2z+2px)} = \frac{dq}{q(2z+2px)} \quad (2) = \frac{dz}{-p(2px^2+2zx) - q(-1)} \quad (3)$$

$$= \frac{dx}{-(2px^2+2zx)} \quad (4) = \frac{dy}{+1} \quad (5) = \frac{df}{0} \quad (6)$$

Q50

DATE

PAGE

taking (2) & (4)

$$\frac{dq}{2q(z+px)} = \frac{dx}{-2x(z+px)}$$

$$\frac{dq}{2q} = \frac{dx}{-2x} \Rightarrow \frac{dq}{q} = -\frac{dx}{x}$$

$$\log q = -\log x + \log a$$

$$\boxed{q = \frac{a}{x}}$$

$$q = (z+px)^2 \Rightarrow \sqrt{q} = (z+px)$$

$$\sqrt{\frac{a}{x}} = (z+px) \Rightarrow px = \sqrt{\frac{a}{x}} - z$$

$$\boxed{p = \frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}}$$

Putting in eq^n

$$dz = p\,dx + q\,dy$$

$$dz = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right)dx + \frac{a}{x}\,dy$$

$$x\,dz = (\sqrt{a}\,(x)^{-1/2} - z)\,dx + a\,dy$$

$$x\,dz + z\,dx = \sqrt{a}\,(x)^{-1/2}\,dx + a\,dy$$

$$\int d(xz) = \int \sqrt{a}\,(x)^{-1/2}\,dx + a\,dy$$

$$xz = 2\sqrt{a}\sqrt{x} + ay + b$$ Ans.

DATE
PAGE

Cauchy Method of characteristics

Consider a 1st order partial differential eq^n :-

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f(x,y) + Ku \quad -(1)$$ such that

$$u(0,y) = h(y)$$

here a, b and f depends on x, y & u

but ~~and~~ not on the derivatives of u.

Let $u(x,y)$ be the sol^n of eq^n (1)

then by chain rule

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}\cdot dy \quad -(2)$$

from 1 & 2 than we get

$$\underbrace{\frac{dx}{a} = \frac{dy}{b}} = \frac{du}{f(x,y)+Ku} \quad -(3)$$

$$\boxed{bx - ay = C} \Rightarrow C \text{ is constant}$$

$$y = \frac{bx - c}{a}$$

DATE
PAGE

$$\frac{dx}{a} = \frac{du}{f(x, y) + Ku}$$

$$\frac{dx}{a} = \frac{du}{\left(x, \frac{bx - c}{a}\right) + Ku} \quad \text{--- (4)}$$

Now eq^n (4) can be re-written as

$$\frac{du}{dx} - \frac{K}{a} u = \frac{1}{a} f\left(x, \frac{bx - c}{a}\right) \quad \text{--- (5)}$$

$$\frac{dy}{dx} + Py = Q \qquad If = e^{\int P dx}$$

Sol^n of (5) is

$$If = e^{-\frac{K}{a} x}$$

$$\boxed{u = G(x, c) + C_1}$$

where $\boxed{C_1 = g(x)}$ where g is an arbitarty function

$$\boxed{u = G(x, c) + g(c)}$$

$g(c)$ can be determine by using of condition.

DATE
PAGE

Use cauchy method of characterstic to solve the partial differential eqⁿ.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x+y \quad ; \quad u(x,0)=0$$

The System of equation

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{x+y}$$

$$\frac{dx}{1} = \frac{dy}{1}$$

$$\boxed{x-y=c}$$

$$x-c=y$$

① & ③

$$dx = \frac{du}{x+y}$$

$$dx = \frac{du}{x+(x-c)}$$

$$dx = \frac{du}{2x-c}$$

$$\int (2x-c)\,dx = \int du$$

$$x^2 - cx + c_1 = u(x,y)$$

Let $c_1 = g(c)$

$$u(x,y) = x^2 - (x-y)x + g(c)$$

$$u(x,y) = x^2 - x(x-y) + g(x-y)$$

$u(x,0)=0$
$y=0$

$$0 = x^2 - x^2 + g(x)$$

DATE

PAGE

$g(x) = 0$ means $c_1 = 0$

$u(x,y) = x^2 - x(x-y) + g(c)$

$= x^2 - x^2 + xy$

$$\boxed{u(x,y) = xy}$$

Q Solve $u_x - u_y = 0$

$u(x,0) = x$

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{0}$$

$dx = -dy$

$dx + dy = 0$

$x + y + c_2 = 0$

$\boxed{x + y = c_2}$

$\boxed{y = c_2 - x}$

$\frac{dx}{1} = \frac{du}{0}$

$0 = du$

$u = c_1$

~~$u = 0$~~

$u(x,y) = c_1 = g(c)$

$u(x,y) = c_1 = g(x+y)$

$\boxed{u(x,0) = g(x)}$

$u(x,0) = x = g(x)$

$g(x) = x$

$g(x+y) = x + y$

$\boxed{u(x,y) = x + y}$

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DATE PAGE

Partial Differential Equation of higher order with constant coefficient

The general form of the partial differential eq^n

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} +$$

$$B_1 \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + \cdots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}} + C_0 \frac{\partial z}{\partial x} + C_1 \frac{\partial z}{\partial y} + P_0 z$$

$$= F(x,y)$$

where $A_0\ A_1\ A_2 \cdots A_2$, $B_0, B_1 \cdots B_n$, $C_0, C_1 \cdots P_0$ are arbitrary constant

Write down: homogeneous linear partial differential eq^n with constant coefficent.

An eq^n of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

where $a_0, a_1, \cdots a_n$ are arbitrary constants is called a homogeneous linear partial differential eq^n, of the nth order with constant coefficient

DATE
PAGE

where all the partial derivative are of nth order.

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n) z = f(x,y)$$

~~f(D) y~~

$$\phi(D, D') z = f(x,y)$$

R|W

$D = \frac{\partial z}{\partial x}$

$D' = \frac{\partial z}{\partial y}$

they

$y + mx$

$z = y + x$

required solⁿ is

$$\boxed{z = CF + PI}$$

for complimentary function

$$\phi(D, D') z = 0$$

Put $D_z = m$, $D'_z = 1$

$$\phi(m) = 0$$

Then we find auxillary equations in m & find its roots

case 1: when auxillary equation have distinct root like m_1, m_2, m_3

Then CF is :-

$$\boxed{CF = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x)}$$

Case I. ***When the A.E. has distinct roots*** *i.e.*, $m_1 \neq m_2$, then (2) can be written as

$$(D - m_1 D')(D - m_2 D')z = 0 \qquad ...(3)$$

Now the solution of $(D - m_2 D')z = 0$ will also be a solution of (3).

But $\qquad (D - m_2 D')z = 0$

$\Rightarrow \qquad p - m_2 q = 0$

which is of Lagrange's form and the auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m_2} = \dfrac{dz}{0}$

The first two members give $dy + m_2 dx = 0$ or $y + m_2 x = a$

Also, $\qquad dz = 0$ or $z = b$

$\therefore \quad z = f_2(y + m_2 x)$ is a solution of $(D - m_2 D')z = 0$

Similarly, (3) will also be satisfied by the solution of

$(D - m_1 D')z = 0$ *i.e.*, by $z = f_1(y + m_1 x)$.

Hence the complete solution of (2) is

$$\boxed{z = f_1(y + m_1 x) + f_2(y + m_2 x)}$$

Case II. ***When the A.E. has equal roots,*** each $= m$, then (2) can be written as

$$(D - mD')(D - mD')z = 0 \qquad ...(4)$$

Let $(D - mD')z = u$, then (4) becomes $(D - mD')u = 0$

Its solution is $u = f(y + mx)$, as proved in Case I.

$\therefore \qquad (D - mD')z = u$ takes the form $(D - mD')z = f(y + mx)$

or $\qquad p - mq = f(y + mx)$ [Lagrange's form]

The Auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{f(y + mx)}$

Now, $\qquad \dfrac{dx}{1} = \dfrac{dy}{-m}$ gives $dy + m\,dx = 0$ or $y + mx = a$

Also, $\qquad \dfrac{dx}{1} = \dfrac{dz}{f(a)}$ gives $dz = f(a)\,dx$

or $\qquad z = f(a)\,x + b$ *i.e.*, $z - xf(y + mx) = b$

$\therefore$ The complete solution of eqn. (2) is

$$z - x f(y + mx) = \phi(y + mx)$$

or

$$\boxed{z = \phi(y + mx) + xf(y + mx)}$$

Note 1. Auxiliary eqn. of $\phi(D, D')z = F(x, y)$ is obtained by putting $D = m$ and $D' = 1$ in $\phi(D, D') = 0$.

Hence the A.E. is $\phi(m, 1) = 0$.

Note 2. Generalising the results of Case I and Case II, we have

(*i*) if the roots of A.E. are $m_1, m_2, m_3, \ldots\ldots$ (all distinct roots), then

$$\text{C.F.} = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \ldots\ldots$$

(*ii*) if the roots of A.E. are $m_1, m_1, m_2, \ldots$ (two equal roots), then

$$\text{C.F.} = f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_2x) + \ldots\ldots$$

(*iii*) if the roots of A.E. are $m_1, m_1, m_1, \ldots$ (three equal roots), then

$$\text{C.F.} = f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \ldots\ldots$$

Note 3. Corresponding to a non-repeated factor D′ on L.H.S., the part of C.F. is taken as $\phi(x)$ and for D'^2 on LHS, the part of C.F. is $y\,\phi(x) + \psi(x)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* 2marks only CF.

(*i*) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = 0$ (*ii*) $(D + 2D')(D - 3D')^2 z = 0$ (*iii*) $4r - 12s + 9t = 0$.

Sol. (*i*) The given equation is

$$(D^2 - DD' - 6D'^2)z = 0, \quad \text{where } D \equiv \frac{\partial}{\partial x} \quad \text{and} \quad D' \equiv \frac{\partial}{\partial y}$$

Auxiliary equation is

$$m^2 - m - 6 = 0$$

$$\Rightarrow \quad (m - 3)(m + 2) = 0 \quad \Rightarrow \quad m = 3, -2$$

$$\therefore \quad \text{C.F.} = f_1(y + 3x) + f_2(y - 2x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + 3x) + f_2(y - 2x)$$

where f_1 and f_2 are arbitrary functions.

(*ii*) Auxiliary equation is

$$(m + 2)(m - 3)^2 = 0 \quad \Rightarrow \quad m = -2, 3, 3$$

$$\therefore \quad \text{C.F.} = f_1(y - 2x) + f_2(y + 3x) + xf_3(y + 3x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y - 2x) + f_2(y + 3x) + xf_3(y + 3x)$$

where f_1, f_2 and f_3 are arbitrary functions.

(*iii*) The given equation is

$$4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$$

$$\Rightarrow \quad (4D^2 - 12DD' + 9D'^2)z = 0$$

Auxiliary equation is $\quad 4m^2 - 12m + 9 = 0$

$$\Rightarrow \quad (2m - 3)^2 = 0 \quad \Rightarrow \quad m = \frac{3}{2}, \frac{3}{2}$$

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{3}{2}x\right) + x f_2\left(y + \frac{3}{2}x\right)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1\left(y + \frac{3}{2}x\right) + x f_2\left(y + \frac{3}{2}x\right)$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve:*

(i) $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 2\dfrac{\partial^3 z}{\partial x \partial y^2} = 0$ (ii) $r = a^2 t$

(iii) $(D^3D'^2 + D^2D'^3)z = 0$ (iv) $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0.$

Sol. (i) The given equation is

$$(D^3 - 3D^2D' + 2DD'^2)z = 0$$

The auxiliary equation is $m^3 - 3m^2 + 2m = 0$

$\Rightarrow$ $m(m-1)(m-2) = 0$

$\Rightarrow$ $m = 0, 1, 2$

$\therefore$ $\text{C.F.} = f_1(y) + f_2(y + x) + f_3(y + 2x)$

$\text{P.I.} = 0$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(y + x) + f_3(y + 2x)$,
where f_1, f_2 and f_3 are arbitrary functions.

(ii) The given equation is $(D^2 - a^2D'^2)z = 0$

The auxiliary equation is $m^2 - a^2 = 0 \Rightarrow m = \pm a$

$\therefore$ $\text{C.F.} = f_1(y + ax) + f_2(y - ax)$

$\text{P.I.} = 0$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + ax) + f_2(y - ax),$$

where f_1 and f_2 are arbitrary functions.

(iii) The given equation is $D^2D'^2(D + D')z = 0$

$\therefore$ $\text{C.F.} = f_1(y) + xf_2(y) + f_3(x) + yf_4(x) + f_5(y - x)$

$\text{P.I.} = 0$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + xf_2(y) + f_3(x) + yf_4(x) + f_5(y - x),$$

where f_1, f_2, f_3, f_4 and f_5 are arbitrary functions.

(iv) The given equation is $(D^4 - D'^4)z = 0$

The auxiliary equation is $m^4 - 1 = 0$

$\Rightarrow$ $(m^2 - 1)(m^2 + 1) = 0$

$\Rightarrow$ $m = \pm 1, \pm i$

$\therefore$ $\text{C.F.} = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$

$\text{P.I.} = 0$

$\therefore$ The complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix),$$

where f_1, f_2, f_3 and f_4 are arbitrary functions.

Q $(D^3 - 3D^2D' + 4D'^3)u = e^{x+2y}$

Solve the differential equation

$-1, 1$

$-2, 2$

$z = CF + PI$

$-3, 3$

for CF $D \to m$ $D' \to 1$

$m^3 - 3m^2 + 4 = 0$

$-1 - 3 + 4$ $m = -1$ $\Rightarrow$ $\boxed{m + 1 = 0}$

$-4 + 4 = 0$

$m+1 \,\big)\, m^3 - 3m^2 + 4$

$(m+1)$ $(m+1)$ $(m+1)$

$m^2(m+1) - 4m(m+1) + 4(m+1)$

$m^3 + m^2 - 4m^2 - \cancel{4m} + \cancel{4m} + 4$

$(m+1)(m^2 - 4m + 4) = 0$

$(m+1)(m-2)^2 = 0$ $\Rightarrow m = -1$

$m = 2$

$CF = f_1(y - x) + f_2(y + 2x) + x f_3(y + 2x)$

$$PI = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

$e^{(x+2y)}$ $a = 1$

$f(ax + by)$ $b = 2$

$D \Rightarrow a = 1$

$D' \Rightarrow b = 2$

DATE ☐☐ ☐☐ ☐☐
PAGE ☐☐

$$= \frac{1}{1-6+32} e^{x+2y}$$

$$= \frac{1}{27} \iiint e^{u} \cdot du\, du\, du$$

$$= \frac{1}{27} \iint e^{u}\, du\, du$$

$$= \frac{1}{27} \int e^{u}\, du$$

$$PI = \frac{1}{27} e^{u} \Rightarrow \frac{1}{27} e^{x+2y}$$

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

Q2 $r + 3s + 2t = x+y$

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x+y$$

$$D^2 + 3DD' + 2D'^2 = x+y$$

$$\boxed{m^2 + 3m + 2 = x+y}$$

$m = -1, -2$

DATE

PAGE

$$\boxed{CF = f_1(y-x) + f_2(y-2x)}$$

$x+y$

$f(ax+by)$ $\quad D = a = 1 \quad ; b = 1 = D'$

$$PI = \frac{1}{D^2 + 3DD' + 2D'^2} \, x+y$$

$$= \frac{1}{1+3+2} \, x+y$$

$$= \frac{1}{6} \iint u \, du \, du$$

$$= \frac{1}{6} \int \frac{u^2}{2} \, du \Rightarrow \frac{1}{6 \cdot 2} \frac{u^3}{3}$$

$$\Rightarrow \frac{1}{12} \frac{u^3}{3} \Rightarrow \frac{1}{36} u^3 \Rightarrow \frac{(x+y)^3}{36}$$

$$Z = f_1(y-x) + f_2(y-2x) + \frac{(x+y)^3}{36}$$

DATE
PAGE

Q3 $r - 2s + t = \sin(2x + 2y)$

$$\frac{\partial^2 z}{\partial x^2} - \frac{2\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$$

$$D^2 - 2DD' + D'^2 = \sin(2x+3y)$$

$D \to m \qquad D' \to 1$

$$m^2 - 2m + 1 = \sin(2x+3y)$$

$D = a = 2$
$D' \; b = 3$

$$PI = \frac{1}{D^2 - 2D'D + D'^2} \sin(2x+3y)$$

$$= \frac{1}{4 + 12 + 9} \sin(2x+3y)$$

$$= \sin(2x+3y)$$

$$= \iint \sin u \, du \, du$$

$$= \int -\cos u \, du \qquad \boxed{u = 2x+3y}$$

$$= -\sin u$$

$$= -\sin(2x+3y)$$

DATE
PAGE

Q.4 $r + s - 2t = \sqrt{2x+y}$

$$D^2 + DD' - 2D'^2 = \sqrt{2x+y}$$

$D = a = 2$; $b = 1 = D'$

$$PI = \frac{1}{D^2 + DD' - 2D'^2} \sqrt{2x+y}$$

$$= \frac{1}{4+2-2} \sqrt{2x+y}$$

Let $u = 2x+y$

$$= \frac{1}{4} \iint \sqrt{u} \; du \, du$$

order 2 due to which $\iint$

$$= \frac{1}{4} \int \frac{2u^{3/2}}{3} \, du$$

$$= \frac{1}{4} \times \frac{2}{3} \times \frac{2u^{5/2}}{5} \Rightarrow \frac{1}{15} u^{5/2}$$

$$\boxed{PI = \frac{1}{15} (2x+y)^{5/2}}$$

Q5 $(D^2 + D'^2) z = \underbrace{\cos mx \cos ny}_{P_1} + \underbrace{30(2x+y)}_{P_2}$

$$PI = \frac{\cos mx \cos ny}{D^2 + D'^2}$$

$$PI = \frac{1}{2} \frac{2 \cos mx \cos ny}{D^2 + D'^2}$$

$$= \frac{1}{2(D^2+D'^2)} [\cos(mx+ny) + \cos(mx-ny)]$$

$u = mx + ny$

$= D = a = m \qquad b = n = D'$

$a = m \; ; b = -n$
$v = mx - ny$

$$\Rightarrow \frac{1}{2(m^2+n^2)} \iint \cos u \, du \, du \quad +$$

$$\frac{1}{2(m^2+n^2)} \iint \cos v \, dv \, dv$$

$$= \frac{-1}{2(m^2+n^2)} [\cos u + \cos v]$$

$$\underline{P_1} = \frac{-1}{2(m^2+n^2)} [\cos(mx+ny) + \cos(mx-ny)]$$

PAGE

$$PI_2 = \frac{30 (2x+y)}{D^2 + D'^2}$$

$$= \frac{1}{4+1} 30 (2x+y)$$

$$= 6 (2x+y)$$

$$= 6 \iint u \, du \, du \Rightarrow 6 \int \frac{u^2}{2} du$$

$$= 3 \int u^2 du \Rightarrow \frac{3}{3} u^3 du \Rightarrow u^3$$

$$PI_2 = (2x+y)^3$$

$$\boxed{PI = \frac{-1}{2(m^2+n^2)} [\cos(mx+ny) + \cos(mx-ny)] + (2x+y)^3}$$

DATE

PAGE

Q6 $(D^2 - 2DD' + D'^2)z = \sin x$

$D = a = 1$

$D' = b = 0$

$$PI = \frac{\sin x}{D^2 - 2DD' + D'^2}$$

$$= \frac{1}{1} \iint \sin u \, du \, du$$

$$= -\sin x$$

Q7 $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 4\sin(2x+y)$

$D^3 - 4D^2D + 4DD'^2 = 4\sin(2x+y)$

$a = 2$ (D) $b = 1$ (D')

$$PI = \frac{4\sin(2x+y)}{(2)^3 - 4(2)^2 + 4(2)}$$

$$PI = \frac{4\sin(2x+y)}{0}$$

$$PI = \frac{x \cdot 4\sin(2x+y)}{3D^2 - 8DD' + 4D'^2}$$

constant

$a = 2$ $b = 1$

DATE

PAGE

$$PI = \frac{4x \sin(2x+y)}{0}$$

$$PI = x \cdot \frac{4x \sin(2x+y)}{6D - 8D'}$$

$$PI = \frac{4x^2 \sin(2x+y)}{6 \times 2 - 8}$$

$$PI = \frac{4x^2 \sin(2x+y)}{4}$$

$$PI = x^2 \int \sin u \, du$$

$$\boxed{PI = -x^2 \cos u}$$

$$\boxed{PI = -x^2 \cos(2x+y)}$$

R|W

$a = 2 \quad b = 1$

$\downarrow \qquad \downarrow$

$D \qquad D'$

$u = (2x+y)$

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DATE
PAGE

Q8 $(D^2 + 5DD' + 6D'^2) z = \dfrac{1}{y-2x}$

$PI = \dfrac{1}{(y-2x)(D^2+5DD'+6D'^2)}$

$x \to a = -2 \to D$ $\quad y \to b = 1 \to D'$

$4 + 5(-2) + 6$
$4 - 10 + 6 = 0$ case fail

$PI = \dfrac{x}{(y-2x)(2D+5)}$

$PI = \dfrac{x}{(y-2x)}$ $\quad \underline{D=-2}$ $\quad 5-4=\underline{1}$

$= x \int \dfrac{1}{u} du$

$= x \log u$

$\boxed{PI = x \log (y-2x)}$

#

Q9) $4r - 4s + t = 16 \log (x+2y)$

$$\boxed{\cancel{PI = \frac{16 \log(x+2y)}{4r-4s+t}}}$$

$$4D^2 - 4DD' + D'^2 = 16 \cdot \log(x+2y)$$

$$PI = \frac{16 \log(x+2y)}{4D^2 - 4DD' + D'^2}$$

$D \to a = 1$ $\quad\quad D' \to b = 2$

$\cancel{16 - 8 + 4 \to 8 = 8 = 0}$ $\quad 4 - 8 + 4 = 0$

$$PI = \frac{16x \log(x+2y)}{8D - 4D}$$

$$8 - 8 = 0$$

$$PI = \frac{16x^2}{8} \log(x+2y)$$

$$\boxed{PI = 2x^2 \log(x+2y)}$$

$CF \Rightarrow 4m^2 - 4m + 1 = 0$

$4m^2 - 2m - 2m + 1 = 0$

$2m(2m-1) - 1(2m-1) = 0$

$m = \frac{1}{2}, \frac{1}{2}$

$$CF = f_1\left(y + \frac{1}{2}x\right) + xf_2\left(y + \frac{x}{2}\right)$$

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Q10 $(D-D')^2 z = x + \phi(x+y)$

$PI = \dfrac{x + \phi(x+y)}{(D-D')^2}$

$PI \Rightarrow \underbrace{\dfrac{x}{(D-D')^2}}_{PI\ 1} + \underbrace{\dfrac{\phi(x+y)}{(D-D')^2}}_{PI\ 2}$

$PI\ 1 \Rightarrow \dfrac{x}{(D-D')^2}$

$D = a = 1 \qquad D' = b = 0$

$\Rightarrow \dfrac{x}{D^2 + D'^2 - 2DD'}$

$\Rightarrow \dfrac{x}{1} \Rightarrow \dfrac{1}{1} \iint u \, du \, du$

$\Rightarrow \int \dfrac{u^2}{2} du \Rightarrow \dfrac{1}{2} \dfrac{u^3}{3} \Rightarrow \dfrac{u^3}{6}$

$PI\ 2 \Rightarrow \dfrac{\phi(x+y)}{(D-D')^2}$

$D = a = 1 \qquad D' = b = 1$

$\Rightarrow \dfrac{1}{D^2 - 2DD' + D'^2} \phi(x+y)$

$$\Rightarrow \frac{x}{2D-2D'} \phi(x+y)$$

$$\Rightarrow \frac{x^2}{2} \phi(x+y)$$

$$PI = \frac{x^3}{6} + \frac{x^2}{2} \phi(x+y)$$

Q11 $(D^2D' - 2DD'^2 + D'^3) z = \frac{1}{x^2}$ | $a = 1 ; b = 0$

$$PI = \cancel{\frac{1}{x^2(D^2D' - 2DD'^2 + D'^3)}}$$

$$\Rightarrow \cancel{\frac{x}{x^2(2DD' - D'^2)}} = \cancel{\frac{x^2}{x^2(2D')}}$$

$m = 0, 1, 1$

$CF = f_1(x) + f_2(x+y) + y f_3(x+y)$

$$PI = \frac{1}{(D^2D' - 2DD'^2 + D'^3)} \cdot \frac{1}{(1\cdot x + 0\cdot y)^2}$$

$a = 1 ; b = 0 \rightarrow D = 1 \quad D' = 0$

$$PI = \frac{y}{D^2 - 4DD' + 3D'^2} \cdot \frac{1}{(x + 0 \cdot y)^2}$$

?

$$= y \iint \frac{1}{u^2} du\, du \Rightarrow -y \log u \Rightarrow -y \log \boxed{\frac{1}{x}}$$

$$\Rightarrow -y(\log x^{-1}) \Rightarrow -y(-1) \log x$$

$$\Rightarrow y \log x$$

DATE
PAGE

Q12 $(D^2 - DD')z = \cos 2y (\sin x + \cos x)$

$$PI = \frac{1}{D^2 - DD'} \times [\cos 2y \sin x + \cos 2y \cos x]$$

$$= \frac{1}{2} [\sin(x+2y) + \cos(x+2y)]$$

$$PI = -\frac{1}{6} \sin(x-2y) + \cos(x-2y)$$

Q13 $(D^2 + 7DD' + 12D'^2)z = \sinh x$

$$m^2 + 7m + 12$$
$$\Rightarrow m = -3, -4$$

$\int \sinh x = \cosh x$ $y \neq x$

$\int \cosh x = \sinh x$

$$cf \Rightarrow f_1(-3x + y) + f_2(-4x + y)$$

$$PI = \frac{1}{(D^2 + 7DD' + 12D'^2)} (\sinh x) -$$

$D = 1$

$$PI = \frac{1}{1+0} \sinh x \Rightarrow PI = \iint \sinh u \, du \, du$$

$$PI = \frac{1}{2} \int e^{+u} - e^{-u} \, du$$

$$PI = \frac{1}{2} \left[\int e^{u} du - \int e^{-u} du \right]$$

$$PI = \frac{1}{2} [e^u + e^u] \Rightarrow \underline{\underline{e^u}}$$

DATE

PAGE

Q14 $(D^2+3DD'-4D'^2)z = x+\sin y$

$$PI = \frac{x+\sin y}{D^2+3DD'-4D'^2}$$

$$PI_1 = \frac{x}{D^2+3DD'-4D'^2} + \frac{\sin y}{D^2+3DD'-4D'^2}$$

$a=1$; $b=0$
D ; D'

$D = x = 0$
$D' = y = 1$

$$PI_1 = \frac{x}{D^2+3DD'-4D'^2}$$

$$PI = \frac{x}{1} \quad ; \quad u = x$$

$$= \iint u \, du \, du$$

$$= \frac{1}{2}\int u^2 \, du$$

$$= \frac{1}{2}\frac{u^3}{3}$$

$$= \frac{u^3}{6}$$

$$PI_1 = \frac{x^3}{6} \quad — (1)$$

$$PI_2 = \frac{\sin y}{D^2+3DD'-4D'^2}$$

$$= \frac{1}{-4} \cdot \sin y$$

$$= -\frac{1}{4}\iint \sin u \, du \, du$$

$$= \frac{1}{4}\sin u$$

$$= \frac{1}{4}\sin y \quad — (2)$$

$$PI = \frac{x^3}{6} + \frac{\sin y}{4}$$

Case $F(x,y) = x^m y^n$

or is rational integral algebraic function of x and y

In this case PI is obtained by expanding $\phi(D, D')$ is an infinite series like $[1 \pm f(D, D')]$ of the ascending power of D or D' or $\frac{D'}{D}$.

Q1 $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$ ✓ ★

$$D^3 - D'^3 = x^3 y^3$$

$$m^3 - 1 = 0$$

$$m^3 = 1 ; \quad \boxed{m = 1, \omega, \omega^2}$$

$$\boxed{CF \Rightarrow f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)}$$

$$PI = \frac{x^3 y^3}{D^3 - D'^3} \Rightarrow \frac{x^3 y^3}{D^3\left(1 - \frac{D'^3}{D^3}\right)}$$

$$PI = \frac{1}{D^3}\left[1 - \frac{D'^3}{D^3}\right]^{-1} x^3 y^3$$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + - -$$

DATE
PAGE

only operate on y.

x will constant

$$= \frac{1}{D^3}\left[1 + \frac{D'^3}{D^3}\right] x^3 y^3$$

$$= \frac{1}{D^3}\left[x^3 y^3 + \frac{1}{D^3} D'^3 (x^3 y^3)\right]$$

$$= \frac{1}{D^3}\left[x^3 y^3 + \frac{1}{D^3} 6x^3\right]$$

$$= \frac{1}{D^3}\left[x^3 y^3 + \frac{x^6}{4\times 5}\right]$$

$$= \frac{1}{D^3}\left[x^3 y^3 + \frac{x^6}{20}\right]$$

$$PI = \left[\iiint x^3 y^3 + \frac{1}{20}\iiint x^6\right]$$

$$PI = \left[\frac{x^6 y^3}{4\times 5\times 6} + \frac{x^9}{20\times 7\times 8\times 9}\right]$$

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DATE
PAGE

Q2

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -4\pi(x^2+y^2)$$

$$D^2 + D'^2 = -4\pi(x^2+y^2)$$

CF : $m^2 + 1 = 0$

$\therefore \boxed{m = +i, -i}$

CF $\Rightarrow f(y+ix) + f_2(y-ix)$

PI $\Rightarrow \dfrac{-4\pi(x^2+y^2)}{D^2+D'^2} \Rightarrow \dfrac{-4\pi(x^2+y^2)}{D^2\left[1+\frac{D'^2}{D^2}\right]}$

$$PI = -\frac{4\pi}{D^2}\left[1+\frac{D'^2}{D^2}\right]^{-1}(x^2+y^2)$$

$$(1+D)^{-1} \Rightarrow 1 - D + D^2 - D^3 + \dots$$

2y

2 baar differentiation

only y

$$= -\frac{4\pi}{D^2}\left[1 - \frac{D'^2}{D^2}\right](x^2+y^2)$$

$$= -\frac{4\pi}{D^2}\left[(x^2+y^2) - \frac{1}{D^2}D'^2(x^2+y^2)\right]$$

$$= -\frac{4\pi}{D^2}\left[(x^2+y^2) - \frac{1}{D^2} \times 2\right]$$

$$\Rightarrow -\frac{4\pi}{D^2}\left[(x^2+y^2) - \iint 2\, dx\, dx\right]$$

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DATE
PAGE

$$-\frac{4\pi}{D^2}\left[x^2+y^2-2\int x\,dx\right]$$

$$-\frac{4\pi}{D^2}\left[x^2+y^2-x^2\right]$$

$$\underbrace{-\frac{4\pi}{D^2}y^2} \Rightarrow -4\pi\iint y^2\,dx \Rightarrow \underline{-2\pi y^2x^2} = PI$$

Q3 $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2\partial y} = 2e^{2x}+3x^2y$

$$D^3 - 2D^2D' = 2e^{2x}+3x^2y$$

$$m^3-2m^2=0 \rightarrow m^2(m-2)=0$$

$$m = 0, 0, 2$$

$$CF = f_1(y) + x f_2(y) + f_3(y+2x)$$

$$PI \Rightarrow \frac{2e^{2x}+3x^2y}{D^3-2D^2D'}$$

$$PI \Rightarrow \frac{1}{D^3\left[1-\frac{2D'}{D}\right]}\left[2e^{2x}+3x^2y\right]$$

$$\Rightarrow \frac{1}{D^3}\left[1-\frac{2D'}{D}\right]^{-1}\left(2e^{2x}+3x^2y\right)$$

$$(1-D)^{-1} \Rightarrow 1+D+D^2$$

PAGE

14 August 2019

Q1 $r + 2s + t = x^2 + y^2 + xy$

$D^2 + 2D'D + D'^2 = x^2 + y^2 + xy$

$D^2 + 2DD' + D'^2 = x^2 + y^2 + xy$

Let $D \to m$; $D' \to 1$

$m^2 + 2m + 1$

$$PI \Rightarrow \frac{x^2 + y^2 + xy}{D^2 + 2DD' + D'^2}$$

$\cancel{x \Rightarrow 1 \quad y \Rightarrow 1}$

$\cancel{1 + 2 + 1 = 4}$

$D' = y$

$D = x$

$$PI \Rightarrow \frac{x^2 + y^2 + xy}{D^2\left[1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right]}$$

$$PI \Rightarrow \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right]^{-1} (x^2 + y^2 + xy)$$

$$\Rightarrow \frac{1}{D^2}\left[1 - \left(\frac{2D'}{D} + \frac{D'^2}{D^2}\right) + \left(\frac{4D'^2}{D^2}\right)\right] (x^2 + y^2 + xy)$$

$$\Rightarrow \frac{1}{D^2}\left[(x^2 + y^2 + xy) - \left(\frac{2}{D}(2y + x) + \frac{1}{D^2}(2)\right) + \frac{4}{D^2}(2)\right]$$

$$\Rightarrow \frac{1}{D^2}\left[(x^2 + y^2 + xy) - \left(2\int(2y + x)\,dx + \iint 2\,dx\,dx\right) + 4\iint 2\,dx\,dx\right]$$

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$$\Rightarrow \frac{1}{D^2}\left[(x^2+y^2+xy) - \left[\left(2\ \frac{x^2}{2}\right) + \cancel{2}4xy\right] + x^2 + 4x^2\right]$$

$$\Rightarrow \frac{1}{D^2}\left[x^2+y^2+xy - x^2 - 4xy - x^2 + 4x^2\right]$$

$$\Rightarrow \frac{1}{D^2}\left[3x^2 - 3xy + y^2\right]$$

$$\Rightarrow \frac{x^4}{4} - \frac{x^3y}{2} + \frac{x^2y^2}{2} \quad \text{Ans}$$

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7 Marks.

General Method for finding PI.

$F(x,y)$ is not always of the form given above. Then general method is applicable to all cases.

Here $\phi(D, D')$ can be factorised into n linear factors

$$\phi(D, D') = (D - m_1 D')(D - m_2 D')(D - m_3 D') \cdots (D - m_n D')$$

$$\boxed{PI = \frac{1}{(D - m_1 D')} f(x,y) = \int f(x, c - m_1 x)\,dx}$$

Ques 1 $r + s - 6t = y \cos x$

$$D^2 + DD' - 6D'^2 = y \cos x$$

CF $\Rightarrow D^2 + DD' - 6D'^2$

$\Rightarrow m^2 + m - 6$

$\Rightarrow m^2 + 3m - 2m - 6$

$\Rightarrow m(m+3) - 2(m+3)$

$\Rightarrow (m-2)(m+3)$

$\Rightarrow \boxed{m = 2}$ $\boxed{m = -3}$

CF $\Rightarrow f(y+2x) + f(y-3x)$

PI:-

$m = -3$, $m + 3 = 0$

$y + mx$

$c - 3x$

$\sin x = \int \cos x$

$-\cos x = \int \sin x$

$\boxed{y = mx + C}$

$y = \ldots$

PI $\Rightarrow \dfrac{y \cos x}{(D-2D')(D+3D')}$

$\Rightarrow \dfrac{1}{(D-2D')} \cdot \dfrac{1}{(D+3D')} y \cos x$

$\dfrac{1}{D - mD'} f(x,y) = \int f(x, c - mx)\, dx$

$D \Downarrow c$, $D' \Downarrow x$

$\Rightarrow \dfrac{1}{(D-2D')} \int \underbrace{\cos x}_{v} \underbrace{(c+3x)}_{u}\, dx$

$\int \underset{\mathrm{I}}{u} \underset{\mathrm{II}}{v}\, dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \ldots$

$\left[\begin{array}{l} 1, 2, 3 \longrightarrow \int dx \\ ', '', ''' \longrightarrow \frac{d}{dx} \end{array}\right]$

$\Rightarrow \dfrac{1}{(D-2D')} \left[(c+3x) \sin x + 3 \cos x \right]$

now put y

$\Rightarrow \dfrac{1}{D-2D'} \left[y \sin x + 3 \cos x \right]$

$D \to C_1 ; 2D' \Rightarrow 2x$

~~$\Rightarrow \dfrac{1}{D-2D'} [\quad] \Rightarrow \int \{ (C_1 - 2x)(-\cos x) -$~~

$\Rightarrow \int \{ \underbrace{(C_1 - 2x)}_{u} \underbrace{\sin x}_{v} + 3 \cos x \}$

$\Rightarrow (C_1 - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x$

DATE PAGE

$\Rightarrow -y\cos x - 2\sin x + 3\sin x$

$PI = \sin x - y\cos x$

Ques 2 $r + s - 2t = (y-1)e^x$

$D^2 + DD' - 2D'^2 = (y-1)e^x$

$\cancel{PI = (y-1)e^x}$

$m^2 + m - 2 = 0$

$m^2 + 2m - 1m - 2 = 0$

$m(m+2) - (m+2) = 0$

$(m-1)(m+2) = 0 \;;\; \boxed{m = 1, -2}$

$PI \Rightarrow \dfrac{(y-1)e^x}{(D-D')(D+2D')}$ $m-1=0$ $D - 1D'$

$m+2=0$ $D+2D'$

$\Rightarrow \dfrac{1}{(D-D')} \dfrac{1}{D+2D'} (y-1)e^x$ $m=1$

$y = mx + c \rightarrow \cancel{y = -x + c}$ $\boxed{y = c + 2x}$

$\Rightarrow \dfrac{1}{(D-D')} \int (c + 2x - 1)e^x \, dx$

$\Rightarrow \dfrac{1}{(D-D')} \left[\int (ce^x) + 2xe^x - e^x \right] dx$

DATE
PAGE

$$\Rightarrow \frac{1}{D-D'}\left[ce^{x} + 2\int x e^{x} - \int e^{x}\right]$$

$$\Rightarrow \frac{1}{D-D'}\left[ce^{x} + 2\left[xe^{x} - e^{x}\right] - e^{x}\right]$$

$$\Rightarrow \frac{1}{D-D'}\left[ce^{x} + 2xe^{x} - 2e^{x} - e^{x}\right]$$

$$\Rightarrow \frac{1}{D-D'}\cdot\left[ce^{x} - 3e^{x} + 2xe^{x}\right]$$

$\cancel{(c+1+2x)\, e^{x}}$

$$\Rightarrow \frac{1}{D-D'}\left[(c+2x-3)\,e^{x}\right]$$

$$\Rightarrow \frac{1}{D-D'}\left[(c+2x)\,e^{x} - 3e^{x}\right]$$

$$\Rightarrow \frac{1}{D-D'}\left[ye^{x} - 3e^{x}\right]$$

$\boxed{y = mx + c}$

$$\Rightarrow \int (c_1 - x)\,e^{x} - 3e^{x}\,dx$$

$$\Rightarrow c_1 e^{x} - (x-1)\,e^{x} - 3e^{x}$$

$$\Rightarrow ((c_1 - x) - 2)\,e^{x}$$

$$\Rightarrow (y-2)\,e^{x}$$

DATE
PAGE

Ques 4 $r + 2s + t = 2\cos y - x\sin y$

$D^2 + 2DD' + D'^2$.

$m^2 + 2m + 1 \Rightarrow m^2 + m + m + 1$ $\qquad \sin x = \int \cos x$

$\Rightarrow m(m+1) + (m+1)$ $\qquad -\cos x = \int \sin x$

$\Rightarrow (m+1)(m+1)$

$(D + D')(D + D')$ $\qquad y = mx + c$

PI $\Rightarrow \dfrac{2\cos y - x\sin y}{(D+D')(D+D')}$ $\qquad \boxed{y = c + x}$

$\Rightarrow \dfrac{1}{(D+D')} \int 2\cos(c+x) - x\sin(c+x)$

$\Rightarrow \dfrac{1}{D+D'}\left[2\int \cos(c+x) - \int x \sin(c+x)\right]$

$\Rightarrow \dfrac{1}{D+D'}\left[2[\sin(c+x)] - [-x\cos(c+x) - (-1)\sin(c+x)]\right]$

$\Rightarrow \dfrac{1}{D+D'}\left[2\sin(c+x) - [-x\cos(c+x) + 1\sin(c+x)]\right]$

$\Rightarrow \dfrac{1}{D+D'}\left[\underline{2\sin(c+x)} + x\cos(c+x) - \underline{\sin(c+x)}\right]$

$\Rightarrow \dfrac{1}{D+D'}\left[\sin[c+x] + x\cos[c+x]\right]$

$$\Rightarrow \frac{1}{D+D'}\left[\sin y + x\cos y\right] \quad -(1)$$

$$\Rightarrow \int \sin(c_1+x) + \underbrace{x}_{(1)}\ \underbrace{\cos(c_1+x)}_{(2)}$$

$$\Rightarrow -\cos(c_1+x) + x\sin(c_1+x) + \cos(c_1+x)$$

$$\Rightarrow \underline{\underline{x\sin y}}$$

Non Homogeneous Partial Differential Eq^n

Non homogenous partial linear differential Eq^n will constant : variable.

→ complete solution in this eq^n is also

$$\phi(D, D')z = F(x, y)$$

$$z = C.F + P.I$$

Method of finding of CF

We resolve $\phi(D, D')$ into linear factors like

Case 1 :

$$(D-m_1D'-a_1)(D-m_2D'-a_2) - - - - (D-m_nD'-a_n)z = 0$$

$$z = \boxed{e^{a_1x} f_1(y+m_1x)} + e^{a_2x} f_2(y+m_2x) + - - -$$

$$- - + e^{a_nx} f_n(y+m_nx) \quad -(1)$$

DATE

PAGE

Case 2 In this case repeated factor like

$$(D - mD' - a)^3 z = 0$$

$$z = e^{ax} f_1(y + mx) + x e^{ax} f_2(y + mx) + x^2 e^{ax} f_3(y + mx)$$

Case 3 when $f(D, D')$ cannot be factorised in linear form in such cases we use a trial solution like :-

$$\boxed{z = A e^{hx + ky}}$$

A, h, K are constant

Then find relation b/w h and K and put in considered final solⁿ.

like that

$$\boxed{z = \Sigma A e^{hx + ky}}$$

DATE
PAGE

Non homogeneous

Q $DD'(D+2D'+1)z=0$

$(D^2D'+2D'^2D+DD')z=0$

$$\frac{\partial^3 z}{\partial x^2\partial y}+2\frac{\partial^3 z}{\partial x\partial y^2}+\frac{\partial^2 z}{\partial x\partial y}=0$$

$y=mx+c$

$D=0 \Rightarrow f_1(y)$

$D'=0 \Rightarrow f_2(x)$

$D+2D'+1 \Rightarrow e^{-x}f_3(y-2x)$

$$\boxed{CF=f_1(y)+f_2(x)+e^{-x}f_3(y-2x)}$$

Q $r+2s+t+2p+2q+z=0$

$\cancel{\frac{\partial^2 z}{\partial x\partial y}}$ $\cancel{D^2}$ $(D^2+2DD'+D'^2+2D+2D'+1)z=0$

$$\Big(\underbrace{(D+D')^2}_{a^2}+\underbrace{2(D+D')}_{2ab}+\underbrace{1}_{b^2}\Big)z=0$$

$$[\underbrace{D+D'}_{a}+1]^2 z=0$$

$(m+1)^2$

$x=-1,-1$

$z=CF=e^{-x}f_1(y-x)+x\,e^{-x}f_2(y-x)$

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DATE
PAGE

$D - D' \quad m - 1 = 0$

$m = 1$

Q $r - t + p - q = 0$

$$(D^2 - D'^2 + D - D')z = 0$$

$$((D - D')(D + D') + D - D')z = 0$$

$$(D - D')(D + D' + 1)z = 0$$

$m^2 - 1 + m - 1$

$m^2 + m - 2$

$m^2 + 2m - m - 2$

$m(m+2) - (m+2)$

$(m+2)(m-1)$

CF $\Rightarrow$ $Z = f_1(y + x) + e^{-x} f_2(y - x)$

Q $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$

$\cancel{D^2 + 2D - DD' - 2D'^2 + 2D'}$

$\cancel{D(D+2) \quad D'(D + 2D' + 2)}$

$D^2 + 2D - DD'$

$m^2 - m - 2$

$m^2 - 2m + 1m - 2$

$m(m-2) + 1(m-2)$

$(m+1)(m-2)$

$m = 2, \quad m = -1$

$m - 2 \Rightarrow D - 2D'$

$m + 1 \Rightarrow D + D'$

$$D^2 + 2D - DD' + 2(D + D')$$

$$(D - 2D')(D + D') + 2(D + D')$$

$$(D + D')[D - 2D' + 2]z = 0$$

$$D + D' = 0$$

$$D - 2D' + 2 = 0$$

$$Z = f_1(y - x) + e^{-2x} f_2(y + 2x)$$

Q $(D^2 + D'^2 - K^2)z = 0$

Since $D^2 + D'^2 - K^2$ cannot be ~~resolved~~ resolved into linear factors in D & D' then in such case assume trial solⁿ.

$$Z = Ae^{hx + my}$$

DATE

PAGE

$$D^2 z + D'^2 z - K^2 z = 0 \quad (1)$$

$$Dz = \frac{d}{dx}\left(Ae^{hx+my}\right) \Rightarrow Dz = Ah\,e^{hx+my}$$

$$\Rightarrow D^2 z = Ah^2 e^{hx+my}$$

$$\rightarrow D'^2 z = Am^2 e^{hx+my}$$

Putting in (1)

$$Ae^{hx+my}(h^2+m^2-K^2) = 0$$

$$h^2+m^2-K^2=0$$

$$\underline{h^2+m^2 = K^2}$$

where h may be taken as $\underline{K\cos\alpha}$ &
m may be taken as $\underline{K\sin\alpha}$

$$h \longrightarrow K\cos\alpha \qquad m \longrightarrow K\sin\alpha$$

hence our solⁿ is

$$K^2\cos^2\alpha + K^2\sin^2\alpha \Rightarrow K^2(\cos^2\alpha + \sin^2\alpha) = K^2$$

$$\rightarrow Z = Ae^{\cancel{K\cos\alpha \cdot x + K\sin\alpha \cdot y}}$$

$$Z = \sum Ae^{K(\cancel{x\cos\alpha + y\sin\alpha})}$$ is Trial Solⁿ

DATE
PAGE

Q $(D^3 - 3DD' + D' + 4)z = 0$

$D^3 z - 3DD'z + D'z + 4z = 0$

Let $z = Ae^{hx+my}$

$Dz = Ah\,e^{hx+my}$

$\rightarrow D^3 z = Ah^3 e^{hx+my}$

$\rightarrow D'z = Am\,e^{hx+my}$

$(Ah^3 e^{hx+my}) - 3\,Ah \cdot e^{hx+my} \cdot Am\,e^{hx+my} + 4z$ ~~$= 0$~~
$+ Am\,e^{hx+my} = 0$

$Ae^{hx+my}[h^3 - 3hm + m + 4] = 0$

$$\boxed{\begin{aligned} z &= \Sigma\, Ae^{hx+my} \\ \text{where} \quad & h^3 - 3hm + m + 4 = 0 \end{aligned}}$$

DATE PAGE

$$PI = \frac{1}{\phi(D, D')} f(x,y)$$

$$\phi(D, D')z = f(x,y)$$

Case 1 when $f(x,y) = e^{ax+by}$

$$PI = \frac{1}{\phi(D, D')} e^{ax+by} = \frac{1}{\phi(a,b)} e^{ax+by}$$

$$\phi(a,b) \neq 0$$

1) $(s + ap + bq + ab)z = e^{mx+ny}$ find complete solⁿ

$$(DD' + aD + bD' + ab)z = e^{mx+ny}$$

CF :> $\underline{DD'} + aD + \underline{bD'} + ab = 0$

$$z[D'(D+b) + a(D+b)] = 0$$

$$(D'+a)(D+b)z = 0$$

$$CF \Rightarrow e^{-ay} f_1(x) + e^{-bx} f_2(y)$$

PI :> $\dfrac{e^{mx+ny}}{(D'+a)(D+b)}$

$$\boxed{PI :> \frac{e^{mx+ny}}{(m+b)(n+a)}}$$

$D' = n$

$D = m$

R|w

$D' + a = 0$

$D + b = 0$

$m = 0$

y

$D+b$

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DATE

PAGE

Q $(D^3 - 3DD'^2 + D'^2 + 4)z = e^{2x+y}$

Assume Trial solⁿ

for CF : $Z = A e^{hx+ky}$

~~$D z = A$~~

$A e^{hx+ky}(h^3 - 3hk^2 + k^2 + 4) = 0$

CF $\Rightarrow$ $Z = \sum A e^{hx+ky}$

where $\boxed{h^3 - 3hk^2 + 4 + k^2 = 0}$

PI : $\dfrac{e^{2x+y}}{D^3 - 3DD'^2 + D'^2 + 4}$ $a = 2$ $b = 1$

$8 - 6 + 5$

$13 - 6 = 7$

$\Rightarrow \dfrac{e^{2x+y}}{7}$

$\boxed{PI \Rightarrow \frac{1}{7} e^{2x+y}}$

$$\boxed{Z = \sum A e^{hx+ky} + \frac{1}{7} e^{2x+y} \quad \text{where } h^3 - 3hk^2 + 4 + k^2}$$

DATE

PAGE

$f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$PI = \frac{1}{\phi(D, D')} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$PI = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by) =$$

$$\frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$\boxed{\phi(-a^2, -ab, -b^2) \neq 0}$$

Q $(D^2 - DD' + D' - 1)z = \sin(x+2y)$

CF : $D^2 - DD' + D' - 1$

$D^2 - 1 - D'(D - 1)$

$(D-1)(D+1) - D'(D-1)$

$(D-1)(D+1-D')$

$m - 1 = 0$

$m = 1$

~~$CF \Rightarrow f_1(y+x) + e^{-ax} f_2(y-x)$~~

$CF \Rightarrow e^{x} f_1(y) + e^{-x} f_2(y+x)$

m

$y + mx$

$$PI = \frac{1}{(D^2 - DD' + D' - 1)} \sin(x+2y)$$

$$= \frac{1}{-1+2+D'-1} \sin(x+2y)$$

DATE
PAGE

$$PI = \frac{1}{D'} \sin(x+2y)$$

$$= \int \sin(x+2y)\, dy$$

$$= -\frac{\cos(x+2y)}{2}$$

① Imp

$$(D-D'-1)(D-D'-2)z = \sin(2x+3y)$$

$$D^2 - DD' - 2D - DD' + D'^2 + 2D' - D + D' + 2$$

$$D^2 + D'^2 - 2DD' - 3D + 3D' + 2$$

$$D^2 + D'^2 - 2DD' - 3D + 3D' + 2$$

$$PI \Rightarrow \frac{\sin(2x+3y)}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2}$$

$$PI \Rightarrow \quad a = 2 \quad b = 3$$

$$\frac{\sin(2x+3y)}{-4 - 2(-6) - (9) - 3D + 3D' + 2}$$

$$\frac{\sin(2x+3y)}{-4 + 12 - 9 - 3D + 3D' + 2}$$

$$12 - 13 + 2$$

$$12 - 11 \Rightarrow \quad \Rightarrow 1$$

$$\frac{\sin(2x+3y)}{+1-3D+3D'} \Rightarrow \cancel{\frac{\sin(2x+3y)}{-(3D-3D'-1)}}$$

$$\Rightarrow \frac{\sin(2x+3y)\,\left((3D'-3D)-1\right)}{\left(\underbrace{(3D'-3D)}_{a}+\underbrace{1}_{b}\right)\left(\underbrace{(3D'-3D)}_{a}-\underbrace{1}_{b}\right)}$$

$$a^2-b^2$$

$$\frac{(3D'-3D)-1}{(3D'-3D)^2-1}\sin(2x+3y)$$

$$\begin{array}{r} 4 \\ 18 \\ 6 \\ \hline 108 \end{array}$$

$$\Rightarrow \frac{(3D'-3D)-1}{9D'^2+9D^2-18DD'-1}\sin(2x+3y)$$

$$D^2=-a^2=-4 \;;\quad DD'=-ab=-6 \;;\quad D'^2=-b^2=-9$$

$$9(-9)+9(-4)-18(-6)-1$$
$$-81-36+108-1$$

$$PI = \frac{1}{\ }\left[(3D'-3D)-1\right]\sin(2x+3y)$$

$$= \frac{-1}{8}\left[9\cos(2x+3y)-6\cos(2x+3y)-\sin(2x+3y)\right]$$

$$= \frac{-1}{10}\left[3\cos(2x+3y)-\sin(2x+3y)\right]$$

DATE
PAGE

Case 3: when $f(x,y) = x^m y^n$ where m & n being (+)ve integer.

$$f(x,y) = x^m y^n$$

$$PI = \frac{1}{\phi(D,D')} x^m y^n = [\phi(D,D')]^{-1} x^m y^n$$

Q $(D+D'-1)(D+2D'-3)z = 4+3x+6y$

Sol $$PI = \frac{1}{(D+D'-1)(D+2D'-3)}(4+3x+6y)$$

$$= \frac{1}{(1-(D+D'))(3-(D+2D'))}(4+3x+6y)$$

$$= \frac{1}{3(1-(D+D'))\left(1-\left(\frac{D+2D'}{3}\right)\right)}(4+3x+6y)$$

$$= \frac{1}{3}\left[[1-(D+D')]^{-1}\left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1}\right](4+3x+6y)$$

$$\Rightarrow \frac{1}{3}\left[(1+(D+D'))\left(1+\left(\frac{D+2D'}{3}\right)\right)\right](4+3x+6y)$$

$$\Rightarrow \frac{1}{3}\left[\left(1+(D+D')\right)\left(1+\frac{D}{3}+\frac{2D'}{3}\right)\right](4+3x+6y)$$

$$\Rightarrow \frac{1}{3}\left[1+\frac{4D}{3}+\frac{5D'}{3}+\frac{D^2}{3}+DD'+\frac{2D'^2}{3}\right](4+3x+6y)$$

$$\Rightarrow \frac{1}{3}\left[4+3x+6y+\frac{4}{3}\cdot 3+\frac{5}{3}\cdot 6\right] \Rightarrow \frac{1}{3}\left[4+3x+6y+4+10\right]$$

$$\Rightarrow \frac{1}{3}\left[18+3x+6y\right]$$

$$= 6+x+2y$$

Q $\quad (D-D')(D+D'-3)z = xy$.

$$PI = \frac{xy}{(D-D')(D+D'-3)}$$

$$PI = \frac{xy}{D\left(1-\frac{D'}{D}\right)(-3)\left[1-\left(\frac{D+D'}{3}\right)\right]}$$

$$PI = \frac{xy}{-3D\left[1-\frac{D'}{D}\right]\left[1-\left(\frac{D+D'}{3}\right)\right]}$$

$$\Rightarrow \frac{-1}{3D}\left[\left(1-\frac{D'}{D}\right)^{-1}\left(1-\left(\frac{D+D'}{3}\right)\right)^{-1}\right]xy$$

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$$= -\frac{1}{3D}\left[\left(\frac{1+D'}{D}\right)\left(1+\frac{D+D'}{3}\right)+\frac{2}{9}DD'\right]xy$$

$$PI = -\frac{1}{3D}\left[1+\frac{D}{3}+\frac{D'}{3}+\frac{D'}{D}+\frac{D'}{3}+\frac{2DD'}{9}+\cdots\right]xy$$

$$= -\frac{1}{3D}\left[xy+\frac{y}{3}+\frac{x}{3}+\frac{x^2}{2}+\frac{x}{3}+\frac{2}{9}\right]$$

$$PI = -\frac{1}{3}\left[\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{6}+\frac{x^3}{6}+\frac{x^2}{6}+\frac{2}{9}x\right]$$

Q Case 4: when $f(x,y) = e^{ax+by}V$ where V is a function of x, y.

$$\boxed{PI = \frac{1}{\phi(D, D')}e^{ax+by}\cdot V \Rightarrow e^{ax+by}\frac{1}{\phi(D+a, D'+b)}V}$$

Q $(D-3D'-2)^2 z = 6e^{2x}\sin(3x+y)$

$CF \Rightarrow e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x)$

$$PI = 6\cdot\frac{1}{(D-3D'-2)^2}e^{2x}\sin(3x+y)$$

$$= \frac{6e^{2x}}{(D+2-3D'-2)^2}\sin(3x+y)$$

DATE PAGE

$$PI = \frac{6e^{2x} \sin(3x+y)}{(D-3D')^2}$$

$$= \frac{6e^{2x} \sin(3x+y)}{(3-3)^2}$$

$$\Rightarrow \frac{6e^{2x}\, x \sin(3x+y)}{2(D-3D')}$$

$$\Rightarrow \frac{6e^{2x}\, x \sin(3x+y)}{2(0)}$$

$$\Rightarrow 6e^{2x} \frac{x^2}{2} \sin(3x+y)$$

$$\boxed{PI = 3e^{2x} x^2 \sin(3x+y)}$$

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DATE PAGE

Q $(D^2 - D')z = x e^{ax + a^2 y}$

$$PI = \frac{x e^{ax + a^2 y}}{D^2 - D'}$$

$$\Rightarrow \frac{x e^{ax + a^2 y}}{(D+a)^2 - (D' + a^2)}$$

$$D^2 + \cancel{a^2} + 2Da - D' - \cancel{a^2}$$

$$D^2 + 2Da - D'$$

$$\Rightarrow \frac{x e^{ax + a^2 y}}{D^2 + 2Da - D'} \Rightarrow e^{ax + a^2 y} \cdot \frac{1}{D^2 + 2Da - D'} \cdot x$$

$$\cancel{a^2 + 2a^2 - a^2 \Rightarrow}$$

$$\cancel{\Rightarrow \frac{1}{2a^2} x e^{ax + a^2 y}}$$

$$e^{ax + a^2 y} \cdot \frac{1}{2Da\left[1 + \frac{D^2 - D'}{2Da}\right]} \cdot x$$

$$\Rightarrow e^{ax + a^2 y} \cdot \frac{1}{2Da\left[1 + \left(\frac{D}{2a} - \frac{D'}{2Da}\right)\right]} \cdot x$$

$$\Rightarrow e^{ax + a^2 y} \cdot \frac{1}{2Da}\left[1 + \left(\frac{D}{2a} - \frac{D'}{2Da}\right)\right]^{-1} x$$

$$\Rightarrow \frac{1}{2Da} e^{ax+a^2y}\left[x - \frac{D(x)}{2a} + \frac{D'(x)}{2Da}\right]$$

$$\Rightarrow \frac{1}{2Da} e^{ax+a^2y}\left[x - \frac{1}{2a}\right] \Rightarrow \frac{1}{2a} e^{ax+a^2y} \frac{1}{D}\left[x - \frac{1}{2a}\right]$$

DATE
PAGE

$$\Rightarrow e^{ax+a^2y} \frac{1}{2Da}\left[1 - \left(\frac{D}{2a} - \frac{D'}{2aD}\right)\right]x$$

$$\Rightarrow e^{ax+a^2y} \frac{1}{2a}\left[\frac{x^2}{2} - \frac{x}{2a}\right]$$

Q. Equation reducible to partial differential Equation with constant coefficient

An equation in which the coefficient of derivative of any order say K is a multiple of the variables of the degree K then it can be reduced to partial differential Equation with constant coefficient in the following way.

Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$.

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} = D'z \qquad \left\{ D' \Rightarrow \frac{\partial}{\partial X}\right\}$$

$$x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)z$$

$$x^3\frac{\partial^3 z}{\partial x^3} = D(D-1)(D-2)z$$

DATE PAGE

$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} = D'z$$

$$y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$$

$$y^3\frac{\partial^3 z}{\partial y^3} = D'(D'-1)(D'-2)z$$

$$\boxed{xy\frac{\partial^2 z}{\partial x \partial y} = DD'z}$$

Q $x^2 r - y^2 t + px - qy = \log x$

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \log x$$

Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$

$$x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)z$$

$$x\frac{\partial z}{\partial x} = Dz$$

$$y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$$

$$y\frac{\partial z}{\partial y} = D'z$$

DATE

PAGE

$$\left(D(D-1) - D'(D'-1) + D - D'\right) z = X$$

$$\left(D^2 - \cancel{D} - D'^2 + \cancel{D'} + \cancel{D} - \cancel{D'}\right) z = X$$

$$(D^2 - D'^2) z = X$$

$$m^2 - 1 = 0 \; ; \; m^2 = 1 \; ; \; m = \pm 1$$

$$CF = f_1(Y + X) + f_2(Y - X)$$

$$PI = \frac{X}{D^2 - D'^2} \Rightarrow \frac{X + 0 \cdot Y}{D^2 - D'^2}$$

$$D \rightarrow m = 1 \quad ; \quad D' = m = 0$$

$$PI = \iint u \, du \, du \Rightarrow \frac{u^3}{6} = \frac{x^3}{6}$$

$$Z = CF + PI$$

$$Z = f_1(Y + X) + f_2(Y - X) + \frac{X^3}{6}$$

$$\boxed{Z = f_1(\log y + \log x) + f_2(\log y - \log x) + \frac{(\log x)^3}{6}}$$

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$$(D - mD' - a) \Rightarrow CF \Rightarrow e^{+ax} f(y + mx)$$

DATE

PAGE

(1) $x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} = x^3 y^4$

$$\left(D(D-1) - 4DD' + 4D'(D'-1) + 6D'\right) z = x^3 y^4$$

$$\left(D^2 - D - 4DD' + 4D'^2 - 4D' + 6D'\right) z = x^3 y^4$$

$$\left(D^2 - D - 4DD' + 4D'^2 + 2D'\right) z = x^3 y^4$$

Let $x = e^X \Rightarrow X = \log x$, $Y = \log y$

$$\left(D^2 - D - 4DD' + 4D'^2 + 2D'\right) z = e^{3X} \cdot e^{3Y}$$

$$y = e^Y \Rightarrow Y = \log y$$

$CF \Rightarrow D^2 - D - 4DD' + 4D'^2 + 2D'$

$$\cancel{D(D-1)} \left[(D^2 - 4DD' + 4D'^2) - (D - 2D')\right] z = e^{X+Y}$$

$$(D - 2D')(D - 2D' - 1)\, z = e^{3X+3Y}$$

$$CF = f_1(Y + 2X) + e^X f_2(Y + 2X)$$

$$PI = \frac{e^{3X+4Y}}{(D - 2D' - 1)(D - 2D')}$$

$$PI = \frac{1}{(D - 2D' - 1)} \quad \frac{e^{3X+4Y}}{D - 2D'}$$

$$PI = \frac{1}{D-2D'-1} \qquad \int \frac{e^{3x+4y}}{(3)-2(+4)}$$

$$= \frac{1}{(D-2D'-1)} \qquad -\frac{1}{5}\int e^{4}\,dy$$

$$\Rightarrow -\frac{1}{5}\,\frac{e^{4}}{(D-2D'-1)} \qquad \Rightarrow -\frac{1}{5}\,\frac{e^{3x+4y}}{(D-2D'-1)}$$

$D = 3$
$D' = 4$

$$\Rightarrow \frac{1}{30}\,e^{3x+4y}$$

$$= \frac{1}{30}\,e^{3\log x + 4\log y} \qquad \Rightarrow \frac{1}{30}\,x^3 y^4$$

$$z = f_1(\log y + 2\log x) + x f_2(\log y + 2\log x) + \frac{1}{30}x^3 y^4$$

Q $\quad x^2 r - y^2 t = xy$

$$x^2 D^2 - y^2 D'^2 = xy$$

$$\big(D(D-1) - D'(D'-1)\big)z = xy$$

$$(D^2 - D - D'^2 + D')z = xy$$

$$(D^2 - D'^2) + (D' - D)\,z = xy$$

$$\big[(D-D')(D+D') - (D-D')\big]z = xy$$

$\Rightarrow \quad (D-D')(D+D'-1)\,z = xy$

$\cancel{x = e^t} \quad x = e^X \quad \Rightarrow \quad X = \log x$

$y = e^Y \quad \Rightarrow \quad Y = \log y$

$(D-D')(D+D'-1)\,z = e^X \cdot e^Y$

$(D-D')(D+D'-1)\,z = e^{X+Y}$

CF $\Rightarrow \quad \underbrace{(D-D')}\,(D+D'-1)\,z$

$m-1=0 \;;\; m=1$

CF $\Rightarrow \quad f_1(y-x) + e^x f_2(y-x)$

$$PI = \frac{e^{X+Y}}{(D-D')(D+D'-1)}$$

$$\Rightarrow \frac{x}{2D - 2D'} \phi(x+y)$$

$$\Rightarrow \frac{x^2}{2} \phi(x+y)$$

$$PI = \frac{x^3}{6} + \frac{x^2}{2} \phi(x+y)$$

Q11 $(D^2D' - 2DD'^2 + D'^3) z = \frac{1}{x^2}$

$$PI = \frac{1}{x^2 (D^2D' - 2DD'^2 + D'^3)} \quad \Big| \quad a = 1 ; b = 0$$

$$\Rightarrow \frac{x}{x^2 (2DD' - D'^2)} = \frac{x^2}{x^2(2D')}$$

$m = 0, 1, 1$

$CF = f_1(x) + f_2(x+y) + y f_3(x+y)$

$$PI = \frac{1}{(D^2D' - 2DD'^2 + D'^3)} \cdot \frac{1}{(1\cdot x + 0\cdot y)^2}$$

$a = 1 ; b = 0 \rightarrow D = 1 \quad D' = 0$

$$PI = \frac{y}{D^2 - 4DD' + 3D'^2} \cdot \frac{1}{(x + 0\cdot y)^2}$$

$$= y \iint \frac{1}{u^2} du\, du \Rightarrow -y \log u \Rightarrow -y \log \boxed{\frac{1}{x}}$$

$$\Rightarrow -y(\log x^{-1}) \Rightarrow -y(-1)\log x$$

$$\Rightarrow y \log x$$