COMPLEMENTED LATTICE :-

Consider a bounded lattice 'L' with the greatest element '1'/I and least element '0'/O. An element a' is called complement of $a \in L$ iff.

$$\boxed{\begin{aligned} x \vee x' &= 1 \\ x \wedge x' &= 0 \end{aligned}}$$

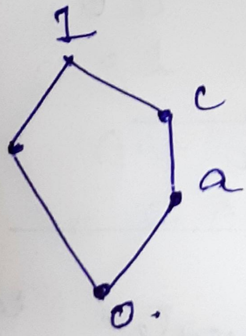
→ complements are symmetric i.e a's comp is a' then a' is comp of 'a'.

→ They need not to be unique.

A lattice 'L' is called complemented if 'L' is bounded and every element has its complement.

$0' = 1$
$1' = 0.$

eg.



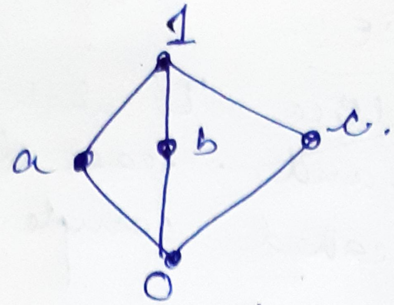
	0	a	b	c	1
comp	1	b	a, c	b	0

$b \wedge a = 0$
$b \vee a = 1.$

$b \wedge c = 0$
$b \vee c = 1.$

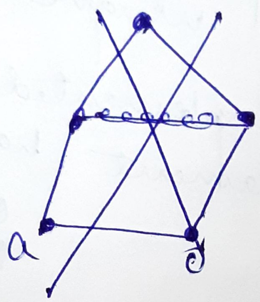
It is a complemented lattice.

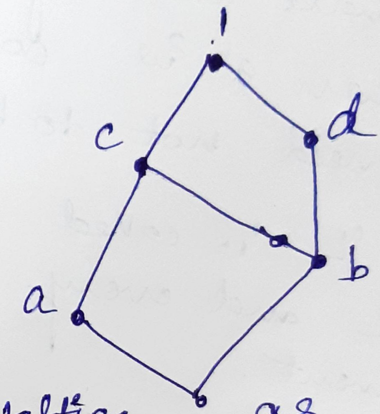
Q)



Elements	0	a	b	c	1
comp.	1	b, c	a, c	a, b	0

Q)



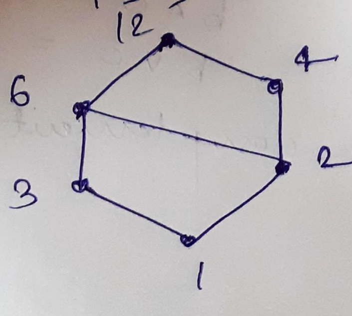


Not a complemented lattice. as

0	1	a	b	c	d
1	0	d	ϕ	ϕ	a

Q) Is D_{12} a complemented lattice?

$D_{12} = \{1, 2, 3, 4, 6, 12\}$



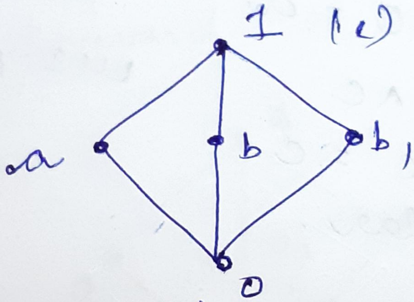
1 – 12
2 – ϕ
3 – 4
4 – 3
6 – ϕ.
12 – 1

(4) MODULAR LATTICE:-

If in a lattice $(L, \wedge, \vee)$ modular inequality becomes equality iff for all $a, b, c \in L$ if $a \leq c$ then

$$\boxed{a \vee (b \wedge c) = (a \vee b) \wedge c}$$

Q) Diamond Lattice is modular or not.



$\underline{LHS}$ $\quad a \vee (b \wedge c)$ $\qquad a \leq c$ i.e $a \leq 1$

$= a \vee b$

$= c = 1$

$\underline{RHS}$ $\quad (a \vee b) \wedge c$ $\qquad$ LHS = RHS.

$= c \wedge c = c = 1$

$\underline{R.}$ $\quad b_1 \leq 1$ $\qquad a = b$, $b = b_1$, $c = 1$.

LHS =

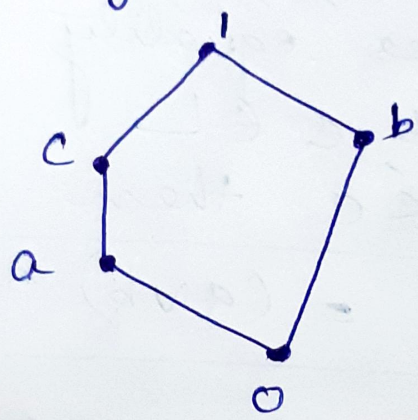
$b \vee (b_1 \wedge 1)$

$= b \vee b_1 = 1.$

$\underline{RHS}$ $\quad (b \vee b_1) \wedge 1$ $\qquad$ Yes it is modular.

$= 1 \wedge 1 = 1$ $\qquad$ LHS = RHS.

8) Find pentagonal is modular or not.



$a, b, c \in L$

if $a \leq c$ then

$a \vee (b \wedge c) = (a \vee b) \wedge c$

LHS $= a \vee (b \wedge c)$

$\Rightarrow a \vee 0 = a$

RHS $= (a \vee b) \wedge c$

$1 \wedge c = c.$

LHS $\neq$ RHS.

NOT Modular.

Distributive lattice :-

A lattice 'L' is called distributive if it holds 2 conditions.

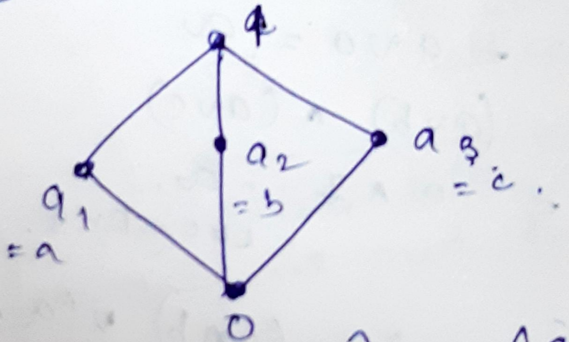
(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\forall, a, b, c \in L$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

→ Every chain is distributive Lattice

→ $(P(S), \subseteq)$ is distributive.

Q) Show diamond lattice is distributive or not.



(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

LHS :- $b \wedge c = 0$

$a \vee 0 = a$.

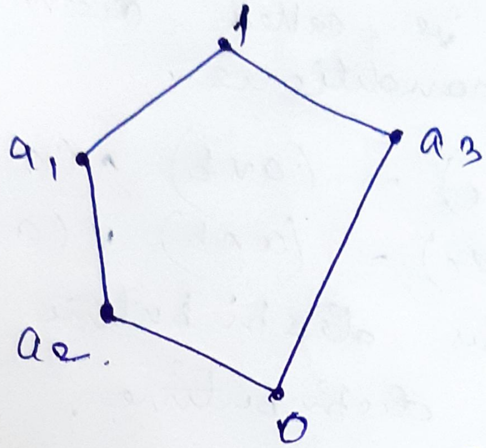
RHS :- $(a \vee b) \wedge (a \vee c)$

$1 \wedge 1 = 1$

LHS $\neq$ RHS.

(Not distributive).

8) Show 5 basic elements of pentagonal



(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

LHS $= a \vee (b \wedge c)$

$= a \vee 0 = a$

RHS $= (a \vee b) \wedge (a \vee c)$

$= a \wedge 1 = a.$

LHS = RHS. (✓)

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

LHS:- $a \wedge (b \vee c)$

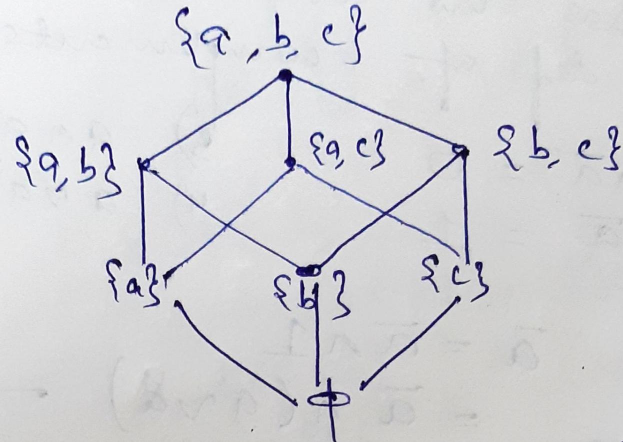
$= a \wedge 1 = a$

RHS $= (a \vee b) \vee (a \wedge c)$

$= b \vee 0$

$= b.$ LHS $\neq$ RHS.

Q) Check $(P(S), \subseteq)$ is complemented bounded lattice or not.



$\{a\} - \{b, c\}$
$\{b\} - \{a, c\}$
$\{c\} - \{a, b\}$.

Yes it is complemented lattice.

Proof :- Every distributive lattice is modular but converse is not true.

Sol^n If $a \le c$ then.
$a \vee c = c$ — (1)

For a distributive lattice,
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad — (2).$$
substituting (1) in (2) we get.

If $a \le c$ then.
$$\boxed{a \vee (b \wedge c) = (a \vee b) \wedge c.}$$
which shows 'L' is modular.

Note:-
Diamond lattice is ~~not~~ modular (By counter example) but not distributive.

Th :- Let $(L, \wedge, \vee)$ is a complemented distributive lattice and $a \in L$ then $\bar{a}$ is unique.

Sol^n Let a has two complement $\bar{a}$, a' then by def. of complements :-

1) $a \wedge \bar{a} = 0$
2) $a \vee \bar{a} = 1$
3) $a \wedge a' = 0$
4) $a \vee a' = 1$.

Now ⇒
$$\bar{a} = \bar{a} \wedge 1$$
$$= \bar{a} \wedge (a \vee a') \quad \text{— from } (4).$$

RHS ⇒ By distributive law :-

$$= (\bar{a} \wedge a) \vee (\bar{a} \wedge a')$$
$$= 0 \vee (\bar{a} \wedge a')$$
$$\Rightarrow (a \wedge a') \vee (\bar{a} \wedge a') \quad \text{from } (3).$$
$$\Rightarrow (a \vee \bar{a}) \wedge a' \quad \text{(Reverse distributive)}$$
$$= 1 \wedge a' \quad \text{from } (2).$$
$$= a'$$

$\bar{a} = a'$ Proved.

Th-2 If a, b are elements of complemented distributive lattice then show

i) $a \vee (a' \wedge b) = a \vee b$

(ii) $a \wedge (a' \vee b) = a \wedge b$.

Sol^n) i) $a \vee (a' \wedge b)$

$= (a \vee a') \wedge (a \vee b)$ (distributive)

$= 1 \wedge (a \vee b)$ (comp. law).

$= (a \vee b) = \text{RHS}.$

(ii) $a \wedge (a' \vee b)$

$= (a \wedge a') \vee (a \wedge b)$ (distributive)

$= 0 \vee (a \wedge b)$ (comp. prop).

$= a \wedge b = \text{RHS}.$

Th-8

Joint cancellation law :-

For any distributive lattice if

$a \vee b = a \vee c$ and

$a \wedge b = a \wedge c$ then

$b = c$

(If sup & inf both are equal then we can apply cancellation law :-.

Sol^n

$a \vee b = a \vee c$ — (1).

$a \wedge b = a \wedge c$ = (2).

By absorption law :-

$b \vee (a \wedge b) = b.$

LHS: $\Rightarrow b \vee (a \wedge b)$

$\Rightarrow b \vee (a \wedge c)$ by eq^n (2).

$\Rightarrow (b \vee a) \wedge (b \vee c)$ by distributive

$\Rightarrow (a \vee c) \wedge (b \vee c)$ by eq^n (1).

$\Rightarrow (a \wedge b) \vee c$ (Reverse distributive)

$\Rightarrow (a \wedge c) \vee c$ by eq^n (2).

$\Rightarrow c$ by absorption law.

Product of Lattice:—

Let L and M are 2 lattices, set of ordered pairs

$\{(x,y) : x \in L, y \in M\}$

with operations $\vee$ and $\wedge$ defined by

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$$
$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$$

in $L \times M$ lattice.

If $(L, \leq)$ $(M, \leq)$

$(x_1, y_1) \leq (x_2, y_2)$ when

$\left\{ \begin{matrix} x_1 \leq x_2 \\ (\text{in } L) \end{matrix} \right\}$ and $\left\{ y_1 \leq y_2 \text{ in } M \right\}$.

$(L \times M)$ will be a lattice if glb, lub exist by the above rule.