Unit-1

Chapter 3

Functions

3.1 Definition

3.1.1 Domain

Set of inputs (also called pre-image) is called domain. It is denoted by D_f.

Here, the domain is $N = \{1, 2, 3, 4, \ldots\ldots\}$

3.1.2 Co-Domain

Set of possible outputs is called co-domain. Here, the co-domain is $N = \{1, 2, 3, 4, \ldots\ldots\}$

3.1.3 Range

Set of actual outputs (also called images) is called Range. It is denoted by R_f.
Here, the range is $\{1, 4, 9, 16, \ldots\ldots\}$. Obviously $R_f \subseteq$ co-domain.

3.2 Function (Mathematical Definition)

If A and B are two non-empty sets then a rule f, under which to every element x of the set A there corresponds one and only one elements of set B then the rule f is called the function from A to B, it is denoted by

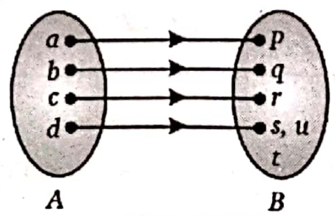
$$f : A \to B$$

If a pre-image is denoted by x and an image is denoted by y then we can write $y = f(x)$

Where y is image $\in B$ and x is pre image $\in A$ and $f(x)$ is called the value of function f at the point x.

3.2.1 Methods of Representing a function

(a) Arrow diagram



$\therefore \; f : A \to B$

Fig. 3.1

$D_f = \{a, b, c, d\}$. Co-domain $= \{p, q, r, s, t, u\}$ and $f(a) = p,\; f(b) = q,\; f(c) = r,\; f(d) = s$

➤ **Vertal Description**

f is a function which assigns to each country in the world and its capital city

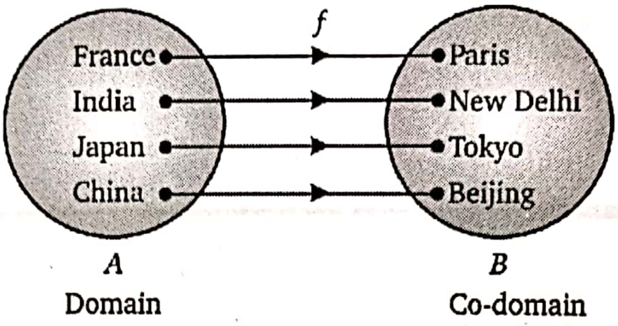


Fig. 3.2

Since every country has one and only one capital. Hence, f is a function.

D_f is the set of the countries of the world. R_f is the set of all the capital cities.

$\therefore\ f: A \to B$ such that $x \in A \Rightarrow f(x) \in B$ i.e., f (India) = New Delhi

➤ **Tabular Form**

A	x	1	2	3	4	5
B	y	a	b	c	d	e

Let $f: A \to B$, such that

$$D_f = \{1, 2, 3, 4, 5\},\ R_f = \{a, b, c, d, e\} \text{ and } f(2) = b,\ f(5) = e \text{ etc.}$$

➤ **Enumerating the Ordered Pairs**

Let $f = \{(1, a),\ (2, a),\ (3, a),\ (4, a)\}$

Then $D_f = \{1, 2, 3, 4\},\ R_f = \{a\}$ and $f(1) = a,\ f(2) = a\ldots\ldots$

This function can be represented by

$$f(x) = a \ \forall\ \{x \in N : 1 \le x \le 4\}$$

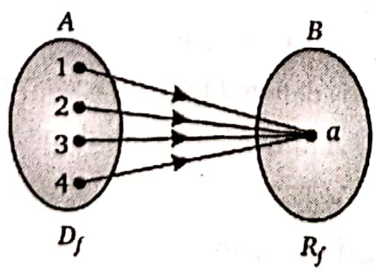


Fig. 3.3

Note:

(i) The set of first elements of ordered pairs of a function is called domain (D_f).

(ii) The set of second elements of the ordered pairs of a function is called range (R_f).

➤ **Formula or an Equation**

Let $$f : R \to R$$

Such that $$f(x) = x^2 \quad \forall x \in R$$

Then $$D_f = R,\ R_f = R^{+} \cup \{0\}$$

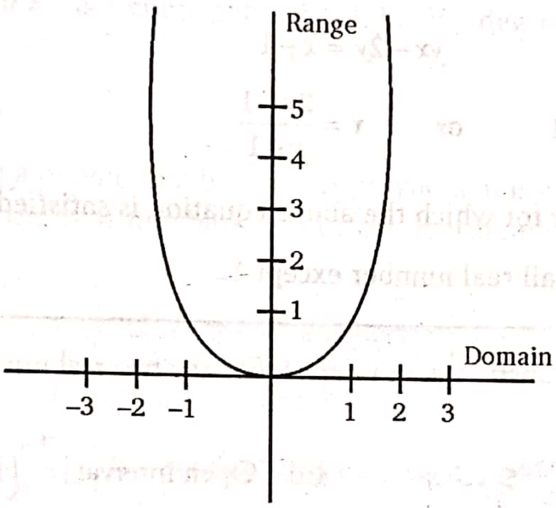


Fig. 3.4

3.3 Classification of Functions or Type of Functions

3.3.1 Real function

A function $f: A \to B$ is called real valued if the image of every element of A under f is a real number *i.e.*,

if $\quad f(x) \in R \ \ \forall x \in A \quad$ or $\quad y = f(x),$

where y is dependent variable and x is independent variable.

The curve $y^2 = 4ax$ and $x^2 = 4ay$ are the functions as they contains two dependent variable value of one gives the result of other.

Note:

(1) The function is not defined if denominator of function is zero.

i.e., $\quad f(x) = \dfrac{1}{x-2}$ then function is not defined at $x = 2$

(2) The function is not defined if function becomes indeterminate for any value of x

i.e., $\quad f(x) = (|x| - 2)^{(x-2)}$ is not defined at $x = 2$ Since $f(2) = 0^0$

➤ **Range**

It is the set of all the images of all the elements of domain, which is obtained as

i.e., Let $f(x) = y$, and $f(x) = \dfrac{x-1}{x-2}$

$\therefore$ $y = \dfrac{x-1}{x-2}$

$\Rightarrow$ $yx - 2y = x - 1$

$\Rightarrow$ $(y-1)x = 2y - 1$ or $x = \dfrac{2y-1}{y-1}$

For $y = 1$ there is no real values of x for which the above equation is satisfied.

Hence $R_f = R - \{1\}$. *i.e.*, the set of all real number except 1.

> **Note:** Usually the domain of a real function is an interval. For any two real numbers a and b where $a < b$, we defined
>
> (i) Closed interval $[a, b] = \{x \in R : a \leq x \leq b\}$
> (ii) Open interval $]a, b[= \{x \in R, a < x < b\}$
> (iii) Left-half open interval $]a, b] = \{x \in R : a < x \leq b\}$
> (iv) Right-half open interval $[a, b[= \{x \in R, a \leq x < b\}$
> (v) $]a, \infty[= \{x \in R : x > a\}$
> (vi) $[a, \infty[= \{x \in R ; x \geq a\}$
> (vii) $]-\infty, a[= \{x \in R : x < a\}$
> (viii) $]-\infty, a] = \{x \in R : x \leq a\}$

3.3.2 Algebraic Functions

The functions consisting of finite number of terms involving different power of independent variable (x) and the operations plus (+), minus (–), multiplication (×) and division (÷) are called **Algebraic functions.**

e.g., $2x^2 + x^{1/3} + 4,\ x^2 + 3x + 9,\ \dfrac{x^3 - 1}{x^3 + 1}$ *etc.*

3.3.3 Polynomial Functions

A function whose domain and co-domain both is the set of real numbers and contains finite number of terms containing natural number powers of x multiplied by real constants is called polynomial function.

If $f : R \to R_1$ such that $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_n x^n$

Where $n \in N$ and $a_1, a_2, \ldots\ldots a_n \in R$ then f is called a polynomial function. Some time $f(x)$ is also called a polynomial in x.

➤ Degree of a Polynomial

The highest power of x having non-zero coefficient is called the degree of the polynomial. Here, if $a_n \neq 0$ then degree of $f(x)$ is n. Thus

(i) $f(x) = ax + b, a \neq 0$ is a linear polynomial, having degree one.

(ii) $f(x) = ax^2 + bx + c, a \neq 0$ is a quadratic polynomial, having degree two.

(iii) $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$ is a cubic polynomial, having degree three.

3.3.4 Rational Functions

A function obtained by dividing a polynomial by another polynomial is called a rational function. Domain of such functions contains all the real numbers except the real numbers for which the polynomial in denominator is zero.

Hence, $$f : A \to R; \quad f(x) = \frac{P(x)}{R(x)}$$

Here, $P(x)$ and $Q(x)$ are the polynomial functions and $A = \{x : x \in R \text{ such that } Q(x) \neq 0\}$.

Thus, the function $f(x) = \left\{\dfrac{x^2+1}{x^2 - 2x + 5}\right\}$ is a rational function, where $x^2 - 2x + 5 \neq 0$

Illustrations:

(i) $f(x) = \dfrac{x^2 + 5x + 7}{x^2 - 3x + 2}$ is function. As $x^2 - 3x + 2 \neq 0$, the domain of function is $R - \{1, 2\}$.

(ii) $f(x) = \dfrac{1}{x^n}, n \in N$ is function whose domain is $R - \{0\}$.

3.3.5 Irrational Functions

The algebraic function containing one more terms having non-integral rational power of x are called irrational functions.

Illustration:

If $g(x) = \dfrac{2x^{3/2} + 5x}{x^2 - 1} = \dfrac{2(x^{1/2})^3 + 5x}{x^2 - 1}$, then g is undefined if $x < 0$ and also if $x^2 - 1 = 0$ *i.e.,* $x = \pm 1$

3.3.6 Modulus Function

$$f : R \to R \text{ such that}$$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

then $f(x)$ is called ***Modulus Function.***

Since the modulus of every real number is a unique non-negative real number, so $D_f = R$. Since $|x|$ is either 0 or a positive real number, we have $R_f = \{|x| : x \in R\}$ = Set of non-negative real numbers.

3.3.7 Signum function

This function is defined by
$$f(x)=\begin{cases}\dfrac{|x|}{x}, & \text{when } x\neq 0\\ 0\ , & \text{when } x=0\end{cases}$$

Thus, we have,
$$f(x)=\begin{cases}1, & \text{when } x>0\\ 0, & \text{when } x=0\\ -1, & \text{when } x<0\end{cases}$$

Then
$$D_f = R \text{ and } R_f = [-1, 0, 1]$$

3.3.8 Constant Function

Let C be fixed real number. Then the function defined by $f(x)=C\ \forall\ x\in R$ is called constant function C.

Clearly, $D_f = R$ and $R_f = (C)$

Here, $A=\{x_1, x_2, x_3\}$ and $B=\{a, b, c\}$ and $f: A\to B$ be such that $f(x_1)=b$, $f(x_2)=b$, $f(x_3)=b$. Then, $D_f=\{x_1, x_2, x_3\}$, $R_f=\{b\}=$ Range of f

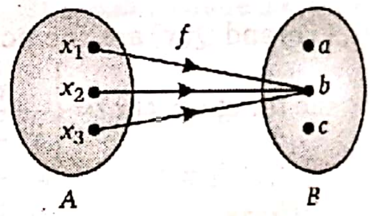


Fig. 3.5

3.3.9 Identity Function

The function defined by $f(x)=x$ for all $x\in R$ is called the identity function clearly
$$D_f = R,\ R_f = R$$

3.3.10 Reciprocal Function

The function defined by $f(x)=\frac{1}{x}$ is called the reciprocal function. The function $f(x)=\frac{1}{x}$ is not defined if $x=0$.

$D_f = R-\{0\},\ R_f = R-\{0\}$

3.3.11 Step Function or the Greatest Integer Function

If $x\in R$ then $[x]$ is defined as the greatest integer not exceeding x.

i.e., $[2.01]=2,\ [2.9]=2,\ [-1.3]=-2,\ [3]=3,\ [-1]=-1$

If we consider $f(x)=[x]$ then $D_f=R$ and $R_f=\{[x]: x\in R\}$

Example 1: *Find a set of all real number x such that* $[x]=2$

Solution: For all x such that $2\le x<3$, we have $f(x)=[x]=2$. So required set $=\{x\in R: 2\le x<3\}=[2, 3[$

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3.3.12 Exponential Function

The function $f(x) = e^x$ is called the exponential function. Since $f(x) = e^x$ is defined for all real value of x. So $D_f = R$, also $y = e^x \Rightarrow x = \log y$

And, we know that $\log y$ is not defined when $y = 0$ or negative so, $R_f =]0, \infty[$

3.3.13 Logarithmic Function

The function $f(x) = \log x$ is called the logarithmic function, $\log x$ is not defined when x is zero or negative.

$\therefore \quad D_f =]0, \infty[, \qquad R_f = \text{Range } (f) = \{\log x : x \in]0, \infty[\ \} = \text{set of all real numbers} = R$

3.3.14 Trigonometric Function

The function $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$ are called the trigonometric functions. The domains and ranges of these functions are given below:

(a) **sin x:** Since $\sin x$ is defined for all value of $x \in R$, its domain $R = R$. As value of $\sin x$ lies between -1 and $+1$, so range $= [-1, 1]$

(b) **cos x:** Since $\cos x$ is defined for all value of $x \in R$, its domain $R = R$. As value of $\cos x$ lies between -1 and $+1$, so range $= [-1, 1]$.

(c) **tan x:** We have $\tan x = \dfrac{\sin x}{\cos x}$ which is not defined when $\cos x = 0$. And $\cos x = 0 \Rightarrow \cos x = \cos (2n+1)\dfrac{\pi}{2} \Rightarrow x = (2n+1)\dfrac{\pi}{2}$. Then domain $R = R - \{(2n+1)\dfrac{\pi}{2},\ n \in I\}$. Since tan x take all value of x so its range $= R$

(d) **sec x:** We have $\sec x = \dfrac{1}{\cos x}$ is not defined when $\cos x = 0$ and $\cos x = 0 \Rightarrow \cos x = \cos (2n+1)\dfrac{\pi}{2} \Rightarrow x = (2n+1)\dfrac{\pi}{2}$. Then domain $= R - \{(2n+1)\dfrac{\pi}{2}, n \in I\}$, Range $= R -]-1, 1[$

(e) **cot x:** We have $\cot x = \dfrac{\cos x}{\sin x}$ is not defined, when

$$\sin x = 0 \Rightarrow \sin x = \sin n\pi \qquad \Rightarrow x = n\pi,\quad n = 0, 1, 2, 3, \ldots\ldots$$

$$\therefore \qquad \text{Domain} = R - \{n\pi : n \in I\} \qquad \text{and} \qquad \text{Range} = R$$

(f) **cosec x:** We know $\operatorname{cosec} = \dfrac{1}{\sin x}$ is not defined,

$$\sin x = 0 \Rightarrow \sin x = \sin n\pi$$

$$\Rightarrow x = n\pi$$

$$\therefore \qquad \text{Domain} = R - \{n\pi : n \in I\}, \qquad \text{Range} = R -]-1, 1[.$$

3.3.15 Inverse Trigonometric Functions

Let $\sin y = x \Rightarrow y = \sin^{-1}(x)$ is called inverse circular function. The domain and range of inverse trigonometric functions are given as:

	Function	Domain	Range
1.	$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
2.	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$\tan^{-1} x$	R	$[-\pi/2, \pi/2]$
4.	$\cot^{-1} x$	R	$]0, \pi[$
5.	$\text{cosec}^{-1} x$	$R -]-1, 1[$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
6.	$\sec^{-1} x$	$R -]-1, 1[$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

3.4 Difference between Function and a Relation

Let A and B be two sets and f be a function or mapping from the set A to the set B. Then by definition of function, F is a subset of A×B, in which $a \in A$ will exists in one and only one ordered pair from all order pairs contained in f as first co-ordinate.

In other words, the mapping f will be a subset of $A \times B$ which satisfy the following two conditions:

(i) For each $a \in A$, $(a, b) \in f$ for some $b \in B$ (ii) If $(a, b) \in f$ and $(a, b') \in f \Rightarrow b = b'$

But from the definition of relation, every subset of $A \times B$ is a relation from the set A to the set B. Hence every mapping is a relation but every relation is not function.

If R is a relation from A to B, then the domain of R may be a subset of A, but if f is a function or mapping from A to B, then the domain f will be A.

In relation R, any element of A can be associated to more than one elements in B and it is also possible that some elements of A are not associated to any element in B. But in mapping f, every element of A is associated to one and only one element in B.

Illustration: Let $A = \{1, 2, 3, 4, 5\}, B = \{a, b, c\}$.

if $R = \{(1, a), (2, b), (3, b), (4, c), (5, c)\}$

Then R is a function from A to B, clearly B is a function from A to B. Again, let S be a subset of $A \times B$,

where $S = \{(1, a), (2, c), (1, b), (4, c), (5, c)\}$

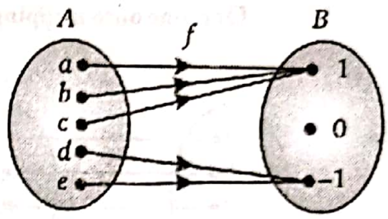
Here S is a relation from A to B, but S is not a function from A to B, because $1 \in A$ is associated with two elements a and b of B.

3.5 Kind of Mappings

3.5.1 Into Mapping

[U.P.T.U. (B.Tech.) 2003]

If $f: A \to B$ be a mapping such that at least one element of B is not a f–image of any element of the set A, then the mapping f is said to be an into mapping or A into B mapping. Symbolically, the above definition can be given as follows: A mapping $f: A \to B$ is said to be into mapping if $\{f(x) : x \in A\} \subset B$.



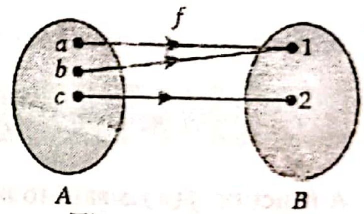
This is into mapping

Fig. 3.6

3.5.2 Onto Mapping

If f be a mapping such that each element of B is f–image of at least one element of A, then the mapping f is said to an onto or subjective mapping.

In onto mapping $f(A) = B$, symbolically–A mapping $f: A \to B$ will be an onto mapping, if $\{f(x) : x \in A\} = B$



This is onto mapping

Fig. 3.7

3.5.3 One-One Mapping

[U.P.T.U. (B.Tech.) 2003]

Let $f: X \to Y$ be a mapping. If all distinct elements of the domain X has distinct f–images in Y, then the mapping f is said to one-one or injective mapping.

Thus, $f: X \to Y$ will be one one or injective mapping,

if $$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

or

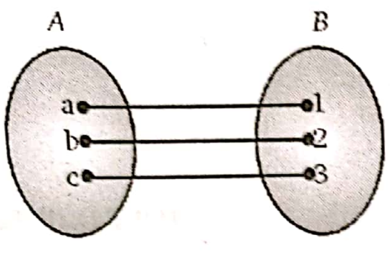
if $$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

3.5.4 Many-One Mapping

Let $f: X \to Y$ by a mapping. If two or more than two elements of the domain X have the same f–image in Y, then the mapping f is said to be many-one mapping.

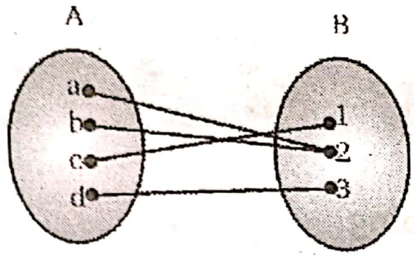
Thus, $f: X \to Y$ will be many one mapping if $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$.

(i)



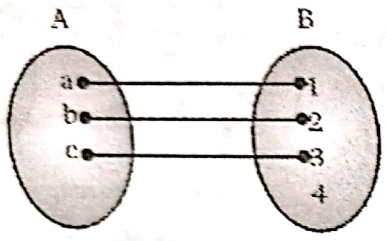
One-one onto mapping

(ii)

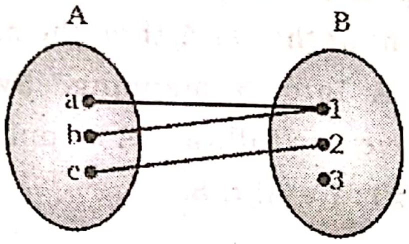


Many-one onto mapping

(iii)



One-one into mapping



Many-one into mapping

Fig. 3.8

3.6 Even and Odd Functions

A function $f(x)$ is said to be

(i) even if $f(-x) = f(x)\,\forall x$ (ii) odd if $f(-x) = -f(x)\,\forall x$

Example 2: *Show that the mapping $f : R \to R$, where $f(x) = -\sin x, x \in R$ is neither one-one nor onto.*

Solution: Here $f(x_1) = -\sin x_1$, $f(x_2) = -\sin x_2$, $x_1, x_2 \in R$.

$\therefore$ $$f(x_1) = f(x_2) \Rightarrow -\sin x_1 = -\sin x_2 \Rightarrow \sin x_1 = \sin x_2.$$

Since $$f(x_1) = f(x_2) \text{ even when } x_1 \neq x_2.$$

$\therefore$ The mapping is not one-one.

Again, since the numerical value of $\sin x$ cannot exceeds 1, and so the mapping is not onto.

Example 3: *Discuss the mapping $f : R \to R$ defined by $f(x) = x^2$, where R is the set of real numbers.*

Solution: Here the domain of the mapping is R and f–image is the set of positive real numbers, because the square of any real number is positive. Thus, f–image is a proper subset of its domain;

i.e., $$\{f(x) : x \in R\} \subset R.$$

Hence it is an into mapping.

Again, $f(x_1) = f(-x_1) = x_1^2$; *i.e.,* f–image of two distinct elements is the same element of R. Thus, it is a many-one mapping. Hence the given mapping is many-one into mapping.

Example 4: *If $X = \{-1, 1\}$ and $f(x) = x^3$ and $f : X \to X$, then prove that f is one-one onto mapping.*

Solution: $\because$ $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$, $x_1, x_2 \in X$.

$\therefore$ mapping is one-one.

Since $f(-1) = (-1)^3 = -1$ and $f(1) = 1^3 = 1$; *i.e.,* each element of X is f–image of some element of X itself.

$\therefore$ mapping f is onto.

Thus, the given mapping is one-one onto mapping.

Example 5: *If N be the set of natural numbers and a mapping $f: N \to N$ is defined by $f(x) = x^2, x \in N$, then prove that f is one–one, but not onto.*

Solution: Let $x_1, x_2 \in N$. Then $f(x_1) = x_1^2$, $f(x_2) = x_2^2$

$$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1 = +x_2$$

[Taking only + ve sign, since x_1, x_2 are positive integers]

$\therefore$ The mapping f is one-one.

$\because$ Each element of N is not f–image of some elements of N itself, for instance, 2, 3, 5, 6, 7, 8,...etc. are not square of any positive integer. Thus, f is not onto.

$\therefore$ f is into mapping.

Hence, the given mapping is one-one into.

Example 6: *Test the mapping $f: R \to R$ defined by $f(x) = \cos x$ for being one-one and onto, where R is the set of real numbers.*

or

Prove that the mapping $f: R \to R$ defined by $f(x) = \cos x, \forall x \in R$ is many-one into mapping.

Solution: Since $f(x) = \cos x$ and $f(-x) = \cos(-x) = \cos x$, so two elements x and $-x$ have the same f–image.

Hence, the mapping f is not one-one; *i. e.*, f is many–one.

Again we know that for all real values of x, the values of $\cos x$ oscillates from -1 to $+1$; *i.e.*, $-1 \le \cos x \le 1 (x \in R)$. Therefore only those elements in R (range set) which lie between -1 and $+1$ are f–images and remaining elements of this set are not associated to any element of the domain set. Thus, the given mapping f is not onto; *i.e.*, f is into. Therefore, it follows that the given mapping f is many-one into.

Example 7: *Show that the mapping $f: R \to R$, $f(x) = \frac{1}{x}$, $x \ne 0$ and $x \in R$ is one-one onto, where R is the set of non-zero real numbers.*

Solution: Let $x_1, x_2 \in R$ be any two non-zero real numbers, then

$$f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \qquad \left[\because f(x) = \frac{1}{x}\right]$$

$$\Rightarrow x_1 = x_2$$

$\therefore$ $$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

$$\Rightarrow f \text{ is one-one mapping.}$$

Let $y \ne 0$ be any real number such that $f(x) = y$;

i.e. $$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}.$$

$\because$ $y \ne 0$, $\therefore$ $\frac{1}{y}$ is also a non-zero real number

Hence, for every $y \neq 0$ there exists its pre-image $\frac{1}{y} \neq 0$ in R as shown below:

$$f\left(\frac{1}{y}\right) = \frac{1}{(1/y)} = y \qquad \left[\because f(x) = \frac{1}{x}\right]$$

It shows that $\{f(x) = \frac{1}{x} : x \neq 0, x \in R\} = R$

$\therefore$ f is onto.

Hence, the given mapping is one-one onto.

Example 8: *Show that the mapping $f : R \to R$, defined by $f(x) = 3x + 5$ is one-one into, where, R is the set of real numbers.*

Solution: f is one-one: Let $x, y \in R$ be such that $f(x) = f(y)$, then

$$f(x) = f(y) \Rightarrow 3x + 5 = 3y + 5 \Rightarrow 3x = 3y \Rightarrow x = y.$$

$\therefore$ f is a one-one mapping.

f is onto: Let $z \in R$ (range of f) be arbitrary, then let $x \in R$ (domain of f) be such that $f(x) = z$, then

$$f(x) = z \Rightarrow 3x + 5 = z \Rightarrow x = \frac{z-5}{3} \in R.$$

Thus, for each member z in the range R of f, $\exists$ a member $\frac{z-5}{3}$ in the domain R of f such that

$$f\left(\frac{z-5}{3}\right) = \frac{3(z-5)}{3} + 5 = z.$$

Hence, f is a mapping from R onto R.

$\therefore$ the given mapping $f : R \to R$ is one-one onto.

3.7 Inverse Function (Inverse Mapping)

Let $f : X \to Y$ be one-one onto mapping. Let $y \in Y$, since f is onto, there exists $x \in X$ such that $f(x) = y$. Again since f is one-one, this is the only element of X such that $f(x) = y$.

Thus, we have seen that for each $y \in Y$, there is a unique $x \in X$ that $f(x) = y$.

This mapping from Y to X is called ***Inverse of*** f. The inverse of the mapping f is usually denoted by f^{-1}.

Thus, $f(x) = y \Leftrightarrow f^{-1}(y) = x$.

Definition: If $f : X \to Y$ is a one-one onto mapping, then the mapping

$$f^{-1} : Y \to X.$$

Which associate each element $y \in Y$ to a unique element $x \in X$ is called inverse mapping of $f : X \to Y$.

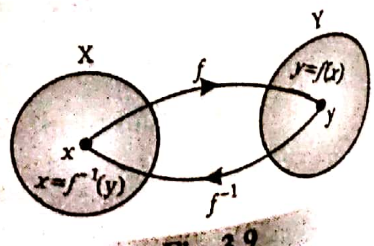


Fig. 3.9

The inverse function is more clear from the ***Adjoining Figure.***

Let $f: X \to Y$ be an one-one onto mapping, then $x \in X, y \in Y$ are associated.

$\therefore$ $$y = f\text{–image of } x = f(x).$$

Thus, $$x = f^{-1}(y).$$

Hence, the mapping $f^{-1}: Y \to X, y \in Y$ is mapped with $f^{-1}(y)$; $i.e,\ x \in X$.

▸ Theorems

Theorem 1: *Prove that if $f: A \to B$ is one-one onto mapping then $f^{-1}: B \to A$ will be one-one onto mapping.*

[U.P.T.U. (B.Tech) 2007]

Proof: Since f is one-one onto mapping, so we have

$$y = f(x) \Rightarrow f^{-1}(y) = x,\ x \in A,\ y \in B.$$

Let $$y_1 = f(x_1), y_2 = f(x_2), x_1, x_2 \in A; y_1, y_2 \in B,$$

Then $$x_1 = f^{-1}(y_1), x_2 = f^{-1}(y_2).$$

Now $$f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow x_1 = x_2 \Rightarrow f(x_1) = f(x_2) \qquad [\because f \text{ is a function}]$$

$$\Rightarrow y_1 = y_2.$$

$\therefore$ The mapping f^{-1} is one-one.

Again, let x be an arbitrary element of A, then for the mapping f there exists an element y in B such that $y = f(x)$.

But $$f^{-1}(y) = x \ \forall x \in A.$$

$\therefore$ $$\{f^{-1}(y): y \in B\} = A.$$

Thus, $f^{-1}: B \to A$ is onto.

Hence, $f^{-1}: B \to A$ is one-one onto mapping.

Theorem 2: *If f is one-one onto (bijective) function from the set A to the set B, then f^{-1} will be a unique function.*

Proof: Since $f: A \to B$ is one-one onto mapping and so its inverse will exists.

Suppose, if possible, there are two inverse g and h of f.

Now let y be an arbitrary element of B, then $g(y) = x_1$ and $h(y) = x_2$, where $x_1, x_2 \in A$.

Hence $$f(x_1) = y \text{ and } f(x_2) = y. \qquad [\because g \text{ and } h \text{ are inverses of } f]$$

Therefore $$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \qquad [\because f \text{ is one-one onto}]$$

$\Rightarrow$ $$g(y) = h(y) \quad \forall y \in B \Rightarrow g = h.$$

Hence, inverse of f is unique.

Theorem 3: *The inverse of an inversible mapping is unique.* **[M.K.U. (B.E.) 2005, 2008]**

Proof: Let $f: A \to B$

By invertible mapping, if possible, let

$$g: B \to A \text{ and } h: B \to A$$

be two different inverse mapping of f. Let $b \in B$ and $g(b) = a_1, a_1 \in A, \; h(b) = a_2, \; a_2 \in A$

Now $\quad g(b) = a_1 \Rightarrow b = f(a_1)$

and $\quad h(b) = a_2 \Rightarrow b = f(a_2)$

$\therefore \quad f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \quad$ [$\because f$ is one-one]

This prove that $g(b) = h(b) \; \forall \; b \in B$

Thus, the inverse of f is unique.

***Example 9:** $f : R \to R$ is defined by $f(x) = ax + b$ where $a, b, x \in R$ and $a \neq 0$. Show that f is invertible and find the inverse of f.* **[U.P.T.U. (B.Tech.) 2007]**

Solution: First we have to show f is one-one

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

then $\quad ax_1 + b = ax_2 + b$

$\Rightarrow \quad ax_1 = ax_2$

$\Rightarrow \quad x_1 = x_2$

$\Rightarrow f$ is one-one.

To show that f is onto

Let $y \in R$ such that $y = f(x)$ or $y = ax + b$ or $ax = y - b$

$\therefore$ given $y \in R$, there exists an element $x = \frac{1}{a}(y - b) \in R$ such that $f(x) = y$

$\Rightarrow f$ is onto

Hence, f is one-one and onto $\Rightarrow f$ is invertible and $f^{-1}(y) = \frac{(y - b)}{a}$

***Example 10:** If $f : I \to I$, where $f(x) = x^2$ and I be the set of integers, then find $f^{-1}(9)$ and $f^{-1}(-2)$.*

Solution: We know that $(-3)^2 = 9$ and $(3)^2 = 9$ and there is no other integer whose square is 9. Hence $f^{-1}(9) = \{-3, 3\}$, but there is no other integer whose square is -2, therefore $f^{-1}(-2) = \phi$, null set.

***Example 11:** If $f : R \to R$ is defined by $f(x) = x^2 + 1$, then find $f^{-1}(3)$, $f^{-1}(\{10, 37\})$.*

Solution:

(i) From $x^2 + 1 = 3$, $x = \pm\sqrt{2}$. Hence $f^{-1}(3) = \{-\sqrt{2}, \sqrt{2}\}$.

(ii) From $x^2 + 1 = 10$, $x = \pm 3$ and from $x^2 + 1 = 37$, $x = \pm 6$.

Hence $f^{-1}(\{10, 37\}) = \{-6, -3, 3, 6\}$.

Remark: f^{-1} is not a mapping, because for that f must be one-one and onto which is not here.

***Example 12:** If $x, y \in R$, then find f^{-1} the inverse mapping of $f = \{(x, y) : y = 3x - 2\}$*

Solution: Let $f : R \to R$, where $y = f(x) = 3x - 2$.

Now, from $y = 3x - 2$, $3x = y + 2$ or $x = \frac{1}{3}(y + 2)$.

By the definition of inverse function, $f^{-1} : R \to R$ where $x = f^{-1}(y) = \frac{1}{3}(y + 2)$ or $f^{-1} = \left\{(y, x) : x = \frac{1}{3}(y + 2)\right\}$

Example 13: *Let $f : Q \to Q$ be defined by $f(x) = 3x + 5 (x \in Q)$, where Q is the set of rational then prove that f is one-one and onto mapping. Find also the formula which define the inverse function f^{-1}.*

Solution: Let $x_1, x_2 \in Q$. Then $f(x_1) = f(x_2) \Rightarrow 3x_1 + 5 = 3x_2 + 5$

$$\Rightarrow x_1 = x_2 \qquad \text{[By cancellation law]}$$

Hence, f is a one-one mapping.

Let $y \in Q$, then $x = \frac{1}{3}(y-5) \in Q$ be such that

$$f(x) = f\left(\frac{1}{3}(y-5)\right) = 3\left\{\frac{1}{3}(y-5)\right\} + 5 = y - 5 + 5 = y.$$

Therefore, the mapping f is onto.

Hence, the given mapping is one-one onto.

Now, $\quad y = f(x) = 3x + 5 \quad \Rightarrow \quad x = \frac{1}{3}(y-5).$

Hence, from the definition of inverse function, $f^{-1} : Q \to Q$ where

$$x = f^{-1}(y) = \frac{1}{3}(y-5);$$

i.e, $$f^{-1} = \left\{(y, x) : x = \frac{1}{3}(y-5)\right\}.$$

Example 14: *If Q is the set of rational numbers and $f : Q \to Q$ is defined by $f(x) = 3x + 2, x \in Q$, then prove that f is one-one and onto. Find also f^{-1}.*

Solution: Let $x_1, x_2 \in Q$, then

$$f(x_1) = f(x_2) \Rightarrow 3x_1 + 2 = 3x_2 + 2 \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2.$$

$\therefore$ f is a one-one mapping.

Again, let $y \in Q$ (range of f) be arbitrary, then

$$y = f(x) = 3x + 2$$

$$\Rightarrow \quad x = \frac{y-2}{3} \in Q.$$

Since for $y \in Q$, $x \in Q$ be such that

$$f(x) = f\left(\frac{y-2}{3}\right) = 3\left(\frac{y-2}{3}\right) + 2 = y.$$

Thus, f is onto mapping.

Hence, f is one-one onto mapping.

Let y be image of x under f, then

$$y = f(x) = 3x + 2,\ y \in Q$$

$$\Rightarrow \quad x = \frac{1}{3}(y-2)$$

$$\Rightarrow \quad f^{-1}(y) = x = \frac{1}{3}(y-2)$$

Hence $$f^{-1} = \left\{(y, x) : x = \frac{1}{3}(y-2)\right\}.$$

Example 15: *Is the inverse function $x^2 - 3 = y$, possible give reasons?*

Solution: Let $x \in R$ (domain) and $y \in R$ (Co-domain)

Such that $f(x) = y$

$\Rightarrow$ $x^2 - 3 = y$

$\Rightarrow$ $x = (3+y)^{1/2}$

thus, $f^{-1} : R \to R$ is defined by $f^{-1}(x) = (x+3)^{1/2}$, $\forall x \in R$.

Example 16: *If $f: X \to Y$ and A and B are two subsets of X, then prove that*

(i) $f(A \cup B) = f(A) \cup f(B)$. (ii) $f(A \cap B) \subseteq f(A) \cap f(B)$.

Solution:

(i) Let y be any arbitrary element of $f(A \cup B)$, then

$$y \in f(A \cup B) \Rightarrow y = f(x), \text{ where } x \in A \cup B$$

$$\Rightarrow y = f(x), \text{ where } x \in A \text{ or } x \in B$$

$$\Rightarrow y = f(x) \in f(A) \text{ or } y = f(x) \in f(B)$$

$$\Rightarrow y = f(x) \in [f(A) \cup f(B)]$$

$$\Rightarrow y \in f(A) \cup f(B)$$

$\therefore$ $f(A \cup B) \subseteq f(A) \cup f(B)$. ...(1)

Again, Let y be any arbitrary element of $f(A) \cup f(B)$, then

$$y \in f(A) \cup f(B) \Rightarrow y \in f(A) \text{ or } y \in f(B)$$

$$\Rightarrow y = f(x), \text{ where } x \in A \text{ or } x \in B$$

$$\Rightarrow y = f(x), \text{ where } x \in A \cup B$$

$$\Rightarrow y = f(x), \text{ where } f(x) \in f(A \cup B)$$

$$\Rightarrow y \in f(A \cup B)$$

$\therefore$ $f(A) \cup f(B) \subseteq f(A \cup B)$ ...(2)

Hence, from (1) and (2), we get $f(A \cup B) = f(A) \cup f(B)$

(ii) Let y be any arbitrary element of $f(A \cap B)$, then

$$y \in f(A \cap B) \Rightarrow y = f(x), \text{ where } x \in A \cap B$$

$$\Rightarrow y = f(x), \text{ where } x \in A \text{ and } x \in B$$

$$\Rightarrow y = f(x), \text{ where } f(x) \in f(A) \text{ and } f(x) \in f(B)$$

$$\Rightarrow y \in f(A) \text{ and } y \in f(B)$$

$$\Rightarrow y \in f(A) \cap f(B)$$

Therefore $f(A \cap B) \subseteq f(A) \cap f(B)$.

Example 17: *If $f: X \to Y$ and A, B are two subset of Y, then prove that*

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ (ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Solution: (i) Let x be any arbitrary element of $f^{-1}(A \cup B)$, then

$$x \in f^{-1}(A \cup B) \Rightarrow f(x) \in A \cup B$$
$$\Rightarrow f(x) \in A \text{ or } f(x) \in B$$
$$\Rightarrow x \in f^{-1}(A) \text{ or } x \in f^{-1}(B)$$
$$\Rightarrow x \in f^{-1}(A) \cup f^{-1}(B)$$

$\therefore \quad f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$ ...(1)

Again, let x be any arbitrary element of $f^{-1}(A) \cup f^{-1}(B)$, then

$$x \in f^{-1}(A) \cup f^{-1}(B) \Rightarrow x \in f^{-1}(A). \text{ or } x \in f^{-1}(B)$$
$$\Rightarrow f(x) \in A \text{ or } f(x) \in B$$
$$\Rightarrow f(x) \in A \cup B \Rightarrow x \in f^{-1}(A \cup B)$$

$\therefore \quad f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B).$ ...(2)

Hence, from (1) and (2), we get

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

(ii) Let x be any arbitrary element of $f^{-1}(A \cap B)$, then

$$x \in f^{-1}(A \cap B) \Rightarrow f(x) \in A \cap B$$
$$\Rightarrow f(x) \in A \text{ and } f(x) \in B$$
$$\Rightarrow x \in f^{-1}(A) \text{ and } x \in f^{-1}(B)$$
$$\Rightarrow x \in f^{-1}(A) \cap f^{-1}(B)$$

$\therefore \quad f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$ ...(1)

Again, let x be any arbitrary element of $f^{-1}(A) \cap f^{-1}(B)$, then

$$x \in f^{-1}(A) \cap f^{-1}(B) \Rightarrow x \in f^{-1}(A) \text{ and } x \in f^{-1}(B)$$
$$\Rightarrow f(x) \in A \text{ and } f(x) \in B$$
$$\Rightarrow f(x) \in A \cap B$$
$$\Rightarrow x \in f^{-1}(A \cap B)$$

$\therefore \quad f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$ ...(2)

Hence, from (1) and (2), we get

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

Example 18: *If $f: X \to Y$ be a mapping and $\{A_\alpha\}_{\alpha \in \Delta}$ be an indexed family of subsets of Y, then prove that*

(i) $f^{-1}\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) = \bigcup_{\alpha \in \Delta} f^{-1}(A_\alpha)$ (ii) $f^{-1}\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) = \bigcap_{\alpha \in \Delta} f^{-1}(A_\alpha)$

Solution: (i) Let x be any arbitrary element of $f^{-1}\left(\bigcup_{\alpha \in \Delta} A_\alpha\right)$ then

$$x \in f^{-1}\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) \Leftrightarrow f(x) \in \bigcup_{\alpha \in \Delta} (A_\alpha)$$
$$\Leftrightarrow f(x) \in A_\alpha \text{ for some } \alpha \in \Delta$$
$$\Leftrightarrow x \in f^{-1}(A_\alpha) \text{ for some } \alpha \in \Delta$$
$$\Leftrightarrow x \in \bigcup_{\alpha \in \Delta} f^{-1}(A_\alpha)$$

$$\therefore \quad f^{-1}\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) \Leftrightarrow \bigcup_{\alpha \in \Delta} f^{-1}(A_\alpha)$$

(ii) Let x be any arbitrary element of $f^{-1}\left(\bigcap_{\alpha \in \Delta} A_\alpha\right)$ then

$$x \in f^{-1}\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) \Leftrightarrow f(x) \in \bigcap_{\alpha \in \Delta} (A_\alpha)$$
$$\Leftrightarrow f(x) \in A_\alpha \; \forall \alpha \in \Delta$$
$$\Leftrightarrow x \in f^{-1}(A_\alpha) \; \forall \alpha \in \Delta$$
$$\Leftrightarrow x \in \bigcap_{\alpha \in \Delta} f^{-1}(A_\alpha)$$

$$\therefore \quad f^{-1}\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) = \bigcap_{\alpha \in \Delta} f^{-1}(A_\alpha)$$

Example 19: *If $f: X \to Y$ and $A, B \subseteq Y$, then prove that*

(i) $A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$.   (ii) $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.   (iii) $f^{-1}(Y - A) = X - f^{-1}(A)$.

Solution:

(i) Let $x \in f^{-1}(A)$ be arbitrary. Then

$$x \in f^{-1}(A) \Rightarrow f(x) \in A$$
$$\Rightarrow f(x) \in B \qquad [\because A \subseteq B]$$
$$\Rightarrow x \in f^{-1}(B)$$

Hence

$$f^{-1}(A) \subseteq f^{-1}(B).$$

(ii) Let x be any arbitrary element of $f^{-1}(A - B)$, then

$$x \in f^{-1}(A - B) \Leftrightarrow f(x) \in A - B$$
$$\Leftrightarrow f(x) \in A \text{ and } f(x) \notin B$$
$$\Leftrightarrow x \in f^{-1}(A) \text{ and } x \notin f^{-1}(B)$$
$$\Leftrightarrow x \in [f^{-1}(A) - f^{-1}(B)]$$

Therefore,

$$f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B).$$

be any arbitrary element of $f^{-1}(Y-A)$, then

$$\begin{aligned} x \in f^{-1}(Y-A) &\Rightarrow f(x) \in Y-A \\ &\Rightarrow f(x) \in Y \text{ and } f(x) \notin A \\ &\Rightarrow x \in f^{-1}(Y) \text{ and } x \notin f^{-1}(A) \\ &\Rightarrow x \in X \text{ and } x \notin f^{-1}(A) \qquad [\because f^{-1}(Y)=X] \\ &\Rightarrow x \in [X - f^{-1}(A)]. \end{aligned}$$

$$f^{-1}(Y-A) \subseteq X - f^{-1}(A) \qquad \ldots(1)$$

let x be any arbitrary element of $X - f^{-1}(A)$, then

$$\begin{aligned} x \in X - f^{-1}(A) &\Rightarrow x \in X \text{ and } x \notin f^{-1}(A) \\ &\Rightarrow x \in f^{-1}(Y) \text{ and } x \notin f^{-1}(A) \qquad [\because f^{-1}(Y)=X] \\ &\Rightarrow f(x) \in Y \text{ and } f(x) \notin A \\ &\Rightarrow f(x) \in Y-A \\ &\Rightarrow x \in f^{-1}(Y-A) \end{aligned}$$

$\therefore$ $$X - f^{-1}(A) \subseteq f^{-1}(Y-A) \qquad \ldots(2)$$

(1) and (2), we have $f^{-1}(Y-A) = X - f^{-1}(A)$

ple 20: *Find the domain and range of the following function:*

$f(x) = \sin x - \cos x$ (ii) $f(x) = |\sin x|$

tion:

Domain of $f(x) = \sin x - \cos x$, the value of $\cos x$ and $\sin x$ lies between -1 and $+1$ for the given function $f(0) = -1$, $f(\pi) = 1$, so domain is R *i.e.,* $D_f = R$.

Range of $f(x) = R_f = \sin x - \cos x$ is $[-1,1]$.

Domain of $f(x) = |\sin x|$. The given function will be positive, zero or negative according as the value of x is > 0, $= 0$, < 0. Thus, domain of $f(x) = |\sin x|$ is R.

Range of $f(x) = R_f = [0, 1]$.

mple 21: *Find the range of the function* $f(x) = \dfrac{1}{2 - \cos 3x}$.

tion: Let $y = \dfrac{1}{2 - \cos 3x}$

$$\Rightarrow 2 - \cos 3x = \frac{1}{y} \quad \text{or} \quad \cos 3x = 2 - \frac{1}{y}$$

$$\Rightarrow -1 \le (2 - \frac{1}{y}) \le 1 \quad (\because -1 \le \cos 3x \le 1)$$

$$\Rightarrow -1 \le 2-\frac{1}{y} \text{ and } 2-\frac{1}{y} \le 1$$

$$\Rightarrow y \ge \frac{1}{3} \text{ and } y \le 1 \text{ i.e., } \frac{1}{3} \le y \le 1$$

$$\Rightarrow R_f = \text{Range }(f) = \left[\frac{1}{3}, 1\right]$$

Example 22: *Find the domain and Range of the function.*

$$f(x) = \begin{Bmatrix} x^2, & \text{when } x < 0 \\ x\ , & \text{when } 0 \le x \le 1 \\ \frac{1}{x}\ , & \text{when } x > 1 \end{Bmatrix}$$

Find (i) $f(-2)$ (ii) $f(1/3)$ (iii) $f(1)$ (iv) $f(\sqrt{2})$ (v) $f(-\sqrt{3})$

Solution: Since, $f(x)$ is defined for all value $x \in R \Rightarrow$ domain $f(x) = D_f = R$

Now $\quad x < 0 \Rightarrow f(x) = x^2 > 0 \Rightarrow f(x) \in]\,0, \infty[$

$$0 \le x \le 1 \Rightarrow f(x) = x \Rightarrow f(x) \in]0, 1[$$

and, $\quad x > 1 \Rightarrow f(x) = \frac{1}{x} \in [0, 1]$

$\therefore \quad$ Range $f = R_f =]0, \infty[\, \cup [\,0, 1] \cup]\,0, 1[= [0, \infty[$

(i) $-2 < 0 \Rightarrow f(-2) = (-2)^2 = 4$

(ii) $f\left(\frac{1}{3}\right) = 1/3$ and

(iii) $f(1) = 1$

(iv) $\sqrt{2} > 1 \Rightarrow f(\sqrt{2}) = \frac{1}{\sqrt{2}}$

(v) $-\sqrt{3} < 0 \Rightarrow f(-\sqrt{3}) = (-\sqrt{3})^2 = 3.$

Example 23: *If* $g(x) = \frac{x^2 + 2x + 3}{x}$ *then find* R_g = *Range of* $g(x)$

Solution: Let $\frac{x^2 + 2x + 3}{x} = y$

$$\Rightarrow x^2 + 2x + 3 = xy \text{ or } x^2 + (2-y)x + 3 = 0\ \forall x \in R$$

$\therefore \quad (2-y)^2 - 4 \times 1 \times 3 = 0 \qquad (B^2 - 4Ac = 0)$

$$\Rightarrow y^2 - 4y + 4 - 12 \ge 0$$

$$\Rightarrow y^2 - 4y - 8 \ge 0$$

$$\Rightarrow (y - (2-2\sqrt{3}))(y - (2+2\sqrt{3})) \ge 0$$

$$\Rightarrow y \in (-\infty, 2-2\sqrt{3}) \cup (2+2\sqrt{3}, \infty)$$

i.e., $\quad y \in R - (2-2\sqrt{3}, 2+2\sqrt{3}) = R_g$

Example 24: *If* $f(x)=\dfrac{x^2-2}{x^2-3}$ *then find* R_f *= Range of* $f(x)$

Solution: Let $y=\dfrac{x^2-2}{x^2-3}$ $\Rightarrow yx^2-3y=x^2-2$

$$\Rightarrow x^2(y-1)=3y-2$$

$$\Rightarrow x^2=\frac{3y-2}{y-1}$$

This is possible only if $\dfrac{3y-2}{y-1}\geq 0$ $\quad\Rightarrow y\in(-\infty,\frac{2}{3})\cup(1,\infty)=R_f$

Example 25: *If* $g(x)=\cos^{-1}\dfrac{6-3x}{4}+\text{cosec}^{-1}\left(\dfrac{x-1}{2}\right)$ *then find* D_g *= domain of g.*

Solution: Let $\phi(x)=\cos^{-1}\dfrac{6-3x}{4}$ and $\psi(x)=\text{cosec}^{-1}\dfrac{x-1}{2}$

Then $\phi(x)$ is defined when $-1\leq\dfrac{6-3x}{4}\leq 1$

i.e., $\quad -4\leq 6-3x\leq 4 \quad$ or $\quad -10\leq -3x\leq -2$

i.e., $\quad \dfrac{10}{3}\geq x\geq\dfrac{2}{3}.$

$\therefore \quad D_\phi=\left(\dfrac{2}{3},\dfrac{10}{3}\right)$

The function $\psi(x)$ is defined when

$$-\infty\leq\frac{x-1}{2}\leq -1 \text{ or } 1\leq\frac{x-1}{2}<\infty$$

i.e., $\quad -\infty<x-1\leq -2 \quad or \quad 2\leq x-1<\infty$

i.e, $\quad -\infty<x\leq -1 \quad or \quad 3\leq x<\infty$

$\therefore \quad D_\psi=(-\infty,-1)\cup(3,\infty)$

$\therefore \quad D_g=D_{(\phi+\psi)}=D_\phi\cap D_\psi=\left[3,\dfrac{10}{3}\right]$

Example 26: *Find the domain and Range of the function.*

(i) $f(x)=x^2$ $\quad$ (ii) $g(x)=\dfrac{1}{x-3}$ $\quad$ (iii) $h(x)=\sqrt{9-x^2}$ $\quad$ (iv) $k(x)=\dfrac{1}{1-x^2}$

Solution:

(i) For each $x\in R$, x^2 is unique real number

$\therefore$ domain $(f(x))=R$

Now $\quad y=x^2\Rightarrow x=\pm\sqrt{y}$

$\therefore$ Range (f) = set of all non-negative real number = $[0,\infty[$

(ii) The function $g(x)$ is not defined when $x-3=0$ *i.e.,* $x=3$

Domain $(f(x)) = R-(3)$

Also $$y=\frac{1}{x-3} \Rightarrow x-3=\frac{1}{y} \Rightarrow x=\frac{1+3y}{y}$$

$\Rightarrow$ x is not defined when $y=0$

$\therefore$ Range of $f(x)=R-\{0\}$.

(iii) $h(x)=\sqrt{9-x^2}$, $h(x)$ is not defined when $(9-x^2)$ is negative.

$\therefore$ $h(x)$ is defined only when $(9-x^2)\geq 0$

But, $(9-x^2)\geq 0 \Rightarrow x^2 \leq 9$

$\Rightarrow x\leq 3$ *or* $x \geq -3$

$\Rightarrow -3\leq x\leq 3$

$\Rightarrow x\in[-3, 3]$

$\therefore$ Domain $(f(x))=[-3, 3]$

For each $x\in[-3, 3]$, $h(x)$ will be unique value in the interval $[0, 3]$

Hence, range $h(x)=[0, 3]$

(iv) $k(x)=\dfrac{1}{1-x^2}$, $k(x)$ is not defined when $(1-x)^2=0$ *i.e.,* $x=\pm 1$

$\therefore$ Domain $k(x)=R-[-1, 1]$.

Also $$y=\frac{1}{1-x^2} \Rightarrow 1-x^2=\frac{1}{y} \quad \text{or} \quad x=\sqrt{1-\frac{1}{y}}$$

Clearly, x is not defined when $\left(1-\dfrac{1}{y}\right)<0$ *i.e.,* when $y<1$

$\therefore$ Range $k(x)=(y\in R: 1\leq y<\infty) = [1, \infty[$.

Example 27: *Find the domain and range of the function*

$$f(x)=1-|x|$$

Solution: The function $f(x)$ is defined for all $x\in R$.

$\therefore$ Domain $(f(x))=R$

Also $|x|\geq 0 \Rightarrow -|x|\leq 0 \Rightarrow 1-|x|\leq 1$

$\therefore$ Range $(f(x))=\{y\in R: y\leq 1\} =]-\infty, 1]$

3.8 Operations on Real Valued Function

(i) **Addition:** $(f+g)x = f(x)+g(x)$, $f+g$ will be defined only for those values of x for which both f and g are defined.

$\therefore \quad D_{(f+g)} = D_f \cap D_g$, D_f = Domain of $f(x)$, $\qquad D_g$ = Domain of $g(x)$

(ii) **Subtraction:** $(f-g)x = f(x)-g(x) \qquad \Rightarrow \qquad D_{(f-g)} = D_f \cap D_g$

(iii) **Multiplication:** $(fg)x = f(x)g(x) \qquad \Rightarrow \qquad D_{(fg)} = D_f \cap D_g$

(iv) **Division:** $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}$

$\left(\frac{f}{g}\right)$ will be defined for those value of x for which both f and g are defined and also $g(x) \neq 0$.

$\therefore \quad D_{(f/g)} = D_f \cap D_g - \{x : x \in D_g \text{ and } g(x) = 0\}$

(v) **Composition:** Let X, Y, Z be three non empty sets, $f: X \to Y$ and $g: Y \to Z$ be two mapping, then the composition of mappings f and g is the composition mapping denoted by 'gof' and is a function.

$$(gof): X \to Z \text{ and defined as}$$

$$(gof)x = g[f(x)] \; \forall x \in X$$

The composition 'gof' is denoted by the following diagram

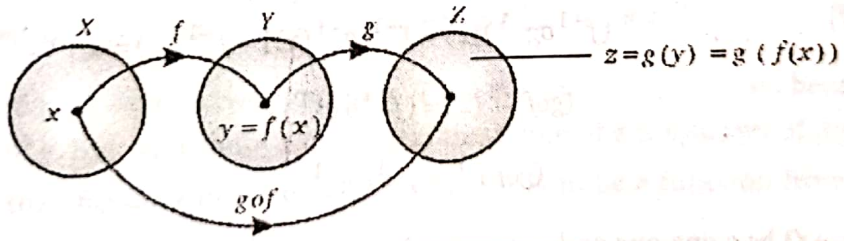


Fig. 3.10

Theorem 4: Let $f: X \to Y$ and $g: Y \to Z$ be one-one and onto maps. Then the composite function

$$gof : X \to Z \text{ is one-one onto}$$

and $\qquad (gof)^{-1} = f^{-1} o g^{-1}$ **[U.P.T.U. (M.C.A.) 2005]**

Proof: Since f and g are both one-one onto and therefore f^{-1} and g^{-1} both exists and both are one-one onto

gof is one-one Let $x_1, x_2 \in X$ then

$$(gof)x_1 = (gof)x_2$$

$$\Rightarrow \qquad g[f(x_1)] = g[f(x_2)]$$

$$g(y_1) = g(y_2) \quad \text{Where } f(x_1) = y_1 \in Y \text{ and } f(x_2) = y_2 \in Y$$

$\Rightarrow$ $$y_1 = y_2 \quad [g \text{ is one-one}]$$

$\Rightarrow$ $$f(x_1) = f(x_2)$$

$\Rightarrow$ $$x_1 = x_2 \quad [f \text{ is one-one}]$$

$\Rightarrow$

It clearly shows that gof is one-one mapping

***gof* is onto:** Let $z \in Z$, since g is onto, hence there definitely exists an element $y \in Y$ such that $g(y) = z$. Again f is onto, there exists an element $x \in X$ such that $f(x) = y$.

$$(gof)x = g[f(x)] = g(y) = z$$

It shows that for each $z \in Z$ there exists an element $x \in X$ such that $(gof)x = z$

$\therefore gof$ is onto mapping. Hence the mapping gof is one-one onto.

To prove that $(gof)^{-1} = f^{-1}og^{-1}$

$\because gof$ is one-one onto mapping

$\therefore$ The inverse mapping $(gof)^{-1}$ exists

If $\quad f: X \to Y$ be given by $f(x) = y$, where $x \in X$, $y \in Y$

$\quad g: Y \to Z$ be given by $g(y) = z$, where $y \in Y, z \in Z$

and $\quad gof: X \to Z$ be given by $(gof)(x) = z$, where $x \in X, z \in Z$

By definition of inverse mapping

$$f^{-1}(y) = x,\ g^{-1}(z) = y,\ (gof)^{-1}(z) = x$$

Now, $$(f^{-1}og^{-1})(z) = f^{-1}[g^{-1}(z)] = f^{-1}(y) = x$$

$\therefore$ $$(gof)^{-1}(z) = (f^{-1}og^{-1})z$$

$\Rightarrow$ $$(gof)^{-1} = f^{-1}og^{-1}$$

Theorem 5: *Let $f: P \to Q$ be a one-one and onto mapping and I_P and I_Q be the identify functions of the sets P and Q respectively. Then show that $f^{-1}of = I_P$ and $fof^{-1} = I_Q$.*

Proof: Note that a function $f: P \to Q$ is called an identity function, if $f(P) = p$, $\forall p \in P$ and is denoted by I_P. Let $f: P \to Q$ be defined as

$$f(p) = q,\ p \in P \text{ and } q \in Q$$

Then, the inverse image of the element q under f is

$$f^{-1}(q) = p$$

Again, since f is one-one onto therefore $f^{-1}: Q \to P$ exists and we have

$$f^{-1}(q) = p,\ p \in P \text{ and } q \in Q$$

Now $$(f^{-1}of)p = f^{-1}(f(p))\ \forall\, p \in P \Rightarrow f^{-1}(q) = p$$

Thus $$(f^{-1}of)\,p = p = I_P(p)\forall p \in P.$$

Hence $$f^{-1}of = I_P$$

Similarly $$(fof^{-1})(q) = f(f^{-1}(q))\forall q \in Q$$

$\Rightarrow$ $$f(p) = q = I_Q(q) \Rightarrow fof^{-1} = I_Q$$

Theorem 6: *Show that the composite of mapping obey associative law.*

or

If X, Y, Z, T are four sets and f, g, h be three mapping given by $f : X \to Y,\ g : Y \to Z,\ h : Z \to T$, then prove that $ho(gof) = (hog)of$

Proof: We have given that

$$f : X \to Y, \quad g : Y \to Z, \quad h : Z \to T$$

Let x be any arbitrary element of X, then

$$\begin{aligned}[ho(gof)]\,x &= h[(gof)(x)]\\ &= h[g(f(x))]\\ &= h[g(y)] && [f(x) = y]\\ &= h(z) && [g(y) = z]\end{aligned}$$

Again $[(hog)of]x = (hog)(f(x)) = (hog)(y) = h[g(y)] = h(z) = t$ ...(2)

From (1) and (2) $$ho(gof) = (hog)of.$$

Theorem 7: *The composition of functions is not commutative i.e. $fog \neq gof$.* **[U.P.T.U. (B.Tech.) 2009]**

Proof: Let $f : P \to Q$ and $g : Q \to R$ be two functions. Then the function gof exists because the range of f is a subset of the domain of g. But fog cannot exist unless the range of g is a subset of domain of f *i.e.* unless $R \subset P$. As such we find that fog does not exist if $R \not\subset P$ but fog will be a function from Q to itself if $P = R$,. Thus, if $P = R$

$$f : P \to Q \text{ and } g : Q \to P \Rightarrow gof : P \to P \text{ and } fog : Q \to Q$$

Now we find that both fog and gof exist but they cannot be equal of P and Q are two distinct sets, which are their domains. However if $P = Q = R$, then both function gof and fog exist and both are from P to itself, even then they may not be equal. Hence, in general the composition of functions is not necessarily commutative.

Example 28: *Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined "x divides y" indicated as $x|y$ $(x|y)$ if there exist an integer z such that $xz = y$.*

(i) *Write R as a set of ordered pairs.*

(ii) *Find the inverse relation R^{-1} and describe R^{-1} in words.* **[U.P.T.U. (B.Tech.) 2003]**

Solution: The ordered pairs in R are those elements in which second element of every ordered pair is divisible by first element.

(i) $R = \{(1, 2), (1, 3), (1, 4), (1, 6), (2, 4), (2, 6), (3, 6)\}$

Here $1 | 2$, $2|4$ and $3|6$

(ii) To find R^{-1}, arrange all the ordered pairs in reverse order

$$R^{-1} = \{(2, 1), (3, 1), (4, 1), (6, 1), (4, 2), (6, 2), (6, 3)\}$$

Now relation R^{-1} can be defined as x is divisible by y or ${}_xR_{y^{-1}}$

Example 29: *Show that there exists one to one mapping from $A \times B$ to $B \times A$. Is it onto also?*

[U.P.T.U. (B.Tech.) 2004]

Solution: Let $(x, y) \in A \times B \Rightarrow (y, x) \in B \times A$

Now, $f : A \times B$ to $B \times A$

Let $x_1, y_1, x_2, y_2 \in A \times B$

and f be defined as $f(x, y) = (y, x)$

If $\quad f(x_1, y_1) = f(x_2, y_2)$

$\Rightarrow \quad (y_1, x_1) = (y_2, x_2)$

$\Rightarrow \quad y_1 = y_2$ and $x_1 = x_2$

Hence it is one-one. It will be onto also every element of co-domain will be f-image of at least one element of a domain.

Example 30: *Let $X = \{1, 2, 3\}$, $Y = \{p, q\}$ and $Z = \{a, b\}$, let $f : X \to Y$ be $f = \{(1, p), (2, p), (3, q)\}$ $g : Y \to Z$ be $g = \{(p, b), (q, b)\}$. Find gof and show it pictorially*

[U.P.T.U. (B.Tech.) 2003]

Solution: $gof = g[f(x)] \Rightarrow (gof)(1) = g[f(1)] = g(p) = b$

$$(gof)(2) = g[f(2)] = g(p) = b$$

$$(gof)(3) = g[f(3)] = g(q) = b$$

$$gof = \{(1, b), (2, b), (3, b)\}$$

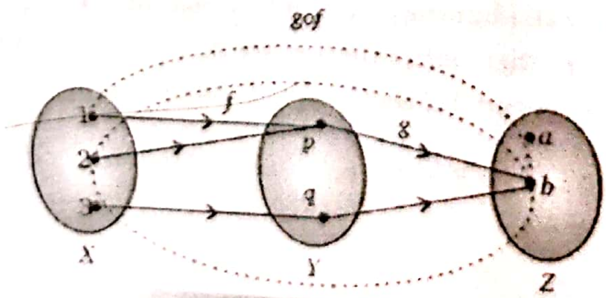


Fig. 3.11

Example 31: *Let $X = \{a, b, c\}$. Define $f : X \to X$ such that*

$$f = \{(a, b), (b, a), (c, c)\}$$

Find (i) f^{-1} (ii) f^2 (iii) f^3 (iv) f^4

[U.P.T.U. (B.Tech.) 2004]

Solution: (i) $f^1 = \{(a, b), (b, a), (c, c)\}$

(ii) $f^2 = fof = (fof)\, a = f\,[f\,(a)] = a$

$$(fof)\, b = f\,[f\,(b)] = b$$

$$(fof)\, c = f\,[f\,(c)] = c$$

$$f^2 = fof = \{(a, a), (b, b), (c, c)\}$$

(iii) $f^3 = f^2of = f^2\,[f\,(a)] = f^2\,[f\,(a)] = b$

$$f^2\,[f\,(b)] = a, \quad f^2\,(f\,(c)] = c$$

$\therefore$ $$f^3 = \{(a, b), (b, a), (c, c)\}$$

(iv) $f^4 = f^3of = f^3\,[f\,(a)] = a$

$$f^3\,[f\,(b)] = b, \quad f^3\,[f\,(c)] = c$$

$$f^4 = \{(a, a), (b, b), (c, c)\}$$

Example 32: *Let f, g and $h : R \to R$; be defined by (R is the set of real numbers)*

Compute (i) $f^{-1}\, g\,(x)$ (ii) $h\,.\,f\,(g\,f^{-1}y).\,(h\,.\,f\,(x)$

Where $$f\,(x) = x + 2,\ g\,(x) = \frac{1}{x^2+1},\ h\,(x) = 3$$

Solution: Let $f\,(x) = x + 2$, Let $y = f\,(x)$ or $x = f^{-1}\,(y)$

$\because$ $$y = x + 2 \Rightarrow x = y - 2$$

$$\Rightarrow f^{-1}\,(y) = y - 2$$

(i) $$f^{-1}\,[g\,(x)] = f^{-1}\left[\frac{1}{x^2+1}\right] = \frac{1}{x^2+1} - 2 = \frac{1 - 2x^2 - 2}{x^2+1} = \frac{-(2x^2+1)}{x^2+1}$$

(ii) $h\{f\,(g\,f^{-1}\,(y)\}\,.\,\{h\,f\,(x)\}$

$$= h\,f\,(g\,(y-2)\,h\,f(x)$$

$$= h\,f\left[\frac{1}{(y-2)^2+1}\right]h\,(x+2), \quad \text{Let } \frac{1}{(y-2)^2+1} = t$$

$$= h\,f\,(t)\,h\,(x+2)$$

$$= h\,(t+2)\,3$$

$$= 3.3 = 9.$$

Example 33: *Show that the function f and g both of which are from $N \times N$ to N given by $f(x, y) = x + y$ and $g(x, y) = xy$ are onto but not one-one.* **[U.P.T.U. (B.Tech.) 2003]**

Solution: For one-one mapping

We have $f(x, y) = x + y$

Let $x_1, y_1, x_2, y_2 \in N$ If $f(x_1, y_1) = f(x_2, y_2)$

$\Rightarrow$ $x_1 + y_1 = x_2 + y_2$

$\Rightarrow$ $x_1 \neq x_2$ and $y_1 \neq y_2$

e.g. $x_1 = 1, y_1 = 4, x_2 = 4, y_1 = 1$

Then $f(x_1, y_1) = f(x_2, y_2)$. But $x_1 \neq x_2,\ y_1 \neq y_2$

Hence f is not one-one mapping

Again, $g(x, y) = xy$

Let $x_1, y_1, x_2, y_2 \in N$

and $g(x_1, y_1) = g(x_2, y_2)$

$\Rightarrow$ $x_1 y_1 = x_2 y_2$

It not necessarily $x_1 = x_2$ and $y_1 = y_2$

e.g. if $x_1 = 1, y_1 = 5, x_2 = 5, y_2 = 1$

Then $g(x_1, y_1) = g(x_2, y_2)$

But $x_1 \neq x_2$ and $y_1 \neq y_2$

Onto f is defined as $f(x, y) = x + y$

Since every element of N can be written as the sum of two elements of N. There will be no element in the co-domain that does not have pre-image in domain. Hence it is onto. Similarly g is also onto.

Example 34: *Let the maps $f : X \to Y$ and $g : Y \to Z$ be defined by the diagram*

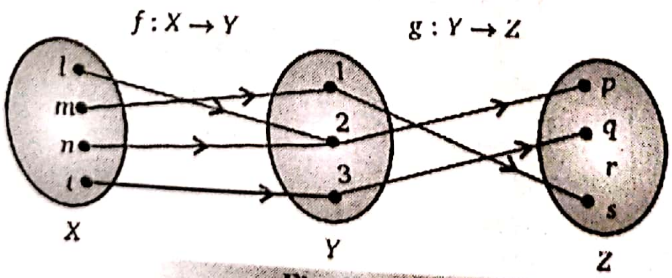


Fig. 3.12

Find $gof : X \to Z$ **[P.T.U. (B.E.) Punjab 2007]**

Solution: We have

$$(gof)(l) = g[f(l)] = g(2) = p$$
$$(gof)(m) = g[f(m)] = g(1) = s$$
$$(gof)(n) = g[f(n)] = g(2) = p$$
$$(gof)(t) = g[f(t)] = g(3) = q$$

Hence, $$gof = \{p, s, p, q\}$$

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Example 35: *Let $X = \{1, 2, 3\}$ and f, g, h, s are functions from X to X given by*

$$f = \{(1, 2), (2, 3), (3, 1)\} \quad g = \{(1, 2), (2, 1), (3, 3)\}$$

$$h = \{(1, 1), (2, 2), (3, 1)\} \quad s = \{(1, 1), (2, 2), (3, 3)\}$$

find $fog, gof, fohog, sog, gos, sos$ *and* fos

Solution: $fog = \{(1, 3), (2, 2), (3, 1)\}$

$$gof = \{(1, 1), (2, 3), (3, 2)\} \neq fog$$

$$fohog = \{(1, 3), (2, 2), (3, 2)\}$$

$$sog = \{(1, 2), (2, 1), (3, 3)\} = g = gos$$

$$sos = \{(1, 1), (2, 2), (3, 3)\} = s$$

$$fos = \{(1, 2), (2, 3), (3, 1)\} = f$$

Example 36: *Let $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$ for $x \in R$, find gof, fog, fof, $fohog$*

Solution: $gof = \{(x, x) : x \in R\} = g[f(x)] = g(x + 2) = x + 2 - 2 = x$

$$fog = \{(x, x) : x \in R\} = f[g(x)] = f(x - 2) = x - 2 + 2 = x = gof$$

$$fof = \{(x, x + 4) : x \in R\} = f[f(x)] = f(x + 2) = x + 4$$

$$(foh)og = fo(hog) = (foh)g(x) = (foh)(x - 2)$$

$$= f[h(x - 2)]$$

$$= f[3(x - 2)]$$

$$= 3(x - 2) + 2 = 3x - 4$$

3.9 Number of Functions

Cardinality of a sets A and B are m and n respectively.

Number of functions from $A \to B = n^m$

Number of functions from $A \to A = n^n$

Example 37: *Prove that the number of functions from $A \to A$ is less than number of relations from $A \to A$*

i.e. $n^n < 2^{n^2}$ **[U.P.T.U. (M.C.A.) 2004-2005]**

Solution: First we prove by induction that

$$n < 2^n \ \forall \text{ positive integers } n.$$

We know that $P(1) : 1 < 2^1$ is true. ...(1)

Let us assume that $P(k) : k < 2^k$, $k \in N$ is true. ...(2)

We shall prove that $P(k)$ is true, then $P(k+1)$ is also true.

i.e. $$(k+1) < 2^{k+1} \quad ...(3)$$

Now $$k < 2^k \quad \text{[from (2)]}$$

$$\Rightarrow \quad k+1 < 2^k + 1 \quad \text{(adding 1 on both sides)}$$

$$\Rightarrow \quad k+1 < 2^k + 2^k \quad (\text{as } b\,1 < 2^k)$$

$$\Rightarrow \quad (k+1) < 2.2^k$$

$$\Rightarrow \quad (k+1) < 2^{k+1}$$

Hence, by principle of induction $P(k)$ is true $k \in N$.

Thus, $$n < 2^n \ \forall\, n \in N$$

Taking n^{th} power on both sides, we find

$$(n)^n < (2^n)^n$$

or $$n^n < 2^{n^2}.$$

Example 38: *If A is finite set and $f : A \to A_1$ is a one-one function, then f is onto also.*

[R.G.P.V. (B.E.) Raipur 2005]

Solution: Let $A = \{a_1, a_2, a_3 \ldots a_n\}$ be finite set. Since f is a one-one function $f(a_1), f(a_2) \ldots f(a_n)$ are all n distinct elements of the set A. But A has only n elements.

Therefore, $$A = \{f(a_1), f(a_2) \ldots f(a_n)\}$$

It means co-domain of f = Range of f. Therefore f is onto function.

Example 39: *List all possible functions from $X = \{a, b, c\}$ to $Y = \{0, 1\}$ and indicate in each case whether the function is one to one, is onto is one to one onto.*

[U.P.T.U. (B.Tech.) 2006]

Solution: We have $X = \{a, b, c\}$ and $Y = \{0, 1\}$

All possible function from X to Y are

$$f_1 = \{(a, 0), (b, 0), (c, 0)\}, \quad f_2 = \{(a, 1), (b, 1), (c, 1)\}$$
$$f_3 = \{(a, 0), (b, 0), (c, 1)\}, \quad f_4 = \{(a, 1), (b, 1), (c, 0)\}$$
$$f_5 = \{(a, 0), (b, 1), (c, 1)\}, \quad f_6 = \{(a, 1), (b, 0), (c, 0)\}$$
$$f_7 = \{(a, 0), (b, 1), (c, 0)\}, \quad f_8 = \{(a, 1), (b, 0), (c, 1)\}$$

The total number of function from $X \to Y$ is $2^3 = 8$

The functions f_3, f_4, f_5, f_7, f_8 are onto functions. No functions is one-one and no function is one-one onto.

All the functions are given below:

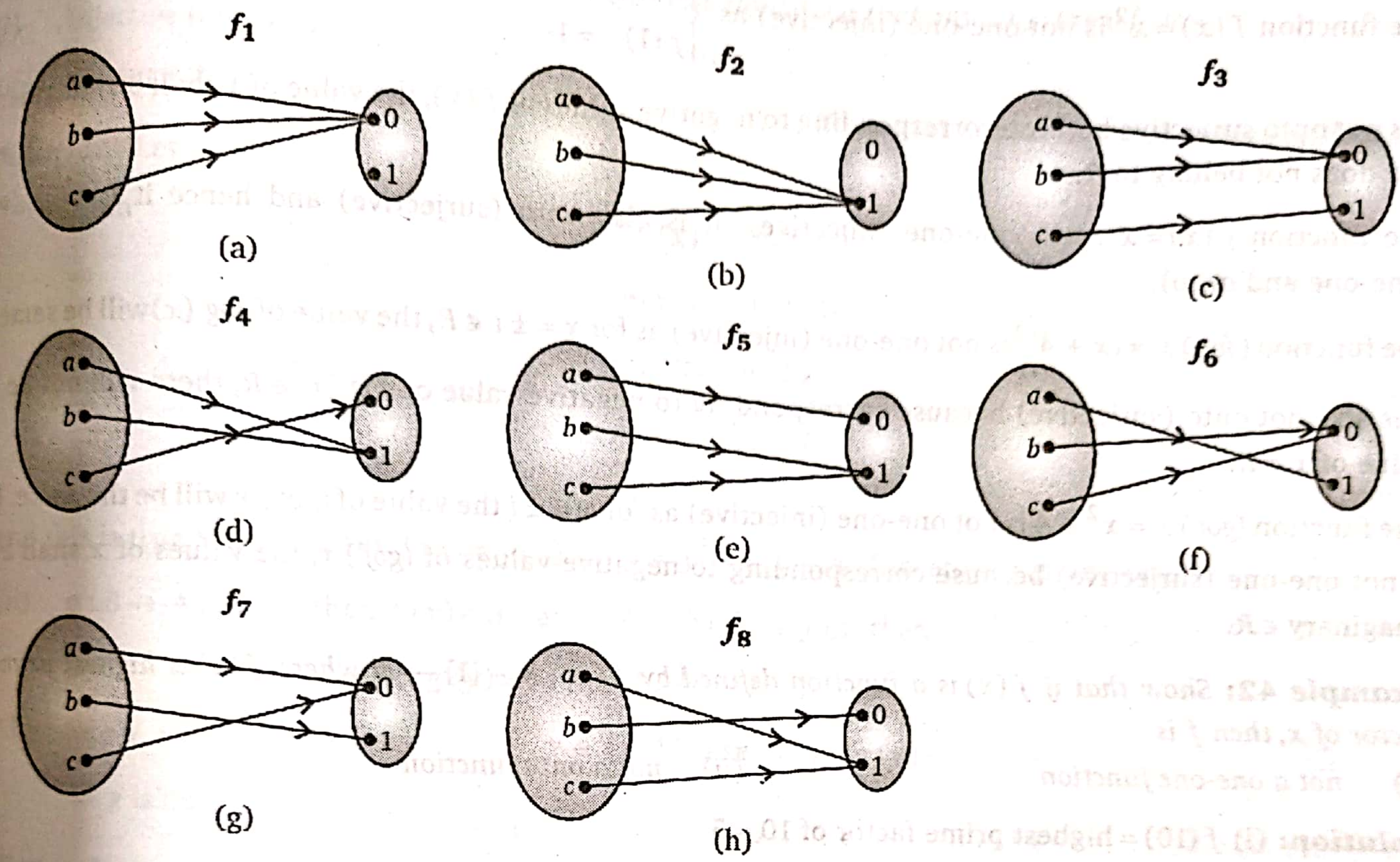

Fig. 3.13

Example 40: *Let N be the set of natural numbers including zero. Determine which of the following functions are one-to-one, which are onto and which are one-to-one onto.*

(i) $f: N \to N$, $f(j) = j^2 + 2$ (ii) $f: N \to N$, $f(j) = \begin{cases} 1 \text{ if } j \text{ is odd} \\ 0 \text{ if } j \text{ is even} \end{cases}$

[U.P.T.U. (B.Tech.) 2006]

Solution: (i) For each value of $j \in N$ its image $f(j) = j^2 + 2 \in N$

If $j = a_1, a_2 \in N$, then $f(a_1) = a_1^2 + 2$ and $f(a_2) = a_2^2 + 2$

As $a_1 \neq a_2$ then $a_1^2 + 2 \neq a_2^2 + 2$

i.e. $f(a_1) \neq f(a_2)$, so f is one-one

There exists value of $f(j) = 0, 1$ for which the pre images $\sqrt{(-2)}, \sqrt{(-1)}$ does not belong to N

(ii) All odd value of $j \in N$ have the same image $(1) \in N$. So f is not one-one

Example 41: *Let $f: R \to R$ and $g: R \to R$, where R be the set of real numbers. Find fog and gof, where $f(x) = x^2$ and $g(x) = x + 4$. State whether these functions are injective, surjective and bijective.*

[U.P.T.U. B.Tech.) 2005]

Solution: We have

$$(fog)\,x = f[g(x)] = f(x+4) = (x+4)^2$$

$$(gof)\,x = g[f(x)] = g[x^2] = x^2 + 4$$

The function $f(x) = x^2$ is not one-one (injective) as $\begin{cases} f(-1) = 1 \\ f(1) \ \ = 1 \end{cases}$

It is not onto surjective because corresponding to negative values of $f(x)$, the value of x shall be imaginary and does not belong to R.

The function $g(x) = x + 4$ is one-one (injective). It is also onto (surjective) and hence it is bijective (one-one and onto).

The function $(fog)\,x = (x+4)^2$ is not one-one (injective) as for $x = \pm i \notin R$, the value of $fog\ (x)$ will be same.

It is also not onto (surjective) because corresponding to negative value of $fog\ (x) \in R$, there will not be a value of $x \in R$.

The function $(gof)\,x = x^2 + 4$ is not one-one (injective) as for $x = \pm i$ the value of $(gof)\,x$ will be the same. It is not one-one (surjective) because corresponding to negative values of $(gof)\,x$, the values of x shall be imaginary $\notin R$.

Example 42: *Show that if $f(x)$ is a function defined by $f(x): N - \{1\} \to N$ where $f(x)$ is highest prime factor of x, then f is*

(i) not a one-one function *(ii) not a onto function*

Solution: (i) $f(10) =$ highest prime factor of $10 = 5$

$f(15) =$ highest prime factor of $15 = 5$

$f(20) =$ highest prime factor of $20 = 5$

Therefore f is a many-one function and is not a one-one function.

(ii) Again image of $x \in N - \{1\} =$ the largest prime number that divide $x =$ a prime number. Therefore the range of f is a set of prime numbers. It means the range of $f \neq N$ so the function f is not an onto function.

Example 43: *Let $f: R \to R$ be a function defined by $f(x) = px + q\ \forall\ x \in R$. Also $fof = I_R$, find the values of p and q.* **[Pune (B.E.) 2004, 2008; Delhi (B.E.) 2007; P.T.U. (B.E.) Punjab 2005, 2009]**

Solution: $(fof)\,x = I_R\ (x)$

$$\Rightarrow \quad f(f(x)) = x$$

$$\Rightarrow \quad f(px+q) = x$$

$$\Rightarrow \quad p(px+q) + q = x$$

$$\Rightarrow \quad p^2x + pq + q - x = 0$$

$$\Rightarrow \quad x(p^2 - 1) + pq + q = 0 \quad \forall\ x \in R$$

$$\Rightarrow \quad p^2 - 1 = 0 \quad \text{and} \quad pq + q = 0$$

$$\Rightarrow \quad p = \pm 1 \quad \text{and} \quad q(p+1) = 0$$

$\Rightarrow$ $p = 1$ and $q = 0$ or $p = -1$ and q is any real value.

Hence, either (i) $p = 1$ and $q = 0$ or (ii) $p = -1$ and $q =$ any real value.

Example 44: *If $f : A \to B$ and $g : B \to A$ are two functions such that $gof = I_A$, then f is*

(i) *injective (one-one)*

(ii) *g is surjective*

[Rohtak (M.C.A.) 2005, 2009; Kurukshetra (B.E.) 2007]

Solution: Let $x, y \in A$

Then $$f(x) = f(y)$$

$$\Rightarrow \quad g(f(x)) = g(f(y))$$

$$\Rightarrow \quad (gof)\, x = (gof)\, y$$

$$\Rightarrow \quad I_A(x) = I_A(y)$$

$$\Rightarrow \quad x = y$$

and this is true $\forall\, x \in A$. Thus $f(x) = f(y) \Rightarrow x = y\ \forall\, x, y \in A$. Therefore f is one-one

(ii) $g : B \to A$ if $x \in A$ then $f(x) \in B$ (as $f : A \to B$) if $y \in f(x)$ then

$$g(y) = g[f(x)] = (gof)\, x = I_A(x)$$

Therefore $\forall\, x \in A$, there exists *a* $y \in f(x) \in B$ such that $g(y) = x$

$\Rightarrow$ g is onto.

Example 45: *Composition of function is commutative. Prove the statement or give counter example.*

[U.P.T.U. (B.Tech.) 2009]

or

Show that the function $f(x) = x^3$ and $g(x) = x^{1/3}$ for all $x \in R$ are inverse to one another.

[Nagpur (B.E.) 2005, 2008]

Solution: Since $(fog)\, x = f[g(x)] = f(x^{1/3}) = x$

and $$(gof)\, x = g[f(x)] = g[x^3] = x$$

$$\therefore \quad fog = gof \text{ then } f = g^{-1} \text{ or } g = f^{-1}$$

Example 46: *If $f(x) = x^3 - \dfrac{1}{x^3}$, find the value of $f(x) + f\left(\dfrac{1}{x}\right)$*

Solution: We have $f(x) = (x^3 - \dfrac{1}{x^3})$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \left(\left(\frac{1}{x}\right)^3 - \frac{1}{\left(\frac{1}{x}\right)^3}\right) = \left(\frac{1}{x^3} - x^3\right)$$

Hence, $$f(x) + f\left(\frac{1}{x}\right) = \left(x^3 - \frac{1}{x^3}\right) + \left(\frac{1}{x^3} - x^3\right) = 0.$$

Example 47: *If* $f(x) = \log\left(\frac{1+x}{1-x}\right)$, *show that* $f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$.

Solution: We have
$$f(x) = \log\left(\frac{1+x}{1-x}\right) \quad \ldots(1)$$

and
$$f(y) = \log\left(\frac{1+y}{1-y}\right) \quad \ldots(2)$$

$\therefore$
$$f(x) + f(y) = \log\left(\frac{1+x}{1-x}\right) + \log\left(\frac{1+y}{1-y}\right)$$
$$= \log\left\{\left(\frac{1+x}{1-x}\right)\left(\frac{1+y}{1-y}\right)\right\} = \log\left\{\frac{1+x+y+xy}{1-x-y+xy}\right\}$$

$\therefore$
$$f\left(\frac{x+y}{1+xy}\right) = \log\left[\frac{1+\left(\frac{x+y}{1+xy}\right)}{1-\left(\frac{x+y}{1+xy}\right)}\right] = \log\left\{\frac{1+x+y+xy}{1+xy-x-y}\right\}$$

$\therefore$
$$f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$$

Example 48: *If* $f(x) = x^2 - \frac{1}{x^2}$, *show that* $f(x) + f\left(\frac{1}{x}\right) = 0$

Solution: We have $f(x) = x^2 - \frac{1}{x^2}$ Put $x = \frac{1}{x}$, we get $f\left(\frac{1}{x}\right) = \frac{1}{x^2} - x^2$

$\therefore$
$$f(x) + f\left(\frac{1}{x}\right) = x^2 - \frac{1}{x^2} + \frac{1}{x^2} - x^2$$
$$= 0$$

Example 49: *If* $y = f(x) = \left(\frac{3x+1}{5x-3}\right)$, *prove that* $f(y) = x$

Solution: Put $x = y$ in given function
$$f(y) = \frac{3y+1}{5y-3} = \frac{3\left(\frac{3x+1}{5x-3}\right)+1}{5\left(\frac{3x+1}{5x-3}\right)-3} = \frac{14x}{(5x-3)} \times \frac{(5x-3)}{14} = x$$

Example 50: $f(x) = \log\left(\frac{1+x}{1-x}\right)$, *Show that* $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$

Solution: We have $f(x) = \log\left(\frac{1+x}{1-x}\right)$

$\therefore$
$$f\left(\frac{2x}{1+x^2}\right) = \log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right) = \log\left(\frac{1+2x+x^2}{1-2x+x^2}\right) = \log\left(\frac{1+x}{1-x}\right)^2 = 2\log\left(\frac{1+x}{1-x}\right) = 2f(x).$$

Example 51: *Define function. Let f and g be two functions defined by*

$$f(x)=\sqrt{x-1} \text{ and } g(x)=\sqrt{4-x^2},$$

then describe the function:

(i) $(f-2g)x$ (ii) $(fg)x$

Solution:

(i) We have, $f(x)=\sqrt{x-1}$, $g(x)=\sqrt{4-x^2}$

$\therefore$ $(f-2g)x = f(x)-2g(x) = \sqrt{x-1}-2\sqrt{4-x^2}$

(ii) We have, $f(x)=\sqrt{x-1}$, $g(x)=\sqrt{4-x^2}$

$\therefore$ $(fg)x=(fog)x=f[g(x)] = f\left[\sqrt{4-x^2}\right] = \sqrt{\sqrt{4-x^2}-1}$

Example 52: *Let $f: R \to R$ be the identity function. What is fof ? What is ff ?*

Solution: Let $f: R \to R$ then f is said to be identify mapping if $f(x)=x$

Now $(fof)x = f[f(x)] = f(x) = x$

and ff is another way of representation of fof

Thus $ff(x)=(fof)x = f[f(x)] = f(x) = x$

Example 53: *If $f: A \to B$ such that $f(x)=x-1$ and $g: B \to C$ such that $g(y)=y^2$ find fog (y).*

Solution: We have $f: A \to B$ such that $f(x)=x-1$.

and $g: B \to C$ such that $g(y)=y^2$

$\therefore$ $(fog)y = f[g(y)] = f[y^2] = y^2-1$

Example 54: *If $f(x)=\dfrac{1}{1-x}$ and $g(x)=\dfrac{x-1}{x}$. Find the value of $g[f(x)]$.*

Solution: We have $f(x)=\dfrac{1}{1-x}$, $g(x)=\dfrac{x-1}{x}$

Then $g[f(x)]=g\left[\dfrac{1}{1-x}\right]=\dfrac{\dfrac{1}{1-x}-1}{\dfrac{1}{1-x}}=\dfrac{1+x-1}{1-x}\times\dfrac{1-x}{1}=x$

Example 55: *Define composite function. Let $f: N \to R$ s.t $f(x)=2x-3$ and $g: Z \to R$ s.t $g(x)=\dfrac{x-3}{2}$, then find formula for $gof: N \to R$.*

Solution: We have $(gof)x = g[f(x)] = g[2x-3] = \dfrac{(2x-3)-3}{2} = x-3$

Example 56: *If* $f(x) = x$ *and* $g(x) = |x|$. *Define* $f+g$ *and* $f-g$.

Solution: $(f+g)x = f(x) + g(x) = x + |x| = \begin{cases} 2x, & \text{when } x \geq 0 \\ 0, & \text{when } x < 0 \end{cases}$

Also $(f-g)x = f(x) - g(x) = x - |x| = \begin{cases} 2x, & \text{when } x < 0 \\ 0, & \text{when } x \geq 0 \end{cases}$

Example 57: *If* $f(x) = x^2$ *and* $g(x) = x^3$, *define* fg

Solution: We have $(fg)x = f(x)g(x) = x^3x^2 = x^5, \forall x \in R$

Example 58: *If* $f(x) = 1 + x^2$ *and* $g(x) = 1 - x$, *define* (f/g).

Solution: We have $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)} = \frac{1+x^2}{1-x}, \forall x \in R.$

Example 59: *Find the domain of the function.* $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$.

Solution: Let $f(x) = g(x) + h(x)$, where $g(x) = \frac{1}{\log_{10}(1-x)}$ and $h(x) = \sqrt{x+2}$

The function $g(x)$ defined for all real value of x for which $1 - x > 0$ and $x \neq 0$ *i.e.,* $x < 1$ and $x \neq 0$.

$\therefore$ Domain $(f) =]-\infty, 1[-\{0\}$

Again, $h(x)$ is defined for those real value of x for which $x + 2 \geq 0$ *i.e.,* $x \geq -2$

$\therefore$ Domain $h(x) = [-2, \infty[$

Now domain f = domain $(f+g)$= domain $(f) \cap$ domain (g)

$=]-\infty, 1[-\{0\} \cap [-2, \infty[= [-2, 1-\{0\}]$

Example 60: *Find the domain of the function.* $f(x) = \frac{\cos^{-1} x}{[x]}$

Solution: Let $f(x) = \frac{g(x)}{h(x)}$, where $g(x) = \cos^{-1} x$ and $h(x) = [x]$ then,

domain $(g) = [-1, 1]$ and domain $(h) = R$

$\therefore$ Domain (f) = domain $(g) \cap$ domain $(h) - \{x : h(x) = 0\}$

$= [-1, 1] \cap R - [0, 1] = [-1, 0[\cup \{1\}$

Example 61: *If* $f(x) = [x]$ *and* $g(x) = |x|$.

Find (*i*) $(gof)(-5/3) - (fog)(-5/3)$ (*ii*) $(gof)(5/3) - (fog)(5/3)$

Solution: We have

$$(gof)(x) = g[f(x)] = g([x]) = |[x]|$$
$$(fog)(x) = f[g(x)] = f(|x|) = [|x|]$$

(i) $(gof)(-5/3) - (fog)(-5/3)$

$$= |[-5/3]| - \left[\left|\frac{-5}{3}\right|\right] = |[-1.66]| - [5/3] = |-2| - [1.66]$$

$$= 2 - 1 = 1$$

(ii) $(gof)(5/3) - (fog)5/3 = |[5/3]| - [|5/3|] = |[1.66....]| - [1.66....]$

$$= 1 - 1 = 0$$

Example 62: *Let $f(x) = x^2 - 1$ and $g(x) = (3x+1)$, describe*

(i) *gof* (ii) *fog* (iii) *fof* (iv) *gog*

Solution: Here domain $(f) = R$, and domain $(g) = R$ and

Range $(f) \subseteq$ Domain (g) and range $(g) \subseteq$ domain (f)

Hence, all the required composite functions are defined and the domain of each is R. Now

(i) $(gof)x = g[f(x)] = g(x^2 - 1) = 3(x^2 - 1) + 1 = 3x^2 - 2$

(ii) $(fog)x = f[g(x)] = g(3x+1) = (3x+1)^2 - 1 = 9x^2 + 6x$

(iii) $(fof)x = f[f(x)] = f(x^2 - 1) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$

(iv) $(gog)x = g[g(x)] = g[3x+1] = 3(3x+1) + 1 = (9x+4)$

Example 63: *If $f : R \to R$, $g : R \to R$ and $h : R \to R$ and defined as*

$$f(x) = x^2, g(x) = \tan x \text{ and } h(x) = \log x.$$

Then show $[ho(gof)]\, x = 0$ if $x = \dfrac{\sqrt{\pi}}{2}$

Solution: $[ho(gof)]x = h[(gof)x]$

$$= h[g(f(x))] \quad = h[g(x^2)] \quad = h(\tan x^2) \quad = \log \tan x^2 \quad = \log \tan\left(\frac{\pi}{4}\right) \quad = \log 1 \quad = 0$$

Example 64: *Let $f : R \to R$ where $f(x) = 2x + 1$, $g : R \to R$ where $g(x) = x^2 - 2$*

Then find, *(i)* $(gof)x$ and $(gof)(1)$ *(ii)* $(gof) \neq fog$

Solution: (i) $(gof)x = g[f(x)] = g[2x+1] = (2x+1)^2 - 2 = 4x^2 + 4x - 1$

And $(gof)(1) = 4 \times (1)^2 + 4 \times (1) - 1 = 7$

(ii) $(gof)x = 4x^2 + 4x - 1$

$(fog)x = f[g(x)] = f(x^2 - 2) = 2(x^2 - 2) + 1 = 2x^2 - 4 + 1 = 2x^2 - 3$

$\therefore \quad (gof) \neq fog$

Example 65: *If mapping $f : R \to R$, $f(x) = \cos x$, $x \in R$ and $g : R \to R$, $g(x) = x^3$, $x \in R$*

Then, find $(gof)x$ and $(fog)x$ and show that $(gof)x \neq (fog)x$

Solution: $(gof)x = g[f(x)] \quad = g[\cos x] \quad = (\cos x)^3 = \cos^3 x$

Again $\quad (fog)x = f[g(x)] = f[x^3] = \cos x^3$

$\because \quad (fog)x \neq (gof)x$

Example 66: *Let $f: A \to A$ and $g: A \to A$ be two function defined on set $A = \{1, 2, 3, 4, 5\}$ such that $f(1) = 3, f(2) = 5, f(3) = 3, \quad f(4) = 1, f(5) = 2$. And $g(1) = 4, g(2) = 1, g(3) = 1, g(4) = 2, g(5) = 3$ find fog and gof.*

Solution: Here $f(1) = 3 \Rightarrow gof = g[f(1)] = g(3) = 1 \qquad f(2) = 5 \Rightarrow gof = g[f(2)] = g(5) = 3$

$f(3) = 3 \Rightarrow gof = g[f(3)] = g(3) = 1 \qquad f(4) = 1 \Rightarrow gof = g[f(4)] = g(1) = 4$

$f(5) = 2 \Rightarrow gof = g[f(5)] = g(2) = 1 \qquad g(1) = 4 \Rightarrow fog = f[g(1)] = f(4) = 1$

$g(2) = 1 \Rightarrow fog = f[g(2)] = f(1) = 3 \qquad g(3) = 1 \Rightarrow fog = f[g(3)] = f(1) = 3$

$g(4) = 2 \Rightarrow fog = f[g(4)] = f(2) = 5 \qquad g(5) = 3 \Rightarrow fog = f[g(5)] = f(3) = 3$

Exercise

1. Let $f(x) = (2x + 5)$ and $g(x) = x^2 + 1$. Describe

   (i) gof (ii) fog (iii) fof (iv) ff

2. If $f(x) = \dfrac{x-1}{x+1}$, when $x \neq 1$ and $x \neq -1$. Show that (fof^{-1}) is an identify function.

3. If $f(x) = \sqrt{x-1}$ and $g(x) = \dfrac{1}{x}$. Describe

   (i) c (ii) $f - g$

   (iii) c (iv) f/g, find the domain in each case.

4. If $f(x) = \dfrac{1}{1-x}$. Show that $f[f(f(x)] = x$

5. Find the domain of the function $f(x) = \dfrac{x}{x^2 - 3x + 2}$

6. Find the domain and range of $f(x) = \dfrac{1}{x^2 - 1}$

7. If $f(x) = x^2 + 3x + 2$, find the value of x for which, $f(x+1) = f(x)$

8. If $f(x) = \dfrac{1 - \tan x}{1 + \tan x}$, show that $f(\cos\theta) = \tan^2(\theta/2)$

9. If $y = f(x) = \left(\dfrac{2x-1}{5x-2}\right)$, prove that $f(y) = x$

10. If $f(x) = \log_e x$ and $x > 0$, prove that $f(\upsilon \nu \omega) = f(\upsilon) + f(\nu) + f(\omega)$

11. (i) If $f(x) = \dfrac{\sin x}{1 + \cos x}$, find $f\left(\dfrac{\pi}{2}\right)$ (ii) If $f(\theta) = \dfrac{1 - 2\tan\theta}{1 + 2\tan\theta}$, find $f\left(\dfrac{\pi}{4}\right)$

    (iii) If $f(x) = \sin^{-1}(\log x)$, find $f(1)$

12. If $f(x)=\begin{cases} 3x-1, & \text{when } x \geq 1 \\ -x, & \text{when } x<1 \end{cases}$ then find $f(0)$ and $f(2)$

13. Let $f: R \to R$, where $f(x)=x^2+1$, find $f^{-1}(-5)$, $f^{-1}(26)$

14. Let R be the set of all real number and $f: R \to R$ be function such that $f(x)=\sin x$, $x \in R$ and $g(x)=x^2, x \in R$. Then show $gof = fog$.

15. If $g(x)=e^x$ and $f(x)=x^2$ then find $(fog)x$ and $(gof)x$.

16. If $f:R \to R$ be given by:

$$f(x)=\sin^2 x+\sin 2\left(x+\frac{\pi}{3}\right)+\cos x \cos\left(x+\frac{\pi}{3}\right), \forall x \in R \text{ and}$$

$g:R \to R$ be such that $g\left(\frac{5}{4}\right)=1$

Then, show that $gof:R \to R$ be a constant function.

17. If $f:R \to R$ where $f(x)=x^2+2$ and $g:R \to R$ where $g(x)=1-\frac{1}{1-x}$, then find

(i) fog (ii) gof

18. Show that $f:R-\{-1\} \to R-\{1\}$ given by $f(x)=\frac{x}{x+1}$ is invertible. Also find f^{-1}.

19. What do you mean by a function? Are $y^2=4ax$ and $x^2=4ay$ functions? Explain.

20. Defined the relation and function. What is the difference between a function and relation?

21. Defined the relation and function with examples.

22. Explain with example that every function is a relation but every relation is not *a* mapping.

23. Let the function f, g and h be defined by

(i) $f(x)=x^2$ where $0 \leq x \leq 2$ (ii) $g(y)=y^2$ where $3 \leq y \leq 10$

(iii) $h(z)=z^2$ where $z \in R$, which of these functions are equal?

24. Consider the function $f: R \to R$ given by $f(x)=x^3+1$. Prove that f is one-to-one and maps R onto R.

25. If $f:N \to N$ is defined by

$$f(n)=\begin{cases} 2n \text{ if } n \text{ is even} \\ n \text{ if } n \text{ is odd} \end{cases}$$ Show that f is one to one.

26. Let $A=\{1, 2, 3\}$ and $B=\{a, b, c, d\}$. In each case, state whether the given function is injective, surjective, bijective

(i) $f=\{(1, a), (2, d), (3, b)\}$ (ii) $g=\{(1, a), (2, a), (3, d)\}$

(iii) $h=\{(1, a), (2, b)\}$

27. Determir which of the following are one-to-one, which are onto and which are one-to-one onto:

(i) $f : N \to N \quad f(x) = x^2 + 2$ (ii) $f : N \to N \quad f(x) = x \pmod 3$

(iii) $f : N \to N \quad f(x) = \begin{cases} 1, & x \text{ is odd} \\ 0, & x \text{ is even} \end{cases}$

28. If $f : R \to R$ and $g : R \to R$ defined by

$$f(x) = x^3 - 4x, \quad g(x) = \frac{1}{x^2 + 1}, \quad h(x) = x^4$$

Find the following composition function

(i) $(goh)\,x$ (ii) $(gog)\,x$ (iii) $(hogof)\,x$ (iv) $(fogoh)\,x$

29. Let $f = \{(1, -1), (4, -2), (9, -3), (16, 4)\}$ and $g = \{(-1, -2), (-2, -4), (-3, -6), (4, 8)\}$ show that gof is defined while fog is not defined. Also, find (gof).

30. Composition of functions is commutative. Prove the statement or give counter example.
[U.P.T.U. (B.Tech.) 2009]

31. Define bijection mapping. Let h be a set of points in the co-ordinate diagram of $E \times F$ which is function of E into F.

(i) If the horizontal line contains at most one point h, what type of function h?

(ii) If each horizontal line contains at least one point of h, what type of function is h?
[U.P.T.U. (B.Tech.) 2005]

32. Let F_x be the set of all one-to-one mapping from X onto X, where $X = \{1, 2, 3\}$. Find all the elements of F_x and find the inverse of each element.
[U.P.T.U. (B.Tech.) 2007]

Answers

1. (i) $(gof)x = 4x^2 + 20x + 26$ (ii) $(fog)x = 2x^2 + 7$

(iii) $(fof)x = 4x + 15$ (iv) $(ff)x = 4x^2 + 20x + 25$

3. Domain in each case is R **5.** $[R - \{1, 2\}]$

6. $[D_f = R - [-1, 1], \quad R_f = R -]-1, 0]$ **7.** $[-2]$

11. (i) 1 (ii) $-1/3$ (iii) 0 **12.** $f(0) = 0, \quad f(2) = 5$

13. $f^{-1}(-5) = \phi, \quad f^{-1}(26) = \pm 5$ **15.** $[e^{2x}, e^{x^2}]$

16. $\left[f^{-1}(x) = \dfrac{x}{1 - x}\right]$ **17.** (i) $\dfrac{3x^2 - 4x + 2}{(1 - x)^2}$ (ii) $\dfrac{x^2 + 2}{x^2 + 1}$

23. none of these function are equal.

26. (i) Injective (ii) not injective (iii) is not a function

27. (i) one-to-one, not onto (ii) neither (iii) neither

28. (i) $(x^8+1)^{-1}$ (ii) $(x^2+1)^2[1+(x+1)^2]$

(iii) $[(x^3-4x)^2+1]^{-4}$ (iv) $(x^8+1)^{-3}-4(x^8+1)^{-1}$

29. $\{(1,-2),(4,-4),(9,-6),(16,8)\}$

31. (i) h is many-one mapping (ii) h is many-to-many mapping

32. $f^{-1}(x) = \dfrac{x-b}{a}$

3.10 Recursively Defined Function

[U.P.T.U. (B.Tech.) 2007]

It is the function in which the definition of function refers to itself. This definition must have the following properties so that it may not be circular.

(i) There must be certain arguments called ***Base Values*** for which the function does not refer to itself.

(ii) Each time the function refers to itself, the argument of function must be closure to a base value.

3.11 Factorial Function

[U.P.T.U. (B.Tech.) 2007]

Recursive definition of factorial function $n!$ can be defined as

If $n=0$ then $n! = 0! = 1$

If $n>0$ then $n! = n.(n-1)!$

This definition is recursive since it refers to itself, 0 is the base value of the function

$3! = 3.2!\ldots$ Level (1)

$2! = 2.1!\ldots$ Level (2)

$1! = 1.0!\ldots$ Level (3)

$0! = 1\ldots$ Level (4)

So depth of level = 4 in the case

3.12 Recursive Definition of Fibonacci Function

This function is defined as if $n=0$ or $n=1$ then $F_n = n$ if $n>1$, then $F_n = F_{n-2} + F_{n-1}$

Here, base values are 0 and 1. Function is 0, 1, 1, 2, 3, 5, 8, 13, ...

It is called ***Fibonacci Function***. Here value of F_n are defined in terms of smaller values of n that are closer to the base values 0 and 1.

Illustration: *Let x and y are positive integer and function f is defined as*

$$f(x, y) = \begin{cases} 0 \text{ if } x < y \\ f(x-y, y) + 1 \text{ if } y \le x \end{cases}$$

Then $f(7, 8) = 0$ as $7 < 8$

and $f(20, 5) = f(20-5, 5) + 1$ (as $y < x$)

$= f(15, 5) + 1 = [f(15-5, 5) + 1] + 1 = f(10, 5) + 2 = [f(10-5, 5) + 1] + 2$

$= f(5, 5) + 3 = [f(0, 5) + 1] + 3 = f(0, 5) + 4 \quad (x < y) = 0 + 4 = 4$

Illustration: *We defined the function in factorial notation as*

$$f(n) = \begin{cases} 0 \text{ if } n = 1 \\ f\left(\left(\frac{n}{2}\right)!\right) + 1 \text{ if } n > 1 \end{cases}$$

where $f\left(\left(\frac{n}{2}\right)!\right)$ means greatest integer $\le \frac{n}{2}$

Then $f(23) = f\left(\left(\frac{23}{2}\right)!\right) + 1 = f(11) + 1 = \left[f\left(\frac{11}{2}\right)! + 1\right] + 1 = f(5) + 2$

$= \left[f\left(\frac{5}{2}\right)! + 1\right] + 2 = f(2) + 3 = f(1) + 4 = 0 + 4 = 4$

Example 67: *Give a recursive or inductive definition of function a_n where a is non-zero real number and n is non-negative integer.*

Solution: $a^n = 1$ if $n = 0$ or $a^\circ = 1$ and $a^{n+1} = a \cdot a^n$ or $a^n = a\, a^{n-1}$

3.13 Ackermann's Function

It can be defined as

(i) If $m = 0$, then $A(m, n) = n + 1$

(ii) If $m \ne 0$, but $n = 0$, then $A(m, n) = A(m-1, 1)$

(iii) If $m \ne 0$ and $n \ne 0$, then $A(m, n) = A(m-1), A(m, n-1)$

$A(m, n)$ is explicitly given only when $m = 0$

Example 68: *Evaluate (i) $A(2, 0)$ (ii) $A(2, 1)$*

Solution: (i) $A(2, 0) = A(1, 1)$

$$A(1, 1) = (A(1-1, A(1, 1-1)) = A(0, A(1, 0))$$
$$A(1, 0) = A(0, 1)$$
$$A(0, 1) = 1 + 1 = 2$$

Use backward process i.e.

$$A(1, 0) = A(0, 1) = 2$$
$$A(1, 1) = (A(0, A(1, 0)) = A(0, 2) = 2 + 1 = 3$$
$$A(2, 0) = A(1, 1) = 3$$

(ii) $A(2, 1) = ((A(1, A(2, 0)) = A(1, 3)$ by (i)

$$= A(0, A(1, 2))$$

$$A(1, 2) = A(0, A(1, 1)) = A(0, 3) = 4$$

$$A(2, 1) = A(0, 4) = 5$$

3.14 Primitive Recursive Function

Primitive recursive functions are functions that can be generated by what is called ***Primitive Recursion*** from a set of initial function, employing method, similar to the ones used in the definition of addition and multiplication. The basic building block of primitive recursive expressions are the constant 0, a set of variables $x_1, x_2 \ldots$ and a set of function symbols. The formation rules for the primitive recursive expression can be formulated as

(i) 0 is expression

(ii) Every variable is an expression

(iii) If f is function symbol of arbitrary d and $t_1, t_2 \ldots t_d$ are expressions, then $f(t_1, t_2, t_3 \ldots t_d)$ is an expression.

3.14.1 Initial Function

(i) ***Zero Function:*** $Z(x) = 0 \ \forall x \in N$ and $Z : N \to N$

(ii) ***Successor Function:*** $S(x) = x + 1 \ \forall x \in N$ and $S : N \to N$

(iii) ***Projection Function:*** $U_i^n(x_1, x_2, x_3, \ldots x_n) = x_i$ for n – tuples $(x_1, x_2, x_3, \ldots x_n)$ for $1 \le i \le n$ and $x_i \in N, i = 1, 2 \ldots n$. So, U_i^n is also called ***generalized identity function***.

In particular $S(4) = 4 + 1$, $Z(8) = 0$

$$U_1^3(3, 5, 8) = 3, \ U_2^3(3, 5, 8) = 5, \ U_3^3(3, 5, 8) = 8$$

All these functions are initial functions. The initial function provides an infinite set of functions from which other function can be built up by using the following strategies.

(i) ***Composition:*** If f is an n-ary function and $g_1, g_2, \ldots g_n$ are k-ary functions, then $f(g_1, g_2, g_3, \ldots g_n)$ is a k-ary function, where $f(g_1, g_2, \ldots g_n)(x_1, x_2, \ldots x_k) = f(g_1(x_1, x_2, \ldots x_k) \ldots g_n(x_1, x_2 \ldots x_k))$

(ii) ***Primitive Recursion:*** Let $g : N^n \to N$ and $h : N^{n+2} \to N$ be functions. We can say that $f : N^{n+1} \to N$ is defined from g and h by primitive recursion if f satisfies the conditions

$$f(x_1, x_2, \ldots x_n, 0) = g(x_1, x_2, \ldots x_n)$$

and $$f(x_1, x_2, \ldots x_n, y + 1) = h(x_1, x_2 \ldots x_n, y, f(x_1, x_2 \ldots x_n, y))]$$

where y is inductive variable. Hence,

A function f is said to be ***Primitive Recursive*** if it can be obtained from the initial functions by a finite number of operations of composition and recursion.

Example 69: *Show that the function $f(x) = k$, where k is constant, is primitive recursive function.* [U.P.T.U. (B.Tech.) 2005]

Solution: Let $k = 0$, then

$$f(x) = 0 = f(y)$$

Otherwise $$f(x+1) = k = f(x) = U_2^2(x_1\ f(x))$$

So, $f(x)$ is primitive recursive function.

Example 70: *Show that the function $f < x, y > = x - y$ is partial recursive.* [U.P.T.U. (B.Tech.) 2003]

Solution: Since every element of a domain maps to at most one element of the co-domain. Now, the predecessor function.

$$p(0) = 0, \quad p(y+1) = y = U_1^2$$

which is recursive function and from it the subtraction is

$$f(x, 0) = x = U_1^1, \quad f(x, y+1) = p(f(x, y)) = g(x, y, f(x, y))$$

where $$g(x, y, z) = p(z) = U_3^3$$

$\Rightarrow f$ is partial recursive function

Example 71: *Show that additive is primitive recursive.*

Solution: We define addition as

$$x + 0 = 0,\ x + (y+1) = (x+y) + 1 \ \ \forall \ \ x, y \in N$$

we define, $f(x, y) = x + y$ such that

$$f(x, y+1) = x + y + 1 = (x+y) + 1 = f(x, y) + 1 = S(f(x, y)$$

Also $$f(x, 0) = x$$

$\therefore$ $$f(x, 0) = x = \cup_1^1(x)$$

$$f(x, y+1) = S(\cup_3^3(x, y, f(x, y))$$

Hence, we find $\cup_1^1, \cup_3^3$ comes from primitive recursion so f is primitive recursive.

Example 72: *Show that multiplication $(*)$ defined by*

$$x * 0 = 0, \ \ x * (y+1) = x * (y + x)$$

is primitive recursive.

Solution: We define $\mu(x, y)$ to be $x * y$, so that $\mu(x, 0) = 0 = Z(x)$

$$\mu(x, y+1) = \mu(x, y) + 1 = f(x, \mu(x, y))$$

$$\mu(x, y+1) = f\{(\cup_3^3(x, y, \mu(x, y)), \cup_1^3(x, y, \mu(x, y))\}$$

where f is additive function.

Example 73: *Show that $f(x, y) = x^y$ is a primitive recursive function.*

Solution: Here $x^\circ = 1$ for $x \neq 0$ we put $x^\circ = 0$ for $x = 0$

Again $$x^{y+1} = x^y * x$$

Hence, $$f(x, y) = y^x \text{ is define as}$$

$$f(x, 0) = 1$$

$$f(x, y+1) = x * f(x, y) = \cup_1^3(x, y, f(x, y) * \cup_3^3(x, y, f(x, y))$$

Therefore, $f(x, y)$ is primitive recursive function.

Example 74: *Consider a recursive function G from Z^+ to Z for all integer $n \geq 1$*

$$g(n) = \begin{cases} 1 \text{ if } n \text{ is } 1 \\ 1 + g\left(\frac{n}{2}\right) \text{ if } n \text{ is even} \\ g(3n-1) \text{ if } n \text{ is odd and } n > 1 \end{cases}$$

Is g well defined? Justify.

Solution: Suppose g is a function. Then by definition of g

$$g(1) = 1$$

$$g(2) = 1 + g(1) = 1 + 1 = 2$$

$$g(3) = g(8) = 1 + g(4) = 1 + (1 + g(2)) = 1 + (1 + 2) = 4$$

$$g(4) = 1 + g(2) = 1 + 2 = 3$$

$$g(5) = g(14) = 1 = g(7) = 1 + g(20)$$

$$= 1 + (1 + g(10)) = 1 + 1(1 + (1 + g(5))) = 3 + g(5)$$

$$\therefore \quad g(5) = 3 + g(5)$$

$$\Rightarrow \quad 3 = 0$$

Hence, g is not well define.

Illustration: *Let $f_1(x, y) = x + y$, $f_2(x, y) = 3x$, $f_3(x, y) = xy$ and $g(x, y, z) = x + y + z$ be function over N. Then* $g(f_1(x, y), f_2(x, y), f_3(x, y)) = g(x + y, 3x, xy)$

$$= x + y + 3x + xy$$

This gives the composition of g with f_1, f_2, f_3.

Example 75: *Show that the proper subtraction is primitive recursive and hence prove that* $5 - 3 = 2$.

Solution: We have $P(0) = 0$

$$P(y+1) = y = \cup_1^2(y, P(y))$$

which is recursive function and from it, the subtraction is $f(x, 0) = x = \cup_1^1(x)$

$$f(x, y+1) = P(f(x, y) = g(x, y, f(x, y)$$

where $\quad g(x, y, z) = P(z) = \cup_3^3(x, y, z)$

Hence, the proper subtraction function is a recursive function

Now, $\quad f(5, 3) = p\,f(5, 2) = p(p\,f(5, 1)) = p(p(p(5, 1))) = p(p(4) = p(3) = 2$

Example 76: *Show that the function $x!$ is primitive recursive, where $0! = 1$ and $n! = n * (n-1)!$*

Solution: Let $f(x) = x$, then

$$f(0) = 0! = 1 = S(0)$$

and

$$f(x+1) = (x+1)!$$

$$= (x+1) * x!$$

$$= (x+1) * f(x)$$

$$= x * f(x) + f(x)$$
$$= \cup_1^2 (x, f(x)) * ?$$
$$= \cup_2^2 (x, f(x)) + ?$$
$$= \cup_2^2 (x, f(x)$$

Since the sum (+) and product (*) are primitive recursive, $f(x)$ is primitive recursive.

3.15 Growth of Functions

The growth of functions is often described using three important notations: the big-oh (o) the big omega (Ω) and beg-theta (θ). They provide a special way to compare relative size of functions that is very useful in the analysis of computer algorithm. It often happens that the time or memory space requirements for the algorithm available to do a certain job differ from each other or such a grand scale that differences of just a constant factor are completely over shadowed. The o-notations makes use of approximation that highlights these large scale difference while ignoring differences of a constant factor and differences that only occur for small set of input data.

"Let f and g be function from the set of integers or the set of real numbers to the set of real numbers. Then f of order g written as $f(x)$ is $o(g(x))$ if there are constants C and K such that $|f(x)| \leq C|g(x)|$ whenever $x > K$ which is read as $f(x)$ is big-oh of $g(x)$.

Remark: To show $f(x)$ is $o(g(x))$ we have to find only one pair of constant C and K such that $f(x) < Cg(x)$ if $x > K$. However a pair C, K that satisfies the definition is never unique. Moreover, if one such pair exists there are infinitely many such pairs. A simple way to see this a note that if C, K is one such pair, any pair C', K' with $C < C'$ and $K < K'$ also satisfies the definition. Since $f(x) < C' g(x)$ whenever $x > K' > K$.

Example 77: *Use O-notation to express*

$$|3x^3 + 2x + 7| \leq |2| x^3|$$

for all real number $x > 1$

Solution: We take $C = 12$ and $K = 1$, the given statement translates to $2x^3 + 2x + 7$ is $o(x^3)$.

Example 78: *Use the definition of order to show that $x^2 + 2x + 1$ is $0(x^2)$.*

Solution: The function f and g referred to in the definition of o-notation are defined as follows. For all real number x

$$f(x) = x^2 + 2x + 1 \text{ and } \quad g(x) = x^2$$

For all real number $x > 1$

$$|x^2 + 2x + 1| = x^2 + 2x + 1 \text{ as } x^2 + 2x + 1 > 0 \text{ for } x > 1$$
$$\leq x^2 + 2x^2 + x^2$$
$$\leq 4x^2$$
$$\leq 4|x^2|$$

Therefore, $\quad |f(x)| \leq 4|g(x)|$ for all $x > 1$

Let $\quad C = 4, K = 1$

$\Rightarrow \quad |f(x)| \leq C|g(x)| \quad \forall x > K$

Hence, $\quad x^2 + 2x + 1$ is $o(x^2)$.

Exercise

Recursively Defined

1. Show that the proper subtraction is primitive recursive and hence prove that $5-3=2$.

2. Show that for any real number $x>1$, $2x^4+4x^3+5 \le 11x^4$.

3. Use the definition of order to show that $5x^3-3x+4$ is $o(x^3)$.

4. Show that $9x^2$ is $o(x^3)$. Is it true that x^3 is $o(9x^2)$?

5. Let $f_1(x, y)=x+y$, $f_2(x, y)=xy+y^2$ and $g(x, y)=xy$ be functions defined over N. Find the compositions of g with f_1 and f_2.

6. Let $f(x, y)=3x^2+y$, $g_1, (x, y, z)=x+y+z$ and $g_2(x, y, z)=x-y-z$ be defined over N. Find the composition of f with g_1 and g_2.

7. Consider the function $f(x, y)=\begin{cases} 1 \text{ if } x=y \\ 0 \text{ if } x \ne y \end{cases}$ show that this function is primitive function.

8. Compute $A(3, 1)$, $A(3, 2)$, $A(2, 3)$ and $A(3, 3)$.

9. Prove the following for the Ackermann function
   (i) $A(1, y)=y+2$ (ii) $A(2, y)=2y+3$

10. A function $f: Z^+ \to Z$ is defined by

$$f(n)=\begin{cases} 1 \\ f(n/2) \text{ if } n \text{ is even} \\ 1+f(5n-9) \text{ if } n \text{ is odd and } n>1 \end{cases}$$

show that f is not well defined.

11. Let x and y be two integers and suppose $g(x, y)$ is defined recursively by

$$g(x, y)=\begin{cases} 5 \text{ if } x<y \\ g(x-y, y+2)+x \text{ if } x \ge y \end{cases}$$

Find $g(2, 7)$, $g(5, 3)$, and $g(15, 2)$

12. Use the definition of o-notation, prove the followings:
   (i) $6x^2+5x$ is $o(x^2)$ (ii) $8x^5-17x^4+80x-24$ is $o(x^5)$
   (iii) $1^2+2^2+3^2+\ldots+x^2$ is $o(x^3)$

13. Give big-oh estimates for the factorial function and the logarithm of the factorial function and the logarithm of the factorial function where the factorial function $a_n = n!$ is defined by $n! = 1, 2, 3 \ldots n$.

14. What do you mean by recursively defined function? Give an example. [U.P.T.U. (M.C.A.) 2008]

15. Show that the function $f(x) = k$, where k is constant is primitive recursive. [U.P.T.U. (B.Tech.) 2005]

16. What is meant by a recursively defined function? Give the recursive definition of factorial function. [U.P.T.U. (B.Tech.) 2002, 2006]

17. Let $f : R \to R$ is defined by $f(x) = x^2 + 1$. Find

(a) $f^{-1}(5)$ (b) $f^{-1}(10)$

18. State whether the following functions are one-one

(a) To each person on the earth assign the number which corresponds to his age.
(b) To each country, assign the number of people living in the country.
(c) To each book written by only one author, assign to author.
(d) To each country having prime-number, assign to prime number.

Answers

5.	$(x + y)(xy + y)$	**6.**	$3(x + y + z)^2 + x - y + z$
8.	13, 17, 9, 37	**11.**	5, 10, 42
17.	(a) (– 2, 2) (b) (–3, 3)	**18.**	(a) not one-one (b) one-one (c) not one-one (d) one-one

Objective Type Quesitons

Multiple Choice Questions

1. Let $f : R \to R$ be given by $f(x) = -x^2$ and $g : R_+ \to R_+$ be given by $g(x) = \sqrt{x}$, where R_+ is the set of non-negative real numbers and R is the set of real numbers. What are fog and gof? [U.P.T.U. (M.C.A) 2008]

(a) $x, -x$ (b) $-x, x$ (c) not defined, $-x$ (d) $-x$, not defined

2. Which of the following sets are equal? [U.P.T.U. (M.C.A.) 2007]

$A = \{x : x^2 - 4x + 3 = 0\}$ $B = \{x : x^2 - 3x + 2 = 0\}$ $C = \{x : x \in N, x < 3\}$

$D = \{x : x \in N, x \text{ is odd}, x < 5\}$ $E = \{1, 2\}$ $F = \{1, 2, 1\}$

$G = \{3, 1\}$ $H = \{1, 1, 3\}$

3. Let A and B be two finite sets having m and n elements respectively. Then the total number of mapping from A to B is [U.P.T.U. (M.C.A.) 2007]

(a) mn (b) 2^{mn} (c) m^n (d) n^m