MODULE 1

Partial Differential Equations

1.1 INTRODUCTION

A differential equation containing partial derivatives of a function of two or more independent variables is called a partial differential equation. *e.g.*,

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

are the partial differential equations.

When we have a function z of two independent variables x and y, we use the alphabets p, q, r, s, t to denote the partial derivatives as follows:

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}$$

Partial differential equations generally occur in the problems of Physics and Engineering. Some of the important partial differential equations are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad ...(2)$$

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad ...(3)$$

Equations (1), (2) and (3) are respectively known as Laplace's equation, wave equation and heat conduction equation.

Sometimes for brevity, the partial differentiation with regard to a variable is denoted by a suffix. *e.g.*, Laplace's equation may be rewritten as $u_{xx} + u_{yy} + u_{zz} = 0$.

If u does not depend on z, then we get two-dimensional Laplace's equation as $u_{xx} + u_{yy} = 0$.

1.2 ORDER AND DEGREE OF PARTIAL DIFFERENTIAL EQUATION

Order of a partial differential equation is the order of the highest ordered derivative present in the equation.

Degree of a partial differential equation is the greatest exponent (power) of the highest ordered derivative present in the equation when it has been made free from radical signs and fractional powers.

1.3 SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

Solution is one which satisfies. The solution of a partial differential equation in a region D is a function having partial derivatives which satisfy the differential equation at every point in D.

The general solution of a p.d.e. contains arbitrary constants or arbitrary functions or both. Consequently, we can say that by the elimination of arbitrary constants or arbitrary functions, partial differential equations can be formed.

1.4 FORMATION OF PARTIAL DIFFERENTIAL EQUATION

(1) By the elimination of arbitrary constants

Let
$$f(x, y, z, a, b) = 0 \qquad \text{...(1)}$$
be the given function, where a, b are arbitrary constants. x and y are independent variables and z is a dependent variable. Differentiating eqn. (1) partially w.r.t. x, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \qquad \text{...(2)}$$

Again differentiating eqn. (1) partially w.r.t. y, we get

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \qquad \text{...(3)}$$

Eliminating a, b from equations (1), (2) and (3), we get
$$F(x, y, z, p, q) = 0$$
which is a partial differential equation of first order.

Note. If the number of arbitrary constants is more than the number of independent variables, then the order of the partial differential equation obtained will be greater than 1.

(2) By the elimination of arbitrary functions

Let u, v be two known functions of x, y, z connected by the relation
$$\phi(u, v) = 0 \qquad \text{...(1)}$$
where ϕ is an arbitrary function.

Differentiating eqn. (1) partially w.r.t. x, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

$$\Rightarrow \quad \frac{\partial \phi/\partial u}{\partial \phi/\partial v}=-\left(\frac{\dfrac{\partial v}{\partial x}+p\dfrac{\partial v}{\partial z}}{\dfrac{\partial u}{\partial x}+p\dfrac{\partial u}{\partial z}}\right) \qquad ...(2)$$

Differentiating eq. (1) partially w.r.t. y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)=0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

$$\Rightarrow \quad \frac{\partial \phi/\partial u}{\partial \phi/\partial v}=-\left(\frac{\dfrac{\partial v}{\partial y}+q\dfrac{\partial v}{\partial z}}{\dfrac{\partial u}{\partial y}+q\dfrac{\partial u}{\partial z}}\right) \qquad ...(3)$$

From eqns. (2) and (3), we get

$$\frac{\dfrac{\partial v}{\partial x}+p\dfrac{\partial v}{\partial z}}{\dfrac{\partial u}{\partial x}+p\dfrac{\partial u}{\partial z}}=\frac{\dfrac{\partial v}{\partial y}+q\dfrac{\partial v}{\partial z}}{\dfrac{\partial u}{\partial y}+q\dfrac{\partial u}{\partial z}}$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \boxed{\mathrm{P}p+\mathrm{Q}q=\mathrm{R}} \qquad ...(4)$$

Eqn. (4) is a partial differential equation of first degree (*i.e.* linear) in p and q.

Note. If the given relation between x, y, z contains two arbitrary functions, then the p.d.e. derived therefrom will be, in general of order greater than one.

ILLUSTRATIVE EXAMPLES

Example 1. *Form partial differential equations from the following equations by eliminating the arbitrary constants:*

(*i*) $z = ax + by + ab$ (*ii*) $z = ax + a^2y^2 + b$

(*iii*) $z = (x + a)(y + b)$ (*iv*) $az + b = a^2x + y$.

Sol. (*i*) Differentiating z partially w.r.t. x and y,

$$p=\frac{\partial z}{\partial x}=a, \quad q=\frac{\partial z}{\partial y}=b$$

Substituting for a and b in the given equation, we get

$$z = px + qy + pq$$

which is a partial differential equation.

(*ii*) Differentiating z partially w.r.t. x and y

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = 2a^2y$$

Eliminating a between these results, we get $q = 2p^2y$
which is a partial differential equation.

(*iii*) Differentiating z partially w.r.t.x, we get

$$\frac{\partial z}{\partial x} = p = y + b \quad ...(1)$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = q = x + a \quad ...(2)$$

Multiplying eqns. (1) and (2), we get

$$pq = (y + b)(x + a)$$

$$\Rightarrow \quad pq = z$$

which is a partial differential equation.

(*iv*) Differentiating the given relation w.r.t. x partially, we get

$$a\frac{\partial z}{\partial x} = a^2$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = p = a \quad ...(1)$$

Again differentiating the given relation w.r.t. y partially, we get

$$a\frac{\partial z}{\partial y} = 1$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = q = \frac{1}{a} \quad ...(2)$$

Multiplying eqns.(1) and (2), we get $pq = 1$
which is a partial differential equation.

Example 2. *Form the partial differential equation by eliminating the arbitrary function(s) from the following :*

(*i*) $z = f(x^2 - y^2)$ (U.P.T.U. 2015) (*ii*) $z = \phi(x) . \psi(y)$
(*iii*) $z = x + y + f(xy)$ (*iv*) $z = f(x + it) + g(x - it)$. (U.K.T.U. 2011)

Sol. (*i*) Differentiating z partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = p = f'(x^2 - y^2) . 2x \quad ...(1)$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = q = f'(x^2 - y^2) . (-2y) \quad ...(2)$$

Dividing eqn. (1) by eqn. (2), we get

$$\frac{p}{q} = \frac{x}{(-y)} \Rightarrow py + qx = 0$$

which is a partial differential equation.

(*ii*) Differentiating z w.r.t. x, partially, we get

$$\frac{\partial z}{\partial x} = p = \phi'(x)\, \psi(y) \qquad ...(1)$$

Differentiating z w.r.t. y partially, we get

$$\frac{\partial z}{\partial y} = q = \phi(x)\, \psi'(y) \qquad ...(2)$$

Differentiating eqn. (1) partially w.r.t. y, we get

$$\frac{\partial^2 z}{\partial y \partial x} = s = \phi'(x)\, \psi'(y) \qquad ...(3)$$

Multiplying eqns. (1) and (2), we get

$$pq = \phi(x)\, \psi(y)\, \phi'(x)\, \psi'(y) = zs \qquad | \text{ Using (3)}$$

$$\Rightarrow \qquad pq - zs = 0$$

which is a partial differential equation.

(*iii*) Differentiating z w.r.t. x, partially, we get

$$\frac{\partial z}{\partial x} = p = 1 + f'(xy)\,.\,y \quad \Rightarrow \quad p - 1 = yf'(xy) \qquad ...(1)$$

Differentiating z w.r.t. y, partially, we get

$$\frac{\partial z}{\partial y} = q = 1 + f'(xy)\,.\,x \quad \Rightarrow \quad q - 1 = x\,f'(xy) \qquad ...(2)$$

Dividing eqn. (1) by eqn. (2), we get

$$\frac{p-1}{q-1} = \frac{y}{x} \quad \Rightarrow \quad px - qy = x - y$$

which is a partial differential equation.

(*iv*) Given $z = f(x + it) + g(x - it)$

Differentiating z twice partially w.r.t. x and t, we have

$$\frac{\partial z}{\partial x} = f'(x + it) + g'(x - it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x + it) + g''(x - it) \qquad ...(1)$$

$$\frac{\partial z}{\partial t} = i\,f'(x + it) - ig'(x - it)$$

$$\frac{\partial^2 z}{\partial t^2} = i^2 f''(x + it) + i^2 g''(x - it)$$

or

$$\frac{\partial^2 z}{\partial t^2} = -f''(x + it) - g''(x - it) \qquad ...(2)$$

Adding (1) and (2), we obtain $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial t^2} = 0$

which is a partial differential equation of second order.

1.5 DEFINITIONS

The solution $f(x, y, z, a, b) = 0$ of a first order partial differential equation, which contains two arbitrary constants is called a *complete solution* or *complete integral*.

If in this solution, we put $b = \phi(a)$ and find the envelope of the family of surfaces $f(x, y, z, a, \phi(a)) = 0$, we get a solution involving an arbitrary function ϕ. This is called the *general solution* or *general integral*.

A solution obtained from the complete integral by giving particular values to the arbitrary constant is called a *particular solution* or *particular integral*.

A partial differential equation may have a large number of entirely different solutions. For example, $u = x^2 - y^2$, $u = \log(x^2 + y^2)$, $u = \sin kx \cosh ky$ are solutions of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. The unique solution of a partial differential equation corresponding to a physical problem must satisfy certain other conditions at the boundary of the region R. These are known as the *boundary conditions*.

If these conditions are given for the time $t = 0$, they are known as the *initial conditions*.

1.6 LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION

A partial differential equation is said to be linear if it is of first degree in the dependent variable and its partial derivatives and also they are not multiplied together.

If, in addition, every term of the equation contains the dependent variable or its derivative, it is called a homogeneous equation.

Note. An equation which is not linear is called a non-linear partial differential equation.

1.7 QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation is said to be quasi-linear if degree of highest ordered derivative is one and no products of partial derivatives of the highest order are present.

e.g., $$z\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial y}\right)^2 = 0 \text{ is a quasi-linear p.d.e. of second order.}$$

The general form of a quasi-linear p.d.e. of the first order is

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z) \qquad \text{...(1)}$$

If P and Q are independent of z and R is linear in z then eqn. (1) is linear partial differential equation.

1.8 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

The equations which contain **only one** partial derivative can be solved by direct integration. However, in place of the constants of integration, we must use arbitrary functions of the variable kept constant.

Example 3. *Solve:* $t = \sin(xy)$.

Sol. The given equation is

$$\frac{\partial^2 z}{\partial y^2} = \sin(xy)$$

Integrating w.r.t. y, we get

$$\frac{\partial z}{\partial y} = -\frac{1}{x}\cos(xy) + f(x)$$

The constant is being taken as a function of x.

Again integrating w.r.t. y, we get

$$z = -\frac{1}{x^2}\sin(xy) + yf(x) + \phi(x).$$

Example 4. *Solve:* $\frac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ *and* $\frac{\partial z}{\partial x} = 1$.

Sol. If z were *a* function of **x alone,** the solution would have been $z = c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are arbitrary constants.

But here z is a function of x and y, therefore, c_1 and c_2 are arbitrary functions of y, the independent variable kept constant.

$\therefore$ The solution of the given equation is

$$z = f(y)\cos x + \phi(y)\sin x \text{ and } \frac{\partial z}{\partial x} = -f(y)\sin x + \phi(y)\cos x$$

When $x = 0$, $z = e^y$ $\quad\therefore\quad e^y = f(y)$

When $x = 0$, $\frac{\partial z}{\partial x} = 1$ $\quad\therefore\quad 1 = \phi(y)$

$\therefore$ The required particular solution is $z = e^y \cos x + \sin x$.

1.9 LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

A differential equation involving first order partial derivatives p and q only is called a *partial differential equation of the first order*. If p and q both occur in the first degree only and are not multiplied together, then it is called *linear partial differential equation of the first order*.

1.10 LAGRANGE'S EQUATION

The partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z is the standard form of a quasi-linear partial differential equation of the first order and is called *Lagrange's Equation*.

Now Lagrange's equation $\quad Pp + Qq = R \quad$ ...(1)

is obtained by eliminating an arbitrary function ϕ from $\quad \phi(u, v) = 0 \quad$ ...(2)

where u, v are some definite functions of x, y, z.

Differentiating (2) partially w.r.t. x and y

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

and $$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$, we get

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)-\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)=0$$

or $$\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

which is the same as (1) with

$$P=\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y},\quad Q=\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z},\quad R=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}.$$

To determine u, v from P, Q, R, suppose $u = a$ and $v = b$, where a, b are constants, so that

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=du=0$$

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=dv=0$$

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}}$$

or $$\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}} \qquad ...(3)$$

The solutions of these equations are $u = a$ and $v = b$. Thus determining u, v from the simultaneous equations (3), we have the solution of the partial differential equation

$$Pp + Qq = R \text{ as } \phi(u, v) = 0 \quad \text{or} \quad v = f(u).$$

Note. Equations (3) are called **Lagrange's auxiliary equations** or **subsidiary equations.**

Remark. In the differential equation $Pp + Qq = R$, there are **two independent variables** x and y. To solve it, we have to find **two independent solutions** satisfying the equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$. This procedure can be extended to linear partial differential equations of the first order involving more than two independent variables.

If $u_1 = c_1, u_2 = c_2, \ldots\ldots, u_n = c_n$ are n independent solutions of the equations

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=\ldots\ldots=\frac{dx_n}{P_n}=\frac{dz}{R}$$

then the general solution of the partial differential equation

$$P_1\frac{\partial z}{\partial x_1}+P_2\frac{\partial z}{\partial x_2}+\ldots\ldots+P_n\frac{\partial z}{\partial x_n}=R$$

is $\phi(u_1, u_2, \ldots\ldots, u_n) = 0$, where ϕ is an arbitrary function.

1.11 WORKING RULE

To solve the equation $$Pp + Qq = R$$

(*i*) Form the auxiliary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

(*ii*) Solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions $u = a$ and $v = b$, where a, b are arbitrary constants.

(*iii*) Then $\phi(u, v) = 0$ or $v = f(u)$ is the general solution of the equation $Pp + Qq = R$.

Example 5. *Solve the following partial differential equations:*

(*i*) $yzp - xzq = xy$ (*ii*) $y^2p - xyq = x(z - 2y)$ (U.P.T.U. 2014) (*iii*) $x^2p + y^2q = (x + y)z$ (U.P.T.U. 2015)

(iv) $yq - xp = z$

Sol. (*i*) The given differential equation is $yzp - xzq = xy$ ...(1)

Comparing equation (1) with $Pp + Qq = R$, we get

$$P = yz,\ Q = -xz \quad \text{and} \quad R = xy$$

Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \quad \frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

From first and second fractions, we get

$$\frac{dx}{yz} = \frac{dy}{-xz} \quad \Rightarrow \quad \frac{dx}{y} = \frac{dy}{-x}$$

$$\Rightarrow \quad xdx + ydy = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2} \quad \Rightarrow \quad x^2 + y^2 = c_1 \qquad ...(2)$$

From first and third fractions, we get

$$\frac{dx}{yz} = \frac{dz}{xy} \quad \Rightarrow \quad \frac{dx}{z} = \frac{dz}{x} \quad \Rightarrow x\,dx - z\,dz = 0$$

Integrating, we get

$$\frac{x^2}{2} - \frac{z^2}{2} = \frac{c_2}{2} \quad \Rightarrow \quad x^2 - z^2 = c_2$$

Hence the general solution is $\phi(x^2 + y^2, x^2 - z^2) = 0$

(*ii*) Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \quad \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)}$$

From first and second fractions, we have

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$\Rightarrow \qquad x\,dx + y\,dy = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2} \quad \Rightarrow \quad x^2 + y^2 = c_1 \qquad \text{...(1)}$$

From second and third fractions, we have

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$\Rightarrow \qquad -zdy + 2ydy = ydz$

$\Rightarrow \qquad 2ydy = ydz + zdy$

Integrating, we get

$$y^2 = yz + c_2 \qquad \text{...(2)}$$

Hence the general solution is $\quad \phi(x^2 + y^2, y^2 - yz) = 0$

(*iii*) Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \qquad \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \qquad \text{...(1)}$$

From first and second fractions, we have

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating, we get

$$-\frac{1}{x} = -\frac{1}{y} - c_1$$

$$\Rightarrow \qquad \frac{1}{x} - \frac{1}{y} = c_1 \qquad \text{...(2)}$$

Again from (1),

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z} \quad \Rightarrow \quad \frac{dx - dy}{x - y} = \frac{dz}{z}$$

Integrating, we get

$$\log(x - y) = \log z + \log c_2$$

$$\Rightarrow \qquad \frac{x-y}{z} = c_2 \qquad \text{...(3)}$$

Hence the general solution is

$$\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{x-y}{z}\right) = 0$$

Example 6. *Solve the following differential equations:*

(*i*) $\frac{y^2 z}{x} p + xzq = y^2$ (*ii*) $pz - qz = z^2 + (x+y)^2$

(*iii*) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

Sol. (*i*) The given equation can be written as $y^2zp + x^2zq = xy^2$

Comparing with $Pp + Qq = R$, we have $P = y^2z$, $Q = x^2z$, $R = xy^2$

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

Taking the first two members, we have $x^2dx = y^2dy$

which on integration gives $x^3 - y^3 = c_1$ ...(1)

Again taking the first and third members, we have $xdx = zdz$

which on integration gives $x^2 - z^2 = c_2$ ...(2)

From (1) and (2), the general solution is

$$\phi(x^3 - y^3, x^2 - z^2) = 0.$$

(*ii*) Here Lagrange's subsidiary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

Taking the first two members, we have $dx + dy = 0$

which on integration gives $x + y = c_1$ ...(1)

Again taking the first and third members, we have

$$dx = \frac{zdz}{z^2 + c_1^2}, \text{ since } x + y = c_1$$

or $$\frac{2z\,dz}{z^2 + c_1^2} = 2dx$$

which on integration gives

$$\log(z^2 + c_1^2) = 2x + c_2$$

or $$\log[z^2 + (x+y)^2] - 2x = c_2$$ ...(2)

From (1) and (2), the general solution is

$$\phi[x + y, \log(x^2 + y^2 + z^2 + 2xy) - 2x] = 0.$$

(*iii*) Here Lagrange's subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\therefore \quad \frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)}$$

$$= \frac{dz - dx}{(z-x)(x+y+z)}$$

Taking the first two members, we have $\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$

which on integration gives

$$\log(x - y) = \log(y - z) + \log c_1$$

or $$\log\left(\frac{x-y}{y-z}\right) = \log c_1$$

or $$\frac{x-y}{y-z} = c_1 \qquad \text{...(1)}$$

Now, using x, y, z as multipliers, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \qquad \text{...(2)}$$

Again, using 1, 1, 1 as multipliers, we get

$$\text{each fraction} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy} \qquad \text{...(3)}$$

Hence, from (2) and (3),

$$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} = \frac{dx+dy+dz}{x^2+y^2+z^2-yz-zx-xy}$$

$$\Rightarrow \quad \frac{x\,dx + y\,dy + z\,dz}{x+y+z} = dx + dy + dz$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = (x+y+z)(dx+dy+dz)$$

Intergration gives,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} - c_2$$

$$= \frac{x^2+y^2+z^2}{2} + yz + zx + xy - c_2$$

$$\Rightarrow \quad xy + yz + yx = c_2 \qquad \text{...(4)}$$

Hence the general solution is

$$\phi\left(\frac{x-y}{y-z},\ xy + yz + zx\right) = 0.$$

Example 7. *Solve the following differential equations:*

(i) $(mz - ny)p + (nx - lz)q = ly - mx$

(ii) $x^2(y-z)p + y^2(z-x)q = z^2(x-y).$ (U.P.T.U. 2013

Sol. *(i)* Here Lagrange's subsidiary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using x, y, z as multipliers, we get

$$\text{each fraction} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore \quad xdx + ydy + zdz = 0$$

which on integration gives $x^2 + y^2 + z^2 = a$ ...(

Again using l, m, n as multipliers, we get

$$\text{each fraction} = \frac{ldx + mdy + ndz}{0}$$

$$\therefore \qquad ldx + mdy + ndz = 0$$

which on integration gives $lx + my + nz = b$ ...(2)

From (1) and (2), the general solution is

$$x^2 + y^2 + z^2 = f(lx + my + nz).$$

(*ii*) Here the auxiliary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \qquad \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

which on integration gives $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$ ...(1)

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \qquad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

which on integration gives

$$\log x + \log y + \log z = \log c_2$$

or
$$\log (xyz) = \log c_2$$

or
$$xyz = c_2 \qquad ...(2)$$

From (1) and (2), the general solution is

$$\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0.$$

Example 8. *Solve the partial differential equation*

$$x(y^2 + z)\, p - y(x^2 + z)\, q = z(x^2 - y^2) \text{ where } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}.$$

Sol. Lagrange's subsidiary equations are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y\,(x^2 + z)} = \frac{dz}{z(x^2 - y^2)} \qquad ...(1)$$

Using $x, y, -1$ as multipliers, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy - dz}{x^2\,(y^2 + z) - y^2\,(x^2 + z) - z\,(x^2 - y^2)} = \frac{x\,dx + y\,dy - dz}{0}$$

$$\therefore \quad x\,dx + y\,dy - dz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2}$$

$$\Rightarrow \quad x^2 + y^2 - 2z = c_1 \qquad \text{...(2}$$

Again, using $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{0}$$

$$\therefore \quad \frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\Rightarrow \quad xyz = c_2 \qquad \text{...(3}$$

Hence the general solution is

$$\phi\,(x^2 + y^2 - 2z,\ xyz) = 0.$$

Example 9. *Solve the following partial differential equation*

$$x(y^2 - z^2)\,p + y(z^2 - x^2)\,q = z(x^2 - y^2) \text{ where } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}.$$

Sol. Lagrange's subsidiary equations are

$$\frac{dx}{x\,(y^2 - z^2)} = \frac{dy}{y\,(z^2 - x^2)} = \frac{dz}{z\,(x^2 - y^2)} \qquad \text{...(1}$$

Using multipliers x, y, z, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore \quad x\,dx + y\,dy + z\,dz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = c_1 \qquad \text{...(2}$$

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\Rightarrow \quad xyz = c_2 \qquad ...(3)$$

Hence the general solution is

$$\phi\,(x^2 + y^2 + z^2, xyz) = 0$$

Example 10. *Solve:*

(*i*) $p + 3q = 5z + \tan(y - 3x)$ (A.K.T.U. 2017)

(*ii*) $(y^2 + z^2 - x^2)\,p - 2xyq + 2xz = 0$

(*iii*) $(y + zx)\,p - (x + yz)q = x^2 - y^2$ (U.P.T.U. 2014)

Sol. (*i*) Lagrange's subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)} \qquad ...(1)$$

From first and second fractions, we have

$$\frac{dx}{1} = \frac{dy}{3}$$

Integrating, we get

$$y - 3x = c_1 \qquad ...(2)$$

Now, from first and last fractions, we have

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$$

Integrating, we get

$$x = \frac{1}{5}\log(5z + \tan c_1) - \frac{c_2}{5}$$

$$\Rightarrow \quad \log\{5z + \tan(y - 3x)\} - 5x = c_2 \qquad ...(3)$$

Hence the general solution is

$$\phi\left[y - 3x, \log\{5z + \tan(y - 3x)\}\right] = 0$$

(*ii*) Lagrange's subsidiary equations are

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} \qquad ...(1)$$

Taking the last two fractions, we have

$$\frac{dy}{-2xy} = \frac{dz}{-2xz} \quad \Rightarrow \quad \frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get

$$\log y = \log z + \log c_1$$

$$\Rightarrow \quad \frac{y}{z} = c_1 \qquad \text{...(2)}$$

Using x, y, z as multipliers, we have

$$\text{each fraction} = \frac{xdx + ydy + zdz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{xdx + ydy + zdz}{-x(x^2 + y^2 + z^2)}$$

Let

$$\frac{xdx + ydy + zdz}{-x(x^2 + y^2 + z^2)} = \frac{dz}{-2xz}$$

$$\Rightarrow \quad \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

Integrating, we get

$$\log\left(x^2 + y^2 + z^2\right) = \log z + \log c_2$$

$$\Rightarrow \quad \frac{x^2 + y^2 + z^2}{z} = c_2$$

Hence the general solution is

$$\phi\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0.$$

(iii) Lagrange's subsidiary equations are

$$\frac{dx}{y + zx} = \frac{dy}{-(x + yz)} = \frac{dz}{x^2 - y^2}$$

Using multipliers x, y and $-z$, we get

$$\text{each fraction} = \frac{xdx + ydy - zdz}{xy + x^2z - yx - y^2z - zx^2 + zy^2} = \frac{xdx + ydy - zdz}{0}$$

$$\therefore \quad xdx + ydy - zdz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = \frac{c_1}{2}$$

$$\Rightarrow \quad x^2 + y^2 - z^2 = c_1 \qquad \text{...(1)}$$

Again, using multipliers $y, x, 1$; we get

$$\text{each fraction} = \frac{ydx + xdy + dz}{y^2 + yzx - x^2 - xyz + x^2 - y^2} = \frac{ydx + xdy + dz}{0}$$

$$\therefore \quad ydx + xdy + dz = 0$$

Integrating, we get

$$xy + z = c_2 \qquad ...(2)$$

Hence the general solution is

$$\phi(x^2 + y^2 - z^2, xy + z) = 0.$$

Example 11. *Solve:*

Try (*i*) $(x^2 - y^2 - yz)\, p + (x^2 - y^2 - zx)\, q = z\,(x - y)$

(*ii*) $(x^2 + xy)\, p - (xy + y^2)\, q = -(x - y)\,(2x + 2y + z)$

Sol. (*i*) Lagrange's subsidiary equations are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z\,(x - y)}$$

Using 1, –1, –1 as multipliers, we have

$$\text{each fraction} = \frac{dx - dy - dz}{x^2 - y^2 - yz - x^2 + y^2 + zx - zx + zy} = \frac{dx - dy - dz}{0}$$

$$\therefore \qquad dx - dy - dz = 0$$

Integrating, we get

$$x - y - z = c_1 \qquad ...(1)$$

Again, from first and second, we have (x, −y, 0)

$$\frac{xdx - ydy}{x^3 - xy^2 - xyz - yx^2 + y^3 + yzx} = \frac{dz}{z\,(x - y)}$$

$$\Rightarrow \qquad \frac{xdx - ydy}{x^3 - xy^2 - yx^2 + y^3} = \frac{dz}{z\,(x - y)}$$

$$\Rightarrow \qquad \frac{xdx - ydy}{(x - y)(x^2 - y^2)} = \frac{dz}{z\,(x - y)}$$

$$\Rightarrow \qquad \frac{xdx - ydy}{x^2 - y^2} = \frac{dz}{z}$$

Integrating, we get

$$\frac{1}{2}\log(x^2 - y^2) = \log z + \frac{1}{2}\log c_2$$

$$\Rightarrow \qquad \frac{x^2 - y^2}{z^2} = c_2 \qquad ...(2)$$

Hence the general solution is

$$\phi\left(x - y - z, \frac{x^2 - y^2}{z^2}\right) = 0$$

(*ii*) Lagrange's subsidiary equations are

$$\frac{dx}{x\,(x + y)} = \frac{dy}{-y\,(x + y)} = \frac{dz}{-(x - y)(2x + 2y + z)}$$

Taking first and second fractions, we have

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating, we get

$$\log x = -\log y + \log c_1$$

$$\Rightarrow \quad xy = c_1 \qquad \ldots(1)$$

Again from first and second,

$$\frac{dx + dy}{(x-y)(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

$$\Rightarrow \quad (2x + 2y + z)(dx + dy) + (x + y)\, dz = 0$$

$$\Rightarrow \quad (x + y + z)\, d(x + y) + (x + y)(dx + dy) + (x + y)\, dz = 0$$

$$\Rightarrow \quad (x + y + z)\, d(x + y) + (x + y)\, d(x + y + z) = 0$$

$$\Rightarrow \quad d\{(x + y)(x + y + z)\} = 0$$

Integrating, we get

$$(x + y)(x + y + z) = c_2$$

Hence the general solution is

$$\phi\{xy, (x + y)(x + y + z)\} = 0$$

Example 12. *Solve the partial differential equations:*

$$(i)\ \left(\frac{y-z}{yz}\right) p + \left(\frac{z-x}{zx}\right) q = \left(\frac{x-y}{xy}\right)$$

(ii) $(y^2 + z^2)\, p - xyq = -zx$ (A.K.T.U. 2016, 20017)

Sol. (i) The given equation can also be written as

$$x(y - z)\, p + y(z - x)\, q = z(x - y) \qquad \ldots(1)$$

Lagrange's subsidiary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Using multipliers as $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y - z + z - x + x - y} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_1 \quad \Rightarrow \quad xyz = c_1 \qquad \ldots(2)$$

Again, using 1, 1, 1 as multipliers, we get

$$\text{each fraction} = \frac{dx + dy + dz}{x(y-z) + y(z-x) + z(x-y)} = \frac{dx + dy + dz}{0}$$

$$\therefore \quad dx + dy + dz = 0$$

Integrating, we get

$$x + y + z = c_2$$

Hence the general solution is

$$\phi(xyz, x + y + z) = 0.$$

(*ii*) Lagrange's subsidiary eqns. are

$$\frac{dx}{y^2+z^2}=\frac{dy}{-xy}=\frac{dz}{-zx} \quad ...(1)$$

From second and third fractions,

$$\frac{dy}{y}=\frac{dz}{z}$$

Integrating, we get $\frac{y}{z}=c_1$

Now, using multipliers as x, y, z, we get

$$\text{each fraction} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$\therefore$ $x\,dx+y\,dy+z\,dz=0$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_2}{2} \Rightarrow x^2+y^2+z^2=c_2$$

Hence the general solution is

$$\phi\left(\frac{y}{z}, x^2+y^2+z^2\right)=0.$$

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $2p+3q=1$
2. $xp+yq=3z$
3. $(y-z)p+(x-y)q=z-x$
4. $p\tan x+q\tan y=\tan z$
5. $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$
6. $(z-y)p+(x-z)q=y-x$
7. $yzp+zxq=xy$
8. $(y+z)p+(z+x)q=x+y$
9. $2yzp+zxq=3xy$
10. $z(xp-yq)=y^2-x^2$
11. $2xzp+2yzq=z^2-x^2-y^2$
12. $(3-2yz)\,p+x(2z-1)\,q=2x\,(y-3)$
13. $x\,(z-3y^3)\,p+y\,(3x^3-z)\,q=3\,(y^3-x^3)\,z$ [**Hint:** Multipliers $\rightarrow (x^2, y^2, 1/3)$ and $(1/x, 1/y, 1/z)$]
14. $(z^2-2yz-y^2)\,p+x(y+z)\,q=x\,(y-z)$
15. $y^2\,(x+y)\,p+x^2\,(x+y)\,q=(x^2+y^2)\,z$ [**Hint:** Multipliers $\rightarrow (1, 1, 0)$ and $(x^2, -y^2, 0)$]
16. $(x+2y^2+z)\,p+y\,(2x-1)\,q+2\,(x^2+y^2+xz)=0$ [**Hint:** Multipliers $\rightarrow (x, -y, 1/2)$ and $(y, x+z, y)$]
17. $yp+xq=xyz^2\,(x^2-y^2)$ [*U.P.T.U. 2015*]
18. $(x-y)\,y^2p+(y-x)\,x^2\,q=(x^2+y^2)z$
19. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=xyz$
20. $x_1p_1+2x_2p_2+3x_3p_3+4x_4p_4=0$

Answers

1. $\phi(3x-2y, y-3z)=0$
2. $\phi\left(\frac{x}{y}, \frac{x^3}{z}\right)=0$
3. $\phi\left(x+y+z, \frac{x^2}{2}+yz\right)=0$
4. $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right)=0$
5. $\sqrt{x}-\sqrt{y}=f(\sqrt{x}-\sqrt{z})$
6. $x^2+y^2+z^2=f(x+y+z)$
7. $\phi(x^2-y^2, x^2-z^2)=0$
8. $\phi\left[\frac{x-y}{y-z}, (x-y)^2(x+y+z)\right]=0$
9. $\phi(x^2-2y^2, 3y^2-z^2)=0$
10. $\phi(xy, x^2+y^2+z^2)=0$
11. $\phi\left(\frac{y}{x}, \frac{x^2+y^2+z^2}{x}\right)=0$
12. $\phi(y^2-6y-z^2+z, x^2+z^2+6y)=0$
13. $\phi(x^3+y^3+z, xyz)=0$
14. $\phi(x^2+y^2+z^2, y^2-2yz-z^2)=0$
15. $\phi\left(x^3-y^3, \frac{x+y}{z}\right)=0$
16. $\phi(x^2-y^2+z, xy+yz)=0$
17. $\phi\left\{x^2-y^2, \frac{x^4}{4}-\frac{y^4}{4}+\frac{1}{z}\right\}=0$
18. $\phi\left\{x^3+y^3, \frac{x-y}{z}\right\}=0$
19. $\phi\left(\frac{x}{y}, \frac{y}{z}, xyz-3u\right)=0$
20. $z=\phi\left(\frac{x_1^2}{x_2}, \frac{x_1^3}{x_3}, \frac{x_1^4}{x_4}\right)$

1.12 NON-LINEAR PDE OF FIRST ORDER

A partial differential equation which involves first order partial derivatives p and q with degree higher than one and the products of p and q is called a non-linear partial differential equation of the first order. The complete solution of such an equation involves only two arbitrary constants (*i.e.*, equal to the number of independent variables). There is a general method for solving such equations. Before giving this method, we shall deal with some special types of such equations which can be solved easily by methods other than the general method.

1.12.1. Equations of the Form $f(p, q)=0$, *i.e.*,

*equations involving only **p** and **q** and no **x, y, z***

The complete solution is $z = ax + by + c$ ...(1)

where a and b are connected by the relation

$$f(a, b) = 0 \quad ...(2) \quad \left[\text{since } p=\frac{\partial z}{\partial x}=a \text{ and } q=\frac{\partial z}{\partial y}=b\right]$$

From (2), we can find b in terms of a. Let $b=\phi(a)$.

Putting this value of b in (1), the complete solution is

$z = ax + \phi(a)y + c$, where a and c are arbitrary constants.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:*

(*i*) $\sqrt{p} + \sqrt{q} = 1$ (*ii*) $pq = p + q$.

Sol. (*i*) The equation is of the form $f(p, q) = 0$

The complete solution is $z = ax + by + c$ ...(1)

where $\sqrt{a} + \sqrt{b} = 1$ or $b = (1 - \sqrt{a})^2$

$\therefore$ From (1), the complete solution is

$$z = ax + (1 - \sqrt{a})^2 y + c$$

(*ii*) The equation is of the form $f(p, q) = 0$

The complete solution is $z = ax + by + c$ ...(1)

where $ab = a + b$ or $b = \dfrac{a}{a-1}$

$\therefore$ From (1), the complete solution is $z = ax + \dfrac{a}{a-1} y + c$.

Example 2. *Solve* $x^2p^2 + y^2q^2 = z^2$.

Sol. The given equation can be written as

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{...(1)}$$

Let $\dfrac{\partial x}{x} = \partial X$, $\dfrac{\partial y}{y} = \partial Y$, $\dfrac{\partial z}{z} = \partial Z$ then $X = \log x$, $Y = \log y$, $Z = \log z$

$$\frac{\partial Z}{\partial X} = \frac{x}{z} \cdot \frac{\partial z}{\partial x} \text{ and } \frac{\partial Z}{\partial Y} = \frac{y}{z} \cdot \frac{\partial z}{\partial y}$$

$\therefore$ Eqn. (1) can be written as $\left(\dfrac{\partial Z}{\partial X}\right)^2 + \left(\dfrac{\partial Z}{\partial Y}\right)^2 = 1$

or $P^2 + Q^2 = 1$, where $P = \dfrac{\partial Z}{\partial X}$ and $Q = \dfrac{\partial Z}{\partial Y}$

It is of the form $f(P, Q) = 0$

Its complete solution is $Z = aX + bY + c$ ...(2)

where $a^2 + b^2 = 1$ or $b = \sqrt{1 - a^2}$

$\therefore$ From (2), the complete solution is $\log z = a \log x + \sqrt{1-a^2} \log y + c$.

Example 3. *Solve* $(y - x)(qy - px) = (p - q)^2$.

Sol. Let $x + y = X$ and $xy = Y$

so that
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + y \frac{\partial z}{\partial Y}$$

and
$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} + x \frac{\partial z}{\partial Y}$$

$$\therefore \qquad qy - px = (y - x)\frac{\partial z}{\partial X} \text{ and } p - q = (y - x)\frac{\partial z}{\partial Y}$$

$\therefore$ The given equation can be written as $\frac{\partial z}{\partial X} = \left(\frac{\partial z}{\partial Y}\right)^2$

or $$P = Q^2, \text{ where } P = \frac{\partial z}{\partial X} \text{ and } Q = \frac{\partial z}{\partial Y}$$

It is of the form $f(P, Q) = 0$

Its complete solution is $z = aX + bY + c$ ...(1)

where $a = b^2$ or $b = \sqrt{a}$

$\therefore$ From (1), the complete solution is $z = a(x + y) + \sqrt{a}\, xy + c.$

1.12.2. Equations of the Form $z = px + qy + f(p, q)$

The complete solution is $z = ax + by + f(a, b)$

obtained by writing a for p and b for q.

Example 4. *Solve:* $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Sol. The given equation is of the form $z = px + qy + f(p, q)$

Its complete solution is $$z = ax + by + \sqrt{1 + a^2 + b^2}.$$

Example 5. *Find the singular integral of* $z = px + qy + pq$.

Sol. The equation is of the form

$$z = px + qy + f(p, q)$$

Its complete integral is

$$z = ax + by + ab \qquad ...(1)$$

Diff. (1) partially w.r.t. a and b,

$$0 = x + b \quad \text{and} \quad 0 = y + a \qquad ...(2)$$

Eliminating a and b between eqns. (1) and (2), we get

$$z = -xy - xy + xy$$

$$\Rightarrow \qquad z = -xy$$

which is the required singular solution, for it satisfies the given equation.

Example 6. *Solve:* $4xyz = pq + 2px^2y + 2qxy^2$.

Sol. Let $x^2 = X$ and $y^2 = Y$

so that $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = 2x\frac{\partial z}{\partial X} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = 2y\frac{\partial z}{\partial Y}$$

$\therefore$ The given equation becomes

$$4xyz = 4xy\frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y} + 4x^3y\frac{\partial z}{\partial X} + 4xy^3\frac{\partial z}{\partial Y}$$

or $$z = x^2\frac{\partial z}{\partial X} + y^2\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y} = X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y}$$

or $$z = PX + QY + PQ, \quad \text{where } P = \frac{\partial z}{\partial X} \quad \text{and} \quad Q = \frac{\partial z}{\partial Y}$$

It is of the form $z = PX + QY + f(P, Q)$

Its complete solution is $z = aX + bY + ab$ or $z = ax^2 + by^2 + ab$.

1.12.3. Equations of the Form $f(z, p, q) = 0$,

i.e., equations not containing x and y.

Assume $z = \phi(x + ay) = \phi(u)$, where $u = x + ay$ as a trial solution of the given equation.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \phi'(x + ay) = \phi'(u) = \frac{dz}{du}$$

$$q = \frac{dz}{dy} = \phi'(x + ay)a = a\phi'(u) = a\frac{dz}{du}$$

Substituting these values of p and q in the given equation, we get $f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$ which is an ordinary differential equation of the first order. Integrating it, we get the complete solution.

Method. (*i*) Assume $u = x + ay$ so that $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

(*ii*) Substitute these values of p and q in the given equation.

(*iii*) Solve the resulting ordinary differential equation in z and u.

(*iv*) Replace u by $x + ay$.

Example 7. *Solve:* $z^2(p^2 + q^2 + 1) = a^2$.

Sol. The given equation is of the form $f(z, p, q) = 0$

Let $u = x + by$ (note the use of b instead of a, since a is a given constant)

so that $$p = \frac{dz}{du} \quad \text{and} \quad q = b\frac{dz}{du}.$$

Substituting these values of p and q in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + b^2\left(\frac{dz}{du}\right)^2 + 1\right] = a^2$$

or $$z^2(1 + b^2)\left(\frac{dz}{du}\right)^2 = a^2 - z^2$$

or $$z\sqrt{1+b^2}\,\frac{dz}{du} = \pm\sqrt{a^2 - z^2}$$

or $$\pm\sqrt{1+b^2}\,.\,\frac{z}{\sqrt{a^2 - z^2}}\,dz = du$$

Integrating, we have

$$\pm\sqrt{1+b^2}\,\sqrt{a^2 - z^2} = u + c$$

or $$(1 + b^2)(a^2 - z^2) = (x + by + c)^2$$

Example 8. *Solve:* $z^2(p^2x^2 + q^2) = 1$.

Sol. The given equation can be written as

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = 1 \quad \text{...(1)}$$

Let $\quad X = \log x$

so that $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}\cdot\dfrac{\partial X}{\partial x} = \dfrac{\partial z}{\partial X}\cdot\dfrac{1}{x} \quad$ *i.e.,* $\quad x\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}$

$\therefore$ Equation (1) reduces to $z^2\left[\left(\dfrac{\partial z}{\partial X}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right] = 1$

or $$z^2(P^2 + q^2) = 1, \text{ where } P = \frac{\partial z}{\partial X} \quad \text{...(2)}$$

Let $\quad u = X + ay \quad$ so that $\quad P = \dfrac{dz}{du} \quad$ and $\quad q = a\dfrac{dz}{du}$

$\therefore$ From (2), we have $z^2\left[\left(\dfrac{dz}{du}\right)^2 + a^2\left(\dfrac{dz}{du}\right)^2\right] = 1$

or $$(1 + a^2)z^2\left(\frac{dz}{du}\right)^2 = 1 \quad \text{or} \quad \sqrt{1+a^2}\,.\,z\,dz = \pm\, du$$

Integrating, $\quad \sqrt{1+a^2}\,.\,\dfrac{z^2}{2} = \pm\, u + b$

or $$\sqrt{1+a^2}\cdot z^2 = \pm\, 2(X + ay) + 2b$$

or $$\sqrt{1+a^2}\cdot z^2 = \pm\, 2(\log x + ay) + c$$

1.12.4. Equations of the Form $f_1(x, p) = f_2(y, q)$

*i.e., equations in which **z** is absent and the terms involving **x** and **p** can be separated from those involving **y** and **q**.*

As a trial solution, let us put each side equal to an arbitrary constant a. Then

$$f_1(x, p) = f_2(y, q) = a$$

Solving these equations for p and q, let $p = F_1(x)$ and $q = F_2(y)$

Since $\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = pdx + qdy$

$\therefore \quad dz = F_1(x)\,dx + F_2(y)\,dy$

Integrating, $\quad z = \int F_1(x)\,dx + \int F_2(y)\,dy + b$

Example 9. *Solve:* $p^2 - q^2 = x - y$.

Sol. The given equation is $p^2 - x = q^2 - y$ which is of the form $f_1(x, p) = f_2(y, q)$

Let $\quad p^2 - x = q^2 - y = a$

then $\quad p^2 = x + a$ and $q^2 = y + a$ *i.e.,* $p = \sqrt{x+a}$ and $q = \sqrt{y+a}$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{x+a}\, dx + \sqrt{y+a}\, dy$$

Integrating, $z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b$

Example 10. *Solve:* $yp = 2yx + \log q$.

Sol. The given equation can be written as

$$p = 2x + \frac{1}{y}\log q \quad \text{or} \quad p - 2x = \frac{1}{y}\log q$$

which is of the form $f_1(x, p) = f_2(y, q)$.

Let $\quad p - 2x = \frac{1}{y}\log q = a$ then $p = 2x + a$ and $\log q = ay$, *i.e.,* $q = e^{ay}$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (2x + a)\, dx + e^{ay}\, dy$$

Integrating, $z = x^2 + ax + \frac{1}{a}e^{ay} + b$

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $p^2 + q^2 = 2$
2. $p^3 - q^3 = 0$
3. $p^2 + p = q^2$
4. $p = e^q$
5. $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$
6. $z = px + qy - 2\sqrt{pq}$
7. $z = px + qy + \sin(p + q)$
8. $(pq - p - q)(z - px - qy) = pq$
9. $p^2 = zq$
10. $p(1 + q^2) = q(z - a)$
11. $p(1 + q) = qz$
12. $z^2(p^2 + q^2 + 1) = 1$
13. $z = p^2 + q^2$
14. $p + q = z$
15. $z^2(p^2z^2 + q^2) = 1$
16. $9(p^2z + q^2) = 4$
17. $p^2 + q^2 = x + y$
18. $p - x^2 = q + y^2$
19. $\sqrt{p} + \sqrt{q} = x + y$
20. $\sqrt{p} + \sqrt{q} = 2x$
21. $pe^y = qe^x$
22. $q = xyp^2$
23. $q(p - \cos x) = \cos y$
24. $yp + xq + pq = 0$
25. $z(p^2 - q^2) = x - y$
26. $p + q = \sin x + \sin y$
27. $z^2(p^2 + q^2) = x^2 + y^2$.

Answers

1. $z = ax + \sqrt{2 - a^2}\, y + c$
2. $z = a(x + y) + c$
3. $z = ax + \sqrt{a^2 + a}\, y + c$
4. $z = ax + y \log a + c$
5. $z = a\sqrt{x + y} + \sqrt{1 - a^2}\sqrt{x - y} + c$
6. $z = ax + by - 2\sqrt{ab}$
7. $z = ax + by + \sin(a + b)$
8. $z = ax + by + \dfrac{ab}{ab - a - b}$
9. $z = be^{ax + a^2y}$
10. $4(bz - ab - 1) = (x + by + c)^2$
11. $\log(az - 1) = x + ay + b$

12. $(1 + a^2)(1 - z^2) = (x + ay + b)^2$

13. $4z(1 + a^2) = (x + ay + b)^2$

14. $(1 + a) \log z = x + ay + b$

15. $(z^2 + a^2)^3 = 9(x + ay + b)^2$

16. $(z + a^2)^3 = (x + ay + b)^2$

17. $z = \frac{2}{3}(x + a)^{3/2} + \frac{2}{3}(y - a)^{3/2} + b$

18. $z = \frac{1}{3}(x^3 - y^3) + a(x + y) + b$

19. $z = \frac{1}{3}(x + a)^3 + \frac{1}{3}(y - a)^3 + b$

20. $z = \frac{1}{6}(2x + a)^3 + a^2 y + b$

21. $z = ae^x + ae^y + b$

22. $z = 2\sqrt{ax} + \frac{1}{2} ay^2 + b$

23. $z = ax + \sin x + \frac{1}{a} \sin y + b$

24. $2z = ax^2 - \frac{a}{a+1} y^2 + b$

25. $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$

26. $z = a(x - y) - (\cos x + \cos y) + c$

27. $z^2 = x\sqrt{x^2 + a} + y\sqrt{y^2 - a} + a \log \dfrac{x + \sqrt{x^2 + a}}{y + \sqrt{y^2 - a}} + c.$

1.13 CHARPIT'S METHOD

This is a general method for finding the complete solution of non-linear partial differential equations of the first order.

Let the given equation be $f(x, y, z, p, q) = 0$ ...(1)

If we can find another relation $F(x, y, z, p, q) = 0$ ...(2)

involving x, y, z, p and q, then we can solve (1) and (2) for p and q and substitute in

$$dz = p\,dx + q\,dy \qquad ...(3)$$

Solution of (3), if it exists, is the complete solution of (1).

To determine F, we differentiate (1) and (2) partially w.r.t. x and y. Thus

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot p + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} = 0 \qquad ...(4)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot p + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} = 0 \qquad ...(5)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot q + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y} = 0 \qquad ...(6)$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot q + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial y} = 0 \qquad ...(7)$$

Eliminating $\frac{\partial p}{\partial x}$ between (4) and (5), we get

$$\left(\frac{\partial f}{\partial x} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial x} \cdot \frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial q} \cdot \frac{\partial f}{\partial p}\right) \frac{\partial q}{\partial x} = 0 \qquad ...(8)$$

Eliminating $\frac{\partial q}{\partial y}$ between (6) and (7), we get

$$\left(\frac{\partial f}{\partial y} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial y} \cdot \frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial q}\right) q + \left(\frac{\partial f}{\partial p} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial p} \cdot \frac{\partial f}{\partial q}\right) \frac{\partial p}{\partial y} = 0 \qquad ...(9)$$

Since $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial p}{\partial y}$, the last terms in (8) and (9) differ in sign only. Adding (8) and (9), we get

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0 \quad \text{...(10)}$$

which is a linear partial differential equation of the first order with x, y, z, p, q as independent variables and F as the dependent variable.

$\therefore$ The auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \quad \text{...(11)}$$

Any integral of (11) which involves p or q or both can be taken as the assumed relation (2). In practice, we choose the simplest of the integrals of (11).

ILLUSTRATIVE EXAMPLES

(ii) $px + qy = pq$

Ans $az = \frac{(y+ax)^2}{2} + b$

Example 1. *Solve: $2zx - px^2 - 2pxy + pq = 0$.*

Sol. Here $f \equiv 2zx - px^2 - 2qxy + pq = 0$ ...(1)

$$\therefore \quad \frac{\partial f}{\partial x} = 2z - 2px - 2qy, \; \frac{\partial f}{\partial y} = -2qx, \; \frac{\partial f}{\partial z} = 2x, \; \frac{\partial f}{\partial p} = -x^2 + q, \; \frac{\partial f}{\partial q} = -2xy + p$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

or

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dF}{0}$$

$$\therefore \quad dq = 0 \quad \text{or} \quad q = a$$

Putting $q = a$ in (1), we get $\quad p = \frac{2x(z - ay)}{x^2 - a}$

$$\therefore \quad dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a}dx + ady$$

or

$$\frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}dx$$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$ or $z - ay = b(x^2 - a)$

$\therefore \quad z = ay + b(x^2 - a)$ which is the required complete integral.

Example 2. *Solve: $(p^2 + q^2)y = qz$.* (M.T.U. 2013)

Sol. Here $f \equiv (p^2 + q^2)y - qz = 0$ ...(1)

$$\therefore \quad \frac{\partial f}{\partial x} = 0, \; \frac{\partial f}{\partial y} = p^2 + q^2, \; \frac{\partial f}{\partial z} = -q, \; \frac{\partial f}{\partial p} = 2py, \; \frac{\partial f}{\partial q} = 2qy - z$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0}$$

or
$$\frac{dp}{-pq}=\frac{dq}{p^2}=\frac{dz}{-qz}=\frac{dx}{-2py}=\frac{dy}{-2qy+z}=\frac{dF}{0}$$

From the first two members, we have $pdp + qdq = 0$

Integrating, $p^2 + q^2 = a^2$ ...(2)

Putting in (1), $q = \dfrac{a^2 y}{z}$

$\therefore$ From (2), $p=\sqrt{a^2-q^2}=\sqrt{a^2-\dfrac{a^4y^2}{z^2}}=\dfrac{a}{z}\sqrt{z^2-a^2y^2}$

$$\therefore \quad dz = pdx + qdy = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

or
$$zdz - a^2ydy = a\sqrt{z^2-a^2y^2}\,dx \text{ or } \frac{\frac{1}{2}d(z^2-a^2y^2)}{\sqrt{z^2-a^2y^2}} = a\,dx$$

Integrating, we get $\sqrt{z^2-a^2y^2} = ax + b$

or
$$z^2 = (ax+b)^2 + a^2y^2$$

TEST YOUR KNOWLEDGE

Solve the following equations by Charpit's method:

1. $pz + qy = pq$
2. $z^2 = pqxy$
3. $pxy + pq + qy = yz$
4. $z = p^2x + q^2y$
5. $2(z + xp + yp) = yp^2$
6. $q + xp = p^2$.

Answers

1. $az = \dfrac{1}{2}(y + ax)^2 + b$
2. $z = ax^b y^{1/b}$
3. $\log(z - ax) = y - a\log(a + y) + b$
4. $\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$
5. $z = \dfrac{ax}{y^2} - \dfrac{a^2}{4y^3} + \dfrac{b}{y}$
6. $z = axe^{-y} - \dfrac{1}{2}a^2e^{-2y} + b$.

1.14 CAUCHY'S METHOD OF CHARACTERISTICS

Consider a first order partial differential equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f(x, y) + ku \quad ; \quad u(0, y) = h(y) \qquad \text{...(1)}$$

Here a, b and f depend on x, y and u but not on the derivatives of u.

Let $u(x, y)$ be the solution of eq. (1) then by chain rule,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \text{...(2)}$$

From (1) and (2), we get

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x, y) + ku} \quad ...(3)$$

Taking,

$$\frac{dx}{a} = \frac{dy}{b} \quad ...(4)$$

We get a solution,

$$bx - ay = C \quad ...(5)$$

Where c is a constant.
Now, consider

$$\frac{dx}{a} = \frac{du}{f(x, y) + ku} \quad ...(6)$$

$$\Rightarrow \quad \frac{dx}{a} = \frac{du}{f\left(x, \frac{bx - c}{a}\right) + ku} \quad ...(7)$$

Eqn. (7) can be written as

$$\frac{du}{dx} - \frac{k}{a}u = \frac{1}{a} f\left(x, \frac{bx - c}{a}\right) \quad ...(8)$$

which is a first order linear ordinary differential equation.

$$\text{I.F.} = e^{-\frac{k}{a}x}$$

Solution of eqn. (8) is of the form,

$$u = G(x, c) + c_1 \quad ...(9)$$

Where $c_1 = g(c)$; g is an arbitrary function.
Hence eqn. (9) is

$$u = G(x, c) + g(c)$$

$g(c)$ can be determined by using the given conditions.

ILLUSTRATIVE EXAMPLES

Example 1. *Use Cauchy's method of characteristics to solve the partial differential equaiton*

$$\frac{\partial u}{\partial x} + \frac{du}{\partial y} = x + y \quad ; \qquad u(x, 0) = 0$$

Sol. The system of equations is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{x + y} \quad ...(1)$$

Taking the first two, we get

$$x - y = c \quad ...(2)$$

Where c is a constant.
Now, taking,

$$\frac{dy}{1} = \frac{du}{x + y}$$

$$\Rightarrow \qquad \frac{dy}{1} = \frac{du}{2y+c}$$

We get,
$$u(x, y) = y^2 + cy + c_1 \qquad \text{...(3)}$$

Let
$$c_1 = g(c) \text{ then,}$$

From (3),
$$u(x, y) = y^2 + cy + g(c)$$
$$= y^2 + y(x-y) + g(x-y) \qquad \text{...(4)}$$

where $g(x - y)$ is an arbitrary function.

Applying the condition $u(x, 0) = 0$ on eq. (4), we get
$$0 = g(x)$$
$$\therefore \qquad g(x-y) = 0$$

Consequently, the solution is,
$$u(x, y) = xy$$

Example 2. *Use the method of characteristics to solve:*
$$u_x - u_y = 0 \quad ; \qquad u(x, 0) = x$$

Sol. We have,
$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{0} \qquad \text{...(1)}$$

From the first two, we get
$$x + y = c \qquad \text{...(2)}$$

where c is a constant.

Also, from eq. (1)
$$du = 0$$
$$\therefore \qquad u(x, y) = c_1$$

Hence,
$$u(x, y) = c_1 = g(c) = g(x + y) \qquad \text{...(3)}$$

where $g(x + y)$ is an arbitrary function.

From the given condition,
$$u(x, 0) = x = g(x)$$
$$\therefore \qquad g(x + y) = x + y$$

Therefore, from eq. (3)
$$u(x, y) = x + y$$

Example 3. *Use the method of characteristics to solve the first order PDE.*
$$u_x - yu = 0 \qquad ; \qquad u(0, y) = 1$$

Sol. We have,
$$\frac{dx}{1} = \frac{du}{yu} = \frac{dy}{0} \qquad \text{...(1)}$$

One solution is,
$$y = c \qquad \text{...(2)}$$

Taking the first two,

$$\frac{dx}{1} = \frac{du}{uc}$$

Integration yields,

$$\log u = cx + g(c)$$

or $$\log u = cx + g(y) \quad ...(3)$$

Applying the condition, we get

$$0 = g(y)$$

Hence the solution is,

$$u = e^{xy}$$

TEST YOUR KNOWLEDGE

Use Cauchy's method of characteristics to solve the following first order partial differential equations:

1. $u_x + u_y = u$, $\quad u(x, 0) = 1 + e^x$
2. $x u_x + u_y = x \sinh y + u$, $\quad u(0, y) = 0$
3. $u_x + yu = 0$, $\quad u(0, y) = 1$
4. $u_x + u_y = 2x + 2y$, $\quad u(x, 0) = x^2$
5. $u_x + u_y = 2u$, $\quad u(x, 0) = e^x$
6. $u_x + yu_y = 2u$, $\quad u(0, y) = y$
7. $x u_x + u_y = 2u$, $\quad u(x, 0) = x$
8. $u_x + u_y = \sinh x + \sinh y$, $\quad u(0, y) = 1 + \cosh y$
9. $u_x + u_y = 1 + \cos y$, $\quad u(0, y) = \sin y$
10. $u_x - u_y = 2$, $\quad u(0, y) = -y$.

Answers

1. $u = e^x + e^y$
2. $u = x \cosh y$
3. $u = e^{-xy}$
4. $u = x^2 + y^2$
5. $u = e^{x+y}$
6. $u = y e^x$
7. $u = x e^y$
8. $u = \cosh x + \cosh y$
9. $u = x + \sin y$
10. $u = x - y$

1.15 PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

An equation is said to be a partial differential equation of the second order when it includes at least one of the partial derivatives r, s, t but none of the higher order.

1.16 LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together is called a linear partial differential equation.

General form of such an equation is

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots\ldots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}}$$
$$+ B_1 \frac{\partial^{n-1} z}{\partial x^{n-2} \partial y} + \ldots\ldots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}} + C_0 \frac{\partial z}{\partial x} + C_1 \frac{\partial z}{\partial y} + P_0 z = F(x, y) \quad ...(1)$$

where the coefficients $A_0, A_1, \ldots\ldots, A_n; B_0, B_1, \ldots\ldots, B_{n-1}; C_0, C_1, P_0$ are constants or functions of x and y.

If these coefficients are constants, then eqn. (1) is called as linear partial differential equation with constant coefficients.

If all the derivatives appearing in eqn. (1) are of the same order then the differential equation is called a linear homogeneous p.d.e. with constant coefficients while if all the derivatives are not of same order, then it is called a non-homogeneous linear p.d.e. with constant coefficients.

1.17 HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form $a_0 \dfrac{\partial^n z}{\partial x^n} + a_1 \dfrac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \dfrac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots\ldots + a_n \dfrac{\partial^n z}{\partial y^n} = F(x, y)$ ...(1)

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants, is called a *homogeneous linear partial differential equation of the n^{th} order with constant coefficients.*

Here, all the partial derivatives are of the n^{th} order

Writing D for $\dfrac{\partial}{\partial x}$ and D′ for $\dfrac{\partial}{\partial y}$, eqn. (1) can be written as

$$(a_0 D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \cdots + a_n D'^n)z = F(x, y)$$

or
$$\boxed{\phi(D, D')z = F(x, y)} \qquad \ldots(2)$$

As in the case of ordinary linear differential equations with constant coefficients, the complete solution of (2) consists of two parts:

***(i)* the complementary function** (C.F.) which is the complete solution of the equation $\phi\,(D, D')z = 0$. It must contain n arbitrary functions, where n is the order of the differential equation.

***(ii)* the particular integral** (P.I.) which is a particular solution (free from arbitrary constants) of $\phi(D, D')\, z = F(x, y)$

The complete solution of (2) is $\boxed{z = \text{C.F.} + \text{P.I.}}$

1.18 RULES FOR FINDING C.F.

We explain the procedure for finding the complementary function of a differential equation of the second order. It can be easily extended to differential equations of higher order.

Consider the equation $\dfrac{\partial^2 z}{\partial x^2} + a_1 \dfrac{\partial^2 z}{\partial x \partial y} + a_2 \dfrac{\partial^2 z}{\partial y^2} = 0$

which in symbolic form is $(D^2 + a_1 DD' + a_2 D'^2)z = 0$ ...(1)

Its symbolic operator equated to zero, *i.e.*, $D^2 + a_1 DD' + a_2 D'^2 = 0$ ...(2)

is called the **auxiliary equation** (A.E.)

Considered as a quadratic in D/D′, let its roots be m_1, m_2.

Case I. *When the A.E. has distinct roots* *i.e.*, $m_1 \neq m_2$, then (2) can be written as

$$(D - m_1D')(D - m_2D')z = 0 \qquad ...(3)$$

Now the solution of $(D - m_2D')z = 0$ will also be a solution of (3).

But $\qquad (D - m_2D')z = 0$

$\Rightarrow \qquad p - m_2q = 0$

which is of Lagrange's form and the auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m_2} = \dfrac{dz}{0}$

The first two members give $dy + m_2dx = 0$ or $y + m_2x = a$

Also, $\qquad dz = 0$ or $z = b$

$\therefore \quad z = f_2(y + m_2x)$ is a solution of $(D - m_2D')z = 0$

Similarly, (3) will also be satisfied by the solution of

$$(D - m_1D')z = 0 \quad i.e., \quad \text{by } z = f_1(y + m_1x).$$

Hence the complete solution of (2) is

$$\boxed{z = f_1(y + m_1x) + f_2(y + m_2x)}$$

Case II. *When the A.E. has equal roots,* each $= m$, then (2) can be written as

$$(D - mD')(D - mD')z = 0 \qquad ...(4)$$

Let $(D - mD')z = u$, then (4) becomes $(D - mD')u = 0$

Its solution is $u = f(y + mx)$, as proved in Case I.

$\therefore \qquad (D - mD')z = u$ takes the form $(D - mD')z = f(y + mx)$

or $\qquad p - mq = f(y + mx) \qquad$ [Lagrange's form]

The Auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{f(y + mx)}$

Now, $\qquad \dfrac{dx}{1} = \dfrac{dy}{-m}$ gives $dy + mdx = 0$ or $y + mx = a$

Also, $\qquad \dfrac{dx}{1} = \dfrac{dz}{f(a)}$ gives $dz = f(a)\,dx$

or $\qquad z = f(a)\,x + b \quad i.e., \quad z - xf(y + mx) = b$

$\therefore$ The complete solution of eqn. (2) is

$$z - x\,f(y + mx) = \phi(y + mx)$$

or

$$\boxed{z = \phi(y + mx) + xf(y + mx)}$$

Note 1. Auxiliary eqn. of $\phi(D, D')z = F(x, y)$ is obtained by putting $D = m$ and $D' = 1$ in $\phi(D, D') = 0$.

Hence the A.E. is $\phi(m, 1) = 0$.

Note 2. Generalising the results of Case I and Case II, we have

(*i*) if the roots of A.E. are $m_1, m_2, m_3, \ldots\ldots$ (all distinct roots), then

$$\text{C.F.} = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \cdots\cdots$$

(*ii*) if the roots of A.E. are $m_1, m_1, m_2, \ldots$ (two equal roots), then

$$\text{C.F.} = f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_2x) + \ldots\ldots$$

(*iii*) if the roots of A.E. are $m_1, m_1, m_1, \ldots$ (three equal roots), then

$$\text{C.F.} = f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \ldots\ldots$$

Note 3. Corresponding to a non-repeated factor D′ on L.H.S., the part of C.F. is taken as $\phi(x)$ and for D'^2 on LHS, the part of C.F. is $y\,\phi(x) + \psi(x)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:*

(*i*) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = 0$ (*ii*) $(D + 2D')(D - 3D')^2 z = 0$ (*iii*) $4r - 12s + 9t = 0$.

Sol. (*i*) The given equation is

$$(D^2 - DD' - 6D'^2)z = 0, \quad \text{where } D \equiv \frac{\partial}{\partial x} \text{ and } D' \equiv \frac{\partial}{\partial y}$$

Auxiliary equation is

$$m^2 - m - 6 = 0$$

$$\Rightarrow \quad (m - 3)(m + 2) = 0 \Rightarrow m = 3, -2$$

$$\therefore \quad \text{C.F.} = f_1(y + 3x) + f_2(y - 2x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + 3x) + f_2(y - 2x)$$

where f_1 and f_2 are arbitrary functions.

(*ii*) Auxiliary equation is

$$(m + 2)(m - 3)^2 = 0 \Rightarrow m = -2, 3, 3$$

$$\therefore \quad \text{C.F.} = f_1(y - 2x) + f_2(y + 3x) + xf_3(y + 3x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y - 2x) + f_2(y + 3x) + xf_3(y + 3x)$$

where f_1, f_2 and f_3 are arbitrary functions.

(*iii*) The given equation is

$$4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$$

$$\Rightarrow \quad (4D^2 - 12DD' + 9D'^2)z = 0$$

Auxiliary equation is $\quad 4m^2 - 12m + 9 = 0$

$$\Rightarrow \quad (2m - 3)^2 = 0 \Rightarrow m = \frac{3}{2}, \frac{3}{2}$$

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{3}{2}x\right) + xf_2\left(y + \frac{3}{2}x\right)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1\left(y + \frac{3}{2}x\right) + x f_2\left(y + \frac{3}{2}x\right)$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve:*

(*i*) $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 2\dfrac{\partial^3 z}{\partial x \partial y^2} = 0$ (*ii*) $r = a^2 t$

(*iii*) $(D^3D'^2 + D^2D'^3)\, z = 0$ (*iv*) $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0.$

Sol. (*i*) The given equation is

$$(D^3 - 3D^2D' + 2DD'^2)z = 0$$

The auxiliary equation is $m^3 - 3m^2 + 2m = 0$

$\Rightarrow$ $m(m-1)(m-2) = 0$

$\Rightarrow$ $m = 0, 1, 2$

$\therefore$ $\text{C.F.} = f_1(y) + f_2(y + x) + f_3(y + 2x)$

$\text{P.I.} = 0$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(y + x) + f_3(y + 2x)$.

where f_1, f_2 and f_3 are arbitrary functions.

(*ii*) The given equation is $(D^2 - a^2D'^2)z = 0$

The auxiliary equation is $m^2 - a^2 = 0 \Rightarrow m = \pm a$

$\therefore$ $\text{C.F.} = f_1(y + ax) + f_2(y - ax)$

$\text{P.I.} = 0$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + ax) + f_2(y - ax).$$

where f_1 and f_2 are arbitrary functions.

(*iii*) The given equation is $D^2D'^2(D + D')z = 0$

$\therefore$ $\text{C.F.} = f_1(y) + xf_2(y) + f_3(x) + yf_4(x) + f_5(y - x)$

$\text{P.I.} = 0$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + xf_2(y) + f_3(x) + yf_4(x) + f_5(y - x).$$

where f_1, f_2, f_3, f_4 and f_5 are arbitrary functions.

(*iv*) The given equation is $(D^4 - D'^4)z = 0$

The auxiliary equation is $m^4 - 1 = 0$

$\Rightarrow$ $(m^2 - 1)(m^2 + 1) = 0$

$\Rightarrow$ $m = \pm 1, \pm i$

$\therefore$ $\text{C.F.} = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$

$\text{P.I.} = 0$

$\therefore$ The complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix).$$

where f_1, f_2, f_3 and f_4 are arbitrary functions.

Example 3. *Solve:*

(*i*) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$ (*ii*) $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$

(*iii*) $r - 4s + 4t = 0$.

Sol. (*i*) Auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\Rightarrow \quad (m-1)(m-2)(m-3) = 0$$

$$\Rightarrow \quad m = 1, 2, 3$$

$$\therefore \quad \text{C.F.} = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x)$$

where f_1, f_2 and f_3 are arbitrary functions.

(*ii*) Auxiliary equation is

$$m^3 - 6m^2 + 12m - 8 = 0$$

$$\Rightarrow \quad (m-2)^3 = 0 \quad \Rightarrow \quad m = 2, 2, 2$$

$$\therefore \quad \text{C.F.} = f_1(y + 2x) + x\,f_2(y + 2x) + x^2 f_3(y + 2x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + 2x) + x\,f_2(y + 2x) + x^2 f_3(y + 2x)$$

where f_1, f_2 and f_3 are arbitrary functions.

(*iii*) Auxiliary equation is

$$m^2 - 4m + 4 = 0 \quad \Rightarrow \quad m = 2, 2$$

$$\therefore \quad \text{C.F.} = f_1(y + 2x) + x\,f_2(y + 2x)$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + 2x) + x\,f_2(y + 2x)$$

where f_1 and f_2 are arbitrary functions.

Example 4. *Solve the linear partial differential equation* $\dfrac{\partial^4 z}{\partial x^4} + \dfrac{\partial^4 z}{\partial y^4} = 0$.

Sol. The given equation is $(D^4 + D'^4)z = 0$

Auxiliary equation is $m^4 + 1 = 0$

$$\Rightarrow \quad m^4 + 1 + 2m^2 = 2m^2$$

$$(m^2 + 1)^2 - (m\sqrt{2})^2 = 0$$

$$\Rightarrow \quad (m^2 + \sqrt{2}m + 1)(m^2 - \sqrt{2}m + 1) = 0$$

$$\Rightarrow \quad m^2 + \sqrt{2}m + 1 = 0 \quad \text{or} \quad m^2 - \sqrt{2}m + 1 = 0$$

so that

$\Rightarrow$ $$m = \frac{-1 \pm i}{\sqrt{2}}, \frac{1 \pm i}{\sqrt{2}}$$

Let $$z_1 = \frac{-1+i}{\sqrt{2}} \quad \text{and} \quad z_2 = \frac{1+i}{\sqrt{2}}$$

then, $$m = z_1, \overline{z_1}, z_2, \overline{z_2}$$

Here $\overline{z_1}$ and $\overline{z_2}$ denote complex conjugate of z_1 and z_2 respectively.

$\therefore$ C.F. $= f_1(y + z_1x) + f_2(y + \overline{z_1}\, x) + f_3(y + z_2x) + f_4(y + \overline{z_2}\, x)$

P.I. $= 0$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + z_1x) + f_2(y + \overline{z_1}x) + f_3(y + z_2x) + f_4(y + \overline{z_2}x)$$

where f_1, f_2, f_3 and f_4 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x\, \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$
2. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2\, \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0$
3. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\, \partial y} = 0$
4. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2\, \partial y} + 3\frac{\partial^3 z}{\partial x \partial y^2} = 0$
5. $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 2\frac{\partial^4 z}{\partial x^2\, \partial y^2}$
6. $\frac{\partial^3 z}{\partial x^3} + 3\frac{\partial^3 z}{\partial x^2\, \partial y} + 2\frac{\partial^3 z}{\partial x \partial y^2} = 0$
7. $\frac{\partial^2 z}{\partial x^2} + a^2\frac{\partial^2 z}{\partial y^2} = 0$
8. $\frac{\partial^2 z}{\partial x^2} - 3a\frac{\partial^2 z}{\partial x \partial y} + 2a^2\frac{\partial^2 z}{\partial y^2} = 0$
9. $25r - 40s + 16t = 0$
10. $r + 2s + t = 0.$

Answers

1. $z = f_1(y - 2x) + f_2\left(y - \frac{x}{2}\right)$
2. $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x)$
3. $z = f_1(y) + f_2(y - x)$
4. $z = f_1(y) + f_2(y + x) + f_3(y + 3x)$
5. $z = f_1(y - x) + x\, f_2(y - x) + f_3(y + x) + xf_4(y + x)$
6. $z = f_1(y) + f_2(y - x) + f_3(y - 2x)$
7. $z = f_1(y + aix) + f_2(y - aix)$
8. $z = f_1(y + ax) + f_2(y + 2ax)$
9. $z = f_1\left(y + \frac{4}{5}x\right) + x\, f_2\left(y + \frac{4}{5}x\right)$
10. $z = f_1(y - x) + x\, f_2(y - x).$

1.19 RULES FOR FINDING P.I.

P.I. of the equation $F(D, D')z = \phi(x, y)$ is given by $\frac{1}{F(D, D')}\phi(x, y)$.

Short methods:

Case: When $\phi(x, y)$ is a function of $ax + by$

To find out P.I. of the equation

$$F(D, D')z = \phi(ax + by);\ F(a, b) \neq 0$$

where F(D, D′) is a homogeneous function of D and D′ of degree n.

Steps involved

Step 1. Replace D by a, D′ by b in F(D, D′) to get $F(a, b)$.

Step 2. Put $ax + by = u$ and integrate $\phi(u)$, n times w.r.t. u.

Then, $$\text{P.I.} = \frac{1}{F(a, b)} \underbrace{\iiint \cdots\cdots \int}_{(n \text{ times})} \phi(u)\, \underbrace{du\, du\, du \cdots\cdots du}_{(n \text{ times})}.$$

Step 3. Replace u by $ax + by$ at last.

Remark. It should be noted that here $F(a, b) \neq 0$. If $F(a, b) = 0$, then method fails.

Cases of failure

(*i*) To find out P.I. of the equation

$$F(D, D')z = \phi(ax + by)\ ;\ F(a, b) = 0$$

where F(D, D′) is a homogeneous function of D and D′ of degree n.

Steps involved

Step 1. Differentiate F(D, D′) partially w.r.t. D and simultaneously multiply the expression by x.

Step 2. Check whether $F'(a, b) \neq 0$?

Then, $$\text{P.I.} = \frac{1}{F(D, D')} \phi(ax + by)\ ;\ F(a, b) = 0$$

$$= x \cdot \frac{1}{\frac{\partial}{\partial D}\{F(D, D')\}} \phi(ax + by)$$

This method fails if $F'(a, b) = 0$.

(*ii*) To find out P.I. of the equation

$$F(D, D')z = \phi(ax + by)\ ;\ F'(a, b) = 0$$

where F(D, D′) is a homogeneous function of D and D′ of degree n.

Steps Involved:

Step 1. Repeat the procedure as again differentiate F′(D, D′) partially w.r.t. D and simultaneously multiply the expression by x.

Step 2. Check whether $F''(a, b) \neq 0$?

Then $$\text{P.I.} = x \cdot \frac{1}{F'(D, D')} \phi(ax + by)$$

$$= x^2 \cdot \frac{1}{\frac{\partial}{\partial D}\{F'(D, D')\}} \phi(ax + by)$$

This method also fails if $F''(a, b) = 0$.

We proceed this method as long as the derivative of F(D, D′) vanishes when D is replaced by a and D′ by b.

If $F^r(a, b) \neq 0$, then $\dfrac{1}{F^r(D, D')}$ can be evaluated for particular integral.

Special Case: As a Particular Case when in F(D, D′), Power of D′ in Highest Degree Terms is Greater than that of D and Case of Failure Occurs:

Step 1. Differentiate F(D, D′) partially w.r.t. D′ and simultaneously multiply the expression by y.

Step 2. Check whether $F'(a, b) \neq 0$?

Then,
$$\text{P.I.} = \frac{1}{F(D, D')} \cdot \phi(ax + by);\ F(a, b) \neq 0$$
$$= y \cdot \frac{1}{\dfrac{\partial}{\partial D'}\{F(D, D')\}}\ \phi(ax + by).$$

Step 3. This method fails if $F'(a, b) = 0$.

In such case, we follow the same procedure as given below:

$$\text{P.I.} = y \cdot \frac{1}{F'(D, D')}\ \phi(ax + by);\ F'(a, b) = 0$$
$$= y^2 \cdot \frac{1}{\dfrac{\partial}{\partial D'}\{F'(D, D')\}}\ \phi(ax + by).$$

Step 4. This method also fails if $F''(a, b) = 0$. We repeat the method as long as derivative of F(D, D′) vanishes when D is replaced by a and D′ by b.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linar partial differential equation*

$$\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2\, \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}.$$

Sol. The given equation is

$$(D^3 - 3D^2D' + 4D'^3)\, u = e^{x+2y} \text{ where } D \equiv \frac{\partial}{\partial x} \text{ and } D' \equiv \frac{\partial}{\partial y}$$

Auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$
$$m^2(m + 1) - 4m(m + 1) + 4(m + 1) = 0$$
$$(m - 2)^2 (m + 1) = 0$$
$$\Rightarrow \quad m = 2, 2, -1$$
$$\therefore \quad \text{C.F.} = f_1(y - x) + f_2(y + 2x) + x f_3(y + 2x)$$

$$\text{P.I.} = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

$$= \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3} \iiint e^u \, du \, du \, du \qquad \text{where } x + 2y = u$$

$$= \frac{1}{27} e^{x+2y}$$

Hence the complete solution is

$$u = \text{C.F.} + \text{P.I.} = f_1(y - x) + f_2(y + 2x) + x f_3(y + 2x) + \frac{1}{27} e^{x+2y}$$

where f_1, f_2 and f_3 are arbitrary functions.

Example 2. *Solve the linear partial differential equation*

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y.$$

Sol. The given equation is

$$(D^2 + 3DD' + 2D'^2) z = x + y.$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow \qquad (m + 1)(m + 2) = 0$$

$$\Rightarrow \qquad m = -1, -2$$

$$\therefore \qquad \text{C.F.} = f_1(y - x) + f_2(y - 2x)$$

$$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$$

$$= \frac{1}{(1)^2 + 3(1)(1) + 2(1)^2} \iint u \, du \, du, \qquad \text{where } x + y = u$$

$$= \frac{1}{6} \cdot \frac{u^3}{6} = \frac{u^3}{36} = \frac{(x + y)^3}{36}$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y - x) + f_2(y - 2x) + \frac{(x + y)^3}{36}$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve the linear partial differential equation*

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x + 3y).$$

Sol. The given equation is

$$(D^2 - 2DD' + D'^2) z = \sin(2x + 3y)$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \qquad (m - 1)^2 = 0 \Rightarrow m = 1, 1.$$