PERMUTATION GROUP

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GROUPS

- **Definition**:-An algebraic structure $(G, *)$where G is a non-empty set with the binary operation $(*)$defined on it is said to be group if following axioms are satisfied.

  1] G1: Closure property

  $a*b \in G$ for all $a, b \in G$

  2] G2: Associative property

  $a*(b*c)=(a*b)*c$ for all $a, b, c \in G$

  3] G3: Existence of identity

  There exist an element $e \in G$ such that

  $e*a=a*e=a$ for all $a \in G$

  Therefore e is called identity element of G

  4] G4:Existance of inverse

  For each $a \in G$ there exist $b \in G$ such that ,

  $a*b=b*a=e$

  Then element b is inverse of a.

Permutation

- **Permutation** A permutation is an arrangement of elements. A permutation of n elements can be represented by an arrangement of the numbers 1, 2, …n in some order. eg. 5, 1, 4, 2, 3.

PERMUTATION GROUP

- Definition:-

Let S be a finite set having n distinct elements . A one-one mapping S to S itself is called a permutation of degree n on set S.

Symbol of permutation :

$$Let\ S = \{a_1, a_2, a_3 ......a_n\} be\ a\ finite\ set$$
$$with\ n\ distinct\ elements.let\ f : S \rightarrow S\ be\ a\ 1-1\ mapping$$
$$of\ S\ on\ to\ itself.$$
$$f(a_1) = b_1, f(a_2) = b_2 ......f(a_n) = b_n,\ then\ written\ as\ follows$$
$$f = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 ......a_n \\ b_1 & b_2 & b_3 & b_4 ....b_n \end{pmatrix}$$

Degree of permutation

The number of elements in a finite set S is called as degree on permutation. If n is a degree of permutation mean having n! permutations

Example: Let S=(1,2,3,4,5)and f is a permutation on set S itself.

5! = 120 permutations

If I is a permutation of degree n such that I replaces each element by itself then I is called identity permutation of degree n.

i.e. $f(a)=a$

Ex. $I = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$

$\therefore$ I is identity permutation.

Identity permutation is always even.

EQUALITY OF TWO PERMUTATIONS

Two permutations f and g with degree n are said to be equal if f(a)=g(a).

Ex. $f=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ $g=\begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

$\therefore$ f(a)=g(a)

Product of two permutations

The product or composition of two permutation f and g with degree n denoted by f. g, obtained by first carrying out operation defined by f and then g.

i.e. $f \,.\, g(x) = f(g(x))$

Ex. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 5 & 3 \end{pmatrix}$ $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}$

$$f \cdot g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}$$

$$f \cdot g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

$$g . f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 5 & 3 \end{pmatrix}$$

$$g . f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\Rightarrow \quad f \cdot g \neq g . f$$

Cyclic Permutation

Cyclic permutations : Let f be a permutation of degree n. If it is possible to arrange m elements of the set S in a row in such a way that the f–image of each element in the row is the element which follows it, the f image of the last element is the first element and the remaining $n - m$ elements are left unchanged by f. Then f is called a cyclic permutation or cycle of length m or an m–cycle.

Example:

for example i) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}$

Here $1 \rightarrow 2 \rightarrow 4 \rightarrow 1$ and $3 \rightarrow 3$ and $5 \rightarrow 5$

$\therefore$ cycle is $(1\ 2\ 4)\ (3)\ (5)$

Example:

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 5 & 6 \end{pmatrix}$ be a cyclic permutation.

Since the elements **1, 2, 3, 4** are such that f(1)=2f(1)=2, f(2)=4f(2)=4, f(4)=3f(4)=3 and f(3)=1f(3)=1 and two remaining elements 5 and 6 remain invariant.

Cycle : (1 2 4 3)

Transposition

- **<u>Transposition</u>:** A cycle of length two is called a transposition.
- Thus the cycle (1,3)(2,3) is a transposition.

- The number of transpositions in a permutation is important as it gives the minimum number of 2 element swaps required to get this arrangement from the identity arrangement: 1, 2, 3, … n. The parity of the number of such 2 cycles represents whether the permutation is even or odd.

Note that a cycle of length $k \geq 2$ can be written as a product of $k-1$ transpositions as follows:

$(a_1 \ldots a_{k-1} a_k) = (a_1 a_k)(a_1 a_{k-1}) \ldots (a_1 a_2)$.

EVEN AND ODD PERMUTATION

The cycle (5, 1, 2, 4, 3) can be written as (5, 3)(5, 4)(5, 2)(5, 1). 4 transpositions (even).

Similarly,

(5, 1, 2) -> (5, 2)(5, 1)

(5, 1, 2)(4, 3) -> (5, 2)(5, 1)(4, 3). 3 transpositions (odd).

It is clear from the examples that the ***number of transpositions from a cycle = length of the cycle – 1.***

EVEN PERMUTATION

If the number of transposition is even then permutation is even.

Example:-

a*)(1,2)(1,3)(1,4)(2,5)*

Given permutation is (1,2)(1,3)(1,4)(2,5)

$\therefore$ Number of transposition= 4 =even number.

Hence the given permutation is an even permutation.

- Inverse of even permutation is even.

ODD PERMUTATION

If the number of transposition is odd then permutation is odd.

Example:-

a) *(1,2,3,4,5)(1,2,3)(4,5)*

Given permutation is (1,2)(1,3)(1,4)(1,5)(1,2)(1,3)(4,5)

$\therefore$ Number of transposition= 7 =odd number.

Hence the given permutation is an odd permutation.

- Inverse of odd permutation is odd.

Inverse of Permutation

10) **Inverse permutations**

Let $f = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{pmatrix}$ then

inverse permutation

is $f^{-1} = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$

Symmetric Group.

- Let A be a nonempty set. The set of all permutations on A with the operation of function composition is called the symmetric group on A, denoted Sa.

- Suppose that A={1,2,3}.A={1,2,3}. There are 3!=63!=6 different permutations on A.A. We will call the set of all 6 permutations S3.
- They are listed in the following table. The matrix form for describing a function on a finite set is to list the domain across the top row and the image of each element directly below it.

$$i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad r_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad r_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Elements of S_3

Ex.5.6.1 : **Express the permutation**

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 4 & 2 \end{pmatrix}$$ **as a product of transpositions**

Sol. : We have

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 4 & 2 \end{pmatrix}$$

$$= (1)\ (2\ 6)\ (3\ 5\ 4)$$

$$= (2\ 6)\ (3\ 5)\ (3\ 4)$$

Ex.5.6.2 : **Determine which of the following are even permutations a) $(1\ 2\ 3)\ (4\ 5)$ b) $(1\ 2\ 3\ 4\ 5\ 6)\ (7\ 8)$**

Sol. : a) $(1\ 2\ 3)\ (4\ 5) = (1\ 2)\ (1\ 3)\ (4\ 5)$

$\therefore$ It is an odd permutation

b) $$f = (1\ 2\ 3\ 4\ 5\ 6)\ (7\ 8) = (1\ 2)\ (1\ 3)\ (1\ 4)\ (1\ 5)\ (1\ 6)\ (7\ 8)$$

$\therefore$ f is an even permutation

Example 50: *Express the following permutation as the product of disjoint cycles*

$$g = (1\,3\,2\,5)(1\,4\,3)(2\,5\,1)$$

Solution: We have $g = (1\,3\,2\,5)(1\,4\,3)(2\,5\,1)$

$$\Rightarrow g = \begin{pmatrix} 1 & 3 & 2 & 5 & 4 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 & 3 & 2 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 5 & 1 & 3 & 4 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 2 & 5 & 4 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 & 4 \\ 5 & 4 & 3 \end{pmatrix} = (1\,2)(3\,5\,4)$$

Example 49: *If* $S = (1, 2, 3, 4, 5, 6)$

Compute $(5\,6\,3)\,0(4\,1\,3\,5)$

Solution: We have $(5\,6\,3)\,0\,(4\,1\,3\,5) = \begin{pmatrix} 5 & 6 & 3 \\ 6 & 3 & 5 \end{pmatrix} \begin{pmatrix} 4 & 1 & 3 & 5 \\ 1 & 3 & 5 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 4 & 5 & 6 & 3 \\ 1 & 2 & 4 & 6 & 3 & 5 \end{pmatrix} \begin{pmatrix} 4 & 1 & 3 & 5 & 2 & 6 \\ 1 & 3 & 5 & 4 & 2 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 4 & 5 & 6 & 3 \\ 3 & 2 & 1 & 6 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 2 \\ 3 & 4 & 1 & 6 & 6 & 2 \end{pmatrix} = (1\,3\,4)$$