Unit - 2

Applications of Partial Differential Equations

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classification of linear partial diff. eqn of Second order.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Where A, B, C are constants or continuous fxn of x & y possessing continuous partial derivatives and $A \longrightarrow +ve$

(i) elliptic if $B^2 - 4AC < 0$
(ii) hyperbolic if $B^2 - 4AC > 0$
(iii) parabolic if $B^2 - 4AC = 0$

Ex-1:-

(i) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

$A = 1, B = 1, C = 1$

$B^2 - 4AC = 1 - 4 = -3 < 0$

So, it is eliptic.

(ii) $4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

$A = 4, B = 4, C = 1$

$B^2 - 4AC = 16 - 4 \times 4 \times 1 = 0$

So, it is parabolic.

(iii) $5\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial t^2}$

$A = 5, B = -9, C = 4$

$B^2 - 4AC = 81 - 4 \times 5 \times 4$
$= 1 > 0.$

So, it is hyperbolic.

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Ex-2:- (i) $x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u$.

$A = x^2$, $B = 0$, $C = -1$

$B^2 - 4AC = 0 - 4 \times x^2 \times -1$

Case I : $4x^2 > 0$, i.e, $x > 0$ (hyperbolic)

Case 2 if $4x^2 = 0$, i.e $x = 0$, (parabolic)

Case 3 → $4x^2$ cannot be negative so cannot form elliptic curve.

(ii) $t \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$

$A = t$, $B = 2$, $C = x$

$B^2 - 4AC = 4 - 4tx$

→ Case I, hyperbolic if $4 - 4tx > 0$

$1 - tx > 0 \Rightarrow tx < 1$

parabolic ~~elliptic~~ if $4 - 4tx = 0$

$4tx = 4$

$tx = 1$.

elliptic if $4 - 4tx < 0$

$4 > 4tx$

$tx > 1$

(iii) $x \frac{\partial^2 u}{\partial x^2} + t \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

$A = x$, $B = t$, $C = 1$

$B^2 - 4AC = t^2 - 4x$ ~~> 0~~

Case I, hyperbolic if $t^2 - 4x > 0$

$t^2 > 4x$

elliptic if $t^2 - 4x < 0$ i.e → $t^2 < 4x$

parabolic if $t^2 - 4x = 0$

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Ex-3 :- In second quad. of xy-plane.

$$\sqrt{y^2+x^2}\, u_{xx} + 2(x-y)\, u_{xy} + \sqrt{y^2+x^2}\, u_{yy} = 0$$

Sol^n:-

$$A = \sqrt{y^2+x^2},\quad B = 2(x-y),\quad C = \sqrt{y^2+x^2}$$

$$B^2 - 4AC = 4(x-y)^2 - 4(y^2+x^2) = \cancel{4x^2} + \cancel{4y^2} - 8xy$$
$$- \cancel{4y^2} - \cancel{4x^2} + 0xy$$

In second quad. y is +ve. & x is -ve.

$$B^2 - 4AC = +ve > 0$$

$\therefore$ Diff. eq^n is hyperbolic.

Ex-4:- $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ (hyperbolic)

Sol^n $\rightarrow$ $\dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0$

$$A = 1,\ B = 0,\ C = -c^2$$

$$B^2 - 4AC = (0)^2 - 4(1)(-c^2) = 4c^2$$

which is always greater than zero.

$$B^2 - 4AC > 0.$$

It is hyperbolic.

Ex-5:- $\dfrac{\partial^2 u}{\partial t^2} + t\dfrac{\partial^2 u}{\partial x \partial t} + x\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + 6u = 0$

$$A = 1,\ B = t,\ C = x$$

$$B^2 - 4AC = t^2 - 4(1)(x) = t^2 - 4x.$$

elliptic if $t^2 - 4x < 0$.

parabolic if $t^2 - 4x = 0$

hyperbolic if $t^2 - 4x > 0$

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Ex-6:- $x^2\frac{\partial^2 u}{\partial t^2} + 3\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + 17\frac{\partial u}{\partial t} = 100u$

Solⁿ:- $A = x^2$, $B = 3$, $C = x$.

$B^2 - 4AC = (3)^2 - 4x^2 \cdot x = 9 - 4x^3$

elliptic if $9 - 4x^3 < 0$

parabolic if $9 - 4x^3 = 0$

hyperbolic if $9 - 4x^3 > 0$

Ex-7:- (i) $u_{xx} + xu_{yy} + u_y = 0$. $\qquad x > 0 \to$ elliptic

$A = 1, B = 0, C = x$ $\qquad x < 0 \to$ hyperbolic

$B^2 - 4AC = -4x$

$B^2 - 4AC < 0$.

$-4x < 0$ if $x > 0$ (elliptic)

$B^2 - 4AC > 0$

$-4x > 0$, if $x < 0$ (hyperbolic)

(ii) $z_{xx} + 2xz_{xy} + (1-y^2)z_{yy} = 0$ is elliptic for value of x & y in region $x^2+y^2<1$, parabolic on boundary & hyperbolic outside this region.

Solⁿ → $B^2 - 4AC = 4x^2 - 4(1-y^2) = 4(x^2+y^2-1)$

elliptic if $\qquad B^2 - 4AC < 0$

$4(x^2+y^2-1) < 0$ $\qquad$ if $x^2+y^2<1$

parabolic if $\qquad B^2 - 4AC \neq 0$

$4(x^2+y^2-1) \neq 0$ $\qquad$ if $x^2+y^2=1$ (on boundary)

hyperbolic if $\qquad B^2 - 4AC > 0$

$4(x^2+y^2-1) > 0$ $\qquad$ if $x^2+y^2>1$

Applications of PDE

① Method of Separation of Variables

Ex-1:- $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ $\{u(x,0) = 6e^{-3x}\}$

given→

Let $u = XT$ —— ①

$\frac{\partial}{\partial x}(XT) = 2\frac{\partial}{\partial t}(XT) + XT$

$T\frac{dX}{dx} = 2X\frac{dT}{dT} + XT$

$TX' = 2XT' + XT$

dividing XT in both sides

$\frac{TX'}{XT} = \frac{2XT'}{XT} + \frac{XT}{XT}$

$\frac{X'}{X} = 2\frac{T'}{T} + 1 = -p^2$

(i) $\frac{X'}{X} = -p^2$

$X' = -p^2 X$ 2) $\frac{dX}{dx} + p^2 X = 0$

$\frac{dX}{dx} = -p^2 X$

$\frac{dX}{X} = -p^2 dx$

Int. $\log X = -p^2 x + \log c_1$

$\boxed{X = c_1 e^{-p^2 x}}$ Ⓐ

Second method

$\frac{X'}{X} = -p^2$

$\frac{dX}{dx} + p^2 X = 0$

$(D + p^2)X = 0$

where $D \to \frac{d}{dx}$

for aux. eqn

$m + p^2 = 0$

$\boxed{m = -p^2}$

$\boxed{X = c_1 e^{-p^2 x}}$

Ans

(ii) $\frac{2T'}{T} + 1 = -p^2$

$$\frac{2T'}{T} = (-p^2 - 1)$$

$$\frac{2T'}{T} = -(p^2+1)$$

$$\frac{dT}{T\,dt} = -\frac{(p^2+1)}{2}$$

$$\frac{dT}{T} = -\frac{(p^2+1)}{2}\,dt$$

Integrating, $\log T = -\left(\frac{p^2+1}{2}\right)t + \log c_2.$

$$\frac{T}{c_2} = e^{-\frac{(p^2+1)}{2}t}$$

$$T = c_2 e^{-\left(\frac{p^2+1}{2}\right)t} \quad \text{—— (B)}$$

$\frac{2T'}{T} + 1 = -p^2$

$2\frac{dT}{dt} = -(p^2+1)T$

$\frac{2dT}{dt} + \frac{(p^2+1)}{2} = 0$

$\left(D + \left(\frac{p^2+1}{2}\right)\right) = 0$

$m = -\left(\frac{p^2+1}{2}\right)$

$T = c_2 e^{-\left(\frac{p^2+1}{2}\right)t}$

from (A) & (B)

$$u = XT = c_1 c_2\, e^{-p^2 x} \cdot e^{-\left(\frac{p^2+1}{2}\right)t}$$

$$u = c_1 c_2\, e^{-p^2 x - \left(\frac{p^2+1}{2}\right)t}$$

$$u(x,t) = c_1 c_2\, e^{-p^2 x - \left(\frac{p^2+1}{2}\right)t} \quad \text{—— (C)}$$

$$u(x,0) = c_1 c_2\, e^{-p^2 x - 0}$$

$$u(x,0) = c_1 c_2\, e^{-p^2 x}$$

$$6e^{-3x} = c_1 c_2\, e^{-p^2 x}$$

Comparing,

$\boxed{c_1 c_2 = 6}$, $\cancel{+}p^2 = \cancel{+}3$ $\quad \boxed{\therefore p^2 = 3}$

from (C)

$$u(x,t) = 6e^{-3x - \left(\frac{3+1}{2}\right)t}$$

$$\boxed{u(x,t) = 6e^{-3x-2t}}$$

Ans

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Ex-2 :- $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$

Solⁿ :- Let $u = XY$.

$$\frac{\partial^2 (XY)}{\partial x^2} - 2\frac{\partial}{\partial x}(XY) + \frac{\partial (XY)}{\partial y} = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} - 2Y\frac{\partial X}{\partial x} + X\frac{\partial Y}{\partial y} = 0$$

$$YX'' - 2YX' + XY' = 0.$$

dividing by XY

$$\frac{\cancel{Y}X''}{X\cancel{Y}} - \frac{2\cancel{Y}X'}{X\cancel{Y}} + \frac{\cancel{X}Y'}{\cancel{X}Y} = 0$$

$$\frac{X''}{X} - 2\frac{X'}{X} + \frac{Y'}{Y} = 0.$$

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = -p^2$$

(i) $\frac{X''-2X'}{X} = -p^2$

$$X'' - 2X' + p^2 X = 0$$

aux. eqn , $m^2 - 2m + p^2 = 0$

$$m = \frac{2 \pm \sqrt{4-4p^2}}{2} = 1 \pm \sqrt{1-p^2}$$

$$C.F = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}$$

$PI = 0$.

$$X = C.F + PI = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}$$

Exercise Q-1 Imp

(ii) $\frac{Y'}{Y} = p^2$

$$\frac{dY}{Y dy} = p^2$$

$$\frac{dY}{Y} = p^2 dy$$

Int. $\log Y = p^2 y + \log C_3$

$$\log \frac{Y}{C_3} = p^2 y$$

$$Y = C_3 e^{p^2 y}$$

$u = XY$

$$u(x,y) = [C_1 e^{(1+\sqrt{1-p^2})x} + C_2 e^{(1-\sqrt{1-p^2})x}] C_3 e^{p^2 y}$$

$$u(x,y) = \underbrace{C_1 C_3 e^{(1+\sqrt{1-p^2})x + p^2 y}}_{A} + \underbrace{C_2 C_3 e^{(1-\sqrt{1-p^2})x) + p^2 y}}_{B}$$

Q-1 $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ $\{u(x,0) = 4e^{-x}\}$

Let $u = XY$ — (1)

$$3\frac{\partial}{\partial x}(XY) + 2\frac{\partial}{\partial y}(XY) = 0$$

$$3YX' + 2XY' = 0$$

dividing by XY

$$\frac{3X'}{X} = \frac{-2Y'}{3Y} = -p^2$$

(i) $\frac{3X'}{X} = -p^2$

$$3\frac{dX}{X} = -\frac{p^2}{3} dx$$

Int, $3 \log X = -\frac{p^2}{3}x + \log C_1$

$$\log \frac{X^3}{C_1} = -\frac{p^2 x}{3}$$

$$X = c_1 e^{-\frac{p^2 x}{}}$$

ii) $\frac{-2Y'}{3Y} = -p^2$

$$\frac{Y'}{Y} = \frac{p^2 3}{2}$$

$$\frac{dY}{Y} = \frac{3p^2}{2} dy$$

Int. $\log Y = \frac{3p^2 y}{2} + \log c_2$

$$\log \frac{Y}{c_2} = \frac{3p^2 y}{2}$$

$$Y = c_2 e^{\left(\frac{p^2 y 3}{2}\right)}$$

from ①

$$u = XY$$

$$u(x,y) = c_1 c_2 e^{-\frac{p^2 x}{} + \frac{3p^2 y}{2}} \quad \text{Ⓐ}$$

$$u(x,0) = c_1 c_2 e^{-\frac{p^2 x}{}}$$

$$4e^{-x} = c_1 c_2 e^{-\frac{p^2 x}{}}$$

Comp.

$c_1 c_2 = 4$, $\quad e^{-x} = e^{-\frac{p^2 x}{}}$

$$\boxed{p^2 = 1}$$

from Ⓐ

$$u(x,y) = 4e^{-x + \frac{3}{2}y}.$$

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(2) $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ $\{u(0,y) = 8e^{-3y}\}$

Let $u = XY$ —— (1)

$\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} = 0$

$\frac{1}{4}\frac{\partial(XY)}{\partial x} - \frac{\partial(XY)}{\partial y} = 0$

$\cancel{\frac{1}{4}\frac{Y\partial X}{\partial}}$ $\frac{1}{4}YX' - XY' = -p^2$

$\cancel{\frac{1}{4}\frac{Y\,dX}{dx}}$ dividing by XY

$= \frac{1}{4}\frac{X'}{X} = \frac{Y'}{Y} = -p^2$

=

(i) $\frac{1}{4}\frac{X'}{X} = -p^2$

$\frac{1}{4}\frac{dX}{Xdx} = -p^2$

$\frac{1}{4}\frac{dX}{X} = -p^2 x$

Int. $\frac{1}{4}\log X = -p^2 x + \log c_1$

$\frac{1}{4}\log\frac{X}{c_1} = -p^2 x$

$X = c_1 e^{-4p^2 x}$

(ii) $\frac{Y'}{Y} = -p^2$

$\frac{dY}{Y} = -p^2 dy$

Int $\log Y = -p^2 y + \log c_1$

$\log\frac{Y}{c_2} = -p^2 y$ $= c_2 e^{-p^2 y}$

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from (1)

$$u = c_1 c_2 e^{-4p^2x - p^2y} \quad \text{—— (A)}$$

$$u(0,y) = c_1 c_2 e^{-p^2y}.$$

$$8e^{-3y} = c_1 c_2 e^{-p^2y}.$$

On comp

$c_1c_2 = 8.$ $\quad 8e^{-3y} = e^{-p^2y}$

$$p^2 = 3$$

from (A)

$$u(x,y) = 8e^{-12x-3y}$$

Ex- 4:- $4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u$, $u = 3e^{-x} - e^{-5x}$, $t = 0$

$= u = XT$

$$4\frac{\partial(XT)}{\partial t} + \frac{\partial(XT)}{\partial x} = 3(XT)$$

$$4XT' + TX' = 3XT.$$

divide by XT

$$4\frac{T'}{T} + \frac{X'}{X} = 3$$

$$\frac{4T'}{T} - 3 = -\frac{X'}{X} = p^2$$

(i) $\frac{4T'}{T} - 3 = p^2$

$$\frac{4T'}{T} = p^2 + 3$$

$$\frac{dT}{T} = \left(\frac{3+p^2}{4}\right) dt$$

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Int, $\log T = \left(\frac{3+p^2}{4}\right)t + \log c_1$

$T = c_1 e^{\left(\frac{3+p^2}{4}\right)t}$

(ii.)

$\frac{-X'}{X} = p^2$

$\frac{dX}{X} = -p^2\, dx$

Int. $\log X = -p^2 x + \log c_2$

$\log \frac{X}{c_2} = -p^2 x$

$X = c_2 e^{-p^2 x}$

from ① $u = XT$

$u = c_1 c_2 e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$

Let $c_1 c_2 = C = A + B$

$u(x,0) = 3e^{-x} - e^{-5x}$

One - D Heat Flow

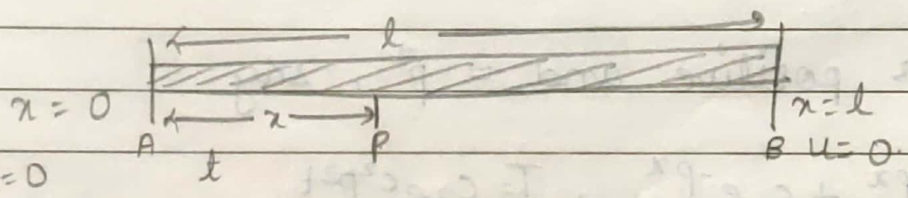
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The amount of heat crossing any section of the Bar per second depends on the area A of the cross-section, the conductivity K of the material of the bar and the temperature gradient $\frac{\partial u}{\partial x}$ i.e, rate of change of temperature w.r.t distance normal to the area.

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$, where $c^2 = \frac{K}{S\rho}$ is known as diffusivity of the material of the bar.

Another method

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$



u = Temperature distribution (Temp. function)
c = Thermal diffusivity (Temperature difference)
x = distance
t = time

B.C $\longrightarrow$ $u(0,t) = u(l,t) = 0$
I.C $\longrightarrow$ $u(x,0) = f(x)$

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By method of variable separable

Solution of the heat equation

Heat equation = $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ —— ①

Let $u = XT$

$\frac{\partial u}{\partial t} = XT'$, $\frac{\partial^2 u}{\partial x^2} = X''T$

put in ①

$XT' = c^2 X''T$

$\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T}$

$\frac{d^2X}{dx^2} - kX = 0$, $\frac{dT}{dt} - kc^2T = 0$ —— ②

solving eqn ②

(i) When k is positive and $= p^2$, say

$X = c_1 e^{px} + c_2 e^{-px}$, $T = c_3 e^{c^2p^2t}$

(ii) When k is negative $= -p^2$, say

$X = c_1 \cos px + c_2 \sin px$

$T = c_3 e^{-c^2p^2t}$

(iii) When $k = 0$

$X = c_1 x + c_2$

$T = c_3$

Various possible solⁿ. s of eqⁿ ① i.e, heat eqⁿ

$$u = (c_1 e^{px} + c_2 e^{-px}) \cdot c_3 e^{c^2p^2t} \quad \text{— ③}$$
$$u = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-c^2p^2t} \quad \text{— ④}$$
$$u = (c_1 x + c_2) \cdot c_3 \quad \text{— ⑤}$$

from ③, ④ & ⑤ eqⁿ, we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solⁿ of the heat eqⁿ is

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t}$$

Case 1:- when $K = 0$

$$\int \frac{1}{X} \frac{d^2X}{dx^2} = 0 \quad , \quad \int \frac{dT}{dt} = 0$$

$$\boxed{X = c_1 + c_2 x} \qquad \boxed{T = c_3}$$

$$\boxed{u(x,t) = (c_1 + c_2 x) c_3}$$

using B.C $\quad u(0,t) = u(l,t) = 0$

at $x = 0$, $\quad u(0,t) = c_1 c_3 \qquad \left\{ \begin{matrix} c_1 = 0 \\ c_3 \neq 0 \end{matrix} \right\}$

$$\boxed{0 = c_1 c_3}$$

at $x = l$

$$u(l,t) = c_2 l c_3 \qquad \left\{ \begin{matrix} c_2 = 0 \\ c_3 \neq 0 \end{matrix} \right\}$$

$c_1 = 0$, $c_2 = 0$ means $u = 0$ which is impossible.
Hence this case is neglected.

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Case - 2 When $k = p^2$ (positive)

$$X = c_1 e^{px} + c_2 e^{-px}, \quad T = c_3 e^{c^2p^2t}$$
$$u = (c_1 e^{px} + c_2 e^{-px}) \cdot c_3 e^{c^2p^2t}$$

B.C

$u(0,t) = u(l,t) = 0$

$x = 0, \quad 0 = (c_1 + c_2) c_3 e^{c^2p^2t}$
$\Rightarrow \boxed{c_1 + c_2 = 0}$

$x = l, \quad 0 = (c_1 e^{pl} + c_2 e^{-pl}) c_3 p^2 c^2 t$

$c_1 + c_2 = 0 \Rightarrow \boxed{c_2 = -c_1}$

$\Rightarrow c_1 e^{pl} + c_2 e^{-pl} = 0$
$\Rightarrow c_1 e^{pl} - c_1 e^{-pl} = 0$
$c_1 (e^{pl} - e^{-pl}) = 0$

$c_1 = 0$ $\quad$ $c_2 = 0$

hence $u = 0$
$\therefore$ There is no heat which is impossible.
$\therefore$ case is neglected.

Case 3 When $k = -p^2$ (negative)

$$u = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-c^2p^2t}$$

BC $\quad u(0,t) = u(l,t) = 0$

at $x = 0, \quad 0 = c_1 c_3 e^{-c^2p^2t}$ $\quad \left\{ \begin{matrix} c_1 = 0 \\ c_3 \neq 0 \end{matrix} \right\}$

at $x = l, \quad 0 = c_2 \sin pl \, (c_3 e^{-p^2c^2t})$ $\quad \left\{ \begin{matrix} c_2 \sin pl = 0 \\ c_3 \neq 0 \end{matrix} \right\}$

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$C_2 \sin pl = 0$ $\{C_2 \neq 0\}$

$pl = n\pi$

$p = \left\{\frac{n\pi}{l}\right\}$

$u(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \, e^{-\frac{c^2 n^2 \pi^2}{l^2} \cdot t}$

from soln.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \, e^{-\frac{c^2 n^2 \pi^2 \cdot t}{l^2}}}$$ → most general.

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \, e^{-c^2 n^2 \pi^2 t / l^2}$

$u(x,0) = f(x)$

$t = 0$

$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$

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Ex-1: A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0°c$ and are kept at that temperature. Find the temp. $f x^n$ $u(x,t)$.

Soln:-
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

~~$u(x,t) = (c_1 \cos px + c_2 \sin px).c_3 e^{-c^2p^2t}$~~

BC → $u(0,t) = u(l,t) = 0$

IC → Initial condition, $u(x,0) = u_0$

~~Since~~, Let $u = XT$

from (1)

$$XT' = c^2 X''T$$
$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = -p^2$$

$$\frac{\partial^2 X}{dx^2 . X} = -p^2 \quad , \quad \frac{1}{c^2}\frac{dT}{dt \cdot T} = -p^2$$

$$\frac{d^2X}{X} = -p^2 x \quad , \quad \frac{dT}{T} = c^2 - p^2 dt.$$

$$\int \frac{dT}{T} = \int -p^2 dt$$

$$\log T = -c^2p^2t.$$
$$T = e^{-p^2c^2t}.$$

$$u(x,t) = (c_1 \cos px + c_2 \sin px).c_3 e^{-c^2p^2t}.$$

$x = 0,$

$$u(0,t) = c_1 c_3 e^{-c^2p^2t} = 0 \quad \Rightarrow \boxed{c_1 = 0}$$

$x = l$

$$u(l,t) = c_2 c_3 \sin pl . e^{-c^2p^2t} = 0 \quad \Rightarrow \boxed{c_2 \neq 0}$$

$\sin pl = 0 \Rightarrow pl = n\pi$

$p = \frac{n\pi}{l}$

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So, $u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2}$ — ~~(m)~~ (A)

$u(x,0) = u_0$.

$$u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

$$b_n = \frac{2}{l} \int_0^l u_0 \sin \frac{n\pi x}{l} dx.$$

$$b_n = \frac{2u_0}{l} \int_0^l \sin \frac{n\pi x}{l} dx = \begin{cases} 0, & n \to \text{even} \\ \frac{4u_0}{n\pi} \end{cases}$$

from (A)

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t / l^2}$$

$$= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l} \cdot e^{-c^2 (2n-1)\pi^2 t / l^2}$$

Ex-2:- Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $\sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$.

Solⁿ:-

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

B.C = $u(0,t) = u(2,t) = 0$

I.C : $u(x,0) = \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$ —— (2)

Solⁿ is $\rightarrow u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t}$ —— (A)

$u(0,t) = c_1 c_3 e^{-c^2p^2t} = 0 \Rightarrow \{c_1 = 0\}$.

$u(x,t) = c_2 c_3 \sin px\, e^{-c^2p^2t}$ —— (B)

$u(2,t) = 0 = c_2 c_3 \sin 2p\, e^{-c^2p^2t}$.

$\sin 2p = 0 = \sin n\pi$

$2p = n\pi$

$\left\{p = \frac{n\pi}{2}\right\}$

from (B)

$$u(x,t) = b_n \sin \frac{n\pi x}{2}\, e^{-\frac{n^2\pi^2c^2t}{4}}$$

General solⁿ,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}\, e^{-\frac{n^2\pi^2c^2t}{4}} \quad \text{—— (A1)}$$

$u(x,0) = \sin\left(\frac{\pi x}{2}\right) + 3 \sin\left(\frac{5\pi x}{2}\right) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$

$= b_1 \sin\left(\frac{\pi x}{2}\right) + b_2 \sin\left(\frac{2\pi x}{2}\right) + b_3 \sin\left(\frac{3\pi x}{2}\right) + b_4 \text{ ''} + b_5 \sin\left(\frac{5\pi x}{2}\right)$

comp.

$b_1 = 1, b_5 = 3$.

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from (A)

$$u(x,t) = \sin\left(\frac{\pi x}{2}\right)e^{-\pi^2 c^2 t/4} + 3\sin\left(\frac{5\pi x}{2}\right)e^{-\frac{\pi^2 c^2 t}{4}}$$ Ans.

Ex-5:- The temp. distribution in a bar of length π which is perfectly insulated at ends $x=0$, $x=\pi$ is governed by partial diff. eqn -
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Soln:-
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

~~u(x,t)~~ =

B.C: Note:- Since both ends of bar are perfectly insulated, no heat can pass from either sides ~~and~~

So, Boundary Conditions =

$$\frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = 0.$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = \pi$$

$$u(x,t) = (C_1 \cos px + C_2 \sin px)\, C_3 e^{-p^2 t} \quad \text{—— (2)}$$

from (2)

$$\frac{\partial u}{\partial x} = C_3 e^{-p^2 t}(-pC_1 \sin px + pC_2 \cos px)$$

At $x=0$

$$0 = C_3 e^{-p^2 t} pC_2 \qquad \Rightarrow \{C_1 = 0\}$$

from (2) :- $u(x,t) = C_2 C_3 e^{-p^2 t} \cos px.$

$x=\pi$
$$\frac{\partial u}{\partial x} = -p C_2 C_3 e^{-p^2 t} \sin px$$

$$0 = -p C_2 C_3 e^{-p^2 t} \sin p\pi$$

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$\sin p\pi = 0 = \sin n\pi$

$p\pi = n\pi$

$\Rightarrow \boxed{p = n}$

$u(x,t) = a_n e^{-n^2 t} \cos nx$, { where $c_2 c_3 = a_n$ }

The most general sol , from half range cosine series,

$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx$

$u(x,0) = \cos 2x = \sum_{n=1}^{\infty} a_n \cos nx + \frac{a_0}{2}$

$\neq$ ~~$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots$~~

comparing

$a_0 = 0, a_n = 1 , n = 2.$

from ② $\boxed{u(x,t) = e^{-4t} \cos 2x}$

imp★

$\boxed{\text{Existence of } a_0}$ ★

If constant $k = 0$

$u(x,t) = (c_1 + c_2 x) c_3.$

Using boundary Condition

$\frac{\partial u}{\partial x} = c_2 c_3. = \frac{a_0}{2}$ (say)

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Ex-8:- $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ under the conditions.

(i) $u \neq \infty$ if $t \to \infty$. (ii) $\frac{\partial u}{\partial x} = 0$ for $x=0$ & $x=l$

(iii) $u = lx - x^2$ for $t=0$, between $x=0$ & $x=l$.

Soln:-

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

$$u(x,t) = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-p^2kt} \quad \text{—— (1)}$$

(i)→ satisfies the condition $u \neq \infty$, if $t \to \infty$.

Applying BC.

$\frac{\partial u}{\partial x} = 0$, at $x=0$ & $x=l$.

$$c_2 = 0$$

at $x=l$

$$\sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$u = c_1 c_3 e^{-\left[\frac{n^2\pi^2 kt}{l^2}\right]} \cdot \cos \frac{n\pi x}{l}$$

$$u = a_n \cos \frac{n\pi x}{l} e^{-\left[\frac{n^2\pi^2 kt}{l^2}\right]}.$$

Again

$$u = (c_1 + c_2 x) \cdot c_3.$$

$$\frac{\partial u}{\partial x} = c_2 c_3 = \frac{a_0}{2} \text{ (say)}.$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \left[e^{-(n^2\pi^2 kt/l^2)}\right] \quad \text{—— (2)}$$

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$u = lx - x^2$ for $t = 0$

$$lx - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l} \int_0^l (lx - x^2)\, dx =$$

$$a_0 = \frac{2}{l} \left[\frac{l x^2}{2} - \frac{x^3}{3} \right]_0^l$$

$$a_0 = \frac{2}{l} \left[\frac{l \cdot l^2}{2} - \frac{l^3}{3} \right]$$

$$a_0 = \frac{2}{l} \left[\frac{l^3}{2} - \frac{l^3}{3} \right] = \frac{\cancel{2}}{\cancel{l}} \cdot \frac{l^{\cancel{3}\,2}}{\cancel{6}\,3} = \frac{l^2}{3}$$

from (B)

$$\cancel{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} \left[e^{-n^2\pi^2 k t / l^2} \right]}$$

$$\cancel{= \frac{l^2}{6} + \sum}$$

$$a_n = \frac{2}{l} \int_0^l (lx - x^2) \cos \frac{n\pi x}{l}\, dx.$$

$$= \begin{cases} -\frac{4l^2}{n^2\pi^2}, & \text{even} \\ 0, & \text{odd.} \end{cases}$$

$$u = \frac{l^2}{6} - \frac{4l^2}{\pi^2} \sum_{n=2,4}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{l} \, e^{-\left(\frac{n^2\pi^2 k t}{l^2}\right)}$$

$n = 2m.$

$$\boxed{u(x,t) = \frac{l^2}{6} - \frac{l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \frac{\cos 2m\pi x}{l} e^{-\left(\frac{4m^2\pi^2 k t}{l^2}\right)}}$$

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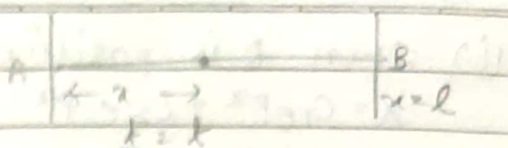
One-D Wave Eqn



$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$y \rightarrow$ virtual $f x^n$ of wave $f x^n$

$x \rightarrow$ distance

$t \rightarrow$ time

$c \rightarrow$ velocity of string.

BC $\rightarrow$ $\boxed{y(0,t) = y(l,t) = 0}$ —— ①

Initial velocity (string released at rest)

$\boxed{\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0}$ —— ②

Initial Condition

$\boxed{y(x,0) = f(x)}$ —— ③

Soln of wave eqn

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— ①}$$

Let $y = XT$ —— ②

$$\frac{\partial^2 y}{\partial t^2} = XT'' \quad , \quad \frac{\partial^2 y}{d x^2} = X''T$$

from ①

$$XT'' = c^2 X''T$$

dividing by XT

$$\frac{T''}{T} = c^2 \frac{X''}{X} \qquad \Rightarrow \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T} = k$$

$$X'' - kX = 0 \ , \quad T'' - kc^2 T = 0$$

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(i) When k is positive $= p^2$.

$X = c_1 e^{px} + c_2 e^{-px}$, $T = c_3 e^{cpt} + c_4 e^{-cpt}$

$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$

(ii) $k = 0$

$X = c_1 x + c_2$, $T = c_3 t + c_4$

$y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

(iii) $k = -p^2$ (negative)

$X = c_1 \cos px + c_2 \sin px$, $T = c_3 \cos pct + c_4 \sin cpt$.

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

It is the only suitable sol^n of wave equation.

$$\boxed{y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)} \quad \text{— (A)}$$

Using BC → $y(0,t) = y(l,t) = 0$

$x = 0$, $y = 0$

$\cancel{y(0,t)} = 0 = c_1 (c_3 \cos cpt + c_4 \sin cpt)$

$\Rightarrow c_1 = 0$

$x = l$, $y = 0$

$0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$

$\Rightarrow c_2 \neq 0$,

$\sin pl = 0$

$\sin pl = \sin n\pi$

$p = \frac{n\pi}{l}$ $\{ n = 1, 2, 3, \text{———} \}$

Using initial velocity

$IV = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ {String is released from rest}

$$\left\{ y(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \right\}$$

$$\frac{\partial y}{\partial t} = C_2 \sin px \, (-cpC_3 \sin cpt + cpC_4 \cos cpt)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \, (0 + cp\,C_4).$$

$$0 = C_4 C_2 \sin px$$

$$\Rightarrow \boxed{C_4 = 0}$$

$$y(x,t) = C_2 \sin px \; C_3 \cos cpt$$

$$y(x,t) = C_2 C_3 \sin px \cos cpt$$

$$y(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{cn\pi t}{l}$$

$$\left\{ y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{cn\pi t}{l} \right\}$$

The most general Sol^n of wave eq^n is

Using IC

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

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Ex-1:- A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form of $y = A\sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Show that the displacement of any point at a distance x from one end at time t is given by — $y(x,t) = A\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$.

Sol^n:-

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \text{——} \quad (1)$$

B.C = $y(0,t) = y(l,t) = 0$

Since, string is released from rest hence its initial velocity will be zero.

$$\frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t=0$$

String is displaced from initial position at time $t=0$.

$$y(x,0) = A\sin\frac{\pi x}{l}$$

Now, let $y = XT$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt}$$

$$\frac{\partial^2 y}{dt^2} = \frac{\partial}{\partial t}\left(X\frac{dT}{dt}\right) = X\frac{d^2T}{dt^2}$$

Similarly → $\frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{dx^2}$

from (1)

$$X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2} \Rightarrow XT'' = c^2TX''$$

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Soln of eqn = $(c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ —— ②

Applying B. C

$y(0,t) = y(l,t) = 0$

at $x=0$ $y(0,t) = 0 = c_1 (c_3 \cos cpt + c_4 \sin cpt)$

$\Rightarrow \boxed{c_1 = 0}$

at $x=l$

$y(l,t) = 0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$

$\Rightarrow c_2 \neq 0$

so, $\sin pl = 0 = \sin n\pi$

$pl = n\pi$

$\boxed{p = \frac{n\pi}{l}}$

from ②

$y(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$

$\frac{\partial y}{\partial t} = 0 = p c_2 \cos px (-c_3 . cp \sin cpt + c_4 . cp \cos cpt)$

at $t=0$

$0 = c_2 \cos px (0 + c_4 1)$

$\Rightarrow \boxed{c_4 = 0}$

Now from ②

$y(x,t) = c_2 c_3 \sin px \cos cpt$

$y(x,t) = \sum_{n=1} b_n \sin \frac{n\pi x}{l} \cos \frac{cn\pi t}{l}$ —— Ⓐ

$y(x,0) = A \sin \frac{\pi x}{l}$

so, on comparing

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$b_n = A \quad , \quad n = 1.$

So, $\boxed{y(x,t) = A \cancel{\cos} \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}}$

Ex-3:- A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially in a position given by $y = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x,t)$.

Soln →

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)}$$

BC → $y(0,t) = y(l,t) = 0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$$

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Sol^n of $eq^n \rightarrow (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$ —— (2)

Now, Applying B.C.

at $x=0$

$$y(0,t) = 0 = C_1 \cos(C_3 \cos cpt + C_4 \sin cpt).$$

$\boxed{C_1 = 0}$

at $x=l$

$$y(l,t) = 0 = C_2 \sin pl (C_3 \cos cpt + C_4 \sin cpt)$$

$$\sin pL = 0 = \sin n\pi$$

$$p = \frac{n\pi}{l}$$

from (2)

$$\boxed{y(x,t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt)}$$

$$\frac{\partial y}{\partial t} = C_2 \sin px (pC_3(-\sin cpt) + C_4 . cp \cos cpt).$$

$$\frac{\partial y}{\partial t} = pC_2 \sin px (cp C_4 \cos cpt - cp C_3 \sin cpt)$$

$$0 = C_2 \sin px (C_4 \cos cpt - C_3 \sin cpt)$$

$$0 = C_2 C_4 \sin px \cos cpt$$

$\Rightarrow \boxed{C_4 = 0}$

from (2)

$$y(x,t) = C_2 C_3 \sin px \cos cpt$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{—— (A)}$$

$$y(x,0) = y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

~~On Comp.~~

$\cancel{\boxed{b_n = y_0}}$

$$\left\{\begin{aligned} \sin 3x &= 3\sin x - 4\sin^3 x. \\ \cos 3x &= 4\cos^3 x - 3\cos x \end{aligned}\right\}$$

$$y(x,0) = y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\cancel{= y_0 [3 \sin}$$

$$y(x,0) = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$$\left\{\begin{aligned} &\sin 3x = 3\sin x - 4\sin^3 x \\ &\sin\frac{3\pi x}{l} = 3\sin\frac{\pi x}{l} - 4\sin^3\frac{\pi x}{l} \\ &\frac{3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}}{4}. \end{aligned}\right.$$

$$\therefore y(x,0) = \frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{\pi x}{l} + b_3 \sin\frac{\pi x}{l} \text{----}$$

$$\therefore b_1 = \frac{3y_0}{4}, \quad b_2 = 0, \quad b_3 = -\frac{y_0}{4}.$$

$\therefore$ from (A)

$$\boxed{y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}}$$ Ans

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Ex-4 A tightly stretched flexible string has its ends fixed at $x=0$ & $x=l$. At time $t=0$, the string is given a shape defined by $F(x) = \mu x(l-x)$, $\mu \to$ constant & then released.
Find the displacement $y(x,t)$ of any point x of the string at any time $t>0$.

Soln:-
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

Soln → $y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$ --- (2)

- BC → $y(0,t) = y(l,t) = 0$
- I.V $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, ~~$y(x,0)=0$~~
- $y(x,0) = \mu x(l-x)$.

using BC.

at $x=0$

$y(0,t) = C_1 (C_3 \cos cpt + C_4 \sin cpt) = 0$
$\Rightarrow C_1 = 0$

at $x=l$

$y(l,t) = 0 = \cancel{C_1 \cos pl}$ & $C_2 \sin pl (C_3 \cos cpt + C_4 \sin cpt)$
$\Rightarrow C_2 \sin pl = 0 \quad \{C_2 \neq 0\}$
$\sin pl = 0$
$\sin pl = \sin n\pi$
$pl = n\pi$
$\left\{ p = \frac{n\pi}{l} \right\}$

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from (2)

$y(x,t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt)$

Now

$\frac{\partial y}{\partial t} = 0 = pC_2 \cos px (- cpC_3 \sin cpt + cpC_4 \cos cpt)$

$0 = C_2 \cos px (- C_3 \sin cpt + C_4 \cos cpt)$

$\underline{t=0}$ $0 = C_2 \cos px (C_4).$

$C_2 C_4 \cos px = 0$

$\Rightarrow \boxed{C_4 = 0}$

from (2)

$C_1 = 0, C_4 = 0$

$y(x,t) = C_2 C_3 \sin px \cos cpt$

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{cn\pi t}{l}.} \quad \text{— (A)}$$

$y(x,0) = \mu x(l-x)$

$y(x,0) = \mu x(l-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$b_n = \frac{2}{l} \int_0^l \mu x(l-x) \sin \frac{n\pi x}{l} dx.$

$= \frac{2\mu}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} dx$

$= \frac{2\mu}{l} \left[(lx-x^2) \left[-\cos \frac{n\pi x}{l} \cdot \frac{l}{n\pi} \right]_0^l - \int_0^l (l-2x) \left[-\cos \frac{n\pi x}{l} \cdot \frac{l}{n\pi} \right] dx \right.$

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$$= \frac{2u}{l}\left\{\left[(l.l-l^2)\left[-\frac{\cos n\pi l}{l}\cdot\frac{l}{n\pi}\right]\right]-0\right\}$$

$$-\left[\int_0^l (l-2x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2u}{l}\left\{\left[\frac{0}{n\pi}\right]-\int_0^l (l-2x)\cos\frac{n\pi x}{l}dx\right\}$$

$$= \frac{2u}{n\pi}\left[(l-2x)\sin\frac{n\pi x}{l}\times\frac{l}{n\pi}\right]_0^l-\int_0^l(-2)\sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\,dx$$

$$= \cancel{\frac{2u}{l}\left[-2l\sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]}$$

$$= \frac{2u}{n\pi}\left[0-\int_0^l(-2)\sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\,dx\right]$$

$$= \frac{4u\cdot l}{n^2\pi^2}\cdot\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4ul}{n^2\pi^2}\left(-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right)_0^l$$

$$= \frac{4ul^2}{n^3\pi^2}\left(-\cos\frac{n\pi l}{l}+\cos 0\right)$$

$$= \frac{4ul^2}{n^3\pi^3}(-\cos n\pi+1)$$

$$= \frac{4ul^2}{n^3\pi^3}(1-(-1)^n).$$

from eqn (A)

$$y(x,t) = \frac{4ul^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\cos\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}$$ Ans

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Note: ① When initial velocity is given as a function of x other than 0 then string must be in equilibrium.

then

IC → $\boxed{y(x,0) = 0}$

(ii) In this case, b_n must be calculated by last condition of initial velocity.

Ex-8:- If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$ Find the displacement $y(x,t)$.

Sol^n →
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——— ①}$$

$$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{—②}$$

B.C. $y(0,t) = y(l,t) = 0$

I.C → $y(x,0) = 0$

$$\left(\frac{\partial y}{\partial t}\right) = b \sin^3 \frac{\pi x}{l} \quad \text{at } t = 0$$

applying BC

at $x=0$ $y(0,t) = 0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$

$\Rightarrow \boxed{C_1 = 0}$

at $x=l$ $y(l,t) = 0 = C_2 \sin pl (C_3 \cos cpt + C_4 \sin cpt)$

$C_2 \neq 0$

$\sin pl = 0 = \sin n\pi$

$pl = n\pi$

$p = n\pi/l$

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from ②

$y(x,t) = C_2 \sin px \,(C_3 \cos pct + C_4 \sin pct)$

using Initial condition

I.C → $y(x,0) = 0 = C_2 \sin px\,(C_3 + 0)$

$0 = C_2 C_3 \sin px$

$\Rightarrow \boxed{C_3 = 0}$

from ②

$y(x,t) = C_2 C_4 \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$

The most general solⁿ.

$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$ —— Ⓐ

Using initial velocity,

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$

diff. eqⁿ partially w.r.t.

$\frac{\partial y}{\partial t} = \sum b_n \sin px \cos cpt \cdot cp$

$b \sin^3 \frac{\pi x}{l} = \sum b_n \sin \frac{n\pi x}{l} \frac{cn\pi}{l}$

t=0 →

$\frac{3b}{4} \sin \frac{\pi x}{l} - \frac{b}{4} \sin \frac{3\pi x}{l} = \sum b_n \sin \frac{n\pi x}{l} \frac{cn\pi}{l}$

$= b_1 \sin \frac{\pi x}{l} \frac{c\pi}{l} + b_2 \sin \frac{2\pi x}{l} \frac{2c\pi}{l} + b_3 \sin \frac{3\pi x}{l} \frac{3c\pi}{l}$

~~$b_1 = \frac{3b}{4}, n=1, b_3 = -\frac{b}{4}, n=3.$~~

~~$b_2 = 0.$~~

$\frac{3b}{4} = b_1 \frac{c\pi}{l} \Rightarrow \boxed{\frac{3bl}{4c\pi} = b_1}$

$n_1 = 1$, $\boxed{b_2 = 0}$

$-\frac{b}{4} = b_3 \frac{3c\pi}{l} \Rightarrow \boxed{b_3 = \frac{-bl}{12c\pi}}$

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from (A)

$$y(x,t) = \frac{3bl}{4\pi c} \sin\frac{\pi x}{l} \sin\frac{\pi ct}{l} - \frac{bl}{12\pi c} \sin\frac{3\pi x}{l} \sin\frac{3\pi ct}{l}$$

$$= \boxed{\frac{bl}{12\pi c}\left[9\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right]}$$

Ex-9 :- A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially at rest in its eq. position. If it is set vibrating by giving to each of its points an initial velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t.

Soln →
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

B.C = $y(0,t) = y(l,t) = 0$

I.C = $y(x,0) = 0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l-x)$$

Soln → $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ --- (2)

BC → $y(0,t) = 0 = c_1(c_3 \cos cpt + c_4 \sin cpt)$

$x=0$ $\Rightarrow$ $c_1 = 0$

$x=l$ $y(l,t) = 0 = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt)$

$\Rightarrow \sin pl = 0 = \sin n\pi$

$pl = n\pi$

$$\left\{p = \frac{n\pi}{l}\right\}$$

from (2)

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$y(x,t) = c_2 \sin px \,(c_3 \cos cpt + c_4 \sin cpt)$

$y(x,0) = 0.$

$\xrightarrow{t=0} y(x,0) = c_2 \sin px \,(c_3 + 0)$

$\Rightarrow \boxed{c_3 = 0}$

from (2)

$y(x,t) = c_2 c_4 \sin px \ \sin cpt.$

The most gen. solⁿ

$y(x,t) = \sum_{n=1}^{\infty} b_n \ \sin px \ \sin cpt$ —— (A)

Apply initial velocity

$\frac{\partial y}{\partial t} = 0 = b_n \ \sin px \ \cos cpt \cdot cp.$

at $t=0$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b_n \sin px \cdot cp.$

$\partial x(l-x) = \sum_{n=1}^{\infty} b_n \ \sin px \cdot cp$

$\partial x(l-x) = \sum_{n=1}^{\infty} b_n \ \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l}.$

$\cancel{\frac{\pi c}{l} b_n = \frac{2}{l}\int_0^l}$

$\partial x(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l}.$

$\frac{\pi c}{l} n b_n = \frac{2}{l}\int_0^l \partial x(l-x) \sin \frac{n\pi x}{l}\, dx$

$\longrightarrow$

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$$= \frac{2\lambda}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\Bigg[(lx-x^2)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\Bigg]_0^l$$

$$= \frac{2\lambda}{l}\left[-2\cdot\frac{l^3}{n^3\pi^3}\left(\cos n\pi - 1\right)\right]$$

$$= \frac{-4\lambda l^{2}}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$= \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} 0, & \text{even} \\ \frac{8\lambda l^2}{n^3\pi^3}, & \text{odd} \end{cases} \quad \text{for odd } n = (2m-1)$$

$$\frac{\pi c}{l}\cdot n\, b_n = \frac{8\lambda l^2}{n^3\pi^3}$$

$$b_n = \frac{8\lambda l^3}{c\pi^4 n^4}$$

$$\boxed{y(x,t) = \frac{8\lambda l^3}{c\pi^4}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)c\pi t}{l}}$$

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Ex-11 :- A tightly stretched string with fixed end points $x=0$ & $x=\pi$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0}$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03\sin x - 0.04\sin 3x.$$

Sol^n → $$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)}$$

BC → $y(0,t) = y(\pi,t) = 0$

IC : $y(x,0) = 0$

I.V = $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03\sin x - 0.04\sin 3x$

Sol^n of eq^n

$$y(x,t) = (C_1\cos px + C_2\sin px)(C_3\cos cpt + C_4\sin cpt) \quad \text{—— (2)}$$

Applying BC

$y(0,t) = 0 = C_1\,(C_3\cos cpt + C_4\sin cpt)$

at $x=0$

$\Rightarrow C_1 = 0$

$y(l,t) = 0 = C_2\sin pl\,(C_3\cos cpt + C_4\sin cpt)$

at $x=l$

$\Rightarrow C_2 \neq 0$

$\sin pl = 0 = \sin n\pi$

$p\cancel{l} = n\pi$

$\left\{ p = \frac{n\pi}{l} \right\}$

~~y(x,0)~~ from (2)

$y(x,t) = C_2\sin px\,(C_3\sin cpt + C_4\sin cpt)$

$y(x,0) = 0 = C_2\sin px\,(C_3 + 0)$

$\Rightarrow \boxed{C_3 = 0}$

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from eqⁿ (2)

$$y(x,t) = C_2 C_4 \sin px \sin cpt$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin px \sin cpt \quad \text{—— (A)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$$

$$\sum_{n=1}^{\infty}\left(b_n \sin px\,(cp) \cos cpt\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$$

$$\sum_{n=1}^{\infty}\left(b_n \sin \frac{n\pi x}{\ell} \cdot \frac{cn\pi}{\ell}\right) = 0.03 \sin x - 0.04 \sin 3x$$

$$b_1 \sin\frac{\pi x}{\ell}\,\frac{c\pi}{\ell} + b_2 \sin\frac{2\pi x}{\ell}\,\frac{2c\pi}{\ell} + b_3 \sin\frac{3\pi x}{\ell}\,\frac{3c\pi}{\ell} = 0.03 \sin x - 0.04 \sin 3x$$

on comp.

$b_1 \frac{c\pi}{\ell} = 0.03$ | $b_2 = 0$ | $b_3 \frac{3c\pi}{\ell} = 0.04$

$\Rightarrow b_1 = \frac{0.03\ell}{c\pi}, n=1$ | | $b_3 = -\frac{0.04\ell}{3c\pi}$

So, from eqⁿ (A)

$$y(x,t) = \sum_{n=1} \frac{0.03\ell}{c\pi} \sin\frac{\pi x}{\ell} \sin\frac{c\pi t}{\ell} - \frac{0.04\ell}{3c\pi} \sin\frac{3\pi x}{\ell} \sin\frac{3c\pi t}{\ell}$$

Ans

$\cancel{y(x,t) = \frac{0.009}{c} \sin}$

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Q. Solve $u_t = a^2 u_{xx}$, BC → $u_x(0,t) = 0 = u_x(\pi,t)$

$u(x,0) = x^2$

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t} \quad (2)$$

BC $\left(\frac{\partial u}{\partial x}\right)_{x=0} = (-c_1 p \sin px + p c_2 \cos px)\, c_3 e^{-c^2p^2t}$

$0 = p c_2 c_3 e^{-c^2p^2t}$

2) $\boxed{c_2 = 0}$

$\left(\frac{\partial u}{\partial x}\right)_{x=\pi} = (-c_1 p \sin p\pi + p c_2 \cos p\pi)\, c_3 e^{-c^2p^2t}$

$0 = -c_1 p \sin p\pi \cdot c_3 e^{-c^2p^2t}$

$c_1 \neq 0$

$\sin p\pi = \sin n\pi$

$p\pi = n\pi$

$\boxed{p = n}$

Now from (2)

$u(x,t) = c_1 c_3 \cos px \; e^{-a^2p^2t}$ — (A)

$u(x,0) = c_1 c_3 \cos px$

$u(x,t) = \sum_{n=1}^{\infty} a_n \cos nx$

$u(x,0) = x^2.$

$x^2 = \sum_{n=1}^{\infty} a_n \cos nx$

$a_n = \frac{2}{l} \int_0^{\pi} x^2 \cos nx \, dx + \frac{a_0}{2}$

$= \frac{2}{l} \left[x^2 \frac{(\sin nx)}{n} - \frac{2x}{n^2}(-\cos nx) + \frac{2}{n^3}(-\sin nx) \right]_0^{\pi}$

$= \frac{2}{l} \left[\frac{2\pi}{n^2} \cos n\pi \right] = \frac{4}{n^2 l}(-1)^n.$

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$$a_n = \frac{4}{n^2}(-1)^n$$

$$a_0 = \frac{\pi^3}{3}$$

$$\cancel{a_0 = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\, dx}$$

$$= \cancel{\frac{2}{\pi}\left[\frac{\sin n}{n}\right]_0^\pi}$$

$$= \frac{2}{\pi}\left[\cancel{x^2 \frac{\sin nx}{n} - \frac{2x}{n^2}(-\cos nx) + \frac{2}{n^3}(-\sin nx)}\right]_0^\pi$$

$$= 0$$

$$\frac{2}{\pi}\left[\frac{x^3}{3}\right] = \frac{2\pi^2}{3}$$

for existence of $\frac{a_0}{2}$, $k=0$

$$u(x,t) = (C_1 + C_2 x) C_3$$

$$\left(\frac{du}{\partial x}\right)_{x=0} = C_2 C_3 = 0 \Rightarrow C_2 = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=\pi} = C_2 C_3$$

So, for $k=0$,
$u(x,t)$ is not zero so, there exist some constant value.

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \cdot e^{-a^2 p^2 t}$$

from ②

$$u(x,t) = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \cdot e^{-a^2 n^2 t} \leftarrow \{p = n\}$$

(11.) $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$ $\frac{\partial \theta}{\partial x} = 0, \; x = 0$

$\theta = 0, \; x = l.$

$\theta = \theta_0$ (at $t = 0$)

$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-kp^2 t}$ —— (1)

$\left(\frac{\partial \theta}{\partial x}\right)_{x=0} = 0$

$0 = (-c_1 . p \sin px + c_2 p . \cos px) c_3 e^{-kp^2 t}$

$0 = c_2 p \cdot c_3 e^{-kp^2 t}$

$\Rightarrow \boxed{c_2 = 0}$

$(\theta = 0)_{x=l} = 0 = c_1 \cos pl \cdot c_3 e^{-kp^2 t}$

$\cos pl = 0$

$\cos pl = \cos (2n-1)\frac{\pi}{2}$

$pl = (2n-1)\frac{\pi}{2}$

$p = \frac{(2n-1)\pi}{2l}$

from (1)

$u(x,t) = c_1 c_3 \cos px \; e^{-kp^2 t}$

$\theta(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{(2n-1)\pi x}{2l} e^{-k\left(\frac{(2n-1)\pi}{2l}\right)^2 t}$ —— (A)

$\theta(x,0) = \theta_0$

$u(x,0) = c_1 c_3 \cos px$

$u(x,0) = \sum_{n=1}^{\infty} a_n \cos px$

$\theta_0 = \sum_{n=1}^{\infty} a_n \cos \frac{(2n-1)\pi}{2l}$

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at $k=0$

$\theta(x,t) = (c_1 + c_2 x) c_3$

$\frac{\partial\theta}{\partial x} = c_2 c_3$ $[c_2 = 0]$

$\theta(x,t) = c_1 c_3 = 0$

$\Rightarrow \boxed{c_1 = 0}$

$a_n = \frac{2}{l}\int_0^l f(x) \cos\frac{(2n-1)\pi x}{2l}$

~~$a_n = \frac{2}{l}\int_0^{\pi} \theta_0 \cos \frac{x}{l}$~~

$a_n = \frac{2}{l}\int_0^l \theta_0 \cos\frac{(2n-1)\pi x}{2l}\, dx.$

$\rightarrow \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Q. (3) $u_t = u_{xx}$, $0 < x < l$, $t > 0$

$u(0,t) = 1$, $u(x,0) = 1 + x + 2\sin \pi x$

$u(l,t) = 2$

Sol$^n \rightarrow$ $u(x,t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}$

$u = u_2(x,t) + u_1(x,t)$

$u = \underset{\text{Initial temp.}}{I.T} + \frac{(\underset{\text{final temp.}}{F.T} - I.T)}{l} x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t}.$

$u = 1 + \frac{(2-1)}{1} x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t}$

$u = 1 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t}$

$u(x,t) = 1 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t}$

$u(0,t) = 1$

$\cancel{1} = \cancel{1} + 0 + \sum_{n=1}^{\infty} a_n e^{-p^2 t}$

$0 = \sum_{n=1}^{\infty} a_n e^{-p^2 t}$ $[a_n = 0]$

$\cancel{2} = \cancel{1} + \cancel{1} + \sum_{n=1}^{\infty} b_n \sin p \, e^{-p^2 t}$

$\sin p = 0 = \sin n\pi$

$p = n\pi$

$u(x,t) = 1 + x + \sum_{n=1}^{\infty} b_n \sin n\pi \, e^{-\pi^2 n^2 t}$

$I.C = u(x,0) = 1 + x + 2\sin \pi x$

$\cancel{1} + \cancel{x} + 2\sin \pi x = \cancel{1} + \cancel{x} + \sum_{n=1}^{\infty} b_n \sin n\pi \, e^{-\pi^2 n^2 t}$

$b_n = 2$ $n = 1$

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$$\left\{ u(x,t) = 1 + x + 2 \sin \pi x \, e^{-\pi^2 t} \right\}$$

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2D- Heat flow

Consider the flow of heat in a metal plate, in the XOY plane. If the temp. at any point is independent of the z-cordinate & depends on x, y and t only then the flow is called 2-D and the heat flow lies in the plane XOY only and is zero along the normal to the plane XOY.

$$c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t} \qquad \left\{c^2 = \frac{K}{s\rho}\right\}$$

Note 1:- In steady state, u is independent of t, so that $\frac{\partial u}{\partial t} = 0$.

$$\text{So, } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \{\text{Laplace eq}^n \text{ in 2-D}\}$$

Note 2:- for 3D →

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \{\text{laplace eq}^n \text{ in 3D}\}$$

Sol^n:- of Laplace eq^n in 2D.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u = XY$

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$YX'' + XY'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = K \text{ (say)}$$

$$\frac{\partial^2 X}{\partial x^2} - KX = 0 \;\&\; \frac{\partial^2 Y}{\partial y^2} + KY = 0.$$

In laplace eq^n either x or y will occur in the form of $\sin x$ or $\cos x$
Acc. to b.c is given in x or y

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① when $k = p^2$

$$m^2 - p^2 = 0$$
$$m = \pm p$$
$$X = C_1 e^{px} + C_2 e^{-px}$$

$$m^2 + p^2 = 0$$
$$m = \pm ip.$$
$$Y = C_3 \cos py + C_4 \sin py \quad (A)$$

② when $K = -p^2$

$$m^2 + p^2 = 0$$
$$m = \pm pi$$
$$X = C_1 \cos px + C_2 \sin px$$

$$m^2 - p^2 = 0$$
$$m = \pm p$$
$$Y = C_3 e^{py} + C_2 e^{-py} \quad (B)$$

③ whe $k = 0$.

$$m^2 = 0$$
$$m = 0,$$

$$\left\{ \begin{matrix} X = C_1 x + C_2, & Y = C_3 y + C_4. \\ \downarrow & \downarrow \\ \text{or } C_1 + C_2 x & \text{or } C_3 + C_4 y \end{matrix} \right\} \quad (C)$$

Of these three sol^n we have to choose that sol^n which is consistent with physical nature of the problem & the given boundary condition solution eq^n B is req. sol^n.

$$u(x,y) = (C_1 \cos px + C_2 \sin px)\,(C_3 e^{py} + C_4 e^{-py}).$$

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Ex-1:- Use separation of variable.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

BC → $u(0,y) = u(l,y) = 0$ —— (1)

$u(x,0) = 0$ —— (2)

$u(x,a) = \sin\frac{n\pi x}{l}$ —— (3)

Soln:- $u = XY$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 (XY)}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 (XY)}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

$\therefore$ $YX'' + XY'' = 0$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

case 1:- $\frac{X''}{X} = -\frac{Y''}{Y} = \cancel{p}\ K$

$K = 0$,

$X = (C_1 + C_2 x)$

$Y = (C_3 + C_4 y)$

$u(x,y) = (C_1 + C_2 x)(C_3 + C_4 y)$

BC $u(0,y)$

$x=0$ $= C_1(C_3 + C_4 y)$

2) $\boxed{C_1 = 0}$

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at $x=l$ = $0 = c_2 l (c_3 + c_4 y)$

$\Rightarrow \boxed{c_2 = 0}$

here, c_1, c_2 are zero.

$\Rightarrow U=0 \Rightarrow u(x,y)=0$.

Hence this case is impossible

This case is neglected.

Case 2 → $\frac{X''}{X} \cancel{=} \frac{-Y''}{Y} = K$

when $K = p^2$

$X = (c_1 e^{px} + c_2 e^{-px})$ $Y = (c_3 \cos py + c_4 \sin py)$

$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

BC

$u(0,y) = 0$ at $x=0$,

$0 = (c_1 + c_2)(c_3 \cos py + c_4 \sin py)$

$\Rightarrow (c_1 + c_2) = 0$

$\boxed{c_1 = -c_2}$

at $x=l$

$0 = [(c_1 e^{pl}) - c_2 e^{-pl})] (c_3 \cos py + c_4 \sin py)$

$c_1 e^{pl} - c_2 e^{-pl} = 0$

$c_1 (e^{pl} + e^{-pl}) = 0$

$\Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

In this case, $c_1, c_2 = 0$

Hence $u(x,y) = 0$

which is impossible (neglected)

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Case 3 → when $\frac{X''}{X} = -\frac{Y''}{Y} = K$

when $K = -p^2$

$X = (c_1 \cos px + c_2 \sin px)$, $Y = (c_3 e^{py} + c_4 e^{-py})$

$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

B.C

$u(0,y) = 0$ | $0 = (c_1)(c_3 e^{py} + c_4 e^{-py})$

$x=0$ | $\Rightarrow \boxed{c_1 = 0}$

$u(l,y) = 0$ | $0 = (c_2 \sin pl)(c_3 e^{py} + c_4 e^{-py})$

$x = l$ | $\Rightarrow c_2 \neq 0$, $\sin pl = 0$

$\sin pl = \sin n\pi$

$pl = n\pi$

$\Rightarrow \left\{ p = \frac{n\pi}{l} \right\}$

$\left\{ c_1 = 0, p = \frac{n\pi}{l} \right\}$

$u(x,y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py})$

using ② condition

$0 = c_2 \sin px (c_3 + c_4)$

$c_3 + c_4 = 0$

$\Rightarrow c_3 = -c_4$ $\{c_4 = -c_3\}$

$u(x,y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py})$

$u(x,y) = c_2 \sin px \, c_3 (e^{py} - e^{-py})$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-n\pi y/l} \right)$$

using I.C.

$u(x,a) = \sin \frac{n\pi x}{l}$

$$\sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \left(\frac{e^{n\pi a/l} - e^{-n\pi a/l}}{2} \right) \times 2$$

comp

$$b_n = \frac{\cancel{\sin n\pi x/l}}{2 \cancel{\sin \frac{n\pi x}{l}} \cdot \sinh \frac{n\pi a}{l}} = \frac{1}{2 \sinh \frac{n\pi a}{l}}$$

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$$u(x,y) = \frac{1}{\sinh \frac{n\pi a}{l}} \cdot \sin \frac{n\pi x}{l} \cdot \sinh \frac{n\pi y}{l}$$

$$\left\{ u(x,y) = \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \cdot \sin\frac{n\pi x}{l} \right\}$$

Ex-8:- $\frac{\partial^2(u)}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u(x,0), = u(x,b) = 0$ — ①, $u(0,y) = 0$ — ②, $u(a,y) = f(y)$ — ③

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad - *$$

$u = XY$

from (*)

$$\frac{\partial^2(XY)}{\partial x^2} + \frac{\partial^2(XY)}{\partial y^2} = 0$$

$$YX'' + XY'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{Y''}{Y} = -\frac{X''}{X} = K$$

case 1:- when $K = 0$

$X = (c_1 x + c_2)$

$Y = (c_3 y + c_4)$

$$u(x,y) = (c_1 x + c_2)(c_3 y + c_4)$$

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~~case 2 :- when K = 0~~

BC =

$u(x,0) = 0$ | $C_4(C_1 x + C_2) = 0$

$y=0$ | $\Rightarrow \boxed{C_4 = 0}$

$u(x,b) = 0$ | $(C_1 x + C_2)(C_3 b) = 0$

$y=b$ | $\Rightarrow \boxed{C_3 = 0}$

So, C_3, C_4 are zero.

$U = (0, 0)$, $u(x,y) = 0$

So, this case is impossible.

case - 2 :- when $K = p^2$

$m^2 - p^2 = 0$ | $m^2 + p^2 = 0$

$m = \pm p$ | $m = \pm ip$

$Y = C_3 e^{py} + C_4 e^{-py}$ | $X = C_1 \cos px + C_2 \sin px$

$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$

BC

$u(x,0) = 0 = \quad 0 = (C_3 + C_4)(C_1 \cos px + C_2 \sin px) = 0$

$y=0$

$\Rightarrow C_3 + C_4 = 0$

$C_4 = -C_3$ or $C_3 = -C_4$

$y=b$ $u(x,b) = 0 = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} - C_3 e^{-py})$

$= C_3(e^{py} - e^{-py})(C_1 \cos px + C_2 \sin px) = 0$

$\Rightarrow \boxed{C_3 = 0}$

$\Rightarrow \boxed{C_4 = 0}$

So, $u(x,y) = 0$

Neglected case.

Case 3 When $K = -p^2$

$X = (c_1 e^{px} + c_2 e^{-px})$

$Y = (c_3 \cos py + c_4 \sin py)$

$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$ — (A)

BC

$u(x,0) = 0$ $= 0 = c_3(c_1 e^{px} + c_2 e^{-px})$

$y = 0$ $\Rightarrow \boxed{c_3 = 0}$

$y = b$ $= 0 = c_4 \sin pb\,(c_1 e^{px} + c_2 e^{-px})$

$= c_4 \neq 0$

so $\sin pb = 0 = \sin n\pi$

$pb = n\pi$

$p = \frac{n\pi}{b}$

$u(x,y) = c_4 \sin py\,(c_1 e^{px} + c_2 e^{-px})$

$u(0,y) = c_4 \sin py\,(c_1 + c_2)$

$0 = c_4 \sin py\,(c_1 + c_2)$

$\Rightarrow c_1 + c_2 = 0$

or $c_2 = -c_1$

Now from eqn (A)

$u(x,y) = c_4 \sin py\,(c_1 e^{px} - c_1 e^{-px})$

$u(x,y) = c_1 c_4 \sin py\,(e^{px} - e^{-px})$

$u(x,y) = b_n \sin \frac{n\pi y}{b} \left[e^{n\pi x/b} - e^{-n\pi x/b}\right]$

$= b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}.$

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Most gen. sol^n.

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi x}{b}$$

$$u(a,y) = f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b} \sin h\frac{n\pi a}{b}.$$

$$\left(\sin h \frac{n\pi a}{b}\right) b_n = \frac{2}{b}\int_0^b f(y) \sin\frac{n\pi y}{b}\, dy.$$

$$\left\{ b_n = \frac{2}{b \cdot \sin h \frac{n\pi a}{b}} \int_0^b f(y) \sin\frac{n\pi y}{b}\, dy \right\}$$

Ex - 8

Ques -:- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0.$

$u(x,b) = u(0,y) = u(a,y) = 0$, $u(x,0) = x(a-x)$ — (3)

BC → $u(0,y) = u(a,y) = 0$ — (1)

I.C = $u(x,b) = 0$ — (2)

Case when $K = -p^2$

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

BC

$u(0,y) = 0$ at $x = 0$ | $c_1 (c_3 e^{py} + c_4 e^{-py}) = 0$

$c_1 = 0$

$x = a$ | $c_2 \sin pa (c_3 e^{py} + c_4 e^{-py}) = 0$

$\sin pa = 0 = \sin n\pi$

$pa = n\pi$

$p = n\pi/a$

$C_2C_3 = A, \; C_2C_4 = B.$

~~u(x,y)~~,

$u(x,y) = \sin px \,(A e^{py} + B e^{-py})$ — (A)

$u(x,b) = \sin px \,(A e^{pb} + B e^{-pb}) = 0$

$A e^{pb} + B e^{-pb} = 0$

$\cancel{C_3 e^{pb} = -C_4 e^{-pb} = -\frac{1}{2} B_n}$

$\cancel{u(x,y) = C_2 \sin \frac{n\pi x}{a}}$

$A e^{pb} = -B e^{-pb} = -\frac{1}{2} B_n$ (say)

So, (A) becomes,

$u(x,y) = \sin \frac{n\pi x}{a} \left[-\frac{1}{2} B_n e^{-\frac{n\pi b}{a}} e^{\frac{n\pi y}{a}} + \frac{1}{2} B_n e^{\frac{n\pi b}{a}} e^{-\frac{n\pi y}{a}} \right]$

$= B_n \sin \frac{n\pi x}{a} \left[\frac{e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)}}{2} \right]$

$= B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a}(b-y)$

$= B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a}(b-y).$

most general sol^n.

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a}(b-y)$$

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Now, $u(x,0) = x(a-x)$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

$$x(a-x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

$$B_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a x(a-x)\sin\frac{n\pi x}{a}\,dx$$

~~$B_n =$~~

$$B_n = \frac{1}{\sinh\frac{n\pi b}{a}} \frac{2}{a}\int_0^a x(a-x)\sin\frac{n\pi x}{a}\,dx$$

$$\left\{ \begin{array}{l} e^{\infty} = 1 \\ e^{-\infty} = \infty \end{array} \right\}$$

rectangular $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

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Ex-2:-

To prove $\rightarrow u(x,y) = 100\, e^{-\pi y/8} \sin \frac{\pi x}{8}$

$u(0,y) = u(8,y) = 0 \quad \Big| \quad u(x,0) = 100 \sin \frac{\pi x}{8}$

Soln:- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{——— (1)}$$

BC $\rightarrow$

$\rightarrow u(0,y) = 0 = u(8,y)$

Infinite length $\lim_{y \to \infty} u(x,y) = 0$ {because here $y \to \infty$ in length

Now,

$u(0,y) = 0$, $x=0$ $\quad \Big| \quad 0 = (c_1 (c_3 e^{py} + c_4 e^{-py})$.

$\Rightarrow \boxed{c_1 = 0}$

$u(8,y)$, $x=8$ $=$ $\quad \Big| \quad 0 = c_2 \sin 8p \,(c_3 e^{py} + c_4 e^{py})$

$\Rightarrow \sin 8p = 0 = \sin n\pi$

$8p = n\pi$

$p = \frac{n\pi}{8}$

$$u(x,y) = c_2 \sin px \,(c_3 e^{py} + c_4 e^{-py}) \quad \text{——— (2)}$$

$\lim_{y\to\infty} u(x,y) = c_2 \sin px \lim_{y \to \infty} (c_3 e^{py} + c_4 e^{-py}) = 0$

$\Rightarrow c_2 \sin px \,(c_3 e^{\infty} + c_4 e^{-\infty}) = 0$

$c_2 \sin px \,(c_3 \infty) = 0$

$\Rightarrow c_3 = 0$

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$u(x,y) = C_2 C_4 \sin px \; \cancel{C_3} \, e^{-py}$

$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{8} \, e^{-n\pi y/8} \longrightarrow$ A

Initial con —

$u(x,0) = 100 \sin \frac{\pi x}{8} = b_n \sin \frac{n\pi x}{8}$

$b_n = 100, \; n = 1$

from A

$u(x,y) = 100 \sin\left(\frac{\pi x}{8}\right) e^{-(\pi y/8)}$

Hence proved.

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Ex-3 :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(0,y) = u(\pi, y) = 0$

$\lim_{y \to \infty} u(x,y) = 0$

$u(x,0) = u_0.$

Sol$^n \rightarrow$ $u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \longrightarrow$ 1

BC

$u(0,y) = 0$, $x = 0$ | $0 = C_1 (C_3 e^{py} + C_4 e^{-py})$

1) $\boxed{C_1 = 0}$

$u(\pi, y) = 0$, $x = \pi$ | $0 = C_2 \sin p\pi \,(C_3 e^{py} + C_4 e^{-py})$

2) $\sin p\pi = 0 = \sin n\pi$

$p\cancel{\pi} = n\cancel{\pi}$

$p = n$

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$u(x,y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py})$ Ⓐ

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$\lim_{y\to\infty} u(x,y) = C_2 \sin px \lim_{y\to\infty} (C_3 e^{py} + C_4 e^{-py})$

$0 = C_2 \sin px\,(C_3 \infty + 0)$

$\Rightarrow \boxed{C_3 = 0}$

from Ⓐ

$u(x,y) = C_2 C_4 \sin px\, e^{-py}$

$u(x,y) = b_n \sin nx\, e^{-ny}$

$u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx\, e^{-ny}$.

$u(x,0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$

$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx$

$= b_n = \frac{2u_0}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi}$

$= \frac{2u_0}{n\pi}\{1-(-1)^n\} = \begin{cases} \frac{4u_0}{n\pi}, & n \to \text{odd} \\ 0, & n \to \text{even.} \end{cases}$

$u(x,y) = \frac{4u_0}{\pi} \sum_{n=1,3,5\ldots} \frac{\sin nx}{n} e^{-ny}$

$$\boxed{u(x,y) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin(2n-1)x\, e^{-(2n-1)y}.}$$

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Q $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u_x(0,y) = u_x(a,y) = 0$, $u_y(x,0) = 0$.
$u_y(x,b) = f(x)$.

Sol^n $\Rightarrow$ $u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

$\left(\frac{\partial u}{\partial x}\right)_{x=0} = (-c_1 . p \sin px + c_2 p \cos px)(c_3 e^{py} + c_4 e^{-py})$

$u_x(0,y)_{x=0} = (c_2 p)(c_3 e^{py} + c_4 e^{-py})$
$\Rightarrow c_2 = 0$

$\left(\frac{\partial u}{\partial x}\right)_{x=a} = (-c_1 . p \sin pa)(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow \sin pa = 0 = \sin n\pi$
$pa = n\pi$
$\left\{ p = \frac{n\pi}{a} \right\}$

~~$u(x,y) = c_1 \cos$~~
$u(x,y) = c_1 \cos px \, (c_3 e^{py} + c_4 e^{-py})$ — (A)

$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$

$c_1 \cos px \, (p c_3 e^{py} - p c_4 e^{-py}) = 0$

$c_1 \cos px \, (p c_3 e^0 - p c_4 e^0) = 0$
$\Rightarrow c_3 - c_4 = 0$
$\Rightarrow c_3 = c_4$

from (A)

$u(x,y) = c_1 c_3 \cos px \left[e^{py} + e^{-py}\right]$

$u(x,y) = a_n \cos \frac{n\pi x}{a} \left[e^{n\pi y/a} + e^{n\pi y/a}\right]$

$u_y(x,y) = a_n \cos \frac{n\pi x}{a} \left[p\, e^{n\pi y/a} + p\, e^{n\pi y/a}\right]$

$u_y($

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Q-10

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

BC → $u(x,0) = u(x,1) = 0$ — ①

$u(\infty, y) = 0$ — ②

$u(0,y) = u_0$ — ③

Solⁿ → $u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

BC
$u(x,0)=0$
$y=0$

$0 = (c_1 e^{px} + c_2 e^{-px})(c_3)$

$\Rightarrow c_3 = 0$

BC
$u(x,1)=0$
$y=1$

$0 = (c_1 e^{px} + c_2 e^{-px})(c_4 \sin p)$

$\sin p = 0 = \sin n\pi$

$p = n\pi$

$u(x,y) = c_4 \sin py\,(c_1 e^{px} + c_2 e^{-px})$ — (A)

$$\lim_{x\to\infty} u(x,y) = \lim_{x\to\infty} (c_1 e^{px} + c_2 e^{-px})(c_4 \sin py)$$

$$= (c_1 e^{\infty} + c_2 e^{-\infty})(c_4 \sin py)$$

$\Rightarrow c_1 = 0$

Now $u(x,y) = c_2 c_4 e^{-px} \sin py.$

$$u(x,y) = \sum_{n=1}^{\infty} b_n e^{-n\pi x} \sin n\pi y, \quad \text{— (B)}$$

$x=0$

$$u(0,y) = \sum_{n=1}^{\infty} b_n \sin n\pi y$$

$$u_0 = \sum_{n=1}^{\infty} b_n \sin n\pi y$$

$$b_n = \frac{2}{1}\int_0^1 u_0 \sin n\pi y.$$

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$$b_n = -240 \left[\frac{\cos n\pi y}{n}\right]_0^1$$

$$b_n = -240\,[\cos n\pi - \cos 0]/n\pi$$

$$b_n = -240\,[(-1)^n - 1]/n\pi$$

$$b_n = \frac{240}{n}\,[1-(-1)^n]/n\pi$$

$$b_n = \begin{cases} 0 & , \text{ even.} \\ \frac{440}{n\pi} & , \text{ odd.} \end{cases} \rightarrow (2n-1)$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin n\pi y.$$

$$= \sum_{m=2n-1}^{\infty} \cancel{b_n} \sin(2n-1)\pi y$$

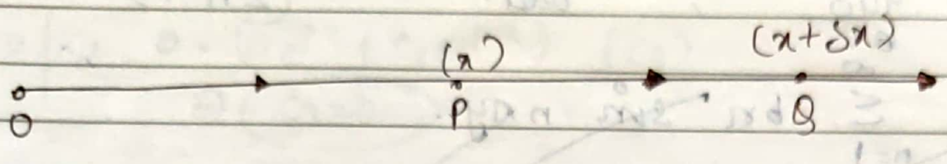
$$\cancel{u(x,y) = \sum_{n=1}^{\infty} 240}$$

$$\left\{ u(x,y) = \frac{240}{\pi} \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n}\right] e^{-n\pi x} \sin n\pi y \right\}$$

Ans

Transmission line Equations.

Consider the flow of electricity in an insulated cable. Let V be the potential and I the current at time t at a point P of the cable at dis. x from a given point O.
Let $V+\delta V$ and $I+\delta I$ be the potential & current at the point $Q(x+\delta x)$ at the same time.



R → resistance
L → inductance
C → capacitance
G → leakance

} per unit length → constant.

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC+LG)\frac{\partial V}{\partial t} + RGV$$

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (LG+RC)\frac{\partial I}{\partial t} + RGI.$$

Telephone Equations.

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(1) If $L = G = 0$.

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t} \qquad \frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t}$$

Telegraph equations. (Same as One D - heat flow)

(2) If $R = G = 0$

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \qquad \frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}$$

Radio equations. (similar to wave equation)

(3) If R & G are negligible

$$\frac{\partial V}{\partial x} = -L\frac{\partial i}{\partial t}, \quad \frac{\partial i}{\partial x} = -C\frac{\partial V}{\partial t}$$

(4) If $L = C = 0$

$$\frac{\partial^2 V}{\partial x^2} = RGV, \quad \frac{\partial^2 I}{\partial x^2} = RGI$$

eqn of Submarine cable.

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Ex-1 find current i & voltage e in line of length l, t sec.

$$i(x,0) = i_0 \ , \quad e(x,0) = e_0 \sin \frac{\pi x}{l}$$

Solⁿ:- Since R & G negligible,

$$\frac{\partial e}{\partial x} = -L\frac{\partial i}{\partial t} \quad —① \qquad \frac{\partial i}{\partial x} = -C\frac{\partial e}{\partial t} \quad —②$$

diff eqⁿ ① partially w.r.t to x

$$\frac{\partial^2 e}{\partial x^2} = -L\frac{\partial^2 i}{\partial x \partial t}$$

diff eqⁿ ② partially w.r.t t

$$\frac{\partial^2 i}{\partial x \partial t} = -C\frac{\partial^2 e}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 e}{\partial x^2} = (-L)(-C)\frac{\partial^2 e}{\partial t^2}$$

$$\left\{ \frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2} \right\}$$

IC → are → $i(x,0) = i_0$, $e(x,0) = e_0 \sin \frac{\pi x}{l}$

Since the ends are suddenly grounded → $e(0,t) = e(l,t) = 0$

BC are →

$i = i_0$ when $t=0$

$\frac{\partial i}{\partial x} = 0 \Rightarrow$ from ①

$$\frac{\partial e}{\partial x} = -L \times 0$$

$$\boxed{\frac{\partial e}{\partial x} = 0}$$

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$$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2}$$

let $e = XT$

$$X''T = LC\, T''. X$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -p^2.$$

$$\frac{d^2X}{dx^2} + p^2X = 0, \quad \frac{d^2T}{dt^2} + \frac{p^2}{LC}T = 0$$

$\sqrt{\beta} = p$ $\qquad \sqrt{\beta} = \frac{p}{\sqrt{LC}}$

$$X = c_1\cos px + c_2\sin px.$$

$$T = c_3\cos\frac{pt}{\sqrt{LC}} + c_4 \cancel{\frac{pt}{\sqrt{LC}}}\sin\frac{pt}{\sqrt{LC}},$$

$$e(x,t) = (c_1\cos px + c_2\sin px)\left(c_3\cos\frac{p}{\sqrt{LC}}t + c_4\sin\frac{pt}{\sqrt{LC}}\right) \quad A$$

applies BC

$e(0,t) = 0$ $\quad = 0 = \cancel{(c_1\cos px} + c_2$

$x=0$ $\quad = 0 = c_1\left(c_3\cos\frac{pt}{\sqrt{LC}} + c_4\sin\frac{pt}{\sqrt{LC}}\right)$

$$c_1 = 0$$

$e(l,t) = 0$
$x = l$ $\quad \left| \; 0 = c_2\sin pl\left(c_3\cos\frac{pt}{\sqrt{LC}} + c_4\sin\frac{pt}{\sqrt{LC}}\right)\right.$

$$\sin pl = 0 = \sin n\pi$$

$$pl = n\pi$$

$$p = n\pi/l.$$

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from (A)

$$e(x,t) = C_2 \sin\frac{n\pi x}{l}\left(C_3 \cos\frac{n\pi t}{l\sqrt{LC}} + C_4 \sin\frac{n\pi t}{l\sqrt{LC}}\right)$$

~~$e(x,t) =$~~

$$\frac{\partial e}{\partial t} = C_2 \sin\frac{n\pi x}{l}\left(-C_3 \sin\frac{n\pi t}{l\sqrt{LC}} \cdot \frac{n\pi}{l\sqrt{LC}} + C_4 \cos\frac{n\pi t}{l\sqrt{LC}} \cdot \frac{n\pi}{l\sqrt{LC}}\right)$$

$$\frac{\partial e}{\partial t} = \sin\frac{n\pi x}{l}\left(-C_2C_3 \sin\frac{n\pi t}{l\sqrt{LC}} \cdot \frac{n\pi}{l\sqrt{LC}} + C_2C_4 \cos\frac{n\pi t}{l\sqrt{LC}} \cdot \frac{n\pi}{l\sqrt{LC}}\right)$$

$$\left(\frac{\partial e}{\partial t}\right)_{t=0} = 0 = \sin\frac{n\pi x}{l}\left(-A \sin\frac{n\pi t}{l\sqrt{LC}} \cdot \frac{n\pi}{l\sqrt{LC}} + B \cos\frac{n\pi t}{l\sqrt{LC}} \cdot \frac{n\pi}{l\sqrt{LC}}\right)$$

where $A = C_2C_3$, $B = C_2C_4$.

$t=0$

$$0 = \sin\frac{n\pi x}{l}(B)$$

$$\frac{\partial e}{\partial t} = 0 \quad (t=0)$$

2) $B = 0$

$B = 0$

2) $e = A \sin\frac{n\pi t}{l\sqrt{LC}}$.

$$e = A \sin\frac{n\pi x}{l} \cos\frac{n\pi t}{l\sqrt{LC}}$$

$$e = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \cos\frac{n\pi t}{l\sqrt{LC}} \longleftarrow (B)$$

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$$e = e_0 \sin \frac{\pi x}{l} \qquad \{at\ t=0)$$

$$e_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{\pi x}{l}$$

$A_1 = e_0$, $n = 1$

from B

$$e = e_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

from eqn (1)

$$\frac{\partial i}{\partial t} = -\frac{1}{L} \frac{\partial e}{\partial x}$$

$$= -\frac{1}{L} \left\{ e_0 \cos \frac{\pi x}{l} \cdot \frac{\pi}{l} \cdot \cos \frac{\pi t}{l\sqrt{LC}} \right\}$$

$$\frac{\partial i}{\partial t} = -\frac{1}{L} \frac{e_0 \cdot \pi}{l} \cdot \cos \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

Int.

$$i = -\frac{e_0 \cancel{\pi}}{L \cancel{l}} \cos \frac{\pi x}{l} \cdot \frac{\cancel{l}\sqrt{LC}}{\cancel{\pi}} \sin \frac{\pi t}{l\sqrt{LC}}$$

$$i = -e_0 \sqrt{\frac{C}{L}} \cdot \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}} + \left\{ \begin{array}{l} M. \to \text{constant} \\ (i_0 = M) \\ i(x,0) = i_0 \end{array} \right\}$$

$$\left\{ i = i_0 - e_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}} \right\} \quad \text{Ans}$$

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Ex-7 $\frac{\partial^2 v}{\partial x^2} = LC \cdot \frac{\partial^2 v}{\partial t^2}$

$\frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t}$

given $\rightarrow \left(\frac{\partial v}{\partial t}\right)_{t=0} = 0.$

$v(0,t) = v(l,t) = 0$

$i(x,0) = i_0.$

$v(x,0) = e_1 \sin \pi x + e_5 \sin \frac{5\pi x}{l}.$

Solⁿ→ Let $v = XT$

$X''T = LC\ T''X$

$\frac{X''}{X} = LC \frac{T''}{T} = -p^2$

$X'' + p^2 X = 0 \Rightarrow X = C_1 \cos px + C_2 \sin px$

$T'' + \left(\frac{p^2}{LC}\right) T = 0 \Rightarrow T = C_3 \cos \frac{p}{\sqrt{LC}} t + C_4 \sin \frac{p}{\sqrt{LC}} t$

$v = XT$

$v(x,t) = (C_1 \cos px + C_2 \sin px)\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{p}{\sqrt{LC}} t\right)$ — (i)

BC

$v(0,t) = 0$ $\Rightarrow 0 = C_1 \left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$

$x=0$

$C_1 = 0$

$v(l,t) = 0$ | $0 = C_2 \sin pl \left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$

$x = l$

$\Rightarrow C_2 \neq 0, \Rightarrow \sin pl = 0 = \sin n\pi$

$pl = n\pi$

$\left\{ p = \frac{n\pi}{l} \right\}$

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from (1)

$$u(x,t) = C_2 \sin\frac{n\pi x}{l} \left(\cancel{C_3 \frac{n\pi t}{}}\ C_3 \cos\frac{n\pi t}{l\sqrt{LC}} + C_4 \sin\frac{n\pi t}{l\sqrt{LC}} \right)$$

$$u(x,t) = \sin\frac{n\pi x}{l} \left\{ a_n \cos\frac{n\pi t}{l\sqrt{LC}} + b_n \sin\frac{n\pi t}{l\sqrt{LC}} \right\} \quad (2)$$

Using IC

$$\cancel{\left(\frac{\partial u}{\partial t}\right)_{t=0} = e_1 \sin\frac{\pi x}{l} + e_5 \sin\frac{5\pi x}{l}}$$

~~from (2)~~

$$\cancel{u(x,0) = \sin\frac{n\pi x}{l} \{ a_n + a_2 \}}$$

$$\Rightarrow b_n = 0$$

from (2)

$$u(x,t) = a_n \sin\frac{n\pi x}{l} \cos\frac{n\pi t}{l\sqrt{LC}}$$

most general solⁿ is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \cos\frac{n\pi t}{l\sqrt{LC}}$$

$$\left(\frac{\partial v}{\partial t}\right)_{t=0} = \sin\frac{n\pi x}{l} \left\{ -a_n \sin\frac{n\pi t}{l\sqrt{LC}} \cdot \frac{n\pi}{l\sqrt{LC}} + b_n \cos\frac{n\pi t}{l\sqrt{LC}} \frac{n\pi}{l\sqrt{LC}} \right\}$$

$$t=0.\ 0 = \sin\frac{n\pi x}{l} \left\{ -a_n \times 0 + b_n \right\}$$

$$\Rightarrow b_n = 0$$

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from (2)

$$u(x,t) = a_n \sin\frac{n\pi x}{l} \cos\frac{n\pi t}{l\sqrt{LC}}$$

~~$u(x,0) = e_1 \sin$~~

most general sol^n

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \cos\frac{n\pi t}{l\sqrt{LC}} \quad \text{—— (A)}$$

$$u(x,0) = e_1 \sin\frac{\pi x}{l} + e_5 \sin\frac{5\pi x}{l}.$$

put $t=0$ in (A)

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$e_1 \sin\frac{\pi x}{l} + e_5 \sin\frac{5\pi x}{l} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$e_1 \sin\frac{\pi x}{l} + e_5 \sin\frac{5\pi x}{l} = a_1 \sin\frac{\pi x}{l} + a_2 \;'' + a_3 + a_4 + a_5 \sin\frac{5\pi x}{l}$$

On comp

$e_1 = a_1$, $e_5 = a_5$ $\boxed{e_2 = e_3 = e_4 = 0}$

from (A)

$$u(x,t) = e_1 \sin\frac{\pi x}{l} \cos\frac{\pi t}{l\sqrt{LC}} + e_5 \sin\frac{5\pi x}{l} \cos\frac{5\pi t}{l\sqrt{LC}}$$

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