6.1 Introduction

- We have studied relation and different types of relation in the previous chapters. A relation which is reflexive, symmetric and transitive is called as an equivalence relation. In practical life. We come across so many examples in which there is no symmetry or symmetric relation. In some cases if symmetric relation exists then the corresponding elements are equal or isomorphic.
- So, in this chapter we define new relation by replacing symmetry by antisymmetric relation we will study Poset, Hasse diagrams and lattices with their examples.

6.2 Partially Ordered Set or Partial Ordering

AKTU : 2012-13, 2016-17

- A relation R on a set A is called a partially ordered relation if

i) R is reflexive, i.e. xRx, $\forall$ $x \in A$.

ii) R is anti-symmetric i.e. if xRy and yRx then $x = y$.

iii) R is transitive i.e. if xRy, yRz then xRz.

- The set A together with partially ordered relation is called a partially ordered set or POSET. (PO-Set)
- It is denoted by (A, R) or $(A, \trianglelefteq)$ where '$\trianglelefteq$' is a partially ordered relation.

Examples :

1) $(\Re, \trianglelefteq)$ $(N, \trianglelefteq)$ are Posets. For any $x, y, z \in \mathbf{R}$, $x \le x$ and if $x \le y$, $y \le x$ then $x = y$ and if $x \le y$, $y \le z$ then $x \le z$.

Therefore '$\trianglelefteq$' is reflexive, antisymmetric and transitive relation. Hence $(\Re, \trianglelefteq)$ is a poset.

2) If $A = P(S)$ where $S = \{a, b, c\}$ and for $X, Y \in A$, Define $X \le Y$ or XRY iff $X \subseteq Y$.

As $X \le X \Rightarrow X \le X$. $\therefore$ '$\le$' is reflexive.

If $X \le Y$, $Y \le Z \Rightarrow X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$

$\therefore$ '$\le$' is antisymmetric relation.

If $X \le Y, Y \le Z \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z \Rightarrow X \le Z$

$\Rightarrow$ '$\le$' is transitive relation.

$\therefore$ $(P(S), \subseteq)$ or $(P(S), \trianglelefteq)$ is a poset.

3) Consider a set $A = \{2, 4, 6, 12\}$ and $a \le b$ iff a divides b.

For any $a, b, c \in A$

$a \mid a \Rightarrow a \le a$

If $a \mid b$ and $b \mid a$ then $a = b$

If $a \mid b$ and $b \mid c$ then $a \mid c$

$\therefore$ '$\le$' is a reflexive, antisymmetric and transitive relation

$\therefore$ $(A, \trianglelefteq)$ is a poset.

I) Comparable elements :

- Let $(A, \trianglelefteq)$ be a poset. Two elements a, b in A are said to be comparable elements if $a \le b$ or $b \le a$. Two elements a and b of a set A are said to be non-comparable if neither $a \le b$ nor $b \le a$.
- In above example (2),
- The comparable elements are

$\{a\} \subseteq \{a, b\}$, $\{b\} \subseteq \{a, b, c\}$, $\{b, c\} \subseteq \{a, b, c\}$

- Non comparable elements are

$\{a\} \not\subseteq \{b\}$ $\{a\} \not\subseteq \{a, c\}$

II) Totally ordered set :

- Let A be any nonempty set. The set A is called linearly ordered set or totally ordered set if every pair of elements in A are comparable.

i.e. for any $a, b \in A$ either $a \le b$ or $b \le a$.

Ex.6.2.1 : Show that the "greater than or equal" relation (> =) is a partial ordering on the set of integers **AKTU : 2016-17**

Sol. : Let $A = \mathbf{Z}$ and for $a, b \in A$, aRb iff $a \ge b$.

i) **Reflexive :** For any $a \in A$, $a \ge a \Rightarrow$ aRa

$\therefore$ R is reflexive

ii) **Antisymmetric relation :** Let $a, b \in A$ and aRb and bRa $\Rightarrow a \ge b$ and $b \ge a \Rightarrow a = b$

$\therefore$ R is antisymmetric relation

iii) **Transitive relation :** If aRb and bRc then $a \ge b$ and $b \ge c$

$\Rightarrow a \ge b \ge c$

$\Rightarrow a \ge c$

$\Rightarrow$ aRc

hus R is a transitive relation.

lence R is a partial ordering relation and (A, ≥) is a oset.

6.3 Hasse Diagram

AKTU : 2012-13, 2015-16

is useful tool, which completely describes the ssociated partially ordered relation. It is also known s ordering diagram.

A diagram of graph which is drawn by considering omparable and non-comparable elements is called Hasse diagram of that relation. Therefore while drawing Hasse diagram following points must be ollowed.

1) The elements of a relation R are called vertices and denoted by points.
2) All loops are omitted as relation is reflexive on poset.
3) If aRb or $a \leq b$ then join a to b by a straight line called an edge such that the vertex b appears above the level of vertex a. Therefore the arrows may be omitted from the edges in Hasse diagram.
4) If $a \not\leq b$ and $b \not\leq a$ i.e. a and b are non - comparable elements, then they lie on same level and there is no edge between a and b.
5) If $a \leq b$ and $b \leq c$ then $a \leq c$. So there is a path $a \rightarrow b \rightarrow c$. Therefore do not join a to c directly i.e. delete all edges that are implied by transitive relation.

Ex.6.3.1 : Draw Hasse diagram of a poset (P(S), ⊆) where S = {a, b, c}.

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Sol. : P(S) = { ϕ {a}, {b}, {c}, {a, b} {a, c} {b, c} {a, b, c}}

Now find the comparable and non comparable elements.

ϕ⊆ {a}, ϕ⊆ {b}, ϕ⊆ {c}, ∴ {a}, {b}, {c} lie above the level of ϕ

{a}⊆ {a,b}, {b}⊆ {a,b}, {c}⊆ {a,c} ∴ {a, b}, {b, c} {a,c} lies above the level of {a}, {b}, {c}.

{a, b} ⊆ S, {b, c} ⊆ S, {a, c} ⊆ S ∴ S lies above the level of {a, b}, {a, c}, {b, c}

But {a}, {b} {c{ are non comparable ∴ {a}, {b}, {c} lie on same level.

{a, b}, {a, c}, {b, c} are non comparable ∴ lie on same level.

By considering the above observations, the Hasse diagram is as follows :

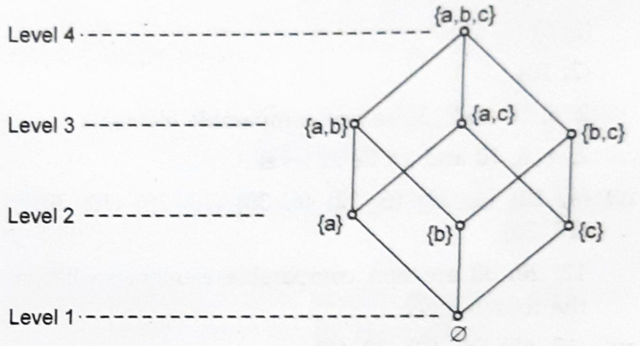


Fig. 6.3.1

Ex.6.3.2 : Consider the set A = { 4, 5, 6, 7}. Let R be the relation '≤ on A. Draw Hasse Diagram.

Sol. : The relation '≤ on the set A is given by,

$$R = \{ (4, 4), (4, 5) (4, 6), (4, 7), (5, 5), (5, 6) (5, 7), (6, 6), (6, 7), (7, 7) \}$$

(A; ≤) is a poset. Consider the following observations

i) Delete all pairs implied by reflexive property i.e. (4, 4), (5, 5), (6, 6) and (7, 7)

ii) Delete all pairs implied by transitive property i.e. As (4, 5), (5, 6) and (5, 7) delete (4, 6) (4, 7)
As (5, 6) and (6, 7) delete (5, 7)

By considering, all above points, Hasse diagram is as follows :

Fig. 6.3.2

Ex. 6.3.3 : Draw the Hasse diagram of D_{60} (divisors of 60).

Sol. : Let A be the set of all divisors of 60

A = {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

Now find comparable and non comparable elements.

i) 1 lies at the first level

ii) (1, 2) (1, 3) (1, 5) and 2, 3, 5 are not comparable elements. ∴ 2, 3, and 5 lies at level 2.

iii) (2, 4) (3, 6) (5, 10) (5, 15)

(2, 6) (3, 15)

(2, 10)

2, 6, 10 and 15 are not comparable elements.

∴ 2, 4, 6, 10 and 15 lie at level 3.

iv) (4, 12) (4, 20) (6, 12) (6, 30) (10, 20) (10, 30) (15, 30)

12, 20, 30 are non comparable element so lie on the fourth level.

v) (12, 60) (20, 60) (30, 60)

∴ 60 lies at the 5th level

∴ By considering these points, the Hasse diagram is as follows.

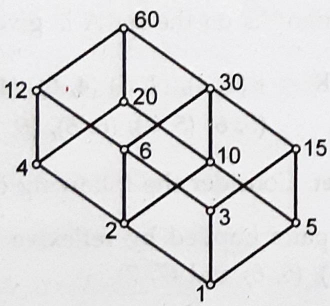


Fig. 6.3.3

Ex.6.3.4 : Draw the Hasse diagram of the set D_{30} (Divisors of 30).

Sol. : Let A be the set of divisors of 30.

∴ A = {1, 2, 3, 5, 6, 10, 15, 30}

∴ It's Hasse diagram is as follows.

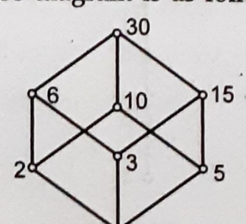


Fig. 6.3.4

Ex.6.3.5 : Let A = {2, 3, 6, 12, 24, 36} and the relation ≤ be such that a ≤ b if a divides b. Draw the Hasse Diagram of (A, ⊴).

Sol. : We have A = {2, 3, 6, 12, 24, 36}

The Hasse diagram is as follows :

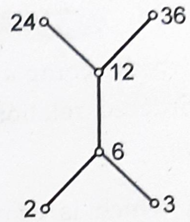


Fig. 6.3.5

Ex. 6.3.6 : Let A = {a, b, c, d} and P(A) is power set of A. Draw Hasse diagram of [P(A) , ⊆]

Sol. : We have A = {a, b, c, d}

and P (A) = {ϕ,{a}, {b}, {c}, {d}

{a, b}, {a, c}, {a, d}, {b, c}, {b, d} {c, d}

{a, b, c} {a, b, d}, {a, c, d}, {b, c, d}

{a, b, c, d}}

Hence the Hasse diagram is as follows :

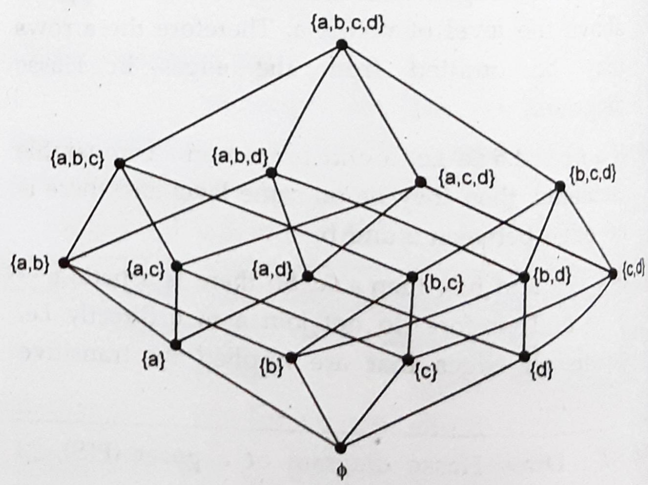


Fig. 6.3.6

6.4 Chains and Antichains

Let (A, ⊴) be a poset. A subset of A is called a chain if every pair of elements in the subset are related.

A subset of A is called antichain if no two distinct elements in a subset are related. e.g. In example (6.3.1)

1) The chains are {ϕ, {a}, {a, b}, {a,b,c}}, {{a},{a,c},{a,b,c}}, {{b,c}, {a,b,c}}

2) Antichains are {{a}, {b}, {c}}

Note :

1) The number of elements in the chain is called the length of chain.

2) If the length of chain is n in a poset (A, ≤) then the elements in A can be partitioned into n disjoint antichains.

6.5 Elements of Poset

AKTU : 2012-13,2015-16

1) Let (A,≤) be a poset. An element $a \in A$ is called a **maximal element** of A if there is no element $c \in A$ such that $a \le c$.

2) An element $b \in A$ is called a **minimal element** of A if there is no element $c \in A$ such that $c \le b$

3) Greatest element : An element $x \in A$ is called a **greatest element** of A if for all $a \in A$, $a \le x$. It is denoted by 1 and is called the unit element.

4) Least element : An element $y \in A$ is called a **least element** of A if for all $a \in A$, $y \le a$. It is denoted by 0 and is called as **zero element.**

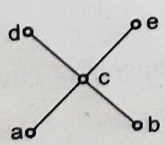
5) Least upper bound (*l*ub) : Let (A,≤) be a poset. For a, b, c ∈ A, an element c is called **upper bound** of a and b if $a \le c$ and $b \le c$. An element c is called as least upper bound of a and b in A if c is an upper bound of a and b and there is no upper bound d of a and b such that $d \le c$. It is also known as **supremum.**

6) Greatest lower bound (glb) : Let (A,≤) be a poset. for a, b, *l* ∈ A, an element *l* is called the **lower bound** of a and b if $l \le a$ and $l \le b$.

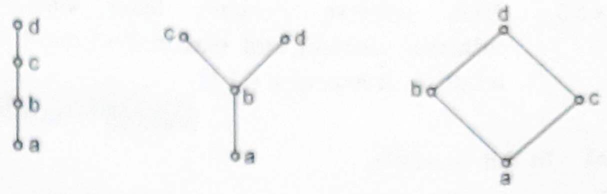
7) An element *l* is called the **greatest lower bound** of a and b if *l* is the lower bound of a and b and there is no lower bound f of a and b such that $l \le f$.

glb is also called as **infimum.**

Ex.6.5.1 : Determine the greatest and least elements of the poset whose Hasse diagrams are shown below.



I



II III IV

Fig. 6.5.1

Sol. : The Poset shown in Fig. 6.5.1 (I) has neither greatest not least element.

The Poset shown in Fig. 6.5.1 (II), has d as greatest and a as least element.

The Poset shown in Fig. 6.5.1 (III), has no greatest element but a is the least element.

The Poset shown in Fig. 6.5.1 (IV), has d as greatest and a as least element.

Ex.6.5.2 : Find glb, lwb, ub, lb, maximal, minimal, of the poset (A, R), Here aRb if a / b where. A = { 2, 3, 5, 6, 10, 15, 30, 45}

Sol. : We have A = {2, 3, 5, 6, 10, 15, 30, 45} and aRb iff a|b.

Hasse diagram is as follows :

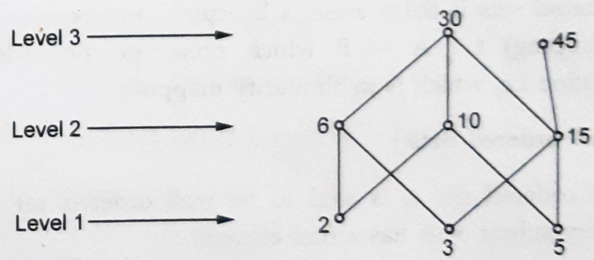


Fig. 6.5.2

1) Here 10 and 30 are upper bounds of 2 and 5, But 10 is the least upper bound of 2 and 5.

2) 5, 15, 3 are lower bounds of 30 and 45. But 15 is the greatest lower bound of 30 to 45.

3) This Poset has neither greatest element nor least element.

4) This poset has two maximal elements 45 and 30 as there is no element c such that $45 \le C$ and $30 \le C$.

5) This poset has three minimal elements 2, 3 and 5. because there is no element $x \in A$ such that $x \le 2$, $x \le 3$ and $x \le 5$

Ex.6.5.3 : **Find greatest element, least element minimal element and maximal element of a lattice in example 6.3.1**

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Sol. : In this example,

greatest element = maximal element = A

and least element = minimal element = ϕ

6.6 Isomorphic Ordered Sets

Let A and B be two partially ordered sets.

A one to one mapping $f : A \to B$ is called a similarity mapping from A into B if f preserves the order relation i.e. if the following two conditions hold for any pair a_1 and a_2 in A.

i) If $a_1 \leq a_2$ then $f(a_1) \leq f(a_2)$

ii) If a_1 and a_2 are non comparable elements then $f(a_1)$ and $f(a_2)$ are also non comparable elements.

Accordingly if A and B are linearly ordered then only (i) is needed for f to be a similarity mapping.

Two ordered sets A and B are said to isomorphic ordered sets if there exists a bijective correspondence (mapping) $f : A \to B$ which preserves the order relation i.e. which is in similarity mapping.

Well ordered Sets :

An ordered set A is said to be well ordered set if every subset of A has a first element.

Note :

1) A well ordered set is linearly ordered.
2) Every subset of a well ordered set is well ordered.
3) If X is well ordered and Y is isomorphic to X then Y is also well ordered.

6.7 Lattices

AKTU : 2002-03, 2004-05, 2011-12, 2013-14, 2014-15, 2016-17

A lattice is a poset in which every pair of elements has a least upper bound (lub) and a greatest lower (glb).

Let (A, $\leq$) be a poset and a, b $\in$ A then lub of a and b is denoted by $a \vee b$. It is called the join of a and b.

i.e. $\quad a \vee b = \text{lub}(a, b)$

The greatest lower bound of a and b is called the meet of a and b and it is denoted by $a \wedge b$

$\therefore \quad a \wedge b = \text{glb}(a, b)$

From the above discussion, it follows that a lattice is a mathematical structure with two binary operations $\vee$ (join) and $\wedge$ (meet). It is denoted by $\{L, \vee, \wedge\}$.

6.7.1 Properties of Lattice

Let (L, $\vee$, $\wedge$) be any lattice. The lattice L for any a, b, and c satisfies the following properties

a) Commutative property :

i) $a \vee b = b \vee a$ ii) $a \wedge b = b \wedge a$

b) Associative property

i) $(a \vee b) \vee c = a \vee (b \vee c)$ ii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

c) Absorption property

i) $a \wedge (a \vee b) = a$ ii) $a \vee (a \wedge b) = a$

Theorem 1 :

Prove that if L is any lattice then $a \wedge b = a$ iff $a \vee b = b$

Proof :

Part I) Assume that $a \wedge b = a$

By absorption property

$$b = b \vee (b \wedge a) = b \vee (a \wedge b) = b \vee a = a \vee b$$

Part II)

Assume that $a \vee b = b$

Again by absorption property

$$a = a \vee (a \wedge b) = (a \vee a) \wedge (a \vee b)$$
$$= a \wedge (b) = a \wedge b$$

Hence $a \wedge b = a$ iff $a \vee b = b$

Theorem 2 :

Prove that for any elements of lattice

i) $a \wedge a = a$ and ii) $a \vee a = a$

Proof :

i) $a \wedge a = a \wedge (a \vee (a \wedge b)) = a$ by absorption property,

ii) $a \vee a = a \vee (a \wedge (a \vee b)) = a$ by absorption property

Theorem 3 :

Prove that the relation $a \leq b$ defined by either $a \wedge b = a$ or $a \vee b = b$ is a partial ordering on lattice L.

Proof :

For any $a \in L$, we have $a \vee a = a$ and $a \wedge a = a$

i.e. $a \leq a$ Therefore, the relation '$\leq$ is reflexive.

Now, assume that $a \leq b$ and $b \leq a$

$\Rightarrow \quad a \wedge b = a$ and $b \wedge a = b$

Thus $\quad a = a \wedge b = b \wedge a = b$

Therefore '$\leq$ is antisymmetric relation.

Now we assume that $a \leq b$ and $b \leq c$

$\therefore \quad a \wedge b = a$ and $b \wedge c = b$

Consider $\quad a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c)$

$= a \wedge b = a$

$a \wedge c = a$

$\Rightarrow \quad a \leq c$

So, the relation '$\leq$ is transitive. Thus '$\leq$ is a partial ordering relation on L.

Theorem 4 :

For any $a, b \in$ Lattice L, prove that

(i) $a \vee (a \wedge b) = a$ and (ii) $a \wedge (a \vee b) = a$

Proof : (i) As $a \vee (a \wedge b)$ is an upper bound of a and $a \wedge b$

$\therefore \quad a \leq a \vee (a \wedge b) \quad \dots (1)$

Now $a \wedge b$ is the lower bound of a and b.

$\therefore \quad a \wedge b \leq a$

We have, $a \leq a$ and $a \wedge b \leq a$

$\therefore$ a is an upper bound for the pair $\{a, a \wedge b\}$

$\therefore$ lub $\{a, a \wedge b\}$

$\Rightarrow$

$a \vee a \; (a \leq b) \leq a \quad \dots$

From equations (1) and (2),

$\boxed{a \vee (a \wedge b) = a}$

Similarly, we can prove $a \wedge (a \vee b) = a$.

Theorem 5 :

Let $[L, \wedge, \vee]$ be a lattice and $a, b, c \in L$. Prove that

i) $a \wedge b \leq a \quad$ (ii) $a \vee b \geq a$

(iii) $a \geq b$ and $a \geq c \Rightarrow a \geq b \vee c$

Proof :

(i) Let $a, b \in L$

$a \wedge b$ is the greatest lower bound of a and b

$\therefore \quad a \wedge b \leq a$ and $a \wedge b \leq b$

(ii) We have $a \vee b$ = Least upper bound of a and b

$a \vee b \geq a$ or b

if $\quad a \geq b \Rightarrow a \vee b \geq a$

and if $b \geq a$ then $a \vee b \geq b \geq a \Rightarrow a \vee b \geq a$.

(iii) Given that $a \geq b$ and $a \geq c$

$\therefore$ a is the greatest among a, b, c

$\therefore$ $a \geq$ Least upper bound of b and c

$\therefore \quad a \geq b \vee c$

Theorem 6 :

In a lattice if $a \leq b \leq c$, then show that

a) $a \vee b = b \wedge c$

b) $(a \vee b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$

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Proof : We have $a \leq b \leq c$

$\therefore \quad a \vee b = b, \quad a \vee c = c, \quad b \vee c = c$

$a \wedge b = a, \quad a \wedge c = a, \quad b \wedge c = b$

a) $\quad$ L.H.S $= a \vee b = b$

R.H.S $= b \wedge c = b$

$\Rightarrow \quad a \vee b = b \wedge c$

b) L.H.S $= (a \vee b) \vee (b \wedge c) = b \vee (b \wedge c)$

$= b \vee b = b$

R.H.S $= (a \vee b) \wedge (a \vee c) = b \wedge (c) = b$

Hence the proof.

Ex.6.7.1 : Let A = {1, 2, 3}
P(A)={ ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2,3}, {1,2,3} }
show that (P(A), $\subseteq$) is a lattice

Sol. : The Hasse diagram of the poset (P(A), $\subseteq$) is given below :

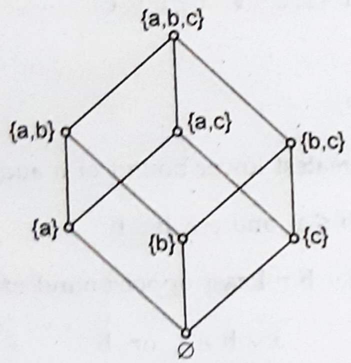


Fig. 6.7.1

Here every pair of elements of a poset has lub and glb. Hence (P(A), $\subseteq$) is a lattice.

Ex.6.7.2 : Determine which of the following posets are lattice.

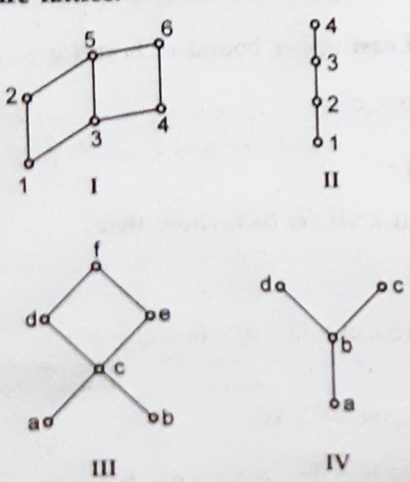


Fig. 6.7.2

Sol. :

I) In Fig. 6.7.2 (I), every pair of elements has lub and glb.

$\therefore$ It is a lattice.

II) In Fig. 6.7.2 (II), every pair of elements has lub and glb.

$\therefore$ It is a lattice.

II) In Fig. 6.7.2 (III), $a \wedge b$ does not exist.

$\therefore$ It is not a lattice.

IV) In Fig. 6.7.2 (IV), $c \vee d$ does not exist.

$\therefore$ It is not a lattice.

Ex.6.7.3 : Let A be the set of positive factors of 15 and R be a relation on A s.t. R = {xRy| x divides y, x y $\in$ A}. Draw Hasse diagram and give $\wedge$ and $\vee$ for lattice.

Sol. : We have A = {1, 3, 5, 15}

R = {(1,1) (1,3) (1,5) (1, 15) (3,15) (5,15) (15,15)}

Hasse diagram of R is :

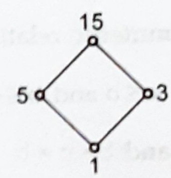


Fig. 6.7.3

Table for $\wedge$ and $\vee$

$\vee$	1	3	5	15
1	1	3	5	15
3	3	3	15	15
5	5	15	5	15
15	15	15	15	15

$\wedge$	1	3	5	15
1	1	1	1	1
3	1	3	1	3
5	1	1	5	5
15	1	3	5	15

Every pair of elements has lub and glb.

$\therefore$ It is a lattice.

Ex.6.7.4 : **Let A = {1, 2, 3, 4, 6, 9, 12} Let a relation R on a set A is R = {(a,b)/ a divides b $\forall$ a, b $\in$ A}. Give list of R. Prove that it is a partial ordering relation. Draw Hasse diagram of the same. Prove or disprove it is a lattice.**

Sol. : We have A = {1, 2, 3, 4, 6, 9, 12}

and

$$R = \left\{\begin{matrix}(1,1),(1,2),(1,3),(1,4)(1,6),(1,9),(1,12),(2,2),(2,4),(2,6),(2,12)(3,3),(3,6),(3,9)\\(3,12),\ (4,4),\ (4,12),\ (6,6),(6,12),(9,9),(12,12)\end{matrix}\right\}$$

We know that for any $a \in A$, $a \mid a$ $\therefore$ aRa

$\therefore$R is a reflexive relation.

As $a \mid b$ and $b \mid a \Rightarrow a = b$ $\therefore$R is antisymmetric relation.

As $a \mid b$ and $b \mid c \Rightarrow a \mid c \Rightarrow$ R is a transitive relation.

$\therefore$R is reflexive antisymmetric and transitive

$\therefore$(A, R) is a poset and R is a partial ordering relation.

Hasse diagram is as follows :

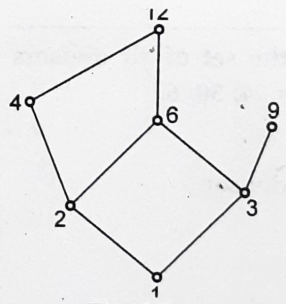


Fig. 6.7.4

In above diagram $6 \vee 9$ does not exist. $\therefore$It is not a lattice.

Ex.6.7.5 : **Determine whether the poset represented by each of the Hasse diagram are lattices. Justify your answer.**

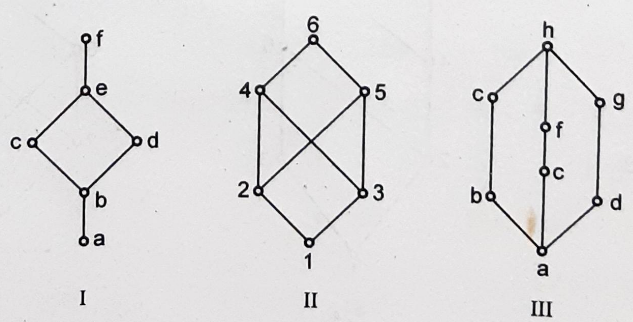


Fig. 6.7.5

Sol. :

I) In Fig. 6.7.5 (I), every pair of element has glb and lub. $\therefore$It is a lattice.

II) In Fig. 6.7.5 (II), every pair of elements has lub and glb. $\therefore$It is a lattice.

III) In Fig. 6.7.5 (III), every pair of elements gas lub and glb. $\therefore$It is a lattice.

Ex.6.7.6 : Show that the set of all divisors of 36 forms a lattice.

Sol. : Let A = {1, 2, 3, 4, 6, 9,12, 18, 36 } and Let '≤ is a divisor of.

It's Hasse diagram is as follows.

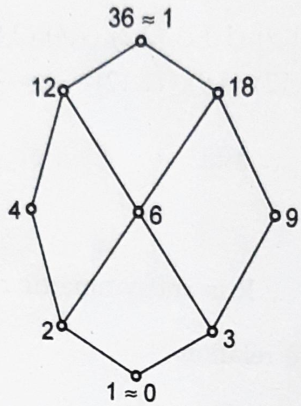


Fig. 6.7.6

The universal upper bound 1 is 36 and lower bound 0 is 1. Every pairs of elements of this poset has lub and glb.

∴It is a lattice.

Ex.6.7.7: Let n be a positive integer, S_n be the set of all divisors of n, Let D denote the relation of divisor Draw the diagram of lattices for n = 24, 30, 6.

Sol. : Given that

i) We have S_6 = {1,2,3,6}, D is the relation of divisor.

ii) S_{24} = {1, 2, 3, 4, 6, 8, 12, 24}

iii) S_{30} = {1, 2, 3, 5, 6, 10, 15, 30}

Diagrams of Lattices are as follows.

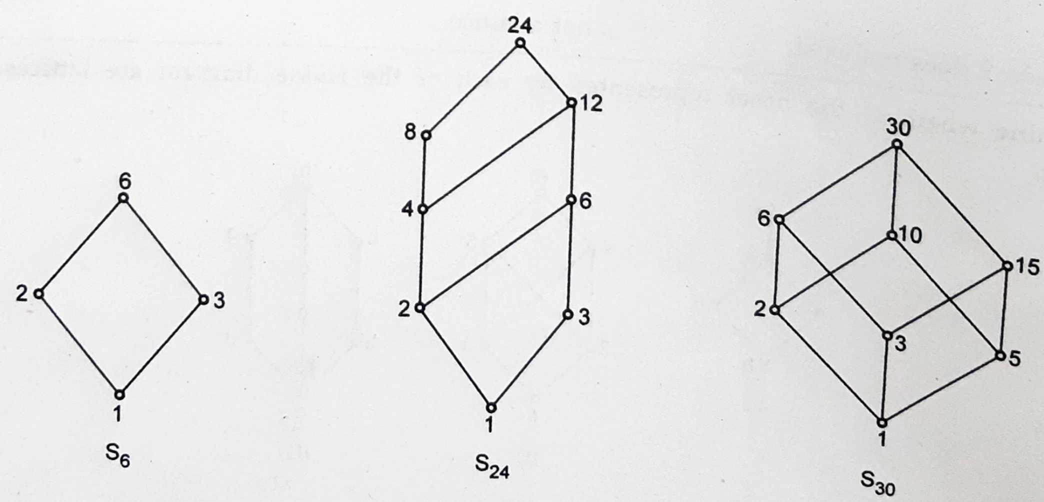


Fig. 6.7.7

Ex.6.7.8 : Show that the set of all divisors of 70 forms a lattice.

Sol. : Let A = {1, 2, 5, 7, 10, 14, 35, 70}

and Let '≤ is "a divisor of".

The universal upper bound 1 is 70 and the lower bound 0 is 1.

It's Hasse diagram is as follows :

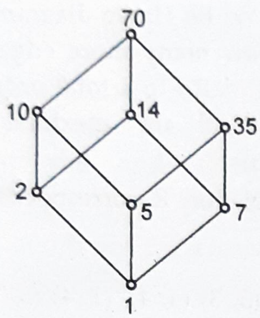


Fig. 6.7.8

Every pair of elements of A has $\wedge$ and $\vee$.

$\therefore$ It is a lattice [write table of $\wedge$ and $\vee$].

Ex. 6.7.9 : Let A = {1, 2, 3, 4 , 5, 6, 7, 8 ,9, 12, 18, 24} be ordered by the relation x divides y. Show that the relation is a partial ordering and draw Hasse diagram.

Sol. : We have $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$

$R = \{(x,y) \mid x \text{ divides } y, \text{ for } x, y \in A\}$

$R = \{(1, 2)\ (1, 3), (1, 4), (1, 5)\ (1, 6)\ (1, 7)\ (1, 8)\ (1, 9)\ (1, 12)\ (1, 18)\ (1, 24)\ (2, 2)\ (2, 4)\ (2, 6)\ (2, 8)\ (2, 12)\ (2, 18)\ (2, 24)\ (5, 5)\ (6, 6)\ (6, 12)\ (6, 18)\ (6, 24), (7, 7)\ (8, 8)\ (8, 24)\ (9, 9)\ (9, 18)\ (12, 12)\ (12, 24)\ (18, 18)\ (4, 24)\}$

We have for any $x \in A$, $x \mid x \Rightarrow R$ is a reflexive for $x \mid y$ and $y \mid x \Rightarrow x = 0 \Rightarrow R$ is antisymmetric. If $x \mid y$ and $y \mid z \Rightarrow x \mid z$ $\therefore$ R is a transitive relation.

$\therefore$ R is a partial ordering relation, It's Hasse diagram is as follows.

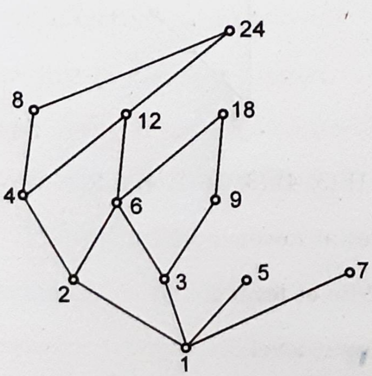


Fig. 6.7.9

Ex.6.7.10 : Let x = {2, 3, 6, 12, 24, 36} and $x \le y$ iff x divides y find

i) Maximal element ii) Minimal element iii) Chain iv) Antichain v) Is Poset lattice

Sol. : We have $x = \{2, 3, 6, 12, 24, 36\}$

The relation 'R' = '$\le$'

$\therefore$ $R = \{(2, 2)\ (2, 6)\ (2, 12)\ (2, 24)\ (2, 36)\ (3, 3)\ (3, 6)\ (3, 12)\ (3, 24)\ (3, 36), (6, 6)\ (6, 12)\ (6, 24)\ (6, 36)\ (12, 12)\ (12, 24)\ (12, 36)\ (24, 24)\ (36, 36)\}$.

It's Hasse diagram is as follows.

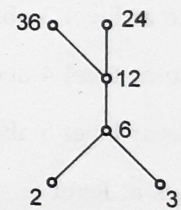


Fig. 6.7.10

i) Maximal elements are 24, 36

ii) Minimal elements are 2, 3

iii) Chain { 2, 6, 12, 24}, { 2, 6, 12, 36}, {3, 6, 12, 24}, {3, 6, 12, 36}

iv) Antichain : {2, 3} {24, 36}

v) The given poset is not a lattice as $2 \wedge 3$ does not exist.

Ex.6.7.11 : Let S be the set of all positive divisors of 120 and $a \le b$ iff a divides b. Is it poset ? If yes draw Hasse diagram.

AKTU : 2013-14

Ans. : We have,

$S = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$

and $a \le b$ iff a divides b.

(i) Reflexive : Every element divides itself

$\therefore$ $a \le a \ \forall a \in S$.

$\therefore$ '$\le$' is reflexive.

(ii) Antisymmetric relation : Let $a, b \in S$. If $a \le b$ and $b \le a$ then $a \mid b$ and $b \mid a \Rightarrow a = b$.

$\therefore$ '$\le$' is antisymmetric relation.

(iii) Transitive relation : If $a \le b$ and $b \le c$ then $a \mid b$ and $b \mid c$

$\Rightarrow$ $b = ak_1$ and $c = bk_2$

$c = (ak_1)\, k_2 = a\,(k_1\, k_2) = ak$

$\therefore$ $a \mid c \Rightarrow a \leq c$

$\therefore$ '$\leq$ is transitive relation.

Thus $(S, \leq)$ is a poset.

Hasse diagram :

1	lies at lower level 1
2, 3, 5	lie at level 2 above level 1
4, 6, 10, 15	lie at level 3 above level 2
8, 12, 20, 30	lie at level 4 above level 3
24, 40, 60	lie at level 5 above level 4.
120	lies at level 6.

$\therefore$ The Hasse diagram is as follows

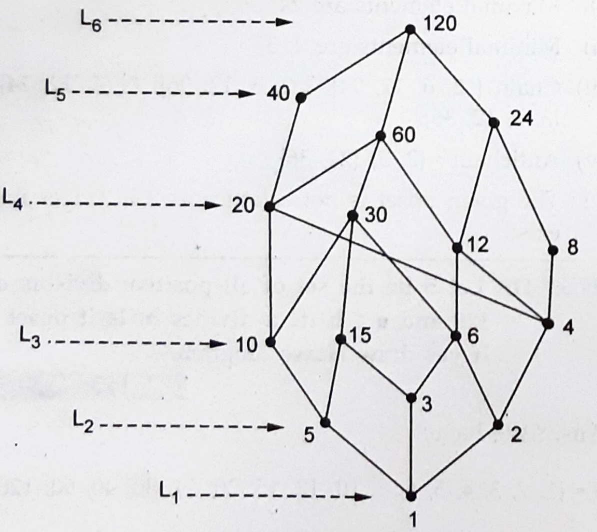


Fig. 6.7.11

Ex.6.7.12 : The directed graph for a relation R on set A = {1, 2, 3, 4} is shown in Fig.6.7.12

AKTU : 2004-05, 2014-15

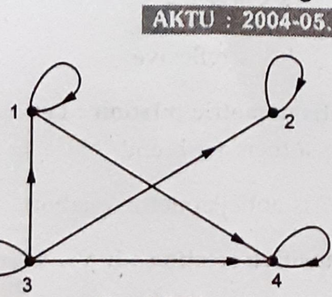


Fig. 6.7.12

(i) Verify that (A, R) is a poset.

(ii) Draw its Hasse diagram

(iii) How many more edges are needed to extend (A.R.) to a total order ?

(iv) What are maximal and minimal elements.

Ans. : (i) The relation R corresponding to the given digraph is

$R = \{(1, 1)\ (2, 2)\ (3, 3)\ (4, 4), (1, 4)\ (3, 1)\ (3, 2)\ (3, 4)\}$

We know that (A, R) is a poset if R is reflexive, antisymmetric and transitive in R.

1. **Reflexive :** $A = \{1, 2, 3, 4\}$ and (1, 1), (2, 2), (3, 3) and (4, 4) R

   $\therefore$ R is reflexive.

2. **Antisymmetric :** $\nexists$ any elements such that (a, b) and $(b, a) \in R$

   $\therefore$R is antisymmetric.

3. **Transitive :**

   $(1, 1)$ and $(1, 4) \in R \Rightarrow (1, 4) \in R$

   $(3, 3)$ and $(3, 1) \in R \Rightarrow (3, 1) \in R$

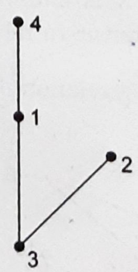
   $(3, 3)$ and $(3, 2) \in R \Rightarrow (3, 2) \in R$

   $(3, 3)$ and $(3, 4) \in R \Rightarrow (3, 4) \in R$

   $\therefore$ R is transitive relation.

   Hence (A, R) is a poset.

(ii) Hasse diagram of poset (A, R) is shown below.



We have $(3, 1)\ (3, 4)\ (3, 2)\ (1, 4) \in R$.

$\therefore$ 3 lies at lower level 1

1, 2 lie at level 2

4 lies at level 3.

(iii) To extend (A, R) to a total ordering relation 2 and 4 must be comparable element.

$\therefore$ Add (2, 4) in relation R.

(iv) From Hasse diagram maximal element = 4 and minimal element = 3.

Ex.6.7.13 : Show that the poset has at most one greatest and and at most one least element. AKTU : 2002-03

Sol. : Suppose a and b are the greatest elements of a poset A. As a is the greatest element and $b \in A$ we have

$$b \leq a \quad \ldots (1)$$

$\Rightarrow$ Now as 'b' is the greatest element and $a \in A$

$$a \leq b \quad \ldots (2)$$

From equation (1) and (2)

$$a = b$$

Hence poset has unique greatest element.

Suppose x and y be two least elements of the poset A.

As x is least element and $y \in A \Rightarrow x \leq y$.

And as y is least element and $x \in A \Rightarrow y \leq x$

$\Rightarrow x = y$ i.e. the poset has unique least element.

Ex.6.7.14 : Define a poset. Suppose that (S, ≤) and (T, ≤) are posets. Show that (S ×T, ≤) is a poset where (s, t) ≤ (u, v) iff s ≤ u in S and t ≤ v in T. AKTU : 2011-12

Sol. : Poset : A set 'A' with a partial order relation R is called a poset. It is denoted as (A, R).

A relation R is said to be partial order relation if R is reflexive, antisymmetric and transitive.

Claim : Show that (S × T, ⪯) is a poset.

(i) **Reflexive** : Let $(s, t) \in S \times T$, $\forall\, s \in S$ and $t \in T$.

$\Rightarrow \quad s \leq S$ in S and $t \leq t$ in T.

$\therefore \quad (s, t) \leq (s, t)$ in $S \times T$

$\therefore$ Relation '⪯' is reflexive.

(ii) **Antisymmetric** : Let (s_1, t_1) and $(s_2, t_2) \in S \times T$ and $(s_1, t_1) \leq (s_2, t_2)$ and $(s_2, t_2) \leq (s_1, t_1)$ in $S \times T$

$\Rightarrow \quad s_1 \leq s_2$ in S and $t_1 \leq t_2$ in T

and $\quad s_2 \leq s_1$ is S

and $\quad t_2 \leq t_1$ is T

As S and T are posets w.r.t. '⪯'

$\therefore \quad s_1 = s_2$

and $\quad t_1 = t_2$

$\Rightarrow \quad (s_1, t_1) = (s_2, t_2)$ in $S \times T$

(iii) Transitivity : Let $(s_1, t_1) \leq (s_2, t_2)$ and $(s_2, t_2) \leq (s_3, t_3)$

Then, $s_1 \leq s_2$ and $t_1 \leq t_2$ and $s_2 \leq s_3$ and $t_2 \leq t_3$

$\Rightarrow s_1 \leq s_2$ and $s_2 \leq s_3$ in S and $t_1 \leq t_2$ and $t_2 \leq t_3$ in T.

$\Rightarrow s_1 \leq s_3$ in S and $t_1 \leq t_3$ in T

$\Rightarrow (s_1, t_1) \leq (s_3, t_3)$ in $S \times T$

$\therefore$ Relation is transitive.

Thus (S × T, ⪯) is a poset.

Ex.6.7.15 : Consider the set A = {1, 2, 3, 4, 5}. Define the relation ≤ on A such that x <= y if (x mod 3) ≤ (y mod 3) AKTU : 2002-03

(i) Prove that (A, < =) is a poset

(ii) Draw Hasse diagram

(iii) Find maximal and minimal elements in (A, < =)

Sol. :

(i) Reflexivity : We have for any $x \in A$.

(x mod 3) = (x mod 3) $\Rightarrow$ x < = x

$\therefore$ R is reflexive on A.

Antisymmetry : Let a < = b and b < = a in (A_<=)

$\Rightarrow$ (a mod 3) < = (b mod 3) and (b mod 3) < = (a mod 3)

$\Rightarrow$ (a mod 3) = (b mod 3)

$\Rightarrow$ a = b

$\therefore$ R is antisymmetric relation.

Transitive : If a < = b and b < = c.

$\Rightarrow$ (a mod 3) < = (b mod 3) and (b mod 3) < = (c mod 3),

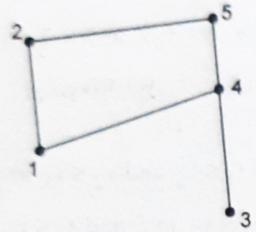
$\Rightarrow$ (a mod 3) < = (c mod 3)

a < = c

$\Rightarrow$ Relaton is transitive.

Thus, (A, ⪯) is a poset.

(ii) It's Hasse diagram is as

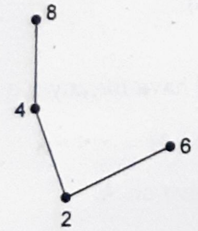


(iii) In (A, < =) Maximal element is 5 and minimal elements are 3 and 1.

Ex.6.7.16 : Consider the partial ordered sat A = {2,4, 6,8} with relation $a \leq b$ iff a | b, a, b $\in$ A. Show with reason whether the following statements are true or false.
(i) Every pair of elements in the poset has a greatest lower bound.
(ii) Every pair of elements in the poset has a least upper bound. AKTU : 2002-03

Sol. : The relation R is R={(2, 2) (4, 4) (6, 6) (8, 8) (2, 4) (2, 6) (2, 8) (4, 8)}

It's Hasse diagram is



(i) **True :** The greatest lower bound table is

glb	2	4	6	8
2	2	2	2	2
4	2	4	2	4
6	2	2	6	2
8	2	4	2	8

$\therefore$ Hence every pair of elements in (A, 1) has greatest lower bound.

(ii) **False :** There is no any upper bound for 6 and 8.

6.8 Bounded Lattices

A lattice L is said to be a bounded lattice if it has a greatest element 1 and a least element 0.

Examples :

i) The power set P(A) of the set A under the operations of intersection and union is a bounded lattice as ϕ is the least element and A is the greatest element of P(A).

ii) If A = set of natural numbers then (A, ⪯) is a lattice. It has a least element 1 but the greatest element does not exist.
$\therefore$ (A, ⪯) is not a bounded lattice.

6.8.1 Properties of Bounded Lattice

1) If L is a bounded lattice then for any element $a \in L$, we have the following identities i) $a \vee 1 = 1$ ii) $a \wedge 1 = a$ iii) $a \vee 0 = a$ iv) $a \wedge 0 = 0$

2) Every finite lattice $L = \{a_1, a_2, a_3, \ldots a_n\}$ is bounded.

$\therefore$ The greatest element of L is $a_1 \vee a_2 \vee a_3 \vee \ldots \vee a_n$ and the least element is $a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n$

Therefore, the greatest and the least elements of finite lattice exist. Hence L is bounded lattice.

6.9 Sublattices

Let $(L, \wedge, \vee)$ be a lattice. A non empty subset L_1 of L is said to be sublattice of L if L_1 it self is a lattice with respect to the operations of L. i.e. if L_1, then $a \wedge b$ and $a \vee b \in L_1$.

Ex.6.9.1 : 1) Consider the lattice of all positive integers I_+ under the operation of divisibility. The lattic D_n of all divisiors of $n > 1$ is a sublattice of I_+

Sol. : This solution is obvious readers are requested to solve it.

Ex.6.9.2 : Determine all the sublattices of D_{30} that contain at least 4 elements,
D_{30} = {1, 2, 3, 5, 6, 10, 15, 30}

Sol.: We have D_{30} = {1, 2, 3, 5, 6, 10, 15, 30} is a lattice with respect to divisibility

∴ It's Hasse diagram is as follows,

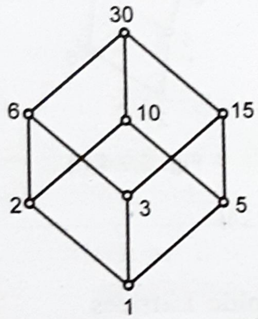


Fig. 6.9.1

The sublattices of D_{30} that contain at least 4 elements are as follows

i) {1, 2, 6, 30} ii) {1, 2, 10, 30} iii) {1, 3, 6, 30}

iv) {1, 3, 15, 30} v) {1, 5, 10, 30} vi) {1, 5, 15, 30}

vii) {1, 2, 3, 6} viii) {1, 3, 5, 15} ix) {1, 2, 5, 10}

x) {3, 6, 15, 30} xi) {2, 6, 10, 30} xii) {5, 10, 15, 30}

xiii) {1, 2, 3, 5, 6, 15, 30} xiv) D_{30}

Ex.6.9.3 : Consider the lattice L = {1, 2, 3, 4, 5} as shown in the following figure. Determine all the sublattices with three or more elements.

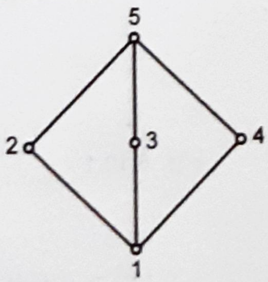


Fig. 6.9.2

Sol. : All the sublattices with three or more elements are those whose ∧ and ∨ exists for every pair of elements which are as follows.

i) {1, 2, 3} ii) {1, 3, 5} iii) {1, 4, 5}

iv) {1, 2, 3, 5} v) {1, 3, 4, 5} vi) {1, 2, 4, 5}

vii) {1, 2, 3, 4, 5}

Ex.6.9.4 : Determine whether or not each of the following is a sublattice of given lattice L.

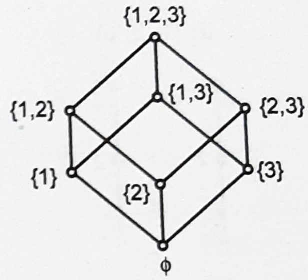


Fig. 6.9.3

A = {ϕ, {1, 2} , {2, 3}, {1, 2, 3} }
B = {ϕ, {1}, {1, 2}, {1, 2, 3} }
C = { {2}, {1, 2}, {2, 3}, {1, 2, 3}
D = {{1}, {3}, {1, 3}, {1, 2, 3}}
E = {{2}, {1, 2} , {2, 3}}

Sol. :

i) As {1, 2} ∧ {2, 3} = {1} ∉ A ∴ A is not a sublattice

ii) B is a sublattice since ∧ and ∨ of every pair of elements exist in B.

iii) C is a sublattice since ∧ and ∨ of every pair of elements exist in C

iv) As {1} ∧ {3} does not exist,

∴ D is not a sublattice

v) As {1,2} ∨ {2,3} = {1, 2, 3} ∉ E ∴ D is not a sublattice

Ex. 6.9.5 If B = Divisors of 24 = D_{24} is a lattice, then find all sublattices of D_{24} with at least 3 elements. Also draw Hasse diagrams.

Sol. : We have B = D_{24} = {1, 2, 3, 4, 6, 8, 12, 24}

It's Masse diagram is as follows

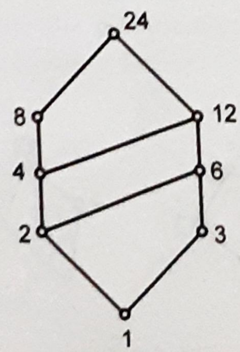


Fig. 6.9.4

Sol. : Consider the following table

Elements	0	1	a	b	c	d
Complement	1	0	d	does not exist	does not exist	a

∴ Given lattice is not complemented lattice.

Ex.6.12.2 : Prove that the lattices shown below are complemented lattices. **AKTU : 2012-13**

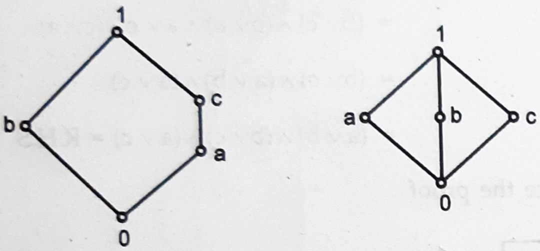


Fig. 6.12.2

Sol. : i) We have

Elements	0	a	b	c	1
Complement	1	b	a and c	b	0

As every element has a complement

∴ Given lattice is a complemented lattice.

ii) We have

Elements	0	a	b	c	1
Complement	1	b, c	a, c	a, b	0

As every element has a complement.

∴ Given lattice is a complemented lattice.

Ex.6.12.3 : Prove that, in a distributive lattice L, the complements are unique if exists. **AKTU : 2015-16**

Sol. :

Proof : Let L be a distributive lattice and $a \in L$. Suppose a_1 and a_2 are complements of a in L

$$\therefore \quad a \vee a_1 = 1, \; a \wedge a_1 = 0$$

$$\text{and} \quad a \vee a_2 = 1, \; a \wedge a_2 = 0$$

consider
$$\begin{aligned} a_1 &= a_1 \vee 0 = a_1 \vee (a \wedge a_2) \\ &= (a_1 \vee a) \wedge (a_1 \vee a_2) \\ &= 1 \wedge (a_1 \vee a_2) \\ &= (a \vee a_2) \wedge (a_1 \vee a_2) \\ &= (a_2 \vee a) \wedge (a_2 \vee a_1) \\ &= a_2 \vee (a \wedge a_1) \\ &= a_2 \vee 0 \\ a_1 &= a_2 \end{aligned}$$

Hence the complements are unique.

Ex.6.12.4 : Is D_{12} a complemented lattice ? **AKTU : 2015-16**

Sol. : We have $D_{12} = \{1, 2, 3, 4, 6, 12\}$

It's Hasse diagram is as follows

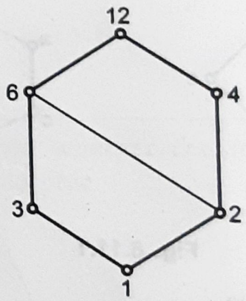


Fig. 6.12.3

Now

Element	1	2	3	4	6	12
Complement	12	No	4	3	No	1

Thus D_{12} is not a complemented lattice

Ex.6.12.5 : Prove that if $a, b \in L$ and L is bounded distributive lattice and a′ is the complement of a, then $a \vee (a' \wedge b) = a \vee b$ **AKTU : 2003-04**

and $a \wedge (a' \vee b) = a \wedge b$ **AKTU : 2004-05**

Sol. : Given that $(L, \vee, \wedge)$ is a bounded distributive lattice and a′ is the complement of a

$$\therefore \quad a \vee a' = 1 \text{ and } a \wedge a' = 0$$

(i) L.H.S. $= a \vee (a' \vee b)$

$= (a \vee a') \wedge (a \vee b)$

$= 1 \wedge (a \vee b) = a \vee b$

= R.H.S.

$\Rightarrow a \vee (a' \wedge b) = a \vee b$

(ii) L.H.S. $= a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$

$= 0 \vee (a \wedge b) = a \wedge b$

= R.H.S.

$\Rightarrow a \wedge (a' \vee b) = a \wedge b$

Ex.6.12.6 : Give an example of a finite lattice where atleast one element has more than one complement and at least one element has no complement. Show that the lattice (P (S), $\subseteq$) where P (S) is the power set of a finite set S is complemented lattice.

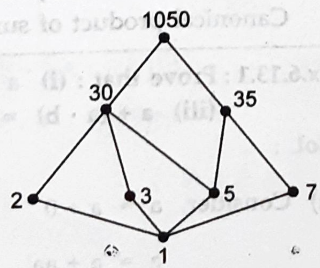
AKTU : 2012-13

Sol. :

Let $L = \{1050, 35, 30, 7, 5, 3, 2, 1\}$

$a \leq b$ iff a divides b $\forall$ a, b $\in$ L

(L, $\trianglelefteq$) is a poset. Its Hasse diagram is



We have,

$7 \vee 3 = 1, \quad 7 \wedge 2 = 1$

$7 \vee 3 = 1050, \ 7 \vee 2 = 1050$

$\therefore$ 7 has two complements 2 and 3.

$5 \vee 30 = 30$

$5 \vee 35 = 35$

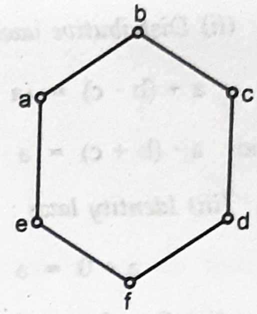
$5 \vee 1050 = 1050$

$\therefore$ There is no any element $a \in L$ such that

$5 \vee a = 1050$ and $5 \wedge a = 1$

$\therefore$ 5 does not have any complement.

Ex.6.12.7 : If the lattice is represented by the Hasse diagram given below :
i) Find all the complements of e
ii) Prove that the given lattice is bounded complemented lattice.



AKTU : 2014-15

Sol. : i) We have 0 = f and 1 = b.

Now $e \wedge d = 0$ and $e \vee d = 1$ and $e \wedge c = 0$,

$e \vee c = b = 1$

Thus e has two complements d and c.

ii) Given lattice has greatest and least elements. So it is bounded lattice.

Now

Element	a	b = 1	c	d	e	f = 0
Complement	c and d	0	a and e	a and e	c and d	1

Thus given lattice is a bounded complemented lattice.

Ex.6.12.8 : Give an example of a lattice which is a modular but not a distributive.

Sol. : Modular Lattice : A lattice L is said to be modular if for every x, y, z $\in$ L with $x \leq z$

$x \vee (y \wedge z) = (x \vee y) \wedge z$

Consider the Diamond latice See Fig. 6.11.1 (L_1). It is a modular lattice but not a distributive

6.13 Boolean Algebra

1. **Boolean algebra :** A non empty set with two binary operations '+' and '·', an unary operation ''' and two distinct elements 0 and 1 is called Boolean algebra, denoted by (B, +, ·, ', 0, 1) iff it satisfies the following properties. Let a, b $\in$ B then

(i) Commutative laws :

$a + b = b + a$ and $a \cdot b = b \cdot a$

(ii) *Distributive laws :*

$$a + (b \cdot c) = (a + b) \cdot (a + c)$$

and $a \cdot (b + c) = a \cdot b + a \cdot c$

(iii) *Identity laws :*

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

(iv) *Complement law :*

$$a + a' = 1 \quad \text{and} \quad a \cdot a' = 0$$

2. **Basic results in Boolean algebra :** Let $a, b, c \in B$ then

(i) *Idempotent laws :* $a + a = a$ and $a \cdot a = a$

(ii) *Boundedness laws :* $a + 1 = 1$ and $a \cdot 0 = 0$

(iii) *Absorption laws :*

$a + (a \cdot b) = a$ and $a \cdot (a + b) = a$

(iv) *Associative laws :* $a + (b + c) = (a + b) + c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(v) *Uniqueness and complements :* $a + x = 1$ and $a \cdot x = 0 \Rightarrow x = a'$

(vi) *Involution laws :* $(a')' = a$, $0' = 1$ and $1' = 0$

(vii) *Demorgan's laws :* $(a + b)' = a' \cdot b'$ and $(a \cdot b)' = a' + b'$

3. **Complement of a function :** The complement of a function F is $\overline{F}$. We can obtain $\overline{F}$ by interchanging 1's for 0's and 0's for 1's.

Demorgan's theorem :

(i) $(\overline{A+B+C}) = \overline{A} \cdot \overline{B} \cdot \overline{C}$

(ii) $(\overline{A \bullet B \bullet C}) = \overline{A} + \overline{B} + \overline{C}$

4. **Simplification of Boolean function :**

(i) *Sum of Products (SOP) :* A Boolean expression E is said to be in a sum of products form if E is a sum of product of variables.

e.g. $$Y = ABC + B\overline{C}D + \overline{A}\,\overline{B}\,\overline{C}$$

(ii) *Product of Sums (POS) :* A Boolean expression E is said to be in a product of sums form if E is a product of sum of terms.

e.g. $$Y = (A + B) + (\overline{A} + B)$$

(iii) *Minterm :* A minterm of n variables is a product of n literals in which each variable appears exactly once in either true or complemented form but not both.

Minterms for two variables are xy, xy', x'y, x'y'.

Minterms for 3 variables are xyz, x'yz, xy'z, xyz', x'y'z, x'yz', xy'z', x'y'z'.

There are 2^n minterms for n variables.

(iv) *Maxterm :* Maxterm of n variabels is the sum of n literals in which each variable appears exactly once in either true or complemented form but not both.

Maxterms for two variables are (x + y), (x' + y), (x + y'), (x' + y')

Maxterms for 3 variables are (x + y + z), (x' + y + z), (x + y' + z), (x + y + z'), (x' + y' + z), (x' + y + z'), (x + y' + z'), (x' + y' + z').

For n variables there will be 2^n maxterms.

(v) When a Boolean expression is written in sum of minterms form, it is referred as minterm expansion or disjunctive normal form. It is also called as canonical sum of products.

(vi) When a Boolean expression is written in product of maxterms form, it is referred as maxterm expansion or conjunction normal form or Canonical product of sums.

Ex.6.13.1 : Prove that : (i) $a + a = a$; (ii) $a + 1 = 1$; (iii) $a + (a \cdot b) = a$

Sol. :

(i) Consider, $a = a + 0$

$a = a + aa'$ ...(By Identity law)

$= (a + a) \cdot (a + a')$ ...(By Complement law)

$= (a + a) \cdot 1$

$a = a + a$

(ii) $a + 1 = a + a + a'$ ...(By Complement law)

$= (a + a) + a'$

$= a + a'$ ($\because$ idempotent)

$= a$

(iii) $a + (a \cdot b) = a \cdot 1 + a \cdot b$

$= a \cdot (1 + b)$

$= a \cdot 1$

$= a$

Ex.6.13.2 : Prove that $(A + B)(A + C) = A + BC$

Sol. : L.H.S. $= (A + B)(A + C)$

$= AA + AC + BA + BC$

$= A + AC + BA + BC$ ...($\because AA = 1$)

$= A(1 + C) + BA + BC$

$= A + AB + BC$ ...($\because 1 + C = 1$)

$= A(1 + B) + BC$

$= A + BC$ ...($\because 1 + B = 1$)

= R.H.S.

Ex.6.13.3 : Prove that $(A+\overline{B}+AB)(A+\overline{B})(\overline{A}B) = 0$

Sol. :

L.H.S. $= (A+\overline{B}+AB)(A+\overline{B})(\overline{A}B)$

$= (A+\overline{B})(A+\overline{B})(\overline{A}B)$

($\because A + AB = A$)

$= (AA+A\overline{B}+\overline{B}A+\overline{B}\overline{B})(\overline{A}B)$

$= (A+\overline{B}(A+A)+\overline{B})(\overline{A}B)$

($\because A + A = A$)

$= (A+\overline{B}A+\overline{B})(\overline{A}B)$

$= [(A(1+\overline{B})+\overline{B}](\overline{A}B)]$

$= (A+\overline{B})(\overline{A}B)$ ($\because 1+\overline{B} = 1$)

$= A\overline{A}B+\overline{B}\overline{A}B$

$= 0 + 0$ ($\because A\overline{A} = 0$ and $B\overline{B} = 0$)

$= 0$

Ex.6.13.4 : Simplify the following expression :

$Y = \overline{(\overline{AB}+\overline{A}+AB)}$

Sol. : $y = \overline{(\overline{AB}+\overline{A}+AB)}$

But $\overline{AB} = \overline{A}+\overline{B}$

...(De Morgan's first theorem)

$y = \overline{(\overline{A}+\overline{B}+\overline{A}+AB)}$

But $\overline{A}+A = \overline{A}$

$\therefore$ $y = \overline{(\overline{A}+\overline{B}+AB)}$

Now use De-Morgan's second theorem which states that,

$\overline{A+B+C} = \overline{A}\cdot\overline{B}\cdot\overline{C}$

$\therefore$ $y = \overline{\overline{A}}\cdot\overline{\overline{B}}\cdot\overline{AB}$

But $\overline{\overline{A}} = A$ and $\overline{\overline{B}} = B$

$y = A\cdot B\cdot\overline{AB}$

But $\overline{AB} = (\overline{A}+\overline{B})$

...(De-Morgan's second theorem)

$\therefore$ $y = A\cdot B(\overline{A}+\overline{B}) = A\overline{A}B+AB\overline{B}$

But $A\overline{A} = 0$ and $B\overline{B} = 0$

$\therefore$ $y = 0 \cdot B + A \cdot 0 = 0 + 0 = 0$

$\therefore$ $y = 0$

Ex.6.13.5 : For the given function, $F = x\overline{y}+x\overline{y}$, find the complement of 'F'.

Sol. : $F = x\overline{y}+x\overline{y}$

$F = x\overline{y}$ ...($A + A = A$)

Take the complement of both sides,

$\therefore$ $\overline{F} = \overline{x\overline{y}}$

Using De morgan's first law, we get,

$\overline{F} = \overline{x}+\overline{\overline{y}}$ ...(as $\overline{A\cdot B} = \overline{A}+\overline{B}$ $\overline{\overline{y}} = y$)

$\overline{F} = \overline{x}+y$

Ex.6.13.6 : Simplify :

$\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}+\overline{A}\,\overline{B}\,\overline{C}\,D+\overline{A}\,\overline{B}\,C\,D+\overline{A}\,\overline{B}\,C\,\overline{D}$
$= \overline{A}\,\overline{B}$

Sol. :

L.H.S. $= \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}+\overline{A}\,\overline{B}\,\overline{C}D+\overline{A}\,\overline{B}CD+\overline{A}\,\overline{B}C\overline{D}$

$= \overline{A}\,\overline{B}\,\overline{C}(D+D+\overline{A}\,\overline{B}C(D+\overline{D})$

But $\overline{D}+D = 1$

L.H.S. $= \overline{A}\,\overline{B}\,\overline{C}+\overline{A}\,\overline{B}C = \overline{A}\,\overline{B}(\overline{C}+C)$

But $\overline{C}+C = 1$

L.H.S. $= \overline{A}\,\overline{B} =$ R.H.S.

Ex.6.13.7 : Find the complement of the following functions :
$F_1 = \overline{A}\,\overline{B}\,\overline{C}+\overline{A}\,\overline{B}\,C$ and $F_2 = A(\overline{B}\,\overline{C}+BC)$

Sol. : (i) $F_1 = \overline{A}\,\overline{B}\,\overline{C}+\overline{A}\,\overline{B}\,C$

$\therefore \quad \overline{F_1} = \overline{(\overline{A}\,\overline{B}\,\overline{C})+(\overline{A}\,\overline{B}\,C)}$

$= \overline{(\overline{A}\,\overline{B}\,\overline{C})}\cdot\overline{(\overline{A}\,\overline{B}\,C)}$

$\therefore \quad \overline{F_1} = (A+\overline{B}+C)\cdot(A+B+\overline{C})$

(ii) $F_2 = A(\overline{B}\,\overline{C}+BC)$

$\overline{F_2} = \overline{[A(\overline{B}\,\overline{C}+BC)]} = [\overline{A}+\overline{(\overline{B}\,\overline{C}+BC)}]$

$= \overline{A}\,(\overline{\overline{B}\,\overline{C}})\cdot(\overline{BC})$

$\therefore \quad \overline{F_2} = \overline{A} + (B+C)(\overline{B}+\overline{C})$

Ex.6.13.8 : For the logic circuit shown in Fig. 6.13.1 Write the Boolean expression and simplify it.

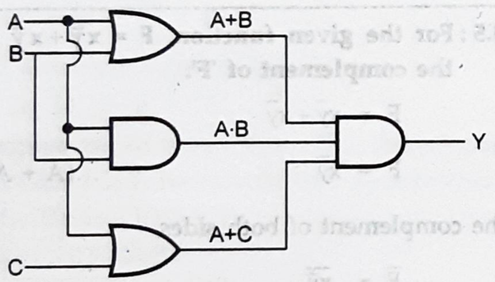


Fig. 6.13.1

Sol. :

Step 1 : Write the Boolean expression.
The expression for output for the given logic circuit is

$$Y = (A + B)\ (AB)\ (A + C)$$

Step 2 : Bring this expression in SOP form.
Multiply the terms to get the expression into SOP form.

$$Y = (A + B)\ (AAB + ABC)$$

$$Y = (A + B)\ (AB + ABC) \quad \dots (AA = A)$$

$$= AAB + AABC + BAB + BABC$$

$$= AB + ABC + AB + ABC$$

$$= AB + AB + ABC + ABC$$

But $AB + AB = AB$ and $ABC + ABC = ABC$

$\therefore \quad Y = AB + ABC = AB\,(1 + C)$

$Y = AB \quad \dots (1 + C = 1)$

This is simplified expression.

Ex.6.13.9 : Convert the following expression into their standard SOP or POS forms.
(a) $Y = AB + AC + BC$
(b) $Y = (A+B)\,(\overline{B}+C)$
(c) $Y = A + BC + ABC$
(d) $(x + y)\,(x' + y')$

Sol. : (a) $Y = AB + AC + BC$

$Y = AB+(C+\overline{C})+AC\,(B+\overline{B})+BC\,(A+\overline{A})$

$= ABC+AB\overline{C}+ACB+AC\,\overline{B}+BCA+BC\overline{A}$

$= ABC+ACB+BCA+AB\overline{C} = AC\overline{B}+BC\overline{A}$

$ABC + ACB + BCA = ABC \quad \dots (\text{As } A + A = A)$

$\therefore \quad Y = ABC+AB\overline{C}+A\overline{B}C+\overline{A}BC$

This is the required expression in standard SOP form.

(b) $Y = (A+B)(\overline{B}+C)$

$= (A+B+C\overline{C})(\overline{B}+C+A\overline{A})$

But $A + BC = (A + B)\,(A + C)$

$Y = (A+B+C)(A+B+\overline{C})(\overline{B}+C+A)(\overline{B}+C+\overline{A})$

This is in the standard POS form.

(c) $Y = A + BC + ABC$

$= A(B+\overline{B})+(C+\overline{C})+BC(A+\overline{A})+ABC$

$= ABC+AB\overline{C}+A\overline{B}C+A\overline{B}\,\overline{C}+BCA+BC\overline{A}+ABC$

$= (ABC+BCA+ABC)+AB\overline{C}+A\overline{B}C+A\overline{B}\,\overline{C}+BC\overline{A}$

as $A + A = A$

then $(ABC + BCA + ABC) = ABC$

$\therefore \quad Y = ABC+AB\overline{C}+A\overline{B}C+A\overline{B}\,\overline{C}+BC\overline{A}$

This expression is in the standard SOP form.

d) We have $(x + y)\,(x' + y')$

$= xx' + xy' + yx' + yy'$

which is the dnf.

Ex.6.13.10 : Explain how to write maxterms and minterms from truth table for three variables.

Sol. : Let A, B, C be three variables and Y be its output. The concept of maxterms and minterms allow us to introduce a very convenient shorthand notation to express logic functions.

Consider the following table

Variables	Minterms	Maxterms
A B C	m_i	M_i
0 0 0	$m_0 = \overline{A}\,\overline{B}\,\overline{C}$	$M_0 = A + B + C$
0 0 1	$m_1 = \overline{A}\,\overline{B}C$	$M_1 = A + B + \overline{C}$
0 1 0	$m_2 = \overline{A}B\overline{C}$	$M_2 = A + \overline{B} + C$
0 1 1	$m_3 = \overline{A}BC$	$M_3 = A + \overline{B} + \overline{C}$
1 0 0	$m_4 = A\overline{B}\,\overline{C}$	$M_4 = \overline{A} + B + C$
1 0 1	$m_5 = A\overline{B}C$	$M_5 = \overline{A} + B + \overline{C}$
1 1 0	$m_6 = AB\overline{C}$	$M_6 = \overline{A} + \overline{B} + C$
1 1 1	$m_7 = ABC$	$M_7 = \overline{A} + \overline{B} + \overline{C}$

Ex.6.13.11 : For the following truth table of 3 variables A, B, C. Write the logic expression in the standard SOP form and POS form.

Sol. : Given truth table is

A	B	C	Y		Product terms
0	0	0	0	$\rightarrow$	$M_0 = A + B + C$
0	0	1	1	$\leftarrow$	$\overline{A}\,\overline{B}\,C(m_1)$
0	1	0	0	$\rightarrow$	$M_2 = A + \overline{B} + C$
0	1	1	0	$\rightarrow$	$M_3 = A + \overline{B} + \overline{C}$
1	0	0	1	$\leftarrow$	$A\,\overline{B}\,\overline{C}(m_4)$
1	0	1	0		$M_5 = \overline{A} + B + \overline{C}$
1	1	0	0		$M_6 = \overline{A} + \overline{B} + C$
1	1	1	1	$\leftarrow$	$ABC(m_7)$

Consider the product terms for which output Y = 1

OR (add) all the product terms

$$Y = \overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + ABC$$

which is the requierd logic expression in standard SOP form.

This expression can also be written as,

$$Y = m_1 + m_4 + m_7 = \sum m(1,4,7)$$

Consider the maxterms for which Y = 0

$$\therefore \quad M_0 = A + B + C,\ M_2 = A + \overline{B} + C,$$

$$M_3 = A + \overline{B} + \overline{C}$$

$$M_5 = \overline{A} + B + \overline{C} \text{ and } M_6 = \overline{A} + \overline{B} + C$$

Therefore, the standard POS form is

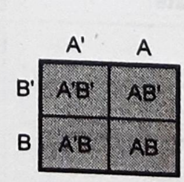
$$Y = (A + B + C)\cdot(A + \overline{B} + C)\cdot(A + \overline{B} + \overline{C})\cdot(\overline{A} + B + \overline{C})\cdot(\overline{A} + \overline{B} + C)$$

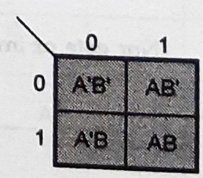
6.14 Karnaugh Map (K-map)

AKTU : 2010-11, 2011-12, 2016-17

Karnaught map method is a graphical technique which provides a simple straight forward procedure for simplification of Boolean expression of two, three or four variables. It can also be extended for five, six or more variables.

1. **Two variable Karnaugh maps :** The number of variables are 2 so the map will have $2^2 = 4$ square. In this case four possible minterms with two variables A and B i.e. AB, AB', A'B , A'B' are represented by four squares in the map labelled below.





The expression can be simplified by properly combining those squares in the K-map which contain 1s. The process for combining 1s is called looping.

2. **Three variable K-map :** The number of variables are 3 so map will have $2^3 = 8$ squaers. The eight possible minterms are labelled as shown in Fig. 6.14.1

Given a minterm expansion of a function, it can be plotted on a map by placing 1s in the square which corresponds to minterms present in the expression and 0s in the remaining squares.

	A'B'	A'B	AB	AB'
C'	A'B'C'	A'BC'	ABC'	AB'C'
C	A'B'C	A'BC	ABC	AB'C

C \ AB	00	01	11	10
0	A'B'C'	A'BC'	ABC'	AB'C'
1	A'B'C	A'BC	ABC	AB'C

Fig. 6.14.1

3. **Four variable Karnaught map :** The number of variables are 4. Hence map will be $2^4 = 16$ squares. Fig. 6.14.2 shows the K-map for four variables A, B, C and D and alternative way of representing four variables.

A'B'C'D'	A'BC'D'	ABC'D'	AB'C'D'
A'B'C'D	A'BC'D	ABC'D	AB'C'D
A'B'CD	A'BCD	ABCD	AB'CD
A'B'CD'	A'BCD'	ABCD'	AB'CD'

CD \ AB	00	01	11	10
	A'B'C'D'	A'BC'D'	ABC'D'	AB'C'D'
	A'B'C'D	A'BC'D	ABC'D	AB'C'D
	A'B'CD	A'BCD	ABCD	AB'CD
	A'B'CD'	A'BCD'	ABCD'	AB'CD'

Fig. 6.14.2

To simplify a sum of product expression in four variable one has to identify groups of minterms of squares 2, 4, 8 of 16 containing 1s that can be combined.

Logic gates :

Logic gates are the devices used as basic building blocks of all the digital circuits. The Boolean algebra developed by Charles Boole way back in 1884 is used for representing, simplifying and analysing the logic circuits. The basic logic gates are NOT, AND and OR along with NOR, NAND, Ex-OR etc.

The relation between the inputs and the outputs of a gate can be expressed mathematically by means of the Boolean expression. In order to understand Boolean algebra, we need to use the gates. So the symbols and Boolean expressions should be known to us which is given as follows :

Various logic gates :

Sr. No	Name of gate	Boolean expression	Truth table	Logical operation
1.	Nor gate or inverter 	$Y = \overline{A}$	A | Y: 0 | 1; 1 | 0	Inversion
2.	AND gate 	Y = AB	A | B | Y: 0 | 0 | 0; 0 | 1 | 0; 1 | 0 | 0; 1 | 1 | 1	Logical multiplication

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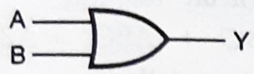
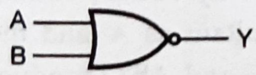
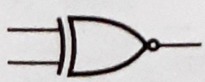
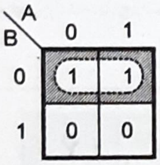
<tr><td>3.</td><td>OR gate
</td><td>$Y = A + B$</td><td><table><tr><th>A</th><th>B</th><th>Y</th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table></td><td>Logical additon</td></tr>
<tr><td>4.</td><td>NAND gate
</td><td>$Y = \overline{AB}$</td><td><table><tr><th>A</th><th>B</th><th>Y</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table></td><td>NOT AND</td></tr>
<tr><td>5.</td><td>NOR gate
</td><td>$Y = \overline{A + B}$</td><td><table><tr><th>A</th><th>B</th><th>Y</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table></td><td>NOT OR</td></tr>
<tr><td>.</td><td>Exclusive OR
</td><td>$Y = A \oplus B$</td><td><table><tr><th>A</th><th>B</th><th>Y</th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table></td><td>Addition/ subtraction</td></tr>
<tr><td>.</td><td>Exclusive NOR X-NOR
</td><td>$Y = \overline{A \oplus B}$</td><td><table><tr><th>A</th><th>B</th><th>Y</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr></table></td><td>NOT EX-OR</td></tr>
</table>

Ex.6.14.1 : Find the K-map and simplify the expression for AB' + A'B'

Sol. : Two adjacent square A'B' and AB' containing 1 have been grouped together. They have been circled. These two terms can be looped that eliminates the A variable since it appears both in complemented and uncomplemented forms.



This can be verified algebraically as follows :

$$AB' + A'B' = (A + A')\,B' = 1 \cdot B' = B'$$

Ex.6.14.2 : Use the K-map to simplify the following :
(i) X = ABC' + ABC ;
(ii) X = A'B'C' + AB'C'

Sol. : (i) The Boolean function

X = ABC' + ABC is shown in Fig. 6.14.3 in the K-map as follows :

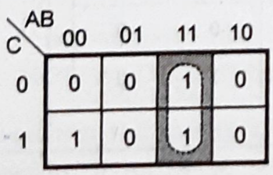


Fig. 6.14.3

The adjacent square representing ABC' and ABC are grouped together. This eliminates the C variable since it appears in both uncomplemented and complemented form. The simplified function will be X = AB.

(ii) The Boolean function X = A'B'C' + AB'C' is shown in Fig. 6.14.4 in the K-map as

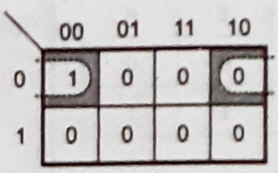
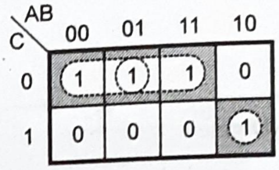


Fig. 6.14.4

Thus the two 1s in this map can be looped to provide a simplified result X = B'C'.

Ex.6.14.3 : Use the K-map to simplify the following
(i) X = A'B'C' + A'BC' + ABC' + AB'C
(ii) X = A'B'C' + A'B'C + A'BC + A'BC' + AB'C + ABC

Sol. : (i) The Boolean function X = A'B'C' + A'BC' + ABC' + AB'C and the K-map of three variable is as follows :



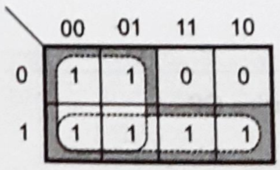
The group of first two horizontal 1 square gives A'C' and the group second and third horizontal 1 square gives BC'.

Hence, the simplfied result is

$$X = A'C' + BC' + AB'C$$

(ii) The K-map of given function

The quad formed by A'B'C', A'BC', A'B'C and A'BC produces the resultant as A' and the quad formed by A'B'C, A'BC, ABC and AB'C produces the resultant as C.

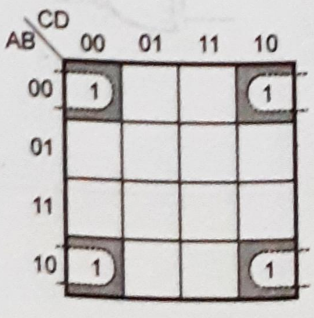


Hence final result is ,

$$X = A' + C$$

Ex.6.14.4 : Use the K-map to simplify
X = A'B'C'D' + AB'C'D' + A'B'CD' + AB'CD'

Sol. : K-map of the Boolean expression is



Sol. : Boolean function of degree n :

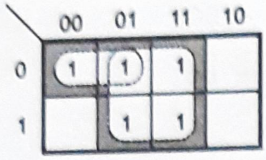


Fig. 6.14.10

Let $B = \{0, 1\}$

Then $B^n = \{(x_1, x_2, x_3 \cdots\cdots x_n) x_i \; B \text{ for } 1 \le i \le n\}$

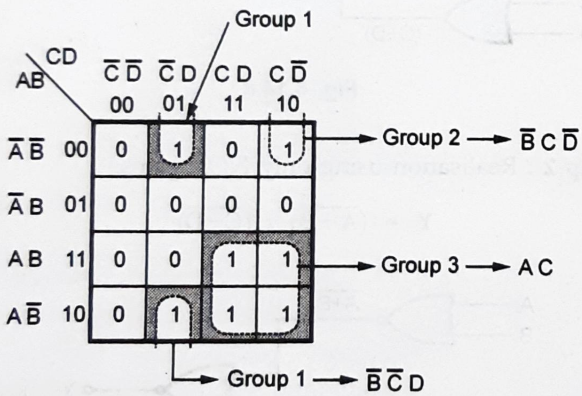
is the set of all possible n-tuples of 0's and 1's. The variable x is called a Boolean variable. B is called Boolean function of degree n.

The K-map for given Boolean function is shown in Fig. 6.14.10

Then the simplified expression is z + x'y'

Ex.6.14.9 : Minimize the following expression using K-map and realize using the basic gates.
$y = \sum m \;(1, 2, 9, 10, 11, 14, 15)$

Sol. :



Minimized experssion :

$$y = \bar{B}\bar{C}D + \bar{B}C\bar{D} + AC$$

$$= \bar{B}(\underbrace{\bar{C}D \oplus C\bar{D}}_{\text{EX-OR gate}}) + AC$$

$$= \bar{B}(C \oplus D) + AC$$

Realisation with minimum number of gates :

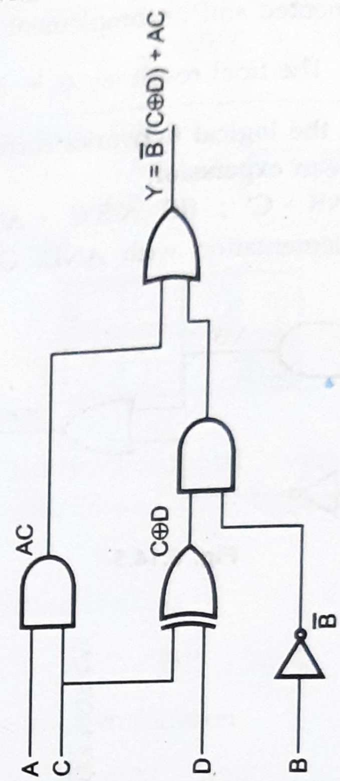


Fig. 6.14.11

Ex.6.14.10 : Describe the Boolean duality principle. Write the dual of each Boolean equations.
(i) $x + \bar{x}y = x + y$; (ii) $(x \cdot 1)(0 + \bar{x}) = 0$

AKTU : 2010-11

Sol. : Boolean duality principle :

In duality of Boolean function, we replace 0 by 1 and 1 by 0 and also · by +

· and · by +

(i) $x + \bar{x}y = x + y$

It's dual is $x \cdot \bar{x} + y = x \cdot y$

(ii) $(x \cdot 1)(0 + \bar{x}) = 0$

It's dual is $(x + 0)(1 \cdot \bar{x}) = 1$

Ex.6.14.11 : Simplify the following Boolean function using K-map f (x, y, z) = S (0, 2, 3, 7)

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Sol. : The given Boolean function can be represented by K-map as shown in Fig. 6.14.12 on simplification.

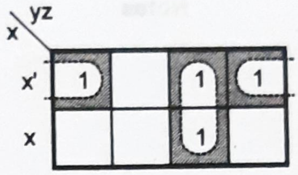

Fig. 6.14.12

The simplified function is f = yz + x' z'

Ex.6.14.12 : Find the Boolean algebra expression for the following system **AKTU : 2016-17**

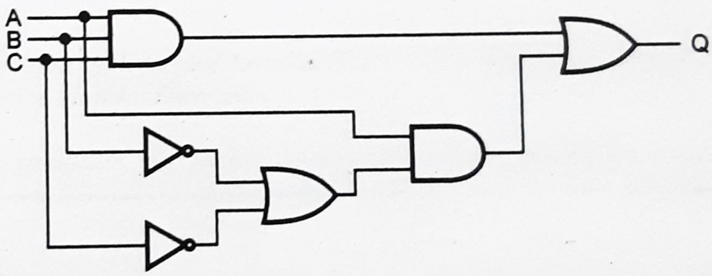

Fig. 6.14.13

Sol. : Given that

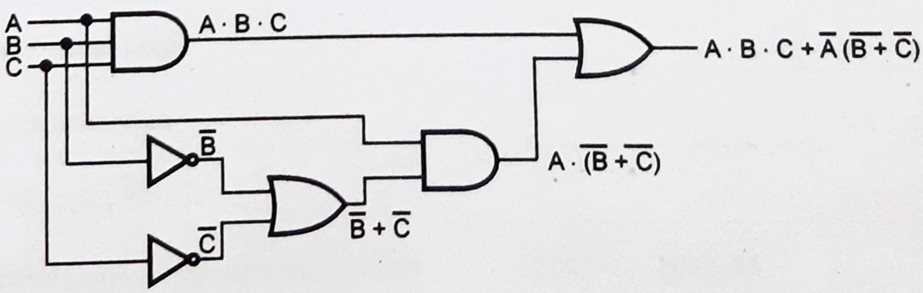

Fig. 6.14.14

From above figure, the required Boolean expression is

$A \cdot B \cdot C + A \cdot (\overline{B} + \overline{C}) = Q$