Discrete Structures and Theory of logic
KCS-303

Unit 1
SET THEORY

Topics Covered

Set Theory - Introduction

Types of Set

Combination of Sets

Multisets

Discrete vs. Continuous Mathematics

Continuous Mathematics

It considers objects that vary continuously;

Example: analog wristwatch (separate hour, minute, and second hands).

From an analog watch perspective, between 1 :25 p.m. and 1 :26 p.m. there are infinitely many possible different times as the second hand moves around the watch face.

Discrete Mathematics

It considers objects that vary in a discrete way.

Example: digital wristwatch.

On a digital watch, there are only finitely many possible different times between 1 :25 P.M. and 1:27 P.M.

A digital watch does not show split seconds: - no time between 1 :25:03 and 1 :25:04. The watch moves from one time to the next.

SET THEORY

Set – Definition

A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

- Some Example of Sets
- A set of all positive integers
- A set of all the planets in the solar system
- A set of all the states in India
- A set of all the lowercase letters of the alphabet

Representation of a Set

Sets can be represented in two ways −

i. Roster or Tabular Form

ii. Set Builder Notation

iii. Descriptive form

i) Roster or Tabular Form

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example 1 − Set of vowels in English alphabet, A={a,e,i,o,u}
Example 2 − Set of odd numbers less than 10, B={1,3,5,7,9}

2. Set Builder Notation

The set is defined by specifying a property that elements of the set have in common. The set is described as

$A=\{x:p(x)\}$

Example 1 – The set vowels in English {a,e,i,o,u} is written as –

$A=\{x:x \text{ is a vowel in English alphabet}\}$

Example 2 – The set {1,3,5,7,9} is written as –

$B=\{x:1\leq x<10 \text{ and } (x\%2)\neq 0\}$

If an element x is a member of any set S, it is denoted by $x \in S$ and if an element y is not a member of set S, it is denoted by $y \notin S$.

Example − If $S=\{1,1.2,1.7,2\}$, $1 \in S$ but $1.5 \notin S$

Descriptive Form:

- Stating in words the elements of a set.

EXAMPLES

- Now we will write the same examples which we write in Tabular Form ,in the Descriptive Form.
- A = set of first five Natural Numbers.(is the Descriptive Form)
- B = set of positive even integers less or equal to fifty. (is the Descriptive Form)
- C = {1, 3, 5, 7, 9, ...} (is the Tabular Form)
- C = set of positive odd integers. (is the Descriptive Form)

Set Theory

- Set: Collection of objects (called elements or members)
- $a \in A$ "a is an element of A"
  "a is a member of A"
- $a \notin A$ "a is not an element of A"
- $A = \{a_1, a_2, \ldots, a_n\}$ "A contains $a_1, \ldots, a_n$"
- Order of elements is insignificant
- e.g. $\{a,b,c\} = \{c,b,a\} = \{a,a,b,c,b\}$
- It does not matter how often the same element is listed (repetition doesn't count).

Basic properties of sets

- Sets are inherently ***unordered***:
  - No matter what objects a, b, and c denote,
    {a, b, c} = {a, c, b} = {b, a, c} =
    {b, c, a} = {c, a, b} = {c, b, a}.
- All elements are ***distinct*** (unequal);
  multiple listings make no difference!
  - {a, b, c} = {a, a, b, a, b, c, c, c, c}.
  - This set contains at most 3 elements!

Examples for Sets

- $A = \varnothing$ "empty set/null set"
- $A = \{z\}$ Note: $z \in A$, but $z \neq \{z\}$
- $A = \{\{b, c\}, \{c, x, d\}\}$ set of sets
- $A = \{x \mid P(x)\}$ "set of all x such that P(x)"

  P(x) is the membership function of set A

- $A = \{x \mid x \in \mathbf{N} \land x > 7\} = \{8, 9, 10, \ldots\}$
  "set builder notation"

Standard Symbols which denote sets of numbers

- **N** : The set of all-natural numbers (i.e., all positive integers)
- **Z** : The set of all integers
- **Z+** : The set of all positive integers
- **Z*** : The set of all nonzero integers
- **E** : The set of all even integers
- **Q** : The set of all rational numbers
- **Q*** : The set of all nonzero rational numbers
- **Q+** : The set of all positive rational numbers
- **R** : The set of all real numbers
- **R*** : The set of all nonzero real numbers
- **R+** : The set of all positive real numbers
- **C** : The set of all complex numbers
- **C*** : The set of all nonzero complex numbers

Cardinality of a Set

- Cardinality of a set S, denoted by |S| or n(S), is the number of elements of the set.
- The number is also referred as the cardinal number.
- If a set has an infinite number of elements, its cardinality is ∞
- **Example** − $|\{1,4,3,5\}|=4$

  $|\{1,2,3,4,5,\ldots\}|=\infty$

Types of Sets

- **Finite and infinite Set**
- **Null Set or Empty Set**
- **Singleton Set**
- **Disjoint Sets**
- **Family of set(set of set)**
- **Subset**
- **Number of Subsets of a set**
- **Super Set**
- **Proper Subset**
- **Equal Set**
- **Universal Set**

TYPES OF SET

1. **Finite Set**
A set which contains a definite number of elements is called a finite set.
Example − $S=\{x|x\in N \text{ and } 70>x>50\}$

2.Infinite Set
A set which contains infinite number of elements is called an infinite set.
Example − $S=\{x|x\in N \text{ and } x>10\}$

3.Subset
A set X is a subset of set Y (Written as $X\subseteq Y$) if every element of X is an element of set Y.

Example 1 − Let, $X=\{1,2,3,4,5,6\}$ and $Y=\{1,2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y\subseteq X$

Example 2 − Let, $X=\{1,2,3\}$ and $Y=\{1,2,3\}$. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y\subseteq X$

4. Proper subset :

The term “proper subset” can be defined as “subset of but not equal to”. A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X|<|Y|$

Example − Let, $X=\{1,2,3,4,5,6\}$ and $Y=\{1,2\}$. Here set $Y \subset X$ since all elements in Y are contained in X too and X has at least one element is more than set Y.

5. Universal Set:

It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.

Example − We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U and so on.

6. Empty Set or Null Set

An empty set contains no elements. It is denoted by $\emptyset$ or $\{\}$. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example −$S=\{x|x\in N \text{ and } 7<x<8\}$

7.Singleton Set or Unit Set

Singleton set or unit set contains only one element. A singleton set is denoted by $\{s\}$.

Example − $S=\{x|x\in N, 7<x<9\} = \{8\}$

8.Equal Set

If two sets contain the same elements, they are said to be equal.

Example − If $A=\{1,2,6\}$and $B=\{6,1,2\}$ they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

9. Family of set(set of set): If a set A contains elements which are itself sets then it is called family of set.

A = { 1, 2, {3, 4}, 9}

10.Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example − If A={1,2,6} and B={16,17,22} they are equivalent as cardinality of A is equal to the cardinality of B. i.e. |A|=|B|=3

Subsets

- $A \subseteq B$ "A is a subset of B"
- $A \subseteq B$ if and only if every element of A is also an element of B.
- We can completely formalize this:
- $A \subseteq B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B)$

- Examples:

A = {3, 9}, B = {5, 9, 1, 3}, A ? B ? true

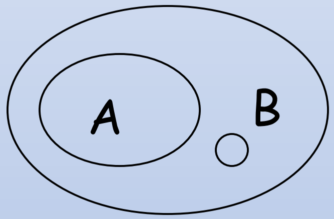
A = {3, 3, 3, 9}, B = {5, 9, 1, 3}, A ? B ? true

A = {1, 2, 3}, B = {2, 3, 4}, A ? B ? false

Proper Subset, Equality

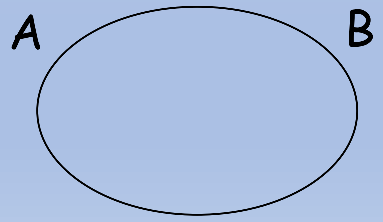
Definition: Given two sets A and B, we say A is a **proper subset** of B, denoted by $A \subset B$, if every element of A is an element of B,

But there is an element in B that is not contained in A.



Fact: If $A \subset B$, then $|A| < |B|$.

Definition: Given two sets A and B, we say A = B if $A \subseteq B$ and $B \subseteq A$.



Fact: If A = B, then $|A| = |B|$.

Improper Subset

If A = B and $A \subseteq B$ Then it is called improper sub set.

Example: A = { 2, 3, 4} and B = { 2, 3, 4}

* Every set is improper subset of itself.

Superset: A set **A** is a superset of another set **B** if all elements of the set **B** are elements of set **A.**

- **Power Set:** The power set of any given set A is the set of all subsets of A and it is denoted by P(A). If A has n elements then P(A) has 2^n elements.

  Exp: A = {1, 2} then

  P(A) = {∅ , {1}, {2}, {1,2}}

- **Cardinality of Finite Set:** The number of elements in the set is cardinality of set. The cardinality of a set A is denoted by |A|.

Exp: A = {1, 2, 3, 4}

|A| = 4

Properties of Subset

- **$\varnothing \subseteq A \subseteq U$**
- **$A \subseteq A$**
- **$A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$**
- **A finite set having n elements has 2^n subset.**

The empty set is a set which has no elements.

- It is also called null set.
- It is denoted by $\emptyset$ or by $\{\}$.

Note the subtlety in $\emptyset \neq \{\emptyset\}$

- The left-hand side is the empty set
- The right hand-side is a singleton set, and a set containing a set

{ } **or Ø**

Empty Set

$$\emptyset = \{\} \qquad \emptyset \neq \{\emptyset\}$$

10. Overlapping Set

Two sets that have at least one common element are called overlapping sets.

Example − Let, A={1,2,6} and B={6,12,42}. There is a common element ‘6’, hence these sets are overlapping sets.

11. Disjoint Set

Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties −

- $n(A \cap B)=\emptyset$
- $n(A \cup B)=n(A)+n(B) n(A \cup B)=n(A)+n(B)$

Example − Let, A={1,2,6} and B={7,9,14} there is not a single common element, hence these sets are overlapping sets.

SUPERSETS AND PROPER SUPERSETS

Definition of superset

A set A is a superset of another set B if all elements of the set B are elements of the set A. The superset relationship is denoted as $A \supset B$.

For example, if A is the set {◊,♡,♣,♠} and B is the set {◊,♣,♠}, then $A \supset B$ but $B \supset A$ Since A contains elements not in B, we can say that A is a proper superset of B.

1.3.1 Super Set

If $A \subseteq B$ then B is called a super set of A and we write $B \supseteq A$.

1.3.2 Proper Subset

If $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B and we write $A \subset B$.

***Illustrations** of Subsets. Super set and proper subset*

(i) Let $A = \{2, 3, 5\}$ and $B = \{2, 3, 5, 7, 9\}$ then, every element of A is an element of B but $A \neq B$

$\therefore \quad A \subset B$, *i.e*, A is proper subset of B

(ii) For every set A, we have $A \subseteq A$, since every element of A is an element of A

(iii) Since ϕ has no element, we agree to say that ϕ is a subset of every set.

(iv) Let $A = \{1, \{2, 3\}, 4\}$

Thus $\{1\} \subset A$, $\{2, 3\} \in A$, and $\{4\} \subset A$.

But $\{2, 3\} \subseteq A$ is a wrong statement in context of this example. The correct statement is $\{\{2, 3\}\} \subseteq A$

Proper superset definition

- A proper superset of a set A is a superset of A that is not equal to A. In other words, if B is a proper superset of A, then all elements of A are in B but B contains at least one element that is not in A.

- For example, if A={1,3,5} then B={1,3,4,5} is a proper superset of A. The set C={1,3,5} is a superset of A, but it is not a proper superset of A since C=A.
- The set D={1,3,7} is not even a superset of A, since D does not contain the element 5

POWER SET

- A Power Set is a set of all the subsets of a set.
- The power set of S is denoted by P(S).
- Notation
- The number of members of a set is often written as |S|, so we can write

$$|P(S)| = 2^n$$

12. Power Set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set $\varnothing$
- What is the power set of $\varnothing$? $P(\varnothing) = \{ \varnothing \}$
- What is the cardinality of $P(\varnothing)$? $| P(\varnothing) | = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

- $A = \{a, b, c, d\}$

The power set of A is $2^4 = 16$

P(A)={}, {a}, {b}, {a, b}, {c}, {a, c}, {b, c}, {a, b, c}, {d}, {a, d}, {b, d}, {a, b, d}, {c, d}, {a, c, d}, {b, c, d}, {a, b, c, d}.

- $B = \{1, 2, 3\}$

The power set of B is $2^3 = 8$

P(B)={}, {1}, {2}, {1, 2}, {3}, {1, 3}, {2, 3}, {1, 2, 3}

- $P(\{1\}) = \{\varnothing, \{1\}\}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$
- $|P(\{1,2\})| = 4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $|P(\{1,2,3\}| = 8$

QUESTION:-

- FIND POWER SET OF

C= {1,a,2,b,c}

- $C=\{a,1,b,2,c\}$

$P(C)=\{\}, \{a\}, \{1\}, \{a, 1\}, \{b\}, \{a, b\}, \{1, b\}, \{a, 1, b\}, \{2\}, \{a, 2\}, \{1, 2\}, \{a, 1, 2\}, \{b, 2\}, \{a, b, 2\}, \{1, b, 2\}, \{a, 1, b, 2\}, \{c\}, \{a, c\}, \{1, c\}, \{a, 1, c\}, \{b, c\}, \{a, b, c\}, \{1, b, c\}, \{a, 1, b, c\}, \{2, c\}, \{a, 2, c\}, \{1, 2, c\}, \{a, 1, 2, c\}, \{b, 2, c\}, \{a, b, 2, c\}, \{1, b, 2, c\}, \{a, 1, b, 2, c\}.$

Combinations of Set

Venn Diagrams:

To visualize the interaction of Sets, John Venn in 1880 thought to use overlapping circles. These illustrations are called Venn Diagrams.

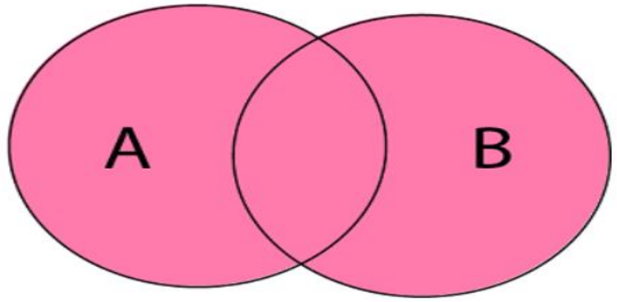
- A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Set Operations

1. Union

Definition: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.

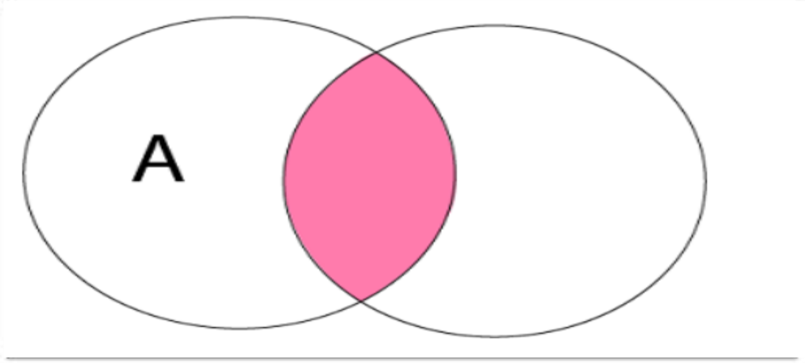


- **Example:**
- $A = \{1,2,3,6\}$ $B = \{ 2,4,6,9\}$
- $A \cup B = \{ 1,2,3,4,6,9 \}$

2.Intersection

Definition: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \wedge x \in B \}$.



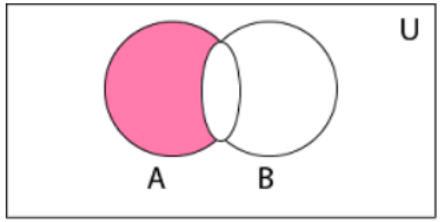
Example:

- $A = \{1,2,3,6\}$ $\qquad$ $B = \{ 2, 4, 6, 9\}$
- $A \cap B = \{ 2, 6 \}$

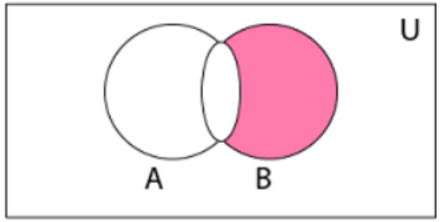
3. Set Difference

Definition: Let A and B be sets. The **difference of A and B**, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{ x \mid x \in A \wedge x \notin B \}$.



A - B



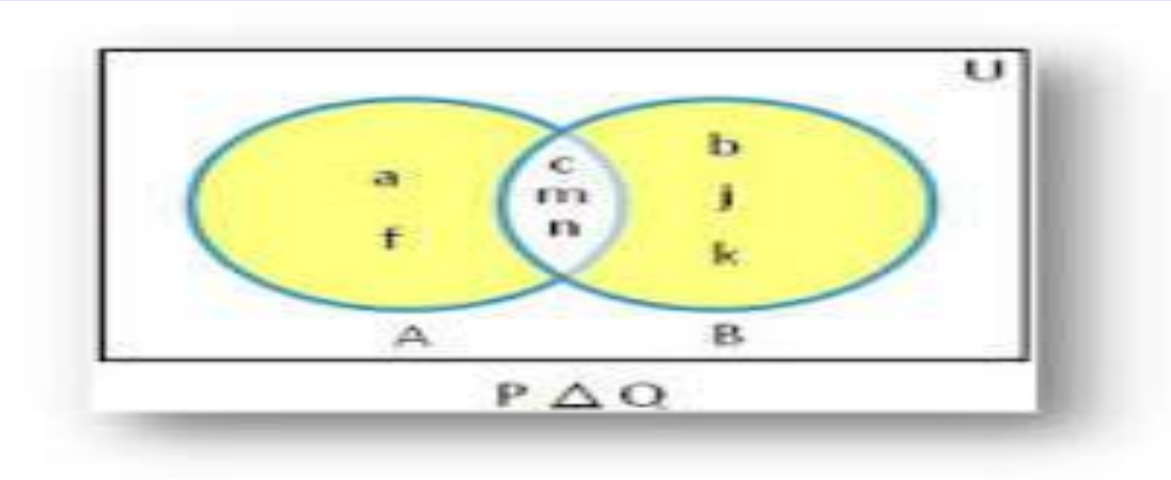
B - A

Example: A= {1,2,3,5,7} B = {1,5,6,8}

- A - B ={2,3,7}

3. Symmetric Difference

- The symmetric difference of set A with respect to set B is the set of elements which are in either of the sets A and B, but not in their intersection. This is denoted as $A \triangle B$ or $A \ominus B$ or $A \oplus B$.
- Using set notation, we can also denote this as $(A \cup B) - (A \cap B)$. Symmetric difference is also known as disjunctive union.
- For example, the symmetric difference of the sets $\{1,2,3\}$and $\{3,4\}$is $\{1,2,4\}$.



4. Cartesian Product

Definition: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where $s \in S$ and $t \in T$. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T \}$.

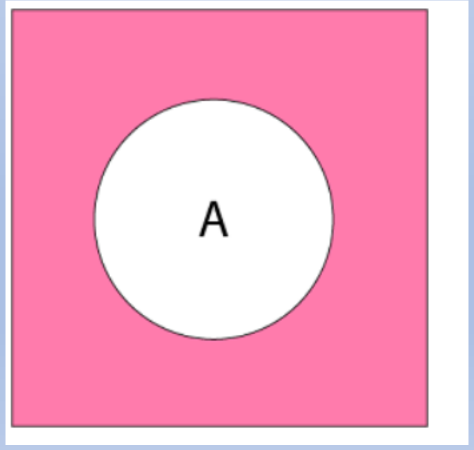
Examples:

- $S = \{1,2\}$ and $T = \{a,b,c\}$
- $S \times T = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) \}$
- $T \times S = \{ (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) \}$
- Note: $S \times T \neq T \times S$!!!!

5. Complement of a Set

Definition : The Complement of a Set A is a set of all those elements of the universal set which do not belong to A and is denoted by A^c.

Example: Let U is the set of all natural numbers.
$A = \{1, 2, 3\}$
$A^c = \{$all natural numbers except 1, 2, and 3$\}$



Multisets

- A multiset is an unordered collection of elements, in which the multiplicity of an element may be one or more than one or zero. The multiplicity of an element is the number of times the element repeated in the multiset. In other words, we can say that an element can appear any number of times in a set.
- Example:

1.A = {l, l, m, m, n, n, n, n}   Multiciplicity of l = 2 m=2 n=4

1.B = {a, a, a, a, a, c}

Operations on Multisets

1. Union of Multisets:

The Union of two multisets A and B is a multiset such that the multiplicity of an element is equal to the maximum of the multiplicity of an element in A and B and is denoted by $A \cup B$.

Example:

1. Let A = {l, l, m, m, n, n, n, n} B = {l, m, m, m, n},

$A \cup B$ = {l, l, m, m, m, n, n, n, n}

2. Intersections of Multisets: The intersection of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the minimum of the multiplicity of an element in A and B and is denoted by $A \cap B$.

Example:

1. Let A = {l, l, m, n, p, q, q, r} B = {l, m, m, p, q, r, r, r, r}
2. $A \cap B$ = {l, m, p, q, r}.

3. Difference of Multisets:

The difference of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the multiplicity of the element in A minus the multiplicity of the element in B if the difference is +ve, and is equal to 0 if the difference is 0 or negative

Example:

1. Let A = {l, m, m, m, n, n, n, p, p, p}   B = {l, m, m, m, n, r, r, r}

A - B = {n, n, p, p, p}

4. Sum of Multisets: The sum of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the sum of the multiplicity of an element in A and B

Example:

1. Let A = {l, m, n, p, r}

B = {l, l, m, n, n, n, p, r, r}

1. A + B = {l, l, l, m, m, n, n, n, n, p, p, r, r, r}

5. Cardinality of Multisets : The cardinality of a multiset is the number of distinct elements in a multiset without considering the multiplicity of an element

Example:

1. A = {l, l, m, m, n, n, n, p, p, p, p, q, q, q}

The cardinality of the multiset A is 5.

PARTITION OF SETS

Partition of a set, say *S*, is a collection of *n* disjoint subsets, say $P_1, P_2, \ldots P_n$ that satisfies the following three conditions −

- P_i does not contain the empty set.

$$[P_i \neq \{\emptyset\} \; for \; all \; 0 < i \leq n]$$

- The union of the subsets must equal the entire original set.

$$[P_1 \cup P_2 \cup \cdots \cup P_n = S]$$

- The intersection of any two distinct sets is empty.

$$[P_a \cap P_b = \{\emptyset\}, \; for \; a \neq b \; where \; n \geq a, \; b \geq 0]$$

Example

Let S={a,b,c,d,e,f,g,h}S={a,b,c,d,e,f,g,h}

One probable partitioning is {a},{b,c,d},{e,f,g,h}{a},{b,c,d},{e,f,g,h}

Another probable partitioning is {a,b},{c,d},{e,f,g,h}{a,b},{c,d},{e,f,g,h}

Bell Numbers

Bell numbers give the count of the number of ways to partition a set. They are denoted by B_n where n is the cardinality of the set.

Example −

Let S={1,2,3}S={1,2,3}, n=|S|=3n=|S|=3

The alternate partitions are −

1. ∅,{1,2,3}∅,{1,2,3}
2. 2_n. {1},{2,3}{1},{2,3}
3. {1,2},{3}{1,2},{3}
4. {1,3},{2}{1,3},{2}
5. {1},{2},{3}{1},{2},{3}

Hence B3=5