HOMOMORPHISM (HOMOMORPHIC MAPPING)

→ Suppose G and G' are two groups. A mapping f of G onto G' is said to be homomorphic if

$$\boxed{f(a \cdot b) = f(a) \cdot f(b)} \quad \forall\, a, b \in G.$$

→ If f is homomorphic mapping of a group 'G' onto group 'G'' so that $\boxed{f(G) = G'}$ then the group G' is called a homomorphic image of group G.

– In general, a mapping $f: G \rightarrow G'$ where $(G, *)$ and $(G', \circ)$ are groups is called homomorphism if for each $a, b \in G$ then,

$$f(a * b) = f(a) \circ f(b).$$

Eg. R is a set of non zero real nos

$$G = (R, +) \qquad G' = (R, \times).$$

$$f(a+b) = f(a) \times f(b).$$

Homomorphism into :- G into G'.

" onto :- G onto G'.

Eg Let G be grp of all real nos under addition and let G' be the grp of non-zero real nos under multiplication and $f: G \to G'$ defined by $f(a) = 2^a$ $\forall\, a \in G$ then it is homomorphic

becoz:—

$$f(a+b) = 2^{a+b} = 2^a\, 2^b = f(a) \cdot f(b)$$

Eg Let G be a grp of integers under addition and $G = G'$ and let $f(x) = 3x$ $\forall\, x \in G$ then

$$\boxed{f(x+y) = 3(x+y) = f(x) + f(y)}$$

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If R is the group of real nos under addition and R^+ be the group of real nos under ($\times$)

$f : R \to R^+$ $(R, +)$ $(R^+, \times)$

$f(x) = e^x$. Show f is homomorphic.

Solⁿ

$$f(a+b) = f(a) \cdot f(b)$$

$$f(a+b) = e^a \cdot e^b.$$

$$e^{a+b} = e^a \cdot e^b$$

$$e^a \cdot e^b = e^a \cdot e^b$$

$\therefore$ f is homomorphic.

ENDOMORPHISM :-

A homomorphism of grp into itself is called endomorphism i.e

$$f : G \to G.$$

Eg if $(G, \cdot)$ is a grp.

(i) $f(a \cdot b) = f(a) \cdot f(b)$

(ii) f is onto.

KERNEL OF HOMOMORPHISM :-

If f be homomorphism of group G into G' then a subset K is said to be kernel if it consist all those elements of G whose image is the identity of G'.

$$\boxed{\ker f = K = \{x : f(x) = e' \ \forall x \in G\}}$$

ISOMORPHISM :-

A mapping $f : G \to G'$ where $(G, *)$ and (G', o) are two groups is an isomorphism if :-

(i) f is one to one

(ii) f is homomorphism $\boxed{f(a * b) = f(a) \, o \, f(b)}$

(iii) f is onto.

$$(G \cong G')$$

Every Isomorphism is Homomorphism but the converse may / may not be true.

AUTOMORPHISM :-

It means isomorphism of G onto itself.

$f : G \xrightarrow[\text{onto}]{\text{one to one}} G$ is an automorphism of G

if $\boxed{f(ab) = f(a) f(b)} \ \forall a, b \in G$.

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Let I^+ be additive group such as.

$$f: I^+ \to I^+ ; \forall x \in I^+$$

$f(x) = x + 2$. Show f is automorphism of I^+.

Soln ① Homomorphism.

$$f: I^+ \to I^+.$$

$$f(x+y) = f(x) + f(y) \quad (\forall x, y \in I^+.)$$

$$= (x+2) + (y+2)$$

$$(x+y+2) \neq x+y+4.$$

NOT HOMOMORPHIC

② one to one :-

$$f(x) = f(y)$$

$$x + \cancel{2} = y + \cancel{2}$$

$$x = y$$

$\therefore f$ is one-one.

③ Onto mapping :-

$$f(x) = y$$

$$x + 2 = y$$

$$x = y - 2$$

$\forall x \in I^+$ there exist image in codomain

Yes it is onto

Q) Show that if $f: G \to G'$ where $f(x) = x^2$ $\forall x \in$ G is homorphic iff G is abelian.

Sol^n Let f is homomorphic, then

$$f(x.y) = f(x) \cdot f(y) \quad \text{--- (1)}$$

$\{(G, \cdot), G = G'\}$ By default

Now we have to prove G is abelian.

i.e $y \cdot x = x \cdot y$

$\because f(x) = x^2$.

$$f(x.y) = x^2 y^2$$

$$f(x) \cdot f(y) = (xy)(xy)$$

$$x^2 y^2 = (xy)(xy)$$

$$\boxed{xy = yx}$$

$\Rightarrow$ G is abelian.

$$xy = yx \quad \text{--- (1)}$$

Now we have to show,

$$f(x.y) = f(x) \cdot f(y)$$

Taking LHS $\Rightarrow f(x.y)$

$$= (xy)^2 = (xy)(xy)$$

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Show that if $f: G \to G'$ where $f(x) = x^{-1}, \forall x \in G$ is Isomorphism if G is Abelian.

Solⁿ Let f is homomorphism

$\Rightarrow f(x.y) = f(x).f(y)$ —— ①

Now show G is abelian

$\because f(x) = x^{-1}$

$\Rightarrow f(x.y) = (xy)^{-1} = y^{-1}x^{-1}$.

① $\rightarrow f(x).f(y) \Rightarrow (xy)^{-1} = y^{-1}x^{-1}$

$$\boxed{x^{-1}.y^{-1} = y^{-1}x^{-1}}$$

Now show f is one-to-one

Let $f(x) = f(y)$

$x^{-1} = y^{-1}$.

Multiply x both sides.

$x.x^{-1} = xy^{-1}$

$e = xy^{-1}$

Multiply y both sides.

$e.y = xy^{-1}.y$

$y = x$

∴ f^n is one-to-one.

Now for onto mapping :-

Let $f(x) = y$

$x^4 = y$

∵ every element has an inverse element and it will be definitely a pre-image in x.

$\Rightarrow f$ is onto.

Hence f is isomorphism.

THEOREM V. imp

Let $(G, *)$ and (G', o) be groups with respect identities e and e'. If $f : G \to G'$ is homomorphism then

(i) $f(e) = e'$ (ii) $f(a^{-1}) = [f(a)]^{-1} \; \forall a \in G$

(iii) The order of f-image of an element is the same as order of the element.

PROOFS (i)

$G \to e$
$G' \to e'$
$e' \in G'$
then $f(e) \in G'$.
∴ $\underbrace{e' o f(e) = f(e)}_{\text{By closure}}$ = Also.

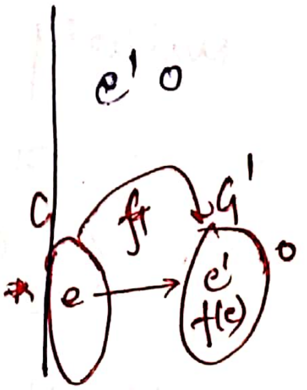
$f(a * b) = f(a) \, o \, f(b)$

$f(e * e) = f(e) \, o \, f(e)$

Also. $f(e * e) = f(e) \, o \, f(e)$

$e' o f(e) = f(e) \, o \, f(e)$

$e' = f(e)$



$$\boxed{f(e) \circ f(e) = f(e)} \quad \text{--- (1)}$$

Now $f(e) \in G'$

$\therefore \quad f(e) = e'$

and $\quad f(e) \circ \cancel{f(e)} = e' \circ \cancel{f(e)}$

(By right cancellation)

$$\boxed{f(e) = e'}$$

Proved

(iii) We know $aa^{-1} = a^{-1}a = e$.

We have $f(e) = e'$

$e' = f(aa^{-1})$

$= f(a) \circ f(a^{-1})$

$e' = f(e) = f(a^{-1}.a) = f(a^{-1}) \circ f(a)$

Thus $f(a) \circ f(a^{-1}) = f(a^{-1}) \circ f(a) = e'$

Hence, $f(a^{-1})$ is the inverse of $f(a)$

i.e $\boxed{f(a^{-1}) = [f(a)]^{-1}}$

Proved