3.1 Introduction

- The concept of relation was defined very generally in the preceeding chapter. We shall now discuss a particular class of relations called functions. Many concepts of computer science can be conveniently stated in the language of functions.

3.2 Function

- Let A and B be non-empty sets. A function f from A to B is denoted by $f : A \to B$ and defined as a relation from A to B such that for every $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$.
- If $(a, b) \in f$ then it can be written as $f(a) = b$. Functions are also called as Mappings or Transformations.

e.g.

1) $f = \{(a, b)\ (b, c)\ (c, a)\}$ is a function on set $A = \{a, b, c\}$
2) $f = \{(a, b)\ (b, c)\ (b, a)\}$ is not a function as (b, c) and $(b, a) \in f$.
3) $f = \{(a, b)\ (b, c)\}$ is not a function on set $A = \{a, b, c\}$ as for $c \in A$ $\nexists$ any element x in A such that $(c, x) \in f$.

3.2.1 Important Definitions

I) Let $f : A \to B$ be a function such that $f(a) = b$ then $b \in B$ is called the image of $a \in A$ and a is called the pre-image of b. The element a is called argument of f.

II) Let $f : A \to B$ be a function. The set A is known as domain set of f and the set B is known as co-domain set of f.

III) The range set of a function $f : A \to B$ is denoted by R(f) and defined as

$$R(f) = \{b / b \in B \text{ and } f(a) = b \text{ for some } a \in A\}$$

In other words, range of f is the set of all images of the elements of A under f.

Examples

Ex.3.2.1 : If f is a function such that $f(x) = x^2 - 1$ where $x \in R$ find the values of f(–1), f(0), f(1), f(3) and range set of f.

Sol. : We have $f(x) = x^2 - 1$

$$\therefore \quad f(-1) = (-1)^2 - 1 = 1 - 1 = 0$$

$$f(0) = 0 - 1 = -1$$

$$f(1) = 1^2 - 1 = 0$$

$$f(3) = 3^2 - 1 = 8$$

For any $x \in \Re$, $x^2 - 1 \in \Re$

$\therefore$ The range of f is $R(f) = \{x / -1 \le x < \infty\}$

Ex.3.2.2 : Let $f : \Re \to \Re$ where f is defined by
a) $f(x) = \sqrt{x}$
b) $f(x) = x^2$
c) $f(x) = \sin x \ \forall\ x \in \Re$

Sol. :

a) We have $f(x) = \sqrt{x} \therefore f(4) = \sqrt{4} = \pm 2$

$\therefore$ f is not a function. Therefore we can not find range of f.

b) We have $f(x) = x^2$; x^2 is positive real number

Hence range of f is $R(f) = \{x \mid 0 \le x < \infty\}$

c) $\forall x \in \Re$, sin x lies between –1 to +1

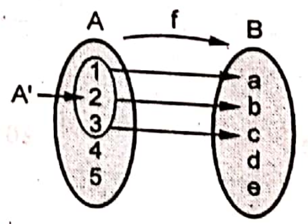
$$\therefore \quad R(f) = \{x \mid -1 \le x \le 1\}$$

3.2.2 Partial Functions

- Let A and B be two non empty sets. A partial function f with domain set A and codomain set B, is any function from A′ to B where $A' \in A$. For any $x \in A - A'$, f(x) is not defined.
- To make the distinction more clear, the function which is not partial is called as a total function.

e.g.

1)



$f : A' \to B$ is a partial function

Fig. 3.2.1

2) The function $f : \Re \to \Re$ defined as $f(x) = \frac{1}{x}$ is a partial function. As it is not defined for $x = 0$.

3) The function $f : \Re \rightarrow \Re$ defined as $f(x) = \sqrt{x}$ is a partial function, as $\sqrt{x}$ is not defined for $x < 0$, in $\Re$

3.2.3 Equality of Two Functions

- Two functions $f : A \rightarrow B$ and $g : A \rightarrow B$ are said to be equal functions or identical functions if and only if

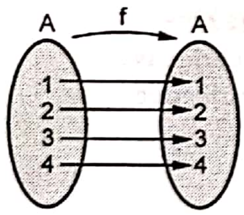
$$f(x) = g(x);\ \forall\ x \in A$$

Note : Two functions $f : \mathbf{Z} \rightarrow \mathbf{Z}$ and $g : \Re \rightarrow \Re$ defined as $f(x) = x,\ \forall\ x \in \mathbf{Z}$ and $g(x) = x,\ \forall\ x \in \Re$ are not identical or equal functions because their domains are not same.

3.2.4 Identity Function

- Let A be any non empty set and function $f : A \rightarrow A$ is said to be the identity function if $f(x) = x,\ \forall\ x \in A$.

e.g.



f : A → A is the identity function

Fig. 3.2.2

3.2.5 Constant Function

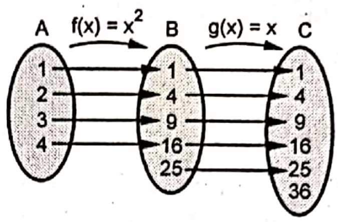
- A function $f : A \rightarrow B$ is said to be a constant function if $f(x)$ = constant = k ; $\forall\ x \in A$.
- The range set of a constant function consists of only one element.

3.2.6 Composite Function

- Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.
- The composite function of f and g is denoted by gof and defined as $gof : A \rightarrow C$ is a function such that $(gof)(a) = g[f(a)];\ \forall\ a \in A$.

Note : gof is defined only when the range of f is a subset of the domain of g.

e.g.



Therefore gof : A → C

$gof(1) = g[f(1)] = g(1) = 1$

$gof(2) = g[f(2)] = g(4) = 4$

$gof(3) = g[f(3)] = g(9) = 9$

$gof(4) = g[f(4)] = g(16) = 16$

Fig. 3.2.3

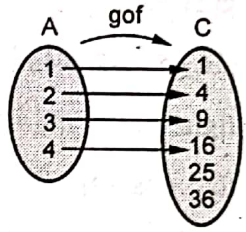


Fig. 3.2.4

Ex.3.2.3 : Let $f(x) = x+2$, $g(x) = x - 2$, $h(x) = 3x$, for $x \in \Re$ Where $\Re$ is the set of real numbers Find i) gof ii) fog iii) fof iv) hog v) gog vi) foh vii) hof viii) fohog ix) gofoh.

Sol. : Let $x \in \Re$ be any real number.

i) $gof(x) = g[f(x)] = g[x + 2] = x + 2 - 2 = x$

ii) $fog(x) = f[g(x)] = f[x - 2] = x - 2 + 2 = x$

iii) $fof(x) = f[f(x)] = f[x + 2] = x + 2 + 2 = x + 4$

iv) $hog(x) = h[g(x)] = h[x - 2] = 3(x - 2) = 3x - 6$

v) $gog(x) = g[g(x)] = g[x - 2] = x - 2 - 2 = x - 4$

vi) $foh(x) = f[h(x)] = f[3\,x] = 3x + 2$

vii) $hof(x) = h[f(x)] = h[x + 2] = 3(x + 2) = 3x + 6$

viii) $fohog(x) = f[h(g(x))] = f[h(x - 2)]$

$= f[3(x - 2)] = f(3\,x - 6)$

$= 3x - 6 + 2 = 3\,x - 4$

ix) $gofoh(x) = g[f(h(x))] = g[f(3\,x)]$

$= g[3\,x + 2] = 3\,x + 2 - 2 = 3\,x$

Ex.3.2.4 Let functions f and g be defined by $f(x) = 2x + 1$, $g(x) = x^2 - 2$

Find a) gof(4) and fog (4)

b) gof(a + 2) and fog(a + 2)

c) fog(5)

d) gof(a + 3)

Sol. :

a)
$$gof(4) = g[f(4)] = g[2(4)+1]$$
$$= g[9] = 9^2 - 2 = 79$$
$$fog(4) = f[g(4)] = f(4^2-2)$$
$$= f(14) = 2(14)+1 = 29$$

b)
$$gof(a+2) = g[f(a+2)] = g[2(a+2)+1]$$
$$= g[2a+5]$$
$$= (2a+5)^2 - 2 = 4a^2+20a+23$$
$$fog(a+2) = f[g(a+2)] = f[(a+2)^2-2]$$
$$= f[a^2+4a+2]$$
$$= 2[(a^2+4a+2]+1 = 2a^2+8a+5$$

c)
$$fog(5) = f[g(5)] = f[25-2]$$
$$= f(23) = 2(23)+1 = 47$$

d)
$$gof(a+3) = g[f(a+3)] = g[2(a+3)+1]$$
$$= g[2a+7] = [2a+7]^2 - 2$$
$$= 4a^2+28a+47$$

3.3 Special Types of Functions

AKTU : 2011-12, 2012-13, 2014-15, 2015-16

I) Let $f : A \rightarrow B$ be a function.

i) Function f is said to be **one to one (or Injective)** function if distinct elements of A are mapped into distinct elements of B.

i.e. f is one to one iff $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

OR $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

ii) Function f is said to be **onto function** if each element of B has at least one preimage in A.

OR

A function $f : A \rightarrow B$ is called **onto (or surjective)** function if the range set of f is equal to B.

iii) A function f is called a **bijective function** if it is both one to one and onto.

iv) A function f is called **into function** if $\exists$ at least one element in B which has no preimage in A.

i.e. A function f is called **into function** if $R(f) \neq B$

2) Let a function $f : A \rightarrow B$ be a bijective function then $f^{-1} : B \rightarrow A$ is called the **inverse mapping** of f and defined as $f^{-1}(b) = a$ iff $f(a) = b$

It is also known as **invertible mapping.**

3) Characteristic function of a set :

Let U be a universal set and A be a subset of U.

Then the function $\psi_A : U \rightarrow \{0, 1\}$ defined by

$$\psi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called a **characteristic function** of the set A.

Examples

Ex.3.3.1 : Give examples of functions of the following types by diagrams.
a) Injective function but not surjective
b) Surjective but not injective
c) Neither injective nor surjective
d) Injective as well as surjective
e) Into function
f) Inverse function

Sol. :

a) Injective but not surjective

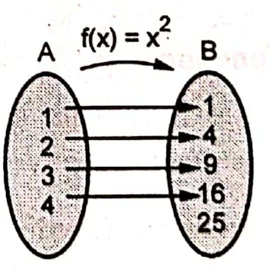


Fig. 3.3.1

b) Surjective but not injective

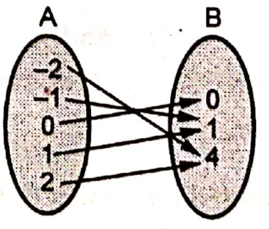


Fig. 3.3.2

c) Neither injective nor surjective

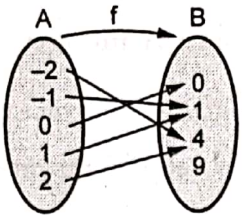


Fig. 3.3.3

d) Injective as well as surjective i.e. bijective

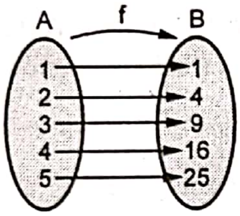


Fig. 3.3.4

e) Into function

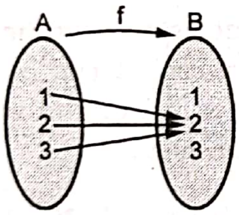


Fig. 3.3.5

f) Inverse function

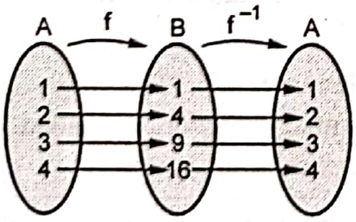


Fig. 3.3.6

Ex.3.3.2 : Determine whether each of the following is a function. If yes, is it injective, surjective, bijective ?

a) Each person in the earth is assigned a number which corresponds to his age.

b) Each student is assigned a teacher.

c) Each country has assigned it's capital.

Sol. :

a) Every person has unique age,

$\therefore$ It is a function. Two person's may have same age. $\therefore$ It is not injective. There is no person whose age is 300 years. $\therefore$ It is not surjective. $\therefore$ Function is not bijective.

b) It is a function. It is not injective. It is not surjective. $\therefore$ It is not bijective.

c) It is a function. It is injective as well as surjective. $\therefore$ It is bijective.

Ex.3.3.3 : Let A = B be the set of real numbers.

$f : A \to B$ given by $f(x) = 2x^3 - 1$

$g : B \to A$ given by $g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$

Show that f is a bijective function and g is also bijective function :

Sol. :

1) Suppose $f(x_1) = f(x_2)$

$$\Rightarrow \quad 2x_1^3 - 1 = 2x_2^3 - 1$$

$$\Rightarrow \quad x_1^3 = x_2^3$$

$$\Rightarrow \quad x_1 = x_2$$

$\therefore$ f is injective mapping

Let $y \in B$ and $f(x) = y \Rightarrow 2x^3 - 1 = y$

$$\therefore \quad 2x^3 = 1 + y \Rightarrow x^3 = \frac{1+y}{2}$$

$$\Rightarrow \quad x = \sqrt[3]{\frac{1+y}{2}} = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} \in A = \Re \text{ for any } y \in B$$

$\therefore$ f is a surjective mapping.

Thus f is injective as well as surjective function.

Hence f is a bijective function.

We have $f(x) = y \Rightarrow x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} \Rightarrow f^{-1}(y) = x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} = g(y)$

Thus $f^{-1} = g$. We know that if f is bijective function then f^{-1} is also bijective. Hence g is bijective function.

Ex.3.3.4 : Determine whether each of these functions is a bijective from $\Re$ to $\Re$

a) $f(x) = x^2 + 1$, b) $f(x) = x^3$,

c) $f(x) = \frac{x^2 + 1}{x^2 + 2}$ AKTU : 2015-16

Sol. :

a) Suppose $f(x_1) = f(x_2)$

$$\Rightarrow \quad x_1^2 + 1 = x_2^2 + 1$$

$$\Rightarrow \quad x_1^2 = x_2^2$$

$$\Rightarrow \quad x_1 = x_2 \quad \text{or} \quad x_1 = -x_2$$

e.g. $x_1 = 2,\ x_2 = -2 \Rightarrow f(x_1) = f(x_2)$

$\therefore$ f is not one to one function. Hence f is not bijective function.

b) Suppose $f(x_1) = f(x_2)$

$$\Rightarrow \quad x_1^3 = x_2^3 \Rightarrow x_1 = x_2$$

$\therefore$ f is one to one mapping

Let $y \in \Re$ and $f(x) = y$

$$\Rightarrow \quad x^3 = y$$

$$\Rightarrow \quad x = (y)^{1/3} \in \Re$$

$\therefore$ f is onto function.

Hence f is bijective function.

c) Suppose $f(x_1) = f(x_2)$

$$\frac{x_1^2+1}{x_1^2+2} = \frac{x_2^2+1}{x_2^2+2}$$

$$\Rightarrow \quad x_1 = x_2 \quad \text{or} \quad x_1 = -x_2$$

$\therefore$ f is not one to one function.

Hence f is not bijective.

Ex.3.3.5 : Explain classification of functions with example.

Sol. : Depending upon the nature of function, there are mainly two functions.

1. **Algebraic function :** A function which consists of a finite number of terms involving powers and roots of the independent variable and four fundamental operations addition, subtraction, multiplication and division, is called an algebraic function.

There are three types of algebraic functions.

(a) Polynomial function : A function of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$, where n is positive integer, $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, is called polynomial function of x with degree n.

e.g. $f(x) = x^3 - 3x^2 + 2x - 5$

(b) Rational function : A function of the form $\frac{f(x)}{g(x)}$, where f(x) and g(x) are polynomial functions and $g(x) \neq 0$, is called rational function e.g. $\frac{x^2-3x+5}{x^2+1}$

(c) Irrational function : A function involving radicals is called irrational function.
e.g. $f(x) = x^{2/3} + 5x^2 + 1, \sqrt[2]{x+1}$

2. **Transcendental function :** A function which is not algebraic is called transcendental function.

e.g. $f(x) = \sin x + x^3 + 5x$

(a) Trigonometric function : The six functions sin x, cos x, tan x, sec x, cosec x, cot x, where x is in radians, are called trigonometric functions.

e.g. $f(x) = \sin x + \tan x.$

(b) Inverse trigonometric function : The six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are called inverse trigonometric functions.

e.g. $f(x) = \cos^{-1} x + 5 \tan^{-1} x$

(c) Exponential function : A function of the form $f(x) = a^x$ $(a > 0)$ satisfying $a^x \cdot a^y = a^{x+y}$ is called exponential fucntion. e.g. $f(x) = 5^x$.

(d) Logarithmic function : The inverse function of the exponential function is called logarithmic function. e.g. $f(x) = \log x$.

Ex.3.3.6 : If f : A $\rightarrow$ B is bijective function then f^{-1} is unique.

Sol. : Let f A $\rightarrow$ B is a bijective function.

Claim : Show that f^{-1} is unique.

Suppose f^{-1} is not unique, so there are two inverse functions say, g and h.

Let $x_1, x_2 \in A$, $\exists$ y in B such that

$$f^{-1}(y) = x_1 \Rightarrow g(y) = x_1 \Rightarrow f(x_1) = y$$

and $$f^{-1}(y) = x_1 \Rightarrow h(y) = x_2 \Rightarrow f(x_2) = y$$

This implies $f(x_1) = f(x_2)$, but f is 1 – 1 function.

$$\therefore \quad x_1 = x_2$$

$$\Rightarrow \quad g(y) = h(y)\ \forall\ y$$

$\Rightarrow \quad x_1^2 = x_2^2$

$\Rightarrow \quad x_1 = x_2 \text{ or } x_1 = -x_2$

e.g. $\quad x_1 = 2, \; x_2 = -2 \Rightarrow f(x_1) = f(x_2)$

$\therefore$ f is not one to one function. Hence f is not bijective function.

b) Suppose $\quad f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

$\therefore$ f is one to one mapping

Let $\quad y \in \Re$ and $f(x) = y$

$\Rightarrow \quad x^3 = y$

$\Rightarrow \quad x = (y)^{1/3} \in \Re$

$\therefore$ f is onto function.

Hence f is bijective function.

c) Suppose $\quad f(x_1) = f(x_2)$

$$\frac{x_1^2+1}{x_1^2+2} = \frac{x_2^2+1}{x_2^2+2}$$

$\Rightarrow \quad x_1 = x_2 \text{ or } x_1 = -x_2$

$\therefore$ f is not one to one function.

Hence f is not bijective.

Ex.3.3.5 : Explain classification of functions with example.

Sol. : Depending upon the nature of function, there are mainly two functions.

1. **Algebraic function :** A function which consists of a finite number of terms involving powers and roots of the independent variable and four fundamental operations addition, subtraction, multiplication and division, is called an algebraic function.

There are three types of algebraic functions.

(a) Polynomial function : A function of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$, where n is positive integer, $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, is called polynomial function of x with degree n.

e.g. $\quad f(x) = x^3 - 3x^2 + 2x - 5$

(b) Rational function : A function of the form $\frac{f(x)}{g(x)}$, where f(x) and g(x) are polynomial functions and $g(x) \neq 0$, is called rational function e.g. $\frac{x^2-3x+5}{x^2+1}$

(c) Irrational function : A function involving radicals is called irrational function.

e.g. $f(x) = x^{2/3} + 5x^2 + 1, \sqrt[2]{x+1}$

2. **Transcendental function :** A function which is not algebraic is called transcendental function.

e.g. $\quad f(x) = \sin x + x^3 + 5x$

(a) Trigonometric function : The six functions sin x, cos x, tan x, sec x, cosec x, cot x, where x is in radians, are called trigonometric functions.

e.g. $\quad f(x) = \sin x + \tan x.$

(b) Inverse trigonometric function : The six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are called inverse trigonometric functions.

e.g. $\quad f(x) = \cos^{-1} x + 5 \tan^{-1} x$

(c) Exponential function : A function of the form $f(x) = a^x$ $(a > 0)$ satisfying $a^x \cdot a^y = a^{x+y}$ is called exponential fucntion. e.g. $f(x) = 5^x$.

(d) Logarithmic function : The inverse function of the exponential function is called logarithmic function. e.g. $f(x) = \log x$.

Ex.3.3.6 : If $f : A \to B$ is bijective function then f^{-1} is unique.

Sol. : Let $f \; A \to B$ is a bijective function.

Claim : Show that f^{-1} is unique.

Suppose f^{-1} is not unique, so there are two inverse functions say, g and h.

Let $x_1, x_2 \in A, \exists$ y in B such that

$f^{-1}(y) = x_1 \Rightarrow g(y) = x_1 \Rightarrow f(x_1) = y$

and $\quad f^{-1}(y) = x_1 \Rightarrow h(y) = x_2 \Rightarrow f(x_2) = y$

This implies $f(x_1) = f(x_2)$, but f is 1 – 1 function.

$\therefore \quad x_1 = x_2$

$\Rightarrow \quad g(y) = h(y) \; \forall \; y$

$\Rightarrow \quad g \equiv h$ i.e. g and h are equal function.

Hence inverse of f is unique.

Ex.3.3.7 : If $f : A \to B$ and $g : B \to C$ are bijective functions then gof is also bijective and $(gof)^{-1} = f^{-1} o g^{-1}$

AKTU : 2012-13, 2014-15

Sol. : Let $f : A \to B$ and $g : B \to C$ be two bijective functions then $gof : A \to C$ is a function.

(i) Let $x_1, x_2 \in A$ and suppose $gof(x_1) = gof(x_2)$

$$g[f(x_1)] = g[f(x_2)]$$

$\Rightarrow \quad f(x_1) = f(x_2)$ ... ($\because$ g is 1 – 1 function)

$\Rightarrow \quad x_1 = x_2$ ... ($\because$ f is 1 – 1 function)

$\Rightarrow$ gof is 1 – 1 function.

(ii) Let $z \in C$.

$\therefore \exists$ y in B such that g (y) = z and $\exists$ x in A such that f(x) = y.

$\therefore g(y) = z \quad \Rightarrow \quad z = g(f(x)) = gof(x)$ where $x \in A$

$\Rightarrow$ gof is onto function.

Hence gof is bijective function.

To prove that,

$$(gof)^{-1} = f^{-1} o g^{-1}$$

As gof is bijective, $(gof)^{-1}$ exists.

If $f : A \to B$ s. t. f (x) = y

$x \in A$ and $y \in B$

$g : B \to C$ s.t. g (y) = z

$y \in B$ and $z \in C$

and $gof : A \to C$ given by

$(gof)(x) = z$ where $x \in A$ and $z \in C$

By the definition of inverse function, $f^{-1}(y) = x$, $g^{-1}(z) = y$

and $(gof)^{-1} : C \to A$ such

$$(gof)^{-1}(z) = x$$

Now $(f^{-1} o g^{-1})(z) = f^{-1}[g^{-1}(z)]$

$= f^{-1}(y) = x$

$= (gof)^{-1}(z)$

Hence $f^{-1} o g^{-1} = (gof)^{-1}$

Ex.3.3.8 : Determine whether the function is bijective.

$f : I \to I$ such that $f(i) = \frac{i}{2}$ if i is even

$= \frac{(i-1)}{2}$ if i is odd

Sol. : Let a and b be any integers such that $a \neq b$.

There are three possibilities

Case 1 : If a and b are odd

$\therefore \quad f(a) = \frac{a-1}{2}, \quad f(b) = \frac{b-1}{2}$

$f(a) = f(b) \Rightarrow \frac{a-1}{2} = \frac{b-1}{2}$

$\Rightarrow a = b$

Case 2 : If a and b are even

then $\quad f(a) = \frac{a}{2}, \quad f(b) = \frac{b}{2}$

$f(a) = f(b) \Rightarrow \frac{a}{2} = \frac{b}{2} \Rightarrow a = b$

Case 3 : If a is odd and b is even

then $\quad f(a) = \frac{a-1}{2}$ and $f(b) = \frac{b}{2}$

Now, $\quad f(a) = f(b) \Rightarrow \frac{a-1}{2} = \frac{b}{2} \Rightarrow \frac{a}{2} - \frac{b}{2} = \frac{1}{2}$

$\Rightarrow \quad a - b = 1 \quad \therefore a \neq b$

In particular a = 7, b = 6

$f(7) = \frac{7-1}{2} = 3 \quad f(6) = \frac{6}{2} = 3$

$f(7) = f(6)$ but $6 \neq 7$

Thus f is not one one function. Hence f is not bijective function.

Ex.3.3.9 : Let $f(x) = ax^2 + b$ and $g(x) = cx^2 + d$, where a, b, c, d are constants. Determine for which values of constants if gof(x) = fog(x).

AKTU : 2011-12

Sol. : Given that, $f(x) = ax^2 + b$ and $g(x) = cx^2 + d$

Suppose $\quad fog(x) = gof(x)$

$$f[g(x)] = g[f(x)]$$

$$f[cx^2+d] = g[ax^2+b]$$

$$\Rightarrow a[cx^2+d]^2+b = c[ax^2+b]^2+d$$

$$\Rightarrow a[c^2x^4+2cdx^2+d^2]+b = c(a^2x^4+2abx^2+b^2)+d$$

$$\Rightarrow ac^2x^4+2acdx^2+ad^2+b = ca^2x^4+2cabx^2+cb^2+d$$

$\Rightarrow$ Coefficient of x^4 = coefficient of x^4

and Coefficient of x^2 = coefficient of x^2

$\Rightarrow$ $ac^2 = ca^2 \Rightarrow a = c$

and $2acd = 2acb \Rightarrow b = d$

Thus $a = c$ and $b = d$

Ex.3.3.10 Show that the functions $f(x) = x^3 + 1$ and $g(x) = (x-1)^{1/3}$ are converse to each other.

Sol. : We have, $g(x) = (x-1)^{1/3}$

Let $y = g(x) = (x-1)^{1/3}$

$\Rightarrow$ $y^3 = x - 1$

$\Rightarrow$ $x = y^3 + 1$ ... (Which is f (x))

Hence f(x) and g(x) are converse to each other.

Ex.3.3.11 : Let $f : X \to Y$ and X and Y are set of real numbers. Find f^{-1} if (i) $f(x) = x^2$; (ii) $f(x) = \frac{2x-1}{5}$

Sol. : Let $f : X \to Y$ and $X, Y \subseteq \Re$

(i) $f(x) = x^2$

Let $f(x) = y$

$x^2 = y$

$\Rightarrow$ $y = x^2$

$\Rightarrow$ $y = \pm\sqrt{x}$

Therefore, given function is not one to one. Hence f^{-1} does not exist.

(ii) Let $f(x) = y$

$$\frac{2x-1}{5} = y$$

$\Rightarrow$ $2x = 5y + 1$

$$x = \frac{5y+1}{2} \quad \forall\ y \in \Re$$

Function f is 1 – 1 and onto.

$\therefore$ The inverse function of f is given as,

$$f^{-1}(x) = \frac{1+5y}{2}$$

Ex.3.3.12 Let X = {a, b, c}. Define $f : X \to X$ such that f = {(a, b) (b, a) (c, c)}. Find (i) f^{-1} ; (ii) f^2 ; (iii) f^3 ; (iv) f

Sol. : Given fucntion f is bijective function.

(i) f^{-1} = {(b, a) (a, b) (c, c)}

(ii) f^2 = fof = {(a, a) (b, b) (c, c)}

fof (a) = f[f (a)] = f(b) = a

fof (b) = f[f (b)] = f (a) = b

fof (c) = f [f (c)] = f (c) = c

(iii) $f^3(a) = f[f^2(a)] = f(a) = b$

$f^3(b) = f[f^2(b)] = f(b) = a$

$f^3(c) = f[f^2(c)] = f(c) = c$

f^2 = {(a, b), (b, a), (c, c)}

(iv) $f^4 = f^2 \circ f^2$

$f^4(a) = f^2[f^2(a)] = f^2(a) = a$

$f^4(b) = f^2[f^2(b)] = f^2(b) = b$

$f^4(c) = f^2[f^2(c)] = f^2(c) = c$

$\therefore$ f^4 = {(a, a) (b, b) (c, c)}

Ex.3.3.13 : Let X = {1, 2, 3} , Y = {p, q} and z = {a, b} Let $f : X \to Y$ be {(1, p), (2, p), (3, q)} and $g : Y \to Z$ be g = {(p, b) (q, b)}. Find gof and show it pictorically.

Sol. : Given that, f = {(1, p), (2, p), (3, q)}

g = {(p, b) (q, b)}

and gof (x) = g [f (x)] and gof : $X \to Z$

gof (1) = g [f (1)] = g (p) = b

gof (2) = g [f (2)] = g (p) = b

gof (3) = g [f (3)] = g (q) = b

gof = {(1, b) (2, b) (3, b)}

It's pictorial representation is

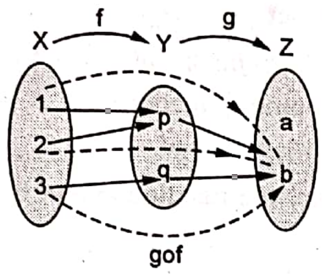


Fig. 3.3.7

Ex.3.3.14 : Let R be the set of real numbers and f, g, h : $\Re \to \Re$ such that
$f(x) = x + 2$, $g(x) = \dfrac{1}{x^2+1}$ $h(x) = 3$.
Compute
(i) $f^{-1} g(x)$; (ii) $hf(gf^{-1})$ [h f (x)]

Sol. : (i) Given that $f(x) = x + 2$

Let $f(x) = y = x + 2 \Rightarrow x = y - 2$

$$f^{-1}(y) = y - 2$$

$$f^{-1}[g(x)] = f^{-1}\left[\frac{1}{x^2+1}\right] = \frac{1}{x^2+1} - 2$$

$$= \frac{1-2x^2-2}{\left(x^2+1\right)} = \frac{-\left(2x^2+1\right)}{x^2+1}$$

(ii) We have $f(x) = x + 2$

$$h\,f(x) = h[f(x)] = h[x+2] = 3$$

$$g\,f^{-1}(x) = g[f^{-1}(x)] = g[x-2]$$

$$= \frac{1}{(x-2)^2+1} = \frac{1}{x^2-2x+5}$$

$$(g\,f^{-1})\,[h\,f(x)] = g\,f^{-1}(3) = \frac{1}{3^2-2(3)+5}$$

$$= \frac{1}{9-6+6} = \frac{1}{8}$$

$$\therefore \quad h\,f\,(g\,f^{-1})\,(h\,f(x)) = h\,f\left[\frac{1}{8}\right] = h\left[f\left(\frac{1}{8}\right)\right]$$

$$= h\left[\frac{1}{8}+2\right] = h\left[\frac{17}{8}\right] = 3$$

i.e. $h\,f\,(g\,f^{-1})\,(h\,f(x)) = 3$

Ex.3.3.15 : If $f : R \to R$ is defined by $f(x) = \cos x$ and the function $g : R \to R$ is defined by $g(x) = x^3$ find (gof) (x) and (fog) x and prove that they are not equal

Sol. : Let $f : R \to R$ and $g : R \to R$ are given by

$$f(x) = \cos x \text{ and } g(x) = x^3$$

Now $(fog)(x) = f[(g(x)] = f[x^3] = \cos(x^3)$

and $(gof)(x) = g[f(x)] = g(\cos x)$

$$= (\cos x)^3 = \cos^3 x$$

$\therefore \quad (gof)\,x \neq (fog)\,x$

Hence gof $\neq$ fog.

Ex.3.3.16 : Show that the mapping $f : z^+ \to z^+$ defined by $f(x) = x^2$ $\forall x \in z^+$ where z^+ is a set of positive integers, is one to one and onto.

Sol. : We have $f : z^+ \to z^+$

i) Let $x_1, x_2 \in z^+$ and $f(x_1) = x_1^2$ and $f(x_2) = x_2^2$

$\therefore \quad f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$

$\therefore$ f is one to one function.

ii) Let y be any arbitrary element of z^+ such that

$$y = f(x) \text{ for some x in } z^+$$

$\therefore \quad y = x^2$

$\Rightarrow \quad \sqrt{y} = x \Rightarrow x = \sqrt{y} \in z^+$

As y is positive integer $\therefore$ take $\sqrt{y} > 0$

Thus $f(x) = f(\sqrt{y}) = y$

i.e. for any $y \in z^+ \ \exists \ \sqrt{y} \in z^+$ such that

$$f(\sqrt{y}) = y$$

$\therefore$ f is onto function.

Ex.3.3.17 : If $f : X \to Y$ and A, B are two subsets of X then i) $f(A \cup B) = f(A) \cup f(B)$
ii) $f(A \cap B) \subseteq f(A) \cap f(B)$.

Sol. : i) Let x be any element in $f(A \cup B)$

i.e. $x \in f(A \cup B) = f^{-1}(x) \in A \cup B$

$\Rightarrow f^{-1}(x) \in A$ or $f^{-1}(x) \in B$

$\Rightarrow \quad x \in f(A)$ or $x \in f(B)$

$\Rightarrow \quad x \in f(A) \cup f(B)$

i.e. $x \in f(A \cup B) \Rightarrow x \in f(A) \cup f(B)$

$\therefore \quad f(A \cup B) \subseteq f(A) \cup f(B)$

Similarly, $f(A) \cup f(B) \subseteq f(A \cup B)$

Hence, $\quad f(A \cup B) = f(A) \cup f(B)$

ii) Let $x \in f(A \cap B)$

$\Rightarrow f^{-1}(x) \in A \cap B$

$\Rightarrow f^{-1}(x) \in A$ and $f^{-1}(x) \in B$

$\Rightarrow x \in f(A)$ and $x \in f(B)$

$\Rightarrow x \in f(A) \cap f(B)$

i.e. $\quad x \in f(A \cap B) \Rightarrow x \in f(A) \cap f(B)$

Hence $\quad f(A \cap B) \subseteq f(A) \cap f(B)$

Ex.3.3.18 : Show that $f, g : N \times N \rightarrow N$ as $f(x, y) = x + y$ and $g(x, y) = xy$ are onto but not one-one

Sol. : (i) Suppose $f(x_1, y_1) = f(x_2, y_2)$

$\Rightarrow \quad x_1 + y_1 = x_2 + y_2$

$\Rightarrow x_1 \neq x_2$ and $y_1 \neq y2$

e.g. $x_1 = 5, x_2 = 2, y_1 = 4, y_2 = 7$.

$\therefore$ f is not one to one.

Now, $g(x_1, y_1) = g(x_2, y_2)$

$\Rightarrow \quad x_1 y_1 = x_2 y_2$

$\Rightarrow \quad x_1$ may or may not be equal to x_2

and y_1 may or may not be equal to y_2

$\therefore$ g is not one to one.

(ii) $\quad f(x, y) = x + y$

$\therefore$ Every element of N can be written as the sum of two elements of N. Hence f is onto.

Now, $\quad g(x, y) = xy$

Every element of can be written as the product of two elements of N.

$\therefore$ g is onto.

3.4 Infinite Sets and Countability

3.4.1 Infinite Set

- A set A is said to be an infinite set if there exists an injective mapping (function) $f : A \rightarrow A$ such that f(A) is a proper subset of A.
- If no such injective function exists, then set is finite.

Examples

1) Let $f : N \rightarrow N$ such that $f(n) = 2n, \forall n \in N$ = Natural number set.

There range set = f(N) = {Set of positive even natural numbers} $\in$ N

$\therefore$ N is an infinite set.

2) Define $f : \mathfrak{R} \rightarrow \mathfrak{R}$ such that $f(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$

f is an injective mapping and for $x \geq 0$, $f(x) = x + 2$

i.e. $f(x) \geq 2$ and for $x < 0$, $f(x) = x < 0$

$\therefore$ Range set = $f(\mathfrak{R}) = \{y \in \mathfrak{R} / f(x) \geq 2 \text{ or } f(x) < 0\}$

$\therefore$ $1 \notin f(\mathfrak{R})$

Thus $f(\mathfrak{R}) \subset \mathfrak{R}$ and $f(R) \neq R$

Hence $\mathfrak{R}$ is an infinite set.

I) Properties of an infinite sets

1) If A is an infinite set, then A × A, P (A) are infinite sets.
2) If A and B are non empty sets and either A or B is an infinite set then A × B is an infinite set.
3) If either A or B is an infinite set then A ∪ B is an infinite set.
4) If $A \subseteq B$ and A is an infinite set then B is also an infinite set.

   i.e. the superset of an infinite set is an infinite.

3.4.2 Countable Sets

- We know that the codinality of a set is the number of elements of that set. If set is finite then, we can list elements as 1, 2, 3, ... Therefore every finite set is countable. As $|\phi| = 0$, the null set is also countable. A question remains same for an infinite sets. Let us define countable infinite sets.

Definition :

- An infinite set A is said to be **countable** if there exists a bijection $f : \mathbf{N} \to A$

$\therefore \quad A = \{ f(1), f(2), f(3), \ldots\}$

- A countably infinite set is also known as a denumarable set.

i.e. If A is a denumerable set then we can least elements of A as $a_1, a_2, a_3, \ldots a_n \ldots$ or $f(1)$, $f(2) \ldots f(n) \ldots$.

e.g.

1) ϕ is countable
2) $A = \{1, 2, 3, 4, \ldots 1000\}$ is countable as $|A| = 1000$.
3) As $f : \mathbf{N} \to \mathbf{N}$ defined as $f(x) = x$: $\therefore$ **N** is countable
4) The set of integers is countable

As $f : \mathbf{N} \to \mathbf{Z}$ such that $f(n) = \frac{n+1}{2}$ if $n = 1, 3, 5, \ldots$

$= \frac{-n}{2}$ if $n = 0, 2, 4, 6, \ldots$

is bijective mapping.

5) The set of rational numbers is countable.
6) The set of real numbers is not countable
7) The set of complex numbers is not countable.
8) The set of real numbers in $[a, b]$, $a < b$ is not countable.
9) The countable union of countable sets is countable.

Properties of countable sets :

1) A subset of a countable set is countable.
2) Let A and B be countable sets then $A \cup B$ is countable.

$\Rightarrow$ Define $f : A \cup B \to \mathbf{N}$ asf $(a_i) = 2i - 1$

and $\quad f(b_i) = 2i$

f is bijective $\therefore A \cup B$ is countable

3) Prove that the set of rational numbers is countable.

Proof : We know that the countable union of countable sets is countable. Therefore it is sufficient to prove that the set of rational numbers in [0, 1] is countable.

We have to prove that $\exists$ at least one function f.

$f : [0, 1] \to \mathbf{N}$ such that f is injective.

We arrange the rational numbers of the interval according to increasing denominators as

$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$

then the one to one correspondence is as follows

$0 \leftrightarrow 1$

$1 \leftrightarrow 2$

$\frac{1}{2} \leftrightarrow 3$

$\frac{1}{3} \leftrightarrow 4$

$\frac{2}{3} \leftrightarrow 5$

$\frac{1}{4} \leftrightarrow 6$ and so on

Hence set of rationals in [0, 1] is countable. Thus the set of rational numbers is countable.

4) The set of irrational numbers is uncountable.

Proof : We know that $\Re = Q \cup \bar{Q}$ where Q = set of rational numbers, $\bar{Q}$ = set of irrational numbers. Suppose $\bar{Q}$ is countable $\Rightarrow Q \cup \bar{Q} = \Re$ is countable which is contradiction.

$\therefore \bar{Q}$ is not countable i.e. uncountable.

5) The set of real numbers in (0, 1) is not countable.

Proof : Assume that the set is countable.

$\therefore \quad A = \{x_1, x_2, x_3, \cdots x_n, \cdots\}.$

- Any real number in (0, 1) can be written in a unique decimal without an infinite string of 9's at the end. i.e. 0.3459999 will be represented as 0.345000. Let the infinite sequence be given by,

$1 \to x_1 = 0 \cdot a_{11}\, a_{12}\, a_{13} \ldots$

$2 \to x_2 = 0 \cdot a_{21}\, a_{22}\, a_{23} \ldots$

$3 \to x_3 = 0 \cdot a_{31}\, a_{32}\, a_{33} \ldots$

$\vdots \qquad \vdots$

$n \to \quad x_n = 0.a_{n1}\, a_{n2}\, a_{n3} \ldots$

$\vdots \qquad \vdots$

Construct a new number $y = 0 \cdot b_1 b_2 b_3 \ldots$

Where $b_i = 0$ if $a_{ii} \neq 0$

$b_i = 1$ if $a_{ii} = 0$

Hence $b_1 \neq a_{11}, b_2 \neq a_{22}, b_3 \neq a_{33} \ldots b_n \neq a_{nn}$

$\therefore b_i \neq a_{ii}, \forall i$

$\therefore y \neq x_1, y \neq x_i, \forall i$

- Hence y is not in the list of numbers $\{x_1, x_2, \cdots x_n \cdots\}$ Thus $y \in (0, 1)$ and it is different from elements in $\{x_1, x_2, \cdots x_n \cdots\}$ which is contradiction that A is countable.
- Hence A is not countable.
- Thus the set of real numbers is not countable.

3.5 Pigeon Hole Principle

AKTU : 2013-14

I) This principle states that if there are n + 1 pigeons and only n pigeon holes then two pigeons will share the same whole.

This principle is stated by using the analogy of the bijective mapping i.e. If A and B are any two sets such that $|A| > |B|$ then there does not exist bijective mapping from A to B.

Examples

Ex.3.5.1 : If 11 shoes are selected from 10 pairs of shoes then there must be a pair i.e. matched shoes among the selection.

Sol. : In the pigeonhole principle, 11 shoes are pigeons and the 10 pairs are the pigeon holes.

Ex.3.5.2 : Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13.

Sol. : We have A = {1, 2, 3, 4, 5, ... 12}

We form the six different sets each containing 2 numbers that add upto 13.

$A_1 = \{1, 12\}, A_2 = \{2, 11\}, A_3 = \{3, 10\}, A_4 = \{4, 9\}, A_5 = \{5, 8\}, A_6 = \{6, 7\}$

Each of the seven numbers chosen must belong to one of these sets. As there are only six sets, by pigeonhole principle two of the chosen numbers must belong to the same set and their sum is 13.

II) The extended pigeon hole principle

If n pigeons are assigned to m pigeon holes, then one of the pigeon holes must be occupied by at least $\left[\frac{n-1}{m}\right] + 1$ pigeons. It is also known as generalized pigeon hole principle. Here $\left[\frac{n-1}{m}\right]$ is the integer division of n – 1 by m. e.g. $\left[\frac{9}{2}\right] = 4, \left[\frac{16}{5}\right] = 3, \left[\frac{8}{3}\right] = 2.$

Examples

Ex.3.5.3 : Show that 7 colours are used to paint 50 bicycles, then at least 8 bicycles will be of same colour.

Sol. : By the extended pigeonhole principle, at least $\left[\frac{n-1}{m}\right] + 1$ pigeons will occupy one piegeonhole.

Here n = 50, m = 7 and m < n then

$$\left[\frac{50-1}{7}\right] + 1 = 7 + 1 = 8$$

Thus, 8 bicycles will be of the same colour.

Ex.3.5.4 : Write generalized pigeonhole principle. Use any form of pigeonhole principle to solve the given problem.

i) Assume that there are 3 mens and 5 womens in a party show that if these people are lined up in a row at least two women will be next to each other.

ii) Find the minimum number of students in the class to be sure that three of them are born in the same month.

Sol. : Please refer section 3.5 (II) for definition.

i) By using analogy of pigeon hole principle, we get 3 men = pigeonholes and 5 women = pigeon

Pigeons are more than pigeon holes.

$\therefore$ At least two pigeons share the same pigeon hole.

Hence at least two women in a row will be next to each other.

ii) Let n = Number of pigeons = Number of students

m = Number of pigeon holes = Number of months = 12

Given that 3 students in the class are born in the same month.

$$\therefore \quad \left[\frac{n-1}{m}\right]+1 = 3 \Rightarrow \frac{n-1}{m} = 2 \Rightarrow \frac{n-1}{12}$$

$$= 2 \Rightarrow n = 25$$

Therefore there are 25 minimum number of students in the class.

Ex.3.5.5 : Prove that among 100,000 people, there are two who are born at exactly the same time (hours, minutes and seconds)

Sol. : Let A be the set of people (pigeons) and B be the set of seconds (pigeon holes) of one day.

$\therefore |A| = 100{,}000 = n, \quad |B| = 24\times 60\times 60 = 86400 = m$

$$\text{Then } K = \left[\frac{n-1}{m}\right]+1$$

$$= \left[\frac{100000-1}{86400}\right]+1 \cong 1+1 \cong 2$$

Hence there are at least two people who are born on the same day same time.

Ex.3.5.6 : Show that there must be at least 90 ways to choose six numbers from 1 to 15 so that all the choices have the same sum.

Sol. : We have $n = 15c_6 = 5005$

The lowest sum of 6 numbers chosen from 1 to 15

$$= 1+2+3+4+5+6 = 21$$

$$\text{Highest sum} = 10+11+12+13+14+15 = 75$$

$$\text{Hence } m = 75-21+1 = 55$$

Hence by pigeon hole principle

$$K = \left[\frac{n-1}{m}\right]+1 = \frac{5005-1}{55}+1 \cong 91$$

Hence in at least 90 ways, we can choose six numbers from 1 to 15 so that all the choices have the same sum.

Ex.3.5.7 : Show that among 13 people, there are at least 2 people who were born in the same month. **AKTU : 2013-14**

Sol. : We have n = 13 and m = 12 = Months in a year.

$$\therefore K = \left[\frac{n-1}{m}\right]+1 = \left[\frac{13-1}{12}\right]+1 = 1+1 = 2$$

Hence there were at least two people who borned in the same month.

❑❑❑