Ch-1

Partial Differential Eqⁿ

P.D. eq^n - An eq^n containing two or more partial derivatives of two or more independent variable, K.a. p.d. eq^n.

$p = \frac{\partial z}{\partial x}$ $\quad q = \frac{\partial z}{\partial y}$ $\quad r = \frac{\partial^2 z}{\partial x^2}$ $\quad s = \frac{\partial^2 z}{\partial x \partial y}$ $\quad t = \frac{\partial^2 z}{\partial y^2}$.

Real life p.d. eq^n.

1. Laplace eq^n - $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$ $\qquad U_{xx} + U_{yy} + U_{zz} = 0.$

2. Wave eq^n :- $\frac{\partial^2 U}{\partial t^2} = c^2 \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right].$

3. Heat eq^n :- $\frac{\partial U}{\partial t} = c^2 \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right]$

Order - Highest ordered derivative is the order of eq^n.

Degree - Exponential power of highest ordered derivative.

Solution of partial Differential eq^n

sol^n is the one which satisfies. sol^n contains arbitary f^n & constant; on removing both p.d. eq^n is obtained.

Formation of partial Differential Eq^n

(i) By elimination of arbitary constant.

Q. (i) $z = ax + by + ab.$

$p = \frac{\partial z}{\partial x}$ $\qquad q = \frac{\partial z}{\partial y}.$

d. w.r.t. x and w.r.t. y

$z = a = p$ $\qquad z = b = q.$

$z = px + qy + pq.$

(ii) $z = ax + a^2 y^2 + b$

$\frac{\partial z}{\partial x} = a = p.$ $\qquad \frac{\partial z}{\partial y} = 2a^2 y = 2p^2 y.$

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(iii) $z = (x+a)(y+b)$

$\frac{\partial z}{\partial x} = p \qquad \frac{\partial z}{\partial y} = q.$

$z = xy + xb + ay + ab.$

$\frac{\partial z}{\partial x} = y+b \qquad \frac{\partial z}{\partial y} = a + x.$

$z = (y+b)(x+a)$

$\boxed{z = pq}$

(iv) $az + b = a^2x + y.$

$az = a^2x + y - b$

$\cancel{a}\frac{\partial z}{\partial x} = a^{\cancel{2}} \qquad a\frac{\partial z}{\partial y} = 1. \Rightarrow q = \frac{1}{a}.$

$p = a \qquad q = \frac{1}{a}.$

$\boxed{pq = 1}$

By eliminating arbitary f^n

Q. $z = F(x^2 - y^2)$

$\frac{\partial z}{\partial x} = F'(x^2-y^2)(2x) \qquad \frac{\partial z}{\partial y} = F'(x^2-y^2)(-2y).$

$\frac{p}{q} = \frac{F'\cancel{(x^2-y^2)}(\cancel{2}x)}{F'\cancel{(x^2-y^2)}(\cancel{-2}y)}$

$\frac{p}{q} = \frac{x}{-y}$

$+py = -qx$

$py + qx = 0.$

(ii) $z = \phi(x)\cdot\psi(y)$

$\frac{\partial z}{\partial x} = \phi'(x)\psi(y) \qquad \frac{\partial^2 z}{\partial x\partial y} = \phi'(x)\psi'(y)$

$\frac{\partial z}{\partial y} = \phi(x)\psi'(y) \qquad z = \phi(x)\psi(y)$

$\boxed{zs = pq}$

(iii) $z = x + y + F(xy)$

$$\frac{\partial z}{\partial x} = 1 + 0 + F'(xy)\cdot y \qquad \frac{\partial z}{\partial y} = 0 + 1 + F'(xy)\cdot x.$$

$$p - 1 = F'(xy)\cdot y \qquad q - 1 = F'(xy)\cdot y$$

$$\frac{p-1}{q-1} = \frac{F'(xy)\cdot y}{F'(xy)\cdot x}$$

$$\frac{p-1}{q-1} = \frac{y}{x}.$$

$$(p-1)x = (q-1)y.$$

$$px - x = qy - y.$$

$$px - qy = x - y.$$

(iv) $z = F(x+it) + g(x+it)$

$$\frac{\partial z}{\partial x} = f'(x+it) + g'(x+it) \qquad \frac{\partial z}{\partial t} = f'(x+it)\cdot i + g'(x+it)\cdot i$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x+it) \qquad \frac{\partial^2 z}{\partial t^2} = f''(x+it)\cdot i^2 + g''(x+it)\cdot (i)^2$$

$$= -f''(x+it) - g''(x+it)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = \cancel{f''(x+it)} + \cancel{g''(x+it)} - \cancel{f''(x+it)} - \cancel{g''(x+it)}$$

$$= 0.$$

Complete Integral :-

The soln $F(x, y, z, a, b) = 0$ of a first order partial diff. eqn, which contains two arbitary constants is called complete soln or complete integral.

Linear Homogenous partial Differential Eqn

A partial differential eqn is said to be linear if it is of first degree in the dependent variable & its partial derivatives and also they are not multiplied together.

Homogenous eqn :- every eqn containing dependent variable or its derivati-ve, called Homogenous eqn.

Quasi-linear partial diff. eqn:-

If the degree of highest ordered derivative is one.

If the degree of highest ordered derivative is one. General form of quasi-linear partial diff. eqn of first order.

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z). \quad \text{--- (1)}$$

P, Q → independent of z. & R is linear in z then eqn (1) is linear pa... differential eqn.

Q. Solve:-

$t = \sin(xy)$

$$\frac{\partial^2 z}{\partial y^2} = \sin xy.$$

on integrating:

$$\frac{\partial z}{\partial y} = -\frac{1}{x}\cos(xy) + F(x)$$

The constant is being taken as a f^n of x.

Again integrating,

$$z = \frac{-1}{x^2}\sin(xy) + yF(x) + \phi(x).$$

Q. $\frac{\partial^2 z}{\partial x^2} + z = 0.$

when $x = 0$; $z = e^y$ $\frac{\partial z}{\partial x} = 1.$

If z were a f^n of x alone, the soln would have been-

$z = c_1 \cos x + c_2 \sin x$; c_1, c_2 → arbitary constant.

The soln of given eqn is

$$z = F(y)\cos x + \phi(y)\sin x.$$

$$\frac{\partial z}{\partial x} = -F(y)\sin x + \phi(y)\cos x.$$

$x = 0$; $z = e^y$ ∴ $e^y = F(y)$

$x = 0$ $\frac{\partial z}{\partial x} = 1$ $1 = \phi(y).$

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Linear partial Differential eqⁿ of first order.

A differential eqⁿ involving first order partial derivative p and q called partial diff. eqⁿ of first order.

i.e. p & q occur in first degree and are not multiplied together.

LAGRANGE's Eqⁿ

$$Pp + Qq = R.$$

1. First write in a form $Pp + Qq = R$.
2. Form auxiliary eqⁿ

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

3. Solve eqⁿ by method of grouping or by multiplier. OR both to get two independent solⁿ $U = a$; $v = b$.
4. then $\phi(U, v) = 0$ or $v = F(u)$

Q. (i) $yzp - xzq = xy$.

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}.$$

$$\frac{dx}{y\cancel{z}} = \frac{dy}{-x\cancel{z}}$$

$$-xdx = ydy.$$

$$xdx + ydy = 0.$$

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$$\frac{dy}{-\cancel{x}z} = \frac{dz}{\cancel{x}y}$$

$$-ydy = zdz.$$

$$ydy + zdz = 0.$$

$$\frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}.$$

$$\phi(x^2+y^2, y^2+z^2) = 0.$$

(ii) $y^2p - xyq = x(z-2y)$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dx}{y^{\cancel{2}}} = \frac{dy}{-x\cancel{y}}$$

$$x\,dx = -y\,dy.$$

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$-(z-2y)dy = ydz.$$

$$-zdy + 2ydy = ydz$$

$$2ydy = ydz + zdy.$$

$$y^2 = yz + c_2.$$

$$\phi(x^2+y^2, y^2 - yz) = 0.$$

Linear Partial Diff. eq^n ~~of partl~~ first order

A diff. eq^n involving first order partial derivative p and q only is called partial diff. eq^n of first order. (only p & q).

If p & q occur only & are not multiplied together then it is called Linear partial Differential eq^n of first order.

Lagrange's eq^n

P.D.E. of form $\underset{f^n \text{ of } x}{P}\overset{\partial z/\partial x}{p} + \underset{f^n \text{ of } y}{Q}\overset{\partial z/\partial y}{q} = R \rightarrow f^n \text{ of } z.$

standard form of quasi-linear partial differential eq^n of first order is called Lagrange's Eq^n.

Working Rule

$$Pp + Qq = R$$

(i) Form auxiliary eq^n $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$.

(ii) Solve this method of grouping. OR
method of multipliers. OR
to get two independent sol^n $u = a$ & $v = b$ where a, b are arbitrary constant.

(iii) Then $\phi(u,v) = 0$ or $v = F(u)$ is general sol^n of eq^n $Pp + Qq = R$.

Q. Solve-

(i) $yzp - xzq = xy$

given diff. eq^n $yzp - xzq = xy$.

Comparing this with $Pp + Qq = R$.

$$P = yz \quad Q = -zx \quad R = xy$$

Lagrange's subsidiary eq^n

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

taking first & second fraⁿ, we get

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\,dx + y\,dy = 0$$

Integrating, we get

$$\frac{x^2}{\cancel{2}} + \frac{y^2}{\cancel{2}} = \frac{c_1}{\cancel{2}}$$

$$x^2 + y^2 = c_1.$$

Now Ist & third

$$\frac{dx}{yz} = \frac{dz}{xy} \Rightarrow \frac{dx}{z} = \frac{dz}{x}$$

$$x\,dx - z\,dz = 0.$$

on integrating, we get

$$\frac{x^2}{2} - \frac{z^2}{2} = \frac{c_2}{2}.$$

$$x^2 - z^2 = c_2$$

In gen. solⁿ is $(x^2+y^2,\ x^2-z^2) = 0.$

Q. $y^2p - xyp = x(z-2y)$

Sol^n comparing with Lagranche's eq^n.

$P = y^2 \quad Q = -xy \quad R = x(z-2y)$

Lagrange's subsidiary eq^n are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}.$$

taking First & second fraction

$$\frac{dx}{y^{\cancel{2}}} = \frac{dy}{-x\cancel{y}}$$

$$\frac{dx}{y} = \frac{dy}{-x}.$$

$$-x\,dx = y\,dy.$$

$$x\,dx + y\,dy = 0.$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$$x^2 + y^2 = c_1$$

taking 2nd & 3rd fraction

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$\Rightarrow \frac{dy}{-y} = \frac{dz}{(z-2y)}$$

$$-z\,dy + 2y\,dy = y\,dz$$

$$2y\,dy = y\,dz + z\,dy$$

on Integrating

$$y^2 = yz + c_2.$$

Hence general eq^n / sol^n is

$$\phi(x^2+y^2,\ y^2-yz) = 0.$$

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(iii) $x^2p + y^2p = (x+y)z$.

$P = x^2 \quad Q = y^2 \quad R = (x+y)z$.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}.$$

taking 1 & 2 fractn.

$$\frac{dx}{x^2} = \frac{dy}{y^2}.$$

on integrating

$$-\frac{1}{x} = -\frac{1}{y} - c_1.$$

$$\frac{1}{x} - \frac{1}{y} = c_1.$$

Now taking again from 1.

$$\frac{dx-dy}{x^2-y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx-dy}{x-y} = \frac{dz}{z}.$$

on integrating, we get

$$\log(x-y) = \log z + \log c_2.$$

$$\frac{x-y}{z} = c_2.$$

Hence general soln is

$$\phi\left[\frac{1}{x} - \frac{1}{y}, \frac{x-y}{z}\right] = 0.$$

(iii) $pz - qz = z^2 + (x+y)^2$.

$P = z \quad Q = -z \quad R = z^2 + (x+y)^2$.

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$

① & ②

$$\frac{dx}{\cancel{z}} = \frac{dy}{-\cancel{z}}$$

$$dx + dy = 0.$$

$$x + y = c_1$$

Now taking ① & ③

$$\frac{dx}{z} = \frac{dz}{z^2+c_1^2}$$

$$\underbrace{(z^2+c_1^2)}_{t}dx = zdz.$$

$$dx = \frac{zdz}{(z^2+c_1^2)}$$

$$2dx = \frac{2zdz}{z^2+c_1^2}.$$

On integration.

$$\log(z^2+c_1^2) = 2x + c_2.$$

$$\log(z^2+(x+y)^2] - 2x = c_2.$$

from ① & ② gen. sol^n is

$$\phi[x+y, \log(x^2+y^2+z^2+2xy) - 2x] = 0.$$

Q7. Solve the diff. eq^n.

(i) $(mz-ny)p + (nz-lz)q = ly - mx$.

$P = mz - ny$

$Q = nz - lz$

$R = ly - mx$.

$$\frac{dx}{mz-ny} = \frac{dy}{nz-lz} = \frac{dz}{ly-mz}$$

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Here grouping is not possible now we'll multiply the term which is not here.

using x, y, z as multiplier, we get. (x, y, z form me hi ata hai).

each fract^n = $\dfrac{x\,dx + y\,dy + z\,dz}{0}$

(we added to make denominator zero)

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$.

which on integration gives-

$x^2 + y^2 + z^2 = a$.

Now, again using l, m, n as multiplier.

each fract^n = $\dfrac{l\,dx + m\,dy + n\,dz}{0}$

which on integrat^n

$lx + my + nz = b$.

from (1) & (2) general sol^n is

$x^2 + y^2 + z^2 = f(lx + my + nz)$

(ii) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multiplier we get

each fract^n = $\dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0.$$

which on integrat^n

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1.$$

Again $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier.

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

On int.

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2.$$

$$xyz = c_2.$$

gen. soln

$$\phi\left[\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right] = 0.$$

Q. solve

$$x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$$

where $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$.

Soln

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

using $x, y, -1$ as multiplier we get

each fractn $\Rightarrow \frac{x\,dx + y\,dy - 1dz}{0}$

$$x\,dx + y\,dy - dz = 0.$$

we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2}.$$

$$x^2 + y^2 - 2z = c_1.$$

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

On integration

$$\log x + \log y + \log z = \log c_2$$
$$xyz = c_2.$$

Hence gen. soln

$$\phi\,(x^2+y^2-2z,\ xyz)=0$$

Q9 Solve following partial eqn

$$x(y^2-z^2)p + y(z^2 - x^2)q = z(x^2-y^2) \quad \text{where } p=\frac{\partial z}{\partial x} \ \& \ q=\frac{\partial z}{\partial y}.$$

Soln multiplier $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ and x, y, z as multiplier.

Q10. solve

$$p + 3q = 5z + \tan(y-3x).$$

$P=1 \quad Q=3 \quad R = 5z + \tan(y-3x).$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$$

First & second

$$\frac{dx}{1} = \frac{dy}{3}.$$

$$3\,dx = 1\,dy.$$
$$3x - y = c_1.$$
$$\text{or } c_1 = y-3x.$$

Second & third or 1st & third

$$\frac{dx}{1} = \frac{dz}{5z+\tan c_1}$$

$$dx\,(5z+\tan c_1) = 1\,dz$$

$$dx = \frac{dz}{5z+\tan c_1} \quad \Rightarrow \cancel{dx} \ =$$

$$x = \frac{1}{5}\log(5z+\tan c_1) - \frac{c_2}{5}.$$

$$\log[5z+\tan(y-3x)] - 5x = c_2.$$

General soln

$$\phi\,[y-3x,\ \log(5z+\tan(y-3x)) - 5x].$$

Q. $(y^2+z^2-x^2)p - 2xyq + 2xz = 0.$

$P = (y^2+z^2-x^2)$

$Q = -2xy.$

$R = -2xz$

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}.$$

taking 2nd & third

$$\frac{dy}{-\cancel{2}xy} = \frac{dz}{-\cancel{2}xz}$$

$$\frac{dy}{y} = \frac{dz}{z}.$$

on integration

$\log y = \log z + \log c_2.$

$\frac{y}{z} = c_1.$

taking first, Now take multipliers x, y, z we have

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$$

$$\frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$$

Now take 2nd or 3rd.

$$\frac{x\,dx + y\,dy + z\,dz}{-\cancel{x}(x^2+y^2+z^2)} = \frac{dz}{-\cancel{2}\cancel{x}z}$$

$$\frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2} = \frac{dz}{z}$$

If $x^2+y^2+z^2 = t.$

$t' = 2x\,dx + 2y\,dy + 2z\,dz.$

On integration forming $\frac{f(x)}{f'(x)}$

Hence $\log(x^2+y^2+z^2) = \log z + \log c_2$.

Q12. Solve

(i) $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \left(\frac{x-y}{xy}\right)$.

multiply by xyz both side.

$x(y-z)p + y(z-x)q = z(x-y)$.

$P = (y-z)x \quad Q = y(z-x) \quad R = z(x-y)$.

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier.

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$$

on integration we get.

$\log x + \log y + \log z = \log c_1$.

$xyz = c_1$.

then $1, 1, 1$ as multiplier

$dx + dy + dz = 0$.

On integration

$x + y + z = c_2$.

General soln

$\phi(xyz, x+y+z) = 0$.

Q. (i) $(x^2-y^2-yz)p + (x^2-y^2-zx)q = z(x-y)$

$P = x^2-y^2-yz$

$Q = x^2-y^2-zx$

$R = z(x-y)$

$$\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y^2-zx} = \frac{dz}{z(x-y)}$$

using 1, -1, -1 as multiplies.

each fraction $= \dfrac{dx-dy-dz}{0}$

$dx-dy-dz=0.$

on integration we get

$x-y-z = c_1.$

from 1st & 2nd, we have.

$$\frac{x\,dx - y\,dy}{x^3-xy^2-xyz-yx^2+y^3+yzx} = \frac{dz}{z(x-y)}$$

$$\frac{x\,dx-y\,dy}{(x-y)(x^2-y^2)} = \frac{dz}{z(x-y)}$$

$$\frac{x\,dx-y\,dy}{(x^2-y^2)} = \frac{dz}{z}.$$

on integrating

$$\frac{1}{2}\log(x^2-y^2) = \log z + \frac{1}{2}\log c_2.$$

$$\frac{x^2-y^2}{z^2} = c_2.$$

Hence gen. soln

$$\phi\left[x-y-z, \frac{x^2-y^2}{z^2}\right].$$

(ii) $(x^2+xy)p - (xy+y^2)q = -(x-y)(2x+2y+z)$.

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

on first & second we have

$$\frac{dx}{x} = \frac{dy}{-y}$$

on integrating we have

$$\log x = -\log y + \log c_1 .$$

from first & second

$$\frac{dx+dy}{(x-y)(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

$$(2x+2y+z)(dz+dy) + (x+y)dz = 0.$$

$$(x+y+z)d(x+y) + (x+y)(dx+dy) + (x+y)dz = 0.$$

$$(x+y+z)d(x+y) + (x+y)d(x+y+z) = 0.$$

$$d\{(x+y)(x+y+z)\} = 0.$$

Non-linear Partial Differential Eq^n of First Order.

Which involves first order partial derivatives p and q with Degree higher than 1 & product is called non-linear partial differential eq^n of first order.

(i) Eq^n of the form $f(p,q)=0$ i.e;
only p and q & no x, y, z ; a, b are conn. by relat^n
where a & b conn. by $f(a,b)=0$

Q. (i) $\sqrt{p} + \sqrt{q} = 1$ compl. sol^n $z = ax+by+c$.
eq^n of form $f(p,q)=0$.
complete sol^n is $x = ax+by+c$
$\sqrt{a} + \sqrt{b} = 1$ or $b = (1-\sqrt{a})^2$
compl. sol^n is $z = ax + (1-\sqrt{a})^2 y + c$. Ans

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(ii). $pq = p + q$.

gen. soln $z = ax + by + c$.

Replace p & q by a & b.

$ab = a + b$

or

$$b = \frac{a}{a-1}$$

Now complete soln

$$z = ax + \frac{a}{a-1} y + c.$$

Q: $x^2p^2 + y^2q^2 = z^2$.

Here degree is 2 hence non-linear

The given eqn can be written as.

$$\left(\frac{x}{z} \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1. \quad (1)$$

Let $\frac{\partial x}{x} = \partial X \quad \frac{\partial y}{y} = \partial Y \quad \frac{\partial z}{z} = \partial Z.$

then $X = \log x$
$Y = \log y$ on integration
$Z = \log z$

$$\frac{\partial Z}{\partial X} = \frac{x}{z} \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial Z}{\partial Y} = \frac{y}{z} \cdot \frac{\partial z}{\partial y}$$

Now eqn (1) can be written as

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1.$$

$$P^2 + Q^2 = 1$$

where;

$$P = \frac{\partial Z}{\partial X} \quad \& \quad Q = \frac{\partial Z}{\partial Y}.$$

classmate
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It is of the form $f(P, Q) = 0$.

its complete sol^n is $z = ax + by + c$.

where $a^2 + b^2 = 1$

or

$b = \sqrt{1 - a^2}$.

Summary

1. p, q ho x, y, z na ho.

$p = \frac{\partial z}{\partial x} = a$; $q = \frac{\partial z}{\partial y} = b$

assume complete sol^n $z = ax + \underbrace{\phi(a)}_{b = \phi(a)} y + c$

Eq^n of Form $z = px + qy + F(p,q)$ (CLAIRAUT TYPE)

Q. $z = px + qy + \sqrt{1 + p^2 + q^2}$

$z = px + qy + \sqrt{1 + p^2 + q^2}$

put $p = a$ $q = b$

$z = ax + by + \sqrt{1 + a^2 + b^2}$.

Q. find singular integral (free from constant) of $z = px + qy + pq$.

$z = px + qy + F(p, q)$

complete integral

$z = ax + by + ab$.

diff. w.r.t. a & b

$0 = x + b$ $\quad 0 = y + a$.

$z = -xy - xy + xy$

$z = -xy$.

singular sol^n of eq^n.

Q. $4xyz = pq + 2px^2y + 2qxy^2$.

$x^2 = X$ $\quad y^2 = Y$

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = 2x \cdot \frac{\partial z}{\partial X}$ & $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = 2y \frac{\partial z}{\partial Y}$

eq^n becomes.

$4xyz = 4xy \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y} + 4x^3y \frac{\partial z}{\partial X} + 4xy^3 \frac{\partial z}{\partial Y}$.

$$z = x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}.$$

$$= X \frac{\partial z}{\partial X} + y \frac{\partial z}{\partial y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial y}.$$

$z = PX + QY + PQ.$

$P = \frac{\partial z}{\partial X} \qquad Q = \frac{\partial z}{\partial Y}.$

$z = PX + QY + f(P, Q).$

complete soln $\quad z = aX + bY + ab$

$z = ax^2 + by^2 + ab.$

Eqn of form $f_1(x, p) = f_2(y, p)$.

z is absent and x's term involving p & can be separated those involving y and q.

$f_1(x, p) = f_2(y, p) - a.$

$p = F_1(x) \qquad q = F_2(y).$

$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} dy = pdx + qdy.$

$dz = F_1(x)dx + f_2(y)dy.$

$z = \int F_1(x)dx + \int F_2(y)dy + b.$

Q. $p^2 - q^2 = x - y.$

p ke sth x.

q ke sth y.

$p^2 - x = q^2 - y$ which is of the form $F_1(x, p) = F_2(y, q)$

$p^2 - x = q^2 - y = a.$

$p^2 = x + a \; ; \quad q^2 = y + a.$

$p = \sqrt{x + a} \; ; \quad q = \sqrt{y + a}.$

put in soln $\quad dz = pdx + qdy.$

$dz = \sqrt{x + a}\, dx + \sqrt{y + a}\, dy$

On Integrating; $z = \frac{2}{3}(x + a)^{3/2} + \frac{2}{3}(y + a)^{3/2} + b.$

Q. Solve : $yp = 2yx + \log q$.

Soln. $p = 2x + \frac{1}{y}\log q$ or $p - 2x = \frac{1}{y}\log q$.

which is of the form $F_1(x,p) = F_2(y,q)$

$$p - 2x = \frac{1}{y}\log q = a$$

then $p = 2x + a$

& $\log q = ay \Rightarrow q = e^{ay}$.

substituting p & q in $dz = pdx + qdy$; we get

$$dz = (2x+a)\,dx + e^{ay}\,dy.$$

On integrating

$$z = x^2 + ax + \frac{1}{a}e^{ay} + b.$$

CHARPIT METHOD

general method for finding soln of non-linear partial diff. eqn of first order.

Let given eqn be $F(x,y,z,p,q) = 0$. — ①

If we can find another relation

$$F(x,y,z,p,q) = 0. \quad — ②$$

involving x, y, z, p & q then solve ① & ② for p & q, & substi-te in $dz = pdx + qdy$. — ③

The auxilary eqn are -

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}} = \frac{dF}{0}.$$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0}.$$

Q. $2zx - px^2 - 2qxy + pq = 0$

$$\frac{\partial f}{\partial x} = 2z - 2px - 2qy$$

$$\frac{\partial F}{\partial y} = -2qx$$

$$\frac{\partial F}{\partial z} = 2x$$

$$\frac{\partial F}{\partial p} = -x^2 + q$$

$$\frac{\partial F}{\partial q} = -2xy + p$$

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}} = \frac{dF}{0}$$

$$\frac{dp}{2z-2py} = \frac{dq}{\cancel{-2qx} + \cancel{q2x}} = \frac{dz}{x^2-q} = \frac{dy}{2xy-p} = \frac{dF}{0}$$

$dq = 0$

$q = a$

$$dz = pdx + qdy = \frac{2x(z-ay)}{x^2-a}\,dx + ady$$

$$\frac{dz - ady}{z-ay} = \frac{2x}{x^2-a}\,dx$$

Integrating

$$\log(z-ay) = \log(x^2-a) + \log b$$

or

$z - ay = b(x^2-a)$ complete integral.

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Q. Solve $(p^2+q^2)y = qz$

Soln

$f = (p^2+q^2)y - qz = 0$

$\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = p^2+q^2$; $\frac{\partial f}{\partial z} = -q$; $\frac{\partial f}{\partial p} = 2py$;

$\frac{\partial f}{\partial q} = 2qy - z$

Apply Auxiliary eqn we get,

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dF}{0}$$

from (1) & (2)

$p\,dp + q\,dq = 0$

integ. $p^2+q^2 = a^2$

put in (1) $q = \frac{a^2 y}{z}$

$p = \sqrt{a^2-q^2}$

$= \sqrt{a^2 - \frac{a^4y^2}{z^2}} - \frac{a}{z}\sqrt{z^2-a^2y^2}$

$dz = p\,dx + q\,dy = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy.$

$z\,dz - a^2y\,dy = a\sqrt{z^2-a^2y^2}\,dz$ or $\frac{\frac{1}{2}d(z^2-a^2y^2)}{\sqrt{z^2-a^2y^2}} = a\,dx$

on integrating, we get,

$\sqrt{z^2-a^2y^2} = ax+b$

$z^2 = (ax+b)^2 + a^2y^2.$

Q. find complete integral of $px+qy=pq$.

Soln Here given eqn is $F(x,y,z,p,q) = px+qy-pq=0$ — ①

∴ Charpit's auxiliary eqn are

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$\frac{\partial F}{\partial x} = p$ $\qquad \frac{\partial F}{\partial y} = q$ $\qquad \frac{\partial F}{\partial p} = x-q$ $\qquad \frac{\partial F}{\partial p} = x-q$

$\frac{\partial F}{\partial z} = 0$ $\qquad \frac{\partial F}{\partial z} = 0$ $\qquad \frac{\partial F}{\partial q} = y-p$ $\qquad \frac{\partial F}{\partial q} = y-p$

$$\frac{dp}{p+0} = \frac{dq}{q+0} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{-x+q} = \frac{dy}{-y+p}$$

$$= \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-px+pq-yq+pq} = \frac{dx}{-x+q} = \frac{dy}{-y+p}$$

$$= \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{2pq-px-yq} = \frac{dx}{-x+q} = \frac{dy}{-y+p} \quad \text{— ②}$$

Taking the last two fraction of (2)

$$\frac{1}{p}\,dp = \frac{1}{q}\,dq$$

Integrating, $\log p = \log q + \log a$

or $p = aq$ — ③

substituting the value of p in ①, we have

$aqx + qy - aq^2 = 0$ or $aq = ax+y$ as $q \neq 0$ — ④

from ③ & ④ $\quad q = (ax+y)/a$ & $p = ax+y$ — ⑤

putting p, q in $dz = pdx + qdy$, we get

$dz = (ax+y)dx + [(ax+y)/a]dy$

or

$a\,dz = (ax+y)\,d(ax+y) = u\,du$ where $u = ax+y$.

Integrating $az = (1/2)u^2 + b = \frac{1}{2}(ax+y)^2 + b$

viz complete integral, a & b being arbitary Const.

Q. Find a complete solⁿ of integral $p^2 - y^2 q = y^2 - x^2$.

$$p^2 - y^2 q - y^2 + x^2 = 0 \quad — (1)$$

$$\frac{dp}{2x} = \frac{dq}{-2qy-2y} = \frac{dz}{-p(2p)-q(-y^2)} = \frac{dx}{-2p} = \frac{dy}{y^2} \quad — (2)$$

$\frac{dp}{2x} = \frac{dq}{}$ taking first & fourth

$$p\,dp + x\,dx = 0.$$

On integrating

$$p^2/2 + x^2/2 = a^2/2$$

or

$$p^2 + x^2 = a^2. \quad — (3)$$

Solving (1) & (3) for p, q &.

$$p = \sqrt{(a^2-x^2)},\; q = a^2 y^{-2} - 1.$$

$$dz = p\,dx + q\,dy$$

$$= \sqrt{a^2-x^2}\,dx + (a^2 y^{-2} - 1)\,dy.$$

Integrating, $z = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) - (a^2/y) - y + b$

cauchy's method

consider first order partial diff. eqn.

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = F(x,y) + ku; \qquad u(0,y) = h(y) \quad — (1).$$

Let $u(x,y)$ be the soln of eqn 1.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy. \quad — (2).$$

from (1) & (2)

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{F(x,y) + ku} \quad — (3)$$

taking (1) & (2)

$$\frac{dx}{a} = \frac{dy}{b} \quad — (4)$$

Now eqn becomes

$$bx - ay = c. \quad — (5)$$

Now

$$\frac{dx}{a} = \frac{du}{F(x,y) + ku} \quad — (6)$$

$$\frac{dx}{a} = \frac{du}{F\left(x \frac{bx-c}{a}\right) + ku} \quad — (7)$$

Now

$$\frac{du}{dx} - \frac{k}{a}u = \frac{1}{a} F\left(x, \frac{bx-c}{a}\right) \quad — (8)$$

which is a first order linear ordin. diff. eqn.

$$I.F. = e^{-k/a\, x}$$

soln of (8) is of the form

$$u = G(x,c) + c_2.$$

where $c_1 = g(c)$; g is an arbitary const f^n

from eq^n (4)

$$U = q(x, c) + g(c)$$

↳ determined by using given conditions.

Q. Use cauchy's method of characteristics to solve partial diff. eq^n.

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = x + y; \quad U(x, 0) = 0.$$

system of eq^n is $\frac{dx}{1} = \frac{dy}{1} = \frac{du}{x+y}$ — (1)

Taking (1) & (2). we get

$$x - y = c. \quad \text{— (2)}$$

$$x = c + y$$

$c \rightarrow$ constant

Now taking $\frac{dy}{1} = \frac{du}{x+y}$

$$(x+y)dy = du$$

$$dy = \frac{du}{2y + c}$$

$$u(x, y) = y^2 + cy + C_1 \text{ — (3)}$$

from (3) $c_1 = g(c)$ then,

$$u(x, y) = y^2 + cy + g(c)$$

$$= y^2 + y(x - y) + g(x - y) \text{ — (4).}$$

where $g(x-y)$ is an arbitary f^n

On applying condition $u(x, 0) = 0$ on eq^n (4) we get

$$0 = g(x)$$

$$g(x - y) = 0$$

consequenty, the sol^n is

$$u(x, y) = xy.$$

Q. use the method of characteristics to solve:-

$U_x - U_y = 0$; $U(x,0) = x$.

$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$; $U(x,0) = x$.

The system of eq^n

$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{0}$. — 1

taking 1 & 2

$\frac{dx}{1} = \frac{dy}{-1}$.

$-1dx = 1dy$.

$-x = y$.

$-y - x = 0$.

$-y - x = c$. — 2

$c = x+y$

$\cancel{y = c + x}$ — 3

Now taking 3 & 3.

$\frac{dy}{-1} = \frac{du}{0}$.

$du = 0$.

$u(x,y) = c_1$.

$u(x,y) = c_1 = g(c) = g(x+y)$.

$g(x+y)$ is an arbit. f^n.

from given conditn

$U(x,0) = x = g(x)$

$g(x+y) = (x+y)$

$\therefore$ from eq^n 3

$u(x,y) = x+y$.

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Partial Diff. eq^n of 2nd order which includes r, s, t but none of higher order.

Linear partial diff. eq^n with constant coefficients

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-1}\partial y} + \dots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}} + C_0 \frac{\partial z}{\partial x} + C_1 \frac{\partial z}{\partial y} + P_0 z = f(x,y) \quad \text{— (1)}$$

$A_0, A_1 \dots A_n, B_0, B_1 \dots B_{n-1}, C_0, C_1, P_0$ are constants or f^n of x and y.

Homogenous linear partial diff. eq^n with constant coefficient.

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad (2)$$

$a_0, a_1, a_2 \dots, a_n \rightarrow$ constant, is called homogenous linear partial differential eq^n of nth order with constant coefficient.

D for $\frac{\partial}{\partial x}$ & D' for $\frac{\partial}{\partial y}$.

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n) z = f(x,y)$$

$$\phi(D, D') z = f(x,y).$$

complete sol^n

$$\boxed{z = C.F. + P.I.}$$

Rules for Finding C.F.

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0.$$

$$(D^2 + a_1 DD' + a_2 D'^2) z = 0.$$

$$D^2 + a_1 DD' + a_2 D'^2 = 0.$$

is called Auxiliary eq^n (A.E.)

Lets's its roots be m_1 & m_2.

case 1:- Distinct root $m_1 \neq m_2$.

$(D - m_1 D')(D - m_2 D') z = 0.$ — ③

Now sol^n of $(D - m_2 D') z = 0$.

$(D - m_2 D') z = 0. \Rightarrow \frac{\partial z}{\partial x} - m_2 \frac{\partial z}{\partial y}$

$\Rightarrow \quad p - m_2 q = 0.$

which is of Lagrange's form and auxiliary eq^n are -

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}.$$

$y + m_2 x = a.$ or $dy + m_2 dx = 0.$

$dz = 0$ or $z = b.$

$\therefore \quad z = F_2(y - m_2 x)$ is a sol^n of $(D - m_2 D') z = 0.$

eq^n ③ will also have soln of

$(D - m_1 D') z = 0$ i.e. $z = F_1(y - m_1 x)$

Hence soln

$$\boxed{z = F_1(y + m_1 x) + F_2(y + m_2 x).}$$

1. m_1, m_2, m_3 roots

then,

$$\boxed{C.F. = f_1(y + m_1 x) + F_2(y + m_2 x) + f_3(y + m_3 x)}$$

2. Equal roots:-

$z = \phi(y + mx) + x F(y + mx)$

or $z = f_1(y + mx) + x f_2(y + mx)$

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Q. $4r - 12s + 9t = 0.$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}.$$

$$4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0.$$

$$\left[4\frac{\partial^2}{\partial x^2} - 12\frac{\partial^2}{\partial x \partial y} + 9\frac{\partial^2}{\partial y^2}\right] z = 0.$$

$$4D^2 - 12DD' + 9D^2 = 0.$$

$$4m^2 - 12m + 9 = 0.$$

$$(2m-3)^2 = 0.$$

$$m = \frac{3}{2}, \frac{3}{2}.$$

$$C.F. = F_1\left(y + \frac{3}{2}x\right) + x F_2\left(y + \frac{3}{2}x\right) \cancel{=0}$$

$$P.I = 0.$$

Q. $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{3\partial^3 z}{\partial x^2 \partial y} + \dfrac{2\partial^3 z}{\partial x \partial y} = 0.$

$$D^3 z - 3D^2D'z + 2DD'^2 z = 0.$$

$$m^3 - 3m^2 + 2m = 0.$$

$$m(m-1)(m-2) = 0.$$

$$m = 0, 1, 2.$$

$$c.f. = F_1(y) + F_2(y+x) + F_3(y+2x)$$

$$P.I = 0.$$

iii) $r = a^2 t$

$$\frac{\partial^2 z}{\partial x^2} = a^2 t$$

$D^2 z = a^2 t \Rightarrow D^2 z - a^2 D'^2 = 0.$

~~$m^2 = 0$~~

~~$m = 0, 0.$~~

~~C.F. = $F_1(y+0) + 0F_2(y+0)$~~

~~C.F. = F_1~~

$m^2 = a^2$

$m^2 - a^2 = 0.$

$m = \pm a.$

C.F. = $F_1(y+ax) + F_2(y+ax)$ ✗

P.I. = 0.

$z = $ C.F. + P.I.

$= F_1(y+ax) + F_2(y+ax).$

Q.(iii) $D^3 D'^2 + D^2 D'^3 = 0.$

$m^3 + m^2 = 0.$

$m^2(m+1) = 0.$

$m = 0, 0, -1.$

C.F. = $F_1(y + 0x) + 0F_2(y + 0x) + F_3(y - x).$

P.I. = 0.

$y = $ C.F. + P.I.

But Order is 5 then how 3 roots

Hence

$D^2 \cdot D'^2 (D + D')z = 0$

$D^2 = 0.$

$m^2 = 0$

$m = 0, 0$

C.F. = $F_1(y) + x f_2(y + x)$

$= f_1(y) + x F_2(y).$

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NOTE-

when power of D' is greater than powes of D; in this situatn formation of auxiliary eqn is

put $D' = m$ $D = 1$

case 1- If roots are distinct like m_1, m_2, m_3.

then C.F. is $F_1(x+m_1y) + F_2(x+m_2y) + F_3(x+m_3y)$.

case 2- If roots are same like m, m, m

C.F. = $F_1(x+my) + yF_2(x+my) + y^2F_3(x+my)$.

$\Rightarrow$ Now back to Questn

$D'^2 = 0$.

$m = 0, 0$.

C.F. = $F_3(x) + yF_4(x)$.

$(D+D')$

$\Rightarrow (m+1) = 0$.

$m = -1$.

C.F. = $F_5(y-x)$.

C.F. = $F_1(y) + xF_2(y) + F_3(x) + yF_4(x) + F_5(y-x)$.

C.F.+P.I = $F_1(y) + xF_2(y) + F_3(x) + yF_4(x) + F_5(y-x) + 0$.

(iv) $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$.

$D^4 z - D'^4 z = 0$.

$m^4 - 1 = 0$. $\Rightarrow (m^2-1)(m^2+1) = 0$.

$m = 1, -1, +i, -i$.

C.F. = $F_1(y+x) + xF_2(y+x) + x^2F_3(y+x) + F_4 x^3(y+x)$.

= $F_1(y+x) + xF_2(y+x) + x^2F_3(y+x) + x^3F_4(y+x) + 0$.

C.F. = $F_1(y+x) + F_2(y-x) + F_3(y+ix) + F_4(y-ix) + 0$

Q. (i) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

(ii) $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$.

(iii) $r - 4s + 4t = 0$.

Q4. Solve $\dfrac{\partial^4 z}{\partial x^4} + \dfrac{\partial^4 z}{\partial y^4} = 0$.

$D^4z + D'^4z = 0$.

$m^4 + 1 = 0$.

$m^4 - (-1) = 0$.

$(m^2 - 1)(m^2 + 1) = 0$.

$m = 1, -1, i, -i$

$C.F. = F_1(y + x) + F_2(y - x) + F_3(y - ix) + F_4(y + ix)$.

$m^4 + 1 = 0$.

$m^4 + 1 + 2m^2 - 2m^2 = 0$.

$m^4 + 1 + 2m^2 = 2m^2$

$(m^2 + 1)^2 - (m\sqrt{2})^2 = 0$.

$(m^2 + \sqrt{2}m + 1)(m^2 - \sqrt{2}m + 1) = 0$

$m^2 + \sqrt{2}m + 1 = 0$.

or

$m^2 - \sqrt{2}m + 1 = 0$.

$m = \dfrac{-1 \pm i}{\sqrt{2}}, \dfrac{1 \pm i}{\sqrt{2}}$

$z_1 = \dfrac{-1 + i}{\sqrt{2}}$ and $z_2 = \dfrac{1 + i}{\sqrt{2}}$.

$m = z_1, \bar{z}_2, z_2, \bar{z}_1$.

$z = C.F. + P.I.$

$= F_1(y + z_1 x) + F_2(y + \bar{z}_1 x) + F_3(y + z_2 x) + F_4(y + \bar{z}_2 x)$.

- Rules for finding P.I.

* P.I. of $F(D,D')z = \phi(x,y)$
  is given by :-
  $$\frac{1}{F(D,D')}\phi(x,y)$$

case :- When $\phi(x,y)$ is a f^n of (x,y) [Right hand wala f^n x,y mt ho].

To find P.I.

$$\underbrace{F(D,D')}_{\text{Hom. } f^n \text{ of degree 'n'}}z = \phi(ax+by);\ F(a,b) \neq 0.$$

Steps :-

1. Replace D by a & D' by b
   $F(D,D')$ to get $F(a,b)$.
2. put $ax+by = u$ now become $\phi(u)$.
   Now integrate $\phi(u)$ untill n times. (integrate times of power/order).

$$P.I. = \frac{1}{F(a,b)}\iiint\cdots\int \phi(u)\,du\,du\,du\ldots\underset{(n\ \text{times})}{du}$$

3. Replace u by $ax+by$ by at last.

* $F(a,b) \neq 0$ if $F(a,b)=0$ then method will fail.

Case of Failures :-

1. Find eqn of $F(a,b)=0$.
   and $F(D,D')z = \phi(ax+by)$; $F(a,b)=0$.
   Now
2. Differentiate $F(D,D')$ partially w.r.t 'D' & simultaneously multiply the expression by x.
3. Check $F'(a,b) \neq 0$?
   then; $P.I. = \frac{1}{F(D,D')}\phi(ax+by)$

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$$= x \cdot \frac{1}{F'(D,D')} \phi(ax+by)$$

Now if it fails

$$= x^2 \cdot \frac{1}{\frac{\partial}{\partial D}(F''(D,D')} \phi(ax+by).$$

If it also fails then $\frac{\partial^3}{\partial D}$ $(F'''(D,D')$

Q. (i) $$\frac{\partial^3 u}{\partial x^3} - \frac{3\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}.$$

Now;

C.F. we'll find

Now;

$$P.I. = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

$$= \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3} \iiint e^u \, du\, du\, du \quad x+2y=u$$

$$= \frac{1}{27} e^{x+2y}.$$

where f_1, f_2, f_3 are arbitrary f^{ns}.

Q. $$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x+y.$$

$D^2 + 3DD' + 2D'^2 = 0.$

$m^2 + 3m + 2 = 0.$

$m^2 + (2+1)m + 2 = 0.$

$m = -1, -2.$

C.F. $= F_1(y-x) + F_2(x-2x).$

Now P.I.

$$= \frac{1}{D^2 + 3DD' + 2D'^2}(x+y).$$

$$= \frac{1}{(1)^2+3(1)(1)+2(1)^2} \iint u\,du\,du \quad \text{2 bar.}$$

$$= \frac{1}{6} \cdot \frac{u^3}{6}$$

$$= \frac{u^3}{36} = \frac{(x+y)^3}{36}.$$

$$z = C.F. + P.I.$$

$$= f_1(y-x) + f_2(y-2x) + \frac{(x+y)^3}{36}.$$

Q. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial x \partial y} = \sin(2x+3y).$

$D^2 - 2DD' + D'^2$

$m^2 - 2m + 1 = 0.$

$m^2 - (1+1)m + 1 = 0.$

$m^2 - m - m + 1 = 0.$

$m(m-1) - 1(m-1) = 0.$

$m = 1, 1.$

$C.F. = f_1(y+\cancel{x}) + x f_2(y+x)$

$P.I. = \sin(2x+3y)$

$$= \frac{1}{D^2 \mp 2DD' + D'^2} (\sin(2x+3y)).$$

$$= \frac{1}{(2)^2 - 2(2)(3) + (3)^2} \iint \sin(2x+3y).$$

$$= \frac{1}{(D-D')^2} \sin(2x+3y)$$

$$= \frac{1}{(2-3)^2} \iint \sin u\,du\,du.$$

$$= -\sin u = -\sin(2x+3y)$$

$$z = C.F. + P.I.$$

$$= f_1(y+x) + x f_2(y+x) - \sin(2x+3y).$$

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Q. $r + s - 2t = \sqrt{2x+y}$

$(D^2 + DD' - 2D'^2)z = (2x+y)^{1/2}$.

$m^2 + m - 2 = 0$.

$(m-1)(m+2) = 0$.

$m = 1, -2$.

$C.F. = F_1(y+x) + F_2(y-2x)$.

$$P.I. = \frac{1}{D^2 + DD' - 2D'^2}(2x+y)^{1/2}.$$

$a = 2 \quad b = 1$.

$$= \frac{1}{D^2 + DD' - 2D'^2}\iint (u)^{1/2}\,du\,du.$$

$$= \frac{1}{(2)^2 + (2)(1) + 2(1)^2}\int \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\,du$$

$$= \frac{1}{4+2+2}\left[\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right].$$

$$= \frac{1}{8}\frac{u^{5/2}}{5/2}.$$

$$= \frac{\cancel{2}}{5}\cdot\frac{1}{\cancel{8}_4}u^{5/2}.$$

$$= \frac{1}{4}\cdot\frac{4}{15}u^{5/2} = \frac{1}{15}(2x+y)^{5/2}.$$

$z = C.F. + P.I.$

$$= F_1(y+x) + F_2(y-2x) + \frac{1}{15}(2x+y)^{5/2}.$$

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Q. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x+y)$.

Sol^n $(D^2 + D'^2)z = \cos mx \cos ny + 30(2x+y)$.

$m^2+1=0$.

$m = \pm i$.

$C.F. = f_1(y+ix) + f_2(y-ix)$.

$$P.I. = \underbrace{\frac{1}{D^2+D'^2} \iint \cos mx \cos ny}_{1} + \underbrace{\frac{1}{D^2+D'^2} \iint 30(2x+y)}_{2}.$$

2 $$= \frac{1}{D^2+D'^2} \iint 30(u)\,du\,du.$$

$$= \frac{1}{(2)^2+(1)^2} \iint 30u\,du\,du.$$

$$= \frac{1}{5} \int 30 \frac{u^{1+1}}{1+1}\,du.$$

$$= \frac{1}{5} \int \frac{\cancel{30}^{15} u^2}{\cancel{2}}\,du.$$

$$= \frac{1}{5}\, \frac{\cancel{15}^{5} u^3}{\cancel{3}}$$

$$= u^3. = (2x+y)^3.$$

1 $\div$ & $\times$ by 2.

$$= \frac{1}{2} \frac{1}{D^2+D'^2} 2\cos(mx+ny).$$

$$= \frac{1}{2}\left[\frac{1}{m+n^2} \iint \cos u\,du\,du\right].$$

$$= \frac{-1}{2(m^2+n^2)} [\cos(mx+ny) + \cos(mx-ny)].$$

$z = C.F. + P.I.$

$$= \frac{-1}{2(m^2+n^2)} \cos(mx+ny) + \cos(mx-ny) + (2x+y)^3.$$

Q. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 4\sin(2x+y)$

$\Rightarrow$ $D^3 - 4D^2D' + 4DD'^2 = 4\sin(2x+y)$.

$m^3 - 4m^2 + 4m = 0$.

$m(m^2 - 4m + 4) = 0$.

$m = 0, 2, 2$.

C.F. = $F_1(y) + F_2(y+2x) + xF_3(y+2x)$.

P.I. = $\frac{1}{D^3 - 4D^2D' + 4DD'^2} \iiint (4\sin(2x+y))\,du\,du\,du$.

$= \frac{1}{(4)^3 - 2(4)^2(1) + 4(2)(1)^2} \iint 4(-\cos(2x+y))\,du\,du$.

$= \frac{4}{D}\left[\frac{1}{(D-2D')^2} \sin(2x+y)\right]$.

$= x \cdot \frac{4}{D}\left[\frac{1}{2(D-2D')} \sin(2x+y)\right] = 4x^2 \cdot \frac{1}{D}\left[\frac{1}{2}\sin(2x+y)\right]$

$= 2x^2 \frac{1}{D} \sin(2x+y)$.

$= -2x^2 \frac{\cos(2x+y)}{2} = -x^2\cos(2x+y)$.

$z = $ C.F. + P.I. $= F_1(y) + F_2(y+2x) + xF_3(y+2x)$
$- x^2\cos(2x+y)$.

Q. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$.

$= (D^2 - DD')z = \sin x \cos y$.

$m^2 - m = 0$ $\quad m = 0, 1$.

C.F. = $f_1(y) + f_2(y+x)$.

P.I. = $\frac{1}{D^2 - DD'} \sin x \cos y$.

$= \frac{1}{2(D^2 - DD')} [\sin(x+y) + \sin(x-y)]$.

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Q. $\frac{\partial^3 z}{\partial x^2 \partial y} - 2\frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = \frac{1}{x^2}$.

$(D^2D' - 2DD'^2 + D'^3)z = \frac{1}{x^2}$.

$m^2 - 2m + 1$
$= m^2 - (1+1)m + 1 = 0$.
$= m^2 - m - m + 1 = 0$.
$= m(m-1) - 1(m-1) = 0$.
$= (m-1)(m-1) = 0$.
$= m = 1, 1$.

$D' = m$.

$D'(D^2 - 2DD' + D'^2)z = \frac{1}{x^2}$.

$D = m$.

$(m-1)(m-1) = 0$.

$m = 1, 1$.

$C.F = f_2(y+x) + xf_3(y+x)$.
$= f_1(x) + f_2(y+x) + xf_3(y+x)$.

$P.I. = \frac{1}{D^2D' - 2DD'^2 + D'^3} \iint \frac{1}{x^2}$

$= \frac{1}{D^2D' - 2DD'^2 + D'^3} \iint (x^{-2})$

$= \frac{1}{D^2D' - 2DD'^2 + D'^3} \iint (1x + 0y)^2$.

$= \frac{1}{0+0+0} \iint u^2 du dw$.

case fail.

power of D' is more hence we'll multiply 'y' in fa case.

diff. w.r. to D' and multiply by 'y'.

$= y \cdot \frac{1}{\frac{\partial}{\partial D'}[F(D, D')]} \phi[ax+by]$.

$= y \cdot \frac{1}{3D'^2 + D^2 - 4DD'} \left[\frac{1}{x^2}\right]$.

$= y \cdot \frac{1}{3(0)^2 + (1)^2 - 4(1)(0)} \iint \frac{1}{u^2} du dw$.

$= y \cdot \int -\frac{1}{u} du = y(-\log u) = -y \log x$.

Hence complete soln is

$z = C.F. + P.I.$

$= f_1(x) + f_2(y+x) + xf_3(y+x) - y\log x$..

where f_1, f_2, f_3 are arbitrary constant.

* when $\phi(x,y)$ is of the form $x^m y^n$.

Q. $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$

soln $(D^3 - D'^3) z = x^3 y^3$.

$m^3 - 1 = 0$.

$m = 1, \omega, \omega^2$.

where ω is one of the cube roots of unity.

$C.F. = f_1(y+x) + f_2(y+\omega x) + f_2(y+\omega^2 x)$.

$P.I. = \frac{1}{D^3 - D'^3}(x^3y^3) = \frac{1}{D^3\left(1 - \frac{D'^3}{D^3}\right)}(x^3y^3)$.

$= \frac{1}{D^3}\left(1 - \frac{D'^3}{D^3}\right)^{-1}(x^3y^3)$

$= \frac{1}{D^3}\left(1 + \frac{D'^3}{D^3}\right](x^3y^3)$.

$= \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^3} D'^3(x^3y^3)\right]$

$= \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^3}(6x^3)\right]$.

$= \frac{1}{D^3}(x^3y^3) + \frac{1}{D^6}(6x^3)$

$= \frac{x^6y^3}{6.5.4} + \frac{\cancel{6}x^9}{4.5.\cancel{6}.7.8.9} = \frac{x^6y^3}{120} + \frac{\cancel{6}\,x^9}{10080}$

$z = f_1(y+x) + f_2(y+\omega x) + f_2(y+\omega^2 x) + \frac{x^6y^3}{120} + \frac{x^9}{10080}$.

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Q. $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$.

$m^2 - 6m + 9 = 0$.

$m = 3, 3$.

C.F. $= F_1(y+3x) + xF_2(y+3x)$

$$P.I. = \frac{1}{D^2 - 6DD' + 9D'^2}(12x^2) + \frac{1}{D^2 - 6DD' + 9D'^2}(36xy) = P_1 + P_2.$$

$$P_1 = \frac{1}{D^2 - 6DD' + 9D'^2}(12x^2)$$

$$= \frac{1}{D^2\left[1 - \frac{6D'}{D} + \frac{9D'^2}{D^2}\right]}(12x^2).$$

$$= \frac{1}{D^2}\left[1 - \frac{6D'}{D} + \frac{9D'^2}{D^2}\right]^{-1}(12x^2)$$

$$= \frac{1}{D^2}\left[1 - \frac{6DD' + 9D'^2}{D}\right]^{-1}(12x^2).$$

$$= \frac{1}{D^2}\left[1 + \left[\frac{6DD' + 9D'^2}{D}\right]\right](12x^2)$$

$$= \frac{1}{D^2} + \frac{1}{D^2}$$

$$= \frac{12}{(1)^2 - 6(1)(0) + 9(0)^2} \iint u^2\,du\,du.$$

$$= \cancel{12} \cdot \frac{u^4}{\cancel{12}} = u^4 = x^4.$$

$$P_2 = \frac{36}{(D - 3D')^2}(xy) = \frac{36}{D^2}\left[1 - \frac{3D'}{D}\right]^{-2}(xy).$$

$$= \frac{36}{D^2}\left[1 + \frac{6D'}{D}\right](xy).$$

$$= \frac{36}{D^2}\left[xy + \frac{6}{D}(x)\right] = \frac{36}{D^2}(xy + 3x^2).$$

$$= 36\left[\frac{x^3}{6}y + \frac{x^4}{4}\right] = 6x^3y + 9x^4$$

$P.I. = P_1 + P_2 = 6x^3y + 10x^4$

$z = f_1(y+3x) + x f_2(y+3x) + 6x^3y + 10x^4.$

General method for P.I.

Q. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x.$

$(D^2 + DD' - 6D'^2)z = y\cos x.$

$m^2 + m - 6 = 0.$

$(m-2)(m+3) = 0.$

$m = 2, -3.$

$C.F. = f_1(y+2x) + f_2(y-3x)$

$P.I. = \frac{1}{D^2 + DD' - 6D'^2} y\cos x = \frac{1}{(D-2D')(D+3D')} y\cos x.$

$\downarrow c \quad \downarrow x.$

$= \frac{1}{(D-2D')}\int (c+3x)\cos x\,dx$; where $y = c+3x.$

[uv valid on AEE; ~~log~~; ~~expo~~; algebric; trigono; expo].

$\left[\int uv\,dx = uv_1 - [u'v_2] + u''v_3 - u'''v_4 + \dots\right]$

$v_1, v_2, v_3 \dots$ denotes integratn

$u', u'', u''' \dots$ denotes differentiation.

$u \rightarrow$ always algebric fn.

$v \rightarrow$ Trigonometric or exponential

$(c+3x) \rightarrow u.$

$\cos x \rightarrow v.$ $\quad u' = 3.$

$v_1 = \sin x$

$= \frac{1}{D-2D'}\left[(c+3x)\sin x - (3)(-\cos x)\right)$

$$= \frac{1}{D-2D'}\left\{c\sin x + 3\int x\cos x\,dx\right\}$$

$$= \frac{1}{D-2D'}\left[c\sin x + 3\left[x\sin x - \int 1\cdot\sin x\,dx\right]\right].$$

$$= \frac{1}{D-2D'}\left[(c+3x)\sin x + 3\cos x\right] \quad \text{where } c = y-3x.$$

$$= \frac{1}{\underset{b \leftarrow}{D} - 2D' \rightarrow x}(y\sin x + 3\cos x) \quad \text{where } b-2x.$$

$$= \int (b-2x)\sin x\,dx + 3\sin x \quad \text{where } y = b-2x.$$

$$= -b\cos x - 2\left[x(-\cos x) - \int 1\cdot(-\cos x)dx\right] + 3\sin x.$$

$$= -b\cos x + 2x\cos x - 2\sin x + 3\sin x.$$

$$= -(b-2x)\cos x + \sin x.$$

$$= -y\cos x + \sin x; \quad \text{where } b = y+2x.$$

$$z = C.F. + P.I.$$

$$= F_1(y+2x) + F_2(y-3x) - y\cos x + \sin x.$$

Q. Solve :- $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\partial y} - \frac{2\partial^2 z}{\partial y^2} = (y-1)e^x$

$$(D^2 + DD' - 2D'^2)z = (y-1)e^x.$$

The auxiliary eqn

$$m^2 + m - 2 = 0.$$

$$(m+2)(m-1) \quad \text{or} \quad (m-1)(m+2) = 0$$

$$m = -2, 1.$$

$$C.F. = F_1(y-2x) + F_2(y+x).$$

$$P.I. = (y-1)e^x.$$

$$= \frac{1}{D^2 + DD' - 2D'}(y-1)e^x = \frac{1}{D^2 + DD' - 2D'}(y-1)e^x.$$

$$= \frac{1}{(D-D')(\underset{\downarrow c}{D} + 2\underset{\downarrow x}{D'})}(y-1)e^x.$$

$$= \frac{1}{(D-D')}\int (c+2x-1)e^x.$$

$$= \frac{1}{(D-D')} \int [\underbrace{(c+2x-1)}_{u} \underbrace{e^x}_{v}] \, dx$$

$$\int uv\,dx = (c+2x-1) \cdot e^x - (2)(e^x)$$

$$= \frac{1}{(D-D')} [(c+2x-1)e^x - 2e^x].$$

$$= \frac{1}{(D-D')} [ce^x + 2xe^x - e^x - 2e^x].$$

$$= \frac{1}{(D-D')} [c + 2x - 1 - 2] e^x.$$

where $y = c + 2x$

$$= \frac{1}{(D-D')} [(c-1)e^x + 2(x-1)e^x].$$

$$= \frac{1}{D-D'} [(c+2x)e^x - 3e^x)$$

$$= \frac{1}{D-D'} (ye^x - 3e^x) \; ; \; c = y - 2x.$$

$$= \int (b-x)e^x \, dx - 3e^x \qquad y = b - x.$$

$$= be^x - (x-1)e^x - 3e^x.$$

$$= (b - x - 2)e^x = (y-2)e^x \quad \text{wh. } b = y + x.$$

$z = C.F. + P.I.$

$$= F_1(y+x) + F_2(y-2x) + (y-2)e^x.$$

$\left.\begin{matrix} f_1 \\ f_2 \\ F_3 \end{matrix}\right\}$ arbitary constant.

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Q. Non-Homogenous partial diff. eqn with constant coefficient:- (D is t in c+).

$f(D, D') = F(x, y)$ Non-Homo. eqn.

$z = C.F. + P.I.$

Methods for finding CF.

Resolve to $D - mD' - a$

Now consider $(D - mD' - a)z = 0$.

$p - mq = az$.

Lagranche's aux. eqn $\frac{dx}{+1} = \frac{dy}{-m} = \frac{dz}{az}$

then find soln. $z = e^{ax} f(y + mx)$.

C.F. of ①

$(D - m_1D' - a_1)(D - m_2D' - a_2) \cdots (D - m_nD' - a_n)z = 0$

$z = e^{a_1x} F_1(y + m_1x) + e^{a_2x} F_2(y + m_2x) + \dots + e^{a_nx} f_n(y + m_nx)$

In case of repeated factor.

$(D - mD' - a)^3 z = 0$.

We have $z = e^{ax} F_1(y + mx) + e^{ax} \cdot x f_2(y - mx) + x^2 e^{ax}(y + mx)$

⇒ $a - 3D' + 2z = 0$

Ans $C.F. = e^{-2x} F_1(y + 3x)$.

$(D + 5D' - 6)z = 0$

$= C.F. = e^{6x} F_1(y - 5x)$.

* When have same factors-

$(D + 3D' - 2)^4 z = 0$.

$= e^{2x} F_1(y - 3x) + xe^{2x} f_2(y - 3x) + x^2 e^{2x} f_3(y - 3x) + x^3 e^{2x} f_4(y - 3x)$.

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* When $F(D, D')$ cannot be factorized into linear factor -
In such cases, we use a trial method

$$(D - D'^2)z = 0. \quad — (1)$$

Let trial soln of (1) be $\quad z = Ae^{hx+ky}$ — (2).
(h, k → constant)

from (2)

$$Dz = \frac{\partial z}{\partial x} = Ahe^{hx+ky}$$

$$D'z = \frac{\partial z}{\partial y} = Ake^{hx+ky}$$

$$D'^2 z = \frac{\partial^2 z}{\partial y^2} = AK^2e^{hx+ky}$$

Putting in (1); we get

$$Ahe^{hx+ky} - AK^2e^{hx+ky} = 0.$$

$$A(h-K^2)e^{hx+ky} = 0.$$

$$h = K^2 \quad — (3)$$

eqn (2) gives,

$$z = Ae^{k^2x+ky} \quad — (4)$$

Now gen. soln.

$$z = \Sigma Ae^{k^2x+ky}.$$

Q. $(D + D' - 1)(D + 2D' - 2)z = 0.$

$(D + D' - 1)(D + 2D' - 2)$

$$z = e^{x}f_1(y-x) + e^{2x}f_2(y-2x). + 0.$$

AS P.I. = 0.

Q. Solve :-

$DD'(D + 2D' + 1)z = 0.$

~~C.F. =~~

$$(D^2D' + 2DD' + DD')z = 0.$$

$$DD'(D + 2D' + 1)z.$$

C.F. = $f_1(y) + e^{-x}f_2(y-2x).$

Q3. Solve $r + 2s + t + 2p + 2q + z = 0$.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} + \frac{2\partial z}{\partial x} + \frac{2\partial z}{\partial y} + z = 0.$$

$$(D^2 + DD' + D'^2 + 2D + 2D' + 1) z = 0.$$

$$(D^2 + DD' + D'^2 + 2\underbrace{(D + D')}_{a} + \underbrace{1}_{b}) z = 0$$

$$= (a+b)^2$$

$$(D + D' + 1)^2 z = 0.$$

$$C.F. = e^{-x} F_1(y-x) + xe^{-x} F_2(y-x)$$

$$P.I. = 0.$$

$$z = e^{-x} F_1(y-x) + xe^{-x} F_2(y-x) + 0.$$

Q (i) $(D^2 - D'^2 + D + 3D' - 2) z = 0$

(ii) $(D^2 - DD' - 2D'^2 + 2D + 2D') z = 0$

(i) $(D^2 - D'^2 + \underbrace{2D - D}_{D} + \underbrace{2D' + D'}_{3D'} - 2 + DD' - DD') z = 0.$

$$= (D^2 - DD' + 2D + DD' - D'^2 + 2D' - D + D' - 2) z = 0.$$

$$(D - D' + 2)(D + D' - 1) z = 0.$$

$$C.F. = e^{-2x} f_1(y+x) + e^{x} f_2(y-x).$$

$$P.I = 0.$$

$$z = e^{-2x} f_1(y+x) + e^{x} f_2(y-x) + 0.$$

OR

$$D^2 - D'^2 + D + 3D' - 2 + \frac{1}{4} - \frac{1}{4}.$$

$$= D^2 + D + \frac{1}{4} - D'^2 + 3D' - 2 - \frac{1}{4}.$$

$$= \left(D + \frac{1}{2}\right)^2 - \left(D'^2 - 3D' - \frac{9}{4}\right)$$

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General method for Homogenous diff. eq^n to find P.I.

ordersame

$$P.I. = \frac{1}{F(D, D')} \phi(x,y).$$

$$= \frac{1}{(D-m_1D')(D-m_2D')\dots(D-m_nD')} \phi(x,y).$$

$$= \frac{1}{D-m_1D'} \cdot \frac{1}{D-m_2D'} \dots \frac{1}{D-m_nD'} \phi(x,y)$$

Let us evaluate $\frac{1}{D-mD'} \phi(x,y)$

consider eq^n

$$(D-mD')z = \phi(x,y)$$

or

$$p - mq = \phi(x,y)$$

subsidiary eq^n are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(x,y)}$$

from first two members

$$y + mx = c.$$

last member.

$$z = \int \phi(x, c-mx)\,dx.$$

$$\frac{1}{D-mD'} \phi(x,y) = \int \phi(x, c-mx)\,dx.$$

where c is replaced by y+mx after integrat^n.

Q $(D^2+2DD'+D'^2)z = 2\cos y - x\sin y$

$m^2+2m+1=0.$

$(m+1)^2=0.$

$m=-1,-1.$

$C.F. = f_1(y-x) + f_2\, x(y-x).$

$P.I. = \dfrac{1}{(D+D')^2} 2\cos y - \dfrac{1}{(D+D')^2}(x\sin y) = P_1 - P_2.$

$P_1 = \dfrac{1}{(D+D')^2} 2\cos y = \dfrac{2}{(0+1)^2} \iint \cos u\, du\, du$ where $y=u.$

$= 2(-\cos y) = -2\cos y.$

$P_2 = \dfrac{1}{(D+D')^2} x\sin y = \dfrac{1}{D+D'} \int x\sin(c+x)dx$; where $y=c+x.$

$= \dfrac{1}{D+D'}\left[x(-\cos(c+x)) - \int 1(-\cos(c+x))dx \right.$

$= \dfrac{1}{D+D'}\left[-x\cos(c+x) + \sin(c+x)\right].$

$= \dfrac{1}{D+D'}\left[-x\cos(c+x) + \sin(c+x)\right].$

$= \dfrac{1}{D+D'}(-x\cos y + \sin y)$

$= \int -x\cos(b+x)dx + \int \sin(b+x)dx$; $y=b+x.$

$= -x\sin(b+x) - 2\cos(b+x).$

$= -x\sin y - 2\cos y.$

Q. $r - 4s + 4t + p - 2q = e^{x+y}$.

The given eqn is.

$(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$.

$(D - 2D')^2 + (D - 2D')z = e^{x+y}$.

$(D - 2D')(D - 2D')$.

C.F. $= F_1(y + 2x) + e^{-x}(y + 2x)$.

$$P.I. = \frac{1}{D - 2D' + 1} \left[\frac{2}{1-2}\right] e^u du.$$

$$= -\frac{1}{D - 2p' + 1} e^{x+y} = -e^{x+y} \cdot \frac{1}{D + 1 - 2(D' + 1) + 1} (1).$$

$$= -e^{x+y}\left[\frac{1}{D - 2D'}(1)\right] = -e^{x+y}\left[\frac{1}{D - 2D'}(e^{0x+0y})\right]$$

$$= -xe^{x+y}.$$

Q. $(D^2 - D')z = xe^{ax+a^2y}$.

$$P.I. = \frac{1}{(D^2 - D')}(xe^{ax+a^2y})$$

$$= e^{ax+a^2y} \cdot \frac{1}{(D+a)^2 - (D' - a^2)} x.$$

$$= e^{ax+a^2y} \cdot \frac{1}{D^2 + 2aD - D'}(x)$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD}\left(1 + \frac{D}{2a} - \frac{D'}{2aD}\right)^{-1}(x).$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD}\left\{1 - \left(\frac{D}{2a} - \frac{D'}{2aD}\right) + \dots\right\} x.$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD}\left(x - \frac{1}{2a}\right).$$

$$= e^{ax+a^2y}\left[\frac{x^2}{4a} - \frac{x}{4a^2}\right].$$

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* Eq^n reducible to partial Diff. eq^n with constant coefficient

$x = e^X$ so that $X = \log x$

$y = e^Y$ $Y = \log y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x}$$

$$= \frac{1}{x} \frac{\partial z}{\partial X}$$

$$x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \Rightarrow \because x \frac{\partial}{\partial x} = D\left(\equiv \frac{\partial}{\partial X}\right)$$

$$x^2 \frac{\partial^2 z}{\partial x^2} = (D-1)\, x \frac{\partial z}{\partial x} = (D-1) D z$$

$$x^3 \frac{\partial^3 z}{\partial x^3} = (D-2)\, x^2 \frac{\partial^2 z}{\partial x^2} = (D-2)(D-1) D z \text{ etc.}$$

$$y \frac{\partial z}{\partial y} = D' z \quad ; \quad y^2 \frac{\partial^2 z}{\partial y^2} = (D'-1) D' z$$

$$y^3 \frac{\partial^3 z}{\partial y^3} = (D'-2)(D'-1) D' z$$

$$xy \frac{\partial^2 z}{\partial x \partial y} = D D' z$$

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