Example 18: *Draw the Hasse diagrams of*

(i) $(D_8, `|')$ (ii) $(D_6, `|')$ [M.C.A. (Uttranchal) 2007]

(iii) $A = \{2, 3, 5, 30, 60, 120, 180, 360, `|'\}$ [I.G.N.O.U. 2004, 2009; R.G.P.V. (B.E.) Bhopal 2005, 2007]

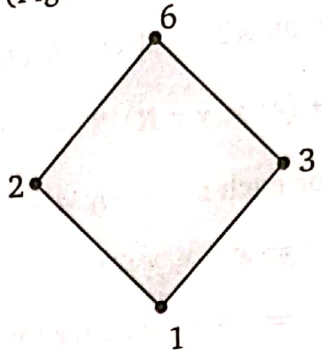
(iv) $h = \{1, 2, 3, 4, 6, 9, `|'\}$ [Rohtak (B.E.) 2008]

Solution: The Hasse diagrams of all these posets are given as

(i) We have $D_8 = \{1, 2, 4, 8\}$, Relation $\leq$ '|' = division (Fig. 6.16)

(D_8, '|')

Fig. 6.16



D_6, '|'

Fig. 6.17

(ii) We have $D_6 = \{1, 2, 3, 6\}$ relation $\leq$ = '|' = divisor (Fig. 6.17)

(iii) We have $A = \{2, 3, 5, 30, 60, 120, 180, 360, \text{'|'}\}$ relation is divisor *i.e.* $a | b$ (Fig. 6.18)

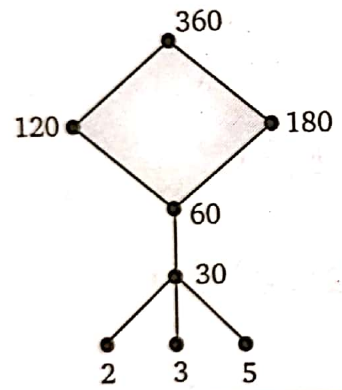


Fig. 6.18

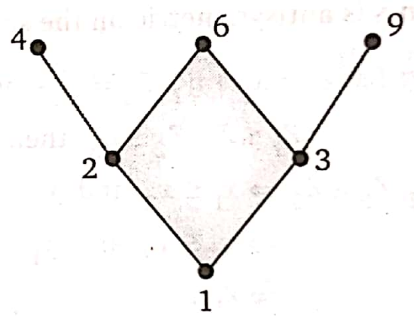


Fig. 6.19

(iv) We have $h = \{1, 2, 3, 4, 6, 9, \text{'|'}\}$ relation is divisor *i.e.* $a | b$ Fig. (6.19)

Example 19: *Show that there are only five distinct Hasse diagrams for partially ordered sets that contain three elements.*

Solution: Let us consider that $a \leq b$ if $a | b$ and consider the set

$A = \{2, 3, 5\}$ $\qquad$ $B = \{2, 3, 4\}$

$C = \{2, 3, 6\}$ $\qquad$ $D = \{2, 4, 6\}$

$E = \{2, 4, 8\}$, then its Hasse diagrams are:

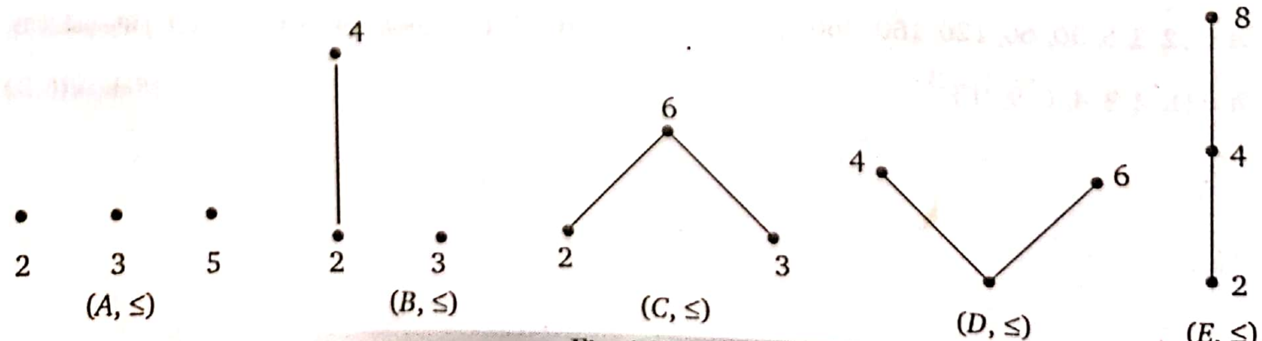


Fig. 6.20

***Example* 20:** *Determine the Hasse diagram of the relation R. $A = \{1, 2, 3, 4\}$*

$R = \{(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)\}$ [Nagpur (B.E.) 2008]

***Solution*:** Diagraph for the given relation set R is

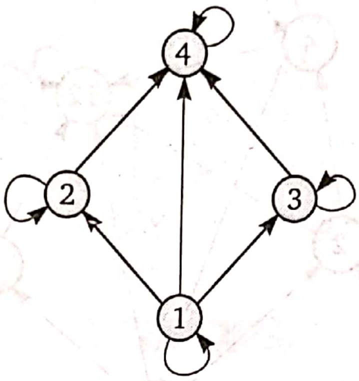


Fig. 6.21

***Step* 1:** Remove Cycles Fig. (6.22)

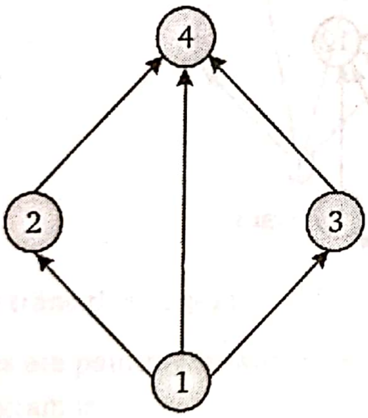


Fig. 6.22

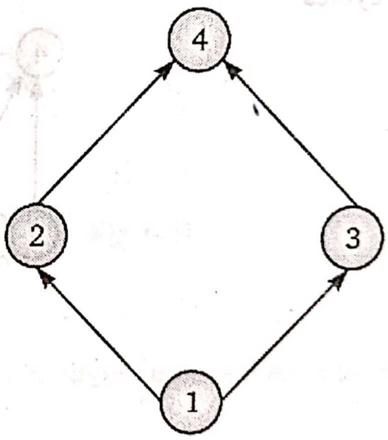


Fig. 6.23

***Step* 2:** Remove transitive edge (Fig. 6.23)

***Step* 3:** All edges are pointing upwards remove arrows from edges, replace circles by dots.

Hence Hasse diagram is

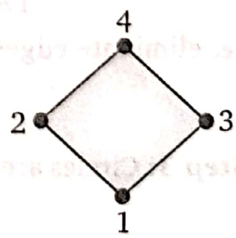


Fig. 6.24

***Example* 21:** *Draw Hasse diagram for the following relations on set*

(i) $A = \{1, 2, 3, 4, 12\}$

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (12, 12), (1, 2), (4, 12), (1, 3), (1, 4),$
$(1, 12), (2, 4), (2, 12), (3, 12)\}$

(ii) $A = \{1, 2, 3, 4, 5\}$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 4), (3, 5), (2, 2), (3, 3), (4, 4), (5, 5)\}$

(iii) $A = \{1, 2, 3, 4, 5\}$

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (4, 5), (1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5)\}$$

Solution: (i) The digraph for the given relation R is

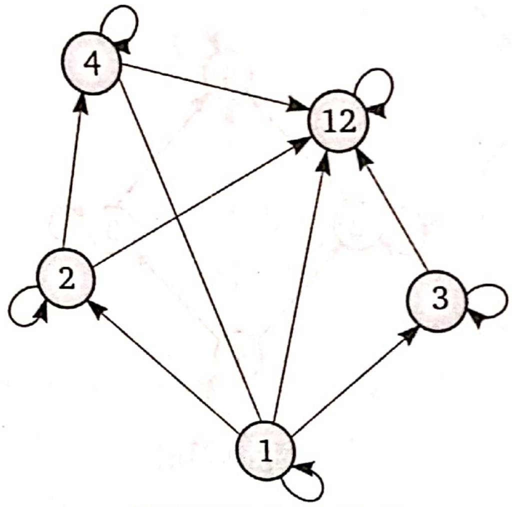


Fig. 6.25

Step 1: Remove cycles

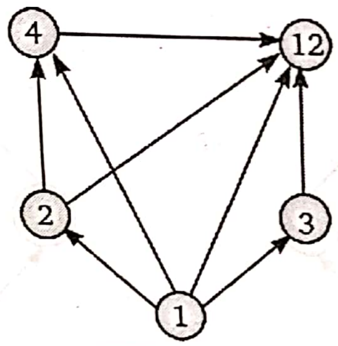


Fig. 6.26

Step 2: Remove transitive edge:

$$1R2,\ 2R4 \Rightarrow 1R4$$
$$2R4,\ 4R12 \Rightarrow 2R12$$
$$1R4,\ 4R12 \Rightarrow 1R12$$

i.e. eliminate edges (1, 4), (2, 12), (1, 12). All arrows are pointing upwards

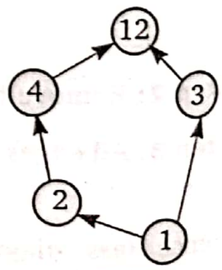


Fig. 6.27

Step 3: Circles are replaced by dots. Arrows are also removed. Hence Hasse diagram

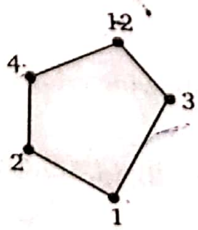


Fig. 6.28

(ii) The diagraph for the given relation R is

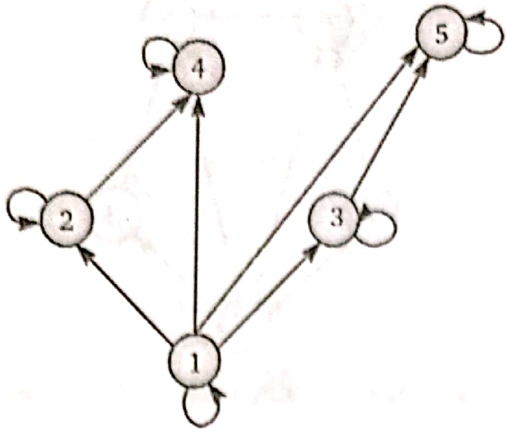


Fig. 6.29

Step 1: Remove Cycles (Fig. 6.30)

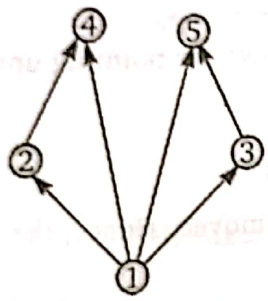


Fig. 6.30

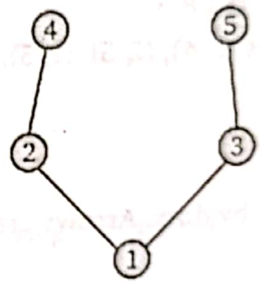


Fig. 6.31

Step 2: Remove transitive edges (1, 4), (1, 5) (Fig. 6.31)

Step 3: All edges are painting upwards, remove arrows from edges replace circles by dots. Hence Hasse diagram is

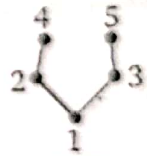


Fig. 6.32

(iii) The diagraph for the given relation set R is

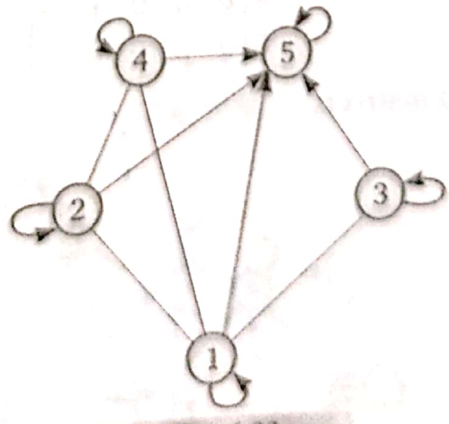


Fig. 6.33

Step 1: Remove Cycles

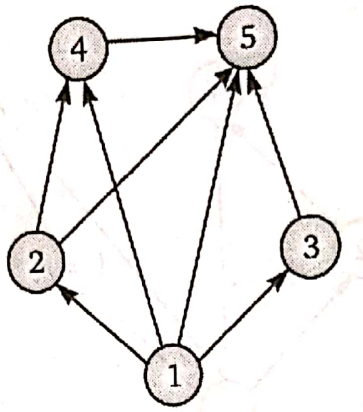


Fig. 6.34

Step 2: Remove transitive edges

$1 R 2, 2 R 4 \Rightarrow 1 R 4$

$2 R 4, 4 R 5 \Rightarrow 2 R 5$

$1 R 4, 4 R 5 \Rightarrow 1 R 5$

i.e. eliminate transitive edges (1, 4), (2, 5), (1, 5). All arrows are pointing upwards.

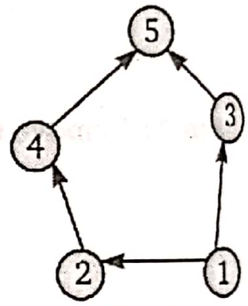


Fig. 6.35

Step 3: Circles are replaced by dots. Arrows are also removed. Hence, the required Hasse diagram is

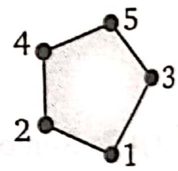


Fig. 6.36

Example 22: *Determine the Hasse diagram of the relation on* $A = \{1, 2, 3, 4, 5\}$, *whose matrix is shown.*

(i) $M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$ (ii) $M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

Solution: (i) The digraph for given matrix is

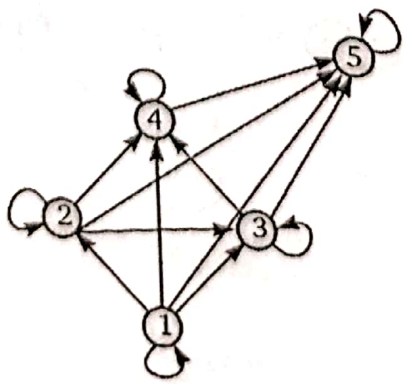


Fig. 6.37

Step 1: Remove Cycles

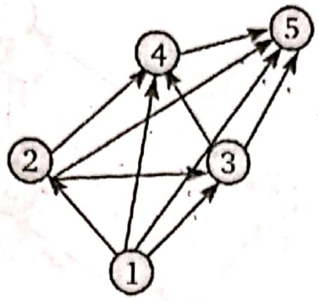


Fig. 6.38

Step 2: Remove transitive edges (2, 5), (1, 3), (2, 4), (1, 5), (1, 4) and (3, 5)

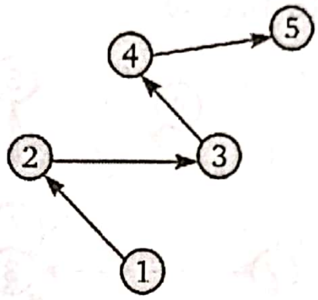


Fig. 6.39

Step 3: Circles are replaced by dots and all edges are pointing upwards. Arrows are removed.

Hence, The required Hasse diagram is

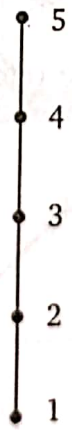


Fig. 6.40

(ii) The digraph for given matrix is

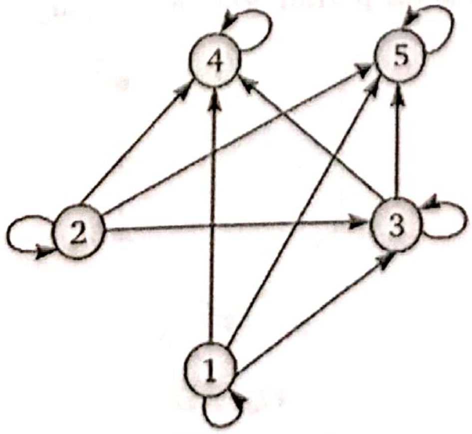


Fig. 6.41

Step 1: Remove Cycles

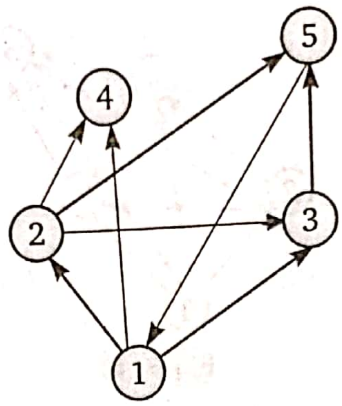


Fig. 6.42

Step 2: Remove transitive edges (2, 4), (1, 4), (1, 5)

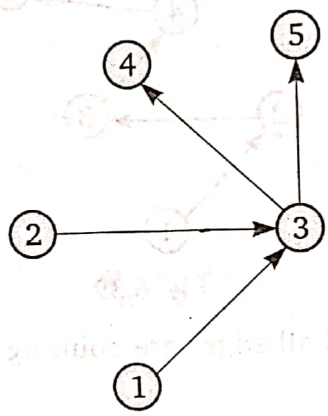


Fig. 6.43

Step 3: Circles are replaced by dots. Arrows are removed. All edges are pointing upwards. Hence the required Hasse diagram is

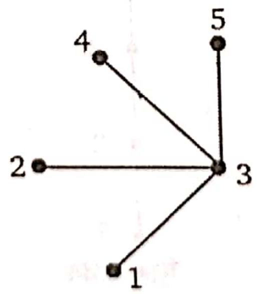


Fig. 6.44

Example 23: *Determine the matrix of the partial order whose diagram is given in fig.* 6.45

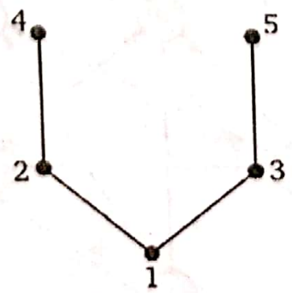


Fig. 6.45

Solution: Step 1: Put arrow on every edge, replace dots by circles

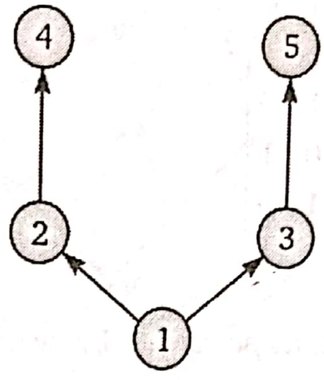


Fig. 6.46

Step 2: Put transitive edges

i.e. $1R2$ and $2R4 \Rightarrow 1R4$

$1R3$ and $3R5 \Rightarrow 1R5$

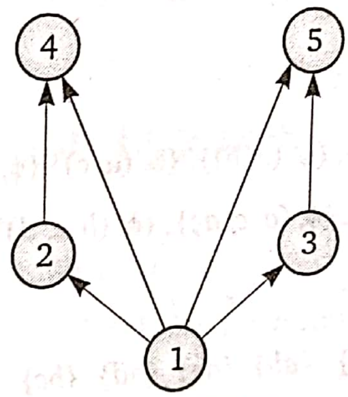


Fig. 6.47

Step 3: Put cycles on all circles

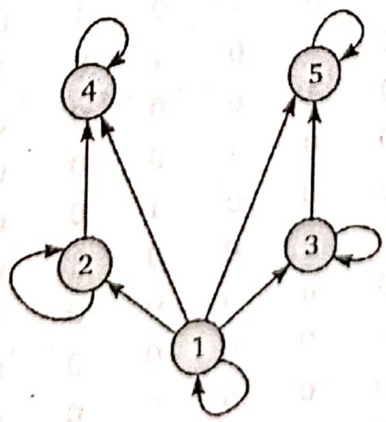


Fig. 6.48

Step 4: Relation set for the above digraph is

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 4), (1, 4), (1, 3), (3, 5), (1, 5)\}$

tep 5: The matrix for above relation set is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example 24: *Let $A = \{a, b, c, d\}$ and $P(A)$ is its power set. Draw diagram for $(P(A), \subseteq)$.*

[R.G.P.V. (B.E.) Bhopal 2003, 2009]

Solution: We have $A = \{a, b, c, d\}$. Then

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\},$$
$$\{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$$

Then $P(A)$ is poset if

The partial order relation R of set $P(A)$

$$R = \{(\phi, \phi), \{\phi, a\}, \{\phi, b\}, \{\phi, c\}, \{\phi, d\}, \{\phi, (a, b)\}, \{\phi, (a, c)\} \{\phi, (a, d)\}, \{\phi, (b, c)\}, \{\phi, b, d\},$$
$$\{\phi, c, d\}, \{\phi, (a, b, c)\}, (\phi, (a, b, d)), \{\phi, (a, c, d)\}, \{\phi, (b, c, d)\}, \{\phi, (a, b, c, d)\} \ldots\}$$

Matrix of Above Relation is given by:

$$MR = \begin{array}{c} \\ \phi \\ \{a\} \\ \{b\} \\ \{c\} \\ \{d\} \\ \{a, b\} \\ \{a, c\} \\ \{a, d\} \\ \{b, c\} \\ \{b, d\} \\ \{c, d\} \\ \{a, b, c\} \\ \{a, b, d\} \\ \{a, c, d\} \\ \{b, c, d\} \\ \{a, b, c, d\} \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} \phi & \{a\} & \{b\} & \{c\} & \{d\} & \{ab\} & \{ac\} & \{ad\} & \{bc\} & \{bd\} & \{cd\} & \{abc\} & \{abd\} & \{acd\} & \{bcd\} & \{abcd\} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The Hasse diagram for given relation R is shown in fig. 6.49

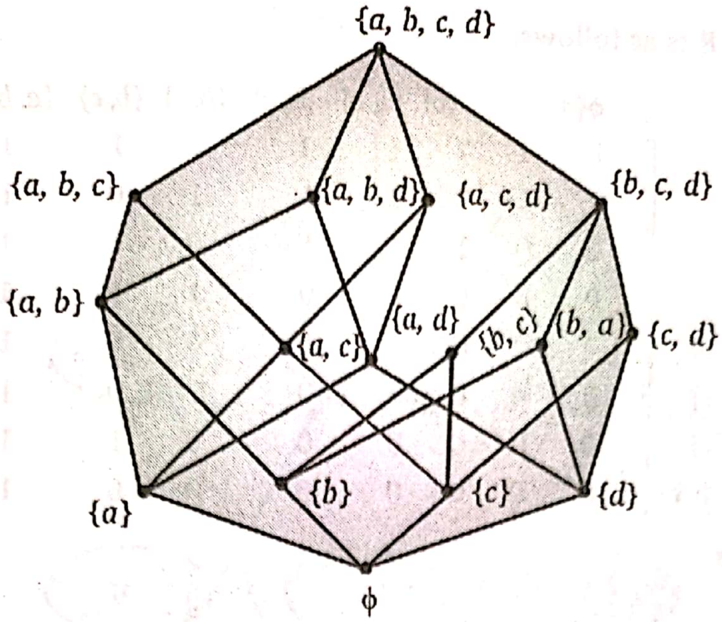


Fig. 6.49

Example 25: *Let $A = \{a, b, c\}$. Show that $(P(A), \subseteq)$ is a poset and draw its Hasse diagram.*

[U.P.T.U. (B.Tech.) 2007]

Solution: We have $A = \{a, b, c\}$. Then

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Then $(P(A), \subseteq)$ will be posets if

(P_1) ***Reflexivity:*** For each set $B \subseteq P(A)$. We have

$$B \subseteq B \quad i.e. \quad B\,R\,B. \text{ So} \subseteq \text{is reflexive}$$

(P_2) ***Antisymmetry:*** For any $B, C \in P(A)$, we have

$$B \subseteq C,\ C \subseteq B \Rightarrow B = C$$

i.e. $\quad B\,R\,B, C\,R\,B \Rightarrow B = C$

So $\quad \subseteq$ is antisymmetric

(P_3) ***Transitivity:*** For any $B, C, D \in P(A)$, we have

$$B \subseteq C, C \subseteq D \Rightarrow B \subseteq D$$

i.e. $\quad B\,R\,C, C\,R\,D \Rightarrow B\,R\,D$

So $\quad \subseteq$ is transitive on $P(A)$

Thus $\subseteq$ is a partial order relation on $P(A)$

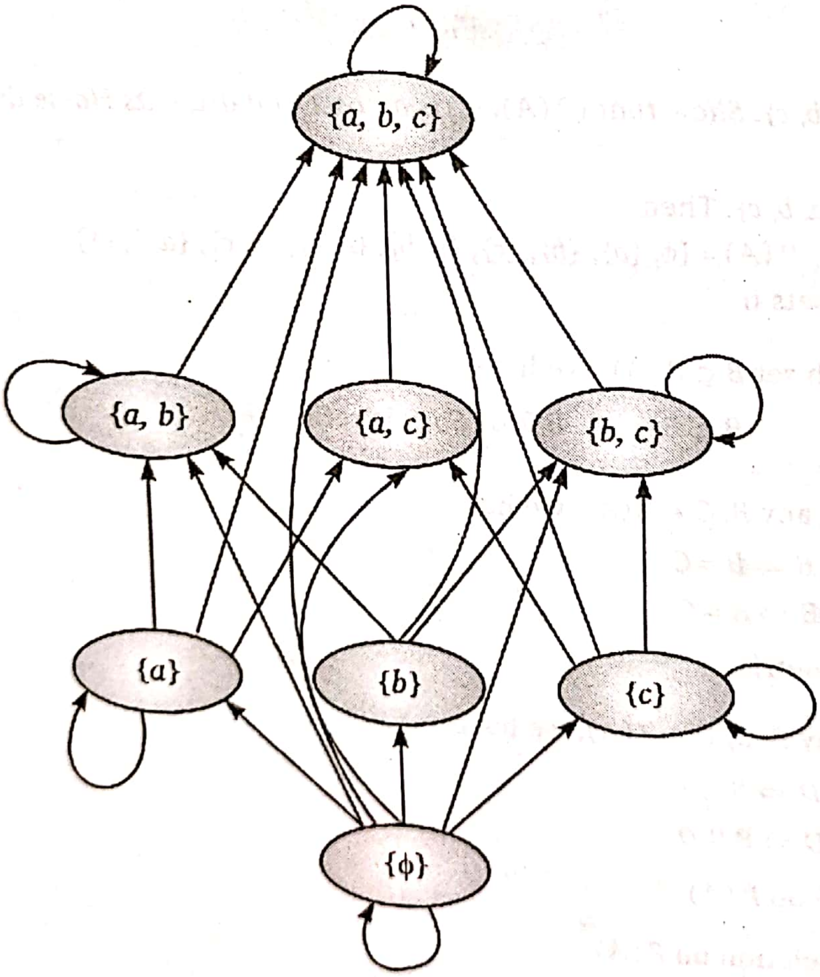
The relation R on $P(A)$ is as

$R = \{(\phi, \phi), (\phi, \{a\}), (\phi, \{b\}), (\phi, \{c\}), (\phi, \{a, b\}), (\phi, \{a, c\}), \{\phi, \{b, c\}, (\phi, \{a, b, c\}, (\{a\}, \{b\}),$
$(\{a\}, \{c\}), (\{a\}, \{a, b\}), (\{a\}, \{a, c\}), (\{a\}, \{b, c\}), (\{a\}, \{a, b, c)\}, (\{b\}, \{c\}), (\{b\}, \{a, b\}),$
$(\{b\}, \{a, c\}), (\{b\} \{b, c\}), (\{b\}, \{a, b, c\}), (\{c\}, \{a, b\}), (\{c\}, \{a, c\}), (\{c\}, \{b, c\}),$
$(\{c\}, \{a, b, d\}), (\{a, b\}, \{a, c\}), (\{a, b\}, \{b, c\}), (\{a, b\}, \{a, b, c\}), (\{a, c\}, \{b, c\}),$
$(\{a, c), \{a, b, c)\}, (\{b, c\}, \{a, b, c\})\}$

The matrix of above relation R is as follows:

$$
M_R = \begin{array}{c} \\ \phi \\ \{a\} \\ \{b\} \\ \{c\} \\ \{a,b\} \\ \{a,c\} \\ \{b,c\} \\ \{a,b,c\} \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \phi\{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{b,c\} & \{a,b,c\} \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Digraph of this matrix M_R is



To convert this digraph into Hasse diagram.

***Step* 1:** Remove Crycles

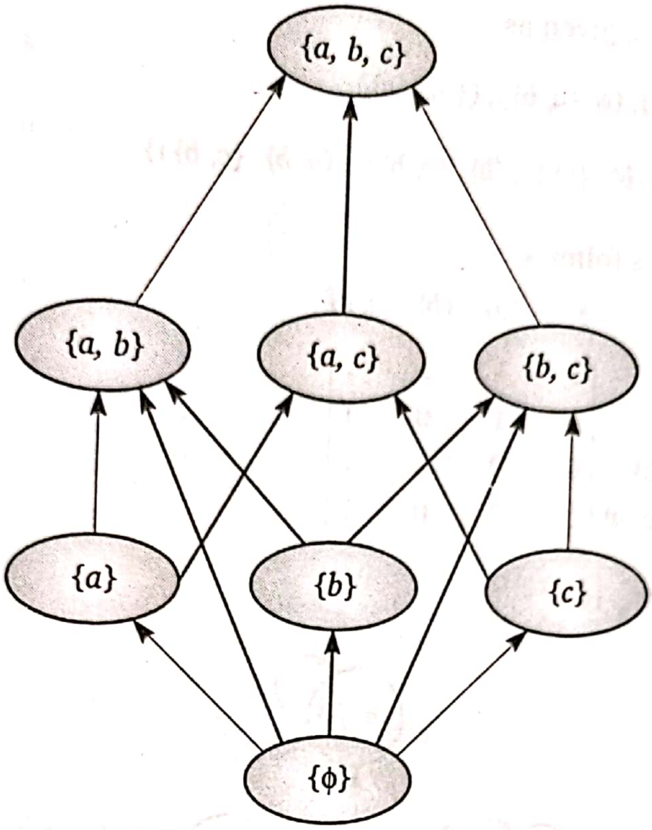


Fig. 6.51

Step 2: Remove transitive edges (ϕ, {a, b}), (ϕ, {a, c}), (ϕ, {b, c}), (ϕ, {a, b, c}), ({a}, {a, b, c}), ({b}, {a, b, c}), ({c}, {a, b, c})

Step 3: All edges are pointing upwards. Now replace circles by dots and remove arrow from edges.

Hence, required Hasse diagram is

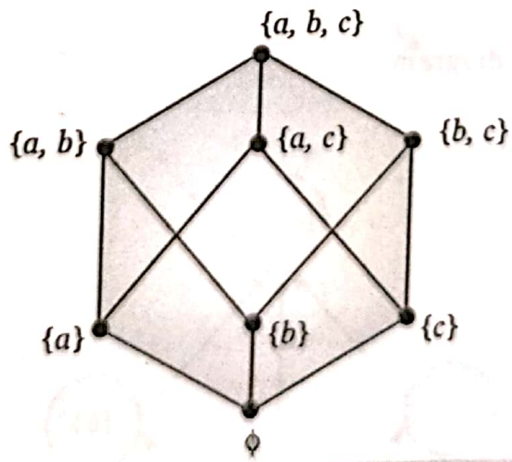


Fig. 6.52

Example 26: *Let $A = \{a, b\}$, Show that $(P(A), \subseteq)$ is poset and draw its Hasse diagram.*

[Raipur (B.E.) 2008; Rohtak (B.E.) 2007]

Solution: We have $A = \{a, b\}$. Then $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

Then $(P(A) \subseteq)$ will be poset. See example 25.

Hence, $(P(A), \subseteq)$ is a poset.

Whose diagram is

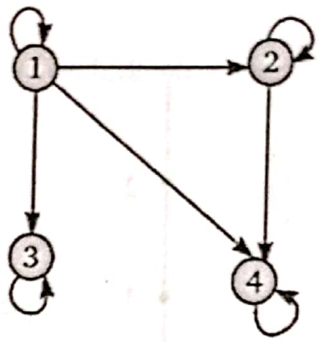


Fig. 6.65

$B = \{0, 1, 2, 3, 4\}$, $\leq$: less than or equal

$R = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

Whose diagram is

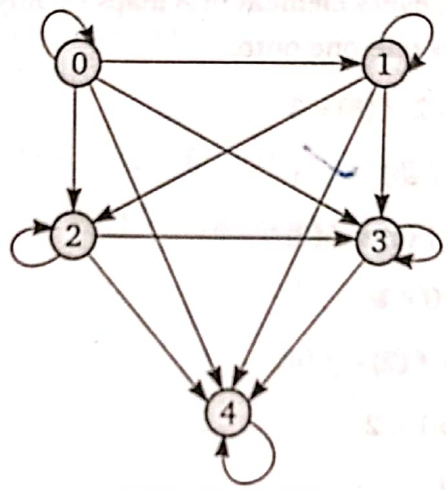


Fig. 6.66

$(A, \leq)$ and $(B, \leq)$ are isomorphic poset because there exist function f which is one-one and onto.

Example 36: *Let $A = \{1, 2, 4, 8\}$ and let $\leq$ be the partial order of divisibility on A. Let $A' = \{0, 1, 2, 3\}$ and let $\leq$ be the usual relation "less than or equal to" on integers. Show that $(A, \leq)$ and $(A', \leq)$ are isomorphic posets.*

[U.P.T.U. (B.Tech.) 2003, 2008]

Solution: The Hasse diagram of $(A, '|')$ is

Fig. 6.67

And the Hasse diagram of $(A', \leq)$ is

Fig. 6.68

The mapping $f : A \rightarrow A'$.

Since both have same Hasse diagram, every element of A maps to only a single element in A and both have same number of elements. Hence it is one-one onto.

If we define $f(1) = 0$, $f(2) = 1$, $f(4) = 2$, $f(8) = 3$

Let $\quad a \text{'|'} b \text{ iff } f(a) \leq f(b)$

$\therefore \quad 1\text{'|'}2 \Rightarrow f(1) < f(2)$

$\Rightarrow 0 < 1$

And $\quad 2\text{'|'}4 \Rightarrow f(2) < f(4)$

$\Rightarrow 1 < 2$

Hence, $f : A \rightarrow A'$ is an isomorphism

Example 37: *Let $A = \{1, 2, 3, 4, 6, 12\}$. Let R be the partial order relation on A given by $x\,R\,y$ iff $x\text{'|'}y$. Then (A, R) is a poset. Consider another poset $(A, \leq)$ in which $\leq$ denotes the usual "less than equal to" relation on A. The (A, R) and $(A, \leq)$ are not isomorphic. The Hasse diagrams of (A, R) and $(A, \leq)$ are given below*

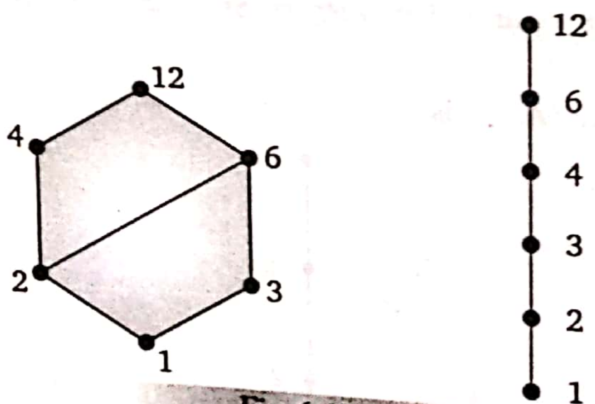


Fig. 6.69

Solution: The poset (A, R) and $(A, \leq)$ are not isomorphic. Let 4 and 6 be two distinct element in (A, R) and $f(4)$, $f(6)$ be two distinct element in $(A, \leq)$. Now $f(4)$ and $f(6)$ are comparable in $(A, \leq)$, while 4 and 6 are non-comparable.

Example 41: *Let S be any non empty set and P (S) be its power set. Show that (P (S), $\subseteq$) is a lattice.*

Solution: We have shown that (P (S), $\subseteq$) is a poset. Also for any two set $A, B \in P(S)$, we have

$$A \vee B = \sup \{A, B\} = A \cup B$$

and

$$A \wedge B = \inf \{A, B\} = A \cap B$$

Since both $A \cup B$ and $A \cap B$ be the element in $P(S)$, therefore $(P(S), \subseteq)$ is a lattice

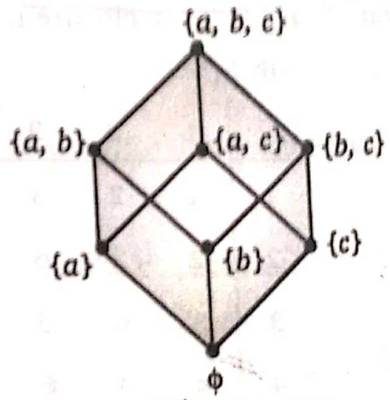


Fig. 6.73

To illustrate this example we consider

$$S = \{a, b, c\}. \text{ Then}$$

$$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The lattice is represented by the following Hasse diagram.

Scanned with CamScanner

Example 43: *Let L be the set of all factor of 12and let '|' be the divisibility relation on L. Show that (L, '|') is a lattice.*

Solution: Construct the closure table for $\vee$ and $\wedge$

where $$a \vee b = \sup\{a, b\} = \text{lcm}(a, b)$$
$$a \wedge b = \inf\{a, b\} = \gcd(a, b)$$

$\vee$	1	2	3	4	6	12
1	1	2	3	4	6	12
2	2	2	6	4	6	12
3	3	6	3	12	6	12
4	4	4	12	4	12	12
6	6	6	6	12	6	12
12	12	12	12	12	12	12

$\wedge$	1	2	3	4	6	12
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	4	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

Since each subset of every two elements in L has $\vee$ and $\wedge$. Then $\leq (L, '|')$ is lattice.

Example 44: *Show that every chain is a lattice.*

Solution: Let $(L, \leq)$ be a chain. Let $a, b \in L$. Then, since L is a chain, we have
$$a, b \in L \Rightarrow a \leq b \text{ or } b \leq a$$
We assume that $a \leq b$. Then
$$a \vee b = b \text{ and } a \wedge b = a$$
i.e. $a \vee b$, $a \wedge b$ both exist in L. Hence every chain is a lattice.

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6.12 Dual Lattice

Let $(L, \leq)$ be a lattice, for any $a, b \in L$, the converse of relation $\leq$, denoted by $\geq$ is defined as

$$a \geq b \Leftrightarrow b \leq a$$

Then $(L, \geq)$ is also a lattice called ***Dual Lattice*** of $(L, \leq)$.

6.12.1 Dual Statement

The dual of any statement in lattice $(L, \leq)$ is defined to be the statement that is obtained by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$, $\leq$ by $\geq$ and $\geq$ by $\leq$. Let L be a lattice and $a, b \in L$, then dual of the statement $a \wedge b \leq b \vee c$ is $a \vee b \geq b \wedge c$.

6.13 Properties of Lattices

Theorem 4: *Let $(L, \leq)$ be a lattice. Then the following results hold.*

(1) *Idempotent Law*: For each $a \in L$

(i) $a \wedge a = a$ (ii) $a \vee a = a$

(2) *Commutative Law*: For each $a, b \in L$

(i) $a \wedge b = b \wedge a$ (ii) $a \vee b = b \vee a$ **[U.P.T.U. (M.C.A.) 2005]**

(3) *Associative Law*: For any $a, b, c \in L$

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

(4) *Absorption Law*: For $a, b \in L$

(i) $a \wedge (a \vee b) = a$ (ii) $a \vee (a \wedge b) = a$ **[R.G.P.V. (B.E.) Bhopal 2005, 2006]**

***Proof*:** (1) Let $a \in L$. Then

(i) $a \wedge a = \inf \{a, a\} = a$. Then $a \wedge a = a$

(ii) $a \vee a = \sup \{a, a\} = a$. Then $a \vee a = a$

(2) Let $a, b \in L$. Then

(i) $a \wedge b = \inf \{a, b\} = \inf \{b, a\} = b \wedge a$

(ii) $a \vee b = \sup \{a, b\} = \sup \{b, a\} = b \vee a$

(3) Let $a, b, c \in L$. Then show

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

Let $x = a \wedge (b \wedge c)$ and $y = (a \wedge b) \wedge c$

Now, $x = a \wedge (b \wedge c) \Rightarrow x \leq a, x \leq b \wedge c$

$\Rightarrow x \leq a, x \leq b, x \leq c$

$\Rightarrow x \leq a \wedge b, x \leq c$

$\Rightarrow x \leq (a \wedge b) \wedge c \leq y$

$\therefore$ $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(ii) By principle of duality $a \vee (b \vee c) = (a \vee b) \vee c$

(4) For $a, b, c \in L$. Then show

(i) $a \wedge (a \vee b) = a$ (ii) $a \vee (a \wedge b) = a$

By definition for any $a \in L$, we have

$$a \leq a,\ a \leq a \vee b \Rightarrow a \leq a \wedge (a \vee b) \quad \ldots(1)$$

Bu $$a \wedge (a \vee b) \leq a \quad \ldots(2)$$

Then from (1) and (2)

$$a \wedge (a \vee b) = a$$

By principle of duality $a \vee (a \wedge b) = a$

Theorem 5: *Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$, the following hold*

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$$ **[U.P.T.U. (B.Tech.) 2009]**

Proof: We know that for any $a, b, c \in L$. Then

$$a \leq c \Leftrightarrow a \vee c = c \quad \ldots(1)$$

and $$\boxed{a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)} \quad \ldots(2)$$

From (1) and (2) we find

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Remark: This inequality is known as ***Modular Inequality***.

Theorem 6: *For any $a, b \in L$, prove that*

$$a \leq a \vee b \text{ and } a \wedge b \leq a$$ **[P.T.U. (Punjab) 2008]**

Proof: We know $a \vee b$ is upper bound of a. Hence $a \leq a \vee b$. Also we know $a \wedge b$ is lower bound of a.

Hence $$a \wedge b \leq a.$$

Theorem 7: *Let $(L, \leq)$ be a lattice and let $\wedge$ and $\vee$ denote the operations of meet and joint in L. Then for any $a, b \in L$.*

(i) $a \leq b \Leftrightarrow a \wedge b = a$ (ii) $a \leq b \Leftrightarrow a \vee b = b$ (iii) $a \wedge b \Rightarrow a \Leftrightarrow a \vee b = b$

Proof: (i) Let $a \wedge b = a$. Since $a \wedge b = \inf\{a, b\} \Rightarrow a \wedge b \leq b$.

Now $$a \wedge b \leq b \Rightarrow a \leq b \quad [a \wedge b = a]$$

Conversly, Let $a \leq b$. By reflexivity of $\leq$ we have

$$a \leq a$$

Now $a \leq b$ and $a \leq a \Rightarrow a$ is lower bound of $\{a, b\}$

$$\Rightarrow a = \inf\{a, b\} = a \wedge b$$

Since $a \wedge b$ infimum of $\{a, b\}$, we have

$$a \wedge b \leq a$$

$\therefore$ $a \leq a \wedge b$ and $a \wedge b \leq a \Rightarrow a \wedge b = a$ (By antisymmetry)

(ii) Let $a \vee b = b$, then show $a \leq b$

Since $$a \vee b = \sup\{a, b\}$$
$$\Rightarrow a \leq a \vee b$$

Now, $a \leq a \vee b$ and $a \vee b = b \Rightarrow a \leq b$

Conversely, Again let $a \le b$ then reflexivity $b \le b$

Now $a \le b, b \le b \Rightarrow b$ is an upper bound of $\{a, b\}$

$$\Rightarrow \sup\{a, b\} \le b$$

$$\Rightarrow a \vee b \le b$$

But from the definition of $a \vee b = \sup\{a, b\} \Rightarrow b \le a \vee b$

Also $a \vee b \le b$ and $b \le a \vee b \Rightarrow a \vee b = b$

(iii) We know $a \wedge b = a \Leftrightarrow a \le b$

$$\Leftrightarrow a \vee b = b$$

Hence, $a \wedge b = a \Leftrightarrow a \vee b = b$.

Theorem 8: *Let $(L, \le)$ be a lattice and $a, b, c \in L$. Then the following implications hold:*

(i) $a \le b$ and $a \le c \Rightarrow a \le b \vee c$ (ii) $a \le b$ and $a \le c \Rightarrow a \le b \wedge c$.

Proof: (i) Suppose $a \le b$ and $a \le c$. From the definition of join operation in lattice $(L, \le)$, we have

$$b \vee c = \sup\{b, c\}$$

$\Rightarrow$ $b \vee c$ is an upper bound of $\{b, c\}$

$\Rightarrow$ $b \le b \vee c$.

Now, by transitivity of the relation $\le$, we have

$$a \le b \text{ and } b \le b \vee c \Rightarrow a \le b \vee c.$$

(ii) Suppose $a \le b$ and $a \le c$. Then

$a \le b$ and $a \le c \Rightarrow a$ is a lower bound of $\{b, c\}$

$$\Rightarrow a \le \text{glb}\{b, c\}$$

$$\Rightarrow a \le b \wedge c \qquad [\because \text{glb}\{b, c\} = b \wedge c]$$

Corollarly: Let $(L, \le)$ be a lattice and $(L, \ge)$ be its dual. Then for $a, b, c \in L$,

(i) $a \ge b$ and $a \ge c \Rightarrow a \ge b \wedge c$ (ii) $a \ge b$ and $a \ge c \Rightarrow a \ge b \vee c$.

Proof: Applying principle of duality on Theorem 8, we get the results.

Theorem 9: *Let $(L, \le)$ be a lattice and $a, b, c, d \in L$. Then the following implications hold:*

(i) $a \le b$ and $c \le d \Rightarrow a \vee c \le b \vee d$ (ii) $a \le b$ and $c \le d \Rightarrow a \wedge c \le b \wedge d$.

Proof: (i) Suppose that $a \le b$ and $c \le d$. By the definition of join operation $\vee$ in lattice $(L, \le)$, we have

$$b \le b \vee d \text{ and } d \le b \vee d.$$

Now, by transitivity of the relation $\le$, we have

$$a \le b \text{ and } b \le b \vee d \Rightarrow a \le b \vee d.$$

Similarly, $c \le d$ and $d \le b \vee d \Rightarrow c \le b \vee d$.

Now, $a \le b \vee d$ and $c \le b \vee d \Rightarrow b \vee d$ is an upper bound of $\{a, c\}$

$$\Rightarrow \text{lub}\{a, c\} \le b \vee d$$

$$\Rightarrow a \vee c \le b \vee d \qquad [\because a \vee c = \text{lub}\{a, c\}]$$

(ii) Suppose that $a \le b$ and $c \le d$. By the definition of meet operation $\wedge$ in lattice $(L, \le)$, we have

$$a \wedge c \le a \text{ and } a \wedge c \le c.$$

By transitivity of the relation $\le$, we have

$$a \wedge c \le a \text{ and } a \le b \Rightarrow a \wedge c \le b.$$

Similarly, $a \wedge c \le c$ and $c \le d \Rightarrow a \wedge c \le d$

Now, $a \wedge c \le b$ and $a \wedge c \le d \Rightarrow a \wedge c$ is a lower bound of $\{b, d\}$

$$\Rightarrow a \wedge c \le \text{glb}\{b, d\} \Rightarrow a \wedge c \le b \wedge d$$

Theorem 10: *Let $(L, \le)$ be a lattice. Then for any $a, b, c \in L$, the following inequalities hold:*

(i) $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$ (ii) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$.

These inequalities are known as distributive inequalities. They are also called **Semi-distributive Laws.**

Proof: (i) By the definition of meet operation $\wedge$ in lattice $(L, \le)$, we have

$$a \wedge b = \inf\{a, b\} \le a \text{ and } a \wedge b = \inf\{a, b\} \le b.$$

Further, by the definition of join operation $\vee$ in lattice $(L, \le)$, we have

$$b \le \sup\{b, c\} = b \vee c \text{ and } c \le \sup\{b, c\} = b \vee c.$$

Also, by transitivity of the relation $\le$, we have

$$a \wedge b \le b \text{ and } b \le b \vee c \Rightarrow a \wedge b \le b \vee c.$$

Now, $a \wedge b \le a$ and $a \wedge b \le b \vee c \Rightarrow a \wedge b$ is a lower bound of $\{a, b \vee c\}$

$$\Rightarrow a \wedge b \le \text{glb}\{a, b \vee c\}$$

$$\Rightarrow a \wedge b \le a \wedge (b \vee c) \quad [\because a \wedge (b \vee c) = \text{glb}\{a, b \vee c\}] \quad ...(1)$$

Again, $a \wedge c \le a$ and $a \wedge c \le b \vee c \Rightarrow a \wedge c \le a \wedge (b \vee c)$. ...(2)

From (1) and (2), we have

$a \wedge (b \vee c)$ is an upper bound of $\{a \wedge b, a \wedge c\}$

Then $\text{lub}(a \wedge b, a \wedge c) \le a \wedge (b \vee c)$

$$\Rightarrow (a \wedge b) \vee (a \wedge c) \le a \wedge (b \vee c)$$

(ii) By principle of duality we get (ii) part.

Theorem 11: *Show that dual of a lattice is a lattice.* [U.P.T.U. (B.Tech.) 2004, 2007]

Solution: Let $(L, \le)$ be a lattice and let $(L, \ge)$ be its dual, where the relation $\ge$ is defined as

$$x \ge y \text{ if and only if } y \le x.$$

We now show that $\ge$ is reflexive, antisymmetric and transitive.

(P_1) $\ge$ is *reflexive*: Let $a \in L$. Since $\le$ is reflexive, we have

$$a \le a \ \forall a \in L \Rightarrow a \ge a \ \forall a \in L \quad \Rightarrow \ \ge \text{is reflexive.}$$

(P_2) $\ge$ is *anti-symmetric*: Let $a, b \in L$ be such that $a \ge b$ and $b \ge a$. Then

$$a \ge b \text{ and } b \ge a \Rightarrow b \le a \text{ and } a \le b$$

$$\Rightarrow a = b \quad [\because \le \text{ is antisymmetric}]$$

Thus $a \ge b$ and $b \ge a \Rightarrow a = b$. Hence, $\ge$ is anti-symmetric.

(P_3) $\ge$ is *transitive*: Let $a, b, c \in L$ be such that $a \ge b$ and $b \ge c$. Then

$$a \ge b \text{ and } b \ge c \Rightarrow b \le a \text{ and } c \le b$$

$$\Rightarrow c \le b \text{ and } b \le a$$

$$\Rightarrow c \le a$$

$$\Rightarrow a \ge c. \quad [\because \le \text{ is transitive}]$$

Thus $a \ge b$ and $b \ge c \Rightarrow a \ge c$. Hence, $\ge$ is transitive.

Therefore, $\ge$ is a partial order relation and L, and so $(L, \ge)$ is a poset.

Let $a, b \in L$. Then since $(L, \le)$ is a lattice, sup $\{a, b\}$ exists in $(L, \le)$.

Let $a \vee b = \sup\{a, b\}$ in $(L, \le)$. Then

$$a \le a \vee b \text{ and } b \le a \vee b.$$

Now $a \le a \vee b$ and $b \le a \vee b$

$$\Rightarrow a \vee b \ge a \text{ and } a \vee b \ge b$$

$$\Rightarrow a \vee b \text{ is a lower bound of } \{a, b\} \text{ in } (L, \geq).$$

We shall show that $a \vee b$ is the greatest lower bound of $\{a, b\}$ in $(L, \geq)$.

Let l be any lower bound of $\{a, b\}$ in $(L, \geq)$. Then

$$l \geq a \text{ and } l \geq b \Rightarrow a \leq l \text{ and } b \leq l$$
$$\Rightarrow l \text{ is an upper bound of } \{a, b\} \text{ in } (L, \leq)$$
$$\Rightarrow \text{lub } \{a, b\} \leq l \text{ in } (L, \leq)$$
$$\Rightarrow a \vee b \leq l \text{ in } (L, \leq)$$
$$\Rightarrow l \geq a \vee b$$
$$\Rightarrow a \vee b \text{ is greatest lower bound of } \{a, b\} \text{ in } (L, \geq).$$

Similarly, we can show that $a \wedge b$ is the least upper bound of $\{a, b\}$ in $(L, \geq)$. Hence $(L, \geq)$ is a lattice.

Theorem 12: *Show that in a lattice $(L, \leq)$ if $a \leq b \leq c$, then*

$$a \vee b = b \wedge c$$

and $$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c).$$

Solution: We know that

$$a \leq b \Rightarrow a \vee b = b \quad \text{...(1)}$$

and $$b \leq c \Rightarrow b \wedge c = b. \quad \text{...(2)}$$

From (1) and (2), we get

$$a \leq b \leq c \Rightarrow a \vee b = b = b \wedge c. \quad \text{...(3)}$$

Thus $$a \leq b \leq c \Rightarrow a \vee b = b \wedge c.$$

Also, $$a \leq b \Rightarrow a \wedge b = a$$

and $$a \leq b \leq c \Rightarrow a \leq c \quad [\because \leq \text{ is transitive}]$$

$$\Rightarrow a \vee c = c. \quad \text{...(4)}$$

Now, replace $a \wedge b$ for a, $b \wedge c$ for b, $a \vee b$ for b and $a \vee c$ for c in R.H.S. of (3) we find

$$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c)$$

Example 45: *Let N denote the set natural numbers. For any $a, b \in N$, show that* max $\{a, \min \{a, b\}\} = a$ and min $\{a, \max \{a, b\}\} = a$

Solution: We know $a \vee b = \max \{a, b\}$ and $a \wedge b = \min \{a, b\}$

By absorption law $$a \wedge (a \vee b) = a \quad \text{...(1)}$$

$$a \vee (a \wedge b) = a \quad \text{...(2)}$$

$$\therefore \quad \min \{a, \max \{a, b\} = \min \{a, a \vee b\} = a \vee (a \wedge b) = a$$

$$\max \{a, \min \{a, b\} = \max \{a, a \wedge b\} = a \vee \{a \wedge b\} = a$$

$\forall$ positive integers a and b.

6.14 Sub-lattice

A non empty subset M of lattice $(L, \leq)$ is said to be a sub-lattice of L if M is closed with respect to meet $(\wedge)$ and joint $(\vee)$ *i.e.*

$$x, y \in M \Rightarrow x \vee y \in M \text{ and } x \wedge y \in M$$

or

A non empty subset of M of lattice $(L, \leq)$ is said to be sub-lattice of L if M itself formed lattice with respect to $\vee$ and $\wedge$ operation.

Example 46: *Consider the lattice of all the integer '|' under the operation of divisibility. The lattice D_n of all divisors of $n > 1$ is a sub-lattice of '|'. Determine all the sub-lattices of D_{30} that contain at least four elements.*

$$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

Solution: The sublattice of D_{30} that solution at least four elements are as follows.

(i) {1, 2, 6, 30} (ii) {1, 2, 3, 30} (iii) {1, 5, 15, 30}
(iv) {1, 3, 6, 30} (v) {1, 5, 10, 30} (vi) {1, 3, 15, 30}

Example 47: *Consider the lattice $L = \{1, 2, 3, 4, 5\}$ as shown in fig. 6.75 Determine all sublattices with three or more elements.*

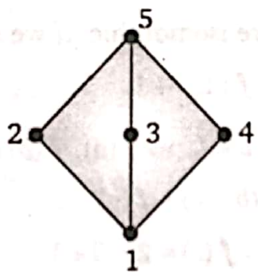

Fig. 6.75

Solution: All the sublattice with three or more elements are those whose least upper bound (lub) and greatest lower bound (glb) exists for every pair of elements which are as.

(i) {1, 2, 5} (ii) {1, 3, 5} (iii) {1, 4, 5} (iv) {1, 2, 3, 5}
(v) {1, 3, 4, 5} (vi) {1, 2, 3, 4, 5} (vii) {1, 2, 4, 5}

6.15 Isomorphic Lattice [U.P.T.U. (B.Tech.) 2007, 2008]

Two lattice L_1 and L_2 are isomorphic if there exists a one-to-one correspondence $f : L_1 \to L_2$ such that $f(a \wedge b) = f(a) \wedge f(b)$

and $f(a \vee b) = f(a) \vee f(b)\ \forall\ a, b \in L_1$ and $f(a), f(b) \in L_2$

Example 48: *Show that the lattice L and L′ given below are not isomorphic?* **[U.P.T.U. (B.Tech.) 2008]**

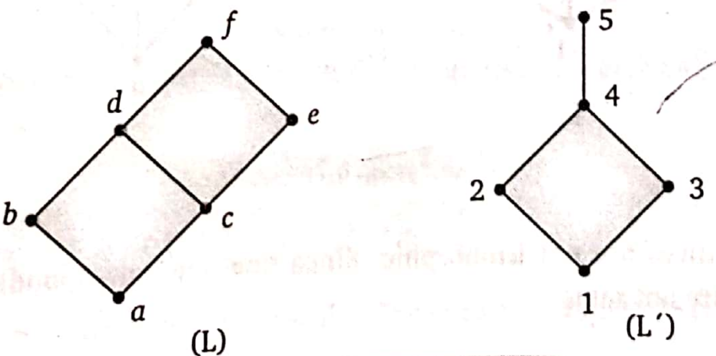

Fig. 6.76

Solution: Consider the mapping

$$f = (a, 1), (b, 2), (c, 3), (d, 4), (e, \text{not defined})$$

Since there is no one-one corresponding between L and L′ so L and L′ are not isomorphic.

Example 49: *Determine whether the lattice shown in Fig. 6.77 are isomorphic.*

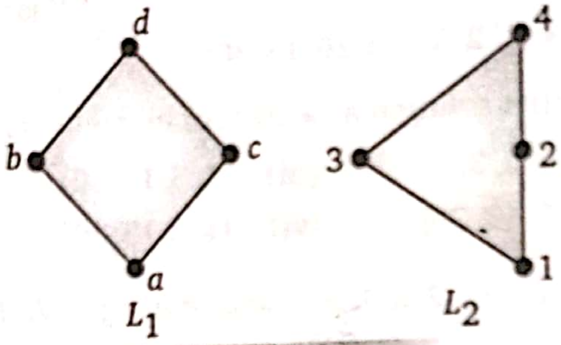

Fig. 6.77

Solution: The lattice shown in fig. 6.77 are isomorphic. If we consider $f = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$

i.e. $$f : L_1 \to L_2$$

$$a, b \in L_1 \Rightarrow f(a), f(b) \in L_2$$

Then $$f(b \wedge c) = f(a) = 1$$

$$f(b) \wedge f(c) = 2 \wedge 3 = 1$$

$\because$ $$f(b \wedge c) = f(b) \wedge f(c)$$

Again $$f(b \vee c) = f(d) = 4$$

$$f(b) \vee f(c) = 2 \vee 3 = 4$$

$\therefore$ $$f(b \vee c) = f(b) \vee f(1)$$

Hence, L_1 and L_2 are isomorphic.

Example 50: *Determine whether the lattice shown in fig. 6.78 are isomorphic.* [R.G.P.V. (B.E.) Bhopal 2009]

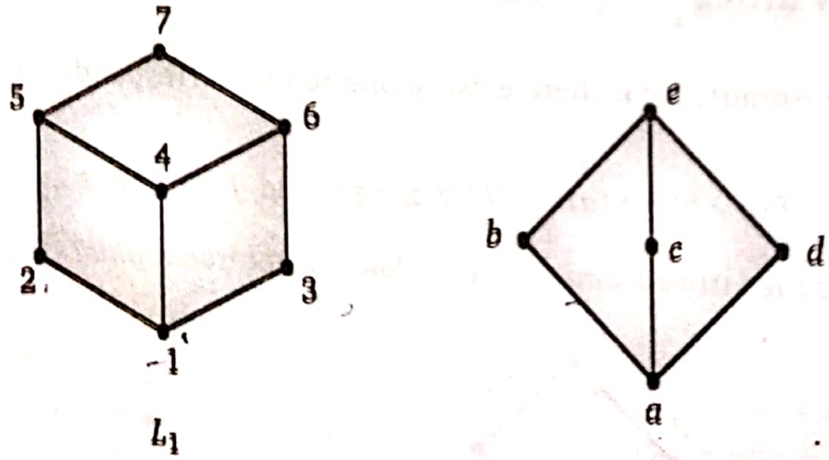

Fig. 6.78

Solution: L_1 and L_2 lattices are not isomorphic. Since one-one corresponding is not possible as the element of two lattices are not same.

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6.16 Distributive Lattice

[U.P.T.U. (B.Tech.) 2009; I.G.N.O.U. (M.C.A.) 2001, 2004, 2006, 2009; R.G.P.V. (B.E.) Bhopal 2005, 2008]

A lattice L is called distributive lattice if for any element a, b and c of L, it satisfies the following properties.

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ (ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Theorem 13: *Let $a, b, c \in L$, where $(L, \leq)$ is a distributive lattice. Then $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow b = c$*

[Rohtak (B.E.) 2008; Raipur (B.E.) 2007]

Proof: We know that

$$\begin{aligned}
b &= b \vee (b \wedge a) && \text{[absorption]}\\
&= b \vee (a \wedge b) && \text{[commutative]}\\
&= b \vee (a \wedge c) && [a \wedge b = a \wedge c]\\
&= (b \vee a) \wedge (b \vee c) && \text{[distributive]}\\
&= (a \vee b) \wedge (c \vee b) && \text{[commutative]}\\
&= (a \vee c) \wedge (c \vee b) && [a \vee b = a \vee c]\\
&= (c \vee a) \wedge (c \vee b) &&\\
&= c \vee (a \wedge c) && [a \wedge b = a \wedge c]\\
&= c \vee (c \wedge a) &&\\
&= c && \text{[absorption]}
\end{aligned}$$

6.17 Complete Lattice

[U.P.T.U. (B.Tech.) 2005]

Let $(L, \leq)$ be lattice. Then L is said to be complete if every subset A of L, $\wedge A$ and $\vee A$ exist in L. Thus, in every complete lattice $(L, \leq)$ there exist a greatest element g and a least element l.

6.18 Complement of an Element in a Lattice

Let $(L, \leq)$ be a lattice and let 0 and 1 be its lower and upper bounds. If $a \in L$ is an element than an element b is called complement of a if

$$a \vee b = 1 \text{ and } a \wedge b = 0$$

Remark: 0 and 1 are called universal bounds. But 64 commutative property we can say if b is complement of a then a is also complement of b.

6.19 Complemented Lattice

Let $(L, \leq)$ be a lattice with universal bounds 0 and 1. The lattice L is said to be complemented lattice if every element in L has a complement *i.e.*

$$\begin{cases} a \vee 1 = 1, a \wedge 1 = a \\ a \wedge 0 = 0, a \vee 0 = a \end{cases}$$

The complement of a is denoted by a' or $\bar{a}$. Then

$$a \wedge a' = 0, a \vee a' = 1$$

6.20 Complemented Complete Lattice

Let $(L, \leq)$ be a complete lattice with greatest and lower elements g and l respectively, then L is called complemented complete lattice. If for each $a \in L$, there exists an element $a' \in L$ such that

$$a \vee a' = g \text{ and } a \wedge a' = l$$

Remark: The complemented distributive lattice is called a ***Boolean Algebra.***

6.21 Bounded Lattice

[U.P.T.U. (B.Tech.) 2008]

Let $(L, \leq)$ be a lattice. Then L is said to be bounded lattice if it has a least element 0 and a greatest element 1. 0 is called the identity of joint and 1 is called the identity of meet in a bounded lattice $(L, \vee, \wedge)$.

6.22 Direct Product of Lattices

Let $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ be two lattices, where $\wedge_1, \vee_1$ are meet and join in L_1 and $\wedge_2, \vee_2$ are meet and join in L_2. The algebraic system $(L_1 \times L_2, \wedge, \vee)$ is called the ***Direct Product*** of the lattices $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ if for any (a_1, a_2) and (b_1, b_2) in $L_1 \times L_2$, so that $a_1, b_1 \in L_1$ and $a_2, b_2 \in L_2$, define $\wedge$ and $\vee$ in $L_1 \times L_2$ by

$$(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge_1 b_1, a_2 \wedge_2 b_2)$$
$$(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee_1 b_1, a_2 \vee_2 b_2)$$

6.23 Modular Lattices

A lattice L is said to be modular lattice if for all $a, b, c \in L$, $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Theorem 14: *Let L $\{a_1, a_2, \ldots a_n\}$ be finite lattice then L is bounded.*

Proof: Let $L = \{a_1, a_2, a_3 \ldots a_n\}$ be any finite lattice. Then we have to show that L having least and greatest element.

Now,
$$b_1 = a_1$$
$$b_2 = a_2 \wedge b_1$$
$$b_3 = a_3 \wedge b_2$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$b_n = a_n \wedge b_{n-1}$$

All $\quad b_1, b_2, \ldots b_n \in L$ and $b_n \leq a_i \ \forall i = 1, 2, 3 \ldots n$

$\Rightarrow$ b_n is the least element of L

$\therefore \quad b_n = a_1 \wedge a_2 \wedge a_3 \ldots \wedge a_n$

Similarly, $a_1 \vee a_2 \vee a_3 \ldots \vee a_n$ is the greatest element of L. Thus, L is bounded lattice.

Theorem 14: *A lattice in which relative complements are unique is distributive.* [Delhi (B.E.) 2009]

Proof: Let L be a lattice in which relative complements are unique. Then L cannot contain a pentagonal sublattice in which relative complements are not unique. Similarly it cannot contain a sublattice isomorphic with M_5. Hence L is distributive.

Theorem 16: *For any a and b in a Boolean algebra B.*

(i) $(a')' = a$ (ii) $(a \vee b)' = a' \wedge b'$

(iii) $(a \wedge b)' = a' \vee b'$ (iv) $\begin{cases} a \vee (a' \wedge b) = a \vee b \\ a \wedge (a' \vee b) = a \wedge b \end{cases}$

[U.P.T.U. (B.Tech.) 2003, 2004, 2005]

Proof: (i) To show $(a')' = a$

Let complement of $a \in B$ be $a' \in B$ then by definition of compliment

$$a \vee a' = 1 \quad \text{and} \quad a \wedge a' = 0 \qquad \text{...(1)}$$

Since the operation $\vee$ and $\wedge$ are commutative then. From (1)

$$a' \vee a = 1 \quad \text{and} \quad a' \wedge a = 0$$

This show $a \in B$ be the complement of a'

i.e. $$(a')' = a$$

(ii) To show $(a \vee b)' = a' \wedge b'$

If $a' \wedge b'$ be complement of $a \vee b$ then show

$$(a \vee b) \vee (a' \wedge b') = 1 \quad \text{and} \quad (a \vee b) \wedge (a' \wedge b') = 0$$

Now
$$\begin{aligned}
(a \vee b) \vee (a' \wedge b') &= [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b'] && \text{[by distributive]} \\
&= [a \vee (b \vee a')] \wedge [a \vee (b \vee b')] && \text{[by associativity]} \\
&= [a \vee (a' \vee b)] \wedge [a \vee 1] && [a \vee b' = 1] \\
&= [(a \vee a') \vee b] \wedge 1 && [a \vee 1 = 1] \\
&= [1 \vee b] \wedge 1 \\
&= 1 \wedge 1 \\
&= 1
\end{aligned}$$

and
$$\begin{aligned}
(a \vee b) \wedge (a' \wedge b') &= [a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')] \\
&= [(a \wedge a') \wedge b'] \vee [b \wedge a'] \wedge b' = (0 \wedge b') \vee (a' \wedge b) \wedge b' \\
&= 0 \vee [a' \wedge (b \wedge b')] = 0 \vee [a' \wedge 0] = 0 \vee 0 = 0
\end{aligned}$$

Thus, $a' \wedge b'$ is the complement of $a \vee b$ i.e.

$$(a \vee b)' = a' \wedge b'$$

(iii) Applying principle of duality on $(a \vee b)' = a' \wedge b'$

We get $$(a \wedge b)' = a' \vee b'$$

(iv) We have $a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b)$ [by distributive]

$$= 1 \wedge (a \vee b) \quad = a \vee b \qquad [a \vee a' = 1]$$

Again, $$a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$$
$$= 0 \vee (a \wedge b) = a \wedge b$$

Theorem 17: *Prove that in a distributive lattice, if an element has a complement then this complement is unique.* **[U.P.T.U. (B.Tech.) 2003, 2008, 2009]**

Proof: Let $(L, \leq)$ be a bounded distributive lattice. Let $a \in L$ having two complement b and c then show $b = c$

Since b and c be complement of a then

$$a \vee b = 1 \qquad a \wedge b = 0$$

$$a \vee c = 1 \qquad a \wedge c = 0$$

Now

$$\begin{aligned} b &= b \wedge 1 \\ &= b \wedge (a \vee c) \\ &= (b \wedge a) \vee (b \wedge c) && \text{[by distributive law]} \\ &= (a \wedge b) \vee (b \wedge c) && [a \wedge b = b \wedge a] \\ &= 0 \vee (b \wedge c) && [a \wedge b = 0] \\ &= (a \wedge c) \vee (b \wedge c) && [0 = a \wedge c] \\ &= (a \vee b) \wedge c && [a \vee b = 1] \\ &= 1 \wedge c = c \end{aligned}$$

Hence, complement of a is unique.

Theorem 23: *Show that every chain is a distributive lattice.* [U.P.T.U. (B.Tech.) 2004]

Solution: Let $(L, \leq)$ be a chain and $a, b, c \in L$. We consider the cases:

(i) $a \leq b$ or $a \leq c$ and (ii) $a \geq b$ or $a \geq c$.

Now we shall show that distributive law is satisfied by a, b, c:

For case (i), we have

$$a \wedge (b \vee c) = a \text{ and } (a \wedge b) \vee (a \wedge c) = a.$$

For case (ii), we have

$$a \wedge (b \vee c) = b \vee c \text{ and } (a \wedge b) \vee (a \wedge c) = b \vee c.$$

Thus, we have $a \wedge (b \vee c) = (a \wedge c) \vee (a \wedge c)$.

This shows that a chain is a distributive lattice.

Theorem 24: *Show that every complete lattice is a bounded lattice.*

Proof: Let L be a complete lattice. Then every non-empty subset of L has least upper bound and greatest lower bound. Therefore L itself has least upper bound and greatest lower bound. Hence L is a bounded lattice.

Theorem 25: *In lattice L with least element 0 and greatest element 1, show that 0 is the unique. Complement of 1 and 1 is the unique complements of 0.* [R.G.P.V. (B.E.) Raipur 2005, 2009; Kurukshetra (B.E.) 2008]

Proof: We have $0 \wedge 1 = 0$ and $0 \vee 1 = 1$

Therefore 0 and 1 are complements of each other. Now we have to show that each of these two complements are unique. Suppose if possible a be any other complement of 0

Then $0 \wedge a = 0$ and $0 \vee a = 1$

But $0 \vee a = a$ Therefore, $a = 1$

This shows that complement of 0 is unique. Similarly we can show that 0 is the only complement of 1.

Theorem 26: *The pentagonal lattice given below is not modular.*

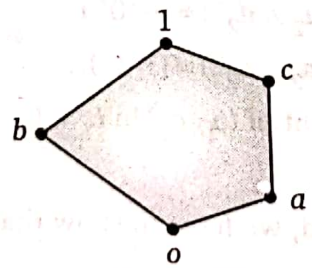

Fig. 6.79

Proof: Let $a \leq c$

and

$$a \vee (b \wedge c) = a \vee 0 = a$$

$$(a \vee b) \wedge c = 1 \wedge c = c$$

$$\therefore \quad a \vee (b \wedge c) \neq (a \vee b) \wedge c$$

Therefore, the pentagonal lattice is not modular.

Theorem 27: *Show that every finite lattice is complete.*

Proof: Let $(L, \wedge, \vee)$ be any finite lattice. And S be any non empty subset of L. Then S is finite set. Let $S = \{a_1, a_2, \ldots a_n\}$. Then $a_1 \wedge a_2 \ldots \wedge a_n$ and $a_1 \vee a_2 \ldots \vee a_n$ are infinum and supremum of S in L. Hence L is complete.

Theorem 28: *Every distributive lattice is modular.* [M.K.U. (B.E.) 2008; P.T.U. (B.E.) Punjab 2008]

Proof: Let L be a distributive lattice and $a, b, c \in L$. Then we have to show

$$a \le c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

Since L is distributive, then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{...(1)}$$

Also $\quad a \le c \Rightarrow a \vee c = c \quad$ ...(2)

Then from (1) and (2) we find

$$a \le c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

This show L is modular lattice.

But converse of above theorem is not true.

Theorem 29: *Show that a lattice L is modular iff for any a, b, c ∈ L, the following condition holds*

$$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$$

Proof: Let L is modular then

$$a \le c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c \quad \text{...(1)}$$

Since $a \le c$ then from (1)

$$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$$

Conversely, Let for all $a, b, c \in L$

$$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c) \quad \text{...(2)}$$

Then show L is modular. Let $a, b, c \in L$, $a \le c$ then $a \le c$

Hence from (2), we have

$$a \le c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

$\Rightarrow L$ is modular.

Example 51: *If $A = \{a, b, c\}$. Prove that lattice $(P(A), \cap, \cup)$ (under $\subseteq$) is distributive when $P(A)$ = power set of A.*

Solution: Let $A = \{a, b, c\}$ then power set

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The Hasse diagram is shown in fig. 6.80

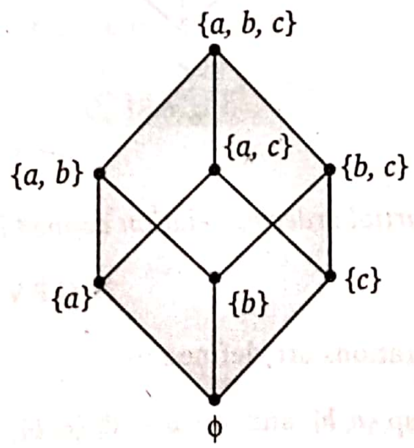

Fig. 6.80

This is distributive lattice under $A \vee B = A \cup B$ and $A \wedge B = A \cap B$, $A, B \in P(A)$

Let $A = \{a\}, B = \{b, c\}, C = \{c, a\}$, then we have

$$B \vee C = \{b, c\} \cup \{c, a\} = \{a, b, c\}$$

$$A \wedge (B \vee C) = \{a\} \cap \{a, b, c\} = a \quad \text{...(1)}$$

Also

$$A \wedge B = A \cap B = \{a\} \wedge \{b, c\} = \phi$$

$$A \wedge C = A \cap C = \{a\} \cap \{c, a\} = \{a\}$$

$\therefore$

$$(A \wedge B) \vee (A \wedge C) = \phi \cup \{a\} = \{a\} \quad \text{...(2)}$$

From (1) and (2) we see that

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

Again

$$B \wedge C = B \cap C = \{b, c\} \wedge \{c, a\} = \{c\}$$

$\therefore$

$$A \vee (B \wedge C) = \{a\} \cup \{c\} = \{a, c\} \quad \text{...(3)}$$

Also

$$A \vee B = A \cup B = \{a\} \cup \{b, c\} = \{a, b, c\}$$

$$A \vee C = A \cup C = \{a\} \cup \{c, a\} = \{a, c\}$$

$\therefore$

$$(A \vee B) \wedge (A \vee C) = \{a, b, c\} \cap \{a, c\} = \{a, c\} \quad \text{...(4)}$$

From (3) and (4), we have

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

Example 52: *The lattice $(L, \leq)$ in fig.* 6.81 is not a distributive lattice.

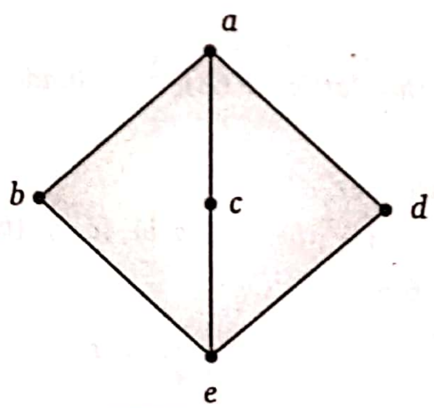

Fig. 6.81

Where $L = \{a, b, c, d, e\}$ and $\leq$ is a partial ordering relation defined on L.

[R.G.P.V. (B.E.) Bhopal 2009; P.T.U. (B.E.) Punjab 2006]

Solution: The joint and meet operations are defined by

$$a \vee b = \text{lup } \{a, b\} \text{ and } a \wedge b = \text{glb } \{a, b\}$$

The joint and meet operation tables are given as.

$\vee$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	b
c	a	a	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

$\wedge$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	e	e	e
c	c	e	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

$\therefore$ $$b \wedge (c \vee d) = b \wedge a = b$$

and $$(b \wedge c) \vee (b \wedge d) = e \vee e = e$$

Thus, $$b \wedge (c \vee d) \neq (b \wedge c) \vee (b \wedge d)$$

Example 53: *Show that the lattice shown in Fig. 6.82 are non-distributive.*

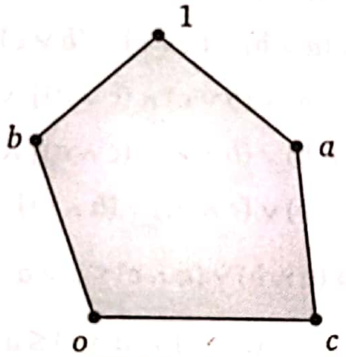

Fig. 6.82

Solution: We have $a \wedge (b \vee c) = a \wedge 1 = a$

and $$(a \wedge b) \vee (a \wedge c) = 0 \vee c = c$$

$\therefore$ $$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$$

Hence, the lattice is not distributive.

Theorem 30: *A lattice $(L, \leq)$ is distributive if and only if*

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \forall\, a, b, c \in L \quad ...(1)$$

Proof: Let $(L, \leq)$ is a distributive lattice, then show

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$$

$$\text{L.H.S.} = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$$

$$= \{a \vee [(b \vee c) \wedge (c \vee a)]\} \vee \{b \wedge [(b \vee c) \wedge (c \wedge a)]\} \quad [L \text{ is distributive}]$$

$$= [(a \wedge (c \vee a)) \wedge (b \vee c)] \vee [(b \wedge (b \vee c) \wedge (c \vee a)]$$

$$= [a \wedge (b \vee c)] \vee [b \wedge (c \vee a)] \quad [\text{by absorption law}]$$

$$= [(a \wedge b) \vee (a \wedge c)] \vee [(b \wedge c) \vee (b \wedge a)]$$

$$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = \text{R.H.S.}$$

Conversely: Let the condition (1) hold, then shown $(L, \leq)$ is a distributive lattice.

We first show that $(L, \leq)$ is a modular lattice.

Let x, y, z be any three elements in A, and let $x \leq z$. Then

$$x \vee (y \wedge z) = [x \vee (x \wedge y)] \vee (y \wedge z) \qquad [x \leq z \Rightarrow x \wedge z$$
$$= [(x \wedge z) \vee (x \wedge y)] \vee (y \wedge z)$$
$$= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$$
$$= (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$$
$$= (x \vee y) \wedge [(y \vee z) \wedge z]$$
$$= (x \vee y) \wedge z$$

Thus, $(L, \leq)$ is a modular lattice.

Now, for any three elements a, b, c in L, we have

$$a \wedge (b \vee c) = [a \wedge (a \vee c)] \wedge (b \vee c) \qquad \text{[By absorption]}$$
$$= a \wedge (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$$
$$= [(a \vee b) \wedge (b \vee c) \wedge (c \vee a)] \wedge a$$
$$= [(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)] \wedge a$$
$$= [(a \wedge b) \vee (c \wedge a)) \vee (b \wedge c)] \wedge a$$

Since $\quad a \wedge b \leq a, a \wedge c \leq a \Rightarrow (a \wedge b) \vee (a \wedge c) \leq a \vee a$

Then

$$(a \wedge b) \vee (a \wedge c) \leq a$$
$$a \wedge (b \vee c) = [(a \wedge b) \vee (c \wedge a)] \vee [(b \wedge c) \wedge a]$$
$$= [(a \wedge b) \vee (c \wedge a) \vee (b \wedge (c \wedge a))]$$
$$= (a \wedge b) \vee (c \wedge a)$$

Hence $(L, \leq)$ is distributive.

Theorem 31: *Every sublattice of a distributive lattice is distributive.*

Proof: Let M be a sublattic of distributive lattice L.

Let $a, b, c \in M$. Then $a, b, c \in L$.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ in } L$$
$$\Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ in } M.$$
$$\Rightarrow M \text{ is distributive.}$$

Example 54: *Let $A = \{1, 2, 3, 5, 30\}$ and $a \le b$ iff a divided b. The Hasse diagram is shown in Fig. 6.83 Find complement of 2.*

[U.P.T.U. (B.Tech.) 2002]

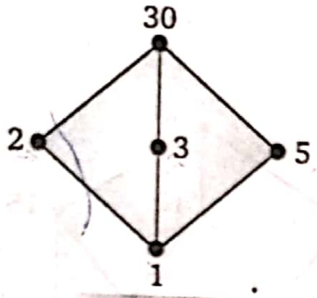

Fig. 6.83

Solution: Since $2 \wedge 3 = 1,\ \ 2 \vee 3 = 30$

$2 \wedge 5 = 1,\ \ 2 \vee 5 = 30$

Hence, 2 has two complement 3 and 5

Example 55: *In the lattice defined by the Hasse diagram given by the following figure.* 6.84.

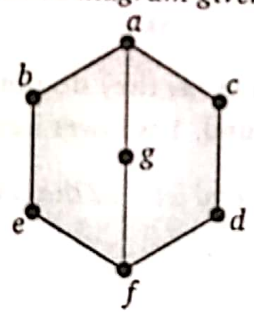

Fig. 6.84

How many complements does the elements 'e' have? Given all **[U.P.T.U. (B.Tech.) 2003, 2006, 2007]**

Solution:

Since $e \wedge g = f,\ \ e \vee g = a$ and $e \wedge d = f \quad e \vee d = a$

where a is universal upper bound and f be universal lower bound. Hence, d, g be two complement of e.

Example 56: *Homomorphism image of a distributive lattice, is distributive.*

Solution: Let $f : L \to M$ be homomorphism and L be a distributive lattice. Then show M is distributive. Let $x, y, z \in M$ as f is onto $\exists\ a, b, c \in L$ such that $f(a) = x,\ f(b) = y,\ f(c) = z$

$$\begin{aligned} x \wedge (y \vee z) &= f(a) \wedge (f(b) \vee f(c)) \\ &= f(a) \wedge (f(b \vee c)) \\ &= f(a) \wedge f(b \vee c) \\ &= f\{a \wedge (b \vee c)\} \\ &= f\{(a \wedge b) \vee (a \wedge c)\} \\ &= f(a \wedge b) \vee f(a \wedge c) \\ &= [f(a) \wedge f(b)] \vee [f(a) \wedge f(c)] = (x \wedge y) \vee (x \wedge z) \end{aligned}$$

Hence, M is distributive.

Example 57: *Consider the lattice $D_{30} = \{1, 2, 3, 5, 6, 15, 30\}$, the divisor of 30 ordered by divisibility.*
(i) *Draw the Hasse diagram of D_{30}* (ii) *Find the complement of 2 and 10, if exists.*

Solution: (i) The Hasse diagram of D_{30} is

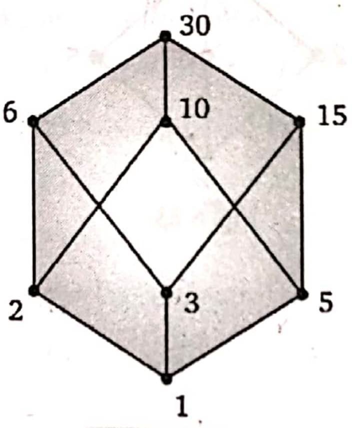

Fig. 6.85

(ii) The elements 2 and 10 has no complement as they do not satisfying the condition of complement $a \vee a' = 30$ and $a \wedge a' = 1$. 30 is upper bound, 1 is lower bound.

Example 58: *Let $(L, \leq)$ be a distributive lattice and let c' be the complement of an element c in L. If $b \wedge c' = 0$ then show that $b \leq c$.*

Solution: Since $b \wedge c' = 0$, then we have

$$\begin{aligned} & (b \wedge c') \vee c = 0 \vee c = c \\ \Rightarrow\quad & (b \vee c) \wedge (c' \vee c) = c \\ \Rightarrow\quad & (b \vee c) \wedge 1 = c \\ \Rightarrow\quad & b \vee c = c \\ \Rightarrow\quad & b \leq c \end{aligned}$$

Example 59: *Let $(L, \leq)$ be a distributive lattice. Show that if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some $a \in L$. Then $x = y$.* [U.P.T.U. (M.C.A.) 2005]

Solution: Let $(L, \leq)$ be a distributive lattice and let

$$a \wedge x = a \wedge y \text{ and } a \vee x = a \vee y \quad \ldots(1)$$

Now

$$\begin{aligned} x &= x \wedge (x \vee a) && \text{[by absorption law]} \\ &= x \wedge (a \vee x) && \text{[by commutative]} \\ &= x \wedge (a \vee y) && \text{[by (1)]} \\ &= (x \wedge a) \vee (x \wedge y) && \text{[by distributive]} \\ &= (a \wedge x) \vee (x \wedge y) && \\ &= (a \wedge y) \vee (x \wedge y) && \text{[by (1)]} \\ &= (a \vee x) \wedge y && \text{[by distributive]} \\ &= y \wedge (a \vee x) && \\ &= y \wedge (a \vee y) && \text{[by (1)]} \\ &= y && \text{[by absorption law]} \end{aligned}$$