4.3 Variant of Induction

Let $P(n)$ be the statement involving natural number n. If

(i) $P(n)$ is true for $n = n_0$ and

(ii) $P(n)$ is true for $n = k + 1$ assuming that the statement is true for $n_0 \le n \le k$ then the statement is true for all $n \ge n_0$

Remark: This is powerful form of mathematical induction because in order to prove that the statement is true for $n = k + 1$. We are allowed to make a stronger (ii)

4.4 Proof Methods or Notion of Proof

In this section we study different types of proof. Direct proof, Indirect proof, Proof by counter example and Proof by cases.

In this method we ~~Const~~ consider a statement P from available information then we construct a chain of implication to available to show prove required statement

4.4.1 Direct Proof in th **[U.P.T.U. (M.C.A.) 2008]**

We consider that P is true, and from the available information the conclusion q is shown to be true by valid reference. In this methods of proof we construct a chain of statements $P, P_1, P_2, \dots P_n \dots q$ where P is either a hypothesis of the theorem or an axiom and each to the implications $p \Rightarrow P_1, P_1 \Rightarrow P_2 \dots P_n \Rightarrow q$ is either an axiom or is implied by the implication preceding it.

Example 28: *If a and b are odd integer, then $a + b$ is an even integer.* **[U.P.T.U. (M.C.A.) 2008]**

Solution: Direct Proof :

An odd integer is of the form $2k + 1$, where k is some integer given that a and b are even integers, then there exist two integer m and n such that

$$a = 2m + 1, \quad b = 2n + 1$$

Then
$$a + b = (2m + 1) + (2n + 1) = 2m + 2n + 2 \quad = 2(m + n) + 2 \quad = 2\{(m + n + 1)\}$$

But $m + n + 1$ is an integer, therefore $a + b$ is an even integer.

Example 29: *Prove that product of two odd integers is an odd integer.* **[U.P.T.U. (B.Tech.) 2005]**

Solution: Direct Proof :

Let a and b be two odd integers, then there exist two integers m and n such that

$$a = 2m + 1, \quad b = 2n + 1$$

$$\therefore \quad ab = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 \quad = 2\{2(mn) + (m + n)\} + 1 \text{ which is odd.}$$

Example 30: *Prove that for every positive integer n, $n^3 + n$ is even if n is even.* **[U.P.T.U. (B.Tech.) 2003]**

Solution: Direct Proof :

We have $n^3 + n = (2k)^3 + (2k) = 8k^3 + 2k = 2\{4k^3 + k\}$ which is even [Let $n = 2k$]

Example 31: *If x is even integer then x^2 is an even integer.*

Solution: *Direct Proof*:

Let $p: x$ is an even integer

$q: x^2$ is an even integer

Consider the hypothesis p, if x is an even integer then by definition of even integer $x = 2k$ for some integer k

$\therefore \quad x^2 = (2k)^2 \quad$ or $\quad x^2 = 4k^2$ is clearly divisible by 2

Therefore x^2 is an even integer. Thus $p \rightarrow q$.

Example 32: *If a number such that $x^2 - 7x + 12 = 0$, then show that $x = 3$, $x = 4$ by direct proof.*

Solution: We have $x^2 - 7x + 12 = 0$

$$\Rightarrow \quad x^2 - 7x + 12 = (x-3)(x-4) = 0$$

$$\Rightarrow \quad (x-3) = 0 \quad \text{or} \quad (x-4) = 0$$

$$\therefore \quad x - 3 = 0 \Rightarrow x = 3,\ x - 4 = 0 \Rightarrow x = 4$$

$$\therefore \quad x = 3 \quad \text{or} \quad x = 4.$$

Example 33: *Prove that if $|x| > |y|$ then $x^2 > y^2$ by direct method.*

Solution: Since $|x| > |y|$ then $|x|^2 > |y|^2$ But $|x|^2 = x^2, |y|^2 = y^2$

Hence $\quad x^2 > y^2$

4.5 Method of Contraposition

4.5.1 Indirect Proof

[U.P.T.U. (M.C.A.) 2008]

This method of proof is very useful and a powerful at all levels of the subject mathematics. Indirect method follows from the tautology

$$(p \Leftrightarrow q) \Leftrightarrow ((\sim q) \Rightarrow (\sim p))$$

This states that the implication $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$. To prove $p \Rightarrow q$ indirectly, we assume that q is false and then show that p is false.

More Generally, to prove the validity of an argument with premises $p_1, p_2, \dots p_n$ and conclusion q by indirect method, we consider second argument with premises

$$\sim q, p_1, p_2, \dots p_{i-1}, p_{i+1}, \dots p_n$$

and conclusion $\sim p_i$ and prove the validity of this second argument.

Example 34: *For any integer $n > 2$, prove that n prime $\Rightarrow n$ odd.* [U.P.T.U. (M.C.A.) 2008]

p q

Solution: Let $p: n$ prime

$q: n$ odd

Then $\sim q: n$ even

$\sim p: n$ is not prime

If n is even number greater then 2, then $n = 2k$ for some integer $k > 1$. Then n is divisible by 2 and $n \neq 2$, therefore n cannot be prime. Thus we have $\sim q \Rightarrow \sim p$

i.e. if n is any number bigger than 2, then n cannot be prime.

n

$\sim q$

$\Rightarrow$ n is even whe $n>$

$\Rightarrow$ $n = 2k,\ k>1$

$\Rightarrow$ n is not prime

Example 35: *If x^2 is an even integer, then x is an even integer.*

Solution: Let p: x^2 is an even integer

q: x is an even integer

let $\sim q$ is true, then x cannot be even integer, therefore x must be odd, x is of the form

$$x = 2k + 1 \text{ for some integer } k$$

$$\therefore \quad x = 2k + 1 \Rightarrow x^2 = (2k+1)^2$$

$$\Rightarrow x^2 = 4k^2 + 2k + 1$$

$$= 2(2k^2 + k) + 1$$

Let $\quad 2k^2 + k = n, \Rightarrow x^2 = 2n + 1$ is odd

Thus we have $\quad \sim q \Rightarrow \sim p$

Hence, by contraposition x is even.

let $\sim q$ is true

$\Rightarrow x$ is not even

$\Rightarrow x$ is odd

$\Rightarrow x = 2k+1$

$\Rightarrow x^2 = 4k^2+1$
$+4k$

$= 2(2k^2+k)+1$

$\Rightarrow x^2$ is odd

$\Rightarrow \sim P$ is true.

Example 36: *Show that the following argument is a valid argument.*

$$\begin{array}{l} p \\ p \wedge q \to r \vee s \\ \quad q \\ \underline{\sim s \qquad\qquad} \\ r \end{array}$$

Solution: We will take as premises for the indirect proof all given premises except $\sim s$ and the negation of conclusion $\sim r$ i.e. we shall show that the following argument is valid.

$$\begin{array}{l} p \\ p \wedge q \to r \vee s \\ \quad q \\ \underline{\sim r \qquad\qquad} \\ \quad s \end{array}$$

Now $\left.\begin{array}{l} p \\ \underline{q} \end{array}\right\}$ premises

$p \wedge q$ a conclusion because $p \wedge q \to q \wedge p$ is always a tautology.

Now, $p \wedge q$ a valid conclusion

$\underline{q \wedge p}$ a premise

$r \vee s$ a valid conclusion by modus ponens

$\underline{\sim r}$ a premise

s a valid conclusion because $(r \vee s) \wedge \sim r \to s$ is tautology.

Example 37: *Prove by indirect method* $p \to (q \wedge r)$, $(q \vee s) \to t$ *and* $p \vee s \Rightarrow t$.

Step 1: $\sim t$ is an additional premise

Step 2:

	1	$p \to (q \wedge r)$	premise
	2	$(q \vee s) \to t$	premise
	3	$\sim t$	additional premise
{1}	4	$p \to q$	premise
{1}	5	$p \to r$	
{2}	6	$s \to t$	premise
{3, 6}	7	$\sim s$	$\sim q,\ p \to q \Rightarrow (\sim p)$
	8	$p \vee s$	premise
{8}	9	s	
	10	$s\,(\sim s) = F$	

Example 38: *Using indirect method show that*

$$p \to q,\ q \to r,\ \sim (p \wedge r),\ p \vee r \Rightarrow r$$

Solution: Step 1: $(\sim r)$ is an additional premise

	1.	$q \to r$	premise
	2	$\sim r$	premise
{1, 2}	3	$\sim q$	$p \to q,\ \sim q \Rightarrow \sim q$
	4	$p \to q$	premise
{3, 4}	5	$\sim p$	$p \to q,\ \sim q \Rightarrow \sim q$
	6	$p \vee r$	premise
{5, 6}	7	r	
	8	$r \wedge (\sim r) = f$	

contradiction, hence premise.

Example 39: *If an integer is divisible by* 10 *then it is divisible by* 2. *If an integer is divisible by* 2, *then it is divisible by* 3. *Prove that an integer divisible by* 10 *is divisible by* 3.

Solution: $D_{10}(x)$: x is divisible by 10

$D_2(x)$: x is divisible by 2

$D_3(x)$: x is divisible by 3

$$\forall x\,[D_{10}(x) \to D_2(x)]$$
$$\forall x\,[D_2(x) \to D_3(x)]$$
$$\overline{\forall x\,[D_{10}(x) \to D_3(x)]}$$

	1	$\forall x[D_{10}(x) \rightarrow D_2(x)]$	premise
	2	$D_{12}(b) \rightarrow D_2(b)$	$\cup S$
	3	$\forall x[D_2(x) \rightarrow D_3(x)]$	premise
	4	$D_2(b) \rightarrow D_3(b)$	$\cup S$
{2, 4}	5	$D_{10}(b) \rightarrow D_3(b)$	$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$
	6	$\forall x[D_{10}(x) \rightarrow D_3(x)]$	conclusion

4.6 Proof by Contradiction

In this method of proof, we assume the opposite of what we are trying to prove and get logical contradiction. Hence our assumption must have been false. Therefore what we were original required to prove must be true.

Example 40: *Show that $\sqrt{2}$ is not rational number.* **[U.P.T.U. (B.Tech.) 2004]**

Solution: Let us consider $\sqrt{2}$ is rational. Then we can find integers such that $\frac{p}{q} = \sqrt{2}$

where p and q have no common factor after cancelling the common factors squaring on both sides

$$\frac{p^2}{q^2} = 2$$

$\Rightarrow$ $p^2 = 2q^2$

$\Rightarrow$ p^2 is even

$\Rightarrow$ p is even

$\Rightarrow$ $p = 2k$ for some integer k

$\Rightarrow$ $(2k)^2 = 2q^2$

$\Rightarrow$ $4k^2 = 2q^2$

$\Rightarrow$ $q^2 = 2k^2$

$\Rightarrow$ q is even

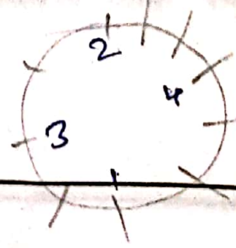
$\therefore$ p and q have common factor of 2

which is contradiction to the statement that p and q have no common factors. Thus $\sqrt{2}$ is irrational.

Example 41: *Suppose that the integers 1, 2, 3, 4 ... 10 are randomly positioned around a circular wheel. Show that sum of same set of 3 consecutively positioned numbers is at least 15.*

Solution: (*Proof by Contradiction*)

Let a_i = the number of position i on the wheel.



Then we have to prove

$$\begin{cases} a_1 + a_2 + a_3 \geq 15 \\ a_2 + a_3 + a_4 \geq 15 \\ \vdots \quad \vdots \quad \vdots \\ a_4 + a_5 + a_6 \geq 15 \\ \vdots \quad \vdots \quad \vdots \\ a_7 + a_8 + a_9 \geq 15 \\ \vdots \quad \vdots \quad \vdots \\ a_{10} + a_1 + a_2 \geq 15 \end{cases} \qquad \ldots(1)$$

where $\qquad a_1 + a_2 + a_3 + \ldots + a_{10} = 1 + 2 + 3 + 4 + \ldots + 10$

Let us consider the given conclusion is false. *i.e.*

$$\begin{cases} a_1 + a_2 + a_3 < 15 \\ a_2 + a_3 + a_4 < 15 \\ a_3 + a_4 + a_5 < 15 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{10} + a_1 + a_2 < 15 \end{cases}$$

We can write these equalities as

$$\begin{aligned} a_1 + a_2 + a_3 &\leq 14 \\ a_2 + a_3 + a_4 &\leq 14 \\ \vdots \quad \vdots \quad \vdots \quad &\vdots \\ a_{10} + a_1 + a_2 &\leq 14 \end{aligned}$$

Adding we get

$$3(a_1 + a_2 + a_3 + \ldots + a_{10}) \leq 10 \times 14 \quad 150$$

$$\Rightarrow \quad 3 \times (1 + 2 + 3 + \ldots + 10) \leq 140 \quad 150$$

$$\Rightarrow \quad 3 . \frac{10\,(10+1)}{2} \leq 140 \qquad \left(\Sigma n = \frac{n\,(n+1)}{2}\right)$$

$$\Rightarrow \quad 3 \times 5 \times 11 \leq 140$$

$$\Rightarrow \quad 165 \leq 140$$ which is contradiction

Hence given statement (1) is true.

165 ≤ 150

×

In opposite direction

4.7 Proof by Counter-Example

this method disprove the statement by some counter example

If a statement claims that a property hold for all objects of a certain type, then to prove it, we must use steps that are valid for all objects of that type. To disprove such a statement, we need only show one counter example. Such a proof is called a proof by counter-example.

Example 42: *Prove or disprove the statement. If x and y are real number*

$$(x^2 = y^2) \Leftrightarrow (x = y)$$

[U.P.T.U. (B.Tech.) 2004]

Solution: – 3, 3 are real numbers and

$$(-3)^2 = (3)^2 \text{ but } -3 \neq 3.$$

$(-3)^2$

Hence, the result is false and implication is false.

Example 43: *Prove that the statement "if n is an integer, then $n^2 - n + 41$ is prime number" is false.*

[U.P.T.U. (M.C.A.) 2005]

Solution: If $n = 41$ then $n^2 - n + 41 = (41)^2$ which is not prime. Hence the statement is false.

4.8 Proof by Cases

In this section we discuss rules of inference. The rules of inference will be given in terms of statement formula. We define consistency.

4.8.1 Consistent

A collection of statements is consistent if the statement can all be true simultaneously.

A set of formulas $H_1, H_2, H_3, \ldots H_n$ is said to be consistent if their conjunction

$$H_1 \wedge H_2 \wedge H_3 \ldots \wedge H_n$$

has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, H_3, \ldots H_n$. And a set of formulae $H_1, H_2, H_3 \ldots H_n$ is said to be inconsistent if their conjunction

$$H_1 \wedge H_2 \wedge H_3 \ldots \wedge H_n \Rightarrow S \wedge (\sim S) \qquad \text{(a contradiction)}$$

where S is any formula.

We use notion of inconsistency in the method of proof called **proof by contradiction** (or **indirect proof**). We now begin our discussion by setting the following two rules of inference.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is Tautologically implied by anyone or more of true preceding formula in the derivation.

4.8.2 Table for the Rule of Inference

$I_1: P \wedge Q \Rightarrow P$ (Simplification)

$I_2: P \wedge Q \Rightarrow Q$ (Simplification)

$I_3: P \Rightarrow P \vee Q$ (Addition)

$I_4: Q \Rightarrow P \vee Q$ (Addition)

$I_5: \sim P \Rightarrow P \rightarrow Q$

$I_6: Q \Rightarrow P \rightarrow Q$

$I_7: \sim(P \rightarrow Q) \Rightarrow P$

$I_8: \sim(P \rightarrow Q) \Rightarrow \sim Q$

$I_9: P, Q \Rightarrow P \wedge Q$

$I_{10}: p \sim P, P \vee Q \Rightarrow Q$ (Disjunction syllogism)

$I_{11}: P, P \rightarrow Q \Rightarrow Q$ (Modus powers)

$I_{12}: \sim Q, P \rightarrow Q \Rightarrow \sim P$ (Modus Tollens)

$I_{13}: P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$

$I_{14}: P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (Dilemma)

Equivalence

$E_1: \sim(\sim p) \Leftrightarrow P$

$E_2: P \wedge Q \Leftrightarrow Q \wedge P$

$E_3: P \vee Q \Leftrightarrow Q \vee P$

$E_4: (P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

$E_5: (P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

$E_6: P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ (Distributive laws)

$E_7: P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ (Distributive laws)

$E_8: \sim(P \wedge Q) \Leftrightarrow (\sim P) \vee (\sim Q)$

$E_9: \sim(P \vee Q) \Leftrightarrow \sim P \wedge (\sim Q)$

$E_{10}: P \vee P \Leftrightarrow P$

$E_{11}: P \wedge P \Leftrightarrow P$

$E_{12}: R \vee (P \wedge (\sim P)) \Leftrightarrow R$

$E_{13}: R \wedge (P \vee (\sim P)) \Leftrightarrow R$

$E_{14}: R \vee (P \vee (\sim P)) \Leftrightarrow T$

$E_{15}: R \wedge (P \wedge (\sim P)) \Leftrightarrow F$

$E_{16}: P \rightarrow Q \Leftrightarrow (\sim P \vee Q)$

$E_{17}: \sim(P \rightarrow Q) \Leftrightarrow P \wedge (\sim Q)$

$E_{18}: P \rightarrow Q \Leftrightarrow \sim Q \rightarrow \sim P$

$E_{19}: P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

$E_{20}: \sim(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows (\sim Q)$

Example 44: *Show that* $(P \rightarrow \rceil Q)$, $R \vee S$, $S \rightarrow \rceil Q$, $P \rightarrow Q \Leftrightarrow \rceil P$ *are in consistent.* **[U.P.T.U. (B.Tech.) 2009]**

Solution

1	P	Assumed
2	$P \rightarrow Q$	Rule P
3	Q	(1) and (2)
4	$S \rightarrow \rceil Q$	P
5	$Q \rightarrow \rceil S$	$P \rightarrow Q \rightleftarrows \rceil Q \rightarrow \rceil P$
6	$\rceil S$	(3), (5)
7	$R \vee S$	P
8	$\rceil R \rightarrow S$	$P \rightarrow Q \rightleftarrows \rceil P \vee \rceil Q$
9	$\rceil S \rightarrow R$	$P \rightarrow Q \rightleftarrows \rceil Q \rightarrow \rceil P$
10	R	(6), (9)
11	$R \rightarrow \rceil Q$	P
12	$\rceil Q$	(10), (11)
13	$Q \wedge \rceil Q$	(3), (12)
Inconsistent		

Example 45: *Prove* $\rceil Q, P \rightarrow Q \Rightarrow \rceil P$ **[U.P.T.U. (B.Tech.) 2005]**

Solution:

1	$P \rightarrow Q$	P
2	$\rceil Q \rightarrow \rceil P$	T, (1)
3	$\rceil Q$	P
4	$\rceil P$	T, (2), (3)

Example 46: *Show that* $\rceil P$ *follows logically form* $\rceil(P \wedge \rceil Q)$, $\rceil Q \vee P$, $\rceil R$

Solution:

1	$\rceil(P \wedge \rceil Q)$	P
2	$\rceil P \vee Q$	$\therefore \rceil(P \wedge Q) \Leftrightarrow \rceil P \vee \rceil Q$
3	$P \rightarrow Q$	$\therefore P \rightarrow Q \Leftrightarrow \rceil P \vee Q$
4	$\rceil Q \vee R$	P
5	$Q \rightarrow R$	
6	$P \rightarrow R$	(3), (5)
7	$\rceil R$	P
8	$\rceil P$	$\rceil Q, P \rightarrow Q \Rightarrow \rceil P$

Exercise

Methods of Proof

1. Give a direct proof that if a and b are odd integers then $a + b$ is even.

2. Prove by direct method

If x and y are rational numbers than $x + y$ is rational.

3. Prove the using direct method

(i) Sum of two even integers is an even integer.

(ii) Sum of an even integer and an odd integer is an odd integer.

(iii) Product of an even integer and an odd integer is an even integer.

4. ***Prove by using Direct Method*:** If an integer a is such that $(a - 2)$ is divisible by 3, then $a^2 - 1$ is divisible by 3.

[Rohtak (B.E.) 2006; Nagpur (B.E.) 2005]

5. Prove that $\sqrt{3}$ is not a rational number (Prove by Contradiction). [U.P.T.U. (M.C.A.) 2004]

6. Prove that $\sqrt{5}$ is not a rational number. [U.P.T.U. (B.Tech.) 2008]

7. If a real number x is such that $|x| > 5$, then $x^2 > 25$ (give the proof by cases).

8. Prove by using contra positive method. If x^2 is an odd integer, then x is an odd integer.

[U.P.T.U. (M.C.A.) 2004]

9. Disprove the proposition (by counter-example) for every integer x there is an integer y where $y^2 = x$.

$2^2 = 5$

[M.K.U. (B.E.) 2004, 2007]

10. Find counter example if $a > b$ then $a^2 > b^2$. [Delhi (B.E.) 2008]

$-3 > -5$

$9 > 25$ not true $\forall a, b$

11. Check the validity of the following arguments

(i)
$$\begin{array}{l} p \to q \\ r \to q \\ \hline p \to r \end{array}$$

(ii)
$$\begin{array}{l} p \\ q \\ \sim p \to r \\ q \to \sim r \\ \hline \sim r \end{array}$$

(iii)
$$\begin{array}{l} p \\ q \\ p \wedge q \to r \vee q \\ p \wedge q \to r \wedge q \\ \hline r \end{array}$$

$\sqrt{5}$ not integer

[Ans. (i) Valid (ii) Valid (iii) Valid **]**

12. Show that the following argument is a fallancy

$$\begin{array}{l} p \to q \\ \sim p \\ \hline \sim q \end{array}$$

13. State and prove Peano's Axiom. Also show how its minimal property forms the basis for mathematical induction? [U.P.T.U. (B.Tech.) 2005]

14. Prove using contrapositive that if $x^2 - 4 < 0$, then $-2 < x < 2$. [P.T.U. (B.E.) Punjab 2007]

$|x| < 2$