Contents

Introduction: Closure of Relations

- The closure of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

- If we add some pairs then we have the desired property.

- The smallest relation S (another relation which has to be add) on set A that contains R and posses the desired property P is called closure of R with respect to that property.

Properties

- Reflexive Closure
- Symmetric Closure
- Transitive Closure

Reflexive Closure with Example

- How to get the reflexive closure of a given relation on particular set?

Theorem: Let R be a relation on a set A. Then:

- $R \cup \Delta_A$ is the reflexive closure of R

Here Δ_A = Identity Relation on Set A

- An identity relation on a set 'A' is the set of ordered pairs (a,a), where 'a' belongs to set 'A'.
- For example, suppose A={1,2,3}, then the set of ordered pairs {(1,1), (2,2), (3,3)} is the identity relation on set 'A'.

Example:

Let A = {k, l, m}. Let R is a relation on A defined by R = {(k, k), (k, l), (l, m), (m, k)}. Find the reflexive closure of R.

Solution:

$\Delta_{A=}$ {(k, k), (l, l), (m,m)}

$R^r = R \cup \Delta_A$ = {(k, k), (k, l), (l, l), (l, m), (m, m), (m, k)}.

Symmetric Closure with Example

- How to get the symmetric closure of a relation on a particular set?
- **Theorem:** Let R be a relation on a set A. Then:

  $R \cup R^{-1}$ is the symmetric closure of R.
- **Example2:** Consider the relation R on A = {4, 5, 6, 7} defined by

  R = {(4, 5), (5, 5), (5, 6), (6, 7), (7, 4), (7, 7)}

  Find the symmetric closure of R.

Solution: The smallest relation containing R having the symmetric property is $R \cup R^{-1}$,i.e.

$R^s = R \cup R^{-1}$ = {(4, 5), (5, 4), (5, 5), (5, 6), (6, 5), (6, 7), (7, 6), (7, 4), (4, 7), (7, 7)}.

Transitive Closure with Example

- How to obtain the transitive closure of a relation?
- Consider a relation R on a set A. The transitive closure R of a relation R of a relation R is the smallest transitive relation containing R.
- Recall that $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$.

  We define by this formula:

$$\boxed{R^* = \bigcup_{i=1}^{\infty} R^i}$$

Contd…

- **Theorem 1:** R^* is the transitive closure of R

  Suppose A is a finite set with n elements.

  $R^* = R \cup R^2 \cup \ldots\ldots \cup R^n$

- **Theorem 2:** Let R be a relation on a set A with n elements. Then

  Transitive $(R^t) = R \cup R^2 \cup \ldots\ldots \cup R^n$

  Example1: Consider the relation R = {(1, 2), (2, 3), (3, 3)} on A = {1, 2, 3}. Then

  Here n=3(no. of elements in set A)

  Transitive $(R) = R \cup R^2 \cup R^3$
  $R^2 = R \circ R$ = {(1, 3), (2, 3), (3, 3)} and $R^3 = R^2 \circ R$ = {(1, 3), (2, 3), (3, 3)}
  Accordingly,
  Transitive (R^t) = {(1, 2), (2, 3), (3, 3), (1, 3)}

Ex.2.10.1 : Let $A = \{1, 2, 3\}$. R_1, R_2 *and* R_3 are relations on set A. Find the reflexive closures of R_1, R_2 *and* R_3. Where $R_1 = \{(1, 1)\ (2, 1)\}$, $R_2 = \{(1, 1)\ (2, 2), (3, 3)\}$, $R_3 = \{(3, 1)\ (1, 3), (2, 3)\}$.

Sol. :

We have. $A = \{1, 2, 3\}$

$\therefore \quad \Delta = \{(1, 1), (2, 2), (3, 3)\}$

Then

i) The reflexive closure of R_1 is $R = R_1 \cup \Delta$

$\therefore \quad R = \{(1, 1)\ (2, 2)\ (3, 3)\ (2, 1)\}$

ii) The reflexive closure of R_2 is $R = R_2 \cup \Delta = R_2$

iii) The reflexive closure of R_3 is $R = R_3 \cup \Delta$

$\therefore \quad R = \{(1, 1)\ (2, 2)\ (3, 3)\ (3, 1)\ (1, 3), (2, 3)\}$

Ex.2.10.2 : Find the symmetric closure of the following relations. On A = {1, 2, 3}.

$R_1 = \{(1, 1)\ (2, 1)\}$

$R_2 = \{(1, 2)\ (2, 1)\ (3, 2)\ (2, 2)\}$

$R_3 = \{(1, 1)\ (2, 2)\ (3, 3)\}$

Sol. : Given that

We have $A = \{1, 2, 3\}$

i) $R_1^{-1} = \{(1, 1)\ (1, 2)\}$

$\therefore$ $R = R_1 \cup R_1^{-1} = \{(1, 1)\ (1, 2)\ (2, 1)\}$

is the symmetric closure of R_1

ii) $R_2^{-1} = \{(2, 1)\ (1, 2)\ (2, 3)\ (2, 2)\}$

$\therefore$ $R = R_2 \cup R_2^{-1}$

$= \{(1, 2)\ (2, 1)\ (3, 2)\ (2, 3)\ (2, 2)\}$

is the symmetric closure of R_2

iii) R_3 is the symmetric relation.

$\therefore$ R_3 itself is the symmetric closure.

Ex.2.10.9 : If A = {1, 2, 3, 4} and Relation R = {(1, 2), (3, 4) (2, 1) (1, 1) (3, 3)} then,

(i) Find reflexive closure of R

(ii) Find symmetric closure of R

Ex.2.10.10 : Find the transitive closure of R where A = {1 2, 3, 4} and R = {(1, 2) (2, 3) (3, 4)}. Draw its digraph.

Ex.2.10.11 : Let R be a relation on $A = \{a, b, c, d\}$

$R = \{(a, b)\ (b, c)\ (d, c)\ (d, a)\ (a, d)\ (d, d)\}$.

Find

(a) Reflexive closure of R

(b) Symmetric closure of R

(c) Transitive closure of R **AKTU : 2003-04**

Solutions

1.

Sol. : Given that

$$A = \{1, 2, 3, 4)$$

and $R = \{(1, 2), (3, 4)\ (2, 1)\ \ (1, 1)\ \ (3, 3)\}$

(i) Reflexive closure :

$$\Delta = \{(1, 1)\ \ (2, 2)\ \ (3, 3)\ \ (4, 4)\}$$

$\therefore$ $R_1 = R \cup \Delta$

$= \{(1, 1)\ (2, 2)\ (3, 3)\ (4, 4)\ (1, 2)\ (3, 4)\ (2, 1)\}$

which is the reflexive closure of R.

(ii) Symmetric closure of R :

Inverse relation $= R^{-1}$

$= \{(2, 1)\ \ (4, 3)\ \ (1, 2)\ \ (1, 1)\ \ (3, 3)\}$

$R^* = R \cup R^{-1}$

$= \{(1, 2), (3, 4)\ (2, 1)\ (1, 1)\ (3, 3)\ (4, 3)\}$

Which is the symmetric closure of R.

2.

$$R = \{(1, 2)\ (2, 3)\ (3, 4)\}$$

$$R^2 = \{(1, 3)\ (2, 4)\}$$

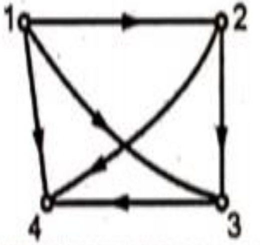
$$R^3 = \{(1, 4)\},$$

$$R^4 = \phi$$

$$R^* = R \cup R^2 \cup R^3 \cup R^4$$

$$= \{(1, 2)\ (2, 3)\ (3, 4)\ \ (1, 3)\ \ (1, 4)\ \ (2, 4)\}$$

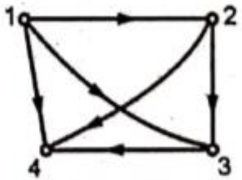
Which is the transitive closure of R. Digraph of R^* is as follows :



3.

$= \{(1, 2)\ (2, 3)\ (3, 4)\ (1, 3)\ (1, 4)\ (2, 4)\}$

Which is the transitive closure of R. Digraph of R* is as follows :



Ex.2.10.11 : Let R be a relation on $A = \{a, b, c, d\}$ $R = \{(a, b)\ (b, c)\ (d, c)\ (d, a)\ (a, d)\ (d, d)\}$.

Find

(a) Reflexive closure of R

(b) Symmetric closure of R

(c) Transitive closure of R AKTU : 2003-04

Sol. :

(a) Reflexive closure $= R \cup \Delta$

$= R \cup \{(a, a)\ (b, b)\ (c, c)\ (d, d)\}$

$= \{(a, a)\ (b, b)\ (c, c)\ (d, d)\ (a, b)$

$(b, c)\ (d, c)\ (d, a)\ (a, d)\}$

(b) Symmetric closure $= R \cup R^{-1}$

$= R \cup \{(b, a)\ (c, b)\ (c, d)\ (a, d)\ (d, a)\ (d, d)\}$

$= \{(a, b)\ (b, c)\ (d, c)\ (d, a)\ (a, d)\ (d, d)$

$(b, a)\ (c, b)\ (c, d)\ (a, d)\}$

(c) $R^2 = \{(a, c)\ (d, b)\ (d, d)\ (a, d)\ (a, a)\ (a, c)$

$(d, c)\ (d, a)\}$

$R^3 = R^2 \cdot R$

$= \{(d, c)\ (d, d)\ (d, a)\ (a,b)\ (a, d)\ (a, c)\ (a, a)$

$(d, b)\} = R^2$

$R^4 = R^3 \cdot R = R^2 \cdot R = R^3$

$R^* = R \cup R^2 \cup R^3 \cup R^4$

$= \{(a, b)\ (b, c)\ (d, c)\ (d, a)\ (a, d)\ (d, d)$

$(a, c)\ (d, b)\ (a, d)\ (a, a)\}$