UNIT -3

POSETS AND LATTICE

Contents

- POSET
- Hasse Diagram
- Minimal, Maximal, Greatest and Least element
- Lattice
- Properties

Partial Orderings: Definitions

- A relation R on a set S is called a partial order if it is

  - Reflexive
  - Antisymmetric
  - Transitive

- A set S together with a partial ordering R is called a partially ordered set or poset and is denoted as "$\prec$" and is not to be mistaken with less than.

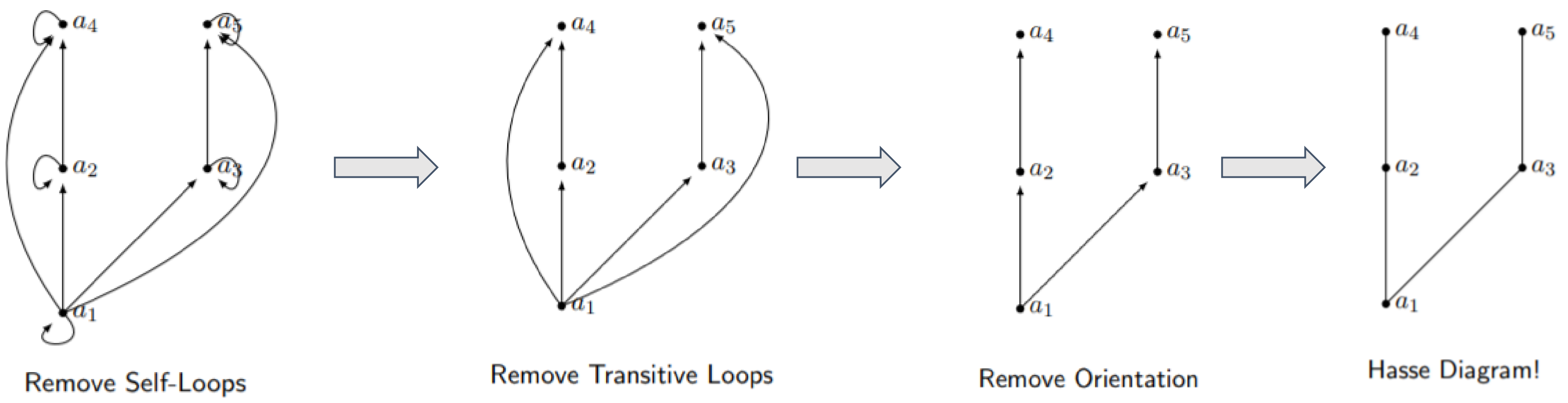
Hasse Diagrams

- ➢ POSET are represented with a graphical structure known as Hasse Diagrams . Consider the digraph representation of a partial order. We can simplify the graph as follows

- Remove all self loops.
- Remove all transitive edges.
- Remove directions on edges assuming that they are oriented upwards.

- ➢ The resulting diagram is far simpler is termed as Hasse Diagram.

Examples:


Remove Self-Loops
Remove Transitive Loops
Remove Orientation
Hasse Diagram!

For regular Hasse Diagram:

- **Maximal elements** are those which are not succeeded by another element.
- **Minimal elements** are those which are not preceded by another element.
- **Greatest element (LAST ELEMENT)** (if it exists) is the element succeeding all other elements.
- **Least element** (FIRST ELEMENT) is the element that precedes all other elements.
- It is sometimes possible to find an element that is greater than or equal to all the elements in a subset of poset . Such an element is called the **upper bound** of . Similarly, we can also find the **lower bound** also.
- The **Least Upper Bound (LUB)(SUPREMUM)** is the smallest element in upper bounds.

Denoted by LUB ({a, b}) by a v b.

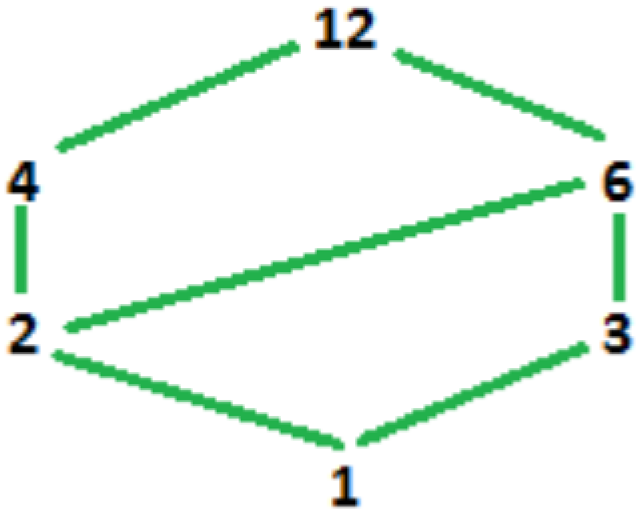
- The **Greatest Lower Bound (GLB)** (INFIMUM) is the largest element in lower bounds. Denoted by GLB ({a, b}) by a ∧ b.

Example 1: Draw Hasse diagram for (D12, /).

Solution – Here, D12 means set of positive integers divisors of 12.

So, D12 = {1, 2, 3, 4, 6, 12}
poset A = {(1 2), (1 3), (1 4), (1 6), (1 12), (2 4), (2 6), (2 12), (3 6), (3 12), (4 12), (6 12)}

So, now the Hasse diagram



Maximal and Greatest element is 12

Minimal and Least element is 1

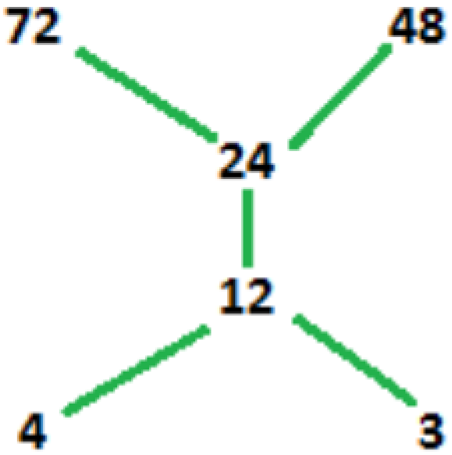
Example-2: Draw Hasse diagram for ({3, 4, 12, 24, 48, 72}, /)

Solution – According to above given question first, we have to find the poset for the divisibility.

Let the set is A.

A={(3 12), (3 24), (3 48), (3 72), (4 12), (4 24), (4 48), (4 72), (12 24), (12 48), (12 72), (24 48), (24 72)}

So, now the Hasse diagram will be:



Maximal elements are: 48 and 72

Minimal elements are: 3 and 4

Greatest: Null

Least : Null

Example – Find the least upper bound and greatest lower bound of the following subsets- {b,c},{g,e,a},{e,f}.

Solutions:

a) For the set {b,c}

Upper Bounds – e,f,h,i

LUB - e

Lower Bounds – a

GLB - a

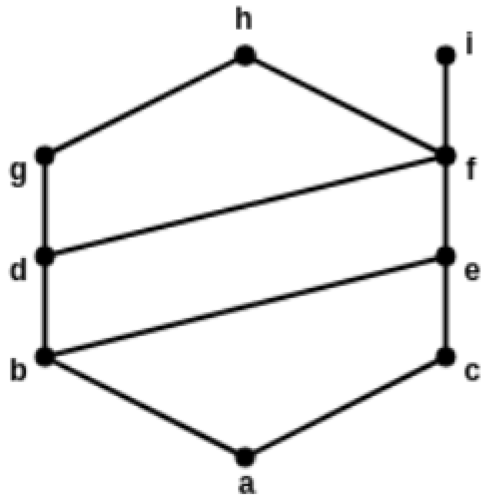
b) For the set {e,f}

Upper Bounds – f,h,i.

LUB- f.

Lower Bounds –e,c,b,a .

GLB- e



c) For the set {g,e,a}

Upper Bounds – h.

LUB - h

Lower Bounds – a

GLB - a.

Lattices

Definition:

A lattice is a partially ordered set or POSET in which every pair of elements has both :

i) a least upper bound(LUB) or $(x \vee y)$ or (called join), and

ii) a greatest lower bound(glb) or $(x \wedge y)$ or (called meet)

Example

Consider the poset (X, ≤) where X = {1, 2, 3, 5, 30} and the partial ordered relation ≤ is defined as "x divides y". Then show that poset (X, ≤) is a lattice.

Solution: Since GLB(x, y) = (x ∧ y) = lcm(x, y) and

LUB(x, y) = (x ∨ y) = gcd(x, y)

Now we can construct the operation table I and table II for GLB and LUB respectively and the Hasse diagram is shown in Fig.

Cont..

1->2,3,5,30

2-> 30

3-> 30

5-> 30

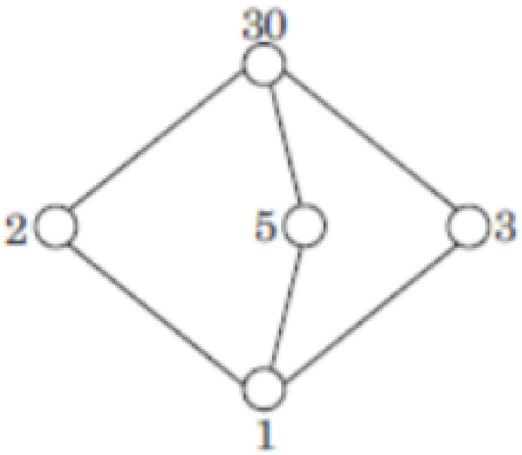


Table I

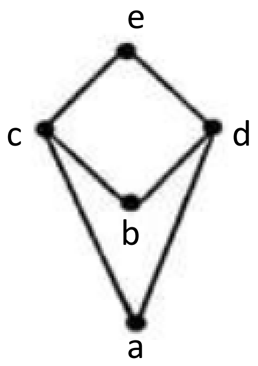
LUB	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30

Table II

GLB	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

It's a Lattice.

Example 2:



LUB	a	b	c	d	e
a	a	c	c	d	e
b	c	b	c	d	e
c	c	c	c	e	e
d	d	d	e	d	e
e	e	e	e	e	e

GLB	a	b	c	d	e
a	a	-	a	a	a
b	-	b	b	b	b
c	a	b	c	b	c
d	a	b	b	d	d
e	a	b	c	d	e

It's not Lattice.

Properties of Lattices

1. Idempotent Properties

a) $a \vee a = a$

b) $a \wedge a = a$.

2. Commutative Properties

a) $a \vee b = b \vee a$

b) $a \wedge b = b \wedge a$

3. Associative Properties

a) $a \vee (b \vee c) = (a \vee b) \vee c$

b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption Properties

a) $a \vee (a \wedge b) = a$

b) $a \wedge (a \vee b) = a$