$$\text{P.I.} = 0$$

$\therefore$ $y = c_3 \cos py + c_4 \sin py$

Now, $X(0) = 0$

$\Rightarrow$ $c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$X(l) = 0$

$\Rightarrow$ $c_1e^{pl} + c_2e^{-pl} = 0 \Rightarrow c_1(e^{pl} - e^{-pl}) = 0$

$\Rightarrow$ $c_1 = 0$ | Since $e^{pl} - e^{-pl} \neq 0$ (as $p \neq 0 \neq l$

$\therefore$ $c_2 = 0$

$\therefore$ $X = 0 \Rightarrow u = XY = 0$ which is impossible

Hence we reject case I.

Case II. When $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = 0$ (say)

(*i*) $\dfrac{X''}{X} = 0$

$\Rightarrow$ $X'' = 0 \Rightarrow X = c_5x + c_6$

(*ii*) $\dfrac{-Y''}{Y} = 0$

$\Rightarrow$ $Y'' = 0 \Rightarrow Y = c_7y + c_8$

Now, $X(0) = 0 \Rightarrow c_6 = 0$

$X(l) = 0$

$\Rightarrow$ $c_5l + c_6 = 0 \Rightarrow c_5l = 0$

$\Rightarrow$ $c_5 = 0$ (Since $l \neq 0$)

$\therefore$ $X = 0$

$\therefore$ $u = XY = 0$ which is impossible

Hence we also reject case II.

Case III. When $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = -p^2$ (say)

(*i*) $\dfrac{X''}{X} = -p^2$

$\Rightarrow$ $X'' + p^2X = 0 \Rightarrow \dfrac{d^2X}{dx^2} + p^2X = 0.$

Auxiliary equation is $m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$$\text{C.F.} = c_9 \cos px + c_{10} \sin px$$

$$\text{P.I.} = 0$$

$$X = c_9 \cos px + c_{10} \sin px$$

(*ii*) $-\dfrac{Y''}{Y} = -p^2$

$\Rightarrow$ $\dfrac{Y''}{Y} = p^2 \Rightarrow \dfrac{d^2Y}{dy^2} - p^2Y = 0.$

Auxiliary equation is

$$m^2 - p^2 = 0$$

$$m = \pm p.$$

$\therefore$ $$\text{C.F.} = c_{11}e^{py} + c_{12}e^{-py}$$

$$\text{P.I.} = 0$$

Hence, $$Y = c_{11}\, e^{py} + c_{12}\, e^{-py}.$$

Now, $$X(0) = 0 \Rightarrow c_9 = 0$$

$\therefore$ $$X = c_{10} \sin px$$

$$X(l) = 0$$

$$c_{10} \sin pl = 0$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi,\ n \in I$$

$\therefore$ $$p = \frac{n\pi}{l}$$

$\therefore$ $$X = c_{10} \sin \frac{n\pi x}{l} \qquad ...(3)$$

Again, $$Y(0) = 0$$

$\Rightarrow$ $$c_{11} + c_{12} = 0 \Rightarrow c_{12} = -c_{11}$$

$$Y = c_{11}(e^{py} - e^{-py}) = c_{11}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right) \qquad ...(4)$$

$\therefore$ $$u = XY = c_{10}c_{11} \sin \frac{n\pi x}{l}\, [e^{(n\pi y/l)} - e^{(-n\pi y/l)}]$$

or $$u(x, y) = b_n \sin \frac{n\pi x}{l}\, [e^{(n\pi y/l)} - e^{(-n\pi y/l)}] \qquad ...(5)$$

Now, $$u(x, a) = \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l}\, [e^{(n\pi a/l)} - e^{-(n\pi a/l)}]$$

$\Rightarrow$ $$b_n = \frac{1}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}} = \frac{1}{2 \sin h\left(\frac{n\pi a}{l}\right)}$$

$\therefore$ $$u(x, y) = \frac{e^{(n\pi y/l)} - e^{-(n\pi y/l)}}{2 \sinh\left(\frac{n\pi a}{l}\right)} \sin \frac{n\pi x}{l} = \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \sin \frac{n\pi x}{l}.$$

Example 2. *A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge y = 0 is given by*

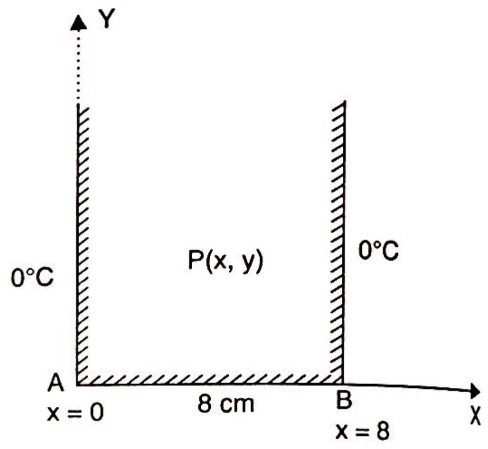
$$u(x, 0) = 100 \sin \frac{\pi x}{8},\ 0 < x < 8$$

while the two long edges x = 0 and x = 8 as well as the other short edge are kept at 0°C, show that the steady state temperature at any point of the plate is given by

$$u(x, y) = 100 e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}.$$ (A. K. T. U. 2018)

Sol. Let $u(x, y)$ be the temperature at any point P of the plate.

Two dimensional heat flow equation in steady state is given by



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(1)$$

Its solution is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad ...(2)$$

Boundary conditions are

$$u(0, y) = 0 = u(8, y)$$

$$\underset{y \to \infty}{\text{Lt}} \; u(x, y) = 0$$

$$u(x, 0) = 100 \sin \frac{\pi x}{8}, \quad 0 < x < 8$$

From (2), $u(0, y) = 0 = c_1 (c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (2), $u(x, y) = c_2 \sin px \, (c_3 e^{py} + c_4 e^{-py})$ ...(3)

$u(8, y) = 0 = c_2 \sin 8p(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $\sin 8p = 0 = \sin n\pi$

$\Rightarrow$ $p = \frac{n\pi}{8} \quad (n \in \text{I})$

$\therefore$ From (3), $u(x, y) = c_2 \sin \frac{n\pi x}{8} (c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}})$ ...(4)

$$\underset{y \to \infty}{\text{Lt}} \; u(x, y) = 0 = c_2 \sin \frac{n\pi x}{8} \lim_{y \to \infty} (c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}})$$

which is satisfied only when

$$c_3 = 0.$$

$\therefore$ From (4), $u(x, y) = c_2 c_4 \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}} = b_n \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}}$ ...(5)

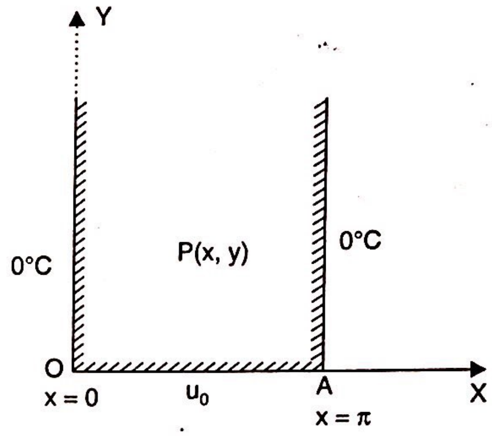
From (5), $u(x, 0) = 100 \sin \frac{\pi x}{8} = b_n \sin \frac{n\pi x}{8}$

$\Rightarrow$ $b_n = 100, n = 1.$

$\therefore$ From (5), $u(x, y) = 100 \sin \left(\frac{\pi x}{8}\right) e^{-(\pi y/8)}$

which is the required steady state temperature at any point of the plate.

Example 3. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.* (U.P.T.U. 2013)



Sol. In steady state, two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{...(1)}$$

Boundary conditions are,

$$u(0, y) = 0 = u(\pi, y)$$

$$\operatorname*{Lt}_{y\to\infty} u(x, y) = 0 \ (0 < x < \pi)$$

and $$u(x, 0) = u_0 \ (0 < x < \pi)$$

Solution to equation (1) is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \text{...(2)}$$

From (2), $u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $c_1 = 0.$

From (2), $u(x, y) = c_2 \sin px(c_3 e^{py} + c_4 e^{-py})$

$u(\pi, y) = 0 = c_2 \sin p\pi(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $\sin p\pi = 0 = \sin n\pi \ (n \in \mathrm{I})$ ...(3)

$\therefore$ $p = n.$

$\therefore$ From (3), $u(x, y) = c_2 \sin nx \,(c_3 e^{ny} + c_4 e^{-ny})$ ...(4)

$$\lim_{y\to\infty} u(x, y) = 0 = c_2 \sin nx \lim_{y\to\infty} (c_3 e^{ny} + c_4 e^{-ny})$$

which is satisfied only when $c_3 = 0$.

$\therefore$ From (4), $u(x, y) = c_2 c_4 e^{-ny} \sin nx = b_n e^{-ny} \sin nx,$ where $c_2 c_4 = b_n$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx \qquad \text{...(5)}$$

$$u(x, 0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx \, dx$$

$$= \frac{2u_0}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{2u_0}{n\pi}\{1-(-1)^n\} = \begin{cases} \dfrac{4u_0}{n\pi}; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ From (5), $$u(x, y) = \frac{4u_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n} e^{-ny} \qquad (n \text{ is odd})$$

or $$u(x, y) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin (2n-1)\, x \, e^{-(2n-1)y}.$$

Example 4. *A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge $y = 0$ is given by*

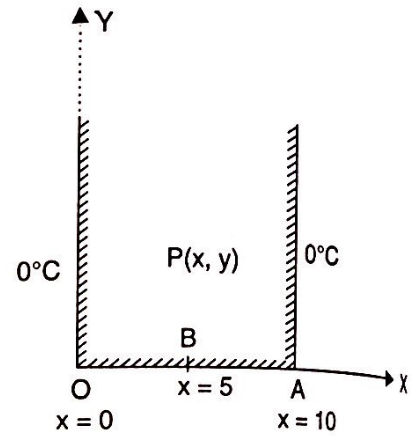
$$u(x, y) = \begin{cases} 20x, & 0 < x \le 5 \\ 20(10 - x), & 5 < x < 10 \end{cases}$$

and

and the two long edges $x = 0$ and $x = 10$ as well as other short edge are kept at 0°C. Find the temperature u at any point $P(x, y)$.

Sol. In steady state, two-dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(1)$$



Its solution is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3\, e^{py} + c_4\, e^{-py}) \quad ...(2)$$

Boundary conditions are $u(0, y) = 0$

$$u(10, y) = 0$$

$$\lim_{y \to \infty} u(x, y) = u(x, \infty) = 0$$

and
$$u(x, 0) = \begin{cases} 20x, & 0 < x \le 5 \\ 20(10 - x), & 5 < x \le 10 \end{cases}$$

From (2), $u(x, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $c_1 = 0$

From (2), $u(x, y) = c_2 \sin px\, (c_3\, e^{py} + c_4\, e^{-py})$ ...(3)

$u(10, y) = 0 = c_2 \sin 10p\, (c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $\sin 10p = 0 = \sin n\pi$

or $10p = n\pi \ (n \in I)$

$\Rightarrow$ $p = \dfrac{n\pi}{10}$.

$\therefore$ From (3), $u(x, y) = c_2 \sin \dfrac{n\pi x}{10} \left(c_3\, e^{\frac{n\pi y}{10}} + c_4\, e^{-\frac{n\pi y}{10}}\right)$ ...(4)

$$\lim_{y \to \infty} u(x, y) = c_2 \sin \frac{n\pi x}{10} \lim_{y \to \infty} \left(c_3\, e^{\frac{n\pi y}{10}} + c_4\, e^{-\frac{n\pi y}{10}}\right)$$

which is satisfied only when $c_3 = 0$.

Hence from (4), $u(x, y) = c_2 c_4 \sin \dfrac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}} = b_n \sin \dfrac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$ ...(5)

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}} \quad ...(6)$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10},$$

where $b_n = \frac{2}{10}\int_0^{10} u(x, 0) \sin \frac{n\pi x}{10}\, dx$

$$= \frac{1}{5}\left[\int_0^5 20x \sin \frac{n\pi x}{10} dx + \int_5^{10} 20\,(10 - x) \sin \frac{n\pi x}{10}\, dx\right]$$

$$= 4\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)\right\}_0^5 - \int_0^5 1\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)dx + \left\{(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)\right\}_5^{10} - \int_5^{10}(-1)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)dx\right]$$

$$= 4\left[\frac{10}{n\pi}(-5)\cos\frac{n\pi}{2} + \frac{10}{n\pi}\left(\frac{\sin\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)_0^5 + \frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{10}{n\pi}\left(\frac{\sin\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)_5^{10}\right]$$

$$= 4\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{100}{n^2\pi^2}\left(0 - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

From (6), $$u(x, y) = \frac{800}{\pi^2}\sum_1^{\infty}\frac{\sin n\pi/2}{n^2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}.$$

Example 5. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < \pi,\ 0 < y < \pi,$ *which satisfies the conditions:*

$u(0, y) = u(\pi, y) = u(x, \pi) = 0$ *and* $u(x, 0) = \sin^2 x.$ (U.K.T.U. 2011)

Sol. The given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ...(1)

Its solution consistent with boundary condition is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad ...(2)$$

From (2), $u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $c_1 = 0.$ ...(3)

$\therefore$ From (2), $u(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py})$

$u(\pi, y) = 0 = c_2 \sin p\pi(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $\sin p\pi = 0 = \sin n\pi\ (n \in I)$

$\therefore$ $p = n.$ ...(4)

Hence from (3), $u(x, y) = c_2 \sin nx\,(c_3 e^{ny} + c_4 e^{-ny}) = \sin nx\,(Ae^{ny} + Be^{-ny})$

where $c_2 c_3 = A$ and $c_2 c_4 = B.$

From (4), $u(x, \pi) = \sin nx(Ae^{n\pi} + Be^{-n\pi})$

$$0 = \sin nx(Ae^{n\pi} + Be^{-n\pi})$$

$\Rightarrow$ $$0 = Ae^{n\pi} + Be^{-n\pi}$$

$\Rightarrow$ $$Ae^{n\pi} = -Be^{-n\pi} = -\frac{1}{2}B_n \text{ (say)}$$

then (4) becomes, $$u(x, y) = \sin nx\left[-\frac{1}{2}B_n e^{-n\pi}e^{ny} + \frac{1}{2}B_n e^{n\pi}e^{-ny}\right]$$

$$= \frac{1}{2}B_n[e^{n(\pi-y)} - e^{-n(\pi-y)}]\sin nx = B_n \sinh n(\pi - y)\sin nx.$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh n(\pi - y)\sin nx \qquad \text{...(5}$$

$$u(x, 0) = \sin^2 x = \sum_{n=1}^{\infty} B_n \sinh n\pi \sin nx$$

where $$B_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(1 - \cos 2x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin nx - \frac{1}{2}\{\sin(n+2)x + \sin(n-2)x\}\right]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n} + \frac{\cos(n+2)x}{2(n+2)} + \frac{\cos(n-2)x}{2(n-2)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{n+2} + \frac{1}{n-2} - \frac{2}{n}\right)\{(-1)^n - 1\}\right], \text{ when } n \neq 2$$

$$B_n \sinh n\pi = \begin{cases} \dfrac{-8}{\pi n(n^2-4)}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even and } \neq 2 \end{cases}$$

when $n = 2$,

$$B_2 \sinh 2\pi = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin 2x\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(1 - \cos 2x)\sin 2x\, dx = \frac{1}{\pi}\int_0^{\pi}\left(\sin 2x - \frac{1}{2}\sin 4x\right)dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2} + \frac{1}{8}\cos 4x\right)_0^{\pi} = 0$$

$\therefore$ $$B_2 = 0.$$

Hence the solution (5) becomes,

$$u(x, y) = \frac{-8}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx \sinh n(\pi - y)}{n(n^2 - 4) \sinh n\pi}$$

or
$$u(x, y) = -\frac{8}{\pi} \sum_{m=1,2,3,\ldots}^{\infty} \frac{\sin (2m-1)x \sinh (2m-1)(\pi - y)}{(2m-1)\{(2m-1)^2 - 4\} \sinh (2m-1)\pi}.$$

Example 6. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, *with the rectangle* $0 \le x \le a$, $0 \le y \le b$ *; given that* $u(x, b) = u(0, y) = u(a, y) = 0$ *and* $u(x, 0) = x(a - x)$.

Sol. The equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

Its solution is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \text{...(2)}$$

$$u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (2), $u(x, y) = c_2 \sin px\,(c_3\, e^{py} + c_4 e^{-py})$ ...(3)

$$u(a, y) = 0 = c_2 \sin ap\,(c_3\, e^{py} + c_4 e^{-py})$$

$\Rightarrow$ $\sin ap = 0 = \sin n\pi \; (n \in \mathrm{I})$

$\Rightarrow$ $ap = n\pi$ or $p = \frac{n\pi}{a}.$

$\therefore$ From (3), $u(x, y) = c_2 \sin \frac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right)$

$$u(x, y) = \sin \frac{n\pi x}{a}\left(\mathrm{A}\, e^{\frac{n\pi y}{a}} + \mathrm{B}\, e^{-\frac{n\pi y}{a}}\right) \qquad \text{...(4)}$$

where $c_2 c_3 = \mathrm{A}$ and $c_2 c_4 = \mathrm{B}$

$$u(x, b) = \sin \frac{n\pi x}{a}\left(\mathrm{A} e^{\frac{n\pi b}{a}} + \mathrm{B} e^{-\frac{n\pi b}{a}}\right)$$

$$0 = \sin \frac{n\pi x}{a}\left(\mathrm{A} e^{\frac{n\pi b}{a}} + \mathrm{B} e^{-\frac{n\pi b}{a}}\right)$$

$\Rightarrow$ $\mathrm{A}\, e^{\frac{n\pi b}{a}} + \mathrm{B} e^{-\frac{n\pi b}{a}} = 0$

$$\mathrm{A}\, e^{\frac{n\pi b}{a}} = -\mathrm{B} e^{-\frac{n\pi b}{a}} = -\frac{1}{2}\mathrm{B}_n \text{ (say)}.$$

Then (4) becomes,

$$u(x, y) = \sin \frac{n\pi x}{a}\left[\frac{-1}{2}\mathrm{B}_n e^{-\frac{n\pi b}{a}} e^{\frac{n\pi y}{a}} + \frac{1}{2}\mathrm{B}_n e^{\frac{n\pi b}{a}} e^{-\frac{n\pi y}{a}}\right]$$

$$= \frac{1}{2} B_n \sin \frac{n\pi x}{a} [e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)}]$$

$$= \frac{1}{2} B_n \sin \frac{n\pi x}{a} . 2 \sinh \frac{n\pi}{a} (b-y) = B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a} (b-y).$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a} (b-y) \quad ...(5)$$

Applying to this the condition $u(x, 0) = x(a-x)$, we get

From (5),
$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

$$\Rightarrow \quad x(a-x) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

where
$$B_n \sinh \frac{n\pi}{a} b = \frac{2}{a} \int_0^a x(a-x) \sin \frac{n\pi}{a} x \, dx$$

$$= \frac{2}{a} \left[\left\{ (ax - x^2) \left(\frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right) \right\}_0^a - \int_0^a (a-2x) . \left(\frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right) dx \right]$$

$$= \frac{2}{a} . \frac{a}{n\pi} \int_0^a (a-2x) . \cos \frac{n\pi}{a} x \, dx$$

$$= \frac{2}{n\pi} \left[\left\{ (a-2x) \frac{\sin \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right\}_0^a - \int_0^a (-2) \left(\frac{\sin \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right) dx \right]$$

$$= \frac{4}{n\pi} . \frac{a}{n\pi} \left(\frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right)_0^a = \frac{4a}{n^2\pi^2} . \frac{a}{n\pi} (1 - \cos n\pi)$$

$$= \frac{4a^2}{n^3\pi^3} [1 - (-1)^n] = \begin{cases} \dfrac{8a^2}{n^3\pi^3} & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$\therefore \quad B_n = \begin{cases} \dfrac{8a^2}{\sinh \left(\frac{n\pi}{a} b \right) (n^3\pi^3)}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$\therefore$ From (5), $$u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin \frac{n\pi x}{a}}{n^3 \sinh \frac{n\pi}{a} b} . \sinh \frac{n\pi}{a}(b - y)$$

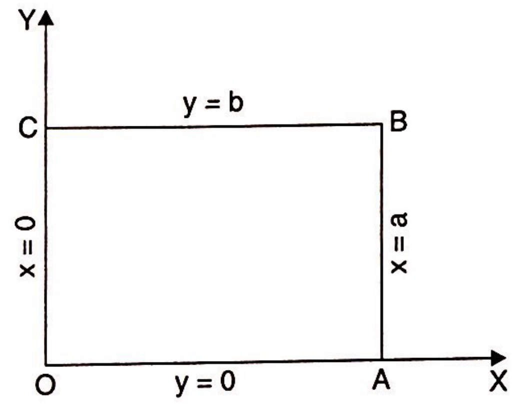
(n is odd)

or $$u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin (2n+1) \frac{\pi x}{a} . \frac{\sinh \frac{(2n+1)\pi}{a}(b-y)}{\sinh \frac{(2n+1)\pi}{a} b}$$

Example 7. *A thin rectangular plate whose surface is impervious to heat flow has at $t = 0$ an arbitrary distribution of temperature $f(x, y)$. Its four edges $x = 0$, $x = a$, $y = 0$, $y = b$ are kept at zero temperature. Determine the temperature at a point of a plate as t increases.*

Sol. Two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}. \quad \ldots(1)$$



Boundary conditions are

$$u(0, y, t) = 0 = u(a, y, t)$$
$$u(x, 0, t) = 0 = u(x, b, t)$$

and $$u(x, y, t) = f(x, y) \text{ at } t = 0.$$

Let the solution be $u = \text{XYT}$

where X is a function of x only, Y is a function of y only and T is a function of t only.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\text{XYT}) = \text{XY} \frac{d\text{T}}{dt}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(\text{XYT}) = \text{YT} \frac{d^2\text{X}}{dx^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(\text{XYT}) = \text{XT} \frac{d^2\text{Y}}{dy^2}.$$

From (1), $\text{YT X}'' + \text{XTY}'' = \frac{1}{c^2}(\text{XYT}')$

$$\Rightarrow \quad \frac{\text{X}''}{\text{X}} + \frac{\text{Y}''}{\text{Y}} = \frac{\text{T}'}{c^2\text{T}} \quad \ldots(2)$$

There are three possibilities :

(i) $\frac{\text{X}''}{\text{X}} = 0$, $\frac{\text{Y}''}{\text{Y}} = 0$, $\frac{\text{T}'}{c^2\text{T}} = 0$

(ii) $\frac{\text{X}''}{\text{X}} = \text{K}_1^2$, $\frac{\text{Y}''}{\text{Y}} = \text{K}_2^2$, $\frac{\text{T}'}{c^2\text{T}} = \text{K}^2$

(iii) $\frac{\text{X}''}{\text{X}} = -\text{K}_1^2$, $\frac{\text{Y}''}{\text{Y}} = -\text{K}_2^2$, $\frac{\text{T}'}{c^2\text{T}} = -\text{K}^2$

where $\text{K}^2 = \text{K}_1^2 + \text{K}_2^2$.

Of these three solutions, we have to select the solution which is consistent with the physical nature of the problem.

The solution satisfying the given boundary conditions will be given by (*iii*).

Then,
$$X = c_1 \cos K_1 x + c_2 \sin K_1 x$$
$$Y = c_3 \cos K_2 y + c_4 \sin K_2 y$$
$$T = c_5 e^{-c^2K^2t}$$
$\therefore$
$$u = XYT$$
$\Rightarrow$
$$u(x, y, t) = (c_1 \cos K_1 x + c_2 \sin K_1 x)(c_3 \cos K_2 y + c_4 \sin K_2 y)(c_5 e^{-c^2K^2t}) \quad \text{...(3)}$$
$$u(0, y, t) = 0 = c_1(c_3 \cos K_2 y + c_4 \sin K_2 y)c_5 e^{-c^2K^2t}$$
$\Rightarrow$
$$c_1 = 0.$$
$\therefore$ From (3),
$$u(x, y, t) = c_2 \sin K_1 x(c_3 \cos K_2 y + c_4 \sin K_2 y)(c_5 e^{-c^2K^2t})$$
$$= c_6 \sin K_1 x\, (c_3 \cos K_2 y + c_4 \sin K_2 y)(e^{-c^2K^2t}) \quad \text{...(4)}$$
where
$$c_2 c_5 = c_6$$
From (4),
$$u(a, y, t) = 0 = c_6 \sin K_1 a(c_3 \cos K_2 y + c_4 \sin K_2 y)e^{-c^2K^2t}$$
$\Rightarrow$
$$\sin K_1 a = 0 = \sin n\pi \ (n \in I)$$
$\therefore$
$$K_1 = \frac{n\pi}{a}.$$
From (4),
$$u(x, y, t) = c_6 \sin \frac{n\pi x}{a} (c_3 \cos K_2 y + c_4 \sin K_2 y)\, (e^{-c^2K^2t}) \quad \text{...(5)}$$
$$u(x, 0, t) = 0 = c_6 \sin \frac{n\pi x}{a} . c_3\, e^{-c^2K^2t}$$
$\Rightarrow$
$$c_3 = 0.$$
$\therefore$ From (5),
$$u(x, y, t) = c_6 c_4 \sin \frac{n\pi x}{a} \sin K_2 y\, e^{-c^2K^2t} \quad \text{...(6)}$$
$$u(x, b, t) = 0 = c_6 c_4 \sin \frac{n\pi x}{a} \sin K_2 b\, e^{-c^2K^2t}$$
$\Rightarrow$
$$\sin K_2 b = 0 = \sin m\pi \ (m \in I)$$
$$K_2 b = m\pi$$
$\Rightarrow$
$$K_2 = \frac{m\pi}{b}.$$
$\therefore$ From (6),
$$u(x, y, t) = c_6 c_4 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\, e^{-c^2K^2t}$$
$$= A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\, e^{-c^2K^2t} \quad \text{...(7)} \quad | \text{ where } c_6 c_4 = A_{mn}$$
But,
$$K^2 = K_1^2 + K_2^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$
or
$$K^2_{mn} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right).$$

By using K_{mn}, equation (7) becomes,

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 K^2_{mn} t} \qquad ...(8)$$

which is the most general solution.

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

which is the double Fourier half-range sine series for $f(x, y)$.

where $$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} f(x, y) dx\, dy.$$

Example 8. *Solve the Laplace equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *in a rectangle in the xy-plane with* $u(x, 0) = 0$, $u(x, b) = 0$, $u(0, y) = 0$ *and* $u(a, y) = f(y)$ *parallel to y-axis.*

(A.K.T.U. 2016)

Sol. The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Let $$u = XY \qquad ...(2)$$

where X is a function of x only and Y is a function of y only. Then,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = YX''$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XY) = XY''$$

$\therefore$ From (1), $YX'' + XY'' = 0 \Rightarrow \frac{Y''}{Y} = -\frac{X''}{X}$ ...(3)

Case I. When $\frac{Y''}{Y} = -\frac{X''}{X} = p^2$ (say)

(*i*) $$\frac{Y''}{Y} = p^2$$

$\Rightarrow$ $$Y'' - p^2 Y = 0$$

Auxiliary equation is

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$\therefore$ $$\text{C.F.} = c_1 e^{py} + c_2 e^{-py}$$

$$\text{P.I.} = 0$$

$\therefore$ $$Y = c_1 e^{py} + c_2 e^{-py}$$

(*ii*) $$-\frac{X''}{X} = p^2$$

$\Rightarrow$ $$X'' + p^2 X = 0$$

Auxiliary equation is

$$m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_3 \cos px + c_4 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_3 \cos px + c_4 \sin px$$

Now, $$Y(0) = 0$$

$\Rightarrow$ $$c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$$

$$Y(b) = 0$$

$\Rightarrow$ $$c_1 e^{pb} + c_2 e^{-pb} = 0$$

$\Rightarrow$ $$c_1(e^{pb} - e^{-pb}) = 0$$

$\Rightarrow$ $$c_1 = 0$$ | Since $e^{pb} - e^{-pb} \neq 0$ (as $p \neq 0 \neq b$)

$\therefore$ $$Y = 0 \quad \Rightarrow \quad u = XY = 0$$ which is impossible.

Hence, we reject case I.

Case II. When $$\frac{Y''}{Y} = -\frac{X''}{X} = 0 \text{ (say)}$$

(*i*) $$\frac{Y''}{Y} = 0$$

$\Rightarrow$ $$Y'' = 0 \quad \Rightarrow \quad Y = c_5 + c_6 y$$

(*ii*) $$-\frac{X''}{X} = 0$$

$\Rightarrow$ $$X'' = 0 \quad \Rightarrow \quad X = c_7 + c_8 x$$

Now, $$Y(0) = 0 \quad \Rightarrow \quad c_5 = 0$$

$$Y(b) = 0 \quad \Rightarrow \quad c_6 b = 0 \quad \Rightarrow \quad c_6 = 0$$ | $\because$ $b \neq 0$

$\therefore$ $$Y = 0$$

$\therefore$ $u = XY = 0$ which is impossible.

Hence, we also reject case II.

Case III. When $$\frac{Y''}{Y} = -\frac{X''}{X} = -p^2 \text{ (say)}$$

(*i*) $$\frac{Y''}{Y} = -p^2$$

$\Rightarrow$ $$Y'' + p^2 Y = 0$$

Auxiliary equation is

$$m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_9 \cos py + c_{10} \sin py$$

$$\text{P.I.} = 0$$

$\therefore$ $$Y = c_9 \cos py + c_{10} \sin py$$

(ii) $$-\frac{X''}{X} = -p^2 \Rightarrow X'' - p^2X = 0$$

Auxiliary equation is

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$$\therefore \quad \text{C.F.} = c_{11}\, e^{px} + c_{12}\, e^{-px}$$

$$\text{P.I.} = 0$$

$$\therefore \quad X = c_{11}\, e^{px} + c_{12}\, e^{-px}$$

Now, $$Y(0) = 0 \Rightarrow c_9 = 0$$

$$Y(b) = 0 \Rightarrow c_{10} \sin bp = 0$$

$$\therefore \quad \sin bp = 0 = \sin n\pi,\ n \in I$$

$$p = \frac{n\pi}{b}$$

Hence, $$u = XY = c_{10} \sin \frac{n\pi y}{b} \left(c_{11}\, e^{\frac{n\pi x}{b}} + c_{12} e^{-\frac{n\pi x}{b}} \right) \quad \text{...(4)}$$

Now, $$u(0, y) = 0 = c_{10} \sin \frac{n\pi y}{b} (c_{11} + c_{12})$$

$$\Rightarrow \quad c_{11} + c_{12} = 0 \Rightarrow c_{12} = -c_{11}$$

$\therefore$ From (4), $$u(x, y) = c_{10} c_{11} \sin \frac{n\pi y}{b} \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}} \right)$$

$$= b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b} \quad \text{...(5)}$$

| where $b_n = 2\, c_{10}\, c_{11}$

Most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b} \quad \text{...(6)}$$

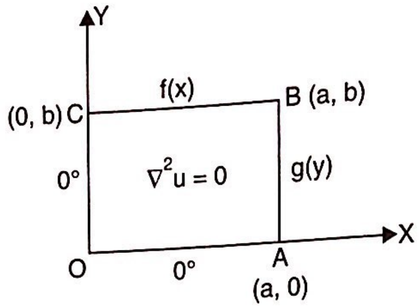
Now, $$u(a, y) = f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi a}{b}$$

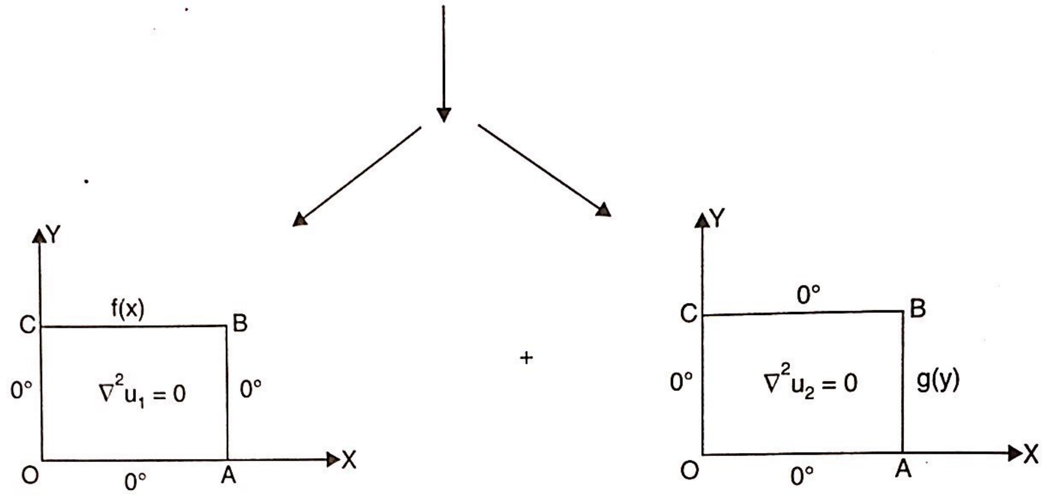
where $$\left(\sinh \frac{n\pi a}{b} \right) b_n = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy$$

$$\Rightarrow \quad b_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy. \quad \text{...(7)}$$

Example 9. *Find the steady state temperature distribution in a rectangular thin plate with its two surfaces insulated and with the conditions $u(0, y) = 0$, $u(x, 0) = 0$, $u(a, y) = g(y)$, $u(x, b) = f(x)$.*

Sol. Superposition applied to boundary conditions dismantles the given problem to solution of two simpler problems each of which can easily be solved by the method of separation of variables.





Now, the following two problems are required to be solved:

Problem 1. $\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} = 0$ subject to conditions

$$u_1(0, y) = 0,\ u_1(a, y) = 0,\ u_1(x, 0) = 0,\ u_1(x, b) = f(x)$$

Problem 2. $\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} = 0$ subject to conditions

$$u_2(0, y) = 0,\ u_2(x, 0) = 0,\ u_2(x, b) = 0,\ u_2(a, y) = g(y)$$

Let us now proceed to solve problem 1.

$$u_1(x, y) = (c_1 \cos px + c_2 \sin px)\,(c_3 e^{py} + c_4 e^{-py}) \quad \text{...(1)}$$

$$u_1(0, y) = 0 = c_1\,(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow$ $$c_1 = 0$$

$\therefore$ $$u_1(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py}) \quad \text{...(2)}$$

$$u_1(a, y) = 0 = c_2 \sin pa\,(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow$ $$\sin ap = 0 = \sin n\pi \quad (n \in I)$$

$$p = \frac{n\pi}{a}$$

Hence, from (2), $$u_1(x, y) = c_2 \sin \frac{n\pi x}{a}\left(c_3\, e^{\frac{n\pi y}{a}} + c_4\, e^{-\frac{n\pi y}{a}}\right) \quad \text{...(3)}$$

$$u_1(x, 0) = 0 = c_2 \sin \frac{n\pi x}{a} (c_3 + c_4)$$

$$\Rightarrow \qquad c_3 + c_4 = 0 \Rightarrow c_4 = -c_3$$

$$\therefore \qquad u_1(x, y) = c_2c_3 \sin \frac{n\pi x}{a} \left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}} \right) = 2\, c_2c_3 \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

$$= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \text{ where } b_n = 2c_2c_3$$

Most general solution to problem 1 is

$$u_1(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \qquad ...(4)$$

Now,
$$u_1(x, b) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}$$

where
$$b_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$$

$$\Rightarrow \qquad b_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx \qquad ...(5)$$

Similarly, the most general solution to problem 2 is

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi x}{b}\right) \qquad ...(6)$$

where
$$B_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g(y) \sin\left(\frac{n\pi y}{b}\right) dy \qquad ...(7)$$

| Interchanging x by y, a by b and $f(x)$ by $g(y)$

Hence, the required solution to the given original problem is

$$u(x, y) = u_1(x, y) + u_2(x, y)$$

$$= \sum_{n=1}^{\infty} \left[b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) + B_n \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi x}{b}\right) \right]$$

where
$$b_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$$

and
$$B_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g(y) \sin \frac{n\pi y}{b}\, dy.$$

TEST YOUR KNOWLEDGE

1. A long rectangular plate of width a cm with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$,

$$\lim_{y \to \infty} v(x, y) = 0 \quad \text{and} \quad v(x, 0) = kx$$

Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\left(\frac{n\pi y}{a}\right)} \sin \frac{n\pi x}{a}.$$

2. A square plate is bounded by the lines $x = 0, y = 0, x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while other three edges are kept at 0°C. Find the steady state temperature in the plate. *(A.K.T.U. 2014, 2017)*

3. A rectangular plate has sides a and b. Let the side of length a be taken along OX and that of length b along OY and the other sides along $x = a$ and $y = b$. The sides $x = 0, x = a$ and $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \frac{\pi x}{a}$. Find the steady-state temperature at any point (x, y).

   [**Hint:** Boundary conditions are $(u_x)_{x=0} = 0$, $(u_x)_{x=a} = 0$, $(u_y)_{y=b} = 0$ and $u(x, 0) = u_0 \cos(\pi x/a)$]

4. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y).

   Hence, show that the temperature at the centre of the plate

$$= \frac{200}{\pi} \left[\frac{1}{\cosh \frac{\pi}{2}} - \frac{1}{3 \cosh \frac{3\pi}{2}} + \frac{1}{5 \cosh \frac{5\pi}{2}} - \dots \right].$$

5. A rectangular plate is bounded by the lines $x = 0, y = 0, x = a, y = b$. Its surfaces are insulated and the temperature along the upper horizontal edge is 100°C while the other three edges are kept at 0°C. Find the steady state temperature function $u(x, y)$ and also the temperature at the point $\left(\frac{1}{2} a, \frac{1}{2} b\right)$.

6. Solve the following Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle with $u(0, y) = 0$, $u(a, y) = 0$, $u(x, b) = 0$ and $u(x, 0) = f(x)$ along x-axis.

7. Solve the boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 \le x \le a, \; 0 \le y \le b$$

   with the boundary conditions:

$$u_x(0, y) = u_x(a, y) = u_y(x, 0) = 0 \text{ and } u_y(x, b) = f(x)$$

   [M.T.U. (SUM) 2011]

8. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 0$ and $u(x, 1) = 100 \sin \pi x$. *(G.B.T.U. 2013)*

9. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0, y) = 0$, $u(a, y) = 0$, $u(x, 0) = 0$ and $u(x, b) = x$

(U.P.T.U. 2015)

10. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(x, 0) = 0$, $u(x, 1) = 0$, $u(\infty, y) = 0$ and $u(0, y) = u_0$.

(U.P.T.U. 2015)

11. The initial temperature distribution in a square plate of unit length is 100°C. Find the temperature distribution $u(x, t)$ if all the sides are maintained at zero degree temperature. *(U.P.T.U. 2014)*

Answers

2. $$u(x, y) = \frac{3200}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{20} \sin \frac{(2n-1)\pi y}{20}}{(2n-1)^3 \sinh (2n-1)\pi}$$

3. $$u(x, y) = u_0 \cos \frac{\pi x}{a} \cosh \frac{\pi}{a}(b - y) \operatorname{sech} \frac{\pi b}{a}$$

5. $$u(x, y) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{\sin \left\{(2m-1)\frac{\pi x}{a}\right\} \sinh \left\{(2m-1)\frac{\pi y}{a}\right\}}{(2m-1) \sinh \left\{(2m-1)\frac{\pi b}{a}\right\}};$$

$$u\left(\frac{1}{2}a, \frac{1}{2}a\right) = \frac{200}{\pi}\left[\frac{1}{\cosh\left(\frac{\pi b}{2a}\right)} - \frac{1}{3\cosh\left(\frac{3\pi b}{2a}\right)} + \ldots\right]$$

6. $$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi}{a}(b - y), \text{ where } B_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$$

7. $$u(x, y) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{a}\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right) \text{ where, } b_n = \frac{1}{n\pi \cosh \frac{n\pi}{a} b} \int_0^a f(x) \cos \frac{n\pi x}{a}\, dx$$

8. $$u(x, y) = 100 \sin \pi x \left(\frac{\sinh \pi y}{\sinh \pi}\right)$$

9. $$u(x, y) = \frac{-2a}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)}{n \sinh\left(\frac{n\pi b}{a}\right)}$$

10. $$u(x, y) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \left\{\frac{1 - (-1)^n}{n}\right\} e^{-n\pi x} \sin n\pi y$$

11. $$u(x, y, t) = \frac{400}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{1 - \cos n\pi}{n}\right)\left(\frac{1 - \cos m\pi}{m}\right) \sin n\pi x \sin m\pi y\, e^{-c^2 k_{mn}^2 t}$$

where $k_{mn}^2 = \pi^2(n^2 + m^2)$.

2.15 LAPLACE EQUATION

Laplace's equation has wide applications in Physics and engineering. The theory of its solutions is called the *potential theory* and its solutions are called *harmonic functions.* The solution of Laplace's equation, subject to certain boundary conditions, is simplified by a proper choice of coordinate system.

Note 1. If the problem involves *rectangular boundaries*, we prefer to take Laplace's equation in *cartesian coordinates given by* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$

Note 2. If the problem involves circular boundaries, we prefer to take Laplace's equation in polar coordinates given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$$

This equations can be obtained from $u_{xx}+u_{yy}=0$ by putting $x=r\cos\theta, y=r\sin\theta$, thus changing the independent variables from (x, y) to (r, θ).

Note 3. If the problem involves *cylindrical boundaries*, we prefer to take Laplace's equation in *cylindrical coordinates* given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

This equation can be obtained from $u_{xx}+u_{yy}+u_{zz}=0$ by putting $x=r\cos\theta, y=r\sin\theta, z=z$, thus changing the independent variables (x, y, z) to (r, θ, z).

Note 4. If the problem involves *spherical boundaries,* we prefer to take Laplace's equation in *spherical polar coordinates* given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=0.$$

This equation can be obtained from $u_{xx}+u_{yy}+u_{zz}=0$ by putting,

$$x=r\sin\theta\cos\phi, y=r\sin\theta\sin\phi, z=r\cos\theta,$$

thus changing the independent variables (x, y, z) to (r, θ, ϕ).

2.16 SOLUTIONS OF LAPLACE'S EQUATION

(*a*) Solution of Laplace's Equation in Two-dimensional Cartesian Form

We have already discussed the solution of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ in Art. 2.14.

(*b*) Solution of Laplace's Equation in Polar Coordinates

Laplace's equation in polar coordinates is $r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0$ ...(1)

Let $u(r, \theta)=R(r)F(\theta)$ or simply $u = RF$ ...(2)

where R is a function of r only and F is a function of θ only, be a solution of (1).

Substituting it in (1), we get

$$r^2R''F + rR'F + RF'' = 0 \quad \text{or} \quad (r^2R'' + rR')F + RF'' = 0$$

Separating the variables, $\frac{r^2R''+rR'}{R}=-\frac{F''}{F}=\text{constant}=k$ (say)

Thus, we get ordinary differential equations

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-kR=0 \qquad ...(3)$$

and

$$\frac{d^2F}{d\theta^2}+kF=0 \qquad ...(4)$$

Now (3) is a homogeneous linear differential equation.

Putting $r = e^z$, (3) reduces to $\frac{d^2R}{dr^2}-kR=0$ ...(5)

Solving (5) and (4), we get

(*i*) *When k is positive and $= p^2$, say*

$$R = c_1e^{pz}+c_2e^{-pz} = c_1r^p + c_2r^{-p},$$
$$F = c_3\cos p\theta + c_4 \sin p\theta.$$

(*ii*) *When k is negative and $= -p^2$, say*

$$R = c_1\cos pz + c_2\sin pz = c_1\cos(p\log r)+c_2\sin(p\log r)$$
$$F = c_3e^{p\theta}+c_4e^{-p\theta}$$

(*iii*) *When $k = 0$*

$$R = c_1z + c_2 = c_1\log r + c_2$$
$$F = c_3\theta + c_4$$

Thus the three possible solution of (1) are

$$u = (c_1r^p + c_2r^{-p})(c_3\cos p\theta + c_4\sin p\theta) \qquad ...(6)$$
$$u = [c_1\cos(p\log r)+c_2\sin(p\log r)]\,(c_3e^{p\theta}+c_4e^{-p\theta}) \qquad ...(7)$$
$$u = (c_1\log r + c_2)(c_3\theta + c_4) \qquad ...(8)$$

Of these solutions, we choose the one which is consistent with the physical nature of the problem.

Note. Usually we require a solution extending up to the origin.

Since u must be finite at the origin, we reject solutions (7) and (8). Also from (6), $c_2 = 0$.

∴ In this case, the solution may be written as

$$u = (A\cos p\theta + B\sin p\theta)r^p$$

The general solution will consist of a sum of similar terms with different (arbitrary) values of A, B and p.

(*c*) Solution of Laplace's Equation in Three-dimensional Cartesian Form

Laplace's equation in three-dimensional cartesian form is

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=0 \qquad ...(1)$$

Let $u(x, y, z) = X(x)\,Y(y)\,Z(z)$ or simply $u = XYZ$ ...(2) be a solution of (1).

Substituting it in (1), we get $X''YZ + XY''Z + XYZ'' = 0$

Dividing to XYZ, we get $\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$

or

$$\frac{1}{X}\cdot\frac{d^2X}{dx^2}+\frac{1}{Y}\cdot\frac{d^2Y}{dy^2}+\frac{1}{Z}\cdot\frac{d^2Z}{dz^2}=0 \qquad ...(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

Since x, y, z are independent, this is possible only when F_1, F_2, F_3 are constants. Assuming these constants to be k^2, l^2 and $-(k^2 + l^2)$ respectively, (3) gives rise to the following equations

$$\frac{d^2X}{dx^2} - k^2X = 0, \quad \frac{d^2Y}{dy^2} - l^2Y = 0, \quad \frac{d^2Z}{dz^2} + (k^2 + l^2)Z = 0$$

Their solutions are $X = c_1e^{kx} + c_2e^{-kx}$, $Y = c_3e^{ly} + c_4e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2 + l^2)}\, z + c_6 \sin \sqrt{(k^2 + l^2)}\, z$$

Hence a solution of (1) is

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly})[c_5 \cos \sqrt{(k^2 + l^2)}\, z + c_6 \sin \sqrt{(k^2 + l^2)}\, z]$$

Since, the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, and alternative solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\, [c_5e^{\sqrt{(k^2 + l^2)}\, z} + c_6e^{-\sqrt{(k^2 + l^2)}\, z}].$$

The choice of the constants and hence the general solution depends on the given initial and boundary conditions.

(*d*) Solution of Laplace's Equation in Cylindrical Coordinates

Laplace's equation in cylindrical coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{...(1)}$$

Let $u(r, \theta, z) = R(r)F(\theta)Z(z)$

or simply $u = RFZ$ ...(2) be a solution of (1).

Substituting it in (1), we get $R''FZ + \frac{1}{r} R'FZ + \frac{1}{r^2} RF''Z + RFZ'' = 0$

Dividing by RFZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2F}\frac{d^2F}{d\theta^2} + \frac{1}{z}\frac{d^2Z}{dz^2} = 0 \qquad \text{...(3)}$$

Assuming

$$\frac{d^2F}{d\theta^2} = -n^2F \text{ and } \frac{d^2Z}{dz^2} = k^2Z \qquad \text{...(4)}$$

Equation (3) reduces to $\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dz}\right) - \frac{n^2}{r^2} + k^2 = 0$

or

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dz} + (k^2r^2 - n^2)R = 0$$

This is Bessel's equation. Its solution is $R = c_1J_n(kr) + c_2Y_n(kr)$

The solutions of equations (3) are $F = c_3 \cos n\theta + c_4 \sin n\theta$, $Z = c_5e^{kz} + c_6e^{-kz}$

Hence a solution of (1) is $u = [c_1J_n(kr) + c_2Y_n(kr)]\,(c_3 \cos n\theta + c_4 \sin n\theta)(c_5e^{kz} + c_6e^{-kz})$

which is known as a *cylindrical harmonic*.

(e) Solution of Laplace's Equation in Spherical Coordinates

Laplace's equation is spherical coordinates is

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=0 \quad ...(1)$$

Let $u(r, \theta, \phi) = R(r)G(\theta)H(\phi)$.

or simply $u = RGH$ ...(2) be a solution of (1).

Substituting it in (1), we get

$$R''GH+\frac{2}{r}R'GH+\frac{1}{r^2}RG''H+\frac{\cot\theta}{r^2}RG'H+\frac{1}{r^2\sin^2\theta}RGH''=0$$

Dividing by RGH, we get

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)+\frac{1}{G}\left(\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}\right)+\frac{1}{H\sin^2\theta}\frac{d^2H}{d\phi^2}=0 \quad ...(3)$$

Putting
$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)=n(n+1) \quad ...(4)$$

and
$$\frac{1}{H}\frac{d^2H}{d\phi^2}=-m^2 \quad ...(5)$$

Equation (3) reduces to $\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}+[n(n+1)-m^2\operatorname{cosec}^2\theta]\,G=0$

This is associated Legendre's equation and its solution is

$$G = c_1P_n^m(\cos\theta)+c_2\,Q_n^m(\cos\theta)$$

The solution of (5) is $H = c_3\cos m\phi + c_4\sin m\phi$

To solve (4), assume that $R = r^k$ so that

$$k(k-1)+2k=n(n+1) \quad \text{or} \quad (k^2-n^2)+(k-n)=0$$

or
$$(k-n)(k+n+1)=0 \quad \therefore \quad k = n \text{ or } -n-1$$

Thus
$$R = c_5r^n + c_6r^{-n-1}$$

Hence, the general solution of (1) is

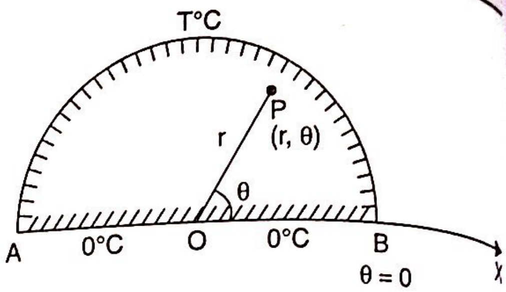
$$u=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[c_1P_n^m(\cos\theta)+c_2Q_n^m(\cos\theta)](c_3\cos m\phi+c_4\sin m\phi)(c_5r^n+c_6r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic.*

ILLUSTRATIVE EXAMPLES

Example 1. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r,\theta)=\frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta. \qquad \text{[G.B.T.U. (C.O.) 2011]}$$



Sol. Take the centre of the circle as the pole and the bounding diameter as the initial line. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad ...(1)$$

Let $$u = \text{RT} \quad ...(2)$$

where R is a function of r only and T is a function of θ only.

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(\text{RT}) = \text{T}\frac{d\text{R}}{dr}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\text{T}\frac{d\text{R}}{dr}\right) = \text{T}\frac{d^2\text{R}}{dr^2}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2}(\text{RT}) = \text{R}\frac{d^2\text{T}}{d\theta^2}.$$

$\therefore$ From (1), $\quad r^2\text{TR}'' + r\text{TR}' + \text{RT}'' = 0$

$$\frac{r^2\text{R}'' + r\text{R}'}{\text{R}} + \frac{\text{T}''}{\text{T}} = 0 \Rightarrow \frac{r^2\text{R}'' + r\text{R}'}{\text{R}} = -\frac{\text{T}''}{\text{T}}$$

Case I. When $$\frac{r^2\text{R}'' + r\text{R}'}{\text{R}} = -\frac{\text{T}''}{\text{T}} = p^2 \text{ (say)}$$

(*i*) $$\frac{r^2\text{R}'' + r\text{R}'}{\text{R}} = p^2 \Rightarrow r^2\text{R}'' + r\text{R}' - p^2\text{R} = 0.$$

Put $r = e^z$ so that $z = \log r$ and let $\text{D} \equiv \frac{d}{dz}$ then above equation reduces to

$$\{\text{D}(\text{D}-1) + \text{D} - p^2\}\,\text{R} = 0$$

$$(\text{D}^2 - p^2)\text{R} = 0$$

Auxiliary equation is $\quad m^2 - p^2 = 0 \Rightarrow m = \pm p$

$$\text{C.F.} = c_1 e^{pz} + c_2 e^{-pz} = c_1 e^{p \log r} + c_2 e^{-p \log r} = c_1 r^p + c_2 r^{-p}$$

$$\text{P.I.} = 0.$$

$\therefore$ $$\text{R} = c_1 r^p + c_2 r^{-p}$$

(*ii*) $$-\frac{\text{T}''}{\text{T}} = p^2$$

$$\frac{d^2\text{T}}{d\theta^2} + p^2\text{T} = 0.$$

Auxiliary equation is $\quad m^2 + p^2 = 0$

$\Rightarrow$ $$m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_3 \cos p\theta + c_4 \sin p\theta$$

$$\text{P.I.} = 0$$

$\therefore$ $$\text{T} = c_3 \cos p\theta + c_4 \sin p\theta$$

$\therefore$ $$u(r, \theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad ...($$

Case II. When $$\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = -p^2 \text{ (say)}$$

(*i*) $$\frac{r^2R'' + rR'}{R} = -p^2$$

$$\Rightarrow \quad r^2R'' + rR' + p^2R = 0$$

Put $r = e^z$ so that $z = \log r$ and let $D \equiv \frac{d}{dz}$ then above equation reduces to

$$[D(D-1) + D + p^2]\,R = 0$$

$$(D^2 + p^2)R = 0.$$

Auxiliary equation is $$m^2 + p^2 = 0$$

$$m = \pm pi$$

$$\text{C.F.} = (c_5 \cos pz + c_6 \sin pz)$$

$$\text{P.I.} = 0$$

$$\therefore \quad R = c_5 \cos pz + c_6 \sin pz$$

$$= c_5 \cos(p \log r) + c_6 \sin(p \log r).$$

(*ii*) $$-\frac{T''}{T} = -p^2$$

$$\Rightarrow \quad \frac{T''}{T} = p^2 \quad \text{or} \quad \frac{d^2T}{d\theta^2} - p^2T = 0.$$

Auxiliary equation is $$m^2 - p^2 = 0$$

$$m = \pm p$$

$$\therefore \quad \text{C.F.} = c_7e^{p\theta} + c_8e^{-p\theta}$$

$$\text{P.I.} = 0.$$

$$\therefore \quad T = c_7e^{p\theta} + c_8e^{-p\theta}.$$

Hence, $$u(r, \theta) = [c_5 \cos(p \log r) + c_6 \sin(p \log r)](c_7e^{p\theta} + c_8e^{-p\theta}) \quad \text{...(4)}$$

Case III. When $$\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = 0 \text{ (say)}$$

(*i*) $$r^2R'' + rR' = 0.$$

Put $r = e^z$ so that $z = \log r$ and let $D \equiv \frac{d}{dz}$, then above equation reduces to

$$[D(D-1) + D]R = 0$$

$$D^2R = 0$$

Auxiliary equation is $$m^2 = 0$$

$$\Rightarrow \quad m = 0, 0$$

$$\therefore \quad \text{C.F.} = (c_9 + c_{10}z)\,e^{0z} = c_9 + c_{10} \log r$$

$$\text{P.I.} = 0$$

$$\therefore \quad R = c_9 + c_{10} \log r.$$

(*ii*)
$$-\frac{T''}{T} = 0$$

$$\Rightarrow \quad T'' = 0$$

$$\Rightarrow \quad T = c_{11} + c_{12}\theta.$$

$$\therefore \quad u(r, \theta) = (c_9 + c_{10}\log r)(c_{11} + c_{12}\theta) \quad \text{...(5)}$$

Of these three solutions (3), (4) and (5), we choose the solution consistent with the given boundary conditions.

Boundary conditions are

$$u(r, 0) = 0 \quad \text{...(6)}$$

$$u(r, \pi) = 0 \quad \text{...(7)}$$

$$u(a, \theta) = T \quad \text{...(8)}$$

and
$$u \to 0 \text{ as } r \to 0 \quad \text{...(9)}$$

Solutions (4) and (5) do not satisfy boundary condition (9).

Hence, the consistent solution is

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad \text{...(10)}$$

From (10), $u(r, 0) = 0 = (c_1 r^p + c_2 r^{-p})\, c_3$

$$\Rightarrow \quad c_3 = 0.$$

From (10),

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\theta \quad \text{...(11)}$$

$$u(r, \pi) = 0 = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\pi$$

$$\Rightarrow \quad \sin p\pi = 0 = \sin n\pi \; (n \in I)$$

$$\therefore \quad p = n.$$

$\therefore$ From (11), $u(r, \theta) = (c_1 r^n + c_1 r^{-n})\, c_4 \sin n\theta$ ...(12)

Condition $u \to 0$ as $r \to 0$ is satisfied only and only when $c_2 = 0$.

Hence from (12), $u(r, \theta) = c_1 c_4 r^n \sin n\theta = b_n r^n \sin n\theta.$

The most general solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad \text{...(13)}$$

$$u(a, \theta) = T = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

where
$$b_n a^n = \frac{2}{\pi}\int_0^{\pi} \{T \sin n\theta \, d\theta\}$$

$$= \frac{2T}{\pi}\left(\frac{-\cos n\theta}{n}\right)_0^{\pi} = \frac{2T}{n\pi}(1 - \cos n\pi)$$

$$= \frac{2T}{n\pi}\{1 - (-1)^n\} = \begin{cases} \dfrac{4T}{n\pi}; & n \text{ is odd} \\ 0\;; & n \text{ is even} \end{cases}$$

$$\therefore \quad b_n = \begin{cases} \dfrac{4T}{n\pi a^n}; & n \text{ is odd} \\ 0\;; & n \text{ is even} \end{cases}$$

From (13),
$$u(r, \theta) = \frac{4T}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n} \cdot \frac{r^n}{a^n} \sin n\theta$$
$$u(r, \theta) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \left(\frac{r}{a}\right)^{2n-1} \sin (2n-1)\theta$$
which is the required steady state temperature in the plate.

Example 2. *Solve:* $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$ *with boundary conditions*

(i) V is finite when $r \to 0$ *(ii)* $V = \Sigma C_n \cos n\theta$ *on* $r = a$.

Sol. Solution to given differential equation is
$$V = \Sigma (A_n r^n + B_n r^{-n}) \cos (n\theta + \alpha)$$
When $r = a$, $V = \Sigma C_n \cos n\theta$
$$\therefore \quad \Sigma C_n \cos n\theta = \Sigma(A_n a^n + B_n a^{-n}) \cos (n\theta + \alpha)$$
$$\therefore \quad C_n = A_n a^n + B_n a^{-n}, \alpha = 0$$
When $r \to 0$, V is finite.
$$\therefore \quad B_n = 0 \qquad \text{(otherwise V becomes } \infty)$$
$$\therefore \quad A_n = \frac{C_n}{a^n}$$
$$\therefore \quad V = \Sigma C_n \left(\frac{r}{a}\right)^n \cos n\theta.$$

Example 3. *The edge* $r = a$ *of a circular plate is kept at temperature* $f(\theta)$*. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution in steady state.*

Sol. Here, we have to take the solution in polar coordinates.

The solution is
$$u = (c_1 \cos p\theta + c_2 \sin p\theta) (c_3 r^p + c_4 r^{-p}) \qquad ...(1)$$
Since, the temperature remains finite at $r = 0$
$$\therefore \quad c_4 = 0 \qquad ...(2)$$
Also, if we increase θ by 2π, we arrive at the same point. So the solution (1) should be periodic with period 2π.

Therefore $p = n$, an integer. Hence, we may write the general solution as
$$u = \sum_{n=0}^{\infty} (c_1 \cos n\theta + c_2 \sin n\theta) c_3 r^n$$
$$= \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) r^n \qquad | \; c_1c_3 = A_n, c_2c_3 = B_n \text{ (say)}$$
Applying to this, the condition
$$u = f(\theta) \quad \text{for} \quad r = a, \text{ we get}$$
$$f(\theta) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) a^n$$

where $$a^n A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

and $$a^n B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta$$

Example 4. *Find a harmonic function ϕ in the semi-circle $r < a$, $0 < \theta < \pi$ which vanishes on $\theta = 0$ and takes the value c on $\theta = \pi$ and on the curved portion $r = a$.*

Sol. A harmonic function is a function satisfying Laplace's equation.

Solution to $r^2 \dfrac{\partial^2\phi}{\partial r^2} + r\dfrac{\partial\phi}{\partial r} + \dfrac{\partial^2\phi}{\partial\theta^2} = 0$ is

$$\phi = (A\cos p\theta + B\sin p\theta)(Cr^p + Dr^{-p}) \qquad ...(1)$$

Since $\phi \to 0$ as $r \to 0$ $\quad\therefore\quad D = 0$

$$\therefore \qquad \phi = (A\cos p\theta + B\sin p\theta)\, Cr^p \qquad ...(2)$$

Imposing on (2), the zero boundary conditions

$\phi(r, 0) = 0$ and $\phi(r, \pi) = 0$, we get

$A = 0$ and $p = n \ (n \in I)$

Putting in (2) and adding up various solutions for $n = 1, 2, 3, \ldots$, we get

$$\phi = \sum_{n=1}^{\infty} B_n r^n \sin n\theta \qquad ...(3)$$

This solution will not satisfy both the given boundary conditions of the problem namely

$$\left.\begin{aligned} &(i)\ \phi = 0 \text{ when } \theta = 0 \\ \text{and } &(ii)\ \phi = c \text{ when } \theta = \pi \end{aligned}\right\} \qquad ...(4)$$

So we add to (3), the solution $\phi = A_0\theta + B_0$ which satisfies Laplace's equation. We choose A_0, B_0 so that (4) is satisfied.

Then, $\qquad B_0 = 0$ and $c = A_0\pi$

Hence, $$\phi = \frac{c\theta}{\pi} + \sum_{n=1}^{\infty} B_n r^n \sin n\theta \qquad ...(5) \quad \text{satisfies (4)}$$

Applying to (5), the condition that $\phi = c$ when $r = a$, we get

$$c = \frac{c\theta}{\pi} + \sum_{n=1}^{\infty} B_n a^n \sin n\theta$$

$$\Rightarrow \qquad c\left(1 - \frac{\theta}{\pi}\right) = \sum_{n=1}^{\infty} B_n a^n \sin n\theta$$

$$\therefore \qquad B_n a^n = \frac{2}{\pi}\int_0^{\pi} c\left(1 - \frac{\theta}{\pi}\right)\sin n\theta\, d\theta$$

$$B_n a^n = \frac{2c}{n\pi} \quad \Rightarrow \quad B_n = \frac{2c}{n\pi a^n} \qquad ...(6)$$

$\therefore$ From (5) and (6),

$$\phi = \frac{c\theta}{\pi} + \frac{2c}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^n \sin n\theta$$

TEST YOUR KNOWLEDGE

1. Show that the steady state temperature distribution in a semi-circular plate of radius a whose bounding diameter is kept at 0°C, while the circumference is kept at 60°C is given by

$$u(r, \theta) = \frac{240}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \left(\frac{r}{a}\right)^{2n-1} . \sin (2n-1)\theta.$$

2. A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta) = k\theta(\pi - \theta)$ while the bounding diameter is kept at zero temperature. Assuming the surfaces of the plate to be insulated, show that the steady-state temperature distribution of the plate is given by

$$u(r, \theta) = \frac{8k}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{2n-1} . \frac{\sin (2n-1)\theta}{(2n-1)^3}.$$

3. The bounding diameter of a semi-circular plate of radius a is kept at 0°C and the temperature along the semi-circular boundary is given by

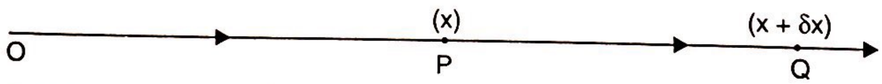
$$u(a, \theta) = \begin{cases} 50\theta, & \text{when } 0 < \theta < \dfrac{\pi}{2} \\ 50(\pi - \theta), & \text{when } \dfrac{\pi}{2} < \theta < \pi \end{cases}.$$

Show that the steady-state temperature distribution given by

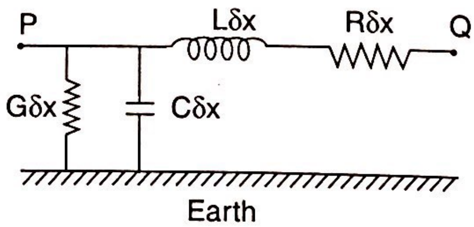
$$u(r, \theta) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \left(\frac{r}{a}\right)^{2n-1} . \sin (2n-1)\theta.$$

2.17 TRANSMISSION LINE EQUATIONS

Consider the flow of electricity in an insulated cable. Let V be the potential and I the current at time t at a point P of the cable at a distance x from a given point O. Let V + δV and I + δI be the potential and the current at the point Q($x + \delta x$) at the same time.



Let R, L, C, G be respectively the resistance, inductance, capacitance and leakance to the ground per unit length of the cable, each assumed to be constant.



Since, the potential drop across the segment PQ = potential drop due to resistance + potential drop due to inductance

$$\therefore \qquad -\delta V = (R\delta x)I + (L\delta x)\frac{\partial I}{\partial t}$$

Dividing by δx and proceeding to the limits as $\delta x \to 0$, we have

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t} \qquad \ldots(1$$

Similarly, the decrease in current in crossing the segment PQ = decrease in current du to capacitance and leakance

$$\therefore \qquad -\delta I = (G\delta x)V + (C\delta x)\frac{\partial V}{\partial t}$$

Dividing by δx and proceeding to the limits as $\delta x \to 0$, we have

$$-\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t} \qquad \ldots(2$$

Differentiating (1) and (2) partially w.r.t. x and t respectively, we have

$$-\frac{\partial^2 V}{\partial x^2} = R\frac{\partial I}{\partial x} + L\frac{\partial^2 I}{\partial x \partial t} \qquad \ldots(3$$

and

$$-\frac{\partial^2 I}{\partial x \partial t} = G\frac{\partial V}{\partial t} + C\frac{\partial^2 V}{\partial t^2} \qquad \ldots(4$$

Eliminating I from (2), (3) and (4), we get

$$-\frac{\partial^2 V}{\partial x^2} = R\left(-GV - C\frac{\partial V}{\partial t}\right) + L\left(-G\frac{\partial V}{\partial t} - C\frac{\partial^2 V}{\partial t^2}\right)$$

or

$$\boxed{\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC + LG)\frac{\partial V}{\partial t} + RGV} \qquad \ldots(5$$

Now, differentiating (1) and (2) w.r.t. t and x respectively, we have

$$-\frac{\partial^2 V}{\partial x \partial t} = R\frac{\partial I}{\partial t} + L\frac{\partial^2 I}{\partial t^2} \qquad \ldots(6$$

and

$$-\frac{\partial^2 I}{\partial x^2} = G\frac{\partial V}{\partial x} + C\frac{\partial^2 V}{\partial x \partial t} \qquad \ldots(7$$

Eliminating V from (1), (6), and (7), we get

$$-\frac{\partial^2 I}{\partial x^2} = G\left(-RI - L\frac{\partial I}{\partial t}\right) + C\left(-R\frac{\partial I}{\partial t} - L\frac{\partial^2 I}{\partial t^2}\right)$$

or

$$\boxed{\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (LG + RC)\frac{\partial I}{\partial t} + RGI} \qquad \ldots(8$$

The equations (5) and (8) are called the **telephone equations.**

2.17.1. If L = G = 0, the equations (5) and (8) become

$$\boxed{\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}} \quad \text{and} \quad \boxed{\frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t}}$$

which are known as the *telegraph equations*. They are similar to the equations in one dimensional heat flow. [*M.T.U. 2013*]

2.17.2. If R = G = 0, the equations (5) and (8) become

$$\boxed{\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}} \quad \text{and} \quad \boxed{\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}}$$

which are known as the *radio equations*. They are similar to the wave equation.

2.17.3. If R and G are negligible, the transmission lines become

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} \quad \text{and} \quad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}$$

2.17.4. If L = C = 0, the equations (5) and (8) become

$$\frac{\partial^2 V}{\partial x^2} = RGV \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = RGI$$

which are *equations for submarine cable*.

Note. We have sofar considered the transient state solutions only. The **steady-state solutions** of transmission lines are however obtained by assuming $v = V e^{i\omega t}$ and $i = I e^{i\omega t}$, where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear diff. eqns. of II order which can be solved at once.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that*

$$i(x, 0) = i_0, \; e(x, 0) = e_0 \sin\frac{\pi x}{l}$$

Also R and G are negligible.

Sol. Since R and G are negligible, transmission line equations become

$$\frac{\partial e}{\partial x} = -L\frac{\partial i}{\partial t} \quad ...(1)$$

and

$$\frac{\partial i}{\partial x} = -C\frac{\partial e}{\partial t} \quad ...(2)$$

For elimination of i, differentiating (1) partially w.r.t. x and (2) partially w.r.t. t, we have

$$\frac{\partial^2 e}{\partial x^2} = -L\frac{\partial^2 i}{\partial x \partial t} \quad \text{and} \quad \frac{\partial^2 i}{\partial t \partial x} = -C\frac{\partial^2 e}{\partial t^2}$$

whence $$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2} \qquad ...(3)$$

The initial conditions are $i(x, 0) = i_0,\ e(x, 0) = e_0 \sin \frac{\pi x}{l}$ ...(4)

Since, the ends are suddenly grounded, the boundary conditions are

$$e(0, t) = e(l, t) = 0 \qquad ...(5)$$

Also $i = i_0$ (constant) when $t = 0$

$\therefore$ $\frac{\partial i}{\partial x} = 0$ which gives $\frac{\partial e}{\partial t} = 0$ when $t = 0$ ...(6) | Using (2)

Now let $e = XT$ be a solution of (3) where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 e}{\partial x^2} = X''T \text{ and } \frac{\partial^2 e}{\partial t^2} = XT''$$

$\therefore$ From (3), $X''T = LCXT''$

Separating the variables $\frac{X''}{X} = LC\frac{T''}{T} = -p^2$ (say)

This leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} + p^2X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + \frac{p^2}{LC}T = 0$$

$\therefore$ $$X = c_1 \cos px + c_2 \sin px$$

$$T = c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}}$$

$\Rightarrow$ $$e = XT = (c_1 \cos px + c_2 \sin px)\left(c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}}\right) \qquad ...(7)$$

Applying the boundary conditions (5), we have

$$c_1 = 0 \quad \text{and} \quad p = \frac{n\pi}{l}, \ n \text{ being an integer.}$$

$\therefore$ Equation (7) becomes

$$e = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi t}{l\sqrt{LC}} + c_4 \sin \frac{n\pi t}{l\sqrt{LC}}\right)$$

or $$e = \sin \frac{n\pi x}{l}\left(A \cos \frac{n\pi t}{l\sqrt{LC}} + B \sin \frac{n\pi t}{l\sqrt{LC}}\right) \qquad ...(8)$$ | where $A = c_2c_3$ and $B = c_2c_4$

$$\frac{\partial e}{\partial t} = \sin \frac{n\pi x}{l}\left(-\frac{An\pi}{l\sqrt{LC}} \sin \frac{n\pi t}{l\sqrt{LC}} + \frac{Bn\pi}{l\sqrt{LC}} \cos \frac{n\pi t}{l\sqrt{LC}}\right)$$

Since $\frac{\partial e}{\partial t} = 0$ when $t = 0$, we get [From (6)]

$$B = 0$$

$\therefore$ From (8), $e = A \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$

By superposition, $e = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$ is also a solution.

But $e = e_0 \sin \frac{\pi x}{l}$ when $t = 0$ [From (4)]

$$\therefore \quad e_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \quad A_1 = e_0 \quad \text{and} \quad A_2 = A_3 = \ldots\ldots = 0$$

Hence, $e = e_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$

Now, $-L \frac{\partial i}{\partial t} = \frac{\partial e}{\partial x}$ [From (1)]

$$\therefore \quad \frac{\partial i}{\partial t} = -\frac{1}{L} \cdot \frac{e_0 \pi}{l} \cos \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

Integrating w.r.t. t, regarding x as constant

$$i = -\frac{e_0 \pi}{Ll} \cos \frac{\pi x}{l} \cdot \frac{l\sqrt{LC}}{\pi} \sin \frac{\pi t}{l\sqrt{LC}} + f(x) \quad \ldots(9)$$

where $f(x)$ is an arbitrary constant function.

Since $i = i_0$ when $t = 0$, we have $i_0 = 0 + f(x)$ or $f(x) = i_0$

$\therefore$ From (9), we have $i = i_0 - e_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}}$.

Example 2. *A transmission line 1000 km long is initially under steady-state conditions with potential 1300 volts at the sending end (x = 0) and 1200 volts at the receiving end (x = 1000). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential E(x, t).*

Sol. Since L and G are negligible, we use the telegraph equation

$$\frac{\partial^2 E}{\partial x^2} = RC \frac{\partial E}{\partial t} \quad \text{or} \quad \frac{\partial E}{\partial t} = \frac{1}{RC} \cdot \frac{\partial^2 E}{\partial x^2} \quad \ldots(1)$$

Here, E_s = initial steady voltage satisfying $\frac{\partial^2 E}{\partial x^2} = 0$

$$= 1300 - \frac{1300 - 1200}{1000} x = 1300 - 0.1x = E(x, 0) \quad \ldots(2)$$

E_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached

$$= 1300 - \frac{1300 - 0}{1000} x = 1300 - 1.3x$$

$\therefore$ $\quad$ $E(x, t) = E_s' + E_r(x, t)$, where $E_r(x, t)$ is the transient part

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2 RC}} \quad \text{...(3) where } l = 1000 \text{ km}$$

Putting $t = 0$ in (3), we have

$$E(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or
$$1300 - 0.1x = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{[Using (2)]}$$

or
$$1.2x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l} \int_0^l 1.2x \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2.4}{l} \left[x \left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(-\frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right) \right]_0^l = \frac{2.4}{l} \cdot \frac{l}{n\pi} [-l \cos n\pi]$$

$$= -\frac{2.4 \times 1000}{n\pi} \times (-1)^n \quad [\because \; l = 1000]$$

$$= \frac{2400}{\pi} \cdot \frac{(-1)^{n+1}}{n}$$

Hence,
$$E(x, t) = 1300 - 1.3x + \frac{2400}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2 RC}}.$$

Example 3. *Neglecting R and G, find the e.m.f. $v(x, t)$ in a line of length l, t seconds after the ends were suddenly grounded, given that $i(x, 0) = i_0$ and $v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l}$.*

(M.T.U 2012)

Sol. Since R and G are negligible, we use the radio equation

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} \quad \text{...(1)}$$

Since, the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, \; v(l, t) = 0 \quad \text{...(2)}$$

Also, the initial conditions are

and
$$\left.\begin{aligned} i(x, 0) &= i_0 \\ v(x, 0) &= e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} \end{aligned}\right\} \quad \text{...(3)}$$

$\therefore$
$$\frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t} \text{ gives}$$

$$\left(\frac{\partial v}{\partial t}\right)_{t=0} = 0 \qquad ...(4)$$

Let $v = XT$ be the solution of (1) where X is a function of x only and T is a function of t only.

$\therefore$ From (1), $\quad TX'' = LCXT''$

$$\Rightarrow \quad \frac{X''}{X} = LC\frac{T''}{T} = -p^2 \quad \text{(say)}$$

$$\therefore \quad X'' + p^2X = 0 \quad \Rightarrow \quad X = c_1 \cos px + c_2 \sin px$$

and $$T'' + \left(\frac{p^2}{LC}\right)T = 0$$

$$\Rightarrow \quad T = c_3 \cos \frac{p}{\sqrt{LC}}t + c_4 \sin \frac{p}{\sqrt{LC}}t$$

$$\therefore \quad v = XT = (c_1 \cos px + c_2 \sin px)\left(c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}}\right) \qquad ...(5)$$

Using the boundary conditions (2), we get

$$c_1 = 0 \text{ and } p = \frac{n\pi}{l}, n \in I$$

$$\therefore \quad v = \sin \frac{n\pi x}{l}\left(a_n \cos \frac{n\pi t}{l\sqrt{LC}} + b_n \sin \frac{n\pi t}{l\sqrt{LC}}\right) \qquad ...(6)$$

Using initial condition (4), we get $b_n = 0$

$\therefore$ From (6), $$v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Thus the most general solution of (1) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}} \qquad ...(7)$$

Finally by the initial condition (3), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$\therefore \quad a_1 = e_1$ and $a_5 = e_5$ while all other a's are zero.

Hence, $$v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

IP **Example 4.** *Solve* $\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}$ *assuming that the initial voltage is* $V_0 \sin \frac{\pi x}{l}$; $V_t(x_0) = 0$ *and* $V = 0$ *at the ends* $x = 0$ *and* $x = l$ *for all* t.

Sol. Let $\quad V = XT \qquad ...(1)$

where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 V}{\partial x^2} = TX'' \quad \text{and} \quad \frac{\partial^2 V}{\partial t^2} = XT''$$

Substituting in the given equations, we get

$$TX'' = LCXT''$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -p^2 \text{ (say)}$$

(*i*) $$\frac{X''}{X} = -p^2 \Rightarrow X'' + p^2X = 0$$

$\therefore$ $$X = c_1 \cos px + c_2 \sin px$$

(*ii*) $$LC\frac{T''}{T} = -p^2 \Rightarrow T'' + \frac{p^2}{LC}T = 0$$

$\therefore$ $$T = c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}}$$

Hence, $$V = XT = (c_1 \cos px + c_2 \sin px)\left(c_3 \cos\frac{pt}{\sqrt{LC}} + c_4 \sin\frac{pt}{\sqrt{LC}}\right) \quad ...(2)$$

Boundary conditions are

$$V(0, t) = 0 = V(l, t) \quad \text{and} \quad \frac{\partial V}{\partial t} = 0 \quad \text{when } t = 0$$

Applying conditions on (2), we get

$$0 = c_1\left(c_3 \cos\frac{pt}{\sqrt{LC}} + c_4 \sin\frac{pt}{\sqrt{LC}}\right)$$

$\Rightarrow$ $$c_1 = 0$$

From (2), $$V = c_2 \sin px\left(c_3 \cos\frac{pt}{\sqrt{LC}} + c_4 \sin\frac{pt}{\sqrt{LC}}\right) \quad ...(3)$$

$$V(l, t) = 0 = c_2 \sin pl\left(c_3 \cos\frac{pt}{\sqrt{LC}} + c_4 \sin\frac{pt}{\sqrt{LC}}\right)$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi \ (n \in I)$$

$\therefore$ $$p = \frac{n\pi}{l}$$

Hence, from (3),

$$V = c_2 \sin\frac{n\pi x}{l}\left[c_3 \cos\frac{n\pi t}{l\sqrt{LC}} + c_4 \sin\frac{n\pi t}{l\sqrt{LC}}\right] \quad ...(4)$$

$$\frac{\partial V}{\partial t} = c_2 \frac{n\pi}{l\sqrt{LC}} \sin\frac{n\pi x}{l}\left[-c_3 \sin\frac{n\pi t}{l\sqrt{LC}} + c_4 \cos\frac{n\pi t}{l\sqrt{LC}}\right]$$

At $t = 0$,

$$0 = c_2 \frac{n\pi}{l\sqrt{LC}} \sin \frac{n\pi x}{l} \cdot c_4$$

$\Rightarrow$ $$c_4 = 0$$

Hence from (4),

$$V = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}} \quad ...(5)$$

Now, $$V(x, 0) = V_0 \sin \frac{\pi x}{l} = c_2 c_3 \sin \frac{n\pi x}{l}$$

Comparing, we get

$$c_2 c_3 = V_0 \quad \text{and} \quad n = 1$$

Hence, the required solution is

$$V(x, t) = V_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}.$$

TEST YOUR KNOWLEDGE

1. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. A time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.

[*M.T.U. 2011*]

2. A line of length l is initially uncharged so that $i(x, 0) = 0$, $v(x, 0) = 0$, $\frac{\partial v}{\partial t} = 0$ when $t = 0$. At $t = 0$, one end $x = l$ is suddenly connected with a constant potential E while the other end is grounded. Neglecting R and G, show that

$$v(x, t) = \frac{Ex}{l} + \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{l} \cos\left(\frac{n\pi t}{l\sqrt{LC}}\right).$$

3. Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with velocity equal to $\frac{i}{\sqrt{LC}}$ where L is self-inductance and C is the capacitance.

4. Derive the partial differential equation for transmission line and also derive the telegraph and radio equations under suitable assumptions.

Answer

1. $$v(x, t) = \frac{20(l - x)}{l} + \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-n^2\pi^2 t/RCl^2}$$

$$i(x, t) = \frac{20}{lR} + \frac{24}{lR} \sum_{n=1}^{\infty} (-1)^n \cos \frac{n\pi x}{l} e^{-n^2\pi^2 t/RCl^2}.$$

ASSIGNMENT-II

(2 Marks Questions for Section–A)

1. Classify the differential equation: $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$.

2. Classify the partial differential equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$. *(M.T.U. 2011)*

3. Classify the partial differential equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Also explain your answer. *(A.K.T.U. 2016)*

4. Classify: $f_{xx} + 2f_{xy} + 4f_{yy} = 0$. *(U.P.T.U. 2014)*

5. Classify: $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$. *[G.B.T.U. (SUM) 2010]*

6. Classify the partial differential equation: $\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$. *(U.P.T.U. 2015)*

7. Explain briefly the method of separation of variables in solving a given partial differential equation. *(M.T.U. 2012)*

8. Write down the two-dimensional wave equation. *(A..K.T.U. 2017)*

9. Mention two applications of partial differential equations in engineering. *[G.B.T.U. (AG) 2012]*

10. Name the following equations: *[G.B.T.U. (AG) 2012]*

    (*i*) $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$ (*ii*) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

11. Apply the method of separation of variables to find the most appropriate solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$. *[G.B.T.U. (A.G.) 2012]*

12. What does the two-dimensional wave equation represent?

13. Write down the partial differential equation for one-dimensional wave equation. *(A.K.T.U. 2015, 2017)*

14. Solve: $4u_x + u_y = 3u$; $u(0, y) = e^{-5y}$.

15. Solve: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ in the steady state. *[G.B.T.U. (A.G.) 2011]*

16. Classify: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

17. Name the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

18. Write down the equation of steady state heat conduction in the rectangular plate. *(G.B.T.U. 2011)*

19. Write down the two-dimensional steady state heat flow equation in polar coordinates.

20. Solve: $3u_x + 2u_y = 0$ where $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$.

21. Classify the following differential equation in the first quadrant:

$$y^2 u_{xx} - x^2 u_{yy} = 0$$ *(G.B.T.U. 2013)*

22. Solve $\frac{\partial u}{\partial x} = 3\frac{\partial u}{\partial t}$ using method of separation of variables. *(G.B.T.U. 2013)*

23. Write the boundary conditions and initial conditions for the displacement of a finite string of length L that is fixed at both ends and is released from rest with an initial displacement $f(x)$. *(G.B.T.U. 2013)*

24. (*i*) Write telegraph equations. *(U.P.T.U. 2014)*
(*ii*) Write two-dimensional heat equation. *(A.K.T.U. 2016)*

25. (*i*) Characterize the following partial differential equation into elliptic, parabolic and hyperbolic equations: $a\frac{\partial^2 z}{\partial x^2} + 2h\frac{\partial^2 z}{\partial x \partial y} + b\frac{\partial^2 z}{\partial y^2} + 2f\frac{\partial z}{\partial x} + 2g\frac{\partial z}{\partial x} + cz = f(x, y)$ where a, b, c, h, f, g are constants. *(M.T.U. 2013)*
(*ii*) Specify with suitable example, the classification of partial differential (PDE) for elliptic, parabolic and hyperbolic differential equations. *(A.K.T.U. 2017)*

26. Find the condition for which the following partial differential equation is parabolic:

$$yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$$ *(U.P.T.U. 2013)*

27. Classify the partial differential equation:

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$ *(U.P.T.U. 2015)*

28. Classify the following partial differential equation along the line $y = x$:

$$yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0.$$ *(U.P.T.U. 2014)*

29. (*i*) Find the steady state temperature distribution in a rod of length L when its one end is kept at 0°C and the other end is kept at 100°C. *(U.P.T.U. 2013)*
(*ii*) Find the steady state temperature distribution in a plate of length of 20 whose ends are kept at 40°C and 100°C respectively. *(U.P.T.U. 2015)*

30. (*i*) Find the steady state temperature distribution in a rod of 2 m whose ends are kept at 30°C and 70°C respectively. *(U.P.T.U. 2015)*
(*ii*) Find the steady state temperature distribution in a rod of length 20 cm, whose ends are kept at 0°C and 60°C. *(U.P.T.U. 2014)*

Answers

1. Elliptic
2. Parabolic
3. Elliptic
4. Elliptic
5. Hyperbolic
6. Hyperbolic

8. $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

10. (*i*) One-dimensional wave equation (*ii*) One-dimensional heat equation

11. $u(x, y, t) = (c_1 \cos k_1 x + c_2 \sin k_1 x)(c_3 \cos k_2 y + c_4 \sin k_2 y)\, c_5 e^{-c^2 k^2 t}$

12. Vibrations of a tightly stretched membrane

13. $\dfrac{\partial y}{\partial t} = c^2 \dfrac{\partial^2 y}{\partial x^2}$

14. $u(x, y) = e^{2x - 5y}$

15. $u = c_1 x + c_2$

16. Parabolic

17. Laplace equation in two dimensions

18. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

19. $r^2 \dfrac{\partial^2 u}{\partial r^2} + r \dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial \theta^2} = 0$

20. $u(x, y) = c\, e^{\frac{k}{6}(2x - 3y)}$

21. hyperbolic

22. $u(x, t) = c_1 c_2\, e^{-p^2(3x+t)}$

23. $y(0, t) = 0 = y(L, t)\}$ boundary conditions

$$\left.\begin{aligned} \left(\frac{\partial y}{\partial t}\right)_{t=0} &= 0 \\ y(x, 0) &= f(x) \end{aligned}\right\} \text{initial conditions}$$

24. (*i*) $\dfrac{\partial^2 V}{\partial x^2} = RC \dfrac{\partial V}{\partial t}$ and $\dfrac{\partial^2 I}{\partial x^2} = RC \dfrac{\partial I}{\partial t}$ (*ii*) $\dfrac{\partial u}{\partial t} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

25. $h^2 < ab \rightarrow$ elliptic
$h^2 = ab \rightarrow$ parabolic
$h^2 > ab \rightarrow$ hyperbolic

26. $y = x$

27. hyperbolic

28. parabolic

29. (*i*) $u(x, 0) = \dfrac{100}{L} x$ (*ii*) $u(x, 0) = 40 + 3x$

30. (*i*) $u(x, 0) = 30 + 20x$ (*ii*) $u(x, 0) = 3x$