$\therefore$ $$\text{C.F.} = f_1(y + x) + xf_2(y + x)$$

$$\text{P.I.} = \frac{1}{D^2 - 2DD' + D'^2} \sin(2x + 3y) = \frac{1}{(D - D')^2} \sin(2x + 3y)$$

$$= \frac{1}{(2-3)^2} \iint \sin u \, du \, du \qquad \text{where } 2x + 3y = u$$

$$= -\sin u = -\sin(2x + 3y).$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + x) + xf_2(y + x) - \sin(2x + 3y)$$

where f_1 and f_2 are arbitrary functions.

Example 4. *Solve:* $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$.

Sol. The auxiliary equation is

$$2m^2 - 5m + 2 = 0$$

$\Rightarrow$ $$(2m - 1)(m - 2) = 0 \Rightarrow m = \tfrac{1}{2}, 2$$

Hence, $$\text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + f_2(y + 2x) = \phi_1(2y + x) + f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2} 24(y - x) \qquad |\ \text{Here } a = -1, b = 1$$

$$= \frac{24}{2(-1)^2 - 5(-1)(1) + 2(1)^2} \iint u \, du \, du \qquad \text{where } y - x = u$$

$$= \frac{4}{9} u^3 = \frac{4}{9}(y - x)^3$$

$\therefore$ Complete solution is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(2y + x) + f_2(y + 2x) + \frac{4}{9}(y - x)^3.$$

where ϕ_1 and f_2 are arbitrary functions.

Example 5. *Solve:* $r + s - 2t = \sqrt{2x + y}$. (U.P.T.U. 2015)

Sol. The given equation is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (2x + y)^{1/2}$$

or $$(D^2 + DD' - 2D'^2)z = (2x + y)^{1/2}$$

Auxiliary equation is

$$m^2 + m - 2 = 0$$

$\Rightarrow$ $$(m - 1)(m + 2) = 0 \Rightarrow m = 1, -2$$

$\therefore$ $$\text{C.F.} = f_1(y + x) + f_2(y - 2x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 2D'^2} (2x + y)^{1/2} \qquad |\ \text{Here } a = 2, b = 1$$

$$= \frac{1}{(2)^2 + (2)(1) - 2(1)^2} \iint \sqrt{u}\, du\, du \qquad \text{where } u = 2x + y$$

$$= \frac{1}{4} \cdot \frac{4}{15} u^{5/2} = \frac{1}{15} (2x + y)^{5/2}$$

Hence the required complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y - 2x) + \frac{1}{15} (2x + y)^{5/2}.$$

where f_1 and f_2 are arbitrary functions.

Example 6. *Solve the linear partial differential equation*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30\,(2x + y). \qquad \text{[G.B.T.U. (AG) 2011, 2012]}$$

Sol. The given equation is

$$(D^2 + D'^2)z = \cos mx \cos ny + 30\,(2x + y).$$

Auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore \quad \text{C.F.} = f_1\,(y + ix) + f_2(y - ix)$$

$$\text{P.I.} = \frac{1}{D^2 + D'^2} (\cos mx \cos ny) + \frac{1}{D^2 + D'^2}\, 30\,(2x + y) = P_1 + P_2$$

where

$$P_1 = \frac{1}{D^2 + D'^2} \cos mx \cos ny$$

$$= \frac{1}{2} \frac{1}{D^2 + D'^2} [\cos (mx + ny) + \cos (mx - ny)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 + D'^2} \cos (mx + ny) + \frac{1}{D^2 + D'^2} \cos (mx - ny)\right]$$

$$= \frac{1}{2}\left[\frac{1}{m^2 + n^2} \iint \cos u\, du\, du + \frac{1}{m^2 + n^2} \iint \cos v\, dv\, dv\right]$$

$$= \frac{1}{2}\left[\frac{-1}{m^2 + n^2} \cos u - \frac{1}{m^2 + n^2} \cos v\right] \text{ where } \quad mx + ny = u \text{ and } mx - ny = v$$

$$= \frac{-1}{2(m^2 + n^2)} [\cos (mx + ny) + \cos (mx - ny)]$$

and

$$P_2 = \frac{1}{D^2 + D'^2}\, 30\,(2x + y) = \frac{30}{(2)^2 + (1)^2} \iint \omega\, d\omega\, d\omega, \qquad \text{where } 2x + y = \omega$$

$$= 6\,\frac{\omega^3}{6} = \omega^3 = (2x + y)^3.$$

$$\therefore \quad \text{P.I.} = -\frac{1}{2(m^2 + n^2)} [\cos (mx + ny) + \cos (mx - ny)] + (2x + y)^3$$

$$= -\frac{1}{m^2 + n^2} \cos mx \cos ny + (2x + y)^3.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad z = f_1(y + ix) + f_2(y - ix) - \frac{1}{m^2 + n^2} \cos mx \cos ny + (2x + y)^3$$

where f_1 and f_2 are arbitrary functions.

Example 7. *Solve :* $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x - 2y).$

Sol. The given equation is

$$(D^2 - 3DD' + 2D'^2)z = e^{2x+3y} + \sin(x - 2y)$$

Auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow \quad (m-1)(m-2) = 0 \quad \Rightarrow \quad m = 1, 2$$

$$\therefore \quad \text{C.F.} = f_1(y + x) + f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D^2 - 3DD' + 2D'^2)} e^{2x+3y} + \frac{1}{(D^2 - 3DD' + 2D'^2)} \sin(x - 2y)$$

$$= P_1 + P_2$$

where

$$P_1 = \frac{1}{D^2 - 3DD' + 2D'^2} e^{2x+3y}$$

$$= \frac{1}{(2)^2 - 3(2)(3) + 2(3)^2} \iint e^u \, du \, du, \qquad \text{where } 2x + 3y = u$$

$$= \frac{1}{4} e^u = \frac{1}{4} e^{2x+3y}$$

$$P_2 = \frac{1}{D^2 - 3DD' + 2D'^2} \sin(x - 2y) = \frac{1}{(1)^2 - 3(1)(-2) + 2(-2)^2} \iint \sin v \, dv \, dv$$

where $x - 2y = v$

$$= \frac{1}{15}(-\sin v) = -\frac{1}{15} \sin(x - 2y)$$

$$\therefore \quad \text{P.I.} = P_1 + P_2 = \frac{1}{4} e^{2x+3y} - \frac{1}{15} \sin(x - 2y)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y + 2x) + \frac{1}{4} e^{2x+3y} - \frac{1}{15} \sin(x - 2y)$$

where f_1 and f_2 are arbitrary functions.

Example 8. *Solve :* $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin x.$ (M.T.U. 2013)

Sol. The given equation is

$$(D^2 - 2DD' + D'^2)z = \sin x$$

Auxiliary equation is

$$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad m = 1, 1$$

$\therefore$ $$\text{C.F.} = f_1(y + x) + x\, f_2(y + x)$$

$$\text{P.I.} = \frac{1}{(D - D')^2} \sin (x + 0.y)$$

$$= \frac{1}{(1 - 0)^2} \iint \sin u \, du \, du, \quad \text{where } x = u$$

$$= - \sin u = - \sin x$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y + x) + x\, f_2 (y + x) - \sin x$$

where f_1 and f_2 are arbitrary functions.

Example 9. *Solve the linear partial differential equation:*

$$\frac{\partial^3 z}{\partial x^3} - 4 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial x \partial y^2} = 4 \sin (2x + y).$$

Sol. The given equation is

$$(D^3 - 4D^2 D' + 4DD'^2)z = 4 \sin (2x + y)$$

The auxiliary equation is $m^3 - 4m^2 + 4m = 0$

$\Rightarrow$ $m (m^2 - 4m + 4) = 0$

$\Rightarrow$ $m = 0, 2, 2.$

$\therefore$ $$\text{C.F.} = f_1 (y) + f_2 (y + 2x) + x\, f_3 (y + 2x)$$

$$\text{P.I.} = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 4 \sin (2x + y)$$

$$= \frac{4}{D}\left[\frac{1}{D^2 - 4DD' + 4D'^2} \sin (2x + y)\right] = \frac{4}{D}\left[\frac{1}{(D - 2D')^2} \sin (2x + y)\right]$$

$$= x \,.\, \frac{4}{D}\left[\frac{1}{2(D - 2D')} \sin (2x + y)\right] = 4x^2 \,.\, \frac{1}{D}\left[\frac{1}{2} \sin (2x + y)\right]$$

$$= 2x^2 \frac{1}{D} \sin (2x + y) = - 2x^2 \frac{\cos (2x + y)}{2} = - x^2 \cos (2x + y).$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y) + f_2 (y + 2x) + x\, f_3 (y + 2x) - x^2 \cos (2x + y)$$

where f_1, f_2 and f_3 are arbitrary functions.

Example 10. *Solve the linear partial differential equation*

$$2 \frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 5 \sin (2x + y).$$

Sol. The given equation is

$$(2D^2 - 5DD' + 2D'^2)\, z = 5 \sin (2x + y)$$

The auxiliary equation is

$$2m^2 - 5m + 2 = 0 \;\Rightarrow\; m = \frac{1}{2}, 2.$$

$\therefore$ $$\text{C.F.} = f_1 \left(y + \frac{1}{2} x\right) + f_2 (y + 2x)$$

$$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2} 5 \sin(2x + y)$$

$$= 5x \cdot \frac{1}{4D - 5D'} \sin(2x + y)$$

$$= 5x \cdot \frac{1}{4(2) - 5(1)} \int \sin u \, du, \quad \text{where } 2x + y = u$$

$$= 5x \cdot \frac{1}{3}(-\cos u) = -\frac{5x}{3}\cos(2x + y).$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1\left(y + \frac{1}{2}x\right) + f_2(y + 2x) - \frac{5x}{3}\cos(2x + y)$$

where f_1 and f_2 are arbitrary functions.

Example 11. *Solve:* $(D^2 + 5DD' + 6D'^2)\, z = \dfrac{1}{y - 2x}$.

Sol. Auxiliary equation is $m^2 + 5m + 6 = 0$

$\Rightarrow$ $(m + 2)(m + 3) = 0$

$\Rightarrow$ $m = -2, -3$

$\therefore$ $\text{C.F.} = f_1(y - 2x) + f_2(y - 3x)$

$$\text{P.I.} = \frac{1}{D^2 + 5DD' + 6D'^2}\left(\frac{1}{y - 2x}\right)$$

$$= x \cdot \frac{1}{2D + 5D'}\left(\frac{1}{y - 2x}\right)$$

$$= x \cdot \frac{1}{2(-2) + 5(1)} \int \frac{1}{u} du, \quad \text{where } y - 2x = u$$

$$= x \log u = x \log(y - 2x)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y - 2x) + f_2(y - 3x) + x \log(y - 2x)$$

where f_1 and f_2 are arbitrary functions.

Example 12. *Solve the linear partial differential equations:*

(*i*) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$

(*ii*) $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$ (A.K.T.U. 2018)

(*iii*) $(D^2 - DD')z = \cos 2y(\sin x + \cos x)$.

Sol. (*i*) The given equation is

$$(D^2 - DD')\, z = \sin x \cos y$$

The auxiliary equation is $m^2 - m = 0 \Rightarrow m = 0, 1.$

$\therefore \quad \text{C.F.} = f_1(y) + f_2(y+x)$

$$\text{P.I.} = \frac{1}{D^2 - DD'} \sin x \cos y$$

$$= \frac{1}{2(D^2 - DD')}[\sin(x+y) + \sin(x-y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\sin(x+y) + \frac{1}{D^2 - DD'}\sin(x-y)\right] = \frac{1}{2}[P_1 + P_2]$$

where $$P_1 = \frac{1}{D^2 - DD'}\sin(x+y)$$

$$= x \cdot \frac{1}{2D - D'}\sin(x+y) = x \cdot \frac{1}{2(1)-(1)}\int \sin u \, du, \qquad \text{where } x + y = u$$

$$= x \cdot (-\cos u) = -x\cos(x+y)$$

$$P_2 = \frac{1}{D^2 - DD'}\sin(x-y)$$

$$= \frac{1}{(1)^2 - (1)(-1)}\iint \sin \omega \, d\omega \, d\omega, \qquad \text{where } x - y = \omega$$

$$= \frac{1}{2}(-\sin \omega) = -\frac{1}{2}\sin(x-y)$$

$$\therefore \quad \text{P.I.} = \frac{1}{2}\left[-x\cos(x+y) - \frac{1}{2}\sin(x-y)\right].$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(y+x) - \frac{x}{2}\cos(x+y) - \frac{1}{4}\sin(x-y)$$

where f_1 and f_2 are arbitrary functions.

(*ii*) The given equation is

$$(D^2 - 2DD')z = \sin x \cos 2y$$

Auxiliary equation is

$$m^2 - 2m = 0 \quad \Rightarrow \quad m = 0, 2$$

$\therefore \quad \text{C.F.} = f_1(y) + f_2(y + 2x)$

$$\text{P.I.} = \frac{1}{D^2 - 2DD'}\sin x \cos 2y$$

$$= \frac{1}{2(D^2 - 2DD')}[\sin(x+2y) + \sin(x-2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 2DD'}\sin(x+2y) + \frac{1}{D^2 - 2DD'}\sin(x-2y)\right]$$

$$= \frac{1}{2}(P_1 + P_2) \qquad \ldots(1)$$

where, $$P_1 = \frac{1}{D^2 - 2DD'}\sin(x+2y)$$

$$= \frac{1}{(1)^2 - 2(1)(2)} \iint \sin u \, du \, du \qquad \text{where } x + 2y = u$$

$$= -\frac{1}{3}(-\sin u) = \frac{1}{3}\sin(x+2y)$$

$$P_2 = \frac{1}{D^2 - 2DD'} \sin(x - 2y)$$

$$= \frac{1}{(1)^2 - 2(1)(-2)} \iint \sin v \, dv \, dv \qquad \text{where } x - 2y = v$$

$$= -\frac{1}{5}\sin v = -\frac{1}{5}\sin(x-2y)$$

$\therefore$ From (1),

$$\text{P.I.} = \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(y+x) + \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)$$

where f_1 and f_2 are arbitrary functions.

(*iii*) Auxiliary equation is

$$m^2 - m = 0 \quad \Rightarrow \quad m = 0, 1$$

$$\therefore \qquad \text{C.F.} = f_1(y) + f_2(y+x)$$

$$\text{P.I.} = \frac{1}{D^2 - DD'} \cos 2y (\sin x + \cos x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\{\sin(x+2y) + \sin(x-2y)\} + \frac{1}{D^2 - DD'}\{\cos(x+2y) + \cos(x-2y)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\{\sin(x+2y) + \cos(x+2y)\} + \frac{1}{D^2 - DD'}\{\sin(x-2y) + \cos(x-2y)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(1)^2 - (1)(2)} \iint (\sin u + \cos u)\, du\, du + \frac{1}{(1)^2 - (1)(-2)} \iint (\sin v + \cos v)\, dv\, dv\right]$$

where $x + 2y = u$ and $x - 2y = v$

$$= \frac{1}{2}\left[\sin(x+2y) + \cos(x+2y) - \frac{1}{3}\sin(x-2y) - \frac{1}{3}\cos(x-2y)\right]$$

$$= \frac{1}{2}[\sin(x+2y) + \cos(x+2y)] - \frac{1}{6}[\sin(x-2y) + \cos(x-2y)]$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(y+x) + \frac{1}{2}[\sin(x+2y) + \cos(x+2y)]$$

$$- \frac{1}{6}[\sin(x-2y) + \cos(x-2y)]$$

where f_1 and f_2 are arbitrary functions.

Example 13. *Solve:*

(*i*) $4r - 4s + t = 16 \log (x + 2y)$ (*ii*) $r + 2s + t = 2(y - x) + \sin (x - y)$

[M.T.U. (SUM) 2011]

(*iii*) $2r - s - 3t = 5 \dfrac{e^x}{e^y}$ (*iv*) $(D - D')^2 z = x + \phi(x + y)$.

Sol. (*i*) The given equation is

$$(4D^2 - 4DD' + D'^2)z = 16 \log (x + 2y)$$

Auxiliary equation is $4m^2 - 4m + 1 = 0 \Rightarrow m = \dfrac{1}{2}, \dfrac{1}{2}$

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$$

$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log (x + 2y)$$

$$= 16 \cdot x \cdot \frac{1}{2(2D - D').2} \log (x + 2y) = 4x \cdot \frac{1}{2D - D'} \log (x + 2y)$$

$$= 4x^2 \cdot \frac{1}{2} \log (x + 2y) = 2x^2 \log (x + 2y)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right) + 2x^2 \log (x + 2y)$$

where f_1 and f_2 are arbitrary functions.

(*ii*) The given equation is

$$(D^2 + 2DD' + D'^2) z = 2(y - x) + \sin (x - y)$$

Auxiliary equation is $m^2 + 2m + 1 = 0$

$\Rightarrow \quad m = -1, -1$

$$\therefore \quad \text{C.F.} = f_1 (y - x) + x f_2 (y - x)$$

$$\text{P.I.} = \frac{1}{(D + D')^2} 2(y - x) + \frac{1}{(D + D')^2} \sin (x - y) = P_1 + P_2$$

where $$P_1 = \frac{1}{(D + D')^2} \cdot 2(y - x) = 2 \cdot x \cdot \frac{1}{2(D + D')} (y - x)$$

$$= x \cdot \frac{1}{D + D'} (y - x) = x^2 \cdot (y - x)$$

$$P_2 = \frac{1}{(D + D')^2} \sin (x - y) = x \cdot \frac{1}{2(D + D')} \sin (x - y) = x^2 \cdot \frac{1}{2} \sin (x - y)$$

$$\therefore \quad \text{P.I.} = x^2 (y - x) + \frac{x^2}{2} \sin (x - y)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y-x) + x f_2(y-x) + x^2(y-x) + \frac{x^2}{2}\sin(x-y)$$

where f_1 and f_2 are arbitrary functions.

(*iii*) The given equation is

$$(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$$

Auxiliary equation is $\quad 2m^2 - m - 3 = 0 \quad \Rightarrow \quad m = -1, 3/2$

$$\therefore \quad \text{C.F.} = f_1(y-x) + f_2\left(y + \frac{3}{2}x\right)$$

$$\text{P.I.} = \frac{1}{2D^2 - DD' - 3D'^2}(5e^{x-y}) = 5x \cdot \frac{1}{4D - D'} e^{x-y}$$

$$= 5x \cdot \frac{1}{4(1)-(-1)} \int e^u \, du = xe^{x-y}, \qquad \text{where } x - y = u.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y-x) + f_2\left(y + \frac{3}{2}x\right) + x\,e^{x-y}$$

where f_1 and f_2 are arbitrary functions.

(*iv*) The given equation is $\quad (D - D')^2 z = x + \phi(x+y)$

Auxiliary equation is $\quad (m-1)^2 = 0, \quad \Rightarrow \quad m = 1, 1$

$$\therefore \quad \text{C.F.} = f_1(y+x) + x f_2(y+x)$$

$$\text{P.I.} = \frac{1}{(D-D')^2}x + \frac{1}{(D-D')^2}\phi(x+y) = P_1 + P_2$$

where

$$P_1 = \frac{1}{(D-D')^2}(x + 0.y) = \frac{1}{(1-0)^2}\iint u \, du\, du, \qquad \text{where } x = u.$$

$$= \frac{u^3}{6} = \frac{x^3}{6}$$

$$P_2 = \frac{1}{(D-D')^2}\phi(x+y) = x \cdot \frac{1}{2(D-D')}\phi(x+y) = \frac{x^2}{2}\phi(x+y)$$

$$\therefore \quad \text{P.I.} = P_1 + P_2 = \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y+x) + x f_2(y+x) + \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)$$

where f_1 and f_2 are arbitrary functions.

Example 14. *Solve the differential equation:*

$$\frac{\partial^3 z}{\partial x^2 \partial y} - 2\frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = \frac{1}{x^2}.$$

Sol. The given equation is

$$(D^2D' - 2DD'^2 + D'^3)\, z = \frac{1}{x^2}$$

$$\Rightarrow \qquad D'(D^2 - 2DD' + D'^2)z = \frac{1}{x^2}$$

Part of C.F. corresponding to $D'\left(\equiv \frac{\partial}{\partial y}\right)$ is $f_1(x)$

Auxiliary equation of the remaining factor is

$$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad m = 1, 1$$

$\therefore$ Another part of C.F. $= f_2(y + x) + x f_3(y + x)$

$\therefore$ Complete C.F. $= f_1(x) + f_2(y + x) + x f_3(y + x)$

$$\text{P.I.} = \frac{1}{D'^3 + D^2D' - 2DD'^2}\left(\frac{1}{x^2}\right) \qquad | \; x^{-2} = (x + 0.y)^{-2}$$

$$= y \cdot \frac{1}{3D'^2 + D^2 - 4DD'}\left(\frac{1}{x^2}\right) \qquad | \text{ Using Special Case of art. 1.19}$$

$$= y \cdot \frac{1}{3(0)^2 + (1)^2 - 4(1)(0)} \iint \frac{1}{u^2}\, du\, du, \qquad \text{where } x = u$$

$$= y \int \frac{-1}{u}\, du = y(-\log u) = -y \log x$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(x) + f_2(y + x) + x f_3(y + x) - y \log x$$

where f_1, f_2 and f_3 are arbitrary functions.

Example 15. *Solve:* $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} - 6\frac{\partial^3 z}{\partial y^3} = \sin(x + 2y) + e^{3x+y}$. (U.P.T.U. 2014)

Sol. The given equation is

$$(D^3 - 7DD'^2 - 6D'^3)z = \sin(x + 2y) + e^{3x+y}$$

Auxiliary equation is

$$m^3 - 7m - 6 = 0 \quad \Rightarrow \quad m = -1, -2, 3$$

$$\therefore \qquad \text{C.F.} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$$

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x + 2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{3x+y}.$$

P.I. corresponding to $\sin(x + 2y)$

$$= \frac{1}{(1)^3 - 7(1)(2)^2 - 6(2)^3}\iiint \sin u\, du\, du\, du, \qquad \text{where } x + 2y = u$$

$$= -\frac{1}{75}\cos u = -\frac{1}{75}\cos(x + 2y)$$

P.I. corresponding to e^{3x+y}

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3} (e^{3x+y})$$

$$= x \cdot \frac{1}{\frac{\partial}{\partial D}(D^3 - 7DD'^2 - 6D'^3)} e^{3x+y} = x \cdot \frac{1}{3D^2 - 7D'^2} e^{3x+y}$$

$$= x \cdot \frac{1}{3(3)^2 - 7(1)^2} e^{3x+y} = x \cdot \frac{1}{20} e^{3x+y}$$

$\therefore$ Required P.I. $= -\frac{1}{75}\cos(2y + x) + \frac{x}{20} e^{3x+y}$

$\therefore$ Complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) - \frac{1}{75}\cos(x + 2y) + \frac{x}{20} e^{3x+y}$$

where f_1, f_2 and f_3 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. (*i*) $\frac{\partial^2 z}{\partial x^2} - 7\frac{\partial^2 z}{\partial x \partial y} + 12\frac{\partial^2 z}{\partial y^2} = e^{x-y}$ (*ii*) $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$ (*U.P.T.U. 2015*)
2. (*i*) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(3x + y)$ (*ii*) $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x + 2y)$ (*U.K.T.U. 2012*)
3. $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$
4. $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 12(x + y)$
5. $(D^2 - 8DD' + 7D'^2)z = \sin(7x + y)$
6. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$
7. $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$
8. $(D^2 - 2DD' + D'^2)z = e^{x+y}$
9. (*i*) $4r + 12s + 9t = e^{3x-2y}$ (*ii*) $(D^2 + 3DD' + 2D'^2)z = e^{2x-3y}$
10. (*i*) $\frac{\partial^3 z}{\partial x^2 \partial y} - 5\frac{\partial^3 z}{\partial x \partial y^2} + 6\frac{\partial^3 z}{\partial y^3} = e^x$ (*ii*) $(D^2 + 7DD' + 12D'^2)z = \sinh x$ (*A.K.T.U. 2017*)
11. (*i*) $r + s - 6t = \cos(2x + y)$ (*ii*) $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$ [*G.B.T.U. (AG) 2011*]
12. (*i*) $(D - D')^2 z = \tan(y + x)$ (*ii*) $(D^2 - DD')z = \sin(x + 2y)$ [*G.B.T.U. (AG) 2012*]
13. $(D^2 + 3DD' - 4D'^2)z = x + \sin y$
14. $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y + x)^{1/2}$ (*A.K.T.U. 2017*)
15. $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + e^{x+y} + \cos(x + 2y)$ (*G.B.T.U. 2012*)

Answers

1. (*i*) $z = f_1(y+3x) + f_2(y+4x) + \frac{1}{20} e^{x-y}$ (*ii*) $z = f_1(y-x) + x f_2(y-x) + \frac{1}{25} e^{2x+3y}$

2. (*i*) $z = f_1(y-3x) + f_2(y+2x) - \frac{1}{6}\cos(3x+y)$ (*ii*) $z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + 2\cos(3x+2y)$

3. $z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$. **4.** $V = f_1(y+ix) + f_2(y-ix) + (x+y)^3$.

5. $z = f_1(y+x) + f_2(y+7x) - \frac{x}{6}\cos(7x+y)$ **6.** $z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$

7. $z = f_1(y+3x) + x f_2(y+3x) + x^2(3x+y)$ **8.** $z = f_1(y+x) + x f_2(y+x) + \frac{1}{2} x^2 e^{x+y}$

9. (*i*) $z = f_1(2y-3x) + x f_2(2y-3x) + \frac{1}{8} x^2 e^{3x-2y}$ (*ii*) $z = f_1(y-x) + f_2(y-2x) + \frac{1}{4} e^{2x-3y}$

10. (*i*) $z = f_1(x) + f_2(y+2x) + f_3(y+3x) + ye^x$ (*ii*) $z = f_1(y-3x) + f_2(y-4x) + \sinh x$

11. (*i*) $z = f_1(y-3x) + f_2(y+2x) + \frac{x}{5}\sin(2x+y)$ (*ii*) $z = f_1(y+x) + f_2(y+4x) - \frac{x}{3}\cos(4x+y)$

12. (*i*) $z = f_1(y+x) + x f_2(y+x) + \frac{x^2}{2}\tan(y+x)$ (*ii*) $z = f_1(y) + f_2(y+x) + \sin(x+2y)$

13. $z = f_1(y+x) + f_2(y-4x) + \frac{x^3}{6} + \frac{1}{4}\sin y$

14. $z = f_1(y+x) + x f_2(y+x) + f_3(y+2x) + x e^{y+x} - \frac{x^2}{3}(y+x)^{3/2}$

15. $z = f_1(y+x) + f_2(y+2x) + \frac{1}{12} e^{2x-y} - x e^{x+y} - \frac{1}{3}\cos(x+2y)$

1.20 WHEN $\phi(x, y)$ IS OF THE FORM $x^m y^n$ OR A RATIONAL INTEGRAL ALGEBRAIC FUNCTION OF x AND y

In this case, P.I. is obtained by expanding F(D, D′) in an infinite series of ascending powers of D or D′ or $\frac{D'}{D}$. It is to be noted that the difference in the two answers of P.I. is immaterial because it can be incorporated in the arbitrary functions occurring in C.F. of the given equation.

5(i) $(D^2+D'^2)z = x^2y^2$
Imp

ILLUSTRATIVE EXAMPLES

Example 1. *Solve*: $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3y^3$.

Sol. The given equation is $(D^3 - D'^3)z = x^3y^3$

Auxiliary equation is $m^3 - 1 = 0$

$\Rightarrow$ $m = 1, \omega, \omega^2$

where ω is one of the cube roots of unity.

$\therefore$ C.F. $= f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$

$$\text{P.I.} = \frac{1}{D^3 - D'^3}(x^3y^3) = \frac{1}{D^3\left(1 - \frac{D'^3}{D^3}\right)}(x^3y^3)$$

$$= \frac{1}{D^3}\left(1 - \frac{D'^3}{D^3}\right)^{-1}(x^3y^3) = \frac{1}{D^3}\left(1 + \frac{D'^3}{D^3}\right)(x^3y^3)$$

$$= \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^3}D'^3(x^3y^3)\right] = \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^3}(6x^3)\right]$$

$$= \frac{1}{D^3}(x^3y^3) + \frac{1}{D^6}(6x^3) = \frac{x^6y^3}{6\cdot5\cdot4} + \frac{6x^9}{4\cdot5\cdot6\cdot7\cdot8\cdot9} = \frac{x^6y^3}{120} + \frac{x^9}{10080}$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y + \omega x) + f_3(y + \omega^2 x) + \frac{x^6y^3}{120} + \frac{x^9}{10080}.$$

where f_1, f_2 and f_3 are arbitrary functions.

Example 2. *Find a real function V of x and y, reducing to zero when y = 0 and satisfying*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi(x^2 + y^2).$$

Sol. The given equation is $(D^2 + D'^2)\,V = -4\pi(x^2 + y^2)$

The auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow$ $m = \pm i$

$\therefore$ C.F. $= f_1(y + ix) + f_2(y - ix)$, which is imaginary.

Now,
$$\text{P.I.} = \frac{1}{D^2 + D'^2}[-4\pi(x^2 + y^2)] = -\frac{4\pi}{D^2}\left(1 + \frac{D'^2}{D^2}\right)^{-1}(x^2 + y^2)$$

$$= -\frac{4\pi}{D^2}\left(1 - \frac{D'^2}{D^2}\right)(x^2 + y^2) = -\frac{4\pi}{D^2}\left[x^2 + y^2 - \frac{1}{D^2}(2)\right]$$

$$= -\frac{4\pi}{D^2}\{x^2 + y^2 - x^2\} = -\frac{4\pi}{D^2}(y^2)$$

$$= -\frac{4\pi}{D}(xy^2) = -4\pi\left(\frac{x^2}{2}y^2\right)$$

$$\text{P.I.} = -2\pi x^2y^2$$

Since V is a real function, hence $V = -2\pi x^2y^2$.

Example 3. *Solve:* $(D^2 - 6DD' + 9D'^2)\,z = 12x^2 + 36xy$.

Sol. Auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$\Rightarrow$ $m = 3, 3$

$\therefore$ C.F. $= f_1(y + 3x) + x f_2(y + 3x)$

$$\text{P.I.} = \frac{1}{D^2 - 6DD' + 9D'^2}(12x^2) + \frac{1}{D^2 - 6DD' + 9D'^2}(36xy) = P_1 + P_2$$

where
$$P_1 = \frac{1}{D^2 - 6DD' + 9D'^2}(12x^2)$$

$$= \frac{12}{(1)^2 - 6(1)(0) + 9(0)^2} \iint u^2 \, du \, du, \qquad \text{where } x = u$$

$$= 12 \frac{u^4}{12} = u^4 = x^4$$

$$P_2 = \frac{1}{D^2 - 6DD' + 9D'^2} (36xy)$$

$$= \frac{36}{(D - 3D')^2}(xy) = \frac{36}{D^2}\left(1 - \frac{3D'}{D}\right)^{-2} (xy)$$

$$= \frac{36}{D^2}\left(1 + \frac{6D'}{D}\right)(xy) \qquad \text{| Leaving higher power terms}$$

$$= \frac{36}{D^2}\left[xy + \frac{6}{D}(x)\right] = \frac{36}{D^2}(xy + 3x^2)$$

$$= 36\left(\frac{x^3}{6} y + \frac{x^4}{4}\right) = 6x^3y + 9x^4$$

$$\therefore \qquad \text{P.I.} = P_1 + P_2 = 6x^3y + 10x^4$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + 3x) + xf_2(y + 3x) + 6x^3y + 10x^4$$

where f_1 and f_2 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve:

1. $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 12\,xy$ *(U.P.T.U. 2015)*

2. $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$

3. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$

4. $(D^2 - a^2 D'^2)z = x$

5. $[D^2 + (a + b) DD' + ab D'^2]z = xy$ *[G.B.T.U. (AG) 2011, G.B.T.U. 2013]*

6. $(D^2 - 2DD' + D'^2)z = e^{x + 2y} + x^3$.

7. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} - 15\dfrac{\partial^2 z}{\partial y^2} = 12xy$ *(U.P.T.U. 2013)*

Answers

1. $z = f_1(y - x) + f_2(y - 2x) + 2x^3y - \dfrac{3}{2}x^4$

2. $z = f_1(y - x) + x f_2(y - x) + \dfrac{1}{4}(x^4 - 2x^3y + 2x^2y^2)$

3. $z = f_1(y) + x f_2(y) + f_3(y + 2x) + \dfrac{1}{4}e^{2x} + \dfrac{1}{60}(3x^5y + x^6)$

4. $z = f_1(y + ax) + f_2(y - ax) + \dfrac{x^3}{6}$

5. $z = f_1(y - ax) + f_2(y - bx) + \dfrac{1}{6}x^3y - \left(\dfrac{a + b}{24}\right)x^4$

6. $z = f_1(y + x) + x f_2(y + x) + e^{x + 2y} + \dfrac{x^5}{20}$

7. $z = f_1(y - 3x) + f_2(y + 5x) + 2x^3y + x^4$.

1.21 GENERAL METHOD TO FIND THE P.I.

$\phi(x, y)$ is not always of the form given above. The general method is applicable to all cases, where $\phi(x, y)$ is not of the form given above.

Now, F(D, D′) can be factorised, in general, into n linear factors.

$$\therefore \quad \text{P.I.} = \frac{1}{F(D, D')}\phi(x, y)$$

$$= \frac{1}{(D - m_1D')(D - m_2D')\dots(D - m_nD')}\phi(x, y)$$

$$= \frac{1}{D - m_1D'} \cdot \frac{1}{D - m_2D'} \cdots\cdots \frac{1}{D - m_nD'}\phi(x, y)$$

Let us evaluate $\dfrac{1}{D - mD'}\phi(x, y)$

Consider the equation,

$$(D - mD')z = \phi(x, y) \quad \text{or} \quad p - mq = \phi(x, y) \qquad \text{[Lagrange's form]}$$

The subsidiary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{\phi(x, y)}$

From the first two members $dy + mdx = 0$ or $y + mx = c$

From the first and last members, we have

$$dz = \phi(x, y)dx = \phi(x, c - mx)dx$$

$$\therefore \quad z = \int \phi(x, c - mx)dx$$

or $$\boxed{\frac{1}{D - mD'}\phi(x, y) = \int \phi(x, c - mx)\,dx}$$

where c is replaced by $y + mx$ after integration.

By repeated application of the above rule, the P.I. can be evaluated.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linear partial differential equation*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x.$$

Sol. The given equation is

$$(D^2 + DD' - 6D'^2)z = y \cos x$$

The auxiliary equation is

$$m^2 + m - 6 = 0$$

$$\Rightarrow \quad (m - 2)(m + 3) = 0$$

$$\Rightarrow \quad m = 2, -3.$$

$$\therefore \quad \text{C.F.} = f_1(y + 2x) + f_2(y - 3x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\, y \cos x = \frac{1}{(D-2D')(D+3D')}\, y \cos x$$

$$= \frac{1}{D-2D'} \int (c + 3x) \cos x \, dx, \quad \text{where } y = c + 3x$$

$$= \frac{1}{D-2D'} \left\{c \sin x + 3 \int x \cos x \, dx\right\}$$

$$= \frac{1}{D-2D'} \left[c \sin x + 3 \left\{x \sin x - \int 1 \,.\, \sin x \, dx\right\}\right]$$

$$= \frac{1}{D-2D'} [(c + 3x) \sin x + 3 \cos x]$$

$$= \frac{1}{D-2D'} (y \sin x + 3 \cos x), \quad \text{where } c = y - 3x$$

$$= \int (b - 2x) \sin x \, dx + 3 \sin x, \quad \text{where } y = b - 2x$$

$$= -b \cos x - 2 \left\{x(-\cos x) - \int 1 \,.\, (-\cos x) \, dx\right\} + 3 \sin x$$

$$= -b \cos x + 2x \cos x - 2 \sin x + 3 \sin x$$

$$= -(b - 2x) \cos x + \sin x$$

$$= -y \cos x + \sin x, \quad \text{where } b = y + 2x$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + 2x) + f_2(y - 3x) - y \cos x + \sin x$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve the linear partial differential equation*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y - 1)\, e^x.$$

Sol. The given equation is

$$(D^2 + DD' - 2D'^2)\, z = (y - 1)\, e^x$$

The auxiliary equation is

$$m^2 + m - 2 = 0$$

$$\Rightarrow \quad (m - 1)(m + 2) = 0$$

$$\Rightarrow \quad m = 1, -2$$

$$\text{C.F.} = f_1(y + x) + f_2(y - 2x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 2D'^2} (y - 1)\, e^x$$

$$= \frac{1}{(D - D')(D + 2D')} (y - 1)\, e^x$$

$$= \frac{1}{D - D'} \int (c + 2x - 1)\, e^x \, dx, \quad \text{where } y = c + 2x$$

$$= \frac{1}{D - D'} [(c - 1)\, e^x + 2(x - 1)\, e^x]$$

$$= \frac{1}{D - D'} [(c + 2x)\, e^x - 3e^x]$$

$$= \frac{1}{D - D'} (ye^x - 3e^x), \qquad \text{where } c = y - 2x$$

$$= \int (b - x) e^x \, dx - 3e^x, \qquad \text{where } y = b - x$$

$$= be^x - (x - 1) e^x - 3e^x$$

$$= (b - x - 2) e^x = (y - 2) e^x, \qquad \text{where } b = y + x$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y + x) + f_2 (y - 2x) + (y - 2) e^x$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve the partial differential equation:*

$$r - t = \tan^3 x \tan y - \tan x \tan^3 y.$$

Sol. The given equation is

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \tan x \tan y (\sec^2 x - \sec^2 y).$$

Auxiliary equation is $\quad m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1.$

$$\text{C.F.} = f_1 (y + x) + f_2 (y - x)$$

$$\text{P.I.} = \frac{1}{D^2 - D'^2} \tan x \tan y (\sec^2 x - \sec^2 y)$$

$$= \frac{1}{D + D'} \left[\frac{1}{D - D'} (\tan x \sec^2 x \tan y - \tan x \tan y \sec^2 y) \right]$$

$$= \frac{1}{D + D'} [\textstyle\int \tan x \sec^2 x \tan (c - x) \, dx - \int \tan x \tan (c - x) \sec^2 (c - x) \, dx]$$

where $y = c - x$

$$= \frac{1}{D + D'} \left[\tan (c - x) \frac{\tan^2 x}{2} + \int \sec^2 (c - x) \frac{\tan^2 x}{2} \, dx + \tan x \,.\, \frac{\tan^2 (c - x)}{2} - \int \sec^2 x \frac{\tan^2 (c - x)}{2} \, dx \right]$$

$$= \frac{1}{2(D + D')} [\tan (c - x) \tan^2 x + \tan x \tan^2 (c - x] + \textstyle\int \sec^2 (c - x) (\sec^2 x - 1) \, dx - \int \sec^2 x \{\sec^2 (c - x) - 1\} \, dx]$$

$$= \frac{1}{2(D + D')} [\tan (c - x) \tan^2 x + \tan x \tan^2 (c - x) + \textstyle\int \{\sec^2 x - \sec^2 (c - x)\} \, dx]$$

$$= \frac{1}{2(D + D')} [\tan (c - x) \tan^2 x + \tan x \tan^2 (c - x) + \tan x + \tan (c - x)]$$

$$= \frac{1}{2(D + D')} [\tan x \sec^2 (c - x) + \tan (c - x) \sec^2 x]$$

$$= \frac{1}{2(D + D')} (\tan x \sec^2 y + \tan y \sec^2 x) \quad \text{where } c = y + x$$

$$= \frac{1}{2} [\int \tan x \sec^2 (b + x)\, dx + \int \tan (b + x) \sec^2 x\, dx], \quad \text{where } y = b + x$$

$$= \frac{1}{2} [\tan x \,.\, \tan (b + x) - \int \sec^2 x \,.\, \tan (b + x)\, dx + \int \tan (b + x) \sec^2 x\, dx]$$

$$= \frac{1}{2} \tan x \tan (b + x) = \frac{1}{2} \tan x \tan y \quad \text{where } b = y - x.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y + x) + f_2 (y - x) + \frac{1}{2} \tan x \tan y$$

where f_1 and f_2 are arbitrary functions.

Example 4. *Solve:* $r - s - 2t = (2x^2 + xy - y^2) \sin xy - \cos xy$.

Sol. The given equation is

$$(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2) \sin xy - \cos xy$$

Auxiliary equation is $\qquad m^2 - m - 2 = 0$

$\Rightarrow \qquad (m + 1)(m - 2) = 0 \quad \Rightarrow \quad m = -1, 2$

$\therefore \qquad \text{C.F.} = f_1(y - x) + f_2(y + 2x)$

$$\text{P.I.} = \frac{1}{(D + D')(D - 2D')} [(2x - y)(x + y) \sin xy - \cos xy]$$

$$= \frac{1}{(D + D')} \int [(4x - c)(c - x) \sin (cx - 2x^2) - \cos (cx - 2x^2)]\, dx, \text{ where } y = c - 2x$$

$$= \frac{1}{(D + D')} \int [(x - c)(c - 4x) \sin (cx - 2x^2) - \cos (cx - 2x^2)]\, dx$$

$$= \frac{1}{(D + D')} \left[(x - c)\{- \cos (cx - 2x^2)\} + \int \cos (cx - 2x^2)\, dx - \int \cos (cx - 2x^2)\, dx \right]$$

$$= \frac{1}{(D + D')} (c - x) \cos \{x (c - 2x)\}$$

$$= \frac{1}{D + D'} (y + x) \cos xy, \qquad \text{where } c = y + 2x$$

$$= \int (b + 2x) \cos (bx + x^2)\, dx, \qquad \text{where } y = b + x$$

$$= \sin (bx + x^2) = \sin xy, \qquad \text{where } b = y - x$$

$\therefore$ Complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y - x) + f_2(y + 2x) + \sin xy$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Solve:* $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y$.

Sol. Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$\Rightarrow \qquad m = -1, -1$

$\therefore \qquad \text{C.F.} = f_1 (y - x) + x f_2 (y - x)$

$$\text{P.I.} = \frac{1}{(D+D')^2} 2\cos y - \frac{1}{(D+D')^2}(x \sin y) = P_1 - P_2$$

where $$P_1 = \frac{1}{(D+D')^2} 2\cos y = \frac{2}{(0+1)^2} \iint \cos u \, du \, du, \text{ where } y = u$$

$$= 2(-\cos y) = -2\cos y$$

$$P_2 = \frac{1}{(D+D')^2} x \sin y = \frac{1}{D+D'} \int x \sin(c+x)\, dx, \text{ where } y = c + x$$

$$= \frac{1}{D+D'}\left[x\{-\cos(c+x)\} - \int 1.\{-\cos(c+x)\}\, dx\right]$$

$$= \frac{1}{D+D'}[-x\cos(c+x) + \sin(c+x)]$$

$$= \frac{1}{D+D'}(-x\cos y + \sin y), \quad \text{where } c = y - x$$

$$= \int -x\cos(b+x)\, dx + \int \sin(b+x)\, dx, \quad \text{where } y = b + x$$

$$= (-x).\sin(b+x) - \int (-1).\sin(b+x)\, dx + \int \sin(b+x)\, dx$$

$$= -x\sin(b+x) - 2\cos(b+x)$$

$$= -x\sin y - 2\cos y, \quad \text{where } b = y - x$$

$\therefore$ P.I. $= P_1 - P_2 = -2\cos y + x\sin y + 2\cos y = x \sin y$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y-x) + x f_2(y-x) + x \sin y$$

where f_1 and f_2 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve:

1. $(D^2 - DD' - 2D'^2)z = (y-1)e^x$ *(U.P.T.U. 2014)*
2. $(D - D')(D + 2D')z = (y+1)e^x$
3. $(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = (y+2)e^x$
4. $r - 4t = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$
5. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$
6. $(D^2 + DD' - 6D'^2)z = y \sin x$ solve in Tute

[*M.T.U. (SUM) 2011; M.T.U. 2012*]

Answers

1. $z = f_1(y-x) + f_2(y+2x) + ye^x$
2. $z = f_1(y+x) + f_2(y-2x) + ye^x$
3. $z = f_1(y+x) + f_2(y-x) + f_3(y-2x) + ye^x$
4. $z = f_1(y+2x) + f_2(y-2x) + x\log y + y\log x + 3x$
5. $z = f_1(y-x) + x f_2(y-x) + f_3(y+x) + \dfrac{e^x}{25}(\cos 2y + 2\sin 2y)$
6. $z = f_1(y+2x) + f_2(y-3x) - y\sin x - \cos x.$

1.22 NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In the equation $\phi(D, D')z = F(x, y)$ ...(1)

if the polynomial $\phi(D, D')$ in D, D′ is not homogeneous, then (1) is called a non-homogeneous linear partial differential equation.

Its complete solution is $z = \text{C.F.} + \text{P.I.}$

1.23 METHODS FOR FINDING OUT C.F.

(*a*) We resolve $\phi(D, D')$ into linear factors of the form $\mathbf{D - mD' - a}$.

Now consider the equation $(D - mD' - a)z = 0$ ...(2)

or $$p - mq = az$$

Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}$$

From the first two members $dy + mdx = 0 \quad \therefore \quad y + mx = b$

From the first and last members $\dfrac{dz}{z} = adx \quad \therefore \quad \log z = ax + \log c$ or $z = ce^{ax}$

$\therefore$ The complete solution of (2) is $z = e^{ax} f(y + mx)$

Hence the C.F. of (1), *i.e.*, the complete solution of

$$(D - m_1D' - a_1)(D - m_2D' - a_2)......(D - m_nD' - a_n)z = 0 \text{ is}$$

$$z = e^{a_1x} f_1(y + m_1x) + e^{a_2x} f_2(y + m_2x) + ... + e^{a_nx} f_n(y + m_nx).$$

1.23.1. In the Case of Repeated Factors, *e.g.*, $(\mathbf{D - mD' - a})^3 z = 0$

We have $z = e^{ax} f_1(y + mx) + xe^{ax} f_2(y + mx) + x^2e^{ax} f_3(y + mx)$.

1.23.2. If the Equation is of the Form

$$(\alpha D + \beta D' + \gamma)z = 0 \qquad ...(3)$$

$$\Rightarrow \qquad \alpha p + \beta q = -\gamma z$$

It is of Lagrange's form.

Lagrange's subsidiary equations are

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{-\gamma z} \qquad ...(4)$$

First two will give $\alpha y - \beta x = c_1$ ...(5)

First and last will give, $\dfrac{dz}{z} = -\dfrac{\gamma}{\alpha} dx$

Integration gives, $\log z = -\dfrac{\gamma}{\alpha} x + \log c_2$

$$\Rightarrow \qquad z = c_2 e^{-\frac{\gamma}{\alpha}x} = \phi(c_1) e^{-\frac{\gamma}{\alpha}x}$$

$$\Rightarrow \qquad z = e^{-\frac{\gamma}{\alpha}x}\, \phi(\alpha y - \beta x)$$

where ϕ is an arbitrary function.

Note. The above result is not applicable in the absence of the first term *i.e.,* D or αD and also when $\alpha = 0$.

Remark 1. Corresponding to each non-repeated factor $(a\mathrm{D}' + b)$, the part of C.F. is $e^{-(by/a)}\, \phi(ax)$, when $a \neq 0$.

Remark 2. Corresponding to repeated factor $(a\mathrm{D}' + b)^r$, the part of C.F. is

$$e^{-\left(\frac{by}{a}\right)}\, [\phi_1(ax) + y\, \phi_2(ax) + y^2\, \phi_3(ax) + \ldots\ldots + y^{r-1}\, \phi_r(ax)].$$

Remark 3. As a particular case of remark 1 with $b = 0$, $a = 1$, corresponding to non-repeated factor D′, the part of C.F. $= \phi_1(x)$.

Remark 4. As a particular case of remark 2 with $b = 0$, $a = 1$, corresponding to repeated factor D'^r, part of C.F. $= \phi_1(x) + y\, \phi_2(x) + y^2\, \phi_3(x) + \ldots\ldots + y^{r-1}\, \phi_r(x)$

(*b*) **When F(D, D′) cannot be factorized into linear factors:** .

In such cases, we use a trial method.

Let the equation be $\qquad (\mathrm{D} - \mathrm{D}'^2)z = 0 \qquad$ ...(1)

Let the trial solution of (1) be $\quad z = \mathrm{A}e^{hx+ky}$, $\quad$ where A, h and k are constants. $\qquad$ ...(2)

From (2),

$$\mathrm{D}z = \frac{\partial z}{\partial x} = \mathrm{A}he^{hx+ky}$$

$$\mathrm{D}'z = \frac{\partial z}{\partial y} = \mathrm{A}\, ke^{hx+ky}$$

$$\mathrm{D}'^2z = \frac{\partial^2 z}{\partial y^2} = \mathrm{A}\, k^2\, e^{hx+ky}$$

Putting in (1), we get $\mathrm{A}he^{hx+ky} - \mathrm{A}\, k^2\, e^{hx+ky} = 0$

$$\Rightarrow \qquad \mathrm{A}(h - k^2)\, e^{hx+ky} = 0$$

or

$$h = k^2 \qquad \text{...(3)}$$

Equation (2) gives, $\qquad z = \mathrm{A}e^{k^2x + ky} \qquad$ ...(4)

Since all values of k will satisfy eqn. (1), a more general solution of (1) is given by

$$z = \Sigma\, \mathrm{A}e^{k^2x + ky}$$

where A and k are arbitrary constants and Σ denotes that any number of terms may be taken.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linear partial differential equation*

$$(D + D' - 1)\, (D + 2D' - 2)z = 0.$$

Sol. The given equation is

$$(\mathrm{D} + \mathrm{D}' - 1)\, (\mathrm{D} + 2\mathrm{D}' - 2)z = 0.$$

Its $\quad$ C.F. $= e^x f_1\, (y - x) + e^{2x} f_2\, (y - 2x)$

and $\quad$ P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{x} f_1 (y - x) + e^{2x} f_2 (y - 2x)$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve: $DD' (D + 2D' + 1)z = 0$.*

Sol. The given equation is

$$DD'(D + 2D' + 1)z = 0$$

Corresponding to the factor D, part of C.F. $= f_1(y)$

Corresponding to the factor D′, part of C.F. $= f_2(x)$

Corresponding to the factor $(D + 2D' + 1)$ part of C.F. $= e^{-x} f_3 (y - 2x)$

Hence combined C.F. $= f_1(y) + f_2(x) + e^{-x} f_3 (y - 2x)$

P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(x) + e^{-x} f_3 (y - 2x)$$

where f_1, f_2 and f_3 are arbitrary functions.

Example 3. *Solve: $r + 2s + t + 2p + 2q + z = 0$.*

Sol. The given equation is

$$[D^2 + 2DD' + D'^2 + 2D + 2D' + 1]z = 0$$

$$\Rightarrow \quad \{(D + D')^2 + 2(D + D') + 1\}z = 0$$

$$\Rightarrow \quad (D + D' + 1)^2 z = 0$$

Its C.F. $= e^{-x} f_1 (y - x) + x\, e^{-x} f_2 (y - x)$

P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{-x} f_1 (y - x) + x\, e^{-x} f_2 (y - x)$$

where f_1 and f_2 are arbitrary functions.

Example 4. *Solve:*

(*i*) $r - t + p - q = 0$ (U.P.T.U. 2013, 2014)

(*ii*) $(D + 4D' + 5)^2 z = 0$ [A.K.T.U. 2018]

(*iii*) $(D^2 - DD' - 2D)z = 0$

(*iv*) $(D + 1) (D + D' - 1)z = 0$.

Sol. (*i*) The given equation is

$$(D^2 - D'^2 + D - D')z = 0$$

$$\Rightarrow \quad (D - D') (D + D' + 1)z = 0$$

Its C.F. $= f_1 (y + x) + e^{-x} f_2 (y - x)$

P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y + x) + e^{-x} f_2 (y - x)$$

where f_1 and f_2 are arbitrary functions.

(*ii*) The given equation is

$$(D + 4D' + 5)^2 z = 0$$

Its C.F. $= e^{-5x} f_1 (y - 4x) + x\, e^{-5x} f_2 (y - 4x)$

P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{-5x} f_1(y - 4x) + x\, e^{-5x} f_2(y - 4x)$$

where f_1 and f_2 are arbitrary functions.

(*iii*) The given equation is

$$(D^2 - DD' - 2D)z = 0$$

$$\Rightarrow \qquad D(D - D' - 2)z = 0$$

Its $\quad$ C.F. $= f_1(y) + e^{2x} f_2(y + x)$

P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + e^{2x} f_2(y + x)$$

where f_1 and f_2 are arbitrary functions.

(*iv*) The given equation is

$$(D + 1)(D + D' - 1)z = 0$$

Its $\quad$ C.F. $= e^{-x} f_1(y) + e^{x} f_2(y - x)$

P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{-x} f_1(y) + e^{x} f_2(y - x)$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Solve:* (*i*) $(D^2 - D'^2 + D + 3D' - 2)z = 0$

(*ii*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0.$

Sol. (*i*) The given equation is

$$(D^2 - D'^2 + D + 3D' - 2)z = 0$$

$$\Rightarrow \qquad (D^2 - D'^2 + 2D - D + 2D' + D' - 2 + DD' - DD')z = 0$$

$$\Rightarrow \qquad [D^2 - DD' + 2D + DD' - D'^2 + 2D' - D + D' - 2]z = 0$$

$$\Rightarrow \qquad (D - D' + 2)(D + D' - 1)z = 0$$

Its $\quad$ C.F. $= e^{-2x} f_1(y + x) + e^{x} f_2(y - x)$

P.I. $= 0$

Hence complete solution is $\quad z = \text{C.F.} + \text{P.I.} = e^{-2x} f_1(y + x) + e^{x} f_2(y - x)$

where f_1 and f_2 are arbitrary functions.

(*ii*) The given equation is

$$(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$$

$$\Rightarrow \qquad (D + D')(D - 2D') + 2(D + D')]z = 0$$

$$\Rightarrow \qquad (D + D')(D - 2D' + 2)z = 0$$

Its $\quad$ C.F. $= f_1(y - x) + e^{-2x} f_2(y + 2x)$

P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y - x) + e^{-2x} f_2(y + 2x)$$

where f_1 and f_2 are arbitrary functions.

Example 6. *Solve:* $(D^2 + D'^2 - p^2)z = 0.$

Sol. Here $D^2 + D'^2 - p^2$ cannot be resolved into linear factors in D and D′.

Let $z = Ae^{hx+ky}$

$\therefore$ $D^2z = Ah^2 e^{hx+ky}$

$D'^2z = Ak^2 e^{hx+ky}$

$\therefore$ $(D^2 + D'^2 - p^2)z = A(h^2 + k^2 - p^2) e^{hx+ky}$

Then, $(D^2 + D'^2 - p^2)z = 0$

$\Rightarrow$ $A(h^2 + k^2 - p^2) e^{hx+ky} = 0$

$\Rightarrow$ $h^2 + k^2 - p^2 = 0$ or $h^2 + k^2 = p^2$

$\therefore$ C.F. $= \Sigma Ae^{hx+ky}$, where $h^2 + k^2 - p^2 = 0$

P.I. $= 0$

$\therefore$ $z = \Sigma Ae^{hx+ky}$, where $h^2 + k^2 - p^2 = 0$

Now, h may be taken as $p \cos \alpha$ and k may be taken as $p \sin \alpha$. Therefore
The complete solution is

$$z = \Sigma Ae^{p(x \cos \alpha + y \sin \alpha)}$$

where A is arbitrary constant and Σ denotes that any number of terms may be taken.

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $(D + D' - 1)(D + 2D' - 3)z = 0$

2. $r - 3s + 2t - p + 2q = 0$

3. $DD'(D - 2D' - 3)z = 0$

4. $t + s + q = 0$

5. $(D^2 - a^2D'^2 + 2abD + 2a^2bD')z = 0$

6. $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$

7. $(2D^4 - 3D^2D' + D'^2)z = 0$

8. $(D^3 - 3DD' + D' + 4)z = 0$

9. $2s + t - 3q = 0$

10. $(DD' + aD + bD' + ab)z = 0$.

Answers

1. $z = e^x f_1(y - x) + e^{3x} f_2(y - 2x)$

2. $z = f_1(y + 2x) + e^x f_2(y + x)$

3. $z = f_1(y) + f_2(x) + e^{3x} f_3(y + 2x)$

4. $z = f_1(x) + e^{-x} f_2(y - x)$

5. $z = f_1(y - ax) + e^{-2abx} f_2(y + ax)$

6. $z = e^x f_1(y + 2x) + \Sigma Ae^{(2k^2+1)x+ky}$

7. $z = \Sigma Ae^{hx+h^2y} + \Sigma A'e^{h'x+h'^2y}$

8. $z = \Sigma A e^{hx+ky}$, where $h^3 - 3hk + k + 4 = 0$

9. $z = f_1(y) + e^{(3x/2)} f_2(2y - x)$.

10. $z = e^{-ay} f_1(x) + e^{-bx} f_2(y)$

1.24 P.I. OF NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Let the given equation be $\phi(D, D')z = F(x, y)$

then
$$\text{P.I.} = \frac{1}{\phi(D, D')} F(x, y)$$

The methods of finding out P.I. of these equations quite resemble to those of ordinary linear differential equation with constant coefficients.

1.24.1. Case I. *When* $F(x, y) = e^{ax+by}$ *and* $\phi(a, b) \neq 0$

$$\text{P.I.} = \frac{1}{\phi(D, D')} e^{ax+by}$$

$$= \frac{1}{\phi(a, b)} e^{ax+by} \quad | \text{ Replacing D by } a \text{ and D}' \text{ by } b$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $s + ap + bq + abz = e^{mx + ny}$.

Sol. The given equation is

$$(DD' + aD + bD' + ab)z = e^{mx + ny}$$

$$\Rightarrow \quad (D + b)(D' + a)z = e^{mx + ny}$$

Its $\quad \text{C.F.} = e^{-bx} f_1(y) + e^{-ay} f_2(x)$

$$\text{P.I.} = \frac{1}{(D+b)(D'+a)} e^{mx + ny} = \frac{e^{mx + ny}}{(m+b)(n+a)}$$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{-bx} f_1(y) + e^{-ay} f_2(x) + \frac{e^{mx + ny}}{(m+b)(n+a)}$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve:* $(D^3 - 3DD' + D' + 4)z = e^{2x + y}$.

Sol. Here $D^3 - 3DD' + D' + 4$ cannot be resolved into linear factors in D and D′.

Let $\quad z = Ae^{hx + ky}$

$\therefore \quad (D^3 - 3DD' + D' + 4)z = A(h^3 - 3hk + k + 4)\, e^{hx+ky}$

Then $\quad (D^3 - 3DD' + D' + 4)z = 0$ iff $h^3 - 3hk + k + 4 = 0$

$\therefore$ The C.F. is $z = \Sigma\, Ae^{hx + ky}$, where $h^3 - 3hk + k + 4 = 0$

$$\text{P.I.} = \frac{1}{D^3 - 3DD' + D' + 4} e^{2x + y} = \frac{e^{2x + y}}{2^3 - 3(2)(1) + 1 + 4} = \frac{1}{7} e^{2x + y}$$

Hence complete solution is $z = \Sigma\, Ae^{hx + ky} + \frac{1}{7} e^{2x + y}$, where $h^3 - 3hk + k + 4 = 0$.

Example 3. *Solve:* $D(D - 2D' - 3)z = e^{x + 2y}$.

Sol. $\quad \text{C.F.} = f_1(y) + e^{3x} f_2(y + 2x)$

$$\text{P.I.} = \frac{1}{D(D - 2D' - 3)} e^{x+2y} = \frac{e^{x + 2y}}{1\{1 - 2(2) - 3\}} = -\frac{1}{6} e^{x+2y}$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + e^{3x} f_2(y + 2x) - \frac{1}{6} e^{x + 2y}$$

where f_1 and f_2 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve:

1. $(D^3 - 3DD' + D + 1)z = e^{2x+3y}$
2. $(D^2 - 4DD' + D - 1)z = e^{3x-2y}$
3. $(D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$
4. $(D - D' - 1)(D + D' - 2)z = e^{2x-y}$
5. $(D^2 - D'^2 + D - D')z = e^{2x+3y}$
6. $(D^2 - 4DD' + 4D'^2 - D + 2D')z = e^{3x+4y}$. *(U.P.T.U. 2013)*

Answers

1. $z = \Sigma\, Ae^{hx+ky} - \frac{1}{7}e^{2x+3y}$, where $h^3 - 3hk + h + 1 = 0$
2. $z = \Sigma\, Ae^{hx+ky} + \frac{1}{35}\, e^{3x-2y}$, where $h^2 - 4hk + h - 1 = 0$
3. $z = f_1(y + x) + e^{3x} f_2(y - x) - \frac{1}{12}e^{x-2y}$
4. $z = e^x f_1(y + x) + e^{2x} f_2(y - x) - \frac{1}{2}e^{2x-y}$
5. $z = f_1(y + x) + e^{-x} f_2(y - x) - \frac{1}{6}\, e^{2x+3y}$
6. $z = f_1(y + 2x) + e^x f_2(y + 2x) + \frac{1}{30}\, e^{3x+4y}$.

1.24.2. Case II. *When $F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$*

$$\text{P.I.} = \frac{1}{\phi(D, D')}\{\sin(ax + by) \text{ or } \cos(ax + by)\}$$

$$= \frac{1}{\phi(D^2, DD', D'^2)}\{\sin(ax + by) \text{ or } \cos(ax + by)\}$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)}\{\sin(ax + by) \text{ or } \cos(ax + by)\}$$

where $\phi(-a^2, -ab, -b^2) \neq 0$. It is to be noted that here D^2 is replaced by $-a^2$, DD' is replaced by $-ab$ and D'^2 is replaced by $-b^2$.

If $\phi(D, D') = \phi(D^2, DD', D'^2, D, D')$, then

$$\text{P.I.} = \frac{1}{\phi(-a^2, -ab, -b^2, D, D')}\{\sin(ax + by) \text{ or } \cos(ax + by)\}$$

which can be evaluated further by operating N^r and D^r by the suitable conjugate operator.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve: $(D^2 - DD' + D' - 1)z = \sin(x + 2y)$.*

Sol. The given equation is $(D^2 - DD' + D' - 1)z = \sin(x + 2y)$

$$\Rightarrow \quad \{(D + 1)(D - 1) - D'(D - 1)\}z = \sin(x + 2y)$$

$$\Rightarrow \quad (D - 1)(D - D' + 1)z = \sin(x + 2y)$$

$$\therefore \quad \text{C.F.} = e^x f_1(y) + e^{-x} f_2(y + x)$$

$$\text{P.I.} = \frac{1}{D^2 - DD' + D' - 1} \sin(x+2y) = \frac{1}{-1+2+D'-1} \sin(x+2y)$$

$$= \frac{1}{D'} \sin(x+2y) = -\frac{\cos(x+2y)}{2}$$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y) + e^{-x} f_2(y+x) - \frac{\cos(x+2y)}{2}$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve : $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$.* (U.K.T.U. 2011)

Sol. C.F. $= e^x f_1(y+x) + e^{2x} f_2(y+x)$

$$\text{P.I.} = \frac{1}{(D-D'-1)(D-D'-2)} \sin(2x+3y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} \sin(2x+3y)$$

$$= \frac{1}{-4+12-9-3D+3D'+2} \sin(2x+3y)$$

$$= \frac{1}{-3D+3D'+1} \sin(2x+3y)$$

$$= -\left[\frac{(3D-3D')+1}{\{(3D-3D')+1\}\{3D-3D'-1\}} \sin(2x+3y)\right]$$

$$= -\left[\frac{(3D-3D')+1}{9D^2+9D'^2-18DD'-1} \sin(2x+3y)\right]$$

$$= -\left[\frac{3D-3D'+1}{-36-81+108-1} \sin(2x+3y)\right]$$

$$= \frac{1}{10}(3D - 3D' + 1) \sin(2x+3y)$$

$$= \frac{1}{10}[6\cos(2x+3y) - 9\cos(2x+3y) + \sin(2x+3y)]$$

$$= \frac{1}{10}[\sin(2x+3y) - 3\cos(2x+3y)]$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{10}[\sin(2x+3y) - 3\cos(2x+3y)]$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Find the particular integral of $2s + t - 3q = 5\cos(3x - 2y)$.*

Sol. The given equation is

$$(2DD' + D'^2 - 3D')z = 5\cos(3x - 2y)$$

$$\text{P.I.} = \frac{1}{2DD' + D'^2 - 3D'} 5\cos(3x - 2y)$$

$$= \frac{1}{2(6) + (-4) - 3D'} 5\cos(3x - 2y) = \frac{1}{(8 - 3D')} 5\cos(3x - 2y)$$

$$= 5\left[\frac{8 + 3D'}{64 - 9D'^2}\cos(3x - 2y)\right] = 5\left[\frac{8 + 3D'}{64 - 9(-4)}\cos(3x - 2y)\right]$$

$$= \frac{1}{20}[8\cos(3x - 2y) + 6\sin(3x - 2y)]$$

$$= \frac{1}{10}[4\cos(3x - 2y) + 3\sin(3x - 2y)].$$

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$
2. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x + y)$ (A.K.T.U. 2016)
3. $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$
4. $(D^2 - DD' - 2D)z = \sin(3x + 4y) - e^{2x+y}$
5. $(D - D'^2)z = \cos(x - 3y)$
6. $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$.

Answers

1. $z = e^x f_1(y) + e^{-x} f_2(y + x) + \frac{1}{2}\sin(x + 2y)$
2. $z = f_1(y - x) + e^{-2x} f_2(2x + y) - \frac{1}{6}\cos(2x + y)$
3. $z = e^{-x} f_1(y) + e^x f_2(y - x) - \frac{1}{10}[2\sin(x + 2y) + \cos(x + 2y)]$
4. $z = f_1(y) + e^{2x} f_2(y + x) + \frac{1}{15}[\sin(3x + 4y) + 2\cos(3x + 4y)] + \frac{1}{2}e^{2x+y}$
5. $z = \Sigma A e^{k^2 x + ky} + \frac{1}{82}[\sin(x - 3y) + 9\cos(x - 3y)]$
6. $z = f_1(y - x) + e^{2x} f_2(y - x) + \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)]$.

1.24.3. Case III. *When $F(x, y) = x^m y^n$ where m and n being positive integers*

$$\text{P.I.} = \frac{1}{\phi(D, D')} x^m y^n = [\phi(D, D')]^{-1}(x^m y^n)$$

which can be evaluated after expanding $[\phi(D, D')]^{-1}$ in ascending powers of $\frac{D'}{D}$ (when $m > n$) or $\frac{D}{D'}$ (when $m < n$) or D or D′ as the case may be.

If a separate constant is present in $\phi(D, D')$ then it should be given preference in taking the term outside the bracket. It is to be noted that if P.I. is obtained by expanding $\phi(D, D')$ in two or more different ways, then difference in P.I. will be immaterial.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linear partial differential equation:*

$$(D - D' - 1)(D - D' - 2)z = e^{3x-y} + x.$$

Sol. $\quad$ C.F. $= e^x f_1(y + x) + e^{2x} f_2(y + x)$

$$\text{P.I.} = \frac{1}{(D - D' - 1)(D - D' - 2)} e^{3x-y} + \frac{1}{(D - D' - 1)(D - D' - 2)}(x) = P_1 + P_2$$

where
$$P_1 = \frac{1}{(D - D' - 1)(D - D' - 2)} e^{3x-y} = \frac{1}{(3 + 1 - 1)(3 + 1 - 2)} e^{3x-y} = \frac{1}{6} e^{3x-y},$$

$$P_2 = \frac{1}{(D - D' - 1)(D - D' - 2)}(x) = \frac{1}{(1 - D + D')(2 - D + D')}(x)$$

$$= \frac{1}{\{1 - (D - D')\}2\left\{1 - \left(\frac{D - D'}{2}\right)\right\}}(x)$$

$$= \frac{1}{2}\left[\{1 - (D - D')\}^{-1}\left\{1 - \left(\frac{D - D'}{2}\right)\right\}^{-1}\right](x)$$

$$= \frac{1}{2}\left[(1 + D - D')\left(1 + \frac{D - D'}{2}\right)\right](x) \qquad \text{| Leaving higher powers}$$

$$= \frac{1}{2}\left[1 + \frac{D}{2} - \frac{D'}{2} + D + \frac{D^2}{2} - \frac{DD'}{2} - D' - \frac{D'D}{2} + \frac{D'^2}{2}\right](x)$$

$$= \frac{1}{2}\left[x + \frac{1}{2} - 0 + 1 + 0 - 0 - 0 - 0 - 0 + 0\right] = \frac{1}{2}\left(x + \frac{3}{2}\right)$$

$\therefore \quad \text{P.I.} = \frac{1}{6} e^{3x-y} + \frac{1}{2}\left(x + \frac{3}{2}\right).$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{6} e^{3x-y} + \frac{1}{2}\left(x + \frac{3}{2}\right)$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve the linear partial differential equation*

$$(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y.$$

Sol. $\text{C.F.} = e^x f_1(y - x) + e^{3x} f_2(y - 2x)$

$$\text{P.I.} = \frac{1}{(D + D' - 1)(D + 2D' - 3)}(4 + 3x + 6y)$$

$$= \frac{1}{(1 - D - D')(3 - D - 2D')}(4 + 3x + 6y)$$

$$= \frac{1}{\{1 - (D + D')\}\, 3 \left\{1 - \left(\frac{D + 2D'}{3}\right)\right\}}(4 + 3x + 6y)$$

$$= \frac{1}{3}\left[\{1 - (D + D')\}^{-1}\left\{1 - \left(\frac{D + 2D'}{3}\right)\right\}^{-1}\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[(1 + D + D')\left(1 + \frac{D}{3} + \frac{2D'}{3}\right)\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[1 + \frac{4D}{3} + \frac{5D'}{3} + \frac{D^2}{3} + DD' + \frac{2D'^2}{3}\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[(4 + 3x + 6y) + \frac{4}{3}(3) + \frac{5}{3}(6) + 0 + 0 + 0\right] = 6 + x + 2y.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y - x) + e^{3x} f_2(y - 2x) + 6 + x + 2y$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve the partial differential equation*

$$(D^2 - D'^2 - 3D + 3D')z = xy + e^{x + 2y}.$$ [G.B.T.U. (C.O.) 2011]

Sol. The given equation is

$$(D^2 - D'^2 - 3D + 3D')z = 0$$

$$(D - D')(D + D' - 3)z = 0$$

$\therefore$ $$\text{C.F.} = f_1(y + x) + e^{3x} f_2(y - x)$$

$$\text{P.I. corresponding to } xy = \frac{1}{(D - D')(D + D' - 3)}(xy)$$

$$= -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 - \frac{D}{3} - \frac{D'}{3}\right)^{-1}(xy)$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\right)\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \ldots\right)(xy)$$

$$= -\frac{1}{3D}\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} + \ldots\right)(xy)$$

$$= -\frac{1}{3D}\left(xy + \frac{y}{3} + \frac{2}{3}x + \frac{x^2}{2} + \frac{2}{9}\right) \qquad | \text{ Leaving higher powers}$$

$$= -\frac{1}{3}\left[\frac{x^2}{2}y + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x\right]$$

P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D-D')(D+D'-3)} e^{x+2y} = \frac{1}{(1-2)(D+D'-3)} e^{x+2y}$$

$$= -e^{x+2y} \cdot \frac{1}{D+1+D'+2-3}(1) = -e^{x+2y} \cdot \frac{1}{D+D'} e^{0x+0y}$$

$$= -e^{x+2y} \cdot x\, e^{0x+0y} = -x\, e^{x+2y}$$

$$\therefore \qquad \text{P.I.} = -\frac{1}{3}\left[\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2x}{9}\right] - x\, e^{x+2y}$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y+x) + e^{3x} f_2(y-x) - \frac{1}{3}\left[\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2x}{9}\right] - x\, e^{x+2y}$$

where f_1 and f_2 are arbitrary funtions.

Example 4. *Solve* $s + p - q = z + xy$.

Sol. The given equation is

$$(DD' + D - D' - 1)z = xy$$

$$\Rightarrow \qquad (D-1)(D'+1)z = xy$$

$$\therefore \qquad \text{C.F.} = e^x f_1(y) + e^{-y} f_2(x)$$

$$\text{P.I.} = \frac{1}{(D-1)(D'+1)}(xy) = -[(1-D)^{-1}(1+D')^{-1}](xy)$$

$$= -[(1 + D + D^2 + \ldots\ldots)(1 - D' + \ldots\ldots)](xy)$$

$$= -(1 + D - D' - DD')\,xy \qquad | \text{ Leaving higher powers}$$

$$= -(xy + y - x - 1) = x + 1 - xy - y$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y) + e^{-y} f_2(x) + x + 1 - xy - y$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Solve:*

(*a*) $r - s + p = 1$ (*b*) $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$.

Sol. (*a*) The given equation is

$$(D^2 - DD' + D)z = 1$$

$$\Rightarrow \qquad D(D - D' + 1)z = 1$$

$$\therefore \qquad \text{C.F} = f_1(y) + e^{-x} f_2(y+x)$$

$$\text{P.I.} = \frac{1}{D(D-D'+1)}(1) = \frac{1}{D}[1+(D-D')]^{-1}(1) = \frac{1}{D}(1) = x$$

$\therefore$ Complete solution is

$$z = \text{C.F} + \text{P.I.} = f_1(y) + e^{-x} f_2(y+x) + x$$

where f_1 and f_2 are arbitrary functions.

(*b*) C.F. $= f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x)$

$$\text{P.I.} = \frac{1}{D(D+D'-1)(D+3D'-2)}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\{1-(D+D')\}^{-1}\left\{1-\frac{D+3D'}{2}\right\}^{-1}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\{1+D+D'+(D+D')^2+\ldots\}.\left\{1+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\ldots\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left[1+\frac{3D}{2}+\frac{5D'}{2}+\frac{7D^2}{4}+\frac{19D'^2}{4}+\frac{11DD'}{2}+\ldots\right](x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left[x^2-4xy+2y^2+3(x-2y)+5(2y-2x)+\frac{7}{2}+19-22\right]$$

$$= \frac{1}{2D}\left(x^2-4xy+2y^2-7x+4y+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{x^3}{3}-2x^2y+2y^2x-\frac{7x^2}{2}+4yx+\frac{x}{2}\right)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x) + \frac{1}{6}x^3 - x^2y + xy^2 - \frac{7}{4}x^2 + 2xy + \frac{x}{4}$$

where f_1, f_2 and f_3 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve:

1. $(D-D'-1)(D-D'-2)z = e^{2x-y} + x$
2. $(D+D'-1)(D+2D'-3)z = 2x+3y$
3. $(D^2-D'-1)z = x^2y$
4. $(D+D'-1)^2 z = xy$ [*G.B.T.U. (AG) 2011*]
5. $(D^2-DD'-2D'^2+2D+2D')z = e^{2x+3y} + \sin(2x+y) + xy.$

Answers

1. $z = e^x f_1(y+x) + e^{2x} f_2(y+x) + \dfrac{e^{2x-y}}{2} + \dfrac{2x+3}{4}$

2. $z = e^x f_1(y-x) + e^{3x} f_2(y-2x) + \dfrac{2}{3}x + y + \dfrac{23}{9}$

3. $z = \Sigma A e^{hx+(h^2-1)y} + x^2 - x^2y - 2y + 4$

4. $z = e^x f_1(y-x) + xe^x f_2(y-x) + xy + 2y + 2x + 6$

5. $z = f_1(y-x) + e^{-2x} f_2(y+2x) - \frac{1}{10} e^{2x+3y} - \frac{1}{6} \cos(2x+y) + \frac{x^2y}{4} - \frac{xy}{4} - \frac{x^3}{12} + \frac{3x^2}{8} - \frac{x}{2}$.

1.24.4. Case IV. When $F(x, y) = e^{ax+by} \cdot V$, where V is a function of x and y

$$\text{P.I.} = \frac{1}{\phi(D, D')} e^{ax+by} \cdot V = e^{ax+by} \cdot \frac{1}{\phi(D+a, D'+b)} V$$

which can be evaluated further by using any one of the previous cases discussed.

Remark. V can be either

(*i*) $e^{a_1x+b_1y}$ (*ii*) $\sin(ax+by)$ or $\cos(ax+by)$

(*iii*) $x^m y^n$ (*iv*) constant (say 1)

or any other function of x and y in some special cases.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve: $(D - 3D' - 2)^3 z = 6e^{2x} \sin(3x+y)$.*

Sol. Here C.F. $= e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x) + x^2e^{2x} f_3(y+3x)$

$$\text{P.I.} = \frac{1}{(D-3D'-2)^3} 6e^{2x} \sin(3x+y)$$

$$= 6e^{2x} \frac{1}{(D+2-3D'-2)^3} \sin(3x+y) = 6e^{2x} \frac{1}{(D-3D')^3} \sin(3x+y)$$

$$= 6e^{2x} \cdot \frac{x^3}{6} \sin(3x+y) = x^3e^{2x} \sin(3x+y)$$

Hence complete solution is

$$z = e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x) + x^2e^{2x} f_3(y+3x) + x^3e^{2x} \sin(3x+y)$$

where f_1, f_2 and f_3 are arbitrary functions.

Example 2. *Solve: $(D - 3D' - 2)^2 z = 2e^{2x} \tan(y+3x)$.*

Sol. C.F. $= e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x)$

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2} 2e^{2x} \tan(y+3x)$$

$$= 2e^{2x} \cdot \frac{1}{(D+2-3D'-2)^2} \tan(y+3x) = 2e^{2x} \cdot \left[\frac{1}{(D-3D')^2} \tan(y+3x)\right]$$

$$= 2e^{2x} \cdot \left[x \cdot \frac{1}{2(D-3D')} \tan(y+3x)\right] = 2e^{2x} \cdot x^2 \cdot \frac{1}{2} \tan(y+3x)$$

$$= x^2 e^{2x} \tan(y+3x)$$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x) + x^2 e^{2x} \tan(y+3x)$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve the linear partial differential equation*

$$(D^2 - DD' + D' - 1)\, z = \cos(x + 2y) + e^y.$$

Sol. The given equation is

$$(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y$$

$$\Rightarrow \quad (D - 1)(D - D' + 1)z = \cos(x + 2y) + e^y$$

$$\text{C.F.} = e^x f_1(y) + e^{-x} f_2(y + x)$$

$$\text{P.I.} = \frac{1}{D^2 - DD' + D' - 1} \cos(x + 2y) + \frac{1}{(D - D' + 1)(D - 1)} e^y = P_1 + P_2$$

where
$$P_1 = \frac{1}{D^2 - DD' + D' - 1} \cos(x + 2y)$$

$$= \frac{1}{-(1)^2 - (-2) + D' - 1} \cos(x + 2y) = \frac{1}{D'} \cos(x + 2y) = \frac{\sin(x + 2y)}{2}$$

and
$$P_2 = \frac{1}{D - D' + 1}\left[\frac{1}{D - 1} e^y\right] = \frac{1}{D - D' + 1}\left[\frac{1}{0 - 1} e^y\right]$$

$$= \frac{1}{D - D' + 1}(-e^y) = -e^y \cdot \frac{1}{(D + 0) - (D' + 1) + 1}(1)$$

$$= -e^y \cdot \frac{1}{D - D'}(1) = -e^y \cdot \frac{1}{D - D'}(e^{0x + 0y}) = -e^y \cdot x \cdot e^{0x + 0y} = -xe^y.$$

$$\therefore \quad \text{P.I.} = \frac{1}{2} \sin(x + 2y) - xe^y.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y) + e^{-x} f_2(y + x) + \frac{1}{2} \sin(x + 2y) - xe^y$$

where f_1 and f_2 are arbitrary functions.

Example 4. *Solve:* $r - 4s + 4t + p - 2q = e^{x+y}$.

Sol. The given equation is

$$(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$$

$$\Rightarrow \quad [(D - 2D')^2 + (D - 2D')]z = e^{x+y}$$

$$\Rightarrow \quad (D - 2D')(D - 2D' + 1)z = e^{x+y}$$

$$\therefore \quad \text{C.F.} = f_1(y + 2x) + e^{-x} f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D - 2D' + 1)} e^{x+y} = \frac{1}{D - 2D' + 1}\left[\frac{1}{D - 2D'} e^{x+y}\right]$$

$$= \frac{1}{D - 2D' + 1}\left[\frac{1}{1 - 2} \int e^u \, du\right] \qquad \text{where } x + y = u$$

$$= -\left[\frac{1}{D-2D'+1}e^{x+y}\right] = -e^{x+y}\cdot\frac{1}{D+1-2(D'+1)+1}(1)$$

$$= -e^{x+y}\left[\frac{1}{D-2D'}(1)\right] = -e^{x+y}\left[\frac{1}{D-2D'}(e^{0x+0y})\right]$$

$$= -xe^{x+y}.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y+2x) + e^{-x} f_2(y+2x) - xe^{x+y}$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Find the particular integral of $(D^2 - D')z = xe^{ax+a^2y}$.*

Sol. $$\text{P.I.} = \frac{1}{D^2 - D'}(xe^{ax+a^2y}) = e^{ax+a^2y}\cdot\frac{1}{(D+a)^2-(D'+a^2)}(x)$$

$$= e^{ax+a^2y}\cdot\frac{1}{D^2+2aD-D'}(x) = e^{ax+a^2y}\cdot\frac{1}{2aD}\left(1+\frac{D}{2a}-\frac{D'}{2aD}\right)^{-1}(x)$$

$$= e^{ax+a^2y}\cdot\frac{1}{2aD}\left\{1-\left(\frac{D}{2a}-\frac{D'}{2aD}\right)+\ldots\right\}x = e^{ax+a^2y}\cdot\frac{1}{2aD}\left(x-\frac{1}{2a}\right)$$

$$= e^{ax+a^2y}\cdot\left(\frac{x^2}{4a}-\frac{x}{4a^2}\right).$$

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x+y)$
2. $(D^2 + DD' + D + D' - 1)z = e^{-2x}(x^2 + y^2)$
3. $(D + D' - 1)(D + D' - 3)(D + D')z = e^{x+y}\sin(2x+y)$
4. $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^x.$

Answers

1. $z = \Sigma\, Ae^{hx+ky} + \frac{4}{3}e^{x+y}\sin(x+y)$, where $3h^2 - 2k^2 + h - 1 = 0$
2. $z = \Sigma\, Ae^{hx+ky} + \frac{1}{27}e^{-2x}(9x^2 + 9y^2 + 18x + 6y + 14)$, where $h^2 + hk + h + k + 1 = 0$
3. $z = e^x f_1(y-x) + e^{3x} f_2(y-x) + f_3(y-x) + \frac{1}{130}[3\cos(2x+y) - 2\sin(2x+y)]\,e^{x+y}$
4. $z = e^x f_1(y) + e^{-x} f_2(y+x) + \frac{1}{2}\sin(x+2y) + \frac{1}{2}xe^x.$

1.25 EQUATIONS REDUCIBLE TO PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation in which the coefficient of derivative of any order say k is a multiple of the variables of the degree k then it can be reduced to partial differential equation with constant coefficients in the following way.

Let
$$x = e^X, y = e^Y \text{ so that } X = \log x, Y = \log y$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial X}$$

or
$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \quad \therefore \quad x\frac{\partial}{\partial x} = D\left(\equiv \frac{\partial}{\partial X}\right)$$

Now $x\dfrac{\partial}{\partial x}\left(x^{k-1}\dfrac{\partial^{k-1} z}{\partial x^{k-1}}\right) = x^k\dfrac{\partial^k z}{\partial x^k} + (k-1)x^{k-1}\dfrac{\partial^{k-1}z}{\partial x^{k-1}}$

or
$$x^k\frac{\partial^k z}{\partial x^k} = \left(x\frac{\partial}{\partial x} - k + 1\right)x^{k-1}\frac{\partial^{k-1}z}{\partial x^{k-1}}$$

Putting $k = 2, 3, \ldots\ldots$, we get

$$x^2\frac{\partial^2 z}{\partial x^2} = (D-1)x\frac{\partial z}{\partial x} = (D-1)Dz$$

$$x^3\frac{\partial^3 z}{\partial x^3} = (D-2)x^2\frac{\partial^2 z}{\partial x^2} = (D-2)(D-1)Dz \text{ etc.}$$

Similarly, $\quad y\dfrac{\partial z}{\partial y} = D'z, \quad y^2\dfrac{\partial^2 z}{\partial y^2} = (D'-1)D'z, \quad y^3\dfrac{\partial^3 z}{\partial y^3} = (D'-2)(D'-1)D'z$ etc.

and
$$xy\frac{\partial^2 z}{\partial x \partial y} = DD'z \ldots\ldots$$

Substituting in the given equation, it reduces to $\psi(D, D')z = V$ which is an equation with constant coefficients.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linear partial differential equation*

$$x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3 y^4.$$

Sol. Put $x = e^X, y = e^Y$ so that $X = \log x$ and $Y = \log y$ and Let $D \equiv \dfrac{\partial}{\partial X}, D' \equiv \dfrac{\partial}{\partial Y}$

and $DD' \equiv \dfrac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$$[D(D-1) - 4DD' + 4D'(D'-1) + 6D']z = e^{3X+4Y}$$

$\Rightarrow$ $$[(D^2 - 4DD' + 4D'^2) - (D - 2D')]z = e^{3X+4Y}$$

$\Rightarrow$ $$(D - 2D')(D - 2D' - 1)z = e^{3X+4Y}$$

Its $$\text{C.F.} = f_1(Y+2X) + e^X f_2(Y+2X)$$
$$= f_1(\log y + 2\log x) + x f_2(\log y + 2\log x)$$
$$= f_1(\log yx^2) + x f_2(\log yx^2) = g_1(yx^2) + x g_2(yx^2)$$

$$\text{P.I.} = \frac{1}{D-2D'-1}\left[\frac{1}{D-2D'} e^{3X+4Y}\right]$$

$$= \frac{1}{D-2D'-1}\left[\frac{1}{3-8}\int e^u\, du\right] \text{ where } 3X+4Y = u$$

$$= \frac{1}{D-2D'-1}\left[-\frac{1}{5} e^{3X+4Y}\right] = -\frac{1}{5}\left[\frac{1}{D-2D'-1} e^{3X+4Y}\right]$$

$$= -\frac{1}{5}\left[\frac{1}{3-8-1} e^{3X+4Y}\right] = \frac{1}{30} e^{3X+4Y} = \frac{1}{30} x^3 y^4.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = g_1(yx^2) + x g_2(yx^2) + \frac{1}{30} x^3 y^4$$

where g_1 and g_2 are arbitrary functions.

Example 2. *Solve:* $x^2 r - y^2 t + px - qy = \log x$.

Sol. Let $x = e^X$, $y = e^Y$ so that $X = \log x$ and $Y = \log y$ and let $D \equiv \frac{\partial}{\partial X}$ and $D' \equiv \frac{\partial}{\partial Y}$, then the given equation reduces to

$$[D(D-1) - D'(D'-1) + D - D']z = X$$

$\Rightarrow$ $$(D^2 - D'^2)z = X \qquad ...(1)$$

which is a homogeneous linear p.d.e. with constant coefficients.

$\therefore$ $$\text{C.F.} = \phi_1(Y+X) + \phi_2(Y-X)$$

and $$\text{P.I.} = \frac{1}{D^2 - D'^2}(X) = \frac{1}{(1)^2 - (0)^2}\iint u\, du\, du \qquad \text{where } X = u$$

$$= \int \frac{u^2}{2} du = \frac{u^3}{6} = \frac{X^3}{6}$$

Hence solution to (1) is

$$z = \phi_1(Y+X) + \phi_2(Y-X) + \frac{X^3}{6}$$

$$= \phi_1(\log y + \log x) + \phi_2(\log y - \log x) + \frac{(\log x)^3}{6}.$$

Therefore, the complete solution to the given differential equation is

$$z = \phi_1(\log xy) + \phi_2\left(\log \frac{y}{x}\right) + \frac{1}{6}(\log x)^3$$

$$z = f_1(xy) + f_2\left(\frac{y}{x}\right) + \frac{1}{6}(\log x)^3$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve:*

(*i*) $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} = xy$ (*ii*) $(x^2 D^2 + 2xy\, DD' + y^2 D'^2)z = x^m y^n$.

Sol. (*i*) Let $x = e^X$, $y = e^Y$ so that $X = \log x$ and $Y = \log y$ and Let $D \equiv \dfrac{\partial}{\partial X}$, $D' \equiv \dfrac{\partial}{\partial Y}$ and $DD' \equiv \dfrac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$$[D(D-1) - D'(D'-1)]z = e^{X+Y}$$

$$\Rightarrow \quad (D^2 - D'^2 - D + D')z = e^{X+Y}$$

$$\Rightarrow \quad (D - D')(D + D' - 1)z = e^{X+Y}$$

$$\text{C.F.} = f_1(Y + X) + e^X f_2(Y - X)$$

$$= f_1(\log y + \log x) + x f_2(\log y - \log x) = g_1(xy) + x g_2\left(\frac{y}{x}\right)$$

$$\text{P.I.} = \frac{1}{(D - D')(D + D' - 1)} e^{X+Y} = \frac{1}{D - D'}\left[\frac{1}{1+1-1} e^{X+Y}\right]$$

$$= \frac{1}{D - D'} e^{X+Y} = X.e^{X+Y} = xy \log x$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = g_1(xy) + x g_2\left(\frac{y}{x}\right) + xy \log x$$

where g_1 and g_2 are arbitrary functions.

(*ii*) Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$ and Let $D \equiv \dfrac{\partial}{\partial X}$, $D' \equiv \dfrac{\partial}{\partial Y}$ and $DD' \equiv \dfrac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$$[D(D-1) + 2DD' + D'(D'-1)]z = e^{mX+nY}$$

$$\Rightarrow \quad (D^2 + 2DD' + D'^2 - D - D')z = e^{mX+nY}$$

$$\Rightarrow \quad [(D + D')^2 - (D + D')]z = e^{mX+nY}$$

$$\Rightarrow \quad (D + D')(D + D' - 1)z = e^{mX+nY}$$

$$\therefore \quad \text{C.F.} = f_1(Y - X) + e^X f_2(Y - X)$$

$$= f_1(\log y - \log x) + x f_2(\log y - \log x)$$

$$= f_1\left(\log \frac{y}{x}\right) + x f_2\left(\log \frac{y}{x}\right) = g_1\left(\frac{y}{x}\right) + x g_2\left(\frac{y}{x}\right)$$

$$\text{P.I.} = \frac{1}{(D + D')(D + D' - 1)} e^{mX+nY}$$

$$= \frac{1}{(m+n)(m+n-1)} e^{mX+nY} = \frac{x^m y^n}{(m+n)(m+n-1)}$$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = g_1(y/x) + x\,g_2(y/x) + \frac{x^m y^n}{(m+n)(m+n-1)}$$

where g_1 and g_2 are arbitrary functions.

ASSIGNMENT-I

(2 Marks Questions for Section–A)

1. Show that $z = f(x^2 + y^2)$ is a solution of $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$. [G.B.T.U. (AG) 2012]

2. Find the particular integral of $(D^2 + DD')z = \sin(x + y)$ where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$. (G.B.T.U. 2012)

3. Solve:
   $(D - 5D' + 1)^2 z = 0$ (U.P.T.U. 2015)

4. Solve:
   $(D - 5D' + 4)^3 z = 0$ (U.P.T.U. 2015)

5. Form the partial differential equation by eliminating a and b from $z = (x^2 + a)(y^2 + b)$. [G.B.T.U. (AG) 2012]

6. Formulate the PDE by eliminating the arbitrary function from $\phi(x^2 + y^2, y^2 + z^2) = 0$.

7. Find the particular integral of $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$ (A.K.T.U. 2016)

8. Solve: $(D^2 - 2DD' + D'^2)z = 0$ (U.P.T.U. 2015)

9. Give two examples of non-linear partial differential equation of the first order. (M.T.U. 2013)

10. Find the P.I. of $(D^2 - D'^2)z = \cos(x + y)$.

11. Find the P.I. of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$. [M.T.U. (SUM) 2011, G.B.T.U. 2011]

42. Find the partial differential equation which is satisfied by the relation $z = c_1xy + c_2$ where c_1 and c_2 are constants. (M.T.U. 2013)

13. Solve: $(D^2 + DD')z = 0$ (U.P.T.U. 2014)

14. Form a partial differential equation from $z = (a + x)^2 + y$.

15. Find the solution of $xp + yq = z$.

16. Solve: $p + q = z$.

17. Solve: $(y - z)p + (z - x)q = x - y$.

18. Solve: $\frac{\partial^3 z}{\partial x^3} = 0$.

19. Find the partial differential equation of all spheres whose centres lie on z-axis and given by equations $x^2 + y^2 + (z - a)^2 = b^2$; a and b being constants. (A.K.T.U. 2017)

20. Write the complementary function of the partial differential equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{1}{x^2}$. [G.B.T.U. (AG) 2011]