$$y(x, t) = \frac{9h}{\pi^2} \sum_{m=1,2,\ldots}^{\infty} \frac{1}{m^2} \sin \frac{2m\pi}{3} \cos \frac{2m\pi ct}{l} \sin \frac{2m\pi x}{l} \quad \ldots(9)$$

(where $n = 2m$)

Putting $x = \frac{l}{2}$ in eqn. (6), we get

$$y\left(\frac{l}{2}, t\right) = \frac{9h}{\pi^2} \sum_{m=1}^{\infty} \sin\left(\frac{2m\pi}{3}\right) \cdot \frac{1}{m^2} \cdot \cos \frac{2m\pi ct}{l} \cdot \sin m\pi = 0.$$

Hence, midpoint of the string is always at rest.

Example 8. *If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity* $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$, *find the displacement y (x, t).*

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots(1)$$

The solution of equation (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) [Refer Sol. of Ex. 1]

Boundary conditions are,

$$y(0, t) = 0 \quad \ldots(3)$$

$$y(l, t) = 0 \quad \ldots(4)$$

$$y(x, 0) = 0 \quad \ldots(5)$$

$$\left(\frac{\partial y}{\partial t}\right) = b \sin^3 \frac{\pi x}{l} \text{ at } t = 0 \quad \ldots(6)$$

From eqn. (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(7)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in I)$

$\therefore$ $p = \frac{n\pi}{l}.$

$\therefore$ From (7), $y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$ ...(8)

$y(x, 0) = 0 = c_1 c_4 \sin \frac{n\pi x}{l}$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (8), $y(x, t) = c_2 c_4 \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$

$$= b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \text{ where } c_2 c_4 = b_n$$

The general solution is

$$y(x, t) = \sum_1^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \text{...(9)}$$

$$\frac{\partial y}{\partial t} = \sum_1^{\infty} b_n . \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_1^{\infty} b_n . \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$b \sin^3 \frac{\pi x}{l} = \sum_1^{\infty} b_n . \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$\frac{b}{4}\left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right] = b_1 \frac{\pi c}{l} \sin \frac{\pi x}{l} + \frac{2b_2 \pi c}{l} \sin \frac{2\pi x}{l} + 3b_3 \frac{\pi c}{l} \sin \frac{3\pi x}{l} + \ldots$$

$\Rightarrow$

$$b_1 \frac{\pi c}{l} = \frac{3b}{4} \Rightarrow b_1 = \frac{3bl}{4\pi c}$$

$$b_2 = 0 \quad \text{and} \quad \frac{3b_3 \pi c}{l} = -\frac{b}{4} \Rightarrow b_3 = -\frac{bl}{12\pi c}$$

Also, $b_4 = 0 = b_5 = \ldots$ etc.

Hence from (9),

$$y(x, t) = \frac{3bl}{4\pi c} \sin \frac{\pi ct}{l} \sin \frac{\pi x}{l} - \frac{bl}{12\pi c} \sin \frac{3\pi ct}{l} \sin \frac{3\pi x}{l}$$

$$= \frac{bl}{12\pi c}\left[9 \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} - \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l}\right].$$

Example 9. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.* [M.T.U. (SUM) 2011]

Sol. Here the boundary conditions are $y(0, t) = y(l, t) = 0$

$$y(x, t) = \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \quad \text{...(1)}$$

| Refer Sol. of Ex. 2

Since the string was at rest initially, $y(x, 0) = 0$

$\therefore$ From (1),

$$0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \Rightarrow a_n = 0$$

$\therefore$

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \text{...(2)}$$

and

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

But

$$\frac{\partial y}{\partial t} = \lambda x(l - x) \quad \text{when } t = 0$$

$$\therefore \qquad \lambda x(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \qquad \frac{\pi c}{l} n b_n = \frac{2}{l} \int_0^l \lambda x(l-x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\left[x(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1-\cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8\lambda l^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases} \quad i.e., \ \frac{8\lambda l^2}{\pi^3(2m-1)^3}, \text{ taking } n = 2m-1$$

$$\Rightarrow \quad b_n = \frac{8\lambda l^3}{c\pi^4(2m-1)^4}$$

$\therefore$ From (2), the required solution is

$$y(x, t) = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \sin\frac{(2m-1)\pi ct}{l} \sin\frac{(2m-1)\pi x}{l}.$$

Example 10. *Transform the equation* $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ *to its normal form using the transformation* $u = x + ct$, $v = x - ct$ *and hence solve it. Show that the solution may be put in the form* $y = \dfrac{1}{2}[f(x+ct) + f(x-ct)]$. (M.T.U. 2013)

Assume initial conditions $y = f(x)$ *and* $(\partial y/\partial t) = 0$ *at* $t = 0$.

Sol. One-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let us introduce two new independent variables

$$u = x + ct \qquad ...(2)$$

and

$$v = x - ct \qquad ...(3)$$

so that y becomes a function of u and v.

Then, $$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \qquad ...(4) \quad [\text{Using (2) and (3)}]$$

Also, $$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad ...(5)$$

$$\therefore \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \qquad ...(6)$$

Also, $$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial t} = c\,\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}(-c) = c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \qquad ...(7)$$

$$\Rightarrow \qquad \frac{\partial}{\partial t} \equiv c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \qquad ...(8)$$

$$\therefore \qquad \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$= c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) \qquad ...(9)$$

From (1), (6) and (9), we have

$$c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

$$\Rightarrow \qquad -4c^2\,\frac{\partial^2 y}{\partial u \partial v} = 0$$

$$\Rightarrow \qquad \frac{\partial^2 y}{\partial u \partial v} = 0 \qquad ...(10) \quad (\because \; c^2 \neq 0)$$

Integrating eqn. (10) partially, w.r.t. u, we get

$$\frac{\partial y}{\partial v} = f_1(v).$$

Integrating again w.r.t. v partially, we get

$$y = \int f_1(v)\,\partial v + \psi(u) = \phi(v) + \psi(u)$$

$$\Rightarrow \qquad y(x, t) = \phi(x - ct) + \psi(x + ct) \qquad ...(11)$$

which is *d'* Alembert's solution of wave equation.

Applying initial conditions $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$ at $t = 0$ in (11), we get

$$f(x) = \phi(x) + \psi(x) \text{ and } 0 = -\phi'(x) + \psi'(x)$$

Hence, $$\phi(x) = \psi(x) = \frac{1}{2} f(x)$$

$$\therefore \qquad y = \frac{1}{2}[f(x + ct) + f(x - ct)].$$

Example 11. *A tightly stretched string with fixed end points $x = 0$ and $x = \pi$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity* $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$

then find the displacement $y(x, t)$ at any point of string at any time t.

Sol. The equation for the vibrations of a string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\,\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Its solution is

$$y(x, t) = (c_1 \cos pct + c_2 \sin pct)(c_3 \cos px + c_4 \sin px) \qquad ...(2)$$

Boundary conditions are

$$y(0, t) = 0 = y(\pi, t)$$

$$y(x, 0) = 0$$

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x.$$

From (2), $y(0, t) = 0 = (c_1 \cos pct + c_2 \sin pct)\, c_3$

$\Rightarrow$ $c_3 = 0$

From (2), $y(x, t) = (c_1 \cos pct + c_2 \sin pct)\, c_4 \sin px$ ...(3)

$y(\pi, t) = 0 = (c_1 \cos pct + c_2 \sin pct)\, c_4 \sin p\pi$

$\Rightarrow$ $\sin p\pi = 0 = \sin n\pi \; (n \in \mathrm{I})$

$\Rightarrow$ $p = n$

From (3), $y(x, t) = (c_1 \cos nct + c_2 \sin nct)\, c_4 \sin nx$ ...(4)

$y(x, 0) = 0 = c_1 c_4 \sin nx$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (4), $y(x, t) = c_2 c_4 \sin nct \sin nx = b_n \sin nct \sin nx$ ...(5)

where $c_2 c_4 = b_n$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin nct \sin nx$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} nc\, b_n \cos nct \sin nx$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} nc\, b_n \sin nx$$

$$0.03 \sin x - 0.04 \sin 3x = cb_1 \sin x + 2cb_2 \sin 2x + 3cb_3 \sin 3x + \ldots$$

$\Rightarrow$ $cb_1 = 0.03 \Rightarrow b_1 = \dfrac{0.03}{c}$

$b_2 = 0$

and $3cb_3 = -0.04 \Rightarrow b_3 = \dfrac{-0.0133}{c}.$

$\therefore$ From (6), $$y(x, t) = \frac{0.03}{c} \sin ct \sin x - \frac{0.0133}{c} \sin 3ct \sin 3x$$

$$= \frac{1}{c}\,[0.03 \sin x \sin ct - 0.0133 \sin 3x \sin 3ct].$$

TEST YOUR KNOWLEDGE

1. Find the deflection $y(x, t)$ of the vibrating string of length π and ends fixed, corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x)$ given $c^2 = 1$.

2. Solve: $y_{tt} = 4y_{xx}$; $y(0, t) = 0 = y(5, t)$, $y(x, 0) = 0$ $\left(\frac{\partial y}{\partial t}\right)_{t=0} = f(x)$

   if (*i*) $f(x) = 5 \sin \pi x$ (*ii*) $f(x) = 3 \sin 2\pi x - 2 \sin 5\pi x$.

3. Find the deflection of the vibrating string which is fixed at the ends $x = 0$ and $x = 2$ and the motion is started by displacing the string into the form $\sin^3\left(\frac{\pi x}{2}\right)$ and releasing it with zero initial velocity at $t = 0$. *(U.P.T.U. 2014)*

4. Find the solution of the equation of a vibrating string of length l satisfying the initial conditions :

   $y = F(x)$ when $t = 0$

   and $\frac{\partial y}{\partial t} = \phi(x)$ when $t = 0$

   It is assumed that the equation of a vibrating string is $y_{tt} = a^2 y_{xx}$.

5. The vibrations of an elastic string is governed by the partial differential equation

   $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

   The length of the string is π and ends are fixed. The initial velocity is zero and the initial deflection is $u(x, 0) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string at any time t.

6. A tight string of length 20 cms fastened at both ends is displaced from its position of equilibrium by imparting to each of its points an initial velocity given by

   $$v = \begin{matrix} x; & 0 \le x \le 10 \\ 20 - x; & 10 \le x \le 20 \end{matrix}\Bigg\} ;$$

   x being the distance from one end. Determine the displacement at any subsequent time.

7. Using d' Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection $f(x) = a(x - x^3)$.

8. Reduce the equation $u_{xx} - 2u_{xy} + u_{yy} = 0$ to its normal form using the transformation $v = x$, $z = x + y$ and solve it.

9. Solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$ using the transformation $v = x + y$, $z = 2x - y$.

10. The ends of a tightly stretched string of length l are fixed at $x = 0$ and $x = l$. The string is at rest with the point $x = b$ drawn aside through a small distance d and released at time $t = 0$. Show that

    $$y(x, t) = \frac{2dl^2}{\pi^2 b(l - b)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi b}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}.$$

11. Find the deflection of the vibrating string of unit length whose end points are fixed if the initial velocity is zero and the initial deflection is given by $u(x, 0) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ -1, & \frac{1}{2} < x \le 1 \end{cases}$ *(U.P.T.U. 2014)*

12. Find the deflection $u(x, t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by

$$\sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x, \quad 0 \le x \le 1$$

(G.B.T.U. 2013)

13. A string is stretched and fastened to two points distance l apart. Find the displacement $y(x, t)$ at any point at a distance x from one end at time t given that:

$$y(x, 0) = A \sin\left(\frac{2\pi x}{l}\right)$$

(M.T.U. 2013)

14. Solve the following initial-boundary value problem (IBVP):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 2, \qquad u(1, t) = 3, \quad t \ge 0$$

$$u(x, 0) = 2 + x + \sin \pi x$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0.$$

15. Solve the following initial-boundary value problem (IBVP):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0$$

$$u_x(0, t) = 1, \qquad u_x(1, t) = 3, \quad t \ge 0$$

$$u(x, 0) = x + x^2, \qquad u_t(x, 0) = \pi \cos \pi x.$$

16. Use D'Alembert's formula to solve the IVP:

$$u_t = 4u_{xx}, \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = \sin x, \quad u_t(x, 0) = 4.$$

17. If $u(x, t) = \sin x \cos t + 2xt$ is a solution of the initial value problem

$$u_{tt} = u_{xx}, \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 2x$$

Use D'Alembert formula to find $f(x)$.

Answers

1. $y(x, t) = k(\cos t \sin x - \cos 2t \sin 2x)$

2. (*i*) $y(x, t) = \frac{5}{2\pi} \sin \pi x \sin 2\pi t$ (*ii*) $y(x, t) = \frac{3}{4\pi} \sin 2\pi x \sin 4\pi t - \frac{1}{5\pi} \sin 5\pi x \sin 10\pi t$

3. $y(x, t) = \frac{3}{4} \sin \frac{\pi x}{2} \cos \frac{\pi ct}{2} - \frac{1}{4} \sin \frac{3\pi x}{2} \cos \frac{3\pi ct}{2}$

4. $y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right)$

where $a_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l}\, dx$ and $b_n = \frac{2}{na\pi} \int_0^l \phi(x) \sin \frac{n\pi x}{l}\, dx$

5. $y(x, t) = 2[\cos t \sin x + \cos 3t \sin 3x]$

6. $y(x, t) = \frac{1600}{a\pi^3} \left[\sin \frac{\pi x}{20} \sin \frac{\pi at}{20} - \frac{1}{3^3} \sin \frac{3\pi x}{20} \sin \frac{3\pi at}{20} + \ldots\ldots \right]$

7. $y(x, t) = ax(1 - x^2 - 3c^2t^2)$

8. $\frac{\partial^2 u}{\partial v^2} = 0$; $u = xf_1(x + y) + f_2(x + y)$.

9. $\frac{\partial^2 u}{\partial v \partial z} = 0$; $u = f_1(x + y) + f_2(2x - y)$

11. $y(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 + \cos n\pi - 2 \cos \frac{n\pi}{2} \right) \sin n\pi x \cos n\pi ct$

12. $u(x, t) = \sin \pi x \cos \pi ct + \frac{1}{3} \sin 3\pi x \cos 3\pi ct + \frac{1}{5} \sin 5\pi x \cos 5\pi ct$

13. $y(x, t) = A \sin \left(\frac{2\pi x}{l} \right) \cos \left(\frac{2\pi ct}{l} \right)$

14. $u(x, t) = 2 + x + \sin \pi x \cos \pi t$

15. $u(x, t) = x + x^2 + t^2 + \cos \pi x \sin \pi t$

16. $u(x, t) = \sin x \cos 2t + 4t$

17. $f(x) = \sin x$

2.8 VIBRATING MEMBRANE—TWO-DIMENSIONAL WAVE EQUATION

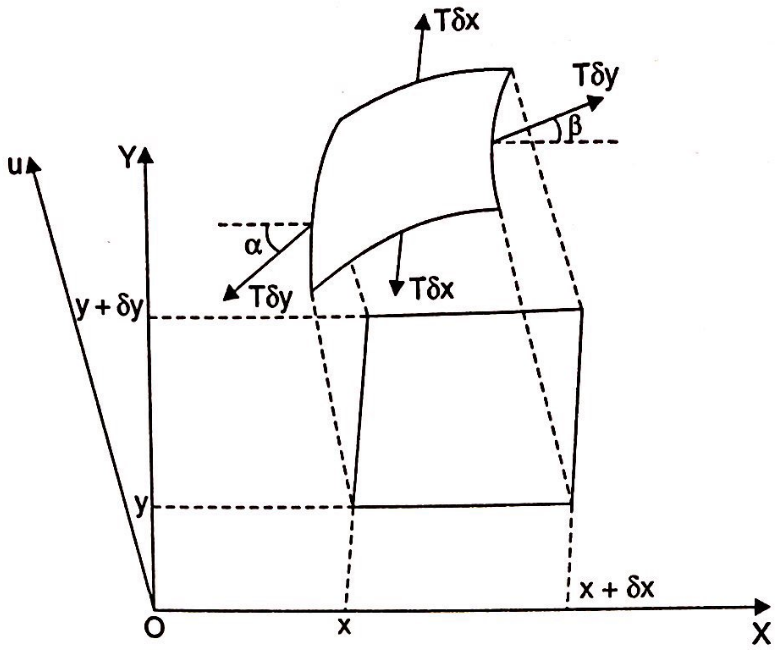
We shall now obtain the equation for the vibrations of a tightly stretched membrane (such as the membrane of a drum). We shall assume that the membrane is uniform and the tension in it per unit length is the same at every point in all directions. Let T be the tension per unit length and m be the mass of the membrane per unit area.

Consider the forces on an element $\delta x \delta y$ of the membrane. Due to its displacement u, perpendicular to the xy-plane, the forces $T\delta y$ (tangential to the membrane) on its opposite edges of length δy act at angles α and β to the horizontal. So their vertical component

$= (T\delta y) \sin \beta - (T\delta y) \sin \alpha = T\delta y(\tan \beta - \tan \alpha)$ approximately, since α and β are small

$$= T\delta y \left[\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right] = T\delta y\, \delta x \left[\frac{\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\delta x} \right]$$

$= T\delta x \delta y \frac{\partial^2 u}{\partial x^2}$ upto a first order of approximation.



Similarly, the forces $T\delta x$ acting on the edges of length δx have the vertical component $T\delta x\delta y\,\dfrac{\partial^2 u}{\partial y^2}$.

Hence the equation of motion of the element is

$$(m\delta x\delta y)\,\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\delta x\delta y$$

or

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)} \quad \text{where } c^2 = \frac{T}{m}.$$

This is the wave equation in two dimensions.

2.9 SOLUTION OF TWO-DIMENSIONAL WAVE EQUATION

Two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad ...(1)$$

Let $u = XYT$ ...(2)

where X is a function of x only, Y is a function of y only and T is a function of t only, be a solution of (1).

Then, $\dfrac{\partial^2 u}{\partial t^2} = XYT''$, $\dfrac{\partial^2 u}{\partial x^2} = X''YT$ and $\dfrac{\partial^2 u}{\partial y^2} = XY''T$

Substituting in (1), we have $\dfrac{1}{c^2}\,XYT'' = X''YT + XY''T$

Dividing by XYT, we have $$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} \quad ...(3)$$

This will be true only when *each member is a constant*. Choosing the constants suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} + l^2Y = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + (k^2 + l^2)\,c^2T = 0$$

The solutions of these equations are respectively

$$X = c_1 \cos kx + c_2 \sin kx$$

$$Y = c_3 \cos ly + c_4 \sin ly$$

and

$$T = c_5 \cos \sqrt{(k^2 + l^2)}\,ct + c_6 \sin \sqrt{(k^2 + l^2)}\,ct$$

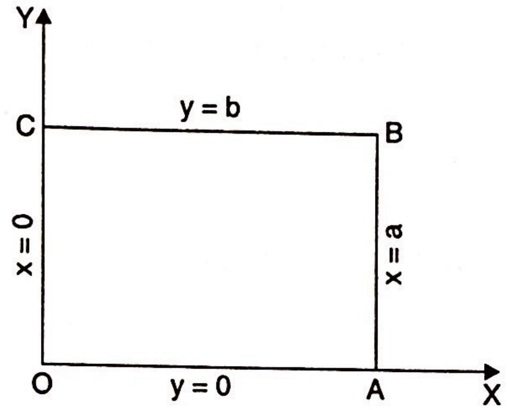
Hence from (2), a solution of (1) is

$$u(x, y, t) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)$$

$$[c_5 \cos \sqrt{(k^2 + l^2)}\,ct + c_6 \sin \sqrt{(k^2 + l^2)}\,ct] \qquad ...(4)$$

Now let us suppose that the membrane is rectangular and is stretched between the lines

$$x = 0, x = a, y = 0, y = b.$$



Then the boundary conditions are:

(*i*) $u = 0$, when $x = 0$

(*ii*) $u = 0$, when $x = a$

(*iii*) $u = 0$, when $y = 0$

(*iv*) $u = 0$, when $y = b$ for all t.

Applying the condition (*i*), we have

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos \sqrt{(k^2 + l^2)}\,ct + c_6 \sin \sqrt{(k^2 + l^2)}\,ct]$$

i.e.,

$$c_1 = 0$$

Putting $c_1 = 0$ in (3) and applying the condition (*ii*), we have $\sin ka = 0$ or $k = \frac{m\pi}{a}$, where m is an integer.

Similarly, applying the conditions (*iii*) and (*iv*), we get

$$c_3 = 0 \quad \text{or} \quad l = \frac{n\pi}{b}, \text{ where } n \text{ is an integer.}$$

Therefore, the solution (3) becomes

$$u(x, y, t) = c_2c_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (c_5 \cos pt + c_6 \sin pt)$$

where $p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$

Replacing the arbitrary constants suitably, we can write the general solution as

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \qquad ...(5)$$

Now suppose the membrane starts from rest from the initial position $u = f(x, y)$ *i.e.*, $u(x, y, 0) = f(x, y)$.

Then applying the condition: $\dfrac{\partial u}{\partial t} = 0$ when $t = 0$, we get $B_{mn} = 0$.

Also using the condition: $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \ldots(6)$$

This is a double Fourier series. Multiplying both sides by $\sin \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b}$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one becomes zero. Thus, we get

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dy\, dx = \frac{ab}{4} A_{mn}$$

i.e.,

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{a}\, dy\, dx \quad \ldots(7)$$

Hence, from (5), the required solution is

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where A_{mn} is given by (7) and $p = \pi c \sqrt{\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the deflection u(x, y, t) of the square membrane with $a = b = c = 1$, if the initial velocity is zero and the initial deflection $f(x, y) = A \sin \pi x \sin 2\pi y$.*

Sol. The vibrations of the square membrane are governed by two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \ldots(1)$$

Here the boundary conditions are,

$$u(0, y, t) = 0,\ u(1, y, t) = 0, \quad u(x, 0, t) = 0,\ u(x, 1, t) = 0$$

and the initial conditions are

$$u(x, y, 0) = f(x, y) = A \sin \pi x \sin 2\pi y \quad \text{and} \quad \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0 \quad \ldots(2)$$

To solve eqn. (1), let $u = XYT$

where X is function of x only, Y is a function of y only, and T is a function of t only.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial t^2} (XYT) = XY \frac{d^2 T}{dt^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} (XYT) = YT \frac{d^2 X}{dx^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(\text{XYT}) = \text{XT}\frac{d^2\text{Y}}{dy^2},$$

From (1), $\quad \text{XYT}'' = (\text{YTX}'' + \text{XTY}'')c^2$

$\Rightarrow \quad \dfrac{1}{c^2}\dfrac{\text{T}''}{\text{T}} = \dfrac{\text{X}''}{\text{X}} + \dfrac{\text{Y}''}{\text{Y}}$

This will be true only when each member is a constant. Choosing the constant suitably, we have

$$\frac{d^2\text{X}}{dx^2} + k^2\text{X} = 0, \quad \frac{d^2\text{Y}}{dy^2} + l^2\text{Y} = 0 \quad \text{and} \quad \frac{d^2\text{T}}{dt^2} + (k^2 + l^2)\,c^2\text{T} = 0.$$

The solutions of these equations are respectively,

$$\text{X} = c_1 \cos kx + c_2 \sin kx$$

$$\text{Y} = c_3 \cos ly + c_4 \sin ly$$

and $\quad \text{T} = c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct.$

$\therefore \quad u(x, y, t) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin y)$

$$[c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct] \qquad ...(3)$$

From (3), $\quad u(0, y, t) = 0 = c_1(c_3 \cos ly + c_4 \sin ly)(c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct)$

$\Rightarrow \quad c_1 = 0.$

$\therefore$ From (3),

$$u(x, y, t) = c_2 \sin kx(c_3 \cos ly + c_4 \sin ly)(c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct) \qquad ...(4)$$

$$u(1, y, t) = 0 = c_2 \sin k(c_3 \cos ly + c_4 \sin ly)(c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct)$$

$\Rightarrow \quad \sin k = 0 = \sin m\pi \; (m \in \text{I})$

$\quad k = m\pi.$

$\therefore$ From (4), $\quad u(x, y, t) = c_2 \sin m\pi x(c_3 \cos ly + c_4 \sin ly)$

$$(c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct) \qquad ...(5)$$

$$u(x, 0, t) = 0 = c_2 \sin m\pi x \,.\, c_3(c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct]$$

$\Rightarrow \quad c_3 = 0.$

$\therefore$ From (5), $\quad u(x, y, t) = c_2 c_4 \sin m\pi x \sin ly\; (c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct) \qquad ...(6)$

$$u(x, 1, t) = 0 = c_2 c_4 \sin m\pi x \sin l\; (c_5 \cos \sqrt{k^2 + l^2}\; ct + c_6 \sin \sqrt{k^2 + l^2}\; ct)$$

$\Rightarrow \quad \sin l = 0 = \sin n\pi \; (n \in \text{I})$

$\Rightarrow \quad l = n\pi.$

$\therefore$ From (6), $u(x, y, t) = c_2 c_4 \sin m\pi x \sin n\pi y \left(c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct\right)$

$$u(x, y, t) = \sin m\pi x \sin n\pi y \,(A_{mn} \cos pt + B_{mn} \sin pt) \quad ...(7)$$

where $p = \pi c \sqrt{m^2 + n^2}$.

The most general solution is

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin m\pi x \sin n\pi y \,(A_{mn} \cos pt + B_{mn} \sin pt) \quad ...(8)$$

Applying the condition,

$$\frac{\partial u}{\partial t} = 0 \text{ at } t = 0$$

we get $B_{mn} = 0$ ...(9)

Also, using the condition $u = f(x, y) = A \sin \pi x \sin 2\pi y$ when $t = 0$, we get

$$A \sin \pi x \sin 2\pi y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin n\pi y$$

This is a double Fourier Series.

$$A_{mn} = \frac{2}{1} \cdot \frac{2}{1} \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y \, dx\, dy$$

Obviously, $A_{m1} = A_{m3} = A_{m4} = ... = 0$

But,
$$A_{m2} = 4A \int_0^1 \int_0^1 \sin \pi x \sin m\pi x \sin^2 2\pi y \, dx\, dy$$
$$= 2A \int_0^1 \int_0^1 \sin \pi x \sin m\pi x \,(1 - \cos 4\pi y) \, dx\, dy$$
$$= 2A \int_0^1 \sin \pi x \sin m\pi x \left(y - \frac{\sin 4\pi y}{4\pi}\right)_0^1 dx$$
$$= 2A \int_0^1 \sin \pi x \sin m\pi x \, dx$$

Again, obviously,

$$A_{22} = A_{32} = A_{42} = ... = 0$$

But,
$$A_{12} = 2A \int_0^1 \sin^2 \pi x \, dx = A \int_0^1 (1 - \cos 2\pi x) \, dx$$
$$= A \left(x - \frac{\sin 2\pi x}{2\pi}\right)_0^1 = A \quad ...(10)$$

$\therefore$ From (8), (9) and (10), we get

$$u(x, y, t) = A \sin \pi x \sin 2\pi y \cos pt \quad ...(11)$$

where $p = \pi c \sqrt{m^2 + n^2} = \pi(1) \sqrt{1 + 4} = \pi\sqrt{5}$.

$\therefore$ From (11), $u(x, y, t) = A \cos \pi\sqrt{5}\, t \sin \pi x \sin 2\pi y$.

Example 2. *Find the deflection $u(x, y, t)$ of a rectangular membrane $(0 \le x \le a,\ 0 \le y \le b)$ whose boundary is fixed; given that it starts from rest and $u(x, y, 0) = xy(a - x)(b - y)$. Show that the deflection u is given by*

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos ckt \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where $\quad A_{mn} = \dfrac{16a^2b^2}{m^3n^3\pi^6}(1 - \cos m\pi)(1 - \cos n\pi)$ *and* $k^2 = \pi^2\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)$.

Sol. Proceeding as in Art. 2.7, we have

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where
$$p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad ...(1)$$

From (1),
$$u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$xy\,(a - x)(b - y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where
$$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b xy\,(a - x)\,(b - y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dy\, dx$$

$$= \frac{4}{ab} \int_0^a x(a - x) \sin \frac{m\pi x}{a}\, dx \,.\int_0^b y(b - y) \sin \frac{n\pi y}{b}\, dy$$

$$= \frac{4}{ab}\left[x\,(a - x)\left(\frac{-\cos \frac{m\pi x}{a}}{\frac{m\pi}{a}}\right) - (a - 2x)\left(\frac{-\sin \frac{m\pi x}{a}}{\frac{m^2\pi^2}{a^2}}\right) + (-2)\left(\frac{\cos \frac{m\pi x}{a}}{\frac{m^3\pi^3}{a^3}}\right)\right]_0^a$$

$$\times \left[y\,(b - y)\left(\frac{-\cos \frac{n\pi y}{b}}{\frac{n\pi}{b}}\right) - (b - 2y)\left(\frac{-\sin \frac{n\pi y}{b}}{\frac{n^2\pi^2}{b^2}}\right) + (-2)\left(\frac{\cos \frac{n\pi y}{b}}{\frac{n^3\pi^3}{b^3}}\right)\right]_0^b$$

$$= \frac{4}{ab}\left[\frac{-2a^3}{m^3\pi^3} \cos m\pi + \frac{2a^3}{m^3\pi^3}\right]\left[\frac{-2b^3}{n^3\pi^3} \cos n\pi + \frac{2b^3}{n^3\pi^3}\right]$$

$$= \frac{4}{ab} \cdot \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3}\, [1 - (-1)^n]\, [1 - (-1)^m]$$

$$A_{mn} = \frac{16a^2b^2}{m^3n^3\pi^6}\, [1 - (-1)^n]\, [1 - (-1)^m].$$

Hence from (1),

$$u(x, y, t) = \frac{16a^2b^2}{\pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[1-(-1)^n][1-(-1)^m]}{m^3n^3} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $$p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad ...(2)$$

Expression (2) may also be put in as,

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos ckt \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where $$A_{mn} = \frac{16a^2b^2}{m^3\pi^6n^3} (1 - \cos m\pi)(1 - \cos n\pi) \quad \text{and} \quad k^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

Example 3. *A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is k sin 2πx sin πy. Show that the deflection at any instant is*

$$k \sin 2\pi x \sin \pi y \cos (\sqrt{5}\, \pi ct).$$

Sol. Here we have to solve the equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

with boundary conditions $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$

and the initial conditions $u(x, y, 0) = f(x, y) = k \sin 2\pi x \sin \pi y$

$$\frac{\partial u}{\partial t} = 0 \quad \text{when } t = 0$$

Proceeding as in Ex. 1, we have

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin n\pi y \cos pt \quad ...(1)$$

Since, $a = b = 1$, where $p = \pi c \sqrt{m^2 + n^2}$

and $$A_{mn} = \frac{4}{1 \times 1} \int_0^1 \int_0^1 k \sin 2\pi x \sin \pi y \sin m\pi x \sin n\pi y \, dy dx$$

$$= 4k \int_0^1 \sin m\pi x \sin 2\pi x \, dx \int_0^1 \sin n\pi y \sin \pi y \, dy$$

$$= 0 \quad \text{for } m \neq 2 \quad \text{or} \quad n \neq 1$$

$$\therefore \quad A_{21} = 4k(\tfrac{1}{2})(\tfrac{1}{2}) = k \quad \text{and} \quad p = \pi c \sqrt{(2)^2 + (1)^2} = \sqrt{5}\, \pi c$$

Hence solution (1) reduces to $u(x, y, t) = k \sin 2\pi x \sin \pi y \cos (\sqrt{5}\, \pi ct)$.

TEST YOUR KNOWLEDGE

1. Find the deflecting $u(x, y, t)$ of a rectangular membrane $(0 < x < 1, 0 < y < 2)$ whose boundary is fixed, given that it starts from rest and $u(x, y, 0) = xy(1-x)(2-y)$.
2. Find the deflection $u(x, y, t)$ of a rectangular membrane $(0 < x < a, 0 < y < b)$ whose boundary is fixed, given that it starts from rest and $u(x, y, 0) = xy(a^2 - x^2)(b^2 - y^2)$.
3. Find the deflection $u(x, y, t)$ of the tightly stretched rectangular membrane with sides a and b having wave velocity $c = 1$ if the initial velocity is zero and the initial deflection is

$$f(x, y) = \sin \frac{2\pi x}{a} \sin \frac{3\pi y}{b}. \qquad \textit{(G.B.T.U. 2011)}$$

Answers

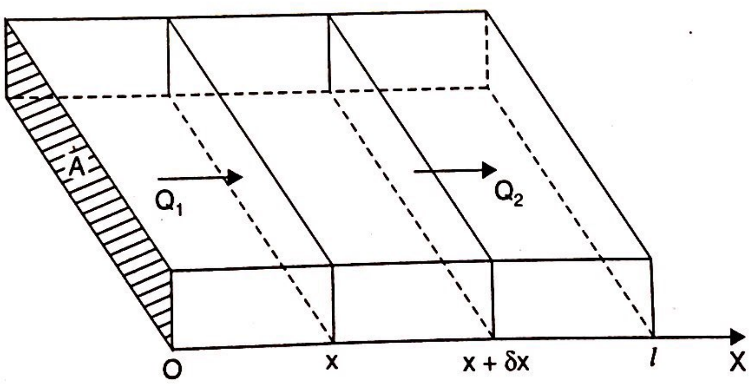
1. $u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin \frac{n\pi y}{2} \cos pt$, where $A_{mn} = \frac{256}{m^3 n^3 \pi^6}$, both m, n odd and $p = \pi c \sqrt{m^2 + \frac{n^2}{4}}$

2. $u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$, where $A_{mn} = \frac{144 a^3 b^3}{m^3 n^3 \pi^6}$ and $p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$.

3. $u(x, y, t) = \sin \frac{2\pi x}{a} \sin \frac{3\pi y}{b} \cos pt$ where $p = \pi \sqrt{\frac{4}{a^2} + \frac{9}{b^2}}$.

2.10 ONE-DIMENSIONAL HEAT FLOW $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Consider the flow of heat by conducting in a uniform bar. It is assumed that the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible. Take one end of the bar as origin and the direction of flow as the positive x-axis. The temperature u at any point of the bar depends on the distance x of the point from one end and the time t. Also, the temperature of all points of any cross-section is the same.



The amount of heat crossing any section of the bar per second depends on the area A of the cross-section, the conductivity K of the material of the bar and the temperature gradient $\frac{\partial u}{\partial x}$ *i.e.*, rate of change of temperature w.r.t. distance normal to the area.

$\therefore$ Q_1, the quantity of heat flowing into the section at a distance x

$$= -KA\left(\frac{\partial u}{\partial x}\right)_x \text{ per sec.}$$

(the negative sign on the right is attached because as x increases, u decreases).

Q_2, the quantity of heat flowing out of the section at a distance $x + \delta x$

$$= -KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ per sec.}$$

Hence the amount of heat retained by the slab with thickness δx is

$$Q_1 - Q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ per sec.} \quad ...(1)$$

But the rate of increase of heat in the slab $= S\rho A\, \delta x \dfrac{\partial u}{\partial t}$ ...(2)

where S is the specific heat and ρ, the density of the material.

$$\therefore \text{ From (1) and (2), } S\rho A\, \delta x \frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

or

$$S\rho \frac{\partial u}{\partial t} = K\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking the limit as $\delta x \to 0$, we have

$$S\rho \frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} = \frac{K}{S\rho}\frac{\partial^2 u}{\partial x^2}$$

or $\boxed{\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}}$, where $c^2 = \dfrac{K}{S\rho}$ is known as diffusivity of the material of the bar.

2.11 SOLUTION OF THE HEAT EQUATION [G.B.T.U. (C.O.) 2011]

The heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

Let

$$u = XT \quad ...(2)$$

where X is a function of x only and T is a function of t only, be a solution of (1).

Then $\dfrac{\partial u}{\partial t} = XT'$ and $\dfrac{\partial^2 u}{\partial x^2} = X''T$

Substituting in (1), we have $XT' = c^2X''T$

Separating the variables, we get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} \quad ...(3)$$

Now the LHS of (3) is a function of x only and the RHS is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. The equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{dT}{dt} - kc^2T = 0 \qquad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1 e^{px} + c_2 e^{-px},\ T = c_3\, e^{c^2p^2t}$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px,\ T = c_3\, e^{-c^2p^2t}$$

(*iii*) *When k = 0*

$$X = c_1 x + c_2,\ T = c_3.$$

Thus the various possible solutions of the heat equation (1) are:

$$u = (c_1 e^{px} + c_2 e^{-px}) \,.\, c_3\, e^{c^2p^2t} \qquad ...(5)$$

$$u = (c_1 \cos px + c_2 \sin px) \,.\, c_3 e^{-c^2p^2t} \qquad ...(6)$$

$$u = (c_1 x + c_2)\, c_3 \qquad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solution of the heat equation is

$$u = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2p^2t}.$$

Solution (5) is rejected since $u \to \infty$ as $t \to \infty$.

Also, solution (7) is rejected as it gives a non-zero temperature at all times.

2.12 INHOMOGENEOUS BOUNDARY CONDITIONS

Now consider the case where the ends of a rod are kept at constant temperatures different from zero.

Consider the IBVP, $u_t = u_{xx},\ 0 < x < L,\ t > 0$

$$u(0, t) = \alpha,\ u(L, t) = \beta;\ t \geq 0$$

$$u(x, 0) = f(x)$$

To convert the inhomogeneous boundary conditions to homogeneous boundary conditions, we use the following transformation formula:

$$u(x, t) = \left[\alpha + \left(\frac{\beta - \alpha}{L}\right)x\right] + v(x, t)$$

We can easily show that now $v(x, t)$ will be governed by the IBVP:

$$v_t = v_{xx}, \quad 0 < x < L, \qquad t > 0$$

$$v(0, t) = 0 = v(L, t), \quad t \geq 0$$

$$v(x, 0) = f(x) - \left[\alpha + \left(\frac{\beta - \alpha}{L}\right)x\right]$$

$v(x, t)$ can now easily be found using method of separation of variables. Consequently $u(x, t)$ can readily be obtained as a final result.

ILLUSTRATIVE EXAMPLES

Example 1. *A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function $u(x, t)$.* (U.P.T.U. 2015)

Sol. The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

As proved in Art. 2.11, we have

$$u(x, t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2 p^2 t} \qquad \ldots(1)$$

Since the ends $x = 0$ and $x = l$ are cooled to 0°C and kept at that temperature throughout, the boundary conditions are $u(0, t) = u(l, t) = 0$ for all t

Also $u(x, 0) = u_0$ is the initial condition.

Since $u(0, t) = 0$, we have from (1),

$$0 = c_1 c_3 e^{-c^2 p^2 t} \quad \Rightarrow \quad c_1 = 0$$

$\therefore$ From (1), $\quad u(x, t) = c_2 c_3 \sin px \,.\, e^{-c^2 p^2 t} \qquad \ldots(2)$

Since $u(l, t) = 0$, we have from (2),

$$0 = c_2 c_3 \sin pl \,.\, e^{-c^2 p^2 t}$$

$\Rightarrow \quad \sin pl = 0 \quad \Rightarrow \quad pl = n\pi$

$\therefore \quad p = \frac{n\pi}{l}$, n being an integer

Solution (2) reduces to $u(x, t) = b_n \sin \frac{n\pi x}{l} \,.\, e^{\frac{-c^2 n^2 \pi^2 t}{l^2}}$ on replacing $c_2 c_3$ by b_n.

The most general solution is obtained by adding all such solutions for $n = 1, 2, 3, \ldots\ldots$

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \,.\, e^{\frac{-c^2 n^2 \pi^2 t}{l^2}} \qquad \ldots(3)$$

Since $u(x, 0) = u_0$, we have $\quad u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

which is half-range sine series for u_0.

$$b_n = \frac{2}{l} \int_0^l u_0 \sin \frac{n\pi x}{l}\, dx = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{4u_0}{n\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Hence the temperature function

$$u(x, t) = \frac{4u_0}{\pi} \sum_{n=1, 3, 5, \ldots\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

$$= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l}\, e^{-\frac{c^2 (2n-1)^2 \pi^2 t}{l^2}}.$$

Example 2. *Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is* $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$.

(U.P.T.U. 2015)

Sol. Let $u(x, t)$ be the temperature in the bar. The boundary conditions are

$$u(0, t) = 0 = u(2, t) \text{ for any } t. \quad \text{...(1)}$$

The initial condition is

$$u(x, 0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} \quad \text{...(2)}$$

One-dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{...(3)}$$

Its solution is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2p^2t} \quad \text{...(4)}$$

$$u(0, t) = 0 = c_1c_3 e^{-c^2p^2t} \quad \text{| Using (1)}$$

$$\Rightarrow \quad c_1 = 0$$

$\therefore$ From (4),

$$u(x, t) = c_2c_3 \sin px\; e^{-c^2p^2t} \quad \text{...(5)}$$

$$u(2, t) = 0 = c_2c_3 \sin 2p\, e^{-c^2p^2t} \quad \text{| Using (1)}$$

$$\Rightarrow \quad \sin 2p = 0 = \sin n\pi$$

$$\therefore \quad p = \frac{n\pi}{2},\ n \in \mathrm{I}$$

Hence from (5),

$$u(x, t) = b_n \sin\frac{n\pi x}{2}\, e^{-\frac{n^2\pi^2c^2t}{4}} \quad \text{| } \because\ c_2c_3 = b_n$$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\, e^{-\frac{n^2\pi^2c^2t}{4}} \quad \text{...(6)}$$

$$u(x, 0) = \sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$= b_1 \sin\left(\frac{\pi x}{2}\right) + b_2 \sin\left(\frac{2\pi x}{2}\right) + \dots + b_5 \sin\left(\frac{5\pi x}{2}\right) + \dots$$

Comparing, we get

$$b_1 = 1 \text{ and } b_5 = 3$$

Hence from (6),

$$u(x, t) = \sin\left(\frac{\pi x}{2}\right) e^{-\pi^2 c^2 t/4} + 3 \sin\left(\frac{5\pi x}{2}\right) e^{-25\pi^2 c^2 t/4}.$$

Example 3. *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.*

[G.B.T.U. (A.G.) 2011, U.K.T.U. 2011]

Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

Sol. Initial temperature distribution in the rod is

$$u_1 = 0 + \left(\frac{100 - 0}{l}\right) x = \frac{100}{l} x$$

Final temperature distribution (*i.e.*, in steady state) is

$$u_2 = 0 + \left(\frac{0 - 0}{l}\right) x = 0$$

To get u in the intermediate period,

$$u = u_2(x) + u_1(x, t)$$

where $u_2(x)$ is the steady state temperature distribution in the rod. $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

$u_1(x, t)$ satisfies one-dimensional heat flow equation

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-c^2 p^2 t} \qquad \text{...(1)}$$

In steady state, $u(0, t) = 0 = u(l, t)$ ...(2)

$$\therefore \text{ From (1),} \quad u(0, t) = 0 = \sum_{n=1}^{\infty} a_n e^{-c^2 p^2 t} \Rightarrow a_n = 0 \qquad \text{...(3)}$$

Also, $$u(l, t) = 0 = \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2 p^2 t} \qquad | \text{ using (3)}$$

$$\Rightarrow \quad \sin pl = 0 = \sin n\pi,\ n \in I$$

or $$p = \frac{n\pi}{l} \qquad \text{...(4)}$$

$$\therefore \text{ From (1), (3) and (4),} \quad u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t} \qquad \text{...(5)}$$

Using initial condition,

$$u(x, 0) = \frac{100}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is half-range sine series for $\frac{100}{l} x$.

$$\therefore \quad b_n = \frac{2}{l}\int_0^l \frac{100}{l}\, x \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[\left\{x.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l 1.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{200}{l^2}\left[\frac{-l^2}{n\pi}\cos n\pi + \frac{l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l\right] = \frac{-200}{n\pi}(-1)^n$$

Hence the temperature function

$$u(x, t) = -\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

In the second part, the initial condition remains the same as in first part *i.e.,*

$$u(x, 0) = \frac{100}{l}x.$$

Boundary conditions are $u(0, t) = 20$ and $u(l, t) = 80$ for all values of t then, final temperature distribution is

$$u_2 = 20 + \left(\frac{80-20}{l}\right)x = 20 + \frac{60}{l}x$$

Then, $\quad u = u_2(x) + u_1(x, t)$

$$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty}(a_n \cos px + b_n \sin px)\, e^{-c^2p^2t} \quad \text{...(6)}$$

$$u(0, t) = 20 = 20 + \sum_{n=1}^{\infty} a_n e^{-c^2p^2t} \quad | \text{ From (6)}$$

$$\Rightarrow \quad a_n = 0$$

$\therefore$ From (6),

$$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin px\, e^{-c^2p^2t} \quad \text{...(7)}$$

$$u(l, t) = 80 = 20 + \frac{60}{l}l + \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2p^2t} \quad | \text{ From (7)}$$

$$\Rightarrow \quad 0 = \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2p^2t}$$

$$\sin pl = 0 = \sin n\pi,\ n \in \mathrm{I}$$

$$\therefore \quad p = \frac{n\pi}{l} \quad \text{...(8)}$$

From (7) and (8),

$$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi c}{l}\right)^2 t} \quad \text{...(9)}$$

Using initial condition,

$$u(x, 0) = \frac{100}{l}x = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \qquad \frac{40}{l}x - 20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l}\int_0^l \left(\frac{40}{l}x - 20\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left\{\left(\frac{40}{l}x - 20\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l \frac{40}{l}\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{2}{l}\left[\frac{-20l}{n\pi}\cos n\pi - \frac{20l}{n\pi} + \frac{40}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l\right]$$

$$= \frac{-40}{n\pi}(1 + \cos n\pi) = \begin{cases} 0, & \text{when } n \text{ is odd} \\ \dfrac{-80}{n\pi}, & \text{when } n \text{ is even} \end{cases}$$

Hence equation (9) becomes,

$$u(x, t) = 20 + \frac{60}{l}x - \frac{80}{\pi}\sum_{\substack{n=2,4,\ldots \\ (n \text{ is even})}}^{\infty} \frac{1}{n}\sin\frac{n\pi x}{l}\,e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

$$= 20 + \frac{60}{l}x - \frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\,e^{-\frac{4c^2m^2\pi^2 t}{l^2}}. \qquad (\text{taking } n = 2m)$$

Example 4. *The ends A and B of a rod of length 20 cm are at temperatures 30°C and 80°C until steady state prevails. Then the temperature of the rod ends are changed to 40°C and 60°C respectively. Find the temperature distribution function u(x, t). The specific heat, density and the thermal conductivity of the material of the rod are such that the combination* $\frac{k}{\rho\sigma} = c^2 = 1$.

Sol. Initial temperature distribution in the rod is

$$u_1 = 30 + \left(\frac{80 - 30}{20}\right)x = 30 + \frac{5}{2}x$$

Final temperature distribution (*i.e.*, in steady state) is

$$u_2 = 40 + \left(\frac{60 - 40}{20}\right)x = 40 + x$$

To get u in the intermediate period,

$$u = u_1(x, t) + u_2(x)$$

where $u_2(x)$ is the steady state temperature distribution in the rod $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

$\because$ $u_1(x, t)$ satisfies one-dimensional heat flow equation.

$$\therefore \qquad u = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-p^2 t} \qquad ...(1)$$

In steady state,

$$u(0, t) = 40 \qquad ...(2)$$

$$u(20, t) = 60 \qquad ...(3)$$

From (1), and (2), $u(0, t) = 40 = 40 + \sum_{1}^{\infty} a_n e^{-p^2 t}$ | From (2)

$$0 = \sum_{1}^{\infty} a_n e^{-p^2 t} \quad \Rightarrow \quad a_n = 0 \qquad ...(4)$$

$\therefore$ From (1),

$$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin px\, e^{-p^2 t}$$

Again, $u(20, t) = 60 = 60 + \sum_{n=1}^{\infty} b_n \sin 20p\, e^{-p^2 t}$

$$\Rightarrow \qquad 0 = \sum_{1}^{\infty} b_n \sin 20p\, e^{-p^2 t}$$

$$\sin 20p = 0 = \sin n\pi, \; n \in \mathrm{I}$$

$$\Rightarrow \qquad p = \frac{n\pi}{20}$$

$$\therefore \qquad u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}\, e^{-\left(\frac{n\pi}{20}\right)^2 t} \qquad ...(5)$$

Using initial condition,

$$u(x, 0) = 30 + \frac{5}{2}x \quad \text{in eqn. (5), we get}$$

$$\Rightarrow \qquad 30 + \frac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

$$\Rightarrow \qquad \frac{3}{2}x - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where $$b_n = \frac{2}{20} \int_0^{20} \left(\frac{3}{2}x - 10\right) \sin \frac{n\pi x}{20}\, dx = -\frac{20}{n\pi}\, [2(-1)^n + 1]$$

From (5), $$u(x, t) = 40 + x - \frac{20}{\pi} \sum_{n=1}^{\infty} \left\{\frac{2(-1)^n + 1}{n}\right\} \sin \frac{n\pi x}{20}\, e^{-\left(\frac{n\pi}{20}\right)^2 t}.$$

Example 5. *The temperature distribution in a bar of length π which is perfectly insulated at ends $x = 0$ and $x = \pi$ is governed by partial differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Assuming the initial temperature distribution as $u(x, 0) = f(x) = \cos 2x$, find the temperature distribution at any instant of time. (M.T.U. 2011)

Sol. $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

Its solution is $$u(x, t) = c_1 e^{-p^2 t} (c_2 \cos px + c_3 \sin px) \quad ...(2)$$

Since ends of bar are insulated, no heat can pass from either sides and boundary conditions are

$$\frac{\partial u}{\partial x} = 0 \quad \text{at } x = 0 \quad ...(3)$$

and $$\frac{\partial u}{\partial x} = 0 \quad \text{at } x = \pi \quad ...(4)$$

From (2), $$\frac{\partial u}{\partial x} = c_1 e^{-p^2 t} (-pc_2 \sin px + pc_3 \cos px)$$

At $x = 0$,

$$0 = c_1 e^{-p^2 t}\, pc_3 \quad \Rightarrow \quad c_3 = 0$$

$\therefore$ From (2), $$u(x, t) = c_1 c_2\, e^{-p^2 t} \cos px \quad ...(5)$$

Again $$\frac{\partial u}{\partial x} = -pc_1 c_2\, e^{-p^2 t} \sin px$$

At $x = \pi$,

$$0 = -pc_1 c_2 e^{-p^2 t} \sin p\pi$$

$\Rightarrow$ $$\sin \cancel{p\pi = 0} = \sin n\pi \ (n \in \text{I})$$

$$p\pi = n\pi \quad \Rightarrow \quad p = n$$

$\therefore$ From (5), $u(x, t) = b_n\, e^{-n^2 t} \cos nx$, where $c_1 c_2 = b_n$

Most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n\, e^{-n^2 t} \cos nx \quad ...(6)$$

$$u(x, 0) = \cos 2x = \sum_{n=1}^{\infty} b_n \cos nx$$

Comparing, we get $b_2 = 1$ and $n = 2$. All other b_i's are zero.

$\therefore$ From (6), $u(x, t) = e^{-4t} \cos 2x$.

Example 6. *Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary condition $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$, $u(l, t) = 0$, where $0 < x < l$.*

Sol. The solution to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

is given by $$u(x, t) = c_1 e^{-p^2 t} (c_2 \cos px + c_3 \sin px) \quad ...(2)$$

From (2), $u(0, t) = c_1 c_2 e^{-p^2 t}$

$\Rightarrow$ $0 = c_1 c_2 e^{-p^2 t}$

$\Rightarrow$ $c_2 = 0.$

$\therefore$ From (2), $$u(x, t) = c_1 c_3 e^{-p^2 t} \sin px \quad ...(3)$$

$u(l, t) = 0 = c_1 c_3 e^{-p^2 t} \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi (n \in \text{I})$

$\therefore$ $p = \dfrac{n\pi}{l}.$

From (3), $u(x, t) = c_1 c_3 e^{-\frac{n^2\pi^2}{l^2}t} \sin \dfrac{n\pi x}{l} = b_n e^{-\frac{n^2\pi^2 t}{l^2}} \sin \dfrac{n\pi x}{l}$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t/l^2)} \sin \frac{n\pi x}{l} \quad ...(4)$$

$\therefore$ From (4), $u(x, 0) = \sum_{1}^{\infty} b_n \sin \dfrac{n\pi x}{l}$

$\Rightarrow$ $3 \sin n\pi x = \sum_{1}^{\infty} b_n \sin \dfrac{n\pi x}{l}.$

Comparison gives, $b_n = 3, l = 1.$

Hence from (4), the required solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

Example 7. *A bar with insulated sides is initially at a temperature 0°C throughout. The end $x = 0$ is kept at 0°C, and heat is suddenly applied at the end $x = l$ so that $\dfrac{\partial u}{\partial x} = A$ for $x = l$, where A is a constant. Find the temperature function $u(x, t)$.*

Sol. One-dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

Its solution is

$$u(x, t) = c_1 e^{-p^2 c^2 t} (c_2 \cos px + c_3 \sin px)$$

or $$u(x, t) = (A_1 \cos px + B \sin px)\, e^{-p^2 c^2 t} \quad ...(2)$$

Applying the zero end conditions as,

$u(0, t) = 0 = A_1 e^{-p^2 c^2 t}$

$\Rightarrow$ $A_1 = 0.$

$\therefore$ From (2), $\quad u(x, t) = B \sin px e^{-p^2c^2t}$ ...(3)

From (3), $\quad \dfrac{\partial u}{\partial x} = pB \cos px e^{-p^2c^2t}$.

At $x = l$, $\quad \left(\dfrac{\partial u}{\partial x}\right)_{x=l} = 0 = pB \cos pl e^{-p^2c^2t}$

$\Rightarrow \quad \cos pl = 0 = \cos\left(n\pi - \dfrac{\pi}{2}\right)$; $n \in I$ or $pl = (2n-1)\dfrac{\pi}{2}$

$\Rightarrow \quad p = (2n-1)\dfrac{\pi}{2l}$.

$\therefore$ From (3), $\quad u(x, t) = B \sin px e^{-p^2c^2t}$ ...(4) where $p = (2n-1)\dfrac{\pi}{2l}$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin px\, e^{-p^2c^2t} \quad \text{...(5) where } p = (2n-1)\frac{\pi}{2l}.$$

The end conditions given for this problem are

(*i*) $u = 0$ at $x = 0$ $\quad$ (*ii*) $\dfrac{\partial u}{\partial x} = A$ at $x = l$ ...(6)

These conditions are different from the zero end conditions. So we add to (5) the solution

$$u = A_1x + B$$

Choosing A_1 and B so that (6) is satisfied.
This gives, $0 = B$ and $A_1 = A$

$$\therefore \quad u(x, t) = Ax + \sum_{n=1}^{\infty} B_n \sin px\, e^{-p^2c^2t} \quad \text{...(7) where } p = (2n-1)\frac{\pi}{2l}.$$

Applying the condition that $u = 0$ at $t = 0$, we have

$$0 = Ax + \sum_{n=1}^{\infty} B_n \sin px$$

or
$$-Ax = \sum_{n=1}^{\infty} B_n \sin px$$

where
$$B_n = \frac{2}{l}\int_0^l (-Ax)\sin px\, dx, \text{ where } p = (2n-1)\frac{\pi}{2l}$$

$$= \frac{-2A}{l}\left[\left\{x\left(\frac{-\cos px}{p}\right)\right\}_0^l - \int_0^l 1.\left(\frac{-\cos px}{p}\right)dx\right]$$

$$= -\frac{2A}{l}\left[\frac{-l\cos pl}{p} + \frac{1}{p}\left(\frac{\sin px}{p}\right)_0^l\right]$$

$$= \frac{-2A}{l}\left[\frac{-l\cos pl}{p} + \frac{1}{p^2}\sin pl\right] = -\frac{2A(2l)^2}{l(2n-1)^2\pi^2}\sin(2n-1)\frac{\pi}{2}$$

$$(\because \quad \cos pl = 0)$$

$$= \frac{-8Al}{\pi^2(2n-1)^2}\sin\left(n\pi - \frac{\pi}{2}\right) = \frac{8Al}{\pi^2(2n-1)^2}(-1)^n \qquad ...(8)$$

$\therefore$ From (7), $$u(x, t) = Ax + \frac{8Al}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n-1)^2}\right]\sin(2n-1)\frac{\pi x}{2l}\, e^{-\left[\frac{(2n-1)^2\pi^2c^2t}{4l^2}\right]}$$

Example 8. *Solve:* $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *under the conditions*

(i) $u \neq \infty$ *if* $t \to \infty$ *(ii)* $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$

(iii) $u = lx - x^2$ *for* $t = 0$ *between* $x = 0$ *and* $x = l$. (U.P.T.U. 2015)

Sol. Solution to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ is

$$u(x, t) = c_1 e^{-c^2kt}(c_2\cos cx + c_3\sin cx) \qquad ...(1)$$

Eqn. (1) satisfies the condition $u \neq \infty$ if $t \to \infty$

Applying $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ to (1), we get

$$c_3 = 0$$

and $$c = \frac{n\pi}{l},\ n \in I$$

$$\therefore \qquad u = c_1c_2\, e^{-\left(\frac{n^2\pi^2kt}{l^2}\right)}\cos\frac{n\pi x}{l} = a_n\cos\frac{n\pi x}{l}\, e^{-\left(\frac{n^2\pi^2kt}{l^2}\right)} \qquad ...(2)$$

Again, the second possible solution is

$$u = c_1(c_2x + c_3) \qquad ...(3) \qquad | \text{ if } c^2 = 0$$

Applying $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ to (3), we get $c_2 = 0$

$$\therefore \qquad u = c_1c_3 = \frac{a_0}{2}\ \text{(say)} \qquad ...(4) \qquad | \text{ From (3)}$$

$\therefore$ The general solution is the sum of solutions (2) and (4) for various n.

$$\therefore \qquad u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\, e^{-\left(\frac{n^2\pi^2kt}{l^2}\right)} \qquad ...(5)$$

Now applying $u = lx - x^2$ for $t = 0$ to eqn. (5), we get

$$lx - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

Here, $$a_0 = \frac{2}{l}\int_0^l (lx - x^2)\,dx = \frac{l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \begin{cases} -\dfrac{4l^2}{n^2\pi^2}; & \text{when } n \text{ is even} \\ 0; & \text{when } n \text{ is odd} \end{cases}$$ | On simplification

$$\therefore \qquad u = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{n^2}\cos\frac{n\pi x}{l}\, e^{-\left(\frac{n^2\pi^2 kt}{l^2}\right)}$$

Put $n = 2m$, we get

$$u(x, t) = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{m^2}\cos\frac{2m\pi x}{l}\, e^{-\left(\frac{4m^2\pi^2 kt}{l^2}\right)}$$

TEST YOUR KNOWLEDGE

1. (*i*) Solve: $\dfrac{\partial u}{\partial t} = \alpha\dfrac{\partial^2 u}{\partial x^2}$; α constant, subject to the boundary conditions $u(0, t) = 0$, $u(\pi, t) = 0$ and the initial condition $u(x, 0) = \sin 2x$. (*M.T.U. 2012*)

   (*ii*) Solve: $\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}$ given that

   (*a*) $u = 0$ when $x = 0$ and $x = l$ for all t (*b*) $u = 3\sin\dfrac{\pi x}{l}$ when $t = 0$ for all x.

   (*iii*) Solve: $u_t = a^2 u_{xx}$ under the conditions $u_x(0, t) = 0 = u_x(\pi, t)$ and $u(x, 0) = x^2 (0 < x < \pi)$.

2. (*i*) Determine the solution of one-dimensional heat equation $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$ where the boundary conditions are $u(0, t) = 0$, $u(l, t) = 0$ $(t > 0)$ and the initial condition $u(x, 0) = x$: l being the length of the bar. (*U.P.T.U. 2015*)

   (*ii*) Find the temperature distribution in a rod of length 2 m whose end points are fixed at temperature zero and the initial temperature distribution is $f(x) = 100x$. (*G.B.T.U. 2012*)

3. The heat flow in a bar of length 10 cm of homogeneous material is governed by partial diff. eqn. $u_t = c^2 u_{xx}$. The ends of the bar are kept at temp. 0°C and initial temp. is $f(x) = x(10 - x)$. Find the temp. in the bar at any instant of time. (*U.P.T.U. 2014*)

4. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length L cm. with its ends kept at zero temperature and initial temperature given by $\dfrac{x(L - x)\,d}{L^2}$.

5. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is $u(x, 0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases}$

   Find the temperature $u(x, t)$ at any time.

6. Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = L$ are kept at zero temperature and whose initial temperature $f(x)$ is given by

$$f(x) = \begin{cases} k, & \text{when } 0 < x < \frac{1}{2}L \\ 0, & \text{when } \frac{1}{2}L < x < L \end{cases}.$$

7. Find the temperature distribution in a rod of length π which is totally insulated including the ends and the initial temperature distribution is $100 \cos x$. *(U.P.T.U. 2015)*

8. Find the temperature in a thin metal rod of length L with both ends insulated (so that there is no passage of heat through the ends) and with initial temperature $\sin \frac{\pi x}{L}$ in the rod.

9. (*i*) The temperature of a bar 50 cm long with insulated sides is kept at 0° at one end and 100° at the other end until steady conditions prevail. The two ends are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

(*ii*) A bar 10 cm long, with insulated sides, has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature at A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t.

(*iii*) A bar of 10 cm length with insulated sides A and B are kept and 20°C and 40°C respectively until steady state conditions prevail. The temperature at A is then suddenly varied to 50°C and the same instant at B, lowered at 10°C. Find the subsequent temperature at any point of the bar at any time. *(A.K.T.U. 2017)*

10. A homogeneous rod of conducting material of length '1' has its ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature distribution. *(U.K.T.U. 2012)*

$$\left[\textbf{Hint: } u(x, 0) = \begin{cases} 2Tx, & 0 \le x \le \frac{1}{2} \\ 2T(1-x), & \frac{1}{2} \le x \le 1 \end{cases}\right]$$

11. Solve $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$, such that

(*i*) θ is finite when $t \to \infty$, (*ii*) $\frac{\partial \theta}{\partial x} = 0$ when $x = 0$ and $\theta = 0$ when $x = l$ for all t,

(*iii*) $\theta = \theta_0$ when $t = 0$ for all values of x between 0 and l.

12. Find a solution of the heat conduction equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ such that

(*i*) u is finite when $t \to \infty$, (*ii*) $u = 100$ when $x = 0$ or π for all values of t,

(*iii*) $u = 0$ when $t = 0$ for all values of x between 0 and π.

(Here, the initially ice-cold rod has its ends in boiling water.)

13. Solve the following IBVP: $u_t = u_{xx}$, $0 < x < 1$, $t > 0$

$u(0, t) = 1$, $u(1, t) = 2$, $t \ge 0$

$u(x, 0) = 1 + x + 2 \sin \pi x$

14. Solve the following IBVP: $u_t = u_{xx}$, $0 < x < 1$, $t > 0$

$u(0, t) = 1$, $u(1, t) = 3$

$u(x, 0) = 1 + 2x + 3 \sin \pi x$

Answers

1. (*i*) $u(x, t) = \sin 2x\, e^{-4\alpha t}$ (*ii*) $u(x, t) = 3 \sin \dfrac{\pi x}{l} e^{-(a^2\pi^2 t/l^2)}$

(*iii*) $u(x, t) = \dfrac{\pi^3}{3} + 4 \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx\, e^{-a^2 n^2 t}$

2. (*i*) $u(x, t) = -\dfrac{2l}{\pi} \sum_{n=1}^{\infty} \dfrac{\cos n\pi}{n} \sin \dfrac{n\pi x}{l} e^{-\left(\frac{c^2 n^2 \pi^2 t}{l^2}\right)}$

(*ii*) $u(x, t) = -\dfrac{400}{\pi} \sum_{n=1}^{\infty} \dfrac{\cos n\pi}{n} \sin \dfrac{n\pi x}{2} e^{-\left(\frac{c^2 n^2 \pi^2 t}{4}\right)}$

3. $u(x, t) = \dfrac{800}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin \dfrac{(2n-1)\pi x}{10} e^{-\left[\frac{(2n-1)^2 \pi^2 c^2 t}{100}\right]}$.

4. $u(x, t) = \dfrac{8d}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin \dfrac{(2n-1)\pi x}{L} e^{-\frac{(2n-1)^2 \pi^2 c^2 t}{L^2}}$.

5. $u(x, t) = \dfrac{400}{\pi^2} \sum_{m=0}^{\infty} \dfrac{(-1)^m}{(2m+1)^2} \sin \dfrac{(2m+1)\pi x}{100} e^{-\left[\frac{(2m+1)c\pi}{100}\right]^2 t}$

6. $u(x, t) = \dfrac{4k}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n} \sin^2 \dfrac{n\pi}{4} \sin \dfrac{n\pi x}{L} e^{-\left(\frac{c^2 n^2 \pi^2 t}{L^2}\right)}$ **7.** $100\, e^{-t} \cos x$

8. $u(x, t) = \dfrac{2}{\pi} - \dfrac{4}{\pi} \sum_{m=1}^{\infty} \dfrac{1}{(4m^2-1)} \cos\left(\dfrac{2m\,\pi x}{L}\right) e^{-\frac{4m^2\pi^2 c^2 t}{L^2}}$

9. (*i*) $u(x, t) = \dfrac{200}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \sin \dfrac{n\pi x}{50} e^{-\left(\frac{n^2\pi^2 kt}{2500}\right)}$

(*ii*) $u(x, t) = 90 - 3x - \dfrac{80}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n} \sin \dfrac{n\pi x}{5} e^{-\left(\frac{c^2 n^2 \pi^2 t}{25}\right)}$

(*iii*) $u(x, t) = 50 - 4x - \dfrac{120}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin \dfrac{n\pi x}{10} e^{-\frac{n^2\pi^2 t}{100}}$

10. $u(x, t) = \dfrac{8T}{\pi^2} \sum_{m=1}^{\infty} \dfrac{(-1)^{m+1}}{(2m-1)^2} \sin (2m-1)\pi x\, e^{-[(2m-1)^2\pi^2 c^2 t]}$

11. $\theta = \dfrac{4\theta_0}{\pi}\left[e^{-(\pi/2l)^2 kt} \cos \dfrac{\pi x}{2l} - \dfrac{1}{3} e^{-\left(\frac{3\pi}{2l}\right)^2 kt} \cos \dfrac{3\pi x}{2l} + \dfrac{1}{5} e^{-\left(\frac{5\pi}{2l}\right)^2 kt} \cos \dfrac{5\pi x}{2l} - \ldots\right]$

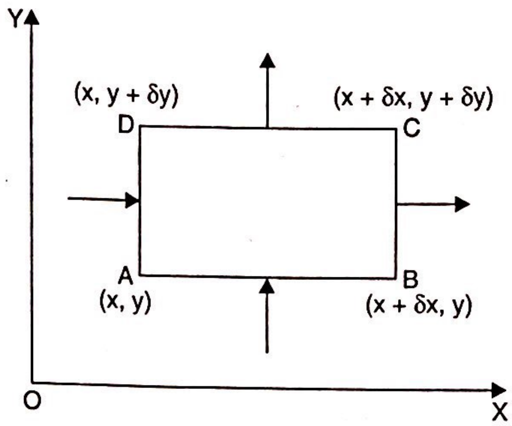
12. $u(x, t) = 100 - \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)x}{2m-1} e^{-(2m-1)^2 \alpha t}$

13. $u(x, t) = 1 + x + 2e^{-\pi^2 t} \sin \pi x$

14. $u(x, t) = 1 + 2x + 3e^{-\pi^2 t} \sin \pi x.$

2.13 TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate, in the XOY plane. If the temperature at any point is independent of the z-coordinate and depends on x, y and t only, then the flow is called two dimensional and the heat-flow lies in the plane XOY only and is zero along the normal to the plane XOY.



Take a rectangular element of the plate with sides δx and δy and thickness α. As discussed in the one-dimensional heat flow along a bar, the quantity of heat that enters the plate per second from the sides AB and AD is given by

$$-k\alpha\, \delta x \left(\frac{\partial u}{\partial y}\right)_y \quad \text{and} \quad -k\alpha\, \delta y \left(\frac{\partial u}{\partial x}\right)_x$$

respectively and that which flows out through the sides CD and BC per second is

$$-k\alpha\, \delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \quad \text{and} \quad -k\alpha\, \delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ respectively.}$$

Therefore, the total gain of heat by the rectangular plate ABCD per second

$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\delta y \left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y} \right] \quad \ldots(1)$$

The rate of gain of heat by the plate is also given by

$$s\rho\delta x\delta y \frac{\partial u}{\partial t} \quad \ldots(2)$$

where s = specific heat and ρ = density of the metal plate.

Equating (1) and (2), we obtain

$$k\alpha\delta x\delta y \left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y} \right] = s\rho\delta x\delta y \frac{\partial u}{\partial t}$$

Dividing both sides by $\alpha\delta x\delta y$ and taking the limit as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = s\rho \frac{\partial u}{\partial t}$$

or $$\boxed{c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}} \quad \text{where } c^2=\frac{k}{s\rho} \qquad ...(3)$$

Equation (3) gives the temperature distribution of the plate in the transient state.

Note 1. In steady state, u is independent of t, so that $\frac{\partial u}{\partial t}=0$ and the above equation reduces to

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad ...(4)$$

which is known as **Laplace's Equation in two dimensions.**

Note 2. The equation of heat flow in a solid (Three-dimensional heat flow) can similarly be derived as

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial u}{\partial t}$$

In the steady state, it reduces to $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

which is **Laplace's Equation in three dimensions.**

2.14 SOLUTION OF LAPLACE'S EQUATION IN TWO DIMENSIONS (*G.B.T.U. 2011*)

Laplace's equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad ...(1)$$

Let $$u = XY \qquad ...(2)$$

where X is a function of x only and Y is a function of y only, be a solution of (1).

Then $$\frac{\partial^2 u}{\partial x^2}=X''Y \text{ and } \frac{\partial^2 u}{\partial y^2}=XY''$$

Substituting in (1), we have $X''Y + XY'' = 0$ or $$\frac{X''}{X}=-\frac{Y''}{Y} \qquad ...(3)$$

Now the LHS of (3) is a function of x only and the R.H.S is a function of y only. Since x and y are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2}-kX=0 \quad \text{and} \quad \frac{d^2Y}{dy^2}+kY=0 \qquad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1e^{px}+c_2e^{-px},\ Y = c_3\cos py + c_4 \sin py$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1\cos px + c_2\sin px,\ Y = c_3e^{py}+c_4e^{-py}$$

(*iii*) *When k = 0*

$$X = c_1x + c_2,\ Y = c_3y + c_4$$

Thus, the various possible solutions of Laplace's equation (1) are:

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(5)$$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(6)$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \qquad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions. Solution (6) is the required solution.

$$\boxed{u(x, y) = (c_1 \cos px + c_2 \sin px)\,(c_3 e^{py} + c_4 e^{-py}).}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Use separation of variables method to solve the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin \frac{n\pi x}{l}$.

[A.K.T.U. 2017]

Sol. The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Let $\qquad u = XY \qquad ...(2)$

where X is a function of x only and Y is a function of y only then,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XY) = X\frac{d^2Y}{dy^2}$$

$\therefore$ From (1), $\qquad YX'' + XY'' = 0$

$$\Rightarrow \qquad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

Case I. When $\qquad \dfrac{X''}{X} = -\dfrac{Y''}{Y} = p^2$ (say)

(*i*) $\qquad \dfrac{X''}{X} = p^2$

$$X'' - p^2 X = 0$$

Auxiliary equation is $\qquad m^2 - p^2 = 0$

$$m = \pm p$$

$\therefore \qquad$ C.F. $= c_1 e^{px} + c_2 e^{-px}$

$$\text{P.I.} = 0$$

$\therefore \qquad X = c_1 e^{px} + c_2 e^{-px}$

(*ii*) $\qquad \dfrac{-Y''}{Y} = p^2 \Rightarrow Y'' + p^2 Y = 0$

Auxiliary equation is $\qquad m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$\therefore \qquad$ C.F. $= c_3 \cos py + c_4 \sin py$