CONTENTS

TYPES AND PROPERTIES OF RELATIONS

1. Reflexive Relation: A relation R on set A is said to be a reflexive if **(a, a) ∈ R** for every **a ∈** A.
Example: If A = {1, 2, 3, 4} then R = {(1, 1) (2, 2), (1, 3), (2, 4), (3, 3), (3, 4), (4, 4)}. Is a relation reflexive?
Solution: The relation is reflexive as for every a ∈ A. (a, a) ∈ R, i.e. (1, 1), (2, 2), (3, 3), (4, 4) ∈ R.

2. **Irreflexive Relation:** A relation R on set A is said to be **irreflexive if (a, a) ∉ R** for every **a ∈ A**.
Example: Let A = {1, 2, 3} and R = {(1, 2), (2, 2), (3, 1), (1, 3)}. Is the relation R reflexive or irreflexive?
Solution: The relation R is not reflexive as for every a ∈ A, (a, a) ∉ R, i.e., (1, 1) and (3, 3) ∉ R. The relation R is not irreflexive as (a, a) ∉ R, for some a ∈ A, i.e., (2, 2) ∈ R.

3. Symmetric Relation: A relation R on set A is said to be symmetric iff (a, b) ∈ R ⟺ (b, a) ∈ R.

Example: Let A = {1, 2, 3} and R = {(1, 1), (2, 2), (1, 2), (2, 1), (2, 3), (3, 2)}. Is a relation R symmetric or not?

Solution: The relation is symmetric as for every (a, b) ∈ R, we have (b, a) ∈ R, i.e., (1, 2), (2, 1), (2, 3), (3, 2) ∈ R but not reflexive because (3, 3) ∉ R.

4. Antisymmetric Relation: A relation R on a set A is antisymmetric iff (a, b) ∈ R and (b, a) ∈ R then a = b.

Example1: Let A = {1, 2, 3} and R = {(1, 1), (2, 2)}. Is the relation R antisymmetric?

Solution: The relation R is antisymmetric as a = b when (a, b) and (b, a) both belong to R.

5. Asymmetric Relation: A relation R on a set A is called an Asymmetric Relation if for every (a, b) ∈ R implies that (b, a) does not belong to R.

6. Transitive Relations: A Relation R on set A is said to be transitive iff (a, b) ∈ R and (b, c) ∈ R ⟺ (a, c) ∈ R.

Example1: Let A = {1, 2, 3} and R = {(1, 2), (2, 1), (1, 1), (2, 2)}. Is the relation transitive?

Solution: The relation R is transitive as for every (a, b) (b, c) belong to R, we have (a, c) ∈ R i.e, (1, 2) (2, 1) ∈ R ⇒ (1, 1) ∈ R.

7. Identity Relation: Identity relation I on set A is reflexive, transitive and symmetric. So identity relation I is an Equivalence Relation.

Example: A= {1, 2, 3} = {(1, 1), (2, 2), (3, 3)}

8. Void/Null/Empty Relation: It is given by R: A →B such that R = ∅ (⊆ A x B) is a null relation.

9. Universal Relation: A relation R: A →B such that R = A x B (⊆ A x B) is a universal relation. Universal Relation from A →B is reflexive, symmetric and transitive. So this is an equivalence relation.

10. Inverse Relation: Inverse relation is seen when a set has elements which are inverse pairs of another set. For example if set A = {(a, b), (c, d)}, then inverse relation will be R^{-1} = {(b, a), (d, c)}. So, for an inverse relation,

$$\mathbf{R^{-1} = \{(b, a): (a, b) \in R\}}$$

FREQUENTLY ASKED QUESTIONS

- Number of different relation from a set with n elements to a set with m elements is 2^{mn}
- Number of Reflexive Relations on a set with n elements : $2^{n(n-1)}$
- Number of Symmetric Relations on a set with n elements : $2^{n(n+1)/2}$
- Number of Anti-Symmetric Relations on a set with n elements: $2^n\ 3^{n(n-1)/2}$.
- Number of Asymmetric Relations on a set with n elements : $3^{n(n-1)/2}$.
- Irreflexive Relations on a set with n elements : $2^{n(n-1)}$.
- Reflexive and symmetric Relations on a set with n elements : $2^{n(n-1)/2}$.

COMPOSITION OF RELATIONS

- Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of A × B and S is a subset of B × C. Then R and S give rise to a relation from A to C indicated by R∘S and defined by:

  Like a (R∘S)c **if for** some b ∈ B we have aRb and bSc.

  is,

  R ∘ S = {(a, c)| there exists b ∈ B **for** which (a, b) ∈ R and (b, c) ∈ S}

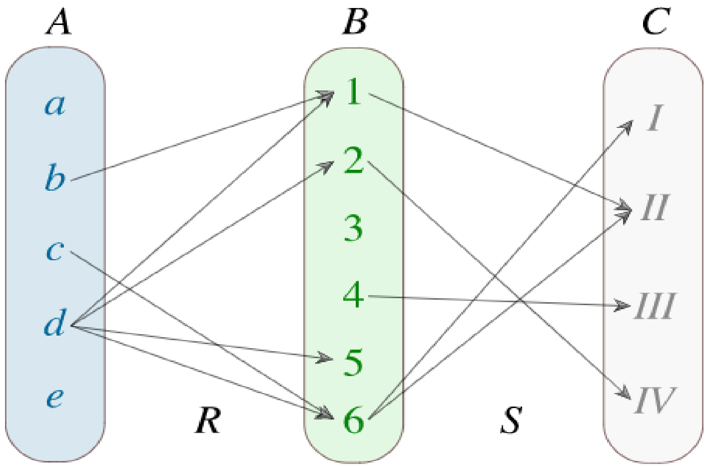
- The relation R∘S is known the composition of R and S; it is sometimes denoted simply by RS.

- Let R is a relation on a set A, that is, R is a relation from a set A to itself. Then R∘R, the composition of R with itself, is always represented. Also, R∘R is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive n.

We assume that the reader is already familiar with the basic operations on binary relations such as the union or intersection of relations. Now we consider one more important operation called the composition of relations.

Definition

Let A, B and C be three sets. Suppose that R is a relation from A to B, and S is a relation from B to C.



The composition of R and S, denoted by $S \circ R$, is a binary relation from A to C, if and only if there is a $b \in B$ such that aRb and bSc. Formally the composition $S \circ R$ can be written as

$$S \circ R = \{(a, c) \mid \exists b \in B : aRb \wedge bSc\},$$

where $a \in A$ and $c \in C$.

The composition of binary relations is associative, but not commutative.

EXAMPLE 1

- Exp 1 : Let X = {4, 5, 6}, Y = {a, b, c} and Z = {l, m, n}. Consider the relation R_1 from X to Y and R_2 from Y to Z.
- R_1 = {(4, a), (4, b), (5, c), (6, a), (6, c)} R_2 = {(a, l), (a, n), (b, l), (b, m), (c, l), (c, m), (c, n)}
- Find the composition of relation **(i)** R_1 o R_2 **(ii)** R_1o R_1^{-1}

Solution:

(i) The composition relation R_1 o R_2 as

R_1 o R_2 = {(4, l), (4, n), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)}

(ii) The composition relation R_1o R_1^{-1}

R_1o R_1^{-1} = {(4, 4), (5, 5), (5, 6), (6, 4), (6, 5), (4, 6), (6, 6)}

EXAMPLE 2

- Let P = {2, 3, 4, 5}. Consider the relation R and S on P defined by

  R = {(2, 2), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (5, 3)}

  S = {(2, 3), (2, 5), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (5, 2), (5, 5)}.

  find the following composition of the relation

  (i)RoS (ii)RoR (iii)SoR

The composition R o S of the relation R and S is

(i) R o S = {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (4, 2), (4, 5), (5, 2), (5, 3), (5, 4), (5, 5)}.

(ii) R o R = {(2, 2), (3, 2), (3, 3), (3, 4), (4, 2), (4, 5), (5, 2), (5, 3), (5, 5)}

(iii) So R = {(2, 4) , (2, 5), (3, 3), (3, 4), (3, 5), (4, 2), (4, 4), (4, 5), (5, 2), (5, 3), (5, 4), (5, 5)}.

EQUIVALENCE RELATIONS

relation R on a set A is said to be an **equivalence relation** if and only if the relation R is reflexive, symmetric and transitive.

Reflexive: A relation is said to be reflexive, if (a, a) ∈ R, for every a ∈ A.

Symmetric: A relation is said to be symmetric, if (a, b) ∈ R, then (b, a) ∈ R.

Transitive: A relation is said to be transitive if (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R.

Question 1:

Let assume that F is a relation on the set **R** real numbers defined by xFy if and only if x-y is an integer. Prove that F is an equivalence relation on **R.**

Solution:

Reflexive: Consider x belongs to **R**,then x – x = 0 which is an integer. Therefore xFx.

Symmetric: Consider x and y belongs to **R** and xFy. Then x – y is an integer. Thus, y – x = – (x – y), y – x is also an integer. Therefore yFx.

Transitive: Consider x and y belongs to **R**, xFy and yFz. Therefore x-y and y-z are integers. According to the transitive property, (x – y) + (y – z) = x – z is also an integer. So that xFz.

Thus, R is an equivalence relation on **R**.

EQUIVALENCE RELATIONS QUESTIONS

2. A relation R is defined on the set Z by "a R b if a – b is divisible by 5" for a, b ∈ Z. Examine if R is an equivalence relation on Z.

Solution:

(i) Let a ∈ Z. Then a – a is divisible by 5. Therefore aRa holds for all a in Z and R is reflexive.

(ii) Let a, b ∈ Z and aRb hold. Then a – b is divisible by 5 and therefore b – a is divisible by 5.

Thus, aRb ⇒ bRa and therefore R is symmetric.

(iii) Let a, b, c ∈ Z and aRb, bRc both hold. Then a – b and b – c are both divisible by 5.

Therefore a – c = (a – b) + (b – c) is divisible by 5.

Thus, aRb and bRc ⇒ aRc and therefore R is transitive.

Since R is reflexive, symmetric and transitive so, R is an equivalence relation on Z.

3. Let m e a positive integer. A relation R is defined on the set Z by “aRb if and only if a – b is divisible by m” for a, b ∈ Z. Show that R is an equivalence relation on set Z.

Solution:

(i) Let a ∈ Z. Then a – a = 0, which is divisible by m

Therefore, aRa holds for all a ∈ Z.

Hence, R is reflexive.

(ii) Let a, b ∈ Z and aRb holds. Then a – b is divisible by m and therefore, b – a is also divisible by m.

Thus, aRb ⇒ bRa.

Hence, R is symmetric.

(iii) Let a, b, c ∈ Z and aRb, bRc both hold. Then a – b is divisible by m and b – c is also divisible by m. Therefore, a – c = (a – b) + (b – c) is divisible by m.

Thus, aRb and bRc ⇒ aRc

Therefore, R is transitive.

Since, R is reflexive, symmetric and transitive so, R is an equivalence relation on set Z

Question 2:

Show that the relation R is an equivalence relation in the set A = { 1, 2, 3, 4, 5 } given by the relation R = { (a, b):|a-b| is even }.

Solutio :

R = { (a, b):|a-b| is even }. Where a, b belongs to A

Reflexive Property :

From the given relation,

$|a - a| = | 0 | = 0$

And 0 is always even.

Thus, |a-a| is even

Therefore, (a, a) belongs to R

Hence R is Reflexive

Symmetric Property :

From the given relation,

$|a - b| = |b - a|$

We know that $|a - b| = |-(b - a)| = |b - a|$

Hence $|a - b|$ is even, Then $|b - a|$ is also even.

Therefore, if $(a, b) \in R$, then (b, a) belongs to R

Hence R is symmetric.

Transitive Property :

If $|a-b|$ is even, then $(a-b)$ is even.

Similarly, if $|b-c|$ is even, then $(b-c)$ is also even.

Sum of even number is also even So, we can write it as $a-b+ b-c$ is even

Then, $a - c$ is also even

So, $|a - b|$ and $|b - c|$ is even , then $|a-c|$ is even.

Therefore, if $(a, b) \in R$ and $(b, c) \in R$, then (a, c) also belongs to R

Hence R is transitive.

Let us assume that R be a relation on the set of ordered pairs of positive integers such that ((a, b), (c, d))∈ R if and only if ad=bc. Is R an equivalence relation?

Reflexive Property

According to the reflexive property, if (a, a) ∈ R, for every a∈A

For all pairs of positive integers,

((a, b),(a, b))∈ R.

Clearly, we can say

ab = ab for all positive integers.

Hence, the reflexive property is proved.

Symmetric Property

From the symmetric property,

if (a, b) ∈ R, then we can say (b, a) ∈ R

For the given condition,

if ((a, b),(c, d)) ∈ R, then ((c, d),(a, b)) ∈ R.

If ((a, b),(c, d))∈ R, then ad = bc and cb = da

since multiplication is commutative.

Therefore ((c, d),(a, b)) ∈ R

Hence symmetric property is proved.

Transitive Property

From the transitive property,

if $(a, b) \in R$ and $(b, c) \in R$, then (a, c) also belongs to R

For the given set of ordered pairs of positive integers,

$((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$,

then $((a, b),(e, f) \in R$.

Now, assume that $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$.

Then we get, $ad = cb$ and $cf = de$.

The above relation implies that $a/b = c/d$ and that $c/d = e/f$,

so $a/b = e/f$ we get $af = be$.

Therefore $((a, b),(e, f)) \in R$.

Hence transitive property is proved

MATRIX REPRESENTATION OF RELATIONS

Let $A = \{a_1, a_2, a_3 \ldots . a_n\}$, $B = \{b_1, b_2, b_3, \ldots . b_m\}$ and $R \subseteq A \times B$. Then the relation matrix of R is denoted by $M_R = [m_{ij}]_{n \times m}$ and defined by

$$m_{ij} = \begin{cases} 0 & \text{if } (a_i, a_j) \notin R \text{ i.e. } a_i R a_j \\ 1 & \text{if } (a_i, a_j) \in R \text{ i.e. } a_i R a_j \end{cases}$$

Ex.2.4.1 : Let $A = \{1, 2, 3, 4\}$ and $R = \{(x, y) / x < y\}$ then find M_R.

Sol. : $R = \{(1, 2)\ (1, 3)\ (2, 3)\ (1, 4)\ (2, 4)\ (3, 4)\}$

$$\therefore \quad M_R = [M_{ij}]_{4 \times 4} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The Matrix of a Relation

Definition: Let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$ and $R \subseteq A \times B$ be a relation. We represent R by the $m \times n$ matrix $\mathbf{M}_R = [m_{ij}]$, which is defined by

$$m_{ij} = \begin{cases} 1, & (a_i, b_j) \in R \\ 0, & (a_i, b_j) \notin R \end{cases}$$

The matrix $\mathbf{M}_R$ is called the matrix of R.

Example: Let $A = \{1,2,3\}$ and $B = \{r, s\}$.

$$R = \{(1, r), (2, s), (3, r)\} \qquad \mathbf{M}_R =$$

Examples of Representing Relations Using Matrices (*cont.*)

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution:

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$

Properties:

1. A relation R is reflexive if the matrix diagonal elements are 1.

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

2. A relation R is irreflexive if the matrix diagonal elements are 0.
3. A relation R is symmetric if the transpose of relation matrix is equal to its original relation matrix. i.e. $M_R = (M_R)^T$.

$$\begin{bmatrix} .. & 1 & & \\ 1 & .. & 0 & \\ & 0 & .. & 1 \\ & & 0 & .. \end{bmatrix}$$

The Digraph of a Relation

Definition: If A is finite and $R \subseteq A \times A$ is a relation. We represent R pictorially as follows:

- Draw a small circle, called a vertex/node, for each element of A and label the circle with the corresponding element of A.
- Draw an arrow, called an edge, from vertex a_i to vertex a_j iff $a_i \; R \; a_j$.

The resulting pictorial representation of R is called a directed graph or digraph of R.

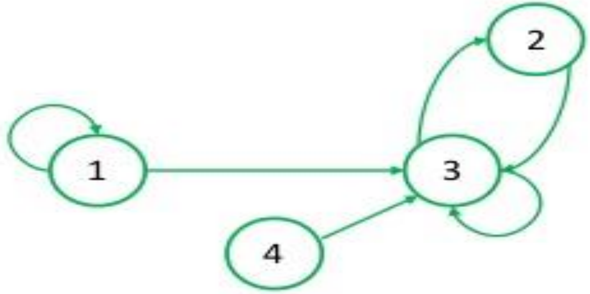
The Digraph of a Relation

Example: Let $A = \{1, 2, 3, 4\}$ and

$$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)\}$$

The digraph of R:

Example: Let $A = \{1, 2, 3, 4\}$ and



Find the relation R:

Properties:

A relation R is reflexive if there is loop at every node of directed graph.

A relation R is irreflexive if there is no loop at any node of directed graphs.

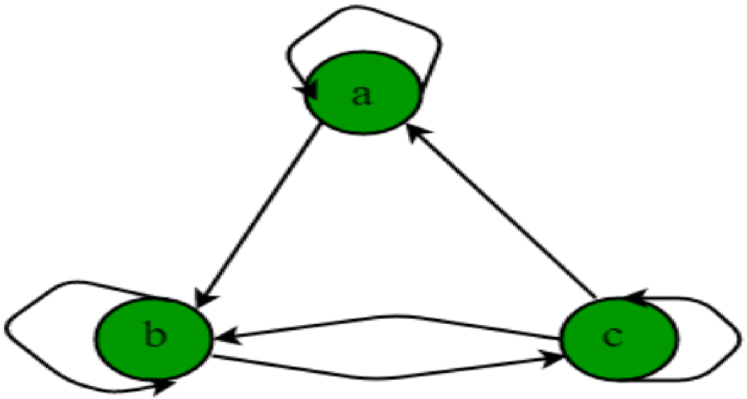
A relation R is symmetric if for every edge between distinct nodes, an edge is always present in opposite direction.

A relation R is asymmetric if there are never two edges in opposite direction between distinct nodes.

A relation R is transitive if there is an edge from a to b and b to c, then there is always an edge from a to c.

Example:

The directed graph of relation R = {(a,a),(a,b),(b,b),(b,c),(c,c),(c,b),(c,a)} is represented as :



Since, there is loop at every node,it is reflexive but it is neither symmetric nor antisymmetric as there is an edge from a to b but no opposite edge from b to a and also directed edge from b to c in both directions. R is not transitive as there is an edge from a to b and b to c but no edge from a to c.