Example 22: *Show that the truth values of the following formulas are independent of their components.*

(i) $(p \wedge (p \to q)) \to q$ (ii) $(p \to q) \Leftrightarrow (\sim p \vee q)$

(iii) $((p \to q) \wedge (q \to r)) \to (p \to r)$

[U.P.T.U. (B.Tech.) 2007]

Solution: Truth value of each of these expression being "Tautology" as shown below is independent of the components.

(i) $(p \wedge (p \to q)) \to q$

p	q	$p \to q$	$p \wedge (p \to q) = A$	$A \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(ii) $(p \to q) \Leftrightarrow (\sim p \vee q)$

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \Leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(iii) $((p \to q) \wedge (q \to r)) \to (p \to r)$

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Example 23: (i) *Consider the conditional statement p. If the floods destroy my house or fires destroy my house, then my insurance company will pay me. Write the converse, inverse and contrapositive of the statement.* **[U.P.T.U. (B.Tech.) 2003]**

(ii) *There are two restaurants next to each other one has sign that says "Good food is not cheap" and other has a sign that says "cheap food is not good". Are the signs saying the same thing?*

[U.P.T.U. (B.Tech.) 2003]

(iii) *Given the following statements as premises, all refering to an arbitrary meal:*

(a) *If he takes coffee, he does not drink milk.*

(b) *He eats crackers only if he drink milk.*

(c) *He does not take soup unless he eats crackers.*

(d) *At noon today, he had coffee.*

Whether he took soup at noon today? If so, what is the correct conclusion? **[U.P.T.U. (B.Tech.) 2004]**

Solution: (i) Let the atomic statements be

P : The floods destroy my house , q :The fires destroy my house

r : My insurance company will pay me

$$(p \vee q) \Rightarrow r$$

Then,

Its inverse is $\sim(p \vee q) \Rightarrow \sim r$ or $\sim p \wedge \sim q \Rightarrow \sim r$

Therefore, the argument will be

If the floods does not destroy my house and fires does not destroy my house, then my insurance company will not pay me. Its converse is $r \Rightarrow p \vee q$

Therefore, argument will be "If my insurance company pay me then the floods will destroy my house or fires will destroy my house.

(ii) Let P : Food is good and q : Food is cheap

Then the argument "Good food is not cheap" is written as

$$p \Rightarrow \sim q$$

and the argument "cheap food is not good" is written as

$$q \Rightarrow \sim p$$

and their truth table is

p	q	$\sim p$	$\sim q$	$p \Rightarrow \sim q$	$q \Rightarrow \sim p$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Since last two columns are same. Hence we can say that the signs are saying the same thing.

(iii) Let p : he takes coffee, q : he drinks milk, r : he eats crackers and s : he takes soup

Then, we see

(a) $p \rightarrow \sim q$ (b) $r \rightarrow q$ (c) $\sim r \rightarrow -s$

(d) $\sim r \rightarrow -s$ (e) by condition (e), we have p

Since implication $r \rightarrow q$ is equivalent to its contrapositive $\sim q \rightarrow \sim r$, we have the following chain of arguments.

$p \rightarrow \sim q$	a premise
$\sim q \rightarrow \sim r$	contrapositive of premise (b)
$p \rightarrow \sim r$	a conclusion of law of syllogism
$\sim r \rightarrow \sim s$	a premise
$p \rightarrow \sim s$	a conclusion by law of syllogism
p	a premise
$\sim s$	a conclusion by modusponen

Hence $\sim s$ is the conclusion, *i.e.* he did not take soup at noon day.

Example 25: *The converse of a statement is given. Write the inverse and contrapositive statements "If I come early, then I can get car".* [Osmania (B.E.) Andhra 2004, 20...]

Solution: Inverse: "If I cannot get car, then I shall not come early".

Contrapositive: If I do not come early, then I can not get the car.

Example 26: *The inverse of statement is given. Write the converse and contrapositive of the statement.*

"If a man is not fisherman, then he is not swimmer".

Solution: Converse: "If he is a swimmer, then the man is a fisherman".

Contrapositive: "If he is not a swimmer, then the man is not a fisherman".

Illustration: *If* $(p \wedge q) \wedge \sim(p \vee q)$ *is a contradiction*

Hence,

$$p \wedge q \Rightarrow p \vee q$$

Illustration: *If* $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ *is a tautology.*

Hence,

$$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$$

Example 27: *The contrapositive of statement is given as*

"If $x < 2$ *Then* $x + 4 < 6$*"*

Write the converse and inverse [U.P. ... (B.Tech.) 2009]

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Solution: Converse: If $x > 2$, then $x + 4 > 6$

Inverse: If $x + 4 > 6$, then $x > 2$ Direct implication

Example 28: *Write the equivalent formula for* $p \wedge (q \rightarrow r) \vee (r \leftrightarrow p)$ *which does not contain bi-conditional.*

[U.P.T.U. (B.Tech.) 2009]

Solution: $p \wedge (q \rightarrow r) \wedge (r \rightarrow q) \vee (r \rightarrow p) \wedge (p \rightarrow r)$

Example 29: *Given that the value of* $p \rightarrow q$ *is true. Can you determine the value of* $\sim p \vee (p \leftrightarrow q)$?

[R.G.P.V. (B.E.) Raipur 2008; P.T.U. (B.E.) Punjab 2007]

Solution: We shall construct the truth table column for $p \rightarrow q$ and $\sim p \vee (p \rightarrow q)$

p	q	$p \rightarrow q$	$\sim p$	$p \leftrightarrow q$	$\sim p \vee (p \leftrightarrow q)$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	F	T
F	F	T	T	T	T

From the table it follows that $p \rightarrow q$ is true then the value of $\sim p \vee (p \leftrightarrow q)$ is true.

We can determine the value of $\sim p \vee (p \leftrightarrow q)$ because corresponding to each possible choice of p and q for which the value of $p \rightarrow q$ is true, the value of $\sim p \vee (p \leftrightarrow q)$ is same as T.

Example 30: *Given that the value of* $p \rightarrow q$ *is false, determine the value of* $(\sim p \vee \sim q) \rightarrow q$

[U.P.T.U. (B.Tech.) 2009; Rohtak (B.E.) 2007]

Solution: We shall construct the truth table column $p \rightarrow q$ and $(\sim p \vee \sim q) \rightarrow q$

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$(\sim p) \vee (\sim q)$	$(\sim p \vee \sim q) \rightarrow q$
T	T	F	F	T	F	T
T	F	F	T	F	T	F
F	T	T	F	T	T	T
F	F	T	T	T	T	F

From the table it follow that $p \rightarrow q$ is false then the value of $(\sim p \vee \sim q) \rightarrow q$ is false.

Example 31: *Prove that* $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ *is a tautology.*

[U.P.T.U. (B.Tech.) 2003]

Solution: For convenience, let

$$p \Leftrightarrow r = A \text{ and } (p \Leftrightarrow q) \wedge (q \Leftrightarrow r) = B$$

p	q	r	$p \Leftrightarrow q$	$q \Leftrightarrow r$	$p \Leftrightarrow r = A$ (suppose)	$(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) = B$ (suppose)	$B \Rightarrow A$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	T	F	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	T	F	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

Last column shows that $B \Rightarrow A$ *i.e.*, $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ is a tautology.

Example 32: *Prove that each of the following statement is a tautology:*

(i) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$

(ii) $[(\sim q \Rightarrow \sim p) \wedge (q \Rightarrow p)] \Rightarrow (p \Leftrightarrow q)$

(iii) $(p \Rightarrow q) \vee (r \Rightarrow p)$

(iv) $(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$

[U.P.T.U. (B.Tech.) 2006]

Solution:

(i) Let the statement patterns $p \Rightarrow q, q \Rightarrow r$ and $p \Rightarrow r$ be denoted by sentence variables P, Q and R respectively.

Truth Table for $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$:

p	q	r	P $p \Rightarrow q$	Q $q \Rightarrow r$	R $p \Rightarrow r$	$P \wedge Q$	$P \wedge Q \Rightarrow R$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Last column show that $(P \wedge Q) \Rightarrow R$ *i.e.*, $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow [p \Rightarrow r]$ is a tautology.

(ii) Let the statement patterns $\sim q \Rightarrow \sim p$, $q \Rightarrow p$ and $p \Leftrightarrow q$ be denoted by sentence variables P, Q and R respectively.

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Truth Table for $[(\sim q \Rightarrow \sim p) \wedge (q \Rightarrow p)] \Rightarrow (p \Leftrightarrow q)$

p	q	$\sim p$	$\sim q$	$(P) \sim q \Rightarrow \sim p$	$(Q)\ q \Rightarrow p$	$P \wedge Q$	$(R)\ p \Leftrightarrow q$	$(P \wedge Q) \Rightarrow R$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T	T

Last column shows that $(P \wedge Q) \Rightarrow R$ i.e., $[(\sim q \Rightarrow \sim p) \wedge (q \Rightarrow p)] \Rightarrow (p \Leftrightarrow q)$ is a tautology.

(iii) **Truth Table for** $(p \Rightarrow q) \vee (r \Rightarrow p)$:

p	q	r	$p \Rightarrow q$	$r \Rightarrow p$	$(p \Rightarrow q) \vee (r \Rightarrow p)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

From the last column, it is clear that the given statement $(p \Rightarrow q) \vee (r \Rightarrow p)$ is a tautology.

(iv) **Truth Table for** $(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$:

p	q	r	$q \wedge r$	$p \Leftrightarrow q \wedge r$	$\sim r$	$\sim p$	$\sim r \Rightarrow \sim p$	$(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$
T	T	T	T	T	F	F	T	T
T	T	F	F	F	T	F	F	T
T	F	T	F	F	F	F	T	T
T	F	F	F	F	T	F	F	T
F	T	T	T	F	F	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	F	T	T	T
F	F	F	F	T	T	T	T	T

From the last column, it is clear that the given statement $(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$ is a tautology.

Example 35: *Prove that $p \Leftrightarrow (\sim p)$ is a contradiction.*

Solution: Truth table for $p \Leftrightarrow (\sim p)$ is:

p	$\sim p$	$p \Leftrightarrow (\sim p)$
T	F	F
F	T	F

Since all entries in the last column are of 'F' s and so it is contradiction.

Example 36: *Prove that $(p \vee q) \wedge (\sim p) \wedge (\sim q)$ is a contradiction.*

Solution: Truth Table for given proposition is:

p	q	$p \vee q$	$\sim p$	$\sim q$	$(p \vee q) \wedge (\sim p)$	$(p \vee q) \wedge (\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	T	F
F	F	F	T	T	F	F

Since all entries in the last column are of 'F' s and so it is contradiction.

Example 37: *Prove that each of the following statement is a contradiction:*

(i) $P = (p \vee q) \wedge (p \vee \sim q \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)$

(ii) $[(p \wedge r) \vee (q \wedge \sim r)] \Leftrightarrow [(\sim p \wedge r) \vee (\sim q \wedge \sim r)]$

(iii) $(p \vee q) \wedge (\sim p) \wedge (\sim q)$

(iv) $[(p \wedge q) \Rightarrow p] \Rightarrow [q \wedge \sim q] = A$

(v) $(p \wedge q \Rightarrow q) \Rightarrow (q \wedge \sim q)$

Solution:

(i) Let $p \vee q = R, p \vee \sim q = S, \sim p \vee q = U$ and $\sim p \vee \sim q = V$.

Then construction of the truth table:

p	q	$\sim p$	$\sim q$	$R = p \vee q$	$S = p \vee \sim q$	$U = \sim p \vee q$	$V = \sim p \vee \sim q$	$P = R \wedge S \wedge U \wedge V$
T	T	F	F	T	T	T	F	F
T	F	F	T	T	T	F	T	F
F	T	T	F	T	F	T	T	F
F	F	T	T	F	T	T	T	F

From the last column of truth table it is clear that P is a contradiction because the truth value of each entry of this column is F.