MODULE 2

Applications of Partial Differential Equations

2.1 INTRODUCTION

Many physical and engineering problems when formulated in the mathematical language give rise to partial differential equations. Besides these, partial differential equations also play an important role in the theory of Elasticity, Hydraulics etc.

Since the general solution of a partial differential equation in a region R contains arbitrary constants or arbitrary functions, the unique solution of a partial differential equation corresponding to a physical problem will satisfy certain other conditions at the boundary of the region R. These are known as *boundary conditions.* When these conditions are specified for the time $t = 0$, they are known as *initial conditions.* A partial differential equation together with boundary conditions constitutes a *boundary value problem.*

In the applications of ordinary linear differential equations, we first find the general solution and then determine the arbitrary constants from the initial values. But the same method is not applicable to problems involving partial differential equations. Most of the boundary value problems involving linear partial differential equations can be solved by the method of separation of variables. In this method, right from the beginning, we try to find the particular solutions of the partial differential equation which satisfy all or some of the boundary conditions and then adjust them till the remaining conditions are also satisfied. A combination of these particular solutions gives the solution of the problem.

Fourier series is a powerful aid in determining the arbitrary functions.

2.2 CLASSIFICATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

2.2.1. Consider the Differential Equation of Second Order in Two Independent Variables x and y as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad ...(1)$$

where A, B, C are constants or continuous functions of x and y possessing continuous partial derivatives and A is positive.

Now equation (1) is

(*i*) **elliptic** if $B^2 - 4AC < 0$

(*ii*) **hyperbolic** if $B^2 - 4AC > 0$

(*iii*) **parabolic** if $B^2 - 4AC = 0$

e.g., one dimensional wave equation $z_{xx} = z_{tt}$ is a hyperbolic equation.

One-dimensional heat flow equation $u_t = u_{xx}$ is parabolic type.
while the two-dimensional heat flow equation in steady state given by $z_{xx} + z_{yy} = 0$ is elliptic in nature.

Remark 1. If A, B, C in the given eqn. (1) are constants, then nature of eqn. (1) will be the same in whole region *i.e.,* for all values of x and y.

Remark 2. If A, B, C are functions of x and y in the given eqn. (1), then nature of eqn. (1) will not be the same in the whole region *i.e.,* for all values of x and y.

ILLUSTRATIVE EXAMPLES

Example 1. *Classify the following operators:*

(*i*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial t} + \dfrac{\partial^2 u}{\partial t^2}$ (*ii*) $4\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial t} + \dfrac{\partial^2 u}{\partial t^2}$ (*iii*) $5\dfrac{\partial^2 u}{\partial x^2} - 9\dfrac{\partial^2 u}{\partial x \partial t} + 4\dfrac{\partial^2 u}{\partial t^2}$.

Sol. (*i*) Here the dependent variable u is a function of two independent variables x and t only.

Here, $\quad A = 1, B = 1, C = 1$

$\therefore \quad B^2 - 4AC = (1)^2 - 4(1)(1) = 1 - 4 = -3 < 0$

Hence the given operator is elliptic.

(*ii*) Here $\quad A = 4, B = 4, C = 1$

$\therefore \quad B^2 - 4AC = (4)^2 - 4(4)(1) = 16 - 16 = 0$

Hence the given operator is parabolic.

(*iii*) Here $\quad A = 5, B = -9, C = 4$

$\therefore \quad B^2 - 4AC = (-9)^2 - 4(5)(4) = 81 - 80 = 1 > 0$

Hence the given operator is hyperbolic.

Note. Coefficient of $\dfrac{\partial^2 u}{\partial t^2}$ or u_{tt} may be taken as A.

Example 2. *Find whether the following operators are hyperbolic, parabolic and elliptic?*

(*i*) $x^2\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + u$ (*ii*) $t\dfrac{\partial^2 u}{\partial t^2} + 2\dfrac{\partial^2 u}{\partial x \partial t} + x\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x}$

(*iii*) $x\dfrac{\partial^2 u}{\partial x^2} + t\dfrac{\partial^2 u}{\partial x \partial t} + \dfrac{\partial^2 u}{\partial t^2}$.

Sol. (*i*) Here $\quad A = x^2, B = 0, C = -1$

Now, $\quad B^2 - 4AC = (0)^2 - 4x^2(-1) = 4x^2$

$\therefore$ The operator is hyperbolic if $4x^2 > 0$ *i.e.,* $x > 0$

parabolic if $4x^2 = 0$ *i.e.,* $x = 0$

Since $4x^2$ being a square, cannot be negative hence operator cannot be elliptic.

(*ii*) Here $\quad A = t, B = 2, C = x$

Now, $\quad B^2 - 4AC = 4 - 4tx$

$\therefore$ The operator is hyperbolic if $4 - 4tx > 0$ *i.e.,* $tx < 1$

elliptic if $4 - 4tx < 0$ *i.e.,* $tx > 1$

and parabolic if $4 - 4tx = 0$ *i.e.,* $tx = 1$.

(*iii*) Here $\quad A = x, B = t, C = 1$

Now $\quad B^2 - 4AC = (t)^2 - 4(x)(1) = t^2 - 4x$

$\therefore$ The operator is hyperbolic if $t^2 - 4x > 0$ *i.e.*, $t^2 > 4x$

elliptic if $t^2 - 4x < 0$ *i.e.*, $t^2 < 4x$

and parabolic if $t^2 - 4x = 0$ *i.e.*, $t^2 = 4x$.

Example 3. *Classify the following differential equation as to type in the second quadrant of xy-plane*

$$\sqrt{y^2 + x^2}\; u_{xx} + 2(x - y)\, u_{xy} + \sqrt{y^2 + x^2}\; u_{yy} = 0.$$

Sol. Here $A = \sqrt{y^2 + x^2}$, $B = 2(x - y)$, $C = \sqrt{y^2 + x^2}$

Now $B^2 - 4AC = 4(x - y)^2 - 4(y^2 + x^2) = -8xy$

In second quadrant, y is +ve while x is –ve

$\therefore$ $B^2 - 4AC = +\text{ve} > 0$

$\therefore$ Differential equation is hyperbolic.

Example 4. *Show that the equation* $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ *is hyperbolic.*

Sol. The given differential equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Here, $A = 1,\ B = 0,\ C = -c^2$

Now, $B^2 - 4AC = (0)^2 - 4\,(1)\,(-c^2) = 4c^2$

which is always greater than zero.

Hence, $B^2 - 4AC > 0$

$\therefore$ The given equation is hyperbolic.

Example 5. *Classify the partial differential equation*

$$\frac{\partial^2 u}{\partial t^2} + t\,\frac{\partial^2 u}{\partial x\,\partial t} + x\,\frac{\partial^2 u}{\partial x^2} + 2\,\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 6u = 0.$$

Sol. Here, $A = 1,\ B = t,\ C = x.$

Now, $B^2 - 4AC = t^2 - 4(1)(x) = t^2 - 4x.$

The equation is elliptic if $t^2 - 4x < 0.$

The equation is parabolic if $t^2 - 4x = 0.$

The equation is hyperbolic if $t^2 - 4x > 0.$

Example 6. *Classify the partial differential equation*

$$x^2\,\frac{\partial^2 u}{\partial t^2} + 3\,\frac{\partial^2 u}{\partial x\,\partial t} + x\,\frac{\partial^2 u}{\partial x^2} + 17\,\frac{\partial u}{\partial t} = 100u.$$

Sol. Here, $A = x^2,\ B = 3,\ C = x.$

Now, $B^2 - 4AC = (3)^2 - 4x^2.x = 9 - 4x^3.$

The equation is elliptic if $9 - 4x^3 < 0.$

The equation is parabolic if $9 - 4x^3 = 0.$

The equation is hyperbolic if $9 - 4x^3 > 0.$

Example 7. (*i*) *Show that the equation* $u_{xx} + xu_{yy} + u_y = 0$ *is elliptic for* $x > 0$ *and hyperbolic for* $x < 0$.

(*ii*) *Show that the equation* $z_{xx} + 2xz_{xy} + (1 - y^2)\, z_{yy} = 0$ *is elliptic for values of x and y in the region* $x^2 + y^2 < 1$, *parabolic on the boundary and hyperbolic outside this region.*

(A.K.T.U. 2018,U.K.T.U. 2011)

Sol. (*i*) Here, $A = 1,\ B = 0,\ C = x$

Now, $B^2 - 4AC = -4x$

Equation is elliptic if $B^2 - 4AC < 0$

i.e., if $-4x < 0$ or if $x > 0$

and equation is hyperbolic if $B^2 - 4AC > 0$

i.e., if $-4x > 0$ or if $x < 0$

(*ii*) Here, $A = 1,\ B = 2x,\ C = 1 - y^2$

Now, $B^2 - 4AC = 4x^2 - 4(1 - y^2) = 4(x^2 + y^2 - 1)$

Equation is elliptic if $B^2 - 4AC < 0$

i.e., if $4(x^2 + y^2 - 1) < 0$ or if $x^2 + y^2 < 1$

Parabolic if $B^2 - 4AC = 0$

i.e., if $4(x^2 + y^2 - 1) = 0$ or if $x^2 + y^2 = 1$ (on the boundary)

and hyperbolic if $B^2 - 4AC > 0$

i.e., if $4(x^2 + y^2 - 1) > 0$ or if $x^2 + y^2 > 1$

TEST YOUR KNOWLEDGE

1. Classify the following operators:

(*i*) $4\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial x\,\partial t} + \dfrac{\partial^2 u}{\partial t^2}$ (*ii*) $t\dfrac{\partial^2 u}{\partial t^2} + 3\dfrac{\partial^2 u}{\partial x\,\partial t} + x\dfrac{\partial^2 u}{\partial x^2} + 17\dfrac{\partial u}{\partial x}$.

2. Classify the following differential equations:

(*a*) $z_{xx} = z_{yy}$ (*b*) $z_{xx} + z_{yy} = 0$ (*c*) $z_{xx} = z_y$ (*d*) $u_{xx} + 3u_{xy} + u_{yy} = 0$. (*M.T.U. 2012*)

3. Classify the equation: $(1 - x^2)\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + (1 - y^2)\dfrac{\partial^2 z}{\partial y^2} - 2z = 0$. (*A.K.T.U. 2016, 2017*)

4. Classify the equation: $y^2 r - 2xys + x^2 t = \dfrac{y^2}{z}p + \dfrac{x^2}{y}q$.

5. Characterize the following partial differential equations into elliptic, parabolic and hyperbolic equations $A\dfrac{\partial^2 u}{\partial x^2} + 2B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} = F(x, y, u, u_x, u_y)$. Here A, B, C may be functions of x and y.

6. Give a consize description of the classification of linear partial differential equations of second order with constant coefficients. Give examples of each type of partial differential equations.

Answers

1. (*i*) parabolic (*ii*) hyperbolic if $xt < \dfrac{9}{4}$, parabolic if $xt = \dfrac{9}{4}$, elliptic if $xt > \dfrac{9}{4}$

2. (*a*) hyperbolic (*b*) elliptic (*c*) parabolic (*d*) hyperbolic

3. hyperbolic in the region $x^2 + y^2 > 1$, parabolic at points on $x^2 + y^2 = 1$, and elliptic in the region $x^2 + y^2 < 1$

4. parabolic.

2.2.2. Consider a Linear P.D.E. of II Order in Three Independent Variables x, y, z as

$$a_{11}\frac{\partial^2 u}{\partial x^2} + a_{12}\frac{\partial^2 u}{\partial x\partial y} + a_{13}\frac{\partial^2 u}{\partial x\partial z} + a_{21}\frac{\partial^2 u}{\partial y\partial x} + a_{22}\frac{\partial^2 u}{\partial y^2} + a_{23}\frac{\partial^2 u}{\partial y\partial z} + a_{31}\frac{\partial^2 u}{\partial z\partial x} + a_{32}\frac{\partial^2 u}{\partial z\partial y}$$

$$+ a_{33}\frac{\partial^2 u}{\partial z^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + b_3\frac{\partial u}{\partial z} + cu = 0 \qquad ...(1)$$

where a_{ij}, b_i and c are constants or some functions of x, y, z.

Eqn. (1) may be rewritten in the form

$$\sum_{i,j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^{3} b_i\frac{\partial u}{\partial x_i} + cu = 0 \qquad ...(2)$$

In general, when there are n independent variables $x_1, x_2, x_3, \ldots\ldots, x_n$; we may write as

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i} + cu = 0 \qquad ...(3)$$

Now, let $\delta_i \equiv \dfrac{\partial}{\partial x_i}$ $(1 \le i \le n)$ and $\delta_i\delta_j \equiv \dfrac{\partial^2}{\partial x_i\partial x_j}$ $(1 \le i \le n, 1 \le j \le n)$

Consider $\phi = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,\delta_i\delta_j$ for all real values of δ_i positive or negative.

At any point $x_1, x_2, \ldots\ldots, x_n$; differential equation (3) is said to be

(*i*) parabolic if Δ vanishes, where $\Delta = \begin{vmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ \vdots & \vdots \quad \vdots & \vdots \\ \vdots & \vdots \quad \vdots & \vdots \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{vmatrix}$.

(*ii*) elliptic if ϕ is positive for all real values of δ_i and it reduces to zero only when all δ_i's are zero.

(*iii*) hyperbolic if ϕ can be both positive or negative.

Note. If a_{ij} are functions of $x_1, x_2, \ldots.., x_n$ the same differential equation can be elliptic, hyperbolic and parabolic at different points while if a_{ij} are constant, the equation will have the same nature throughout.

ILLUSTRATIVE EXAMPLES

Example 1. *Classify the following equation:* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{c^2}\dfrac{\partial u}{\partial t}$.

Sol. Here, $a_{11} = 1, a_{22} = 1, a_{33} = 1, a_{44} = 0$

$a_{13} = a_{23} = a_{31} = a_{24} = a_{34} = 0$, etc.

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Hence the given differential equation is parabolic.

Example 2. *Classify the equation:* $u_{xx} + 2u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$.

Sol. Here, $a_{11} = 1, a_{12} = -2, a_{13} = 0, a_{21} = 0, a_{22} = 2,$
$a_{23} = -2, a_{31} = 0, a_{32} = 0, a_{33} = 1$

$$\therefore \quad \Delta = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

$$\therefore \quad \phi = \delta_1^2 - 2\delta_1\delta_2 + 2\delta_2^2 - 2\delta_2\delta_3 + \delta_3^2 = (\delta_1 - \delta_2)^2 + (\delta_2 - \delta_3)^2$$

which is definite positive for all real values of $\delta_1, \delta_2, \delta_3$.

Hence the equation is elliptic.

TEST YOUR KNOWLEDGE

Classify the following equations:

1. $u_{xx} + u_{yy} + u_{zz} = 0$
2. $u_{xx} + u_{yy} = u_z$
3. $u_{xx} + u_{yy} = u_{zz}$
4. $u_{xx} + u_{yy} + u_{zx} + 2u_{yz} = 0$.

Answers

1. Elliptic
2. Parabolic
3. Hyperbolic
4. Parabolic.

2.3 METHOD OF SEPARATION OF VARIABLES

(*M.T.U. 2012*)

The most important feature of the method of separation of variables is that it successively replaces the partial differential equation by a system of ordinary differential equations that are usually easy to handle. This method requires specific assumptions and transformation formulae in handling partial differential equations. It is well known that this method is applicable if the equation and the boundary conditions are linear and homogeneous. For inhomogeneous boundary conditions, a transformation formula should be employed to transform inhomogeneous boundary conditions to homogeneous boundary conditions.

In this method, we assume the solution to be the product of two functions, each of which involves only one of the variables. The following examples explain the method.

Q $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \quad u(x,0) = 4e^{-x}$

ILLUSTRATIVE EXAMPLES

Example 1. *Use the method of separation of variables to solve the equation*

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \text{ given that } u(x, 0) = 6e^{-3x}.$$

[G.B.T.U. (AG) 2011]

Sol. The given equation is

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad ...(1)$$

Let $u = XT$ ...(2)

where X is a function of x only and T is a function of t only

Then, $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XT) = T\frac{dX}{dx}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\text{XT}) = \text{X}\frac{d\text{T}}{dt}.$$

Substituting in eqn. (1), we get

$$\text{T}\frac{d\text{X}}{dx} = 2\text{X}\frac{d\text{T}}{dt} + \text{XT}$$

$\Rightarrow$ $$\text{TX}' = 2\text{XT}' + \text{XT}$$

$\Rightarrow$ $$\text{TX}' = \text{X}(2\text{T}' + \text{T})$$

$\Rightarrow$ $$\frac{\text{X}'}{\text{X}} = \frac{2\text{T}'}{\text{T}} + 1 = -p^2 \text{ (say)}$$

(*i*) $$\frac{\text{X}'}{\text{X}} = -p^2$$

$$\frac{d\text{X}}{dx} + p^2\text{X} = 0$$

$$\frac{d\text{X}}{\text{X}} = -p^2\, dx$$

Integration yields

$$\log \text{X} = -p^2x + \log c_1$$

$\Rightarrow$ $$\text{X} = c_1\, e^{-p^2x} \qquad \text{...(3)}$$

(*ii*) $$\frac{2\text{T}'}{\text{T}} = -(p^2 + 1)$$

$$\frac{d\text{T}}{\text{T}} = -\left(\frac{p^2+1}{2}\right)dt$$

Integration yields, $\log \text{T} = -\left(\frac{p^2+1}{2}\right)t + \log c_2$

$$\text{T} = c_2\, e^{-\left(\frac{p^2+1}{2}\right)t} \qquad \text{...(4)}$$

From (2), (3) and (4), we get

$$u = \text{XT} = c_1c_2\, e^{-p^2x - \left(\frac{p^2+1}{2}\right)t}$$

or $$u(x, t) = c_1c_2\, e^{-p^2x - \left(\frac{p^2+1}{2}\right)t} \qquad \text{...(5)}$$

$$u(x, 0) = c_1c_2\, e^{-p^2x}.$$

$\Rightarrow$ $$6e^{-3x} = c_1c_2\, e^{-p^2x}$$

Comparison gives, $c_1c_2 = 6$ and $p^2 = 3$.

Hence from (5), we get

$$u(x, t) = 6e^{-3x-2t}.$$

Example 2. *Use the method of separation of variables to solve the equation*

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$ [G.B.T.U. (A.G.) 2012]

Sol. Let $u = XY$ ...(1)

where X is a function of x only and Y is a function of y only.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XY) = Y\frac{dX}{dx}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = X\frac{dY}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2}.$$

Substituting these values in (1), we get

$$Y\frac{d^2X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0$$

$$\Rightarrow \quad YX'' - 2YX' + XY' = 0$$

$$\Rightarrow \quad \frac{X'' - 2X'}{X} + \frac{Y'}{Y} = 0$$

$$\Rightarrow \quad \frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = -p^2 \text{ (say)}$$

(*i*) $$\frac{X'' - 2X'}{X} = -p^2$$

$$\Rightarrow \quad X'' - 2X' + p^2X = 0.$$

Auxiliary equation is $m^2 - 2m + p^2 = 0$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4 - 4p^2}}{2} = 1 \pm \sqrt{1 - p^2}$$

$$\therefore \quad \text{C.F.} = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}$$

$$\text{P.I.} = 0.$$

Hence, $$X = \text{C.F.} + \text{P.I.} = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}$$ ...(2)

(*ii*) $$-\frac{Y'}{Y} = -p^2 \Rightarrow \frac{dY}{dy} = p^2 Y \Rightarrow \frac{dY}{Y} = p^2 dy.$$

Integration yields, $\log Y = p^2 y + \log c_3$

$$Y = c_3 e^{p^2 y}$$ ...(3)

$$\therefore \quad u(x, y) = [c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}]\, c_3 e^{p^2 y}.$$

Example 3. *Use the method of separation of variables to solve the equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u.$$ (U.P.T.U. 2014)

Sol. Let $$u = XY \quad ...(1)$$
where X is a function of x only and Y is a function of y only.

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = X\frac{dY}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2}.$$

From (1), $$Y\frac{d^2X}{dx^2} = X\frac{dY}{dy} + 2XY$$

$\Rightarrow$ $$YX'' = XY' + 2XY = X(Y' + 2Y)$$

$\Rightarrow$ $$\frac{X''}{X} = \frac{Y'}{Y} + 2 = -p^2 \text{ (say)}$$

(*i*) $$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0$$

Auxiliary eqn. is $m^2 + p^2 = 0$

$$m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_1 \cos px + c_2 \sin px$$
$$\text{P.I.} = 0$$

$\therefore$ $$X = c_1 \cos px + c_2 \sin px \quad ...(2)$$

(*ii*) $$\frac{Y'}{Y} + 2 = -p^2$$

$$\frac{Y'}{Y} = -(p^2 + 2)$$

$$\frac{dY}{Y} = -(p^2 + 2)\, dy.$$

Integration yields, $\log Y = -(p^2 + 2)\,y + \log c_3$

$\Rightarrow$ $$Y = c_3\, e^{-(p^2+2)y} \quad ...(3)$$

$\therefore$ $$u(x, y) = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-(p^2+2)y}.$$

Example 4. *Solve by the method of separation of variables:*

$$4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u, \quad u = 3e^{-x} - e^{-5x}, \quad \text{when } t = 0.$$

Sol. Let $$u = XT \quad ...(1)$$
where X is a function of x only and T is a function of t only.

$\therefore$ $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt} = XT' \text{ (say)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XT) = T\frac{dX}{dx} = TX' \text{ (say)}$$

$\therefore$ From the given equation

$$4XT' + TX' = 3XT$$

$$\frac{4T'}{T} + \frac{X'}{X} = 3$$

$$\frac{4T'}{T} - 3 = -\frac{X'}{X} = p^2 \text{ (say)} \quad ...(2)$$

(i) $$\frac{4T'}{T} = p^2 + 3$$

$$\frac{dT}{T} = \left(\frac{3+p^2}{4}\right) dt$$

Integration yields,

$$\log T = \left(\frac{3+p^2}{4}\right) t + \log c_1 \quad \Rightarrow \quad T = c_1 e^{\left(\frac{3+p^2}{4}\right)t} \quad ...(3)$$

(ii) $$\frac{-X'}{X} = p^2 \quad \Rightarrow \quad \frac{X'}{X} = -p^2 \quad \Rightarrow \quad \frac{dX}{X} = -p^2\, dx$$

Integration yields,

$$\log X = -p^2 x + \log c_2$$

$$X = c_2\, e^{-p^2 x} \quad ...(4)$$

From (1), we get

$$u = XT = c_1 c_2\, e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$$

or $$u(x, t) = b_n\, e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$$

Most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n\, e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t} \quad ...(5)$$

when $t = 0$, $$u(x, 0) = 3e^{-x} - e^{-5x} = \sum_{n=1}^{\infty} b_n\, e^{-p^2 x}$$

Comparing, when $p^2 = 1, b_1 = 3$ and when $p^2 = 5, b_2 = -1$

Hence, from (5), general solution is

$$u(x, t) = 3e^{-x+t} - e^{-5x+2t}.$$

Example 5. *Use the method of separation of variables to solve the equation* $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}$ *given that $v = 0$ when $t \to \infty$ as well as $v = 0$ at $x = 0$ and $x = l$.* (G.B.T.U. 2013)

Sol. Let $v = XT$ ...(1)

where X is a function of x only and T is a function of t only.

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt} = XT' \text{ (say)}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XT) = T\frac{d^2 X}{dx^2} = TX'' \text{ (say)}$$

From the given equation

$$TX'' = XT'$$

$$\Rightarrow \qquad \frac{T'}{T} = \frac{X''}{X} = -p^2 \text{ (say)} \qquad ...(2)$$

(i) $$\frac{T'}{T} = -p^2 \Rightarrow \frac{dT}{T} = -p^2 dt$$

Integration yields,

$$\log T = -p^2 t + \log c_1 \Rightarrow T = c_1 e^{-p^2 t} \qquad ...(3)$$

(ii) $$\frac{X''}{X} = -p^2 \Rightarrow X'' + p^2 X = 0$$

Auxiliary eqn. is $m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$$\text{C.F.} = c_2 \cos px + c_3 \sin px$$

$$\text{P.I.} = 0$$

$$\therefore \qquad X = c_2 \cos px + c_3 \sin px \qquad ...(4)$$

Hence $$v = XT = c_1 e^{-p^2 t}(c_2 \cos px + c_3 \sin px) \qquad ...(5)$$

Putting $x = 0$, $v = 0$ in (5), we get

$$0 = c_1 e^{-p^2 t} . c_2$$

$$\therefore \qquad c_2 = 0 \qquad (\because c_1 \neq 0)$$

Hence from (5), $$v = c_1 c_3 e^{-p^2 t} \sin px \qquad ...(6)$$

At $x = l$, $v = 0$

$\therefore$ From (6), $$0 = c_1 c_3 e^{-p^2 t} \sin pl$$

$$\Rightarrow \qquad \sin pl = 0 = \sin n\pi \qquad (\because c_3 \neq 0)$$

$$p = \frac{n\pi}{l}, n \in I$$

$\therefore$ From (6), $$v = c_1 c_3 e^{-\frac{n^2\pi^2 t}{l^2}} \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l} . e^{-\left(\frac{n^2\pi^2}{l^2}\right)t} \qquad ...(7)$$

The above equation (7) satisfies the given conditions for all values of n. Hence the most general solution is

$$v = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n^2\pi^2}{l^2}\right)t} \sin \frac{n\pi x}{l}.$$

Example 6. *Solve the following equation by the method of separation of variables*

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$$

given that $u = 0$ when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ when $x = 0$. (U.P.T.U. 2013)

Sol. Let $$u = XT \qquad ...(1)$$

where X is a function of x only and T is a function of t only.

Then, $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt}$$

Solution let $u = XT$

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x}\left(X\frac{dT}{dt}\right) = \frac{dT}{dt}\cdot\frac{dX}{dx} \qquad ...(2)$$

Substituting (2) in the given equation, we get

$$\frac{dT}{dt}\,\frac{dX}{dx} = e^{-t}\cos x$$

$$e^t\frac{dT}{dt} = \frac{\cos x}{\left(\frac{dX}{dx}\right)} = -p^2 \text{ (say)} \qquad ...(3)$$

Now, $$e^t\frac{dT}{dt} = -p^2$$

$\Rightarrow$ $$dT = -p^2e^{-t}\,dt$$

Integration yields,

$$T = p^2e^{-t} + c_1 \qquad ...(4)$$

Also, $$\frac{dX}{dx} = -\frac{1}{p^2}\cos x$$

$$dX = -\frac{1}{p^2}\cos x\,dx$$

Integration yields,

$$X = -\frac{1}{p^2}\sin x + c_2 \qquad ...(5)$$

Using (4) and (5), we get from (1),

$$u(x, t) = XT = \left(-\frac{1}{p^2}\sin x + c_2\right)(p^2e^{-t} + c_1) \qquad ...(6)$$

Applying the condition $u = 0$ when $t = 0$ in (6), we get

$$0 = \left(-\frac{1}{p^2}\sin x + c_2\right)(p^2 + c_1)$$

$\Rightarrow$ $$p^2 + c_1 = 0 \quad \Rightarrow \quad c_1 = -p^2$$

From (6), $$\frac{\partial u}{\partial t} = \left(-\frac{1}{p^2}\sin x + c_2\right)(-p^2e^{-t}) \qquad ...(7)$$

Applying the condition $\frac{\partial u}{\partial t} = 0$ when $x = 0$ in (7), we get

$$0 = c_2(-p^2e^{-t})$$

$\Rightarrow$ $$c_2 = 0$$

Substituting the values of c_1 and c_2 in (6), we get

$$u(x, t) = -\frac{1}{p^2}\sin x\,(p^2e^{-t} - p^2) = \sin x\,(1 - e^{-t})$$

Example 7. *Solve the P.D.E. by separation of variables method,*

$$u_{xx} = u_y + 2\,u,\ u(0, y) = 0,\ \frac{\partial}{\partial x}\,u(0, y) = 1 + e^{-3y}. \qquad \text{(M.T.U. 2013)}$$

Sol. Let
$$u = XY \qquad ...(1)$$
where X is a function of x only and Y is a function of y only.

$$\therefore \qquad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = X\frac{dY}{dy} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} = YX''$$

From the given equation,

$$YX'' = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y' + 2Y}{Y}$$

$$\Rightarrow \qquad \frac{X''}{X} = \frac{Y'}{Y} + 2 = k(\text{say}) \qquad ...(2)$$

(*i*)
$$\frac{X''}{X} = k$$

$$\Rightarrow \qquad X'' - kX = 0$$

Auxiliary equation is

$$m^2 - k = 0$$

$$\Rightarrow \qquad m = \pm\sqrt{k}$$

$$\therefore \qquad \text{C.F.} = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$
$$\text{P.I.} = 0$$

$$\therefore \qquad X = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x} \qquad ...(3)$$

(*ii*)
$$\frac{Y'}{Y} + 2 = k$$

$$\Rightarrow \qquad \frac{Y'}{Y} = k - 2$$

$$\Rightarrow \qquad \frac{dY}{Y} = (k - 2)\,dy$$

Integration yields,

$$\log Y = (k - 2)y + \log C_3$$

$$\Rightarrow \qquad Y = C_3 e^{(k-2)y} \qquad ...(4)$$

Hence from (1),

$$u(x, y) = (C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x})\,C_3 e^{(k-2)y} \qquad ...(5)$$

Applying the condition $u(0, y) = 0$ in (5), we get

$$u(0, y) = 0 = (C_1 + C_2)\,C_3 e^{(k-2)y}$$

$$\Rightarrow \qquad C_1 + C_2 = 0 \ \Rightarrow\ C_2 = -C_1 \qquad ...(6)$$

From (5), most general solution is

$$u(x, y) = \Sigma\, C_1C_3(e^{\sqrt{k}x} - e^{-\sqrt{k}x})\, e^{(k-2)y} \qquad \ldots(7)$$

$$\frac{\partial u}{\partial x} = \Sigma\, C_1C_3\sqrt{k}(e^{\sqrt{k}x} + e^{-\sqrt{k}x})\, e^{(k-2)y}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 1 + e^{-3y} = \Sigma C_1C_3\sqrt{k}(2)\, e^{(k-2)y} = \sum_{n=1}^{\infty} b_n\, e^{(k-2)y}$$

Comparing the coefficients, we get

(*i*) $b_1 = 1, \quad k - 2 = 0$

$$2C_1C_3\sqrt{k} = 1, \quad k = 2$$

$$\therefore \qquad C_1C_3 = \frac{1}{2\sqrt{2}}$$

(*ii*) $b_3 = -1, \quad k - 2 = -3$

$$2C_1C_3\sqrt{k} = 1, \quad k = -1$$

$$\therefore \qquad C_1C_3 = \frac{1}{2i}$$

Hence from (7), the particular solution is

$$u(x, y) = \frac{1}{2\sqrt{2}}\,(e^{\sqrt{2}x} - e^{-\sqrt{2}x}) + \frac{1}{2i}(e^{ix} - e^{-ix})\, e^{-3y}$$

$$\Rightarrow \qquad u(x, y) = \frac{1}{\sqrt{2}}\, \sinh \sqrt{2}\, x + e^{-3y} \sin x.$$

TEST YOUR KNOWLEDGE

Solve by the method of separation of variables (1–9):

1. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0;\ u(x, 0) = 4e^{-x}$ (*A.K.T.U. 2017*)

2. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y};\ u(0, y) = 8e^{-3y}$

3. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u;\ u(0, y) = 4e^{-y} - e^{-5y}$ (*A.K.T.U. 2018*)

4. $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - 2u;\ u(x, 0) = 10e^{-x} - 6e^{-4x}$ (*A.K.T.U. 2018*)

5. (*i*) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ (*U.P.T.U. 2015*) (*ii*) $y^3\dfrac{\partial u}{\partial x} + x^2\dfrac{\partial u}{\partial y} = 0$ (*G.B.T.U. 2011*)

6. (*i*) $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u;\quad u(x, 0) = 6e^{-5x}$ (*U.P.T.U. 2015*) (*ii*) $x\dfrac{\partial^2 u}{\partial x \partial y} + 2yu = 0$ (*U.P.T.U. 2015*)

7. (*i*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y}$ (*G.B.T.U. 2012*) (*ii*) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} = 0.$ [*U.P.T.U. (SUM) 2007*]

8. Solve the partial differential equation:

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0; \qquad z(0, y) = 2e^{-y}$$

by the method of separation of variables. (*A.K.T.U. 2017*)

9. $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0$; $z(x, 0) = 0$, $z(x, \pi) = 0$, $z(0, y) = 4 \sin 3y$ *(U.P.T.U. 2014)*

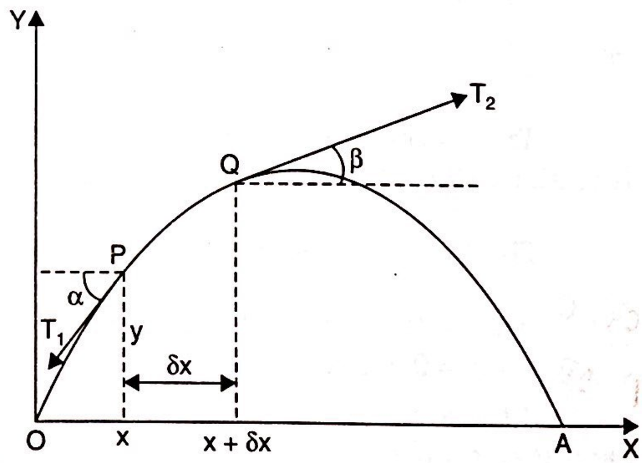
10. Solve $\frac{\partial^2 u}{\partial x^2} = 2u + \frac{\partial u}{\partial y}$ using method of separation of variables subject to the conditions $u = 0$ and $\frac{\partial u}{\partial x} = e^{-3y}$ when $x = 0$ for all values of y. *(G.B.T.U. 2013)*

Answers

1. $u(x, y) = 4e^{-x + \frac{3}{2}y}$
2. $u(x, y) = 8e^{-3y-12x}$
3. $u(x, y) = 4e^{x-y} - e^{2x-5y}$
4. $u(x, t) = 10\, e^{-(x+3t)} - 6e^{-2(2x+3t)}$
5. (i) $u(x, y) = c_1 c_2 \left(\frac{x}{y}\right)^k$ (ii) $u(x, y) = c_1 c_2\, e^{p^2\left(\frac{y^4}{4} - \frac{x^3}{3}\right)}$
6. (i) $u(x, t) = 6e^{-5x-3t}$ (ii) $u(x, y) = c_1 c_2 x^k e^{-(y^2/k)}$
7. (i) $u(x, y) = c_1 c_2 e^{k(x+y)}$ (ii) $u(x, y) = c_1\, e^{-p^2 y}(c_2 \cos px + c_3 \sin px)$
8. $z = 2\, e^{-(x+y)}$
9. $z(x, y) = 4\, e^{9x} \sin 3y$
10. $u(x, y) = e^{-3y} \sin x$.

2.4 VIBRATIONS OF A STRETCHED STRING, ONE DIMENSIONAL WAVE EQUATION $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Consider a uniform elastic string of length l stretched tightly between two points O and A and displaced slightly from its equilibrium position OA. Taking the end O as the origin, OA as the x-axis and a perpendicular line through O as the y-axis, we shall find the displacement y as a function of the distance x and the time t.



We shall obtain the equation of motion for the string under the following assumptions :

(i) The motion takes place entirely in the xy-plane and each particle of the string moves perpendicular to the equilibrium position OA of the string.

(ii) The string is perfectly flexible and does not offer resistance to bending.

(iii) The tension in the string is so large that the forces due to weight of the string can be neglected.

(iv) The displacement y and the slope $\frac{\partial y}{\partial x}$ are small, so that their higher powers can be neglected.

Let m be the mass per unit length of the string. Consider the motion of an element PQ of length δs. Since the string does not offer resistance to bending (by assumption), the tensions T_1 and T_2 at P and Q respectively are tangential to the curve.

Since, there is no motion in the horizontal direction, we have

$$T_1 \cos \alpha = T_2 \cos \beta = T \text{ (constant)} \qquad ...(1)$$

Mass of element PQ is $m\delta s$. By Newton's second law of motion, the equation of motion in the vertical direction is

$$m\delta s \frac{\partial^2 y}{\partial t^2} = T_2 \sin \beta - T_1 \sin \alpha$$

or $$\frac{m\delta s}{T} \frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} \qquad \text{[By using (1)]}$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s} (\tan \beta - \tan \alpha)$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta x} \left[\left(\frac{\partial y}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial y}{\partial x} \right)_x \right]$$

[Since $\delta s = \delta x$ to a first approximation and $\tan \alpha$ and $\tan \beta$ are the slopes of the curve of the string at x and $x + \delta x$]

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \left[\frac{\left(\frac{\partial y}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial y}{\partial x} \right)_x}{\delta x} \right] = \frac{T}{m} \frac{\partial^2 y}{\partial x^2}, \text{ as } \delta x \to 0$$

or $$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \qquad \text{where } c^2 = \frac{T}{m}.$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the **one-dimensional wave equation.**

The *boundary conditions* which the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ has to satisfy are :

$c_3 = 0$

(*i*) $y = 0$ when $x = 0$
(*ii*) $y = 0$ when $x = l$ $\Big]$. These should be satisfied for every value of t.

$p = \frac{n\pi}{l}$

If the string is made to vibrate by pulling it into a curve $y = f(x)$ and then releasing it, the *initial conditions* are

(*i*) $y = f(x)$ when $t = 0$ (*ii*) $\frac{\partial y}{\partial t} = 0$ when $t = 0$.

at last — $c_2 = 0$

2.5 SOLUTION OF THE WAVE EQUATION (*U.P.T.U. 2015*)

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let $$y = XT \quad \text{...(2)}$$
where X is a function of x only and T is a function of t only, be a solution of (1)

Then, $$\frac{\partial^2 y}{\partial t^2} = XT'' \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = X''T$$

Substituting in (1), we have $$XT'' = c^2X''T$$

Separating the variables, we get $$\frac{X''}{X} = \frac{1}{c^2}\cdot\frac{T''}{T} \quad \text{...(3)}$$

Now the L.H.S of (3) is a function of x only and the RHS is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary linear differential equations.

$$X''k - X = 0 \quad \text{and} \quad T'' - kc^2T = 0 \quad \text{...(4)}$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1e^{px} + c_2e^{-px}, \; T = c_3e^{cpt} + c_4e^{-cpt}$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px, \quad T = c_3 \cos cpt + c_4 \sin cpt$$

(*iii*) *When $k = 0$*

$$X = c_1x + c_2 \qquad T = c_3t + c_4$$

Thus the various possible solutions of the wave equation (1) are:

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt})$$
$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$
$$y = (c_1x + c_2)(c_3t + c_4)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, y must be a periodic function of x and t. Therefore, the solution must involve trigonometric terms.

Accordingly $$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{...(5)}$$
is the only suitable solution of the wave equation and it corresponds to $k = -p^2$.

Now, applying boundary conditions that

$$y = 0 \text{ when } x = 0 \quad \text{and} \quad y = 0 \text{ when } x = l, \text{ we get}$$
$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \quad \text{...(6)}$$
and $$0 = (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{...(7)}$$

From (6), we have $c_1 = 0$ and equation (7) reduces to

$$c_2 \sin pl \,(c_3 \cos cpt + c_4 \sin cpt) = 0$$

which is satisfied when $\sin pl = 0$ or $pl = n\pi$ or $p = \dfrac{n\pi}{l}$, where $n = 1, 2, 3, \ldots\ldots$

$\therefore$ A solution of the wave equation satisfying the boundary conditions is

$$y = c_2\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$
$$= \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

on replacing c_2c_3 by a_n and c_2c_4 by b_n.

Adding up the solutions for different values of n, we get

$$y = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad ...(8)$$

which is also a solution.

Now applying the initial conditions

$$y = f(x) \quad \text{and} \quad \frac{\partial y}{\partial t} = 0, \text{ when } t = 0, \text{ we have}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad ...(9)$$

and
$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin \frac{n\pi x}{l} \quad ...(10)$$

Since equation (9) represents Fourier series for $f(x)$, we have

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx \quad ...(11)$$

From (10), $b_n = 0$, for all n

Hence (8) reduces to
$$y = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad ...(12)$$

where a_n is given by (11) when $f(x)$ *i.e.*, $y(x, 0)$ is known.

2.6 INHOMOGENEOUS BOUNDARY CONDITIONS

Here we will discuss wave equations where Dirichlet boundary conditions and Neumann boundary conditions are not homogeneous. It is normal to seek transformation formulae to convert these inhomogeneous boundary conditions to homogeneous conditions.

2.6.1. Dirichlet Boundary Conditions

In this first type of boundary conditions, the displacements $u(0, t) = \alpha$ and $u(\mathrm{L}, t) = \beta$ of a vibrating string of length L are given. Consider an initial–boundary value problem (IBVP) as,

$$u_{tt} = c^2 u_{xx},\ 0 < x < \mathrm{L},\ t > 0$$

$$u(0, t) = \alpha,\ u(\mathrm{L}, t) = \beta, \quad t \geq 0$$

and
$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

To convert the above inhomogeneous boundary conditions to homogeneous boundary conditions, we use the conversion formula

$$u(x, t) = \left[\alpha + \left(\frac{\beta - \alpha}{\mathrm{L}} \right) x \right] + v(x, t)$$

Substituting it in above IBVP, $v(x, t)$ is governed by IBVP:

$$v_{tt} = c^2 v_{xx}, 0 < x < \mathrm{L}, \quad t > 0$$

$$v(0, t) = 0 = v(\mathrm{L}, t)$$

$$v(x, 0) = f(x) - \left[\alpha + \left(\frac{\beta - \alpha}{\mathrm{L}} \right) x \right],\ v_t(x, 0) = g(x)$$

Now using method of separation of variables, $v(x,.t)$ can be found out. Subsequently, $u(x, t)$ is obtained as final result.

2.6.2 Neumann Boundary Conditions

In this second type of boundary conditions, $u_x(0, t) = \alpha$ and $u_x(L, t) = \beta$ are given. It is noted here that the above transformation formula cannot be used for these boundary conditions although it works effectively for first kind of boundary conditions. Consider the initial boundary value problem (IBVP):

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & 0 < x < L,\ t > 0 \\ u_x(0, t) &= \alpha, & u_x(L, t) = \beta, \quad t \geq 0 \\ u_x(x, 0) &= f(x), & u_t(x, 0) = g(x) \end{aligned}$$

To convert the inhomogeneous boundary conditions to homogeneous boundary conditions, we use the transformation formula

$$u(x, t) = \alpha x + \left(\frac{\beta - \alpha}{2L}\right)x^2 + c^2\left(\frac{\beta - \alpha}{2L}\right)t^2 + v(x, t)$$

Using above, it is shown that $v(x, t)$ is now governed by the IBVP:

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, & 0 < x < L,\ t > 0 \\ v_x(0, t) &= 0, & v_x(L, t) = 0 \end{aligned}$$

$$v(x, 0) = f(x) - \left[\alpha x + \left(\frac{\beta - \alpha}{2L}\right)x^2\right]$$

$$v_t(x, 0) = g(x)$$

Now, using method of separation of variables, $v(x, t)$ is readily obtained. Consequently $u(x, t)$ is also found as final result.

2.7 WAVE EQUATION IN AN INFINITE DOMAIN: D'ALEMBERT SOLUTION

The physical model that controls the wave motion of a very long string is governed by a PDE and initial conditions only. The method of separation of variables is not applicable in this case.

However, D'Alembert solution allows us to solve the initial value problem on an infinite domain.

Let,

$$\begin{gathered} y_{tt} = c^2 y_{xx}, \ -\infty < x < \infty, \quad t > 0 \\ y(x, 0) = f(x), \quad y_t(x, 0) = g(x) \end{gathered} \qquad \text{...(1)}$$

To derive D'Alembert's solution, we consider two new variables ξ and η defined by

$$\left.\begin{aligned} \xi &= x + ct \\ \text{and} \quad \eta &= x - ct \end{aligned}\right\} \qquad \text{...(2)}$$

Using chain rule, we get

$$\left.\begin{aligned} y_{xx} &= y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta} \\ \text{and} \quad y_{tt} &= c^2(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta}) \end{aligned}\right\} \qquad \text{...(3)}$$

Substituting (3) in (1), we get

$$y_{\xi\eta} = 0 \qquad \text{...(4)}$$

General solution is

$$y(\xi, \eta) = F(\xi) + G(\eta)$$

where F and G are arbitrary functions. Using (2), we write,

$$y(x, t) = F(x + ct) + G(x - ct) \quad ...(5)$$

Using the initial condition $u(x, 0) = f(x)$ in (5), we get

$$F(x) + G(x) = f(x) \quad ...(6)$$

Again, using the initial condition $u_t(x, 0) = g(x)$ in (5), we get

$$cF'(x) - cG'(x) = g(x)$$

Integration yields,

$$F(x) - G(x) = \frac{1}{c}\int_0^x g(r)\,dr + K \quad ...(7)$$

where K is a constant of integration.

Solving (6) and (7), we get

$$F(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(r)\,dr + \frac{1}{2}K$$

$$G(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(r)\,dr - \frac{1}{2}K$$

This means that

$$F(x + ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c}\int_0^{x+ct} g(r)\,dr + \frac{1}{2}K$$

$$G(x - ct) = \frac{1}{2} f(x - ct) - \frac{1}{2c}\int_0^{x-ct} g(r)\,dr - \frac{1}{2}K$$

Hence by (5),

$$\boxed{y(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(r)\,dr}$$

This completes the formal derivation of D'Alembert's solution.

ILLUSTRATIVE EXAMPLES

Example 1. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A\sin\frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$y(x, t) = A\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$. (A.K.T.U. 2018, 2013, 2017; U.K.T.U. 2011, 2012)

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

Since, the string is stretched between two fixed points $(0, 0)$ and $(l, 0)$ hence the displacement of the string at these points will be zero

$$\therefore \quad y(0, t) = 0 \quad ...(2)$$

and $$y(l, t) = 0 \quad ...(3)$$

Since, the string is released from rest hence its initial velocity will be zero

$$\therefore \qquad \frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t = 0 \qquad ...(4)$$

Since, the string is displaced from its initial position at time $t = 0$ hence the initial displacement is

$$y(x, 0) = A \sin \frac{\pi x}{l} \qquad ...(5)$$

Conditions (2), (3), (4) and (5) are the boundary conditions.

Let us now proceed to solve equation (1),

Let $\qquad y = XT. \qquad ...(6)$

where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(X\frac{dT}{dt}\right) = X\frac{d^2T}{dt^2}.$$

Similarly, $\qquad \dfrac{\partial^2 y}{\partial x^2} = T\dfrac{d^2X}{dx^2}.$

Substituting the above in equation (1), we get

$$X\frac{d^2T}{dt^2} = c^2T\frac{d^2X}{dx^2} \qquad \Rightarrow \quad XT'' = c^2TX''$$

Case I. $\qquad \dfrac{1}{c^2}\dfrac{T''}{T} = \dfrac{X''}{X} = -p^2$ (say)

(i) $\qquad \dfrac{1}{c^2}\dfrac{T''}{T} = -p^2$

$$\frac{d^2T}{dt^2} + c^2p^2T = 0.$$

Auxiliary equation is $\qquad m^2 + c^2p^2 = 0$

$$m^2 = c^2p^2i^2$$

$$m = \pm cpi$$

$$\therefore \qquad \text{C.F.} = c_1 \cos cpt + c_2 \sin cpt$$

$$\text{P.I.} = 0$$

$$\therefore \qquad T = \text{C.F.} + \text{P.I.} = c_1 \cos cpt + c_2 \sin cpt \qquad ...(7)$$

(ii) $\qquad \dfrac{X''}{X} = -p^2 \quad \Rightarrow \quad \dfrac{d^2X}{dx^2} + p^2X = 0.$

Auxiliary equation is $\qquad m^2 + p^2 = 0$

$$m = \pm pi$$

$$\text{C.F.} = c_3 \cos px + c_4 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$\text{X} = c_3 \cos px + c_4 \sin px \qquad ...(8)$$

Hence, $$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad ...(9)$$

Case II. $$\frac{1}{c^2}\frac{\text{T}''}{\text{T}} = \frac{\text{X}''}{\text{X}} = p^2 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{\text{T}''}{\text{T}} = p^2 \Rightarrow \frac{d^2\text{T}}{dt^2} - p^2c^2\text{T} = 0$$

Auxiliary equation is $m^2 - p^2c^2 = 0 \Rightarrow m = \pm pc$

$\therefore$ $$\text{C.F.} = c_5 e^{pct} + c_6 e^{-pct}$$

$$\text{P.I.} = 0$$

$\therefore$ $$\text{T} = c_5 e^{pct} + c_6 e^{-pct}.$$

(*ii*) $$\frac{\text{X}''}{\text{X}} = p^2 \Rightarrow \frac{d^2\text{X}}{dx^2} - p^2\text{X} = 0$$

Auxiliary equation is

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$\therefore$ $$\text{C.F.} = c_7 e^{px} + c_8 e^{-px}$$

$$\text{P.I.} = 0$$

$\therefore$ $$\text{X} = c_7 e^{px} + c_8 e^{-px}$$

Hence, $$y(x, t) = (c_5 e^{pct} + c_6 e^{-pct})(c_7 e^{px} + c_8 e^{-px}) \qquad ...(10)$$

Case III. $$\frac{1}{c^2}\frac{\text{T}''}{\text{T}} = \frac{\text{X}''}{\text{X}} = 0 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{\text{T}''}{\text{T}} = 0 \Rightarrow \text{T}'' = 0 \text{ or } \frac{d^2\text{T}}{dt^2} = 0$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0, 0$$

$\therefore$ $$\text{C.F.} = c_9 + c_{10} t$$

$$\text{P.I.} = 0$$

$\therefore$ $$\text{T} = c_9 + c_{10} t$$

(*ii*) $$\frac{\text{X}''}{\text{X}} = 0 \Rightarrow \text{X}'' = 0 \text{ or } \frac{d^2\text{X}}{dx^2} = 0$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0, 0$$

$\therefore$ $$\text{C.F.} = c_{11} + c_{12}x$$

$$\text{P.I.} = 0$$

$\therefore$ $$\text{X} = c_{11} + c_{12}x$$

Hence, $$y(x, t) = (c_9 + c_{10}\, t)\,(c_{11} + c_{12}x) \qquad ...(11)$$

Out of these three above solutions (9), (10) and (11), we have to choose the solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, the solution must contain periodic functions. Hence the solution which contains trigonometric terms must be the required solution.

Hence solution (9) is the general solution of one-dimensional wave equation given by equation (1).

Now, $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$

Applying the boundary condition,

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (9), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(12)

Again, $y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \; (n \in \text{I})$

$\therefore$ $p = \dfrac{n\pi}{l}.$

Hence from (12), $y(x, t) = \left(c_1 \cos \dfrac{n\pi ct}{l} + c_2 \sin \dfrac{n\pi ct}{l}\right) c_4 \sin \dfrac{n\pi x}{l}$ ...(13)

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$$

$\Rightarrow$ $c_2 = 0,$

$\therefore$ From (13), $y(x, t) = c_1 c_4 \cos \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}$ $= b_n \cos \frac{n\pi ct}{l} . \sin \frac{n\pi x}{l}$ ...(14)

Most General Solution is

$y(x,t) = \Sigma\, b_n \cos \frac{n\pi ct}{l} . \sin \frac{n\pi x}{l}$

$y(x, 0) = \text{A} \sin \dfrac{\pi x}{l} = c_1 c_4 \sin \dfrac{n\pi x}{l}$

$t = 0$

$\Rightarrow$ $c_1 c_4 = \text{A},\; n = 1.$ | Comparing

$A \sin \frac{\pi x}{l} = \Sigma\, b_n \sin \frac{n\pi x}{l}$

Hence from (14), $y(x, t) = \text{A} \cos \dfrac{\pi ct}{l} \sin \dfrac{\pi x}{l}$

$b_1 = A$

which is the required solution.

$y(x,t) = A \cos \frac{n\pi ct}{l} . \sin \frac{n\pi x}{l}$

Example 2. *Show how the wave equation* $c^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$ *can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched between x = 0 and x = l are given by* $y = f(x)$ *and* $\dfrac{\partial y}{\partial t} = g(x)$, *determine the constants in the series solution.*

Sol. The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(1)

Let $y = \text{XT}$ ...(2)

where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(\text{XT}) = \text{X}\frac{d\text{T}}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = X \frac{d^2 T}{dt^2}$$

Similarly, $$\frac{\partial^2 y}{\partial x^2} = T \frac{d^2 X}{dx^2}.$$

Substituting in (1), we get

$$X\frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2} \quad \Rightarrow \quad XT'' = c^2 TX''$$

$$\Rightarrow \quad \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} \qquad ...(3)$$

Case I. When $\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = p^2$ (say)

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = p^2 \quad \Rightarrow \quad \frac{d^2 T}{dt^2} - p^2 c^2 T = 0.$$

Auxiliary equation is

$$m^2 - p^2 c^2 = 0$$

$$m = \pm pc$$

$$\text{C.F.} = c_1 e^{pct} + c_2 e^{-pct}$$

$$\text{P.I.} = 0$$

$$\therefore \quad T = \text{C.F.} + \text{P.I.} = c_1 e^{pct} + c_2 e^{-pct}$$

(*ii*) $$\frac{X''}{X} = p^2 \quad \Rightarrow \quad \frac{d^2 X}{dx^2} - p^2 X = 0.$$

Auxiliary equation is

$$m^2 - p^2 = 0$$

$$m = \pm p$$

$$\text{C.F.} = c_3 e^{px} + c_4 e^{-px}$$

$$\text{P.I.} = 0.$$

$$\therefore \quad X = \text{C.F.} + \text{P.I.} = c_3 e^{px} + c_4 e^{-px}.$$

Hence, the solution is

$$y = XT = (c_1 e^{pct} + c_2 e^{-pct})(c_3 e^{px} + c_4 e^{-px}). \qquad ...(4)$$

Case II. When

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -p^2 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = -p^2 \quad \Rightarrow \quad \frac{d^2 T}{dt^2} + p^2 c^2 T = 0.$$

Auxiliary equation is

$$m^2 + p^2 c^2 = 0 \quad \Rightarrow \quad m = \pm pci$$

$$\therefore \quad \text{C.F.} = (c_5 \cos pct + c_6 \sin cpt)$$

$$\text{P.I.} = 0.$$

$$\therefore \quad T = \text{C.F.} + \text{P.I.} = c_5 \cos cpt + c_6 \sin cpt$$

(ii) $$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0.$$

Auxiliary equation is

$$m^2 + p^2 = 0 \Rightarrow m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_7 \cos px + c_8 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_7 \cos px + c_8 \sin px.$$

Hence, the solution is

$$y = XT = (c_5 \cos cpt + c_6 \sin cpt)(c_7 \cos px + c_8 \sin px) \quad ...(5)$$

Case III. When, $\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = 0$

(i) $$\frac{1}{c^2}\frac{T''}{T} = 0 \Rightarrow \frac{d^2T}{dt^2} = 0$$

$\Rightarrow$ $$T = c_9 + c_{10}t$$

(ii) $$\frac{X''}{X} = 0 \Rightarrow \frac{d^2X}{dx^2} = 0$$

$\Rightarrow$ $$X = c_{11} + c_{12}x.$$

Hence, the solution is

$$y(x, t) = (c_9 + c_{10}t)(c_{11} + c_{12}x) \quad ...(6)$$

Of the above three solutions given by (4), (5) and (6), we have to choose the solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, y must be a periodic function of x and t therefore the solution must involve trigonometric terms hence solution (5) is the required solution.

Boundary conditions are

$$y(0, t) = 0, \qquad y(l, t) = 0$$

$$y = f(x) \qquad \text{when } t = 0$$

$$\frac{\partial y}{\partial t} = g(x) \qquad \text{when } t = 0$$

From equation (5), $y(0, t) = (c_5 \cos cpt + c_6 \sin cpt)\, c_7$

$$0 = (c_5 \cos cpt + c_6 \sin cpt)\, c_7$$

$\Rightarrow$ $$c_7 = 0.$$

Hence from (5), $$y(x, t) = (c_5 \cos cpt + c_6 \sin cpt)\, c_8 \sin px \quad ...(7)$$

$$y(l, t) = 0 = (c_5 \cos cpt + c_6 \sin cpt)\, c_8 \sin pl$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi \ (n \in I) \Rightarrow p = \frac{n\pi}{l}.$$

$\therefore$ From (7), $$y(x, t) = \left(c_5 \cos \frac{n\pi ct}{l} + c_6 \sin \frac{n\pi ct}{l}\right) c_8 \sin \frac{n\pi x}{l} \quad ...(8)$$

$$= \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

where $c_5c_8 = a_n$ and $c_6c_8 = b_n$

The general solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad ...(9)$$

$$y(x, 0) = f(x) = \sum_{1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

Half Range Fourier sine series in $0<x<l$

where $$a_n = \frac{2}{l} \int_0^l f(x) . \sin \frac{n\pi x}{l} \, dx \quad ...(10)$$

$f(x) = \Sigma b_n \sin \frac{n\pi x}{l}$

From (9), $$\frac{\partial y}{\partial t} = \frac{\pi c}{l} \sum_{1}^{\infty} \left(-n \, a_n \sin \frac{n\pi ct}{l} + n \, b_n \cos \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

$b_n = \frac{2}{l} \int_0^l f(x) \sin$

At $t = 0$, $$\left(\frac{\partial y}{\partial t} \right)_{t=0} = g(x) = \frac{\pi c}{l} \sum_{1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

where $$\frac{n\pi c}{l} b_n = \frac{2}{l} \int_0^l g(x) . \sin \frac{n\pi x}{l} \, dx$$

$\Rightarrow$ $$b_n = \frac{2}{n\pi c} \int_0^l g(x) . \sin \frac{n\pi x}{l} \, dx . \quad ...(11)$$

Hence, the required solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

where $$a_n = \frac{2}{l} \int_0^l f(x) . \sin \frac{n\pi x}{l} \, dx$$

and $$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx.$$

Example 3. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.* [G.B.T.U. (C.O.) 2011]

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \quad ...(2)$$

| Refer Sol. of Ex. 1

Boundary conditions are

$$y(0, t) = 0 \quad ...(3)$$

$$y(l, t) = 0 \quad ...(4)$$

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad ...(5)$$

$$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} \quad ...(6)$$

Applying boundary condition in (2),

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $c_3 = 0$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(7)

Again, $y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \quad (n \in \mathrm{I})$

$\therefore$ $p = \dfrac{n\pi}{l}$

Hence, from (7),

$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad \text{...(8)}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $c_2 = 0$.

$\therefore$ From (8),

$$y(x, t) = c_1 c_4 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Most general solution is

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{...(9)}$$

$$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \quad y_0 \left(\frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4}\right) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

Comparing, we get

$$b_1 = \frac{3y_0}{4},\ b_2 = 0,\ b_3 = \frac{-y_0}{4},\ b_4 = b_5 = \ldots = 0$$

Hence, from (9),

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

Example 4. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, μ is a constant and then released. Find the displacement $y(x, t)$ of any point x of the string at any time $t > 0$.*

Sol. The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{dx^2}$ ...(1)

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The solution of equation (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \quad ...(2)$$ (Refer sol. of Ex. 1)

Boundary conditions are $y(0, t) = 0$ ...(3)

$$y(l, t) = 0 \quad ...(4)$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0 \quad ...(5)$$

and $y(x, 0) = \mu x(l - x)$ ...(6)

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(7)

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in \text{I})$

$$p = \frac{n\pi}{l}.$$

From (7),
$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad ...(8)$$

Now from (7),
$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] . c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $c_2 = 0.$

$\therefore$ From (8),
$$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$\Rightarrow$
$$y(x, t) = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \text{ where } c_1 c_4 = b_n.$$

The most general solution is

$$y(x, t) = \sum_{1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad ...(9)$$

$$y(x, 0) = \mu(lx - x^2) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l}\int_0^l \mu\,(lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\mu}{l}\left[\left\{(lx - x^2)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l (l - 2x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{2\mu}{l}\left[\frac{l}{n\pi}\int_0^l (l - 2x) \cos \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2\mu}{n\pi}\left[\left\{(l-2x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l - \int_0^l (-2)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\,dx\right] = \frac{2\mu}{n\pi}\cdot\frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4\mu l}{n^2\pi^2}\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l = \frac{4\mu l^2}{n^3\pi^3}(-\cos n\pi + 1) = \frac{4\mu l^2}{n^3\pi^3}[1-(-1)^n].$$

$\therefore$ From (9),
$$y(x,t) = \frac{4\mu l^2}{\pi^3}\sum_1^\infty \frac{[1-(-1)^n]}{n^3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$= \frac{8\mu l^2}{\pi^3}\sum_{n=1}^\infty \frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}.$$

Example 5. *A string is stretched between two fixed points (0, 0) and (l, 0) and released at rest from the initial deflection given by*

and
$$f(x) = \begin{cases}\left(\dfrac{2k}{l}\right)x, & 0 < x < \dfrac{l}{2}\\[2mm] \left(\dfrac{2k}{l}\right)(l-x), & \dfrac{l}{2} < x < l\end{cases}$$

Find the deflection of the string at any time.

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{dx^2} \qquad \text{...(1)}$$

The solution of eqn. (1) is

$y(x,t) = (c_1\cos cpt + c_2\sin cpt)(c_3\cos px + c_4\sin px)$ ...(2) [Refer Sol. of Ex. 1]

Boundary conditions are, $y(0,t) = 0,\ y(l,t) = 0$

$$\frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t = 0$$

$$y(x,0) = \begin{cases}\dfrac{2k}{l}x, & 0 < x < \dfrac{l}{2}\\[2mm] \dfrac{2k}{l}(l-x), & \dfrac{l}{2} < x < l\end{cases}$$

From (2),
$$y(0,t) = (c_1\cos cpt + c_2\sin cpt)\,c_3$$
$$0 = (c_1\cos cpt + c_2\sin cpt)\,c_3$$

$\Rightarrow$
$$c_3 = 0.$$

$\therefore$ From (2),
$$y(x,t) = (c_1\cos cpt + c_2\sin cpt)\,c_4\sin px \qquad \text{...(3)}$$
$$y(l,t) = 0 = (c_1\cos cpt + c_2\sin cpt)\,c_4\sin pl$$

$\Rightarrow$
$$\sin pl = 0 = \sin n\pi\ ;\ n \in I$$
$$p = \frac{n\pi}{l}.$$

$\therefore$ From (3), $$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad ...(4)$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[- c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$, $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$$

$\Rightarrow$ $$c_2 = 0.$$

$\therefore$ From (4), $$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$= b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad (\text{where } c_1 c_4 = b_n) \quad ...(5)$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad ...(6)$$

$$y(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{[From (6)]}$$

where $$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2k}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} \frac{2k}{l}(l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} - \int_0^{l/2} 1.\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right.$$
$$\left. + \left\{(l - x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/2}^{l} - \int_{l/2}^{l} (-1).\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{4k}{l^2}\left[-\frac{l}{n\pi}.\frac{l}{2}\cos \frac{n\pi}{2} + \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} + \frac{l}{2}.\frac{l}{n\pi}\cos \frac{n\pi}{2} - \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/2}^{l}\right]$$

$$= \frac{4k}{l^2}\left[\frac{l^2}{n^2\pi^2}\sin \frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\sin n\pi - \sin \frac{n\pi}{2}\right)\right]$$

$$= \frac{4k}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$\therefore$ From (6), $$y(x, t) = \frac{8k}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$$

Example 6. *A tightly stretched violin string of length l and fixed at both ends is plucked at* $x = \frac{l}{3}$ *and assumes initially the shape of a triangle of height a. Find the displacement y at any distance x and any time t after the string is released from rest.*

Sol. One-Dimensional wave equation is

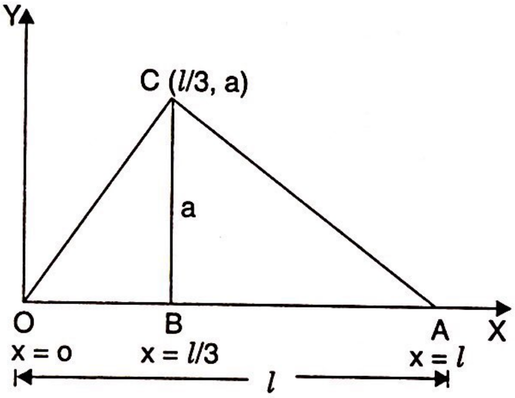
$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of eqn. (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) (Refer Sol. of Ex. 1)

Eqn. of line OC is $$y - 0 = \frac{a - 0}{\frac{l}{3} - 0}(x - 0)$$

$$y = \frac{3a}{l}x \qquad ...(3)$$



Eqn. of line CA is $$y - a = \frac{0 - a}{l - l/3}\left(x - \frac{l}{3}\right)$$

$$y - a = \frac{-a}{\left(\frac{2l}{3}\right)}\left(x - \frac{l}{3}\right) = -\frac{3a}{2l}\left(x - \frac{l}{3}\right)$$

$$y - a = -\frac{3ax}{2l} + \frac{a}{2}$$

$$y = -\frac{3ax}{2l} + \frac{3a}{2} = \frac{3a}{2}\left(1 - \frac{x}{l}\right) \qquad ...(4)$$

Hence the boundary conditions are

$$y(0, t) = 0 \qquad ...(5)$$

$$y(l, t) = 0 \qquad ...(6)$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0 \qquad ...(7)$$

and $$y(x, 0) = \begin{cases} \frac{3ax}{l}, & 0 < x < l/3 \\ \frac{3a}{2}\left(1 - \frac{x}{l}\right), & \frac{l}{3} < x < l \end{cases} \qquad ...(8)$$

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(9)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$$\Rightarrow \qquad \sin pl = 0 = \sin n\pi (n \in I).$$

$$\Rightarrow \qquad p = \frac{n\pi}{l}.$$

$$\therefore \qquad y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \qquad \text{...(10)}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}.$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$$

$$\Rightarrow \qquad c_2 = 0.$$

$$\therefore \qquad y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

The most general solution is

$$y(x, t) = \sum_1^\infty b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(11)}$$

From (11), $\qquad y(x, 0) = \sum_1^\infty b_n \sin \frac{n\pi x}{l}$, where

$$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^l \frac{3a}{2}\left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\left[\frac{3a}{l}\int_0^{l/3} x \sin \frac{n\pi x}{l}\, dx + \frac{3a}{2}\int_{l/3}^l \left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{6a}{l^2}\left[\left(x\left\{\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}\right)_0^{l/3} - \int_0^{l/3} 1.\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$+ \frac{3a}{l}\left[\left(\left(1 - \frac{x}{l}\right)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right)_{l/3}^l - \int_{l/3}^l \left(-\frac{1}{l}\right)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{6a}{l^2}\left[-\frac{l}{n\pi}.\frac{l}{3}\cos \frac{n\pi}{3} + \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/3}\right] + \frac{3a}{l}\left[\frac{l}{n\pi}.\frac{2}{3}\cos \frac{n\pi}{3} - \frac{1}{n\pi}.\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/3}^l\right]$$

$$= \frac{6a}{l^2}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right]+\frac{3a}{l}\left[\frac{2l}{3n\pi}\cos\frac{n\pi}{3}-\frac{l}{n^2\pi^2}\left(0-\sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{6a}{n\pi}\left[\frac{-1}{3}\cos\frac{n\pi}{3}+\frac{1}{n\pi}\sin\frac{n\pi}{3}\right]+\frac{6a}{n\pi}\left[\frac{1}{3}\cos\frac{n\pi}{3}\right]+\frac{3a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$\Rightarrow \qquad b_n = \frac{9a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$\therefore$ From (11), $\displaystyle y(x,t) = \frac{9a}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$

Example 7. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.*

Sol. The equation for the vibration of the string is

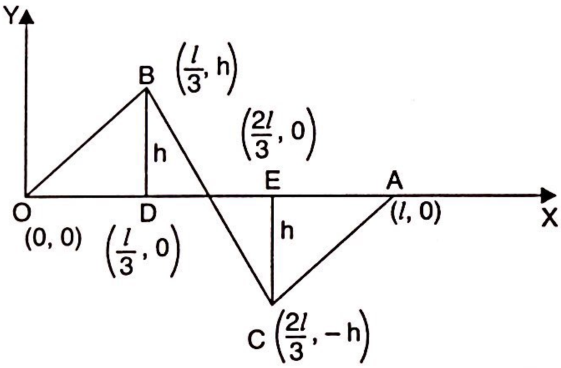
$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of eqn. (1) is

$$y(x,t) = (c_1\cos cpt + c_2\sin cpt)(c_3\cos px + c_4\sin px) \qquad \text{...(2)}$$

Let l be the length of string (Refer Sol. of Ex. 1)

Eqn. of OB is,



$$y - 0 = \frac{h-0}{\frac{l}{3}-0}(x-0)$$

$$\Rightarrow \qquad y = \frac{3h}{l}x \qquad \text{...(3)}$$

Eqn. of BC is,

$$y - h = \frac{-h-h}{\frac{2l}{3}-\frac{l}{3}}\left(x-\frac{l}{3}\right)$$

$$= \frac{-2h}{\left(\frac{l}{3}\right)}\left(x-\frac{l}{3}\right) = -\frac{6h}{l}\left(x-\frac{l}{3}\right)$$

$$y - h = -\frac{6hx}{l} + 2h$$

$$y = 3h - \frac{6hx}{l} = 3h\left(1-\frac{2x}{l}\right) \qquad \text{...(4)}$$

Eqn. of CA is, $\displaystyle y + h = \frac{0+h}{l-\frac{2l}{3}}\left(x-\frac{2l}{3}\right) = \frac{3h}{l}\left(x-\frac{2l}{3}\right) = \frac{3hx}{l} - 2h$

$$y = \frac{3hx}{l} - 3h = 3h\left(\frac{x}{l}-1\right) \qquad \text{...(5)}$$

Hence Boundary conditions are

$$y(0, t) = 0,\ y(l, t) = 0$$

$$\frac{\partial y}{\partial t} = 0 \qquad \text{when } t = 0$$

and

$$y(x, 0) = \begin{cases} \dfrac{3h}{l}x, & 0 \le x \le l/3 \\ \dfrac{3h}{l}(l - 2x), & \dfrac{l}{3} \le x \le \dfrac{2l}{3} \\ \dfrac{3h}{l}(x - l), & \dfrac{2l}{3} \le x \le l \end{cases}$$

From (2), $\quad y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow \quad c_3 = 0.$

$\therefore$ From (2),

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad \text{...(6)}$$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow \quad \sin pl = 0 = \sin n\pi \ (n \in \mathrm{I})$

$\therefore \quad p = \dfrac{n\pi}{l}.$

$\therefore$ From (6),

$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \qquad \text{...(7)}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}.$$

At $t = 0$, $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0 = \dfrac{n\pi c}{l} c_2 c_4 \sin \dfrac{n\pi x}{l}$

$\Rightarrow \quad c_2 = 0$

$\therefore$ From (7), $\quad y(x, t) = c_1 c_4 \cos \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l} = b_n \cos \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}.$

The most general solution is

$$y(x, t) = \sum_1^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(8)}$$

$$y(x, 0) = \sum_1^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3h}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3h}{l}(l - 2x) \sin \frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3h}{l}(x - l) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\cdot\frac{3h}{l}\int_0^{l/3} x\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\cdot\frac{3h}{l}\int_{l/3}^{2l/3}(l-2x)\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\cdot\frac{3h}{l}\int_{2l/3}^{l}(x-l)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6h}{l^2}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^{l/3} - \int_0^{l/3} 1.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$+ \frac{6h}{l^2}\left[\left\{(l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/3}^{2l/3} - \int_{l/3}^{2l/3}(-2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$+ \frac{6h}{l^2}\left[\left\{(x-l)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{2l/3}^{l} - \int_{2l/3}^{l} 1.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{6h}{l^2}\left[\frac{-l}{n\pi}\cdot\frac{l}{3}\cos\frac{n\pi}{3} + \frac{l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/3}\right]$$

$$+ \frac{6h}{l^2}\left[\frac{-l}{3}\cdot\frac{l}{n\pi}\left(-\cos\frac{2n\pi}{3}\right) + \left(\cos\frac{n\pi}{3}\right)\frac{l}{3}\cdot\frac{l}{n\pi} - \frac{2l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/3}^{2l/3}\right]$$

$$+ \frac{6h}{l^2}\left[\frac{-l}{3}\cdot\frac{l}{n\pi}\cos\frac{2n\pi}{3} + \frac{l}{n\pi}\left(\frac{\sin n\pi x/l}{n\pi/l}\right)_{2l/3}^{l}\right]$$

$$= \frac{-2h}{n\pi}\cos\frac{n\pi}{3} + \frac{6h}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{2h}{n\pi}\cos\frac{2n\pi}{3}$$

$$+ \frac{2h}{n\pi}\cos\frac{n\pi}{3} - \frac{12h}{n^2\pi^2}\left(\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3}\right) - \frac{2h}{n\pi}\cos\frac{2n\pi}{3} + \frac{6h}{n^2\pi^2}\left(0 - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{18h}{n^2\pi^2}\sin\frac{2n\pi}{3} = \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{18h}{n^2\pi^2}\sin\left(n\pi - \frac{n\pi}{3}\right)$$

$$= \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3}\cos n\pi$$

$$= \begin{cases} \dfrac{36h}{n^2\pi^2}\sin\dfrac{n\pi}{3}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$$

$\therefore$ From (8), $$y(x,t) = \frac{36h}{\pi^2}\sum_{n=2,4,\ldots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$