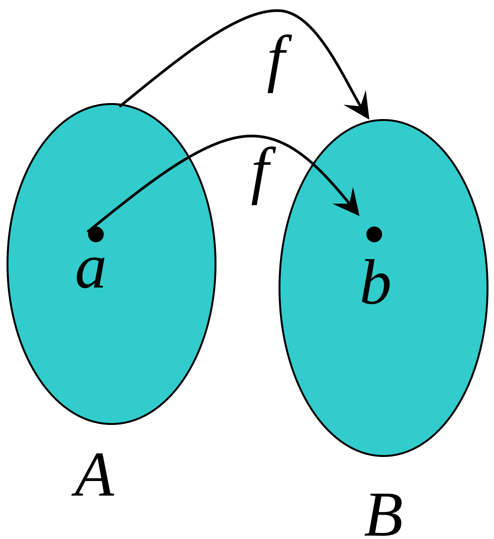
Functions

Definition of Functions

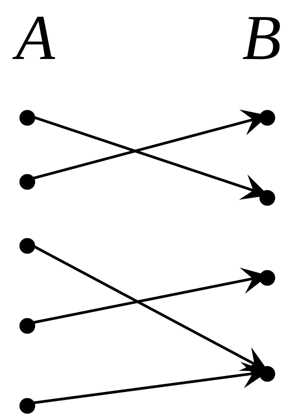
- Given any sets A, B, a function f from *(or "mapping") A to B* (f:$A \rightarrow B$) is an assignment of **exactly one** element $\boldsymbol{f(x) \in B}$ to each element $\boldsymbol{x \in A}$.

Graphical Representations

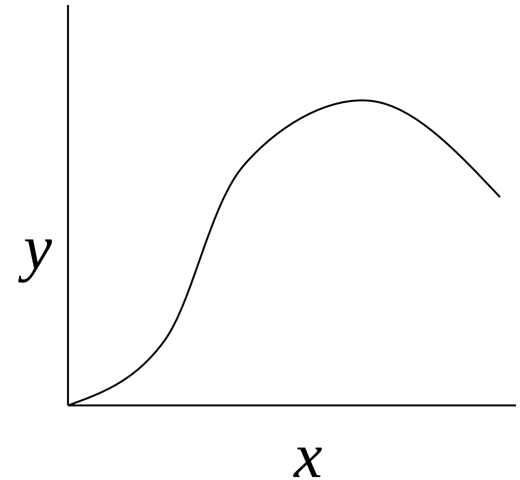
- Functions can be represented graphically in several ways:



Like Venn diagrams



Graph



Plot

Definition of Functions (cont'd)

- Formally: given f:$A \rightarrow B$

  "x is a function" :≡ ($\neg\exists x,y$: $x=y \wedge f(x) \neq f(y)$) or

  "x is a function" :≡ ($\forall$ x,y: $\neg(x=y) \vee \neg(f(x) \neq f(y))$) or

  "x is a function" :≡ ($\forall$ x,y: $\neg(x=y) \vee (f(x) = f(y))$) or

  "x is a function" :≡ ($\forall$ x,y: $(x=y) \rightarrow (f(x) = f(y))$) or

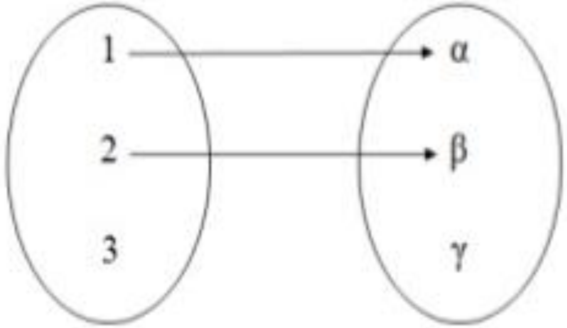
  "x is a function" :≡ ($\forall$ x,y: $(f(x) \neq f(y)) \rightarrow (x \neq y)$)

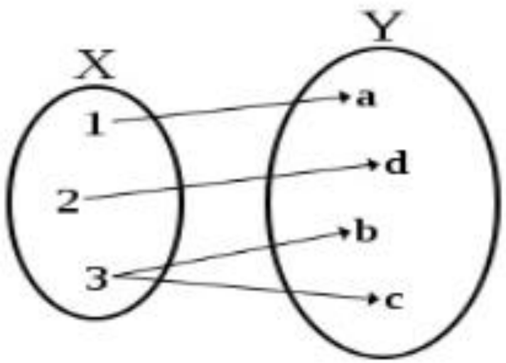
Functions

- **Definition :**
- Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.
- We write $f(a)=b$ if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write $f: A \rightarrow B$.
- Note :Functions are sometimes also called mappings or transformations.

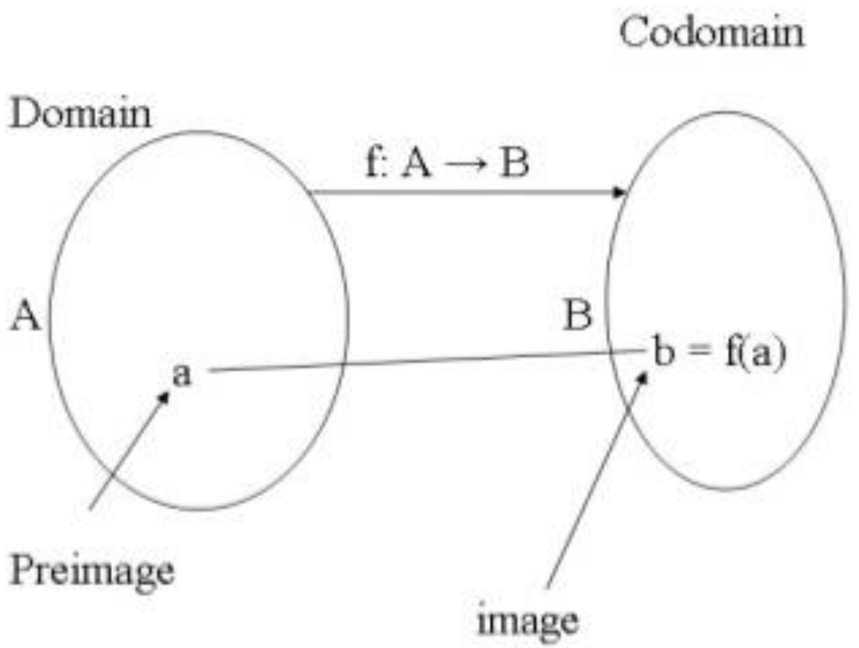
Functions

- Not Function Example

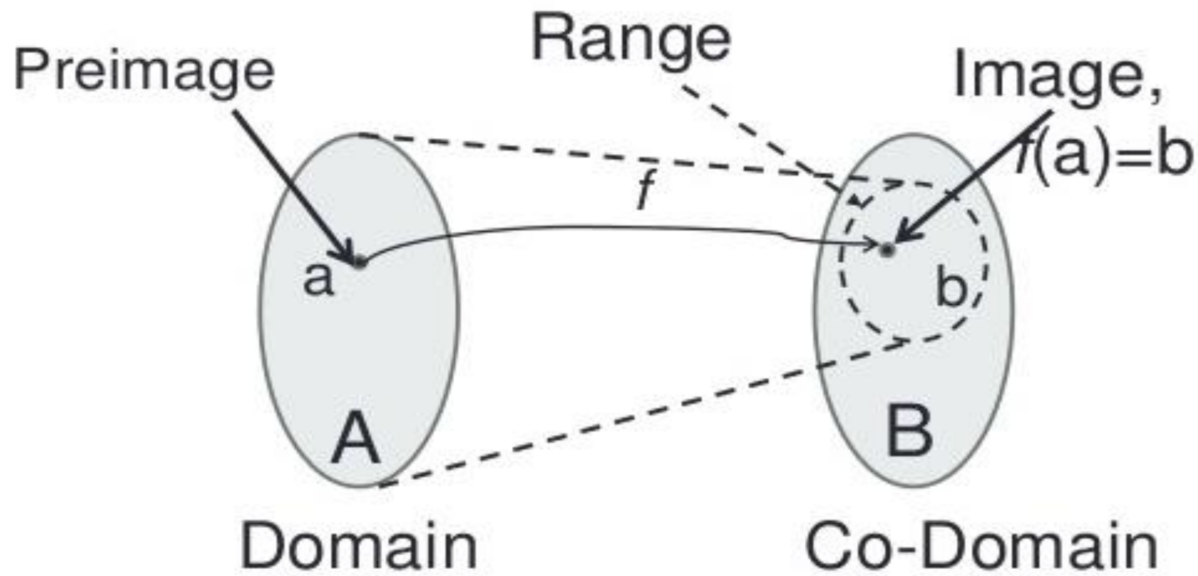




Example



Function: Visualization



A function, f: A $\rightarrow$ B

Some Function Terminology

- If f:$A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than one pre-image.
  - The *range* $R \subseteq B$ of f is $\{b \mid \exists a\ f(a)=b\}$.

Range vs. Codomain - Example

- Suppose that: "*f is a function mapping students in this class to the set of grades {A,B,C,D,E}.*"
- At this point, you know *f*'s codomain is: {A,B,C,D,E}, and its range is unknown!.
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A,B}, but its codomain is still {A,B,C,D,E}!.

Functions

- **Function Addition/Multiplication**
- Let f1 and f2 be functions from A to R. Then f1+f2 and f1 f2 are also functions from A to R defined for all x∈A by
- (f1+f2)(x)=f1(x)+f2(x)
- (f1f2)(x)=f1(x)f2(x).
- **Example:**
- Let $f_1(x)=x^4+2x^2+1$ and $f_2(x)=2-x^2$
  - $(f_1+f_2)(x) = x^4+2x^2+1+2-x^2 = x^4+x^2+3$
  - $f_1 f_2(x) = (x^4+2x^2+1)(2-x^2)= -x^6+3x^2+2$

Function Addition/Multiplication

- We can add and multiply *functions*

$$f,g:\mathbf{R}\rightarrow\mathbf{R}:$$

  - $(f+g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$
  - $(f \times g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f \times g)(x) = f(x) \times g(x)$

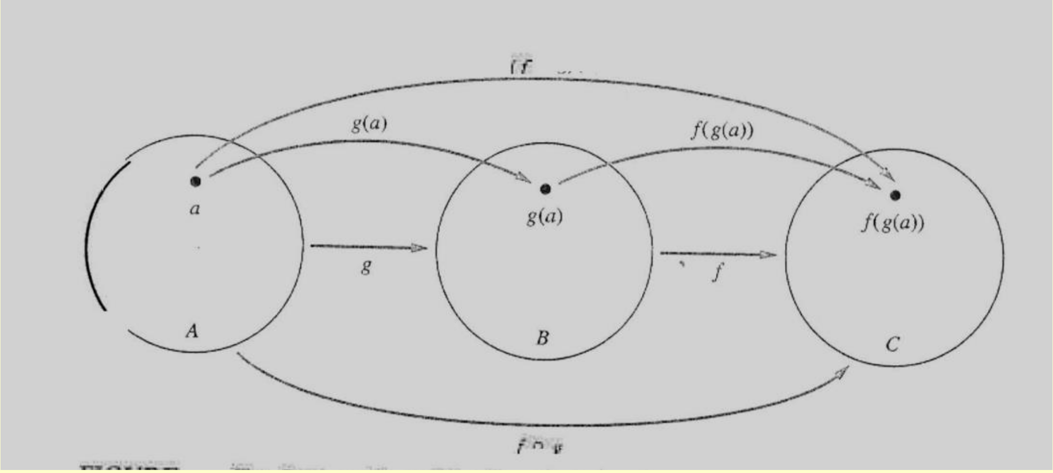
Function Composition

- For functions $g{:}A \rightarrow B$ and $f{:}B \rightarrow C$, there is a special operator called *compose* ("$\bigcirc$").
  - It composes (i.e., creates) a new function out of f,g by <u>applying f to the result of g</u>.

$$(f \bigcirc g){:}A \rightarrow C, \text{ where } (f \bigcirc g)(a) = f(g(a)).$$

  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
  - **The range of *g* must be a subset of *f*'s domain!!**
  - Note that $\bigcirc$ (like Cartesian $\times$, but unlike $+,\wedge,\cup$) is non-commuting. (In general, $f \bigcirc g \neq g \bigcirc f$.)

Function Composition



Examples

Composite Functions

Combination of more than one function.

If $f(x) = 4x$ and $g(x) = x^2$ $\quad g(\ f(3)\ )$?

$f(3) = 12$

so $g(\ f(3)\ )$

$= \quad g\ (12\ ) = 12^2$

$= 144$

$f(x) = 2x + 1$ $\quad\quad$ $g(x) = 4x$

Composite functions $\quad\quad$ $f(g(x))$ or $g(f(x))$

$f(g(x))$	$g(f(x))$
$= f(4x) = 2(4x) + 1$	$= g(2x + 1) = 4(2x + 1)$
$= 8x + 1$	$= 8x + 4$

Images of Sets under Functions

- Given f:$A \rightarrow B$, and $S \subseteq A$,
- The *image* of S under f is simply the set of all images (under f) of the elements of S.
  $f(S) :\equiv \{f(s) \mid s \in S\}$
  $:\equiv \{b \mid \exists\, s \in S: f(s)=b\}$.
- Note <u>the range of f can be defined as simply the image (under f) of f's domain</u>!

One-to-One Functions

- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has **only one** pre-image.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have same cardinality. What about codomain? May Be Larger

Functions

- **One-to-One Function**
- A function is one-to-one if each element in the co-domain has a unique pre-image
- A function *f* from **A** to **B** is called **one-to-one** (or 1-1) if whenever $f(a) = f(b)$ then $a = b$. No element of **B** is the image of more than one element in **A.**
- In a one-to-one function, given any *y* there is only one *x* that can be paired with the given *y*.
- A function is said to be injective if it is one-to-one.

One-to-One Functions (cont'd)

- Formally: given $f{:}A \rightarrow B$

“x is injective” $:\equiv (\neg\exists x,y{:}\ x \neq y \wedge f(x)=f(y))$ or

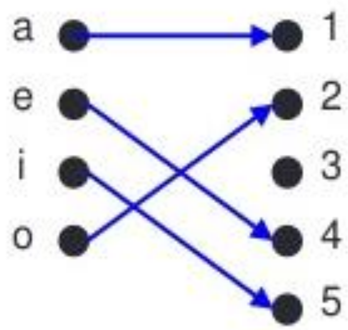
“x is injective” $:\equiv (\forall\ x,y{:}\ \neg(x \neq y) \vee \neg(f(x)=f(y)))$ or

“x is injective” $:\equiv (\forall\ x,y{:}\ \neg(x \neq y) \vee (f(x) \neq f(y)))$ or

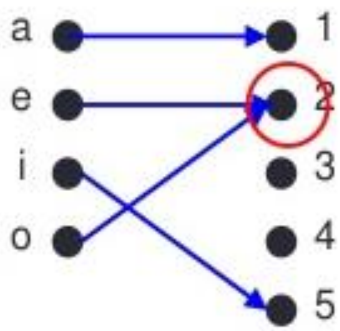
$\boxed{\text{“}x\text{ is injective”} :\equiv (\forall\ x,y{:}\ (x \neq y) \rightarrow (f(x) \neq f(y)))}$ or

$\boxed{\text{“}x\text{ is injective”} :\equiv (\forall\ x,y{:}\ (f(x)=f(y)) \rightarrow (x = y))}$

Example



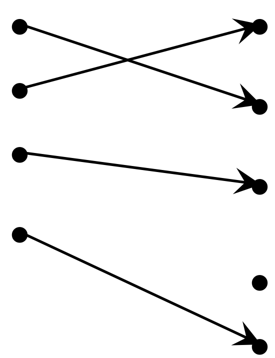
A one-to-one function



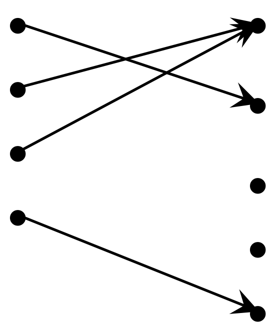
A function that is
not one-to-one

One-to-One Illustration

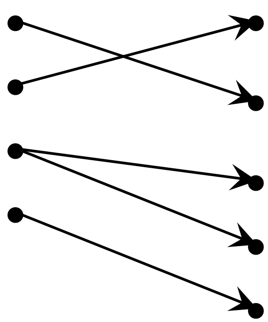
- Graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

Examples of one to one functions

1. $f(x) = x^3$

This cubic function possesses the property that each *x*-value has one unique *y*-value that is not used by any other *x*-element. This characteristic is referred to as being **1-1**.

2. $f(x) = 3x + 5$

3. $f(x) = \sqrt{x}$

4. All odd functions

Examples of functions that are **not** one to one

1. $f(x) = |x|$

2. $f(x) = x^2$

3. $f(x) = \frac{1}{x^2}$

Note that $\boldsymbol{f(x) = x^2}$ is not one-to-one if it is from the set of integers(negative as well as non-negative) to $\boldsymbol{N}$, because for example $\boldsymbol{f(1) = f(-1) = 1}$.

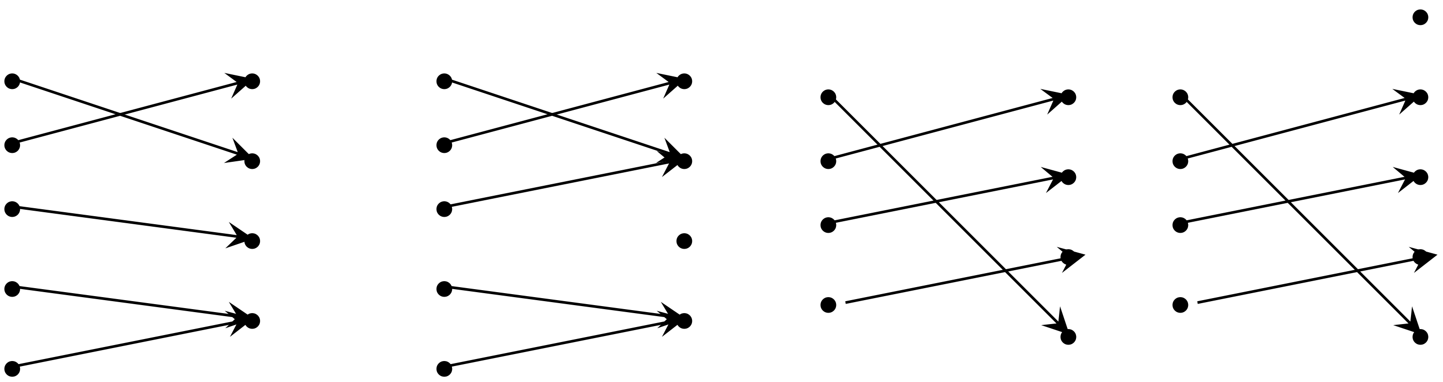
The function $\boldsymbol{f(x) = x^2}$ from the set of natural numbers $\boldsymbol{N}$ to $\boldsymbol{N}$ is a one-to-one function

Onto (Surjective) Functions

- A function $f:A\rightarrow B$ is *onto* or *surjective* or *a surjection* iff its **range is equal to its codomain** ($\forall b\in B, \exists a\in A: f(a)=b$).
- An *onto* function maps the set A onto (over, covering) the *entirety* of the set B, not just over a piece of it.
  - *e.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are or are not *onto* their codomains:



Onto (but not 1-1)

Not Onto (or 1-1)

Both 1-1 and onto

1-1 but not onto

Functions

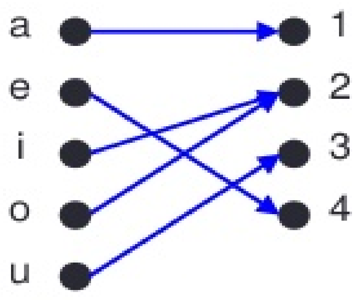
- **Onto Function**
- A function is onto if each element in the co-domain is an image of some pre-image

- A function f: $A \rightarrow B$ is subjective (onto) if the image of f equals its range.

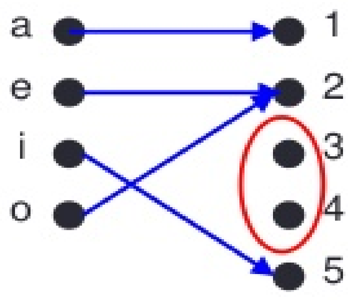
- A function f from A to B is called onto if for all b in B there is an a in A such that $f(a) = b$. All elements in B are used.

- A function is said to be subjective if it is onto function.

Example

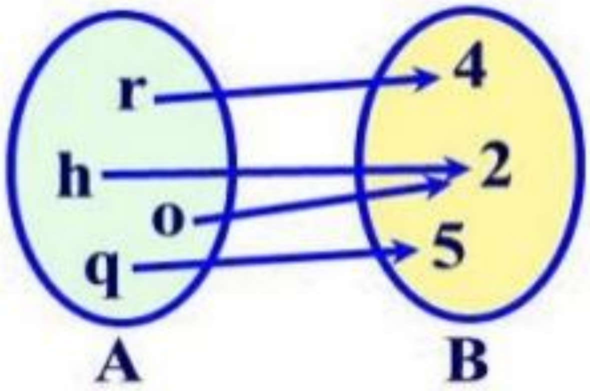


An onto function

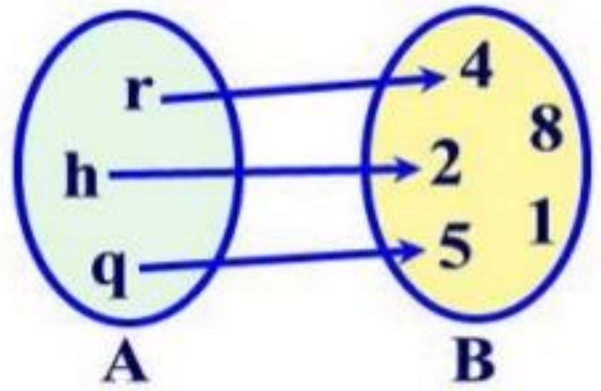


A function that
is not onto

Another Example



"Onto"
(all elements in B are used)



NOT "Onto"
(the 8 and 1 in Set B are not used)

Examples of onto functions

1. $\mathbf{Z} \to \mathbf{Z}$ by $f(n) = n-2$

2. $f(x) = 3x - 4$

As you progress along the line, every possible y-value is used

3. $f(x) = x^2$ from z+ to z+

4. $f: R \to R$ by $f(x) = x^3$

Examples of functions that are **not** onto

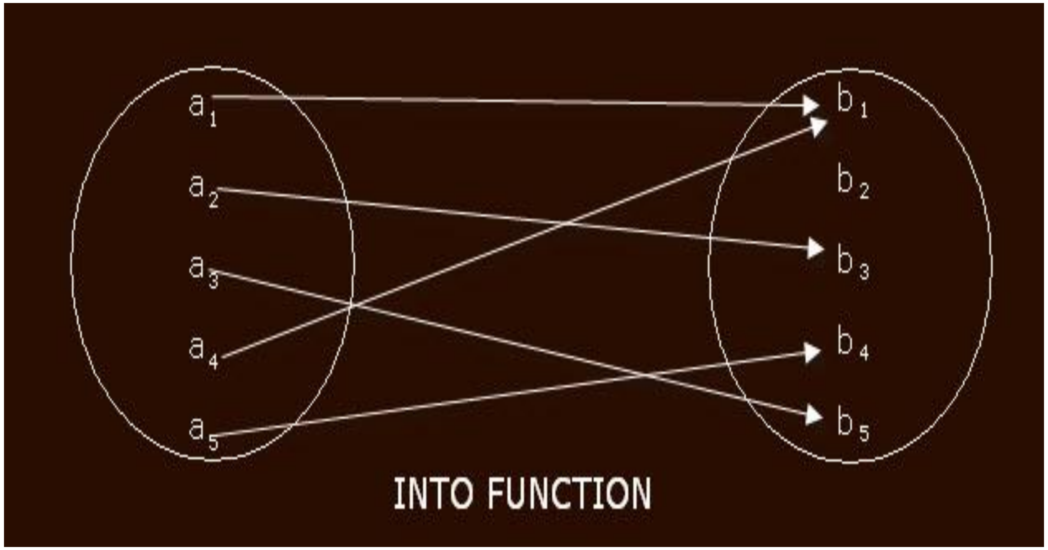
1. g: R → R, by $g(x) = x^2$
2. $f(x) = 2x$
3. $f : Z \to Z$ by $f(n)=2n+3$
4. $f(x) = x+5$

f*(*x*) = *2x from the set of natural numbers ***N*** to ***N*** is not onto, because, for example, nothing in ***N*** can be mapped to ***3*** by this function.

: The function ***f*(*x*) = *2x*** from the set of natural numbers ***N*** to the set of non-negative even numbers ***E*** is an onto function

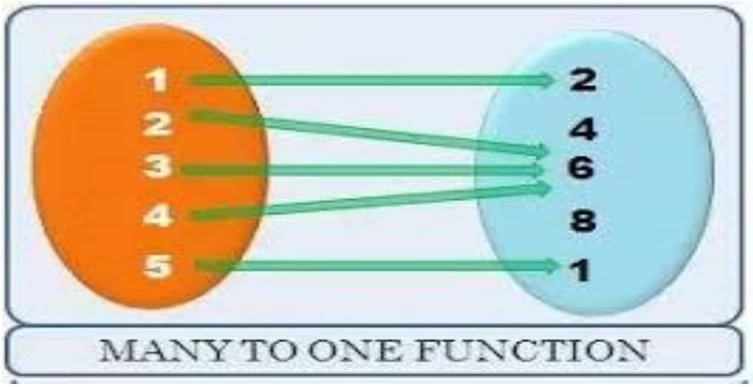
Into Mapping

Function f from set A to set B is Into function if at least set B has a element which is not connected with any of the element of set A.

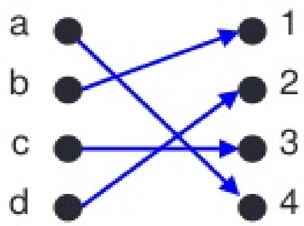


Many to One function

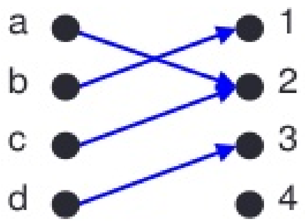
- If any two or more elements of set A are connected with a single element of set B, then we call this function as Many one function.



Example



Both 1-to-1 and onto



Neither 1-to-1 nor onto

Bijections

- A function *f* is *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff **it is both one-to-one and onto**.

Examples of bijective function

1. f: R→R defined by $f(x) = 2x - 3$

2. $f(x) = x^5$

3. $f(x) = x^3$

Examples of functions that are **not** bijective

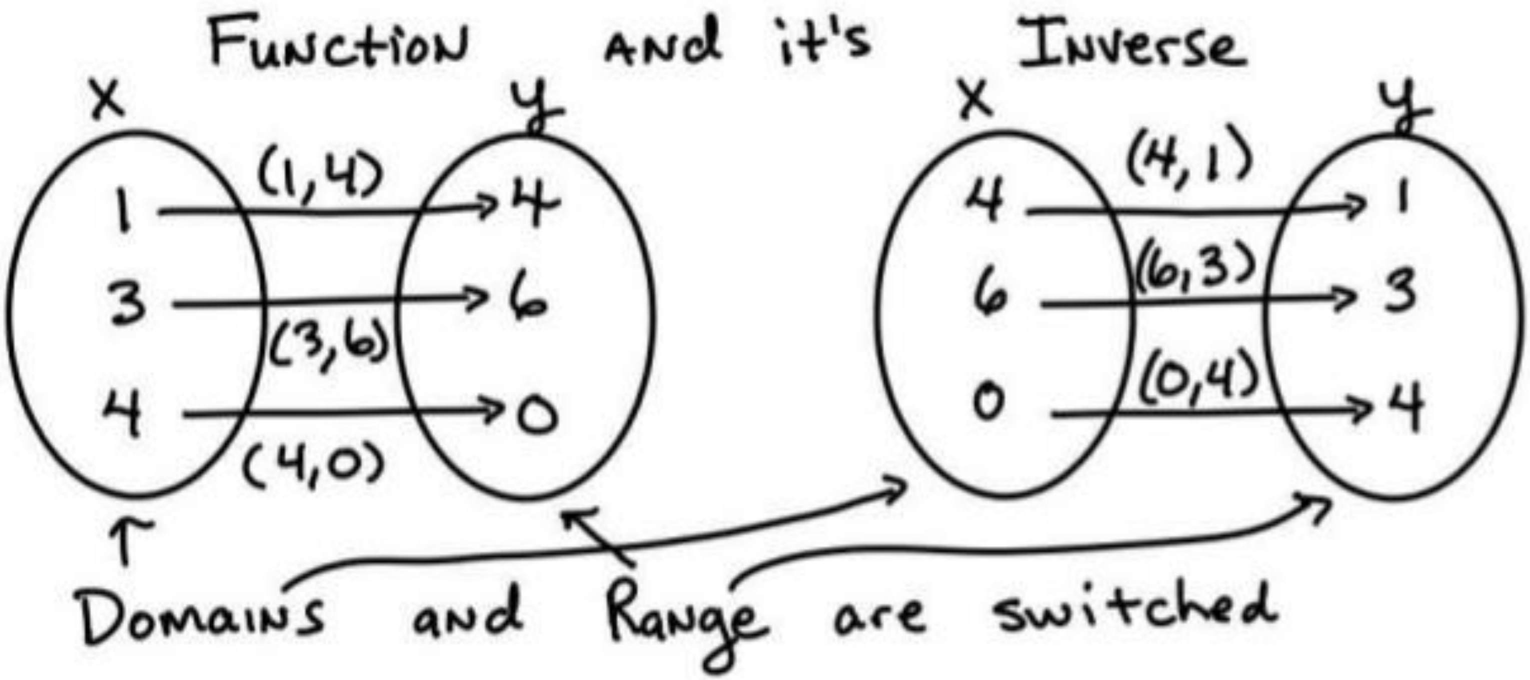
1. $f : \mathbf{Z}$ to $\mathbf{R}$, $f(x) = x^2$

Functions

- **Inverse of a Function**
- Given any function, f, the inverse of the function, f^{-1}, is a relation that is formed by interchanging each (x, y) of f to a (y, x) of f^{-1}.
- The function g would be denoted as f^{-1} and read as "f inverse".

Functions

- **Inverse of a Function Examples**

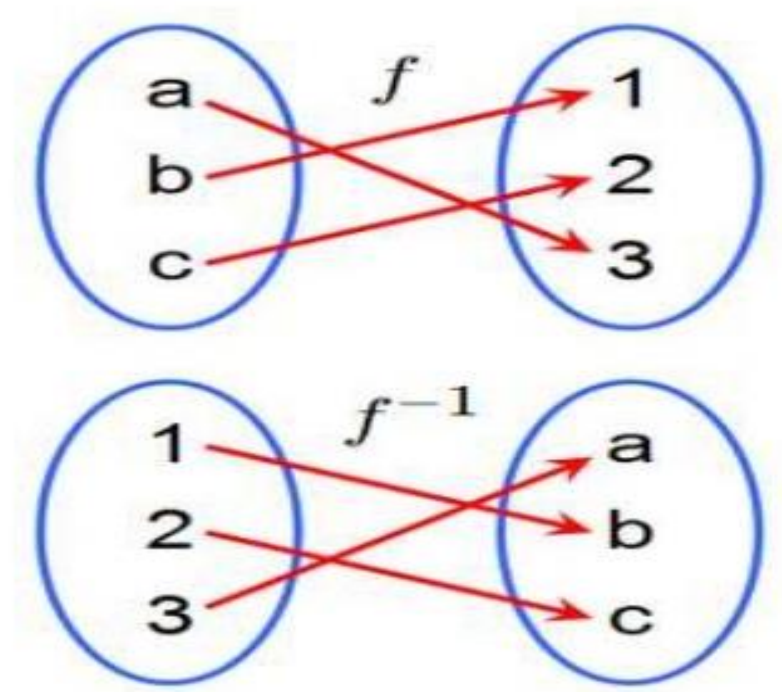


Inverse of a Function

- For bijections $f{:}A\rightarrow B$, there exists an *inverse* of f, written $f^{-1}{:}B\rightarrow A$, which is the unique function such that:

$$f^{-1} \circ f = I$$

Inverse of the Function



Inverse of a function (cont’d)

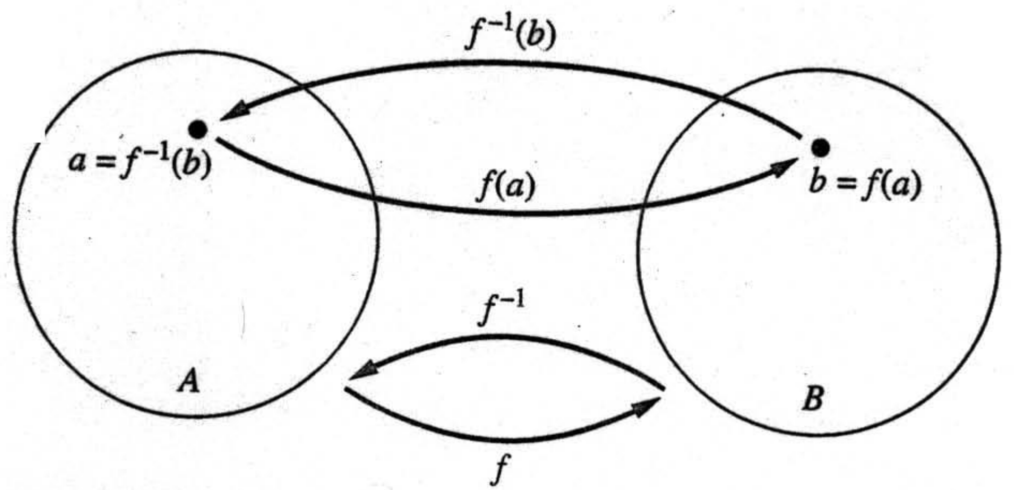


FIGURE 6 **The Function f^{-1} Is the Inverse of Function f.**

Other Example

Let $y = \dfrac{x}{5x-3}$

To find the inverse, switch x and y, $x = \dfrac{y}{5y-3}$

$$x(5y-3) = y$$

Solve for y: $5xy - 3x = y$

$$5xy - y = 3x$$

$$y(5x-1) = 3x$$

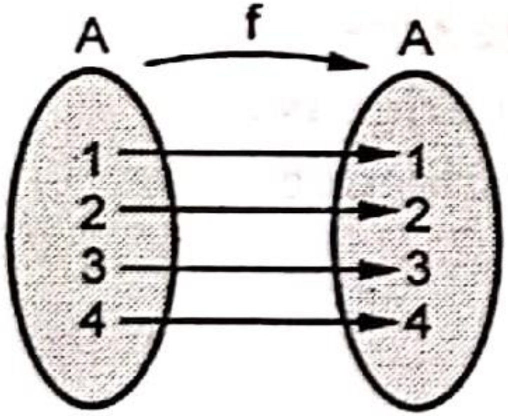
$$y = \frac{3x}{5x-1}$$

So the inverse of $f(x):\ y = \dfrac{x}{5x-3}$ is $f^{-1}(x):\ y = \dfrac{3x}{5x-1}$

IDENTITY FUNCTION

- Let A be any non empty set and function $f : A \rightarrow A$ is said to be the identity function if $f(x) = x, \forall\ x \in A$.

e.g.



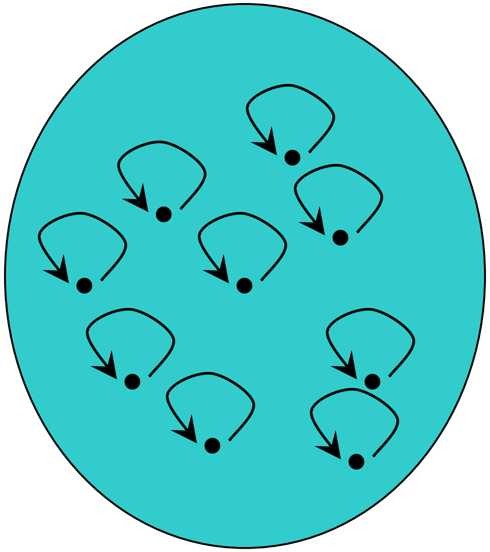
f : A → A is the identity function

The Identity Function

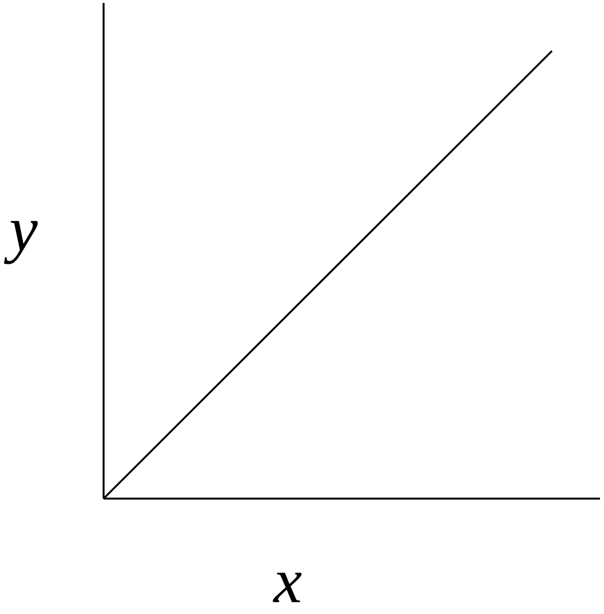
- For any domain A, the *identity function* $I:A\rightarrow A$ (variously written, I_A, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a\in A$: $I(a)=a$.
- Some identity functions you've seen:
  - $\wedge$ing with **T**, $\vee$ing with **F**, $\cup$ing with $\varnothing$, $\cap$ing with U.
- Note that the identity function is both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range



Graphs of Functions

- We can represent a function $f{:}A \to B$ as a set of ordered pairs $\{(a,f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only one pair $(a, f(a))$.
- For functions over numbers, we can represent an ordered pair (x,y) as a point on a plane. A function is then drawn as a curve (set of points) with only one y for each x.

Graphs of Functions

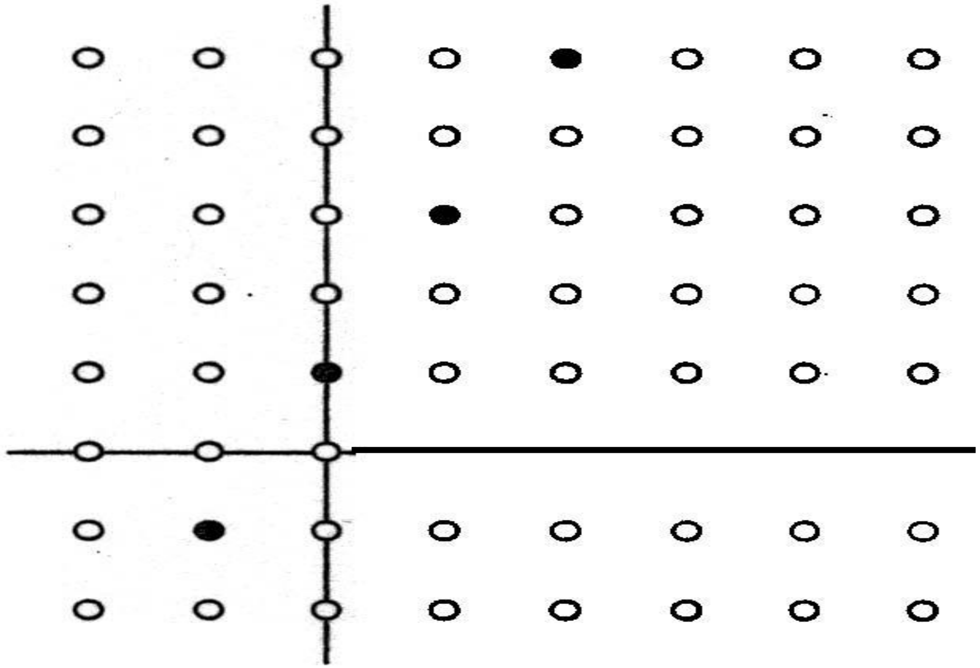


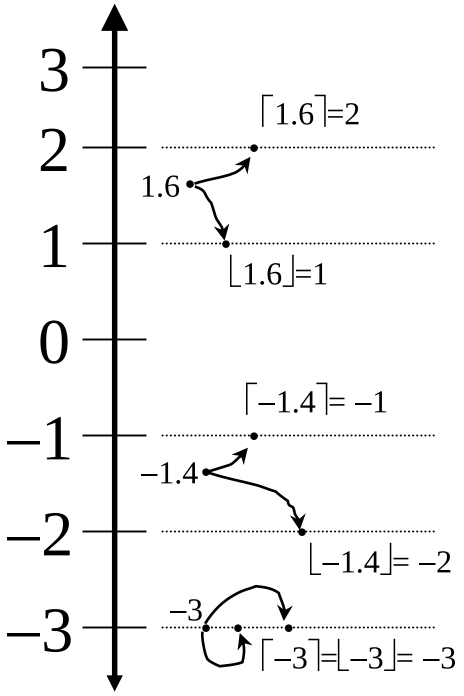
FIGURE 8 The Graph of the Function $f(n) = 2n + 1$ from Z to Z.

A Couple of Key Functions

- In discrete math, we frequently use the following functions over real numbers:
  - $\lfloor x \rfloor$ ("floor of x") is the largest integer $\leq x$.
  - $\lceil x \rceil$ ("ceiling of x") is the smallest integer $\geq x$.

Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$, $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ & $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$, $\lfloor x \rfloor = \lceil x \rceil = x$.



Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$: