7.5.2 Boolean Functions

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Let $(B, +, ., ')$ be a Boolean-algebra with binary operation '+' (,) '.' and a unary operation ','.

(i) *Monomial*: Monomial is any element of Boolean algebra. Also an element obtained by applying the operation '.' on two or more elements is ***called monomial***. For example $x, x', y, x'.y, x'.y.z$ are monomials.

(ii) *Polynomial*: Polynomial is an expression of Boolean algebra containing two or more than two monomials with operation '+' in between them. In other words, if the operation '+' is applied to two or more than two monomials then the expression so obtained is called a polynomial. For example: $x + y' + x.y'$ is a polynomial. Its ***First Term*** is x, ***Second Term*** is y' and ***Third Term*** is $x.y'$.

Factor: Consider an expression $x.(y' + x)$. Then we say that x and $y' + x$ are two factors of this ession. If an expression is composed of several expressions with '.' operation then each of these ressions is called a **factor**.

) **Constant:** By a constant we mean any symbol which represents a specified element of Boolean-ebra B.

ustration: 0 and 1 are constants of B, since 0 and 1 are identity elements for '+' and '.' in B. Hence 0 d 1 represents two specified elements of B.

) **Variable:** By variable we mean a symbol which represents an arbitrary element for B. Usually ıriables are denoted by the symbols $x, y, z, x', y', z', \ldots$

vi) Boolean Function (or Boolean Polynomial)

efinition: An expression obtained by the application of binary operation '+' and '.' and a unary peration ',' on finite number of elements of Boolean algebra $(B,+,.,')$ is called a **Boolean Function** or **Boolean Polynomial.**

Illustration: $x.y' + x'y, x.y.z. + x'.y.z + x.y'.z$ are Boolean functions

> Note: The number of variables in any Boolean function is the number of distinct letters appearing irrespective of whether the letter is primed or unprimed.

Illustrations: **1.** $x + x'$ *is a function of one variables* 'x'. **2.** $x.y'$ *is a function of two variables* x *and* y.

3. $x.y.z + x'.y.z + x.y'.z'$ is a function of three variables x, y and z.

(vii) *Power and Multiple of a Variable in a Boolean Function:* Let $(B,+,.)$ be a Boolean- algebra and $x \in B$ be any arbitrary variable, then

$$x^2 = x.x = x$$

$$x^3 = x^2.x = x.x = x$$

$$x^4 = x^3.x = x.x = x \text{ etc.}$$

$$x^n = x^{n-1}.x = x.x = x, \text{ when } n \text{ is a positive integer.}$$

Again

$$2x = x + x = x$$

$$3x = x + x + x = x + x = x, \text{ etc.}$$

$$nx = x + x + x + \ldots \text{ up to } n \text{ terms, where } n \text{ is +ve integer.}$$

$$= x.$$

Hence, the powers and multiples of any variable do not exist in a Boolean function.

For each positive integer n, we have $x^n = x$.

7.5.3 Minimal (maximal) Boolean Functions or Minterm

A minimal (maximal) Boolean function in n variables $x_1, x_2 \ldots, x_n$ is the product of n letters $\tilde{x}_1, \tilde{x}_2 \ldots, \tilde{x}_n$. In other words a minimal (maximal) Boolean function in n variables $x_1, x_2 \ldots, x_n$ is $\tilde{x}_1, \tilde{x}_2 \ldots, \tilde{x}_n$. Where $\tilde{x}_i$ denotes either x_i or x_i'. All the letters or some of the letters may be primed.

7.5.4 Boole's Theorem

Statement: The number of minimal Boolean function in n-variables are 2^n.

Proof: Suppose $x_1, x_2 \ldots, x_n$ are n variables in a Boolean algebra $(B, +, .)$ and let their complements be respectively $x_1', x_2' \ldots, x_n'$. Now there are two ways of selecting the first variable, x_1 or x_1' in a minimal Boolean function; there are two ways of selecting the second variable, x_2 or x_2'... there are two ways selecting the n^{th} variable, x_n or x_n'. Thus we see that there are two ways of selecting each of the n variables. Hence the number of minimal Boolean functions in n variables.

$$= 2 \times 2 \times 2 \times \ldots \times 2 \ (n \text{ factors}) = 2^n$$

7.6 Simplification of Boolean Function

7.6.1 Disjunctive Normal Form or Canonical Form or Sum of Products (SOP) Form

A Boolean polynomial which can be written as sum of the minimal Boolean functions, is called ***Disjunction Normal Form (or Canonical Form)*** of the Boolean function.

Suppose $x_1, x_2, \ldots, x_n$ are n variables in a Boolean algebra $(B, +, .)$ and their complements are respectively $x_1', x_2', \ldots, x_n'$ and suppose $f(x_1, x_2, \ldots, x_n)$ is an arbitrary Boolean function of these n variables. The function f is called in ***Disjunctive Normal Form*** if the function f can be written as the sum of terms of the type $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$ where $f_i(x_i) = x_i$ or x_i' $\forall i = 1, 2, \ldots, n$ and no two terms are identical. In addition to it, 0 and 1, for $n \geq 0$ are called disjunctive normal form in n variables.

> Note: We know that there are 2^n minimal Boolean functions in n variables. Hence maximum number of distinct terms in a disjunctive normal form of a Boolean function in n variables are 2^n.

7.6.2 Complete Disjunctive Normal Form or Complete Canonical Form

If the number of distinct terms in disjunctive normal form of Boolean function in n variables are 2^n, then it is called ***Complete Disjunctive Normal Form.***

Illustration: *A Boolean function $f(x_1, x_2)$ in two variables x_1, x_2 in **Disjunctive Normal Form** may be given by*

$$f(x_1, x_2) = x_1 . x_2 + x_1' . x_2 + x_1 . x_2'$$

But if Boolean function $f(x_1 . x_2)$ in two variable is ***Complete Disjunctive Normal Form,*** then

$$f(x_1, x_2) = x_1 . x_2 + x_1' . x_2 + x_1 . x_2' + x_1' . x_2'$$

Illustration: *A Boolean function $f(x_1, x_2, x_3)$ in three variables x_1, x_2, x_3 in **Disjunctive Normal Form** may be given by*

$$f(x_1, x_2, x_3) = x_1 . x_2 . x_3' + x_1' . x_2 . x_3 + x_1 . x_2' . x_3$$

Now assigning values 0 or 1 to the variables, we have the following tables:

x_1	x_2	x'_1	x'_2	$x_1 . x_2$	$x'_1 . x_2$	$x_1 . x'_2$	$x'_1 . x'_2$
1	0	0	1	0	0	1	0
0	1	1	0	0	1	0	0

From the above table we observe that by giving values 0 or 1 to the variables only one term, in both distributions, has value 1 and other terms have value 0.

Hence the theorem is verified.

Example 18: *Find complete disjunctive normal form in one variable x_1.*

Solution: $f(x_1) = x_1 + x'_1 = 1.$

Example 19: *Find complete disjunctive normal form in two variables x_1, x_2 and show that its value is 1.*

Solution: $f(x_1, x_2) = x_1 . x_2 + x_1 . x'_2 + x'_1 . x_2 + x'_1 . x'_2$

$= x_1(x_2 + x'_2) + x'_1(x_2 + x'_2)$ [By Distributive Law]

$= x_1 . 1 + x'_1 . 1$ [$\because x_2 + x_2 = 1$]

$= x_1 + x'_1 = 1$

Example 20: *Find complete disjunctive normal form in three variables, and show that its value is 1.*

Solution: $f(x_1, x_2, x_3) = x_1 . x_2 . x_3 + x_1 . x_2 . x'_3 + x_1 . x'_2 . x_3 + x'_1 . x_2 . x_3 + x_1 . x'_2 . x'_3 + x'_1 . x_2 . x'_3$

$+ x'_1 . x'_2 . x_3 + x'_1 . x'_2 . x'_3$

$= x_1 . x_2(x_3 + x'_3) + x_1 . x'_2(x_3 + x'_3) + x'_1 . x_2(x_3 + x'_3) + x'_1 . x'_2[(x_3 + x'_3)$

$= x_1 . x_2 . 1 + x_1 . x'_2 . 1 + x'_1 . x_2 . 1 + x'_1 x'_2 . 1$ [$\because x_3 + x'_3 = 1$]

$= x_1 . x_2 + x_1 . x'_2 + x'_1 . x_2 + x'_1 . x'_2$

$= x_1(x_2 + x'_2) + x'_1(x_2 + x'_2)$

$= x_1 . 1 + x'_1 . 1 = 1$

7.6.3 Complement Function of a Boolean Function in Disjunctive Normal Form

Suppose f is a Boolean function expressed in disjunctive normal form. Then complement function of f is a Boolean function which is the sum of all those terms of complete disjunctive normal form which are not present in disjunctive normal form of f. The complement function of f is denoted by f'.

Illustration: *Let* $f(x, y) = x . y + x' . y'$

Now complete disjunctive normal form in two variables is

$$x . y + x' . y + x . y' + x' . y'$$

Hence

$$f'(x, y) = x' . y + x . y'$$

7.7.3 Boolean Expression

A Boolean expression or form, in n variables $x_1, x_2, \dots x_n$ is any finite string of symbols formed as

(i) 0 and 1 are Boolean expressions.

(ii) If α and β are Boolean expression, then $\alpha \cdot \beta$ or $\alpha * \beta$ and $\alpha + \beta$ or $\alpha \oplus \beta$ are also Boolean expressions.

(iii) $x_1, x_2, x_3 \dots x_n$ are Boolean expressions.

(iv) If α is Boolean expression then $(\alpha)'$ is also a Boolean expression.

(v) If α is a Boolean expression in n variables $x_1, x_2 \dots x_n$ then α can be written as $\alpha\,(x_1, x_2, x_3 \dots x_n)$.

***Remark*:** Consider the Boolean expression

$$\alpha\,(x_1, x_2, \dots x_n) = [x_1 \cdot x_2 + x_3]'$$

if $x_1 = 1, x_2 = 0$ and $x_3 = 0$, then

$$\alpha\,(x_1, x_2, x_3) = \alpha\,(1, 0, 0) = [1.0 + 0]' = (0 + 0)' = 0' = 1$$

7.7.4 Equivalent Expression

Two Boolean expression $\alpha\,(x_1, x_2, x_3 \dots x_n)$ and $\beta\,(x_1, x_2, x_3 \dots x_n)$ are said to be equal or equivalent expression if one can be obtained from the other by finite numbers of application of the identities of Boolean algebra.

7.7.5 Literal

A literal is defined be a Boolean variable or its complement. For example x_1 and x_1' are literals.

Example 22: *Find the complement of the following functions*

(i) $f = a' b + ab'$ (ii) $f = a' bc + abc' + a' b' c + a' b' c'$.

Solution: (i) The complete disjunctive normal form in two variables a, b is

$$F = ab + ab' + a' b + a' b'$$

The given *DN* form is $f = a' b + ab'$

Then

$$f' = F - f$$

$$= ab + ab' + a' b + a' b' - a' b - ab' = ab + a' b'$$

(ii) The complete disjunctive normal form in three variables a, b and c is

$$F = abc + a' bc + ab' c + abc' + a' b' c + a' bc' + a' bc' + a' b' c'$$

The given *DN* form is $f = a' bc + abc' + a' b' c + a' b' c'$

Then

$$f' = F - f = abc + ab' c + a' bc' + ab' c'$$

Example 23: *Find the complement of* $f = (x + y')(x + y)$.

Solution: The given function f is written in *CN* form of two variables.

The complete *CN* form in two variables is $F = (x + y)(x' + y)(x + y')(x' + y')$

The complement of f is $f' = F - f = (x' + y)(x' + y')$

Example 24: *Change the function $f = uv + u'v + u'v'$ from DN form to CN form.*

Solution: The given *DN* form is $f = uv + u'v + u'v'$

The complete *DN* form for two variables u, v is

$$F = uv + u'v + uv' + u'v'$$

The complement of *DN* form

$$f' = F - f = uv'$$

We know the complement of complement *DN* form is *CN* form *i.e.*

$$f = (f')'$$

$$f = (uv')' \qquad \text{[By De Morgan's law]}$$

$$f = u' + (v')' = u' + v \qquad [(v')' = v]$$

Example 25: *Change the function $f = (x + y')(x' + y)(x' + y')$ from CN form to DN form.*

Solution: The given *CN* form is

$$f = (x + y')(x' + y)(x' + y')$$

The complete *CN* form for two variables x, y is

$$F = (x + y')(x' + y)(x' + y')(x + y)$$

The complement of *CN* form $f' = F - f$

$$f' = (x + y)$$

We know the complement of complement *CN* form is *DN* form *i.e.*

$$f = (f')'$$

$$f = (x + y)'$$

$$f = x'y'$$

Example 26: *Simplify the Boolean function by using laws of Boolean algebra*

$$xy + x'y + x'y'$$

Solution: Let $f = xy + x'y + x'y'$

$$= xy + x'(y + y') \qquad [y + y' = 1]$$

$$= xy + x' \qquad \text{[By Commutative]}$$

$$= (x' + xy)$$

$$= (x' + x)(x' + y) \qquad \text{[By Distributive]}$$

$$= 1 \cdot (x' + y) \qquad [x' + x = 1]$$

$$f = x' + y \qquad [1. x = x]$$

Example 27: *Simplify the Boolean function by using the law of Boolean algebra $xy + xyz + yz$.*

Solution: Let $f = xy + xyz + yz$

$$= xy(1+z) + yz$$

$$= xy + yz \qquad [1+z=1]$$

$$= xy + zy \qquad \text{[By Commutative]}$$

$$= (x+z)y$$

Example 28: *Express Boolean expression $(x+y).(z'+w)$ in polynomial.*

Solution: $(x+y)(z'+w) = (x+y)z' + (x+y)w$ [By Distributive Laws of '.' over '+']

$$= z'(x+y) + w(x+y) \qquad \text{[By Commutative Law]}$$

$$= z'.x + z'.y + w.x + w.y \qquad \text{[By Distributive Law]}$$

Example 29: *Express the polynomial $x.z + x.w + y.z + y.w$ into factors.*

Solution: $x.z + x.w + y.z + y.w = x(z+w) + y(z+w)$ [Distributive Law]

$$= (z+w)x + (z+w)y \qquad \text{[Commutative Law]}$$

$$= (z+w)(x+y) \qquad \text{[Distributive Law]}$$

$$= \mathbf{(x+y)(z+w)} \qquad \text{[By Distributive Law]}$$

Example 30: *Obtain the sum of products canonical forms of Boolean expression $x_1x_2' + x_3$.*

Solution: We know

$$f(x_1, x_2, x_3) = x_1x_2' + x_3$$

$$= x_1x_2'(1) + x_3 1.1 \qquad [x_1 + x_1' = x_2 + x_2' = x_3 + x_3' = 1]$$

$$= x_1x_2'(x_3 + x_3') + x_3(x_1 + x_1')(x_2 + x_2')$$

$$= x_1x_2'x_3 + x_1x_2'x_3' + (x_1x_3 + x_1'x_3)(x_2 + x_2')$$

$$= x_1x_2'x_3 + x_1x_2'x_3' + x_1x_2x_3 + x_1x_2'x_3 + x_1'x_2x_3 + x_1'x_2'x_3$$

$$= x_1x_2'x_3 + x_1x_2'x_3' + x_1x_2x_3 + x_1'x_2x_3 + x_1'x_2'x_3$$

This is the sum of products canonical forms.

Example 31: *Write the Boolean expression $f(x_1, x_2, x_3) = (x_1'x_2)'(x_1 + x_3)$ in both disjunctive and conjunctive normal forms.*

Solution: We have

$$f(x_1, x_2, x_3) = (x_1'x_2)'(x_1 + x_3)$$

$$= [(x_1')' + (x_2')](x_1 + x_3)$$

$$= (x_1 + x_2')(x_1 + x_3) \qquad [(x_1')' = x_1]$$

$$= x_1 + x_2'x_3$$

$$= x_1 1.1 + x_2'x_3.1$$

$$= x_1(x_2 + x_2')(x_3 + x_3') + x_2'x_3(x_1 + x_1')$$

$$= (x_1x_2 + x_1x_2')(x_3 + x_3') + x_1x_2'x_3 + x_1'x_2'x_3 \qquad [x_1 + x_1' = x_2 + x_2' = x_3 + x_3' = 1]$$

$$= x_1x_2x_3 + x_1x_2x_3' + x_1x_2'x_3 + x_1x_2'x_3' + x_1x_2'x_3 + x_1'x_2'x_3$$

$$= x_1x_2x_3 + x_1x_2x_3' + x_1x_2'x_3 + x_1x_2'x_3' + x_1'x_2'x_3 \qquad [x_1x_2'x_3 + x_1x_2'x_3 = x_1x_2'x_3]$$

This is the disjunctive normal form.

Now the conjunctive normal form.

$$f(x_1, x_2, x_3) = (x_1' x_2)'(x_1 + x_3)$$
$$= (x_1 + x_2')(x_1 + x_3) \qquad [(x_1')' = x_1)]$$
$$= (x_1 + x_2' + 0)(x_1 + x_3 + 0)$$
$$= (x_1 + x_2' + x_3 x_3')(x_1 + x_3 + x_2 x_2')$$
$$= (x_1 + x_2' + x_3)(x_1 + x_2' + x_3')(x_1 + x_3 + x_2)(x_1 + x_3 + x_2')$$

Example 32: *Change the function $x'yz + xyz + x'yz' + xyz'$ into disjunctive normal form of two variables.*

Solution:

$$x'yz + xyz + x'yz' + xyz' = (x'yz + x'yz') + (xyz + xyz')$$
$$= x'y(z + z') + xy(z + z')$$
$$= x'y(1) + xy(1) \qquad [\because z + z' = 1]$$
$$= x'y + xy = \mathbf{xy + x'y}.$$

Example 33: *Express the following function in disjunctive normal form in the smallest possible number of variables:*

$$f(x, y, z) = xy' + xz + xy.$$

Solution: $f(x, y, z) = xy' + xz + xy$

$$= xy' + xy + xz$$
$$= x(y' + y) + xz = x.1 + xz = x + xz = x(1 + z) = x.1 = \mathbf{x}.$$

Example 34: *Change the following functions to disjunctive normal forms of three variables x, y, z:*

(i) $x + y'$ (ii) $x'z + xz'$ (iii) $(x + y)(x' + y')$ (iv) x

Solution:

(i) $x + y' = x(y + y')(z + z') + y'(x + x')(z + z')$

$$= x(yz + yz' + y'z + y'z') + y'(xz + xz' + x'z + x'z')$$
$$= xyz + xyz' + xy'z + xy'z' + y'xz + y'xz' + y'x'z + y'x'z'$$
$$= xyz + xyz' + (xy'z + xy'z) + (xy'z' + xy'z') + x'y'z + x'y'z'$$
$$= \mathbf{xyz + xyz' + xy'z + xy'z' + x'y'z + x'y'z'}$$

(ii) $x'z + xz' = x'z(y + y') + (xz')(y + y') = x'zy + x'zy' + xz'y + xz'y'$

$$= x'yz + x'y'z + xyz' + xy'z'.$$

(iii) $\mathbf{(x + y)(x' + y')} = xy' + yx'$

$$= xy'(z + z') + yx'(z + z') = \mathbf{xy'z + xy'z' + yx'z + yx'z'}$$

(iv) $x = x(y + y')(z + z')$

$$= x(yz + yz' + y'z + y'z') = \mathbf{xyz + xyz' + xy'z + xy'z'}.$$

Example 35: *Write the following functions into disjunctive normal forms of 3 variables x, y, z:*

(i) $x' + y'$ (ii) $xy' + x'y$ (iii) $(x+y)(x'+z')$

Solution: (i) $x' + y' = x'(y+y')(z+z') + y'(x+x')(z+z')$ $[\because x + x' = 1, \text{etc.}]$

$= x'(yz + yz' + y'z + y'z') + y'(xz + xz' + x'z + x'z')$

$= x'yz + x'yz' + x'y'z + x'y'z' + xy'z + xy'z' + x'y'z + x'y'z'$

$= x'yz + x'yz' + 2x'y'z + 2x'y'z' + xy'z + xy'z'$

$= x'yz + x'yz' + x'y'z + x'y'z' + xy'z + xy'z'$ $[\because nx = x, \forall n \in N]$

(ii) $xy' + x'y = xy'(z+z') + x'y(z+z') = \mathbf{xy'z + xy'z' + x'yz + x'yz'}.$ $[\because z + z' = 1]$

(iii) $(x+y)(x'+z') = (x+y)x' + (x+y)z'$

$= xx' + yx' + xz' + yz'$

$= 0 + x'y + xz' + yz'$ $[\because xx' = 0]$

$= x'y(z+z') + xz'(y+y') + yz'(x+x')$

$= x'yz + x'yz' + xyz' + xy'z' + xyz' + x'yz'$

$= x'yz + 2x'yz' + 2xyz' + xy'z' = \mathbf{x'yz + x'yz' + xyz' + xy'z'}.$

Example 36: *Find a function of three variables x, y, z which has value 1, when either $x = y = 1$ and $z = 0$ or $x = z = 1$ and $y = 0$, otherwise it has value 0.*

Solution:

Case I: *When* $x = y = 1$ *and* $z = 0 \Rightarrow z' = 1$

Function xyz' has value 1, since $xyz' = 1.1.1 = 1$

Case II: *When* $x = z = 1$ *and* $y = 0 \Rightarrow y' = 1$

Function $xy'z$ has value 1, since $xy'z = 1.1.1 = 1$

Hence the required function is the sum of above two functions and is given by $\mathbf{xyz' + xy'z}$

Now if x,y,z are assigned values 0 and 1 in other distinct ways, the above function has value 0.

Example 37: *Express the following function in conjunctive normal form:*

$$f(x, y, z) = (xy' + xz)' + x'$$

Solution: $f(x, y, z) = (xy' + xz)' + x' = (xy')'.(xz)' + x'$

$= (x' + y)(x' + z') + x'$

$= (x' + x' + y)(x' + x' + z')$ $[\because a + bc = (a+b)(a+c)]$

$= (x' + y)(x' + z')$

$= (x' + y + zz')(x' + z' + yy')$

$= (x' + y + z)(x' + y + z')(x' + z' + y)(x' + z' + y')$

$= (x' + y + z)(x' + y + z')(x' + y' + z')$ $[\because aa =$

Example 38: *What are Boolean functions? Simplify the given Boolean expression.* $(a\,.\,b)' \oplus (a \oplus b)'$

Solution: A Boolean function is an expression which is formed with binary variables. The two binary operator OR and AND, the unary operator NOT parenthesis and equal sign.

$$(a\,.\,b)' \oplus (a \oplus b)' = [(a\,.\,b)']'(a \oplus b)' + (a\,.\,b)'((a \oplus b)')' \qquad [a \oplus b = ab' + a'b]$$
$$= (a\,.\,b)(a'b + ab') + (ab)'(a \oplus b)$$
$$= ab(a'b + ab') + (ab)'(a'b + ab')$$
$$= (a'b + ab')1 = a'b + ab'$$

Example 39: *Is the statement always true? Justify your answer:*

"If $x(y + z') = x(y + w')$"

Solution: If $x(y + z') = x(y + w')$

$$\Rightarrow xy + xz' = xy + xw'$$
$$\Rightarrow xz' = xw'$$
$$\Rightarrow z' = w'$$
$$\Rightarrow z = w$$

The statement is true only when $z = w$

Example 40: *If* $f(x, y, z) = xy' + xyz' + x'yz'$ *show that:* $f(x, y, z) + z' \neq f(x, y, z)$

Solution: $f(x, y, z) + z' = xy' + xyz' + x'yz' + z'$

$$= xy' + xyz' + z'(x'y + 1)$$
$$= xy' + xyz' + z' \qquad [\because x'y + 1 = 1,\ z'1 = z']$$
$$= xy' + z'(xy + 1) = xy' + z' \neq \boldsymbol{f(x, y, z)}$$

Example 41: *Simplify the Boolean expression:*

$$(a \wedge b) \vee (a \wedge b \wedge c) \vee (b \wedge c)$$

Solution: $(a \wedge b) \vee (a \wedge b \wedge c) \vee (b \wedge c)$ $\qquad [T = b \wedge c]$

$$= (a \wedge b) \vee (b \wedge c) \wedge (a \vee T) = (a \wedge b) \vee (b \wedge c) \wedge T = \mathbf{(a \wedge b) \vee (b \wedge c)}$$

Example 42: *The function* $xy + x'y + x'y'$ *is given in disjunctive normal form, change it into conjunctive normal form.*

Solution: $xy + x'y + x'y' = xy + x'(y + y')$

$$= xy + x' \qquad [\because y + y' = 1]$$
$$= x' + xy$$
$$= (x' + x)(x' + y) \qquad [\because a + ab = (a + b)(a + c)]$$
$$= \mathbf{(x' + y)}$$

Example 43: *The function* $(x + y')(x' + y)(x' + y')$ *is given in conjunctive normal form, change it into disjunctive normal form.*

Solution: $(x + y')(x' + y)(x' + y')$

$$= (x + y')(x' + yy') \qquad \text{[By Distributive Law]}$$
$$= (x + y')x' \qquad [\because yy' = 0]$$
$$= xx' + x'y' = \mathbf{0 + x'y' = x'y'}$$

Example 44: *Change the following Boolean function to disjunctive normal form*

$$f(x, y, z) = [x + (x' + y)'][x + (y'.z')']$$

Solution: $f(x, y, z) = [x + (x' + y)'][x + (y'.z')']$

$= [x + (x')'y'][x + (y')' + (z')']$ [By De Morgan Law]

$= [x + x.y'][x + y + z]$ [$\because (a')' = a$]

$= (x + x.y')x + (x + x.y')y + (x + x.y')z$ [. is Distributive over +]

$= x.x + x.y'.x + x.y + x.y'.y + x.z + x.y'.z$

$= x + x.y' + x.y + 0 + x.z + x.y'.z$ [$\because x.x = x$]

$= x.1.1 + x.y'.1 + x.y.1 + x.1.z + x.y'.z$

$= x(y + y')(z + z') + x.y'(z + z') + x.y(z + z') + x(y + y')z + x.y'.z$

$= x.y.z + x.y.z' + x.y'.z + x.y'.z' + x.y'.z + x.y'.z' + x.y.z + x.y.z' + x.y.z + x.y'.z + x.y'.z$

$= x.y.z + x.y.z' + x.y'.z + x.y'.z'$

Example 45: *Change the following Boolean function to disjunctive normal form*

$f(x, y, z, t) = (x'.y + x.y.z' + x.y'.z + x'.y'.z'.t + t')'$.

Solution: Applying general De-Morgan Law, we have

$f(x, y, z, t) = (x'.y)'(x.y.z')'(x.y'.z)'(x'.y'.z'.t)'(t')'$

$= (x + y')(x' + y' + z)(x' + y + z')(x + y + z + t')t$

$= [x.x' + x.y' + x.z + y'.x' + y'.y' + y'.z](x' + y + z')(x.t + y.t + z.t + t'.t)$

$= [x.y' + x.z + x'.y' + y' + y'.z](x' + y + z')(x.t + y.t + z.t)$ [$\because a.a' = 0, a.b = b.a$, etc.]

$= [x.z + \{y' + y'(x + x' + z)\}(x' + y + z')(x.t + y.t + z.t)]$

$= (x.z + y')(x' + y + z')(x.t + y.t + z.t)$ [By Absorption Law: $y' + y'.u = y'$]

$= (x.z.x' + x.z.y + x.z.z' + y'.x' + y'.y + y'.z')(x.t + y.t + z.t)$

$= (x.y.z + x'.y' + y'.z')(x.t + y.t + z.t)$ [$\because x.x' = 0$]

$= x.y.z.x.t + x.y.z.y.t + x.y.z.z.t + x'.y'.x.t + x'.y'.y.t + x'.y'.z.t + y'.z'.x.t + y'.z'.y.t + y'.z'.z.t$

$= \mathbf{x.y.z.t + x'.y'.z.t + x.y'.z'.t}$ [$\because a.a' = 0, a + a = a$]

Example 46: *Change the following Boolean function to conjunctive normal form:*

$$f(x, y, z, t) = (x'.y + x.y.z' + x.y'.z + x'.y'.z'.t + t')'$$

Solution: By general De-Morgan's Law, we have:

$f(x, y, z, t) = (x'.y)'(x.y.z')'(x.y'.z)'(x'.y'.z'.t)'(t')'$

$= (x + y')(x' + y' + z)(x' + y + z')(x + y + z + t')t$

$= (x + y')(x' + y' + z)(x' + y + z)t$ [By Absorption Law]

$= (x + y' + 0 + 0)(x' + y' + z + 0)(x' + y + z + 0)(0 + 0 + 0 + t)$

$= (x + y' + z.z' + t.t')(x' + y' + z + t.t')(x' + y + z + t.t')(x.x' + y.y' + z.z' + t)$

$= (x + y' + z + t)(x + y' + z + t')(x + y' + z' + t)(x + y' + z' + t')(x' + y' + z + t)(x' + y' + z + t')$
$(x' + y + z + t)(x' + y + z + t')(x + y + z + t)(x + y + z' + t)(x + y' + z + t)(x + y' + z' + t)$
$(x' + y + z + t)(x' + y + z' + t)(x' + y' + z + t)(x' + y' + z' + t)$ [$\because$ + is Distributive over.]

$= (x + y + z + t)(x + y + z + t')(x + y + z' + t)(x + y + z' + t')(x' + y' + z + t)(x' + y' + z + t')$
$(x' + y + z + t)(x + y' + z' + t)(x + y' + z' + t')(x + y' + z + t')(x + y' + z + t)(x + y' + z' + t)$
$(x' + y + z' + t)(x' + y' + z' + t)$

Example 47: *Evaluate the function f from the following table:*

a	b	c	f	T
1	1	1	0	$a'.b'.c'$
1	1	0	1	$a'.b'.c$
1	0	1	1	$a'.b.c'$
0	1	1	0	$a.b'.c'$
1	0	0	1	$a'.b.c$
0	1	0	0	$a.b'.c$
0	0	1	1	$a.b.c'$
0	0	0	0	$a.b.c$

Where T represents the terms of function f.

Solution: From Bool's expansion theorem, the function f is given by

$f(a.b.c) = 0(a'.b'.c')+1(a'.b'.c)+1(a'.b.c')+0(a.b'.c')+1(a'.b.c)+0(a.b'.c)+1(a.b.c')+0(a.b.c)$

$= a'.b'.c+a'.b.c'+a'.b.c+a.b.c'$

$= a'.b'.c+a'.b(c'+c)+a.b.c'$ [By Distributive Law]

$= a'.b'.c+a'.b.1+a.b.c'$ $[\because c'+c=1]$

$= a'.b'.c+a'.b+a.b.c'$

$= a'(b'.c+b)+a.b.c'$ [By Distributive Law]

$= a'(b+b'.c)+a.b.c'$

$= a'(b+b')(b+c)+a.b.c'$ [By Distributive Law]

$= a'.1(b+c)+a.b.c'$

$= a'(b+c)+a.b.c'$

$= a'.b+a'.c+a.b.c'$

$= a'.b+a.c'.b+a'.c$

$= (a'+ac')b+a'.c$

$= (a'+a)(a'+c')b+a'.c$ [By Distributive Law]

$= 1(a'+c')b+a'.c = (a'+c')b+a'.c = a'.b+c'.b+a'.c$

$= \mathbf{a'.b+b.c'+a'.c}$

Example 48: *Determine the functions f_1 and f_2 from the following table:*

x_1	x_2	x_3	T_1	f_1	f_2	T_2
1	1	1	$x_1 . x_2 . x_3$	1	0	$x_1' . x_2' . x_3'$
1	1	0	$x_1 . x_2 . x_3'$	0	0	$x_1' . x_2' . x_3$
1	0	1	$x_1 . x_2' . x_3$	1	0	$x_1' . x_2 . x_3'$
0	1	1	$x_1' . x_2 . x_3$	0	1	$x_1 . x_2' . x_3'$
1	0	0	$x_1 . x_2' . x_3'$	0	0	$x_1' . x_2 . x_3$
0	1	0	$x_1' . x_2 . x_3'$	0	1	$x_1 . x_2' . x_3$
0	0	1	$x_1' . x_2' . x_3$	1	0	$x_1 . x_2 . x_3'$
0	0	0	$x_1' . x_2' . x_3'$	1	0	$x_1 . x_2 . x_3$

Where T_1 and T_2 represents the terms of f_1 and f_2 respectively.

Solution: From Bool's expansion theorem, f_1 is given by

$$f_1(x_1, x_2, x_3) = 1(x_1 . x_2 . x_3) + 0(x_1 . x_2 . x_3') + 1(x_1 . x_2' . x_3) + 0(x_1' . x_2 . x_3) + 0(x_1 . x_2' . x_3') + 0(x_1' . x_2 . x_3')$$
$$+1(x_1' . x_2' . x_3) + 1(x_1' . x_2' . x_3')$$
$$= x_1 . x_2 . x_3 + x_1 . x_2' . x_3 + x_1' . x_2' . x_3 + x_1' . x_2' . x_3' = x_1(x_2 + x_2')x_3 + x_1' . x_2'(x_3 + x_3')$$
$$= x_1 . 1 . x_3 + x_1' . x_2' . 1 = x_1 . x_3 + x_1' . x_2'$$

Similarly f_2 is given by

$$f_2(x_1, x_2, x_3) = 0(x_1' . x_2' . x_3') + 0(x_1' . x_2' . x_3) + 0(x_1' . x_2 . x_3') + 1(x_1 . x_2' . x_3') + 0(x_1' . x_2 . x_3) + 1(x_1 . x_2' . x_3)$$
$$+ 0(x_1 . x_2 . x_3') + 0(x_1 . x_2 . x_3)$$
$$= x_1 . x_2' . x_3' + x_1 . x_2' . x_3 = x_1 . x_2' . (x_3' + x_3) = x_1 . x_2' . 1 = \mathbf{x_1 . x_2'}$$

Example 49: Determine the function *f* defined by the following table and simplify it:

x	y	z	$f(x, y, z)$	T
1	1	1	0	$x . y . z$
1	1	0	1	$x . y . z'$
1	0	1	1	$x . y' . z$
1	0	0	0	$x . y' . z'$
0	1	1	0	$x' . y . z$
0	1	0	0	$x' . y . z'$
0	0	1	1	$x' . y' . z$
0	0	0	0	$x' .$

Where T represents the terms of f in disjunctive normal form.

Solution: We know that for f to be in disjunctive normal form, the value of only one term is 1 while the value of others is 0, when each variable is assigned arbitrarily the value 0 or 1. Consequently, the function f in disjunctive normal form is given by

$$f(x, y, z) = x.y.z' + x.y'.z + x'.y'.z$$

Verification: From the table, we have

$$\begin{aligned}\Sigma f.T &= 0(x.y.z) + 1(x.y.z') + 1(x.y'.z) + 0(x.y'.z') + 0(x'.y.z) + 0(x'.y.z') + 1(x'.y'.z) + 0(x'.y'.z')\\ &= x.y.z' + x.y'.z + x'.y'.z\\ &= x.y.z' + (x + x').y'.z\\ &= x.y.z' + 1.y'.z = x.y.z' + y'.z = \boldsymbol{f(x, y, z)}\end{aligned}$$

Which is the required function.

Example 50: *Find the complement of the following Boolean functions which are in conjunctive normal forms:*

(i) $F = (x + y + z')(x + y' + z')(x + y + z')$

(ii) $F = (x + y + z)(x' + y' + z')(x + y' + z)(x + y + z').(x + y' + z')$

Solution: Complete conjunctive normal form in three variables is

$$(x + y + z)(x + y + z')(x + y' + z)(x + y' + z')(x' + y + z)(x' + y + z')(x' + y' + z)(x' + y' + z')$$

Let complement of given function F be F', then, we have

(i) $F' = (x + y + z)(x + y' + z)(x' + y + z)(x' + y + z')(x' + y' + z)(x' + y' + z')$

(ii) $F' = (x' + y' + z)(x' + y + z')(x' + y + z)$

Exercise

1. Define the following:
   (i) Boolean Function (ii) Minimal Boolean Function
2. Show that the number of minimal Boolean function in n variables are 2^n.
3. State and prove Boole's Expansion theorem.
4. Show that complete canonical form in n variables is always equal to 2^n. Explain it by giving examples in 2 and 3 variables.
5. Change the following function to disjunctive normal form
   $F(x, y, z) = x.y' + x.z + z.y.$
6. Change the following functions to disjunctive normal form:
   (i) $f(x, y, z) = [(x + y')(x, y', z)']'$ (ii) $f(x, y, z) = (x + y')(y + z')(z + x')(x' + y')$
   (iii) $f(u, v, w) = (u + v + w)(u.v + u'.w)'$ (iv) $f(x, y, z) = x'.y'.z + x'.z$
   (v) $F(x, y, z) = (x.y' + x.z')' + x'$ (vi) $F(x, y, z) = [(x + y') + (y + z')']' + y.z.$
   (vii) $F(x, y, z) = (x + y)(x + z') + (y + z')$ (viii) $F(x, y, z) = [(x.y')' + z'][z + x']'$

7. Show that every Boolean function can be written as the sum of minimal Boolean functions.

8. Change the following function to conjunctive normal forms in which minimum number of variables are used:

(i) $f = (x+y')(y+z')(z+x')(x'+y')$. (ii) $f = (x+y)(x+y')(x'+z)$

(iii) $f = x'.y.z + x.y'.z' + x'.y'.z + x'.y.z' + x.y'.z + x'.y'.z'$.

9. Change the following functions to conjunctive normal forms:

(i) $F = [x+(x'+y)'][x+(y'+z')]$. (ii) $F = (x+y)(x+y')(x'+z)$.

(iii) $F = [(x+y')(x.y'.z)']'$. (iv) $F = x'.y.z + x.y'.z'$

(v) $F(x,y,z) = [x+(x'+y)]'[x+(y'.z')']$

10. Find the complement of the following function

(i) $(x+y)(x'+y)(x'+y')$ (ii) $(x+y+z)(x'+y'+z)(x'+y'+z')$

11. *Express the following functions into disjunctive normal form:*

(i) $x.y$ (ii) $x + x'.y$

(iii) $f(x,y,z) = x.y' + x.z + x.y$ (iv) $f(x,y,z) = (x.y' + x.z)' + x'$

(v) $f(x,y,z) = x.y.z + (x+y)(x+z)$ (vi) $f(x,y,z) = (x+y)(x+y')(x'+z)$

(vii) $f(x,y,z) = (x+y+z)(xy+x'z)'$ (viii) $f(x,y,z) = [(xy')' + z'][z+x']'$

(ix) $f(x,y,z) = (x'+y)'(x+z)' + (y.z)'$

(x) $f(x,y,z) = (x+y')(y+z')(z+x')(x'+y')$.

Answer

5. $[x.y.z + x.y'.z + x.y.z' + x.y'.z']$

6. (i) $x'.y.z + x'.y.z' + x.y'.z$

(ii) $x'.y'.z'$

(iii) $u.v'.w + u.v'.w' + u'.v.w'$

(iv) $x'.y'.z + x'.y.z' + x'.y'.z$

(v) $x.y.z + x'.y.z + x'.y'.z' + x.y.z' + x'.y'.z'$

(vi) $x.y.z + x'.y.z + x'.y.z'$

(vii) $x.y.z + x'.y.z + x.y'.z + x'.y.z' + x'.y.z' + x.y'.z' + x'.y'.z'$

(viii) $x.y.z' + x.y'.z'$

8. (i) $(x+y'+z)(x+y'+z')(x+y+z')(x'+y+z')(x'+y+z)(x'+y'+z)(x'+y'+z')$

(ii) $(x+z)(x+z')(x'+z)$

9. (i) $(x+y+z)(x+y+z')(x+y'+z)(x+y'+z')$

(iii) $(x'+y')]$

(ii) $(x+y+z)(x+y+z')(x+y'+z)(x'+y+z)(x+y'+z')(x'+y'+z)$

(iii) $(x+y+z)(x'+y+z)(x+y+z')(x'+y'+z)(x'+y'+z')$

(iv) $(x+y+z)(x+y'+z)(x+y+z')(x'+y'+z)(x'+y+z')(x'+y'+z')$

(v) $(x+y+z)(x+y+z')(x'+y+z)(x'+y+z')(x+y'+z)(x+y'+z')(x'+y'+z)(x'+y'+z')$

10.	(i) $(x + y')$	(ii)	$(x + y + z)(x' + y' + z)(x' + y' + z')$
11.	(i) $xyz + xyz'$	(ii)	$xy + xy' + x'y$
	(iii) $xyz + xy'z + xyz' + xy'z'$	(iv)	$x'yz + x'yz' + x'y'z + x'y'z' + xyz'$
	(v) $xyz + xyz' + xy'z + xy'z' + x'yz$	(vi)	$xyz + xy'z$
	(vii) $xy'z + xy'z' + x'yz'$	(viii)	$xyz' + xy'z'$
	(ix) $xy'z + xy'z' + x'y'z + x'y'z + xyz' + x'yz' + x'y'z'$		
	(x) $x'y'z'$		

7.8 Minimization of Boolean Functions

Karnaugh map (or map method)

7.8.1 Karnaugh Maps

The Karnaugh map is a pictorial representation of truth table of the Boolean functions . This method is easy to use when Boolean function has six or fewer variables . Since function of one variable can be simplified easily. We illustrate the method when number of variables in a function is 2,3 and 4.

(I) *Case of Two Variables*: We consider the case when the Boolean function f is of two variables x and y. We have constructed a 2×2 matrix of square with each square containing one possible input combination of variable x and y.

The Karnaugh map of the function is the 2×2 matrix obtained by placing 0's and 1's in the square according to whether the functional value is 0 or 1 for the input combination associated with square. Let $f = xy + x'y$ be Boolean function is represented by the Karnaugh map in Fig. 7.1.

	x	x'
y	xy	$x'y$
y'	xy'	$x'y'$

	x	x'
y	1	1
y'	0	0

Fig. 7.1

Now consider the method to obtain minimal form of the function by using Karnaugh map. The application of the Boolean law $xy + x'y = x$, when seen in the context of a Karnaugh map, because the replacement of two adjacent squares containing 1's by a rectangle containing two squares. The absorption law $x + xy = x$ has its counterpart on a Karnaugh map as well. It is simply the group of adjacent squares into the largest possible rectangle of such squares and we will use the largest rectangle instead of individual squares. To find minimal form of the function we first consider all largest rectangles composed of the adjacent squares with 1's in them. From the set of these largest rectangles, the minimum number of rectangles are taken such that every square with 1's is part of at least one such rectangle.

Example 51: *Using the Karnaugh map method to find a minimal DN form of the following functions*

(a) $f(x, y) = xy + xy'$ (b) $f(x, y) = xy + x'y + x'y'$ (c) $f(x, y) = xy + x'y'$

Solution: (a) We first represent $f(x, y)$ by a Karnaugh map. The Karnaugh map representation of $f(x, y) = xy + xy'$ is shown in Fig. 7.2.

	x	x'
y	1	0
y'	1	0

Fig. 7.2

We have represented two adjacent squares with 1's in them by a rectangle this rectangle represents x. Hence.

$$f(x, y) = x$$

(b) We have $f(x, y) = xy + x'y + x'y'$

The Karnaugh map is represented in Fig. 7.3.

	x	x'
y	1	1
y'	0	1

Fig. 7.3

The function $f(x, y)$ contains two pairs of adjacent squares with 1 which includes all the squares of $f(x, y)$ which contain 1. Hence $f(x, y) = y + x' = x' + y$ is its minimal form.

(c) We have the function $f(x, y) = xy + x'y'$

The Karnaugh map is represented in Fig. 7.4.

Thus, $f(x, y) = xy + x'y'$ is the minimal form.

	x	x'
y	1	0
y'	0	1

Fig. 7.4

(ii) ***Case of Three Variables:*** The Karnaugh map corresponding to Boolean function $f(x, y, z)$ is as in (a) part of Example 51.

	xy	xy'	$x'y'$	$x'y$
z				
z'				

Fig. 7.5

Each square represents the minterm corresponding to the column and row intersecting in that square. In order that every pair of adjacent products in figure are geometrically adjacent the right and left edges must be identified. By a basic rectangle in Karnaugh map three variables, we mean a square, two adjacent squares or four squares which form a 1×4 of 2×2 rectangle.

Suppose that the Boolean function $f(x, y, z)$ has been represented in Karnaugh map by placing 0's and 1's in the appropriate squares.

Example 52: *Find using Karnaugh maps a minimal form for each of the following Boolean functions.*

(a) $f(x, y, z) = xyz + xyz' + x'yz' + x'y'z'$ (b) $f(x, y, z) = xyz + xyz' + xy'z + x'yz + x'y'z'$

(c) $f(x, y, z) = xyz + xyz' + x'yz' + x'y'z' + x'y'z$

Solution: (a) The Karnaugh map, corresponding to the given function is given in Fig. 7.6.

	xy	xy'	$x'y'$	$x'y$
z	1	0	1	0
z'	1	0	0	0

Fig. 7.6

From the Karnaugh map, we see that $f(x, y, z)$ has three maximal basic rectangles containing squares with 1 which are shown by rectangles. Observe the squares corresponding to xyz' and $x'yz'$ are adjacent. Thus the symbols are left open to signify that they join in one rectangle. The resulting minimal Boolean function is $xy + yz' + x'y'z$.

(b) The Karnaugh map corresponding to the function

$$f = xyz + xyz' + xy'z + x'yz + x'y'z$$

is given in Fig. 7.7 which has five squares with 1's in them corresponding to the five minterms of f.

	xy	xy'	$x'y'$	$x'y$
z	1	1	1	1
z'	1	0	0	0

Fig. 7.7

From the Karnaugh map, we see that $f(x, y, z)$ has two maximal basic rectangles containing all squares with 1, which are shown by rectangles. One of the maximal rectangle is the two adjacent squares which represents xy and the other is the 1×4 square which represents z. Both are needed to cover all the squares with 1. So the minimal form of $f(x, y, z)$ is given by

$$f(x, y, z) = xy + z$$

(c) The Karnaugh map corresponding to the function $f(x, y, z) = xyz + xyz' + x'yz' + x'y'z' + x'y'z$ is given by

	xy	xy'	$x'y'$	$x'y$
z	1	0	1	0
z'	1	0	1	1

Fig. 7.8

Thus $f(x, y, z)$ has two minimal forms.

$$f(x, y, z) = xy + x'y' + x'z' \quad \text{and} \quad f(x.y.z) = xy + x'y' + yz'$$

(iii) Case of Four Variables: The Karnaugh map corresponding to Boolean function $f(x, y, z, w)$ with four variables x, y, z and w. Each of the 16 squares corresponding to one of the 16 minterms with four variables $xyzw$, $xyzw'$...

as indicated by the labels of the row and column of the square.

	xy	xy'	$x'y'$	$x'y$
zw				
zw'				
$z'w'$				
$z'w$				

Fig. 7.9

Now we consider the first and last column to be adjacent and the first and last rows to be adjacent. both by wrap around. A basic rectangle in a four variable Karnaugh map is a square, two adjacent squares, four squares which form 1×4 or 2×2 rectangle or eight squares which form a 2×4 rectangle. The minimization method is same as three variables functions.

Example 53: *Use Karnaugh maps to find a minimal form for the following Boolean functions.*

(a) $f(x, y, z, w) = x'yzw + xy'zw' + x'y'zw' + xyzw' + xy'z'w'$

(b) $f(x, y, z, w) = xy' + xyz + x'y'z' + x'yzw'$

Solution: (a) The Karnaugh map corresponding (a) is as

	xy	xy'	$x'y'$	$x'y$
zw	0	0	0	1
zw'	0	1	1	0
$z'w'$	1	1	0	0
$z'w$	0	0	0	0

Fig. 7.10

This has five squares with 1's in them corresponding to the five min-terms of f. A minimal cover of all 1's of the map consists of the three maximal rectangles as $f(x, y, z, w) = y'zw' + xz'w' + x'y\,zw$.

(b) The Karnaugh map corresponding to given function (See Fig. 7.11).

The minimum number of maximal basic rectangles to cover all 1's of the map is 3. Thus the minimal form is $f(x, y, z) = xz + y'z' + yzw'$

observe that the upper left 2×2 rectangle represents xz while the other 2×2 rectangle represents $y'z'$.

	xy	xy'	$x'y'$	$x'y$
zw	1	1		
zw'	1	1		1
$z'w'$		1	1	
$z'w$		1	1	

Fig. 7.11

Example 54: *Use a Karnaugh map to find a minimal form of the function*

$$f(x, y, z, w) = xyzw + xyzw' + xy'zw' + x'y'zw + x'y'zw'$$

Solution: The Karnaugh map corresponding to given function (See Fig. 7.12)

Hence we obtain two minimal forms,

$$f(x, y, z, w) = xyz + x'y'z + xzw'$$

and

$$f(x, y, z, w) = xyz + x'y'z + y'zw'$$

	xy	xy'	$x'y'$	$x'y$
zw	1	0	1	0
zw'	1	1	1	0
$z'w'$	0	0	0	0
$z'w$	0	0	0	0

Fig. 7.12

Example 55: *Simplify the following Boolean expressions by using map method and show circuit diagrams for original expression and reduced expression:*

(a) $x'y + xy' + xy$ (b) $x'yz + x'yz' + x'y'z + xy'z$

Solution: (a) Let $F = x'y + xy' + xy'$

Using K map this equation can be reduced as shown in Fig. 7.13. ...(1)

This K map contain two pairs of 1's that are horizontally and vertically adjacent and each pair is reducing a variable.

Thus, the reduce expression is $f = x + y$

$x \backslash y$	0	1
0	0	1
1	1	1

Fig. 7.13

(b) $F = x'yz + x'yz + x'y'z + xy'z$

Using K map method this equation can be reduced as shown in Fig. 7.14.

$x \backslash y'z$	00	01	11	10
0	0	1	1	1
1	0	1	0	0

Fig. 7.14

This K map contain three pairs of 1's the horizontally and vertically adjacent and each pair is reducing a variable. Thus the reduce expression is $F = x'z + x'y + y'z$

7.8.2 Representation of Logical Expressions Using Minterms and Maxterms

We can represent the logical expression using the minterm and maxterms as follows:

(i) $Y = \underbrace{A\,B\,C}_{m_7} + \underbrace{\bar{A}\,B\,C}_{m_3} + \underbrace{A\,\bar{B}\,\bar{C}}_{m_4}$ = given logic expression
← corresponding minterms

$\therefore \quad Y = m_7 + m_3 + m_4 \quad = \Sigma\, m\,(3, 4, 7)$

This is other way of representation, where Σ denotes sum of products

(ii) $Y = \underbrace{(A + \bar{B} + C)}_{M_2} . \underbrace{(A + B + C)}_{M_0} \underbrace{(\bar{A} + \bar{B} + C)}_{M_6}$ = given logic expression
← corresponding minterms

$\therefore \quad Y = M_2\, M_0\, M_6$

$\therefore \quad Y = \pi\, M\ (0, 2, 6)$

This is other way of representation, where π denotes product of sums.

7.8.3 Fundamental Product

A literal or product of two or more literals in which no two literals involve the same variable is called **a fundamental product.**

For example $x_1, x_2', x_3', x\,y'$ are fundamental products.

Example 56: *Write the following Boolean expressions in an equivalent sum of canonical form in three variables x_1, x_2 and x_3.*

(i) $x_1 * x_2$ or $x_1 . x_2$ **[U.P.T.U. (B.Tech.) 2006, 2007]**

(ii) $x_1\,(x_2'\,x_3)'$ **[PTU (B.E.) Punjab 2005, 2008]**

(iii) $x_1 + x_2$ or $x_1 \oplus x_2$ **[U.P.T.U. (B.Tech.) 2006, 2007]**

Solution: (i) $x_1 \cdot x_2 = x_1 \cdot x_2 \cdot 1$

$= x_1 \cdot x_2 \cdot (x_3 + x_3')$ $\qquad [x_3 + x_3' = 1]$

$= x_1\, x_2\, x_3 + x_1\, x_2\, x_3'$

$= M_6 + M_7$

$\therefore \quad x_1 \cdot x_2 = M_6 + M_7$

(ii) $x_1\,(x_2'\, x_3)' = x_1\,[(x_2')' + x_3']$

$= x_1\,[x_2 + x_3']$ $\qquad [(x_2')' = x_2]$

$= x_1\, x_2 + x_1\, x_3'$

$= x_1\, x_2 \cdot 1 + x_1\, x_3' \cdot 1$

$= x_1\, x_2\,(x_3 + x_3') + x_1\, x_3'\,(x_2 + x_2')$

$= x_1\, x_2\, x_3 + x_1\, x_2\, x_3' + x_1\, x_2\, x_3' + x_1\, x_2'\, x_3'$

$= x_1\, x_2\, x_3 + x_1\, x_2\, x_3' + x_1\, x_2'\, x_3'$ $\qquad [x_1\, x_2\, x_3' + x_1\, x_2\, x_3' = x_1\, x_2\, x_3']$

$= m_7 + m_6 + m_4$

$= m_4 + m_6 + m_7$

$= \Sigma\, 4, 6, 7 \leq M\,(4, 6, 7)$

$x_1 + x_2 = (x_1 + x_2) \cdot 1$

$= (x_1 + x_2) \cdot (x_3 + x_3')$ $\qquad [x_3 + x_3' = 1]$

$= x_1\, x_3 + x_1\, x_3' + x_2\, x_3 + x_2\, x_3'$

$= (x_1\, x_3 + x_1\, x_3') \cdot 1 + (x_2\, x_3 + x_2\, x_3') \cdot 1$

$= (x_1\, x_3 + x_1\, x_3') \cdot (x_2 + x_2') + (x_2\, x_3 + x_2\, x_3') \cdot (x_1 + x_1')$

$= x_1\, x_3\, x_2 + x_1\, x_3\, x_2' + x_1\, x_3'\, x_2 + x_1\, x_3'\, x_2' + x_2\, x_3\, x_1 + x_2\, x_3\, x_1' + x_2\, x_3'\, x_1 + x_2\, x_3'\, x_1'$

$= x_1\, x_2\, x_3 + x_1\, x_2'\, x_3 + x_1\, x_2\, x_3' + x_1\, x_2'\, x_3' + x_1'\, x_2\, x_3 + x_1'\, x_2\, x_3'$

Example 57: *Write the following Boolean function in m-notation*

(i) $f\,(x_1, x_2, x_3) = x_1'\, x_2\, x_3 + x_1\, x_2'\, x_3 + x_1\, x_2\, x_3' + x_1\, x_2\, x_3 + x_1\, x_2'\, x_3'$ **[M.K.U. (B.E.) 2005, 2007]**

(ii) $f\,(x_1, x_2, x_3, x_4) = x_1'\,(x_2' + x_4) + x_1\, x_3\, x_4'$ **[Osmania (B.E.) Andhra 2008]**

(iii) $f\,(x_1, x_2, x_3) = (x_1 + x_2 + x_3)(x_1 + x_2 + x_3')(x_1 + x_2'\, x_3)$ **[U.P.T.U. (B.Tech.) 2004]**

Solution: (i) $f = x_1'\, x_2\, x_3 + x_1\, x_2'\, x_3 + x_1\, x_2\, x_3' + x_1\, x_2\, x_3 + x_1\, x_2'\, x_3'.$

$= \text{min}_3 + \text{min}_5 + \text{min}_6 + \text{min}_7 + \text{min}_4$

$= \text{min}_3 + \text{min}_4 + \text{min}_5 + \text{min}_6 + \text{min}_7$

$= \Sigma\, m\,(3, 4, 5, 6, 7)$

(ii) $f = x_1'\,(x_2' + x_4) + x_1\, x_3\, x_4'$

$= x_1'\, x_2' + x_1'\, x_1'\, x_4 + x_1\, x_3\, x_4'$

$= x_1\, x_2'\,(x_3 + x_3')(x_4 + x_4') + x_1\,(x_2\, x_2')(x_3 + x_3')\, x_4 + x_1\,(x_2 + x_2')\, x_3\, x_4'$

$= x_1'\, x_2'\, x_3\, x_4 + x_1'\, x_2'\, x_3'\, x_4 + x_1'\, x_2'\, x_3\, x_4' + x_1'\, x_2'\, x_3'\, x_4' + x_1\, x_2\, x_3\, x_4' + x_1\, x_2'\, x_3\, x_4$

$= m_0 + m_1 + m_3 + m_5 + m_7 + m_{10} + m_{14}$

$= \Sigma\, m\,(0, 1, 3, 5, 7, 10, 14)$

(iii) $f = (x_1 + x_2 + x_3)(x_1 + x_2 + x_3')(x_1 + x_2'\, x_3)$

$= \max\,(x_0)\ \max\,(x_1)\ \max\,(x_2)$

$= \prod M\,(0, 1, 2)$

7.8.4 Alternate Way of Representation

The alternate way of representing the sum of products expression is explain with the help of example.

Let us consider the Boolean function

$$X = A'\,B'\,C' + A\,B\,C' + A'\,B\,C$$

This function can be represent as

$$X\,(A, B, C) = \Sigma\,(0, 6, 3) \text{ or } \Sigma\, m\,(0, 6, 3)$$

where m = minterms

The Minterms Corresponding to square 0, 6, 3 of the Karnaugh map are present in the Boolean function and marked by 1. Other squares are marked by 0. It is explain as

C \ AB	00	01	11	10
0	1	0	1	0
1	0	1	0	0

Fig. 7.15

7.8.5 The Use of Karnaugh Map in Boolean Algebra

[R.G.P.V. (B.E.) Bhopal 2005, 2008; Rohtak (B.E.) 2009]

This widely used method of representation and simplification of Boolean functions is popular among electrical engineers and is suitable for hand computations and is most efficient for function involving upto 4-5 variables. A Karnaugh map is a plane area subdivided into 2^n equal cells each representing a point B_2^n for function of n variables. The cells corresponding to the arguments for which the function has the value 1 are shaded or darkened or a, 1 is written on it.

Example 58: *Use Karnaugh map representation to find minimal sum of products expression for the following Boolean function:*

(i) $F\,(A, B, C, D) = \Sigma\,(0, 2, 7, 8, 10, 15)$ **[U.P.T.U. (B.Tech.) 2009]**

(ii) $F\,(A, B, C, D) = \Sigma\,(0, 1, 2, 3, 4, 5, 7, 6, 8, 9, 11)$ **[U.P.T.U. (B.Tech.) 2003]**

(iii) $F\,(A, B, C) = \Sigma\,(0, 1, 2, 4, 5, 6)$ **[U.P.T.U. (B.Tech.) 2003]**

(iv) $F\,(A, B, C) = \Sigma\,(0, 2, 4)$ **[R.G.P.V. (B.E.) Bhopal 2009]**

(v) $F\,(A, B, C) = \Sigma\,(0, 2, 3, 4, 7)$ **[R.G.P.V. (B.E.) Raipur 2008]**

(vi) $F\,(A, B, C) = \Sigma\,(0, 4, 7)$ **[Rohtak (B.E.) 2007]**

(vii) $F\,(A, B, C) = \Sigma\,(1, 2, 5, 6)$ **[I.G.N.O.U. 2007]**

(viii) $F(A, B, C) = \Sigma\,(1, 9, 11, 13, 15)$ [I.G.N.O.U. (M.C.A.) 2008]

(ix) $F(A, B, C, D) = \Sigma\,(0, 1, 2, 3, 5, 7, 8, 10, 14)$ [R.G.P.V. (B.E.) Bhopal 2006]

(x) $F(A, B, C, D) = \Sigma\,(0, 1, 3, 5, 7, 9, 11, 14, 15)$ [I.G.N.O.U. 2009]

(xi) $F(A, B, C, D) = \Sigma\,(0, 2, 4, 6)$ R.G.P.V. (B.E.) Raipur 2004, 2007, 2009]

(xii) $F(A, B, C, D) = \Sigma\,(0, 2, 4, 5, 6, 7, 8, 10, 13, 15)$ [Osmania (B.E.) 2004, 2006, 2007]

(xiii) $F(A, B, C, D) = \Sigma\,(0, 2, 4, 5, 6, 8, 9, 10, 13)$ [U.P.T.U. (M.C.A.) 2005]

(xiv) $F(A, B, C, D) = \Sigma\,(0, 2, 8, 9, 10, 11, 14, 15)$ [Kurukshetra (B.E.) 2004, 2008]

(xv) $F(A, B, C, D) = \Sigma\,(5, 6, 7, 13, 14, 15)$ [Rohtak (B.E.) 2007, 2009]

(xvi) $F(A, B, C, D) = \Sigma\,(4, 5, 6, 7, 13, 15)$ [U.P.T.U. 2004, 2008]

Solution: (i) $F(A, B, C, D) = \Sigma\,(0, 2, 7, 8, 10, 15)$

The minterms function F are

$$M_0 = 0000 = \bar{A}\,\bar{B}\,\bar{C}\,\bar{D}$$

$$M_2 = 0010 = \bar{A}\,\bar{B}\,C\,\bar{D}$$

$$M_7 = 0111 = \bar{A}\,B\,C\,D$$

$$M_8 = 1000 = A\,\bar{B}\,\bar{C}\,\bar{D}$$

$$M_{10} = 1010 = A\,\bar{B}\,C\,\bar{D}$$

$$M_{15} = 1111 = ABCD$$

Karnaugh map of the given function is shown in fig. 7.16

AB \ CD	$\bar{C}\,\bar{D}$ [0 0]	$\bar{C}\,D$ [0 1]	CD [11]	$C\bar{D}$ [10]
[0 0] $\bar{A}\,\bar{B}$	1 0	1	3	1 2
[0 1] $\bar{A}\,B$	4	5	1 7	6
[11] $A\,B$	12	13	1 15	14
[10] $A\,\bar{B}$	1 8	9	11	1 10

Fig. 7.16

The Karnaugh map has one pair, and one quad. There are no over lapping.

Consider the pair $m_7 + m_{15}$. A is the only variable which changes its form, so A is to be removed

The reduced expression for the pair $m_7 + m_{15}$ is BCD quad is $m_0 + m_3 + m_8 + m_{10}$ in the map moving horizontally we observe that the variable C change its form and then moving vertically. We get A changes its form. Therefore A and C are removed. The reduced expression for quad

$$m_0 + m_3 + m_8 + m_{10} \text{ is } \bar{B}\,\bar{D}$$

Then $$F(A\,B\,C\,D) = B\,C\,D + \bar{B}\,\bar{D}$$

(ii) $F(A, B, C, D) = \Sigma\,(0, 1, 2, 3, 4, 5, 7, 6, 8, 9, 11)$

The looping of one octet and two quad as shown in map. The looping octet will give the minterm as A'. The looping quad formed by 0, 1, 8, 9 produces the minterm as $B'\,C'$ and the quad formed by 1, 3, 9, 11 gives the minterm as $B'\,D$. Hence simplified function is

$$F = A' + B\,C' + B'\,D$$

Karnaugh map of the given function is shown in fig. 7.17

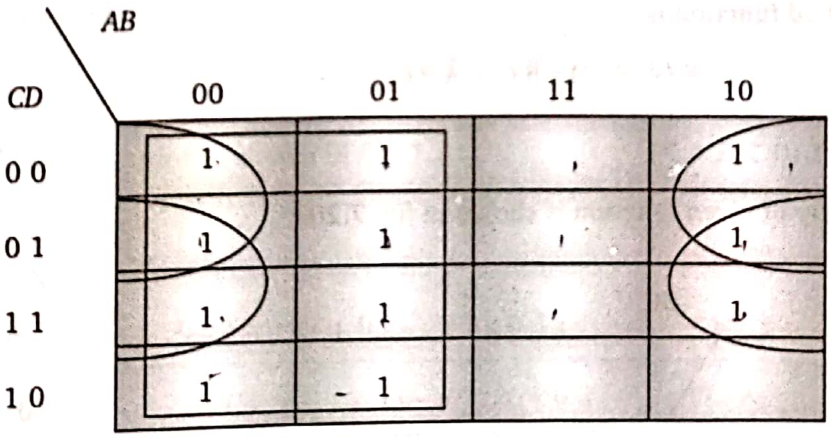

Fig. 7.17

(iii) $F(A, B, C) = \Sigma\,(0, 1, 2, 4, 5, 6)$

Karnaugh map of given function is shown in fig. 7.18

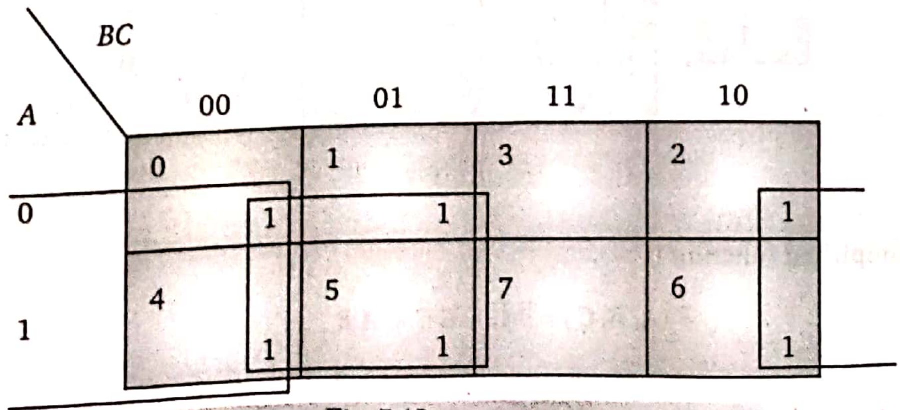

Fig. 7.18

Hence, the simplifying function is $F(A, B, C) = \bar{B} + \bar{C}$

(iv) $F(A, B, C) = \Sigma(0, 2, 4)$

Karnaugh map of given function is shown in fig. 7.19

$A \backslash BC$	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$	1	1 1	3	2 $\boxed{1}$
A	1	5 1	7	6

Fig. 7.19

Hence, the simplified function is

$$F(A, B, C) = \bar{B}\bar{C} + \bar{A}B\bar{C}$$

(v) $F(A, B, C) = \Sigma(0, 2, 3, 4, 7)$

Karnaugh map of given function is shown in fig. 7.20

$A \backslash BC$	00	01	11	10
0	0 1	1	3 1	2 1
1	4 1	5 7	1	6

Fig. 7.20

Hence, the simplified function is

$$F(A, B, C) = \bar{B}\bar{C} + BC + \bar{A}B$$

(vi) $F(A, B, C) = \Sigma(0, 4, 7)$

Karnaugh map of given function is shown in fig. 7.21

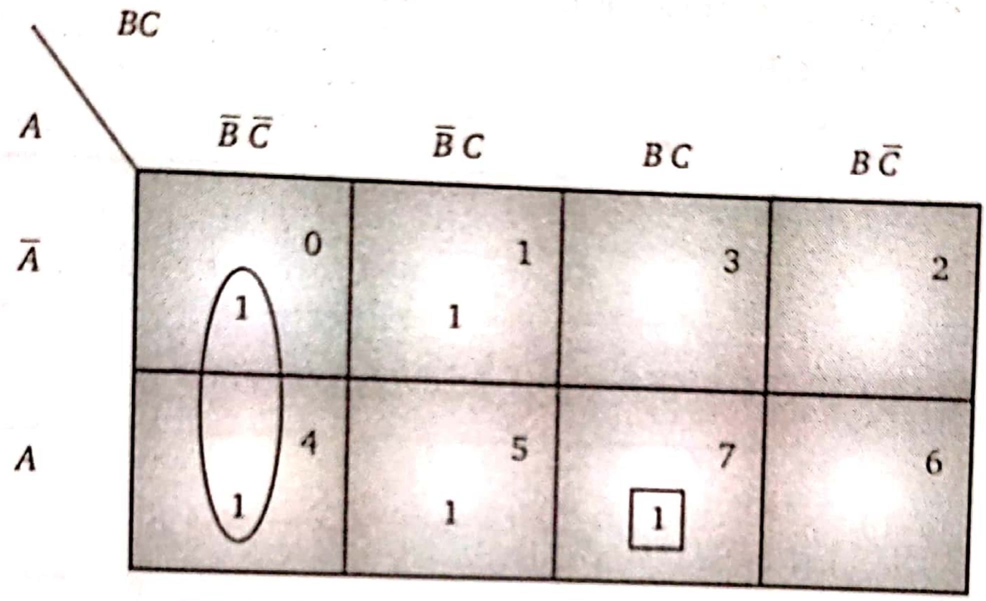

Fig. 7.21

Hence, the simplified function is

$$F(A, B, C) = ABC + \bar{B}\bar{C}$$

(vii) $F(A, B, C) = \Sigma(1, 2, 5, 6)$

Karnaugh map of given function is shown in fig. 7.22

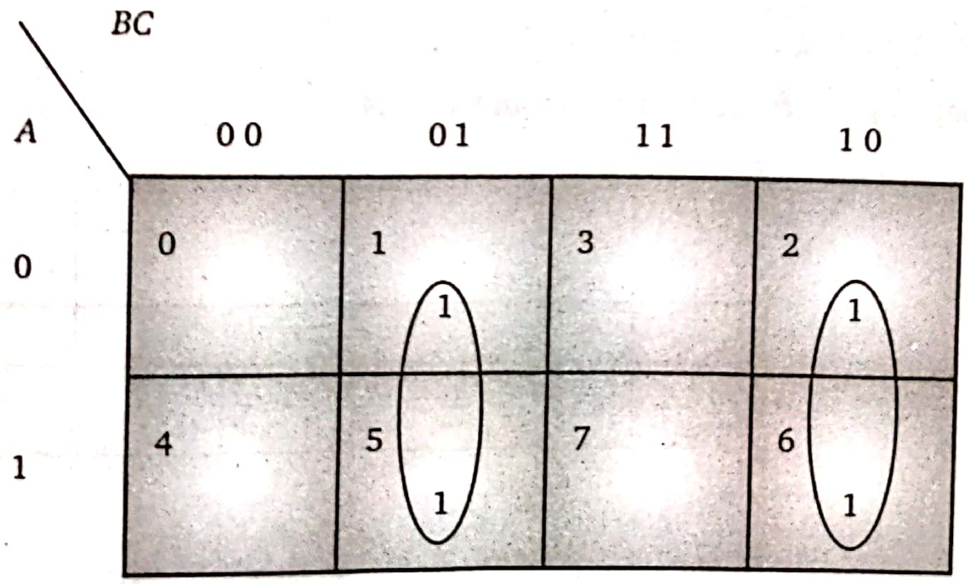

Fig. 7.22

Hence, the simplified function is

$$F(A, B, C) = \bar{B}C + B\bar{C}$$

(viii) $F(A, B, C, D) = \Sigma(1, 9, 11, 13, 15)$

Karnaugh map of given function is shown in fig. 7.23

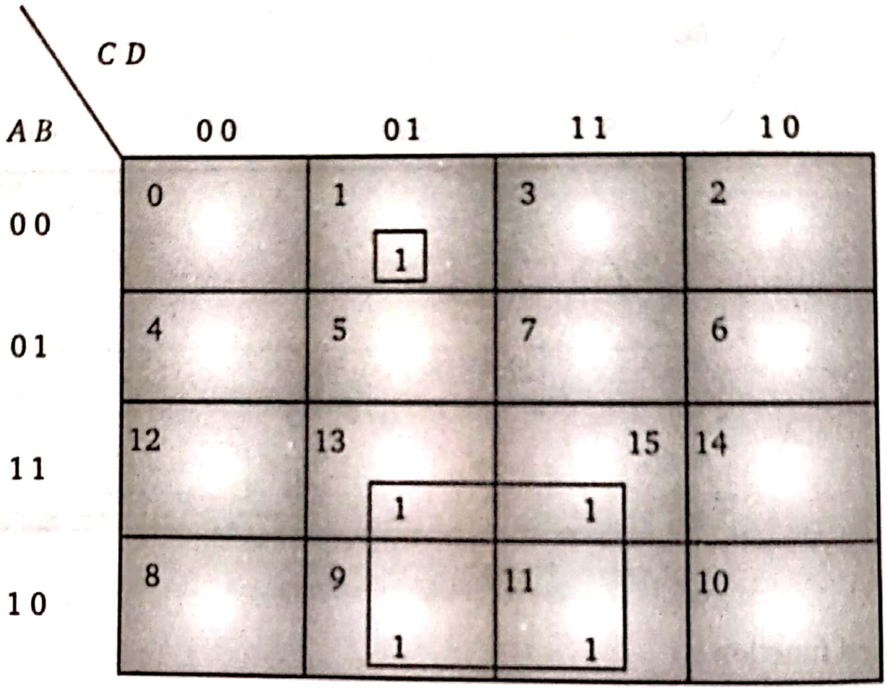

Fig. 7.23

Hence, the simplified function is

$$F(A, B, C, D) = A\,D + \overline{B}\,\overline{C}\,D$$

(ix) $F(A, B, C, D) = \Sigma(0, 1, 2, 3, 5, 7, 8, 10, 14)$

Karnaugh map of given function is shown in fig. 7.24

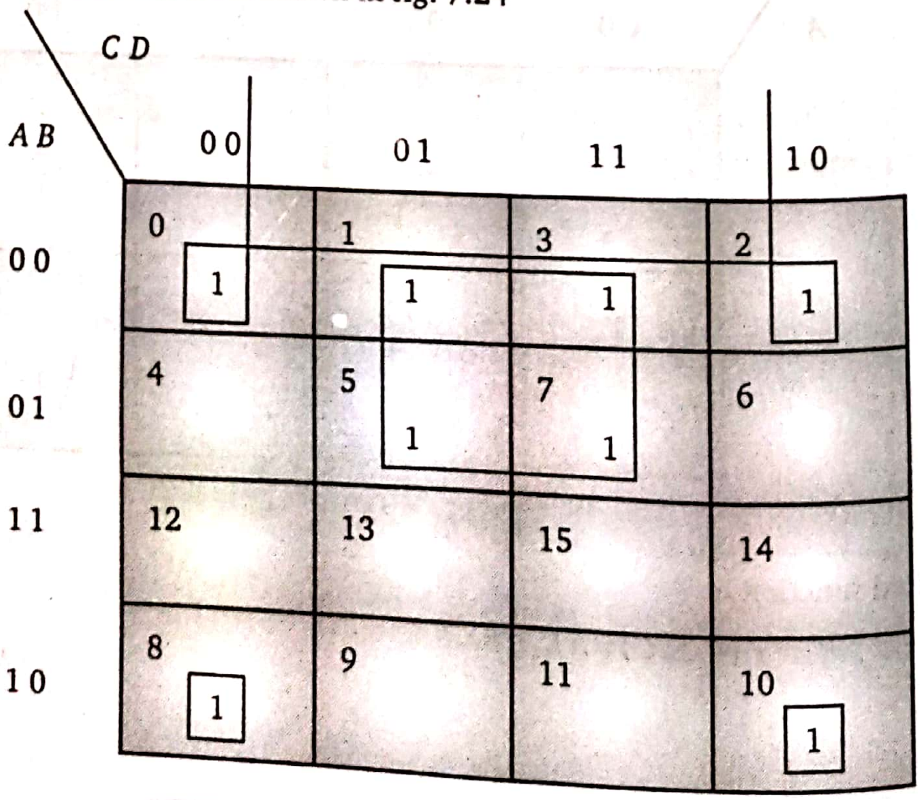

Fig. 7.24

Hence, the simplified function is

$$F(A, B, C, D) = \bar{A}\,\bar{D} + \bar{A}\,\bar{B} + \bar{B}\,\bar{D} + A\,C\,\bar{D}$$

(x) $F(A, B, C, D) = \Sigma\,(0, 1, 3, 5, 7, 9, 11, 14, 15)$

Karnaugh map of given function is shown in fig. 7.25

AB \ CD	00	01	11	10
00	0: 1	1: 1	3: 1	2:
01	4:	5: 1	7: 1	6:
11	12:	13:	15: 1	14: 1
10	8:	9: 1	11: 1	10:

Fig. 7.25

Hence simplifying function is

$$F(A, B, C, D) = \bar{A}\,D + C\,D + A\,B\,C + A\,\bar{B}\,D + \bar{A}\,\bar{B}\,\bar{C}$$

(xi) $F(A, B, C, D) = \Sigma\,(0, 2, 4, 6)$

Karnaugh map of given function is shown in fig. 7.26

AB \ CD	$\bar{C}\,\bar{D}$	$\bar{C}\,D$	$C\,D$	$C\,\bar{D}$
$\bar{A}\,\bar{B}$	0: 1	1: 0	3: 0	2: 1
$\bar{A}\,B$	4: 1	5: 0	7: 0	6: 1
$A\,B$	12: 0	13: 0	15: 0	14: 0
$A\,\bar{B}$	8: 0	9: 0	11: 0	10: 0

Fig. 7.26

Hence, simplified function is

$$F(A, B, C, D) = \bar{A}\,\bar{D}$$

(xii) $F(A, B, C, D) = \Sigma(0, 2, 4, 5, 6, 7, 8, 10, 13, 15)$

Karnaugh map of the given function is shown in fig. 7.27

AB \ CD	00	01	11	10
00	1			1
01	1	1	1	1
11		1	1	
10	1			

Fig. 7.27

Hence, simplified function is

$$F(A, B, C, D) = \bar{B}\,\bar{D} + \bar{A}\,B + B\,D$$

(xiii) $F(A, B, C, D) = \Sigma(0, 2, 4, 5, 6, 8, 9, 10, 13)$

Karnaugh map of the given function is shown in fig. 7.28

AB \ CD	00	01	11	10
00	1			1
01	1	1		1
11		1		
10	1	1		1

Fig. 7.28

Hence, simplified function $= F(A, B, C, D) = \bar{A}\,\bar{B} + B\,\bar{D} + B\,\bar{C}\,D + A\,\bar{C}\,D$

(xiv) $F(A, B, C, D) = \Sigma\,(0, 2, 8, 9, 10, 11, 14, 15)$

Karnaugh map of the given function is shown in fig. 7.29

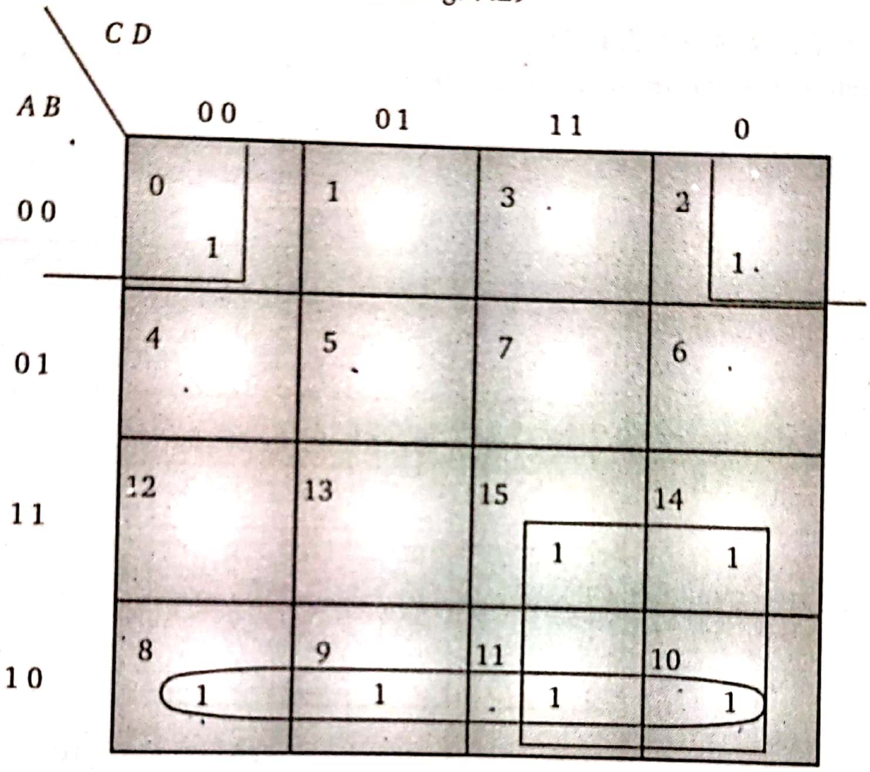

Fig. 7.29

Hence, the simplified function is

$$F = \overline{B}\,\overline{D} + A\overline{B} + AC$$

(xv) $F(A, B, C, D) = \Sigma\,(5, 6, 7, 13, 14, 15)$

Karnaugh map of the given function is shown in fig. 7.30

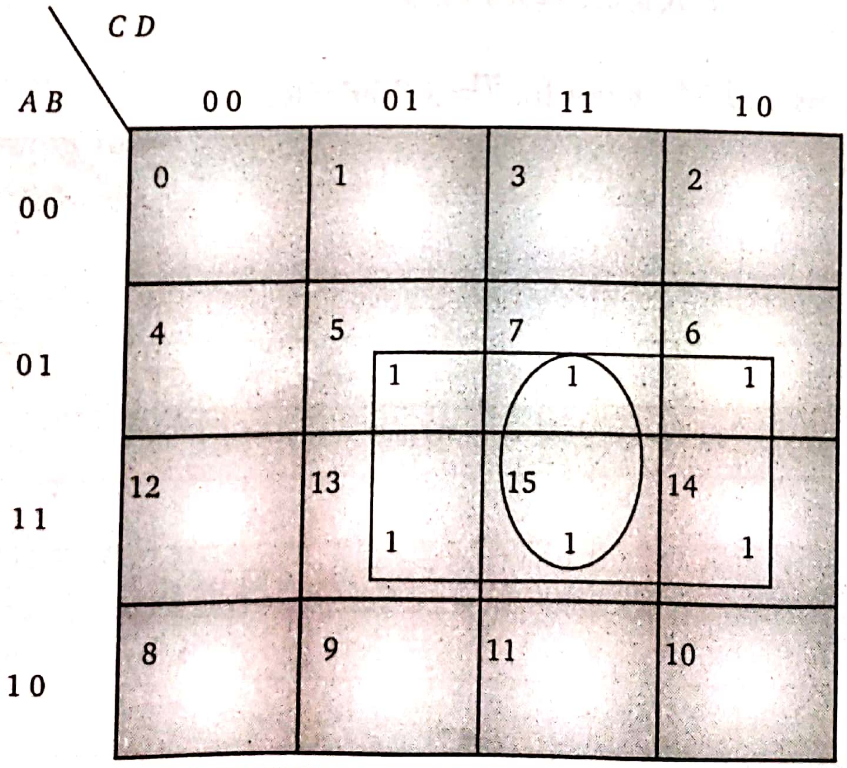

Fig. 7.30

Hence, the simplified function is

$$F = B\,D + B\,C$$

(xvi) $F(A, B, C, D) = \Sigma(4, 5, 6, 7, 13, 15)$

Karnaugh map of the given function is shown in fig. 7.31

$AB \backslash CD$	$\bar{C}\,\bar{D}$	$\bar{C}\,D$	$C\,D$	$C\,\bar{D}$
$\bar{A}\,\bar{B}$	0: 0	1: 0	3: 0	2: 0
$\bar{A}\,B$	4: 1	5: 1	7: 1	6: 1
$A\,B$	12: 0	13: 1	15: 1	14: 0
$A\,\bar{B}$	8: 0	9: 0	11: 0	10: 0

Fig. 7.31

Hence, simplified function is

$$F(A, B, C, D) = \bar{A}\,\bar{B} + B\,D$$

7.8.6 Minterms and Maxterms for Three Variables

Variables			Minterms	Maxterms
A	B	C	m_i	M_i
0	0	0	$\bar{A}\,\bar{B}\,\bar{C} = m_0$	$A + B + C = M_0$
0	0	1	$\bar{A}\,\bar{B}\,C = m_1$	$A + B + \bar{C} = M_1$
0	1	0	$\bar{A}\,B\,\bar{C} = m_2$	$A + \bar{B} + C = M_2$
0	1	1	$\bar{A}\,B\,C = m_3$	$A + \bar{B} + \bar{C} = M_3$
1	0	0	$A\,\bar{B}\,\bar{C} = m_4$	$\bar{A} + B + C = M_4$
1	0	1	$A\,\bar{B}\,C = m_5$	$\bar{A} + B + \bar{C} = M_5$
1	1	0	$A\,B\,\bar{C} = m_6$	$\bar{A} + \bar{B} + C = M_6$
1	1	1	$A\,B\,C = m_7$	$\bar{A} + \bar{B} + \bar{C} = M_7$

Each minterm is represented by m_i, where $i = 1, 2, 3 \ldots 2^n - 1$ and each maxterm is represented by

M_i, where $i = 1, 2, 3 \ldots 2^n - 1$

Minterm $= \bar{A}\,B\,C$ corresponding to $A\,B\,C = 011$

Maxterm $= (A + \bar{B} + \bar{C})$ corresponding to $ABC = 011$

Minterms and Maxterms for two Variables

Variables/literals		Minterms (m_i)	Maxterms M_i
A	B	m_1	M_1
0	0	$\bar{A}\,\bar{B} = m_0$	$A + B = M_0$
0	1	$\bar{A}\,B = m_1$	$A + \bar{B} = m_1$
1	0	$A\,\bar{B} = m_2$	$\bar{A} + B = M_2$
1	1	$A\,B = m_3$	$\bar{A} + \bar{B} = M_3$

Example 59: *$f = \Sigma\, m\,(0, 1, 2, 5)$ is a three input function, transform into its canonical sum of products form.*

Solution: Let x_1, x_2 and x_3 denote the three inputs

Then,
$$m_0 = 000 = x_1'\,x_2'\,x_3'$$
$$m_1 = 001 = x_1'\,x_2'\,x_3$$
$$m_2 = 010 = x_1'\,x_2\,x_3'$$
$$m_3 = 101 = x_1\,x_2'\,x_3$$

$\therefore$ Canonical sum of products form of the expression is

$$f = x_1'\,x_2'\,x_3' + x_1'\,x_2'\,x_3 + x_1'\,x_2\,x_3' + x_1\,x_2'\,x_3$$

Example 60: *Obtain the three variable product of sums canonical form of the Boolean expression*

$$x_1 \cdot x_2 \text{ or } x_1 * x_2$$

[U.P.T.U. (B.Tech.) 2006]

Solution: Let x_3 denote the variable then

$$x_1 \cdot x_2 = (x_1 + x_2\,x_2')(x_2 + x_1\,x_1')$$
$$= (x_1 + x_2)(x_1 + x_2')(x_2 + x_1)(x_2 + x_1')$$
$$= (x_1 + x_2)(x_1 + x_2')(x_2 + x_1') \qquad [(x_1 + x_2)(x_1 + x_2) = (x_1 + x_2)]$$
$$= (x_1 + x_2 + x_3\,x_3')(x_1 + x_2' + x_3\,x_3')(x_1' + x_2 + x_3\,x_3')$$
$$= (x_1 + x_2 + x_3)(x_1 + x_2 + x_3')(x_1 + x_2' + x_3)(x_1 + x_2' + x_3')\;(x_1' + x_2 + x_3)(x_1' + x_2 + x_3')$$

Example 61: *Draw a Karnaugh map for the table*

A	B	C	Z
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

[U.P.T.U. (B.Tech.) 2004]

Solution: First we draw the blank map of fig. Output 1 appears ABC inputs of 000, 001, 010 and 110. The fundamental products for these conditions are $\bar{A}\,B\,\bar{C}$, $\bar{A}\,\bar{B}\,C$, $\bar{A}\,B\,C$ and $A\,B\,\bar{C}$. Enter 1^s for these products on the Karnaugh map Fig. Finally enter 0^s in the remaining spaces shown in fig. 7.32

C \ AB	00	01	11	10
0				
1				

(a)

C \ AB	00	01	11	10
0	1	1	1	
1	1			

(b)

C \ AB	00	01	11	10
0	1	1	1	0
1	1	0	0	0

(c)

This is the Karnaugh map for Given Table

Fig. 7.32

Example 62: *Use the Karnaugh map method to find a minimal DN form (sum-of-products form) of the following functions.*

(i) $Y = AB + AB'$

(ii) $Y = AB + A'B + A'B'$ **[P.T.U. (Punjab) 2009]**

(iii) $Y = AB + A'B'$ **[R.G.P.V. (B.E.) (Raipur) 2008]**

[R.G.P.V. (B.E.) Bhopal 2009]

Solution: The representation of $Y = AB + AB'$ by Karnaugh map is as follows

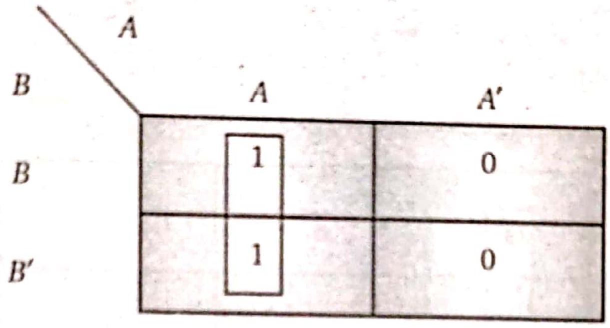

Fig. 7.33

We have represented two adjacent squares with 1s in them by rectangle. This rectangle represents A.

Hence, $$Y = A$$

(ii) The representation of $Y = AB + A'B + A'B'$ by Karnaugh map is as follows

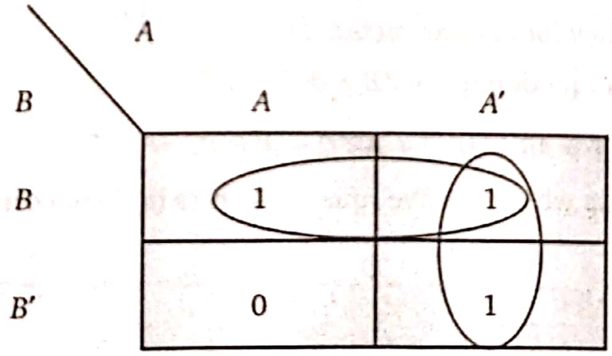

Fig. 7.3

The function Y contains two pairs of adjacent squares with 1 which includes all the squares of Y which contain 1. The horizontal pair (rectangle) represents Y with vertical pair (rectangle) represents A.

Hence, $Y = B + A' = A' + B$ is its minimal form.

(iii) The representation of $Y = AB + A'B'$ by Karnaugh map is as follows.

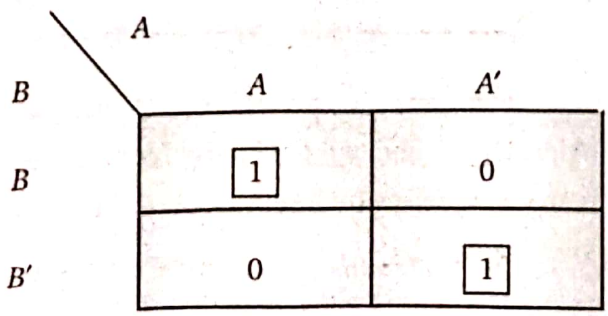

Fig 7.35

The function Y consists of two rectangles as shown in the figure.

Hence $Y = AB + A'B'$ is the minimal form.

Example 63: *Find using Karnaugh maps a minimal form of each of the following Boolean functions*

(i) $Y = ABC + ABC' + A'BC' + A'B'C'$ R.G.P.V. (Bhopal) 2006, 2009]

(ii) $Y = ABC + ABC' + AB'C + A'B'C$ [U.P.P.V. (B.Tech.) 2004]

(iii) $Y = ABC + ABC' + A'BC' + A'B'C' + A'B'C$ [Kurukshetra (B.E.) 2005, 2008]

Solution: (i) The representation of $Y = ABC + ABC' + A'BC' + A'B'C'$ by Karnaugh map is as follows

C \ AB	AB	AB'	$A'B'$	$A'B$
C	1	0	1	0
C'	1	0	0	1

Fig. 7.36

We see from Karnaugh map that Y has three maximal basic rectangles containing squares with 1 which are shown by rectangles. The squares corresponding to ABC' and $A'BC'$ are adjacent. Thus the symbols are left open ended to signify that they join in one rectangle.

Hence, the minimal Boolean function is $Y = AB + BC' + A'B'C$

(ii) The representation of $Y = AB + ABC' + AB'C + A'B'C$ by Karnaugh map is as follows which has five squares with 1s in them corresponding to the five minterms of Y

C \ AB	AB	AB'	$A'B'$	$A'B$
C	1	1	1	1
C'	1	0	0	0

Fig. 7.37

From the Karnaugh map we find that Y has two maximal basic rectangles containing all the squares with 1s, which are shown by rectangles. One of the maximal baisc rectangle in the two adjacent squares which represents AB and the other is (1×4) square which represents C. Both are needed to cover all the squares with 1. So, the minimal form of Y is given by

$$A = AB + C$$

(iii) The representation of $Y = ABC + ABC' + A'BC' + A'B'C' + A'B'C$ by Karnaugh map is as follows

C \ AB	AB	AB'	$A'B'$	$A'B$
C	1	0	1	0
C'	1	0	1	1

Fig. 7.38

which has five squares with 1's' in them corresponding to the five minterms of Y. It has four maximal basic rectangles to cover all squares with 1's' in them. Hence we include basic rectangles which represent AB and $A'B'$ and only one of two rectangles which correspond to $A'C'$ and BC'. Thus Y has two minimal forms

$$Y = AB + A'B' + A'C' \qquad \text{and} \qquad Y = AB + A'B' + BC'$$

Example 64: *Draw a Karnaugh map to represent the following Boolean function*

$$Y = A'B'C'D + ACD + BD' + AB + BC$$

Solution:

AB \ CD	$C'D'$ 00	$C'D$ 01	CD 11	CD' 10
$A'B'$ 00	0	1 1	3	2
$A'B$ 01	1 4	5	1 7	1 6
AB 11	1 12	1 13	1 16	1 16
AB' 10	8	9	1 11	10

Fig. 7.39

The first term represents a four variable term, from which it is clear that its minterm will be in square 1. It has therefore been assigned to square 1. The term ACD represents a pair. Square 11 and 15 satisfy the requirement of its position. There are three quartets, each of which yield a product of two variables. The term BD' points, for which the proper squares are 12, 13, 14 and 15. The last quartet yields the product BC. Its proper position is in squares 6, 7, 14 and 15.

Example 65: *Use a Karnaugh map to find a minimal sum for*

$$E = y't' + y'z't + x'y'zt + yzt'$$

[U.P.T.U. (B.Tech.) 2005; R.G..P.V. (B.E.) Bhopal 2007; U.P.T.U. (M.C.A.) 2008]

Solution: Draw a Karnaugh map to represent the following Boolean function

$$E = y't' + y'z't + x'y'zt + yzt'$$

xy \ zt	zt	zt'	$z't'$	$z't$
xy	0	1	0	0
xy'	0	1	1	1
$x'y'$	1	1	1	1
$x'y$	0	1	0	0

Fig. 7.40

which represent nine squares with 1's' in them corresponding to the four minterms of E. To cover all squares with 1's' in them.

Thus E has three mimal forms

$$E = x'y' + zt' + y'z'$$

Exercise

1. Simplify the following Boolean expressions by using map method and show circuit diagram of original expression and reduced expression

   (a) $A'BC + AB'C' + ABC + ABC'$ (b) $A'C + A'B + AB'C + BC$

2. Simplify the following Boolean expressions by using only laws of Boolean and then construct the circuit diagrams of reduced Boolean expression

   (a) $AB' + C'D'$ (b) $(BC' + A'D)(AB' + CD')$ (c) $(BC + AC' + AB + BCD)$

3. Find the minimal form of the Boolean function of four variables represented by the Karnaugh map given by

	xy	xy'	$x'y'$	$x'y$
zw	1	0	0	1
zw'	0	0	0	0
$z'w'$	0	0	0	0
$z'w$	1	0	0	1

Fig. 7.41

4. Use the Karnaugh map representation to find a minimal form each of the following functions

   (a) $f(x, y) = x'y + xy$

   (b) $f(x, y, z) = xyz + xy'z + x'yz + x'y'z$

   (c) $f(x, y, z) = xyz' + xy'z' + x'y'z' + x'y'z + x'yz + x'yz'$

   (d) $f = xyzw' + xy'zw' + xy'z'w' + xy'z'w + x'y'zw' + x'y'z'w' + x'y'zw'$

5. Draw Karnaugh map and simplify the following Boolean expressions

   (i) $F(A, B, C, D) = \Sigma(2, 3, 6, 7)$ (ii) $F(A, B, C) = \Sigma(0, 1, 3, 4, 5)$

   (iii) $F(A, B, C, D) = \Sigma(7, 13, 14, 15)$ (iv) $F(A, B, C, D) = \Sigma(0, 2, 6, 8, 10, 12, 14, 15)$

   (v) $F(A, B, C) = \Pi(0, 1, 4, 5)$ (vi) $F(A, B, C, D) = \Pi(1, 3, 5, 7, 13, 15)$

6. Define Boolean algebra and give a list of its important properties. Give some examples of Boolean algebra

   [Hint: $B_2 = \{0, 1\}$, $D(6)$, $D(10)$, $D(15)$, $D(30)$ are Boolean algebra]

7. In Boolean algebra, for x_1, x_2 and $x_3 \in B$, find whether the following expressions are equivalent or not

   (a) $(x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ (b) $(x_1 \vee x_2) \wedge (x_1' \vee x_3) \wedge (x_2 \vee x_3)$ (c) $(x_1 \vee x_2) \wedge (x_1' \vee x_3)$

8. State the dual of $a \vee \overline{(\bar{b} \vee a \wedge b)} = 1$.

Answers

1.	(a)	$BC + AC'$	(b)	$C + A'B$		
2.	(a)	$AB' + C'D'$	(b)	0	(c)	$BC + AB + AC'$
3.	$f = yw$					
4.	(a) $f = y$	(b) $f = z$	(c) $f = z' + x'z$	(d) $f = xzw' + xy'z' + x'y'z + x'z'w$		
5.	(i)	B	(ii)	$B' + A'C$	(iii)	$ABD + ABC + BCD$
	(iv)	$D' + ABC$	(v)	B	(vi)	$BC' + B'D + AB'C$
7.	(i)	yes	(ii)	yes	(iii)	yes
8.	$a \wedge (\overline{b \wedge a \vee b}) = 0$					

Objective Type Quesitons

Multiple Choice Questions

1. $a \wedge (a' \vee b)$ is
   (a) $a \vee b$ (b) $a \wedge b$ (c) $a \vee a$ (d) $b \wedge b$
2. Let $(R, +, .)$ be Boolean ring then R is commutative and $2a$ is
   (a) 0 (b) b (c) 1 (d) none of these
3. The number of elements of any finite Boolean algebra is
   (a) 2^n (b) $2n$ (c) n^2 (d) none of these
4. If x is any element of a Boolean algebra B, then $x + x + x + x$ is
   (a) $4x$ (b) $3x$ (c) $2x$ (d) x
5. The dual of $a + 1 = 1$ is
   (a) $a.0 = 0$ (b) $a.0 = 1$ (c) $a.a' = 0$ (d) $a.a = a$
6. A Boolean function is in disjunctive normal form, if it is expressed as
   (a) sum of the minimal Boolean polynomials (b) product of minimal Boolean functions
   (c) in both forms (d) none of these
7. Disjunctive normal form of $a + a'b$ is
   (a) $a.b' + a'b$ (b) $ab + a'b$ (c) $ab + ab' + a'b$ (d) $ab' + a'b + a'b'$
8. Idempotent law in Boolean algebra is
   (a) $(a')' = a$ (b) $a + a.b = a$ (c) $a + a = a$ (d) $a + 1 = a$
9. In Boolean algebra, which of the following statements is true for $x, y \in B$
   (a) $(xy)' = x' + y'$ (b) $(xy)' = x' - y'$ (c) $(xy)' = x'y'$ (d) none of these
10. A Boolean algebra cannot have
    (a) 2 elements (b) 3 elements (c) 4 elem...

Answers

1.	(a)	$BC + AC'$	(b)	$C + A'B$		
2.	(a)	$AB' + C'D'$	(b)	0	(c)	$BC + AB + AC'$
3.	$f = yw$					
4.	(a) $f = y$	(b) $f = z$	(c) $f = z' + x'z$	(d) $f = xzw' + xy'z' + x'y'z + x'z'w$		
5.	(i)	B	(ii)	$B' + A'C$	(iii)	$ABD + ABC + BCD$
	(iv)	$D' + ABC$	(v)	B	(vi)	$BC' + B'D + AB'C$
7.	(i)	yes	(ii)	yes	(iii)	yes
8.	$a \wedge (\overline{\bar{b} \wedge a \vee b}) = 0$					

Objective Type Quesitons

Multiple Choice Questions

1. $a \wedge (a' \vee b)$ is

(a) $a \vee b$ (b) $a \wedge b$ (c) $a \vee a$ (d) $b \wedge b$

2. Let $(R, +, .)$ be Boolean ring then R is commutative and $2a$ is

(a) 0 (b) b (c) 1 (d) none of these

3. The number of elements of any finite Boolean algebra is

(a) 2^n (b) $2n$ (c) n^2 (d) none of these

4. If x is any element of a Boolean algebra B, then $x + x + x + x$ is

(a) $4x$ (b) $3x$ (c) $2x$ (d) x

5. The dual of $a + 1 = 1$ is

(a) $a.0 = 0$ (b) $a.0 = 1$ (c) $a.a' = 0$ (d) $a.a = a$

6. A Boolean function is in disjunctive normal form, if it is expressed as

(a) sum of the minimal Boolean polynomials (b) product of minimal Boolean functions

(c) in both forms (d) none of these

7. Disjunctive normal form of $a + a'b$ is

(a) $a.b' + a'b$ (b) $ab + a'b$ (c) $ab + ab' + a'b$ (d) $ab' + a'b + a'b'$

8. Idempotent law in Boolean algebra is

(a) $(a')' = a$ (b) $a + a.b = a$ (c) $a + a = a$ (d) $a + 1 = a$

9. In Boolean algebra, which of the following statements is true for $x, y \in B$

(a) $(xy)' = x' + y'$ (b) $(xy)' = x' - y'$ (c) $(xy)' = x'y'$ (d) none of these

10. A Boolean algebra cannot have

(a) 2 elements (b) 3 elements (c) 4 elements (d) 5 elements

State True or False

11. Given any finite set Boolean Algebra B, there exists a set S such that B and $P(S)$, the Boolean ring of subsets of S, are isomorphic as Boolean algebra.

12. Every Boolean expression is equivalent to a unique expression in disjunctive normal form.

13. The poset (D_n, divisor) forms a Boolean algebra iff n cannot be written as a product of distinct primes.

Fill in the Blank(s)

14. The simplified form of the Boolean function $a + a\bar{b}$ is ....

15. The graph of the Boolean algebra of order 2^n is ....

16. The disjunctive normal form of the Boolean function $f(A, B) = A + B$ is ....