9.13 Predicate or Propositional Function or Open Sentence

In the previous articles we have discussed the simple statements and the logical techniques to combine simple statements into compound statements. We cannot apply those techniques to the arguments of the following forms:

All human are mortal. Newton is human. Therefore, Newton is mortal.

The validity of such type of argument depends upon the inner logical structure of simple statements it contains.

The second premise of the above argument is a *singular proposition*; it states that the individual Newton has the characteristic of being a human. We say 'Newton' the *subject* term and 'human' the *predicate* term. The individuals are not necessarily persons, but may be any *things*, such as planets, stars, cities, animals etc. of which the characteristics may be predicated. Similarly the characteristic can be designated by not only adjectives but also by nouns, pronouns or verb.

In symbolic notation, we shall use small letters to denote the individuals such as the individual 'Newton' will be denoted by the small letter 'n' [the first letter of the name], and is called *individual constant*. The characteristic will be denoted by the capital letter such as 'mortal' by M and 'human' by H. The symbolic notation for 'Newton is mortal' will be used as $M(n)$ and for 'Newton is human' as $H(n)$. Some other examples of singular propositions with the same characteristic are 'Brijpal is human'. 'Lakshay is human'. and their symbolic formulation are respectively '$H(b)$', '$H(l)$'. Let us use the symbolic representations $H(x)$ to denote the pattern common to all such singular propositions. The small letter 'x' is called '*individual variable*', and takes values from the set of individual constants. When an individual constant is substituted for x, a singular proposition, is produced, such as $H(n)$, $H(b)$, $H(l)$.

predicate variable

The singular propositions $H(n)$, $H(b)$, $H(l)$are either true or false *i.e.*, they have their truth values but three is no truth value (true or false) or $H(x)$since $H(x)$is not a proposition. The expressions of the type $H(x)$ are called **'Propositional Functions'** or **'Open Sentences'**. Thus propositional functions are expressions that contain individual variables and become propositions when their individual constants are replaced by individual variables.

Definition: Let A be a given set. An expression denoted by $P(x)$ is called a *propositional function* or, simply, *an open sentence* or, a *predicate* on A if $P(a)$ is true or false for each $a \in A$ i.e. $P(a)$ has a truth value for each $a \in A$. In other words, $P(x)$ is called a *propositional function* or, *an open sentence* if $P(x)$ becomes a statement whenever any element $a \in A$ is substituted for the variable x.

Illustration: *Let* $P(x)$ *be* '$x + 4 < 9$',

then $P(x)$ *is a propositional function on the set of natural numbers* $\mathbf{N}$.

Clearly $P(x)$ *is true for* $n = 1, 2, 3, 4$ *and false for* $x = 5, 6, 7, \ldots$ *Hence* $P(x)$ *becomes a statement whenever any element* $a \in \mathbf{N}$ *is substituted for* x.

Illustration: *Let* $P(x)$ *be* '$x + 3 > 6$', *then* $P(x)$ *is a propositional function on the set of natural numbers* $\mathbf{N}$.

Illustration: *Let* $P(x)$ *be* '$x + 3 > 6$', *then* $P(x)$ *is not a propositional function on the set of complex numbers* $\mathbf{C}$, *because inequalities are not defined on* $\mathbf{C}$.

9.13.1 Truth Set Definition

Let $P(x)$ be a propositional function and D be its domain. The set of elements $d \in D$ with the property that $P(d)$ is true is called the ***Truth Set*** $T(P)$ of $P(x)$. Symbolically:

$$T(P) = \{x : x \in D,\ P(x) \text{ is true}\} \text{ or } T(P) = \{x : P(x)\}.$$

Illustration: *If* '$x + 3 > 6$' *be a propositional function defined on* N, *then find the truth set of* $P(x)$.

We have $P(x) \equiv$ '$x + 3 > 6$', $D \equiv \mathbf{N}$.

$\therefore$ $T(P) = \{x : x + 3 > 6,\ x \in \mathbf{N}\} = \{4, 5, 6, \ldots\}$.

Illustration: *If* $P(x)$ *be* '$x + 2 < 3$' *and* $D \equiv \mathbf{N}$, *then find the truth set.*

$$T(P) = \{x : x + 2 < 3,\ x \in \mathbf{N}\} = \phi.$$

Illustration: *If* $P(x)$ *be* '$x < 5$' *and* $D \equiv \{5, 6, 7, 8, \ldots\}$, *then*

$$T(P) = \{x : x < 5,\ x \in D\} = \phi.$$

Illustration: *If* $P(x)$ *be* '$x > 5$' *and* $D \equiv \mathbf{N}$, *then*

$$T(P) = \{x : x > 5,\ n \in \mathbf{N}\} = \{6, 7, 8, \ldots\}.$$

Illustration: *If* $P(x)$ *be* '$x > 2$' *and* $D \equiv \{3, 4, 5, \ldots\}$, *then*

$T(P) = \{3, 6, 5, \ldots\} = D$ *i.e.,* $P(x)$ *is true for every* $x \in D$.

9.14 Quantifiers

The restrictions namely 'for every' and 'for some' are called ***Quantifiers.***

There are two types of quantifiers.

9.14.1 Universal Quantifier

Definition: *The symbol $\forall$ which is read as* ***'For Every'*** *or* ***'For All'*** *is called the universal quantifier.*

Let $P(x)$ be a propositional function defined on the set D. If 'for every $x \in D$, $P(x)$ is a true statement' then by the use of universal quantifier $(\forall)$ it is written as

$$(\forall\, x \in D)\, P(x) \text{ or } \forall\, x\, P(x) \text{ or } \forall\, x,\, P(x). \quad \ldots(1)$$

We see that the truth set of $P(x)$ is the entire set D,

i.e.,
$$T(P) = \{x : x \in D,\, P(x)\} = D \quad \ldots(2)$$

Illustration: The statement $(\forall\, n \in \mathbf{N})(n + 2 > 1)$ is true since its truth set

$$T(P) = \{n : n \in \mathbf{N},\, n + 2 > 1\} = \{1, 2, 3, 4, \ldots\} = \mathbf{N}.$$

Illustration: *The statement $(\forall\, n \in \mathbf{N})(n + 1 > 5)$ is false since its truth set*

$$T(P) = \{5, 6, 7, \ldots\} \neq \mathbf{N}.$$

Illustration: *The statement $(\forall\, x \in \mathbf{R})(x^2 > -1)$ where $\mathbf{R}$ is the set of real numbers, is true since its truth set*

$$T(P) = \{x : x \in \mathbf{R},\, x^2 > -1\} = \mathbf{R}$$

Illustration: *Let D be the set of all men, then the statement 'All men are mortal' is written as*

$$(\forall\, x \in D) \quad (x \text{ is Mortal}).$$

Illustration: *Let $\{A_t\}_{t \in T}$ be a family of sets, where T is the index set. The intersection of family of sets denoted by $\bigcap_{t \in T} A_t$ may also be written as (by the use of $\forall$) follows*

$$\bigcap_{t \in T} A_t = \{x : \forall\, t \in T,\, x \in A_t\}$$

Note: The truth set T(P) is also called **'Universe of discourse'**.

9.14.2 Existential Quantifiers

Definition: *The symbol $\exists$ which is read as* ***'There Exists'*** *or* ***'For Some'*** *or* ***'For at Least One'*** *is called the* ***Existential Quantifier.***

Let $P(x)$ be a propositional function defined on the set D. If 'there exists an $x \in D$, such that $P(x)$ is a true statement' or 'for some $x \in D$, $P(x)$ is a true statement' or 'for at least one $x \in D$, $P(x)$ is a true statement' then it is written as

$$(\exists\, x \in D)\, P(x) \qquad \text{or} \qquad \exists\, x\, P(x). \qquad \ldots(1)$$

Hence, we find that the truth set (or universe of discourse) is not the empty set, *i.e.*,

$$T(P) = \{x : x \in D,\, P(x)\} \neq \phi.$$

Therefore, we may state that

(i) if $T(P) \neq \phi$ then $\exists\, x\, P(x)$ is true,

and (ii) if $T(P) = \phi$ then $\exists\, x\, P(x)$ is false.

Illustration: *The statement* $(\exists\, n \in \mathbf{N})\,(n + 7 < 6)$ *is false since*

$$T(P) = \{n : n + 7 < 6\} = \phi.$$

Illustration: *The statement* $(\exists\, x \in \mathbf{R})\,(x^2 - 1 = 0)$, $\mathbf{R}$ *is the set of real numbers, is true since*

$$T(P) = \{x : x^2 - 1 = 0\} = \{1, -1\} \neq \phi.$$

We see the variable x can take only two real values $1, -1$ for which $x^2 - 1 = 0$ is true.

Illustration: *Let* $\{A_t\}_{t \in T}$ *be a family of sets. The union of family of sets denoted by* $\underset{t \in T}{\cup} A_t$ *may also be written as (by the use of* $\exists$*) follows:*

$$\underset{t \in T}{\cup} A_t = \{x : \exists\, t \in T,\, x \in A_t\}.$$

9.15 Negation of A Quantifier

Let us consider the proposition. 'All Indians are honest'. If M denotes the set of Indians, then it can be is written as

$$(\forall\, x \in M)\ (x \text{ is honest}).$$

This statement will become false, if we say that 'there is an Indian who is not honest' or in symbolic form

$$(\exists\, x \in M)\ (x \text{ is not-honest}).$$

Therefore, the negation of the statement 'All Indians are honest' will be 'there is an Indian who is not honest' or in symbolic form $(\exists\, x \in M)\,(x$ is not honest$)$.

Thus the negation of $\forall\, x\, P(x)$ is $\exists\, x\,[\sim P(x)]$; where $P(x)$ denotes 'x' is honest'.

Hence, we see that the negation of a statement with 'universal quantifier' is a statement with 'existential quantifier'.

Again we consider the proposition. 'Some Indian are honest'.

If M denotes the set of Indians, then this statement in symbolic form can be written as

$$(\exists\, x \in M)\,(x \text{ is honest}).$$

This statement will become false, if we say that 'No India is honest' or in symbolic form

$$(\forall\, x \in M)\,(x \text{ is not honest}).$$

Therefore, the negation of the statement 'some Indians are honest' will be 'No Indian is honest'. In symbolic form, the negation of $(\exists\, x \in M)\, P(x)$ is $(\forall\, x \in M)\,(\sim P(x))$ where $P(x)$ denotes 'x is honest'.

Hence, we see that the negation of a statement with 'existential quantifier' is a statement with 'universal quantifier'.

9.16 De-Morgan Laws

If $P(x)$ is a propositional function defined on the domain D, then negation of $(\forall\, x \in D)\, P(x)$ is $(\exists\, x \in D) \sim P(x)$, which can be written as

$$\sim(\forall\, x \in D)\, P(x) \equiv (\exists\, x \in D) \sim P(x) \qquad \text{...(1)}$$

Similarly
$$\sim(\exists\, x \in D)\, P(x) \equiv (\forall\, x \in D) \sim P(x) \qquad \text{...(2)}$$

(1) and (2) are called De-Morgan Laws.

Note: The laws which hold for propositions also hold for propositional functions, for example:

(i) $\sim(P(x) \vee Q(x)) \equiv \sim P(x) \wedge \sim Q(x)$.

(ii) $\sim(P(x) \wedge Q(x)) \equiv \sim P(x) \vee \sim Q(x)$ etc.

Example 94: *Use quantifiers to say that $\sqrt{3}$ is not a rational number.* **[U.P.T.U. (B.Tech.) 2008]**

Solution: $A(x)$: x is prime, $B(x)$: $\sqrt{x}$, x is prime and $C(x)$: x is rational

$\forall\, x,\, x \in A(x),\, B(x) \to \sim C(x)$ i.e., square root of every prime number is not rational

Example 95: *Let $M(x)$ be "x is mammal". Let $A(x)$ be "x is an animal". Let $W(x)$ be "x is warm blooded". Translate into a formula; Every mammal is warm, blooded. Translate into English.*

$$(\exists\, x)\,(A(x) \wedge (\sim M(x)))$$

[U.P.T.U. (B.Tech.) 2008]

Solution: Formula for "every mammal is warn blooded" is $\forall\, x\, (M(x) \to W(x))$

$$(\exists\, x)\,(A(x) \wedge (\sim M(x))$$

Translated into English is there are some animals there are not mammals.

Example 96: *The proposition:*

(i) *There is a dog without a toil can be written as ($\exists$ a dog) (the dog without toil).*

(ii) *There is an integer between 2and 8 inclusive may be written as ($\exists$ an integer) (the integer is between 2and 8).*

Example 97: *Nagate the proposition "All integers are greater then 8".*

[Nagpur (B.E.) 2005; Pune (B.E.) 2004, 2007; M.K.U. (B.E.) 2008; Rohtak (M.C.A.) 2009]

Solution: The given proposition can be written as ($\forall$ integer x) $(x > 8)$

The negation is ($\exists$ an integer x) $(x \leq 8)$ i.e., There is an integer less than or equals to 8

Remark: In negation a proposition "for all" becomes "there is" and "there is" becomes "for all".

The symbol $\forall$ becomes $\exists$ and $\exists$ becomes $\forall$.

Example 98: *Translate the following statements given in English into equivalent statements of Propositional/Predicate Calculus, after introducing appropriate symbolism.*

(i) *Some pet dogs are dangerous*
(ii) *Some physicists are not good in chemistry*
(iii) *Americans will stop driving big cars only if there are comfortable small cars.*
(iv) *Not all birds can fly*
(v) *Himadari, who is a doctor, is a good sports man also*
(vi) *Sum of two positive integers is greater than either of integers.*
(vii) *Some cats are black but all buffaloes are black*
(viii) *Some patients like all doctors*
(ix) *Some people have six fingers on one hand*
(x) *Some people help every body*
(xi) *Some mangoes are green*
(xii) *Some real numbers are integers*
(xiii) *Some mathematicians are not good in computer science*
(xiv) *All lions have tails*
(xv) *All cows are not white*
(xvi) *All frogs are brown*
(xvii) *All lions are dangerous animals*
(xviii) *All integers are either odd integers or even integers*
(xix) *All cows have four legs*
(xx) *All fish except shark are kind to children*
(xxi) *All men are mortal*
(xxii) *Any integer is either positive or negative*
(xxiii) *Every integer is also a real number*
(xxiv) *Nine plus ten equal nineteen*
(xxv) *Let A subset of B. Express this fact in predicate calculus.*
(xxvi) *Some irrational numbers x and y, the number x^y is a rational numbers*
(xxvii) *Couple's only surviving child is male*

Solution: (i) Some Pets dogs are dangerous

Let

$P(x)$: x is pet dog and $D(x)$: x is dangerous

∴ In predicate calculus we have

$$x\, P(x) \rightarrow D(x)$$

(ii) Some physicists are not good in chemistry

Let

$P(x)$: x is a physicist and $C(x)$: x is good in chemistry

$$\exists\, x\, P(x) \rightarrow C(x)$$

(iii) Americans will stop driving big cars only if there are comfortable small cars.

Let $A(x)$: x is American will stop driving big cars and $C(x)$: x is comfortable small cars

In predicate calculus, we have

$$C(x) \to A(x)$$

(iv) Not all birds can fly.

Let $B(x)$: x is bird and $F(x)$: x is fly

In predicate calculus, we have

$$\sim(\forall B(x)) \to F(x)$$

(v) Himadari, who is a doctor, is a good sports person also

Let x : denote Himadari, $D(x)$: is doctor and $S(x)$: x is sport person

In predicate calculus, we have

$$(\exists x) D(x) \wedge S(x)$$

(vi) Sum of two positive integers is greater than either of the integers

Let $I(x)$: x is positive integer, $GT(x, y)$: x is greater then y and $Su(x, y)$: sum of x and y

Thus, in predicate calculus, we have

$$(\forall x)(\forall y)(I(x) I(y))(GT (Su(x, y) GTSu(x, y), y)))$$

(vii) Some cats are black but all buffaloes are black

Let $C(x)$: x is a cat, $B(x)$: x is a black, and $BF(x)$: x is buffalo

$\therefore$ $\exists x \in C(x) \Rightarrow B(x)$

and $\forall x\, BF(x) \Rightarrow B(x)$

(viii) Some patients like all doctors

Let $P(x)$: x is patient, and $D(y)$: y is doctor, (x, y): x likes by y

$\therefore$ $x\, P(x)\, y$

(ix) Some people have six fingers on one hand

Let $P(x)$: x is people, $S(x)$: x is a people having six fingers on one hand

Therefore in predicate calculus, we have

$$\exists x \in P(x) \to S(x)$$

(x) Some people help everybody

Let $P(x)$: x is a people, $HB(y)$: y is human being, and $H(x, y)$: x helps y

$\therefore$ The given statement can be expressed as

$$\exists x \in P(x) \to H(x, y) \forall x \in (P(x) \wedge HB(y))$$

or $\exists x \in P(x) \to \forall y \in (P(x) \wedge HB(y)) \to H(x, y)$

(xi) Some mangoes are green

Let $M(x)$: x are mangoes and $G(x)$: x are green

$\therefore$ In predicate calculus, we have

$$\exists x (M(x) \to G(x))$$

(xii) Some real numbers are integers

Let $R(y)$: y is real number and $I(y)$: y is an integer

$\therefore$ In predicate calculus, we have

$$\exists y \in R(y) \rightarrow I(y)$$

(xiii) Some mathematicians are not good in computer science

Let $M(x)$: x is a mathematician and $G(x)$: x is good in computer science

$\therefore$ The given statement can be expressed in predicate calculus as

$$\exists x \in M(x) \rightarrow \sim G(x)$$

(xiv) All lions have tails

Let $L(x)$: x is lions and $T(x)$: x is tails

In predicate calculus, we have

$$(\forall x)(L(x) \rightarrow T(x))$$

(xv) All cows are not white

Let $C(x)$: x is cow and $W(x)$: x is white

$\therefore$ In predicate calculus, we have

$$\forall x\, C(x) \rightarrow \sim W(x)$$

(xvi) All frogs are brown

Let $F(x)$: x is frogs, $B(x)$: x is an brown frog

$\therefore$ In predicate calculus, we have

$$\forall x \in F(x) \rightarrow B(x)$$

(xvii) All lions are dangerous animals

Let $L(x)$: x is a lion, $D(x)$: x is a dangerous animal

$\therefore$ In predicate calculus, we have

$$\forall x \in L(x) \rightarrow D(x)$$

(xviii) All integers are either odd integers or even integers

Let $I(x)$: x are integers, $O(x)$: x is odd, $E(x)$: x is even

$\therefore$ In predicate calculus, we have

$$\forall x\, I(x) \rightarrow \sim (O(x) \vee E(x))$$

(xix) All cows have four legs

Let $C(x)$: x is cow, $F(x)$: x has four legs

$\therefore$ In predicate calculus, we have

$$\forall x\,(C(x) \rightarrow F(x))$$

(xx) All fish except shark are kind to children

Let $F(x)$: x is a fish, $S(x)$: x is a shark, $K(x)$: x is kind to child

$\therefore$ In predicate calculus, we have

$$\forall x\,(F(x) \neq S(x)) \rightarrow K(x)$$

(xxi) All men are mortal

Let $H(x)$: x is man, $M(x)$: x is mortal

$\therefore$ In predicate calculus, we have

$$(\forall x)(C(x) \rightarrow M(x))$$

(xxii) Any integer is either positive or negative

Let $I(x)$: x is an integer, $P(x)$: x is positive integer

$\therefore$ In predicate calculus, we have

$$\forall x \in I(x) \rightarrow \{P(x) \vee N(x)\}$$

(xxiii) Every integer is also a real number

Let $I(x)$: x is an integer

$R(x)$: x is real number

$\therefore$ In predicate calculus, we have

$$\exists x \in I(x) \rightarrow R(x)$$

(xxiv) Nine plus ten equal nineteen

Let $N(x)$: x is nine, $T(x)$: x is ten, $C(x, y)$: sum of x and y

$NT(x)$: is nineteen

$\therefore$ In predicate calculus, we have $xy\,(N(x)T(y)\,\text{sum}\,(x, y)\,NT(x)$

(xxv) Let A be subset of a set B. Express this fact in predicate calculus.

Let $A(x)$: x is an element of A, $B(x)$: x is an element of B

$\therefore$ In predicate calculus, we have

$$\forall x \in A(x) \wedge B(x)$$

(xxvi) Let $I(x)$: x is an irrational number, $Q(x)$: x is an rational number, $EX\ P(x, y)$: x^y

$\therefore$ In predicate calculus, we have

$$(\exists x, y)[I(x) \wedge I(y) \rightarrow Q\,(EX\ P(x, y)]$$

(xxvii) Couple's only surviving child is a male

Let $SC(x)$: x is couple's surviving child, $M(x)$: x is male

$\therefore$ In predicate calculus, we have

$$\exists x \in SC(x) \rightarrow M(x)$$

Example 99: *Let $Q(x) \equiv x$ is a rational number, $R(x) \equiv x$ is a real number*

$E(x, y) \equiv$ '$x = y$', $G(x, y) \equiv$ '$x > y$'.

Translate the following sentences into symbols:

(i) *π is a real number.*

(ii) *e is a real number.*

(iii) *4/5 is a rational number.*

(iv) *$\sqrt{3}$ is an irrational number.*

(v) *Every rational number is a real number.*

(vi) *Some real numbers are rational.*

(vii) *The square of every real number is not negative.*

Solution: (i) $R(\pi)$,

(ii) $R(e)$, (iii) $Q(4/5)$, (iv) $\sim Q(\sqrt{3})$,

(v) $\forall x\,[Q(x) \Rightarrow R(x)]$, (vi) $\exists x\,[R(x) \wedge Q(x)]$, (vii) $\forall x\, R(x) \Rightarrow \sim G(0, x^2)]$.

Note that $G(0, x^2)$ means $0 > x^2$ *i.e.*, x^2 is negative and therefore $\sim G(0, x^2)$ means x^2 is not negative.

Example 100: *Negate each of the following statements:*

(i) $\forall x\,(|x| = x)$, (ii) $\forall x\,(x + 1 > x)$,

(iii) $\forall x\,(x \neq 1, x \neq 2)$, (iv) $\exists x\,(x^2 < 0)$,

(v) $\forall x\,(x \neq 0) \Rightarrow (x^2 > 0)$, (vi) $\exists x\,|x| = 0)$,

(vii) $\exists x\,(x^2 = x)$, (viii) $\exists x\,(x + 2 = x)$,

(ix) $\exists x\,(x^2 = 1$ *and* $x^2 - 2x + 3 = 0)$, (x) *If there is a will there is a way,*

(xi) *All men are mortal.* (xii) *All integer are greater than zero.*

[U.P.T.U. (B.Tech.) 2008]

Take the universal set as set of real numbers from (i) to (ix).

Solution: (i) $\exists x \sim (|x| = x)$ or $\exists x\,(|x| \neq x)$.

(ii) $\exists x \sim (x + 1 > x)$ or $\exists x\,(x + 1 \leq x)$.

(iii) $\exists x\,((x-1)(x-2) = 0)$ or $\exists x\,(x^2 - 3x + 2 = 0)$.

(iv) $\forall x \sim (x^2 < 0)$ or $\forall x\,(x^2 \geq 0)$.

(v) $\exists x\,[(x \neq 0) \wedge (x^2 < 0)]$.

(vi) $\forall x \sim (|x| = 0)$ or $\forall x\,(|x| \neq 0)$.

(vii) $\forall x \sim (x^2 = x)$ or $\forall x\,(x^2 \neq x)$.

(viii) $\forall x\,(x + 2 \neq x)$.

(ix) $\forall x \sim (x^2 = 1$ and $x^2 - 2x + 3 = 0)$.

or $\forall x\,(x^2 \neq 1$ or $x^2 - 2x + 3 \neq 0)$.

(x) We know that $\sim (p \Rightarrow q) \equiv p \wedge \sim q$. Thus required negation is:

There is a will and there is no way.

(xi) Some men are not mortal.

(xii) Let $A(x)$ denote "x is an integer"

and $(B)x$ denoted "x is greater than 0"

Then given proposition can be written as $(\exists x \in A(x)(-B(x)))$

Example 101: *Let* $D = \{1, 2, 3, 4, 5, 6 \ldots\}$, *negate the following statements:*

(i) $(\forall x \in D)(x + 4 \geq 8)$, (ii) $(\forall x \in D)(x + 2 < 9)$, (iii) $(\exists x \in D)(x + 5 = 9)$,

(iv) $(\exists x \in D)(x + 1 > 6)$, (v) $(\exists x \in D)(x^2 = 6)$, (vi) $(\forall x \in D)(x^2 > 25)$,

Solution: (i) $(\exists x \in D)(x + 4 < 8)$,

(ii) $(\exists x \in D)(x + 2 \geq 9)$, (iii) $(\forall x \in D)(x + 5 \neq 9)$, (iv) $(\forall x \in D)(x + 1 \leq 6)$,

(v) $(\forall x \in D)(x^2 \neq 6)$, (vi) $(\exists x \in D)(x^2 \leq 25)$,

Example 102: *Negate the statement 'He is poor and laborious'.*

Solution: We know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Hence the negation of the given statement is:

'It is false that he is poor and laborious'. ≡ 'He is not poor or he is not laborious'.

Example 103: *Negate the statement 'It is daylight and all the people have arisen'.*

Solution: We know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Hence the negation of the given statement is:

'It is false that it is daylight and all the people have arisen'.

≡ 'It not daylight or it is false that all the people have arisen'.

≡ 'It is not daylight or some one has not arisen'.

Example 104: *Negate the statements:*

(i) $\exists\, x\, P(x) \vee \forall\, y\, Q(y)$. (ii) $\forall\, x\, P(x) \wedge \exists\, y\, Q(y)$.

Solution: (i) We know that $\sim(p \vee q) = \sim p \wedge \sim q$.

$\therefore\ \sim[\exists\, x\, P(x) \vee \forall\, y\, Q(y)] \equiv \sim\exists\, x\, P(x) \wedge \sim\forall\, y\, Q(y) \equiv \forall\, x \sim P(x) \wedge \exists\, y \sim Q(y)$.

$\because\ \sim\exists\, x\, P(x) \equiv \forall\, x \sim P(x)$ and $\sim\forall\, y\, Q(y) \equiv \exists\, y \sim Q(y)$

(ii) We know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

$\therefore\ \sim[\forall\, x\, P(x) \wedge \exists\, y\, Q(y)] \equiv \sim\forall\, x\, P(x) \vee \sim\exists\, y\, Q(y) \equiv \exists\, x \sim P(x) \vee \forall\, y \sim Q(y)$.

Example 105: *Write the three properties of equivalence relation in symbolic notations.*

Solution: Let $R(x, y)$ denote that x has a relation R with y, then the three properties for R to be an equivalence relation are:

(i) ***Reflexivity:*** $\forall\, x\, R(x, x)$.

(ii) ***Symmetric:*** $\forall\, x, y\, [R(x, y) \Rightarrow R(y, x)]$.

(iii) ***Transitivity:*** $\forall\, x, y, z\, [R(x, y) \wedge R(y, z) \Rightarrow R(x, y)]$.

Example 106: *Negate the statement* $\exists\, x\, \forall\, y\, [P(x) \vee \sim Q(y)]$.

Solution: $\sim\exists\, x\, \forall\, y\, [P(x) \vee Q(y)] \equiv \forall\, x\, \exists\, y\, [\sim P(x) \wedge Q(y)]$.

Example 107: *Negate the statement* $\forall\, x\, \exists\, y\, [P(x, y) \Rightarrow Q(y)]$.

Solution: $\sim\forall\, x\, \exists\, y\, [P(x, y) \Rightarrow Q(y)] \equiv \exists\, x\, \forall\, y\, [P(x, y) \wedge -Q(y)]$ $[\because\ \sim(p \Rightarrow q) \equiv p \wedge \sim q]$.

Example 108: *Write the following statement into symbols (using quantifier and the symbol '<' for 'less than'):*

(i) A number x, is less than 7 and greater than 5.

(ii) For a given number x there is a greater number y.

(iii) For two given numbers x and y, there is a number z such that the difference of x and y is less than the product of x^2 and z.

(iv) The numbers x, y, z are such that x + y is greater than xz.

Solution:

(i) $\exists x[x < 7 \wedge 5 < x]$.

(ii) $\forall x \exists y(x < y)$.

(iii) $\forall x \forall y \exists z(|x - y| < x^2 z)$.

(iv) $(\exists x)(\exists y)(\exists z)(xz < x + y)$.

Example 109: *Write the following sentences into symbols:*

(i) *The square of any rational number is not 2.*

(ii) *Two non-parallel coplanar straight lines have a common point.*

(iii) *If there is no prize, then a person does not purchase a ticket.*

Solution: (i) Let $Q(x) = x$ is a rational number, $E(x, y) \equiv$ '$x = y$'.

The given sentence in symbols, using quantifiers, is

$$\exists x[Q(x) \wedge E(x^2, 2)].$$

(ii) Let $P(x) = x$ is a point, $L(x) = x$ is a straight line, $I(x, y) =$ the point x is on y, $D(x, y) = x, y$ are non-parallel and coplanar.

The given sentence in symbols is

$$\forall mn[L(m) \wedge L(n) \wedge D(m\, n) \Rightarrow \exists x\{P(x) \wedge I(x, m) \wedge I(x, n)\}].$$

(iii) Let $P(x) =$ a prize, $T(x) =$ a ticket, $M(x) =$ a person, $A(x, y) = x$ purchases y.

The given sentence is symbols is:

$$\{\sim \exists x\, P(x)\} \Rightarrow \forall x \forall y\{M(x) \wedge T(y) \Rightarrow \sim A(x, y)\}.$$

Example 110: *'If the product of two rational numbers is zero, then at least one factor is zero'. Write this statement into symbols.*

Solution: Let $P(x) \equiv$ product of rational numbers is x and $F(x, y) \equiv x$ in a factor of y.

The given sentence is symbols is

$$\forall x[P(x) \wedge (x = 0) \Rightarrow \exists y[F(x, y) \Rightarrow (y = 0)\}].$$

Example 111: *Show that the negation of $(\forall x)(P(x))$ is $(\exists x)(\sim P(x))$.*

Or

If the projection of $P(x)$ on D is a propositional function, then show that

$$\sim(\forall x \in D : P(x)) \equiv (\exists x \in D : \sim P(x)).$$

Solution: Here following two cases arise:

Case (i): *The truth value of $(\forall x \in D : P(x))$ is T. In this case $P(x)$ is satisfied for all $x \in D$. Therefore, there is an $x \in D$ for which $\sim P(x)$ is not satisfied.*

So the truth value of $(\exists x \in D : \sim P(x))$ is F.

Case (ii): *The truth value of $(\forall x \in D : P(x))$ is F. In this case $P(x)$ is not satisfied for some $x \in D$, therefore, $\sim P(x)$ is satisfied for some $x \in D$.*

So the truth value of $(\exists x \in D : \sim P(x))$ is T. Hence

$$\sim(\forall x \in D : P(x)) \equiv (\exists x \in D : \sim P(x)).$$

Example 112: *Write the following predicate into symbolic language:*

(i) *For every real number there is a greater real number.*

(ii) *Every irrational number is a real number.*

(iii) *The number divisible by an even number is even.*

(iv) *Every teacher of a college is learned.*

(v) *All students are not wise.*

(vi) *Some people like to listen only instrumental music.*

(vii) *If the product of a finite number of numbers is zero, then at least one factor is zero.*

(viii) *The function $f(x)$ is continuous at $x = a$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - f(a)| < \varepsilon \quad \text{where} \quad |x - a| < \delta.$$

Solution: (i) Let $R(x) \equiv x$ is a real number, $G(x, y) \equiv x > y$.

Then the given statement in symbols is

$$\forall x [R(x) \Rightarrow \exists y [\{R(y) \wedge G(y, x)\}].$$

(ii) Let $I(x) \equiv x$ is irrational number, $R(x) \equiv x$ is real number.

Then $\quad \forall x [I(x) \Rightarrow R(x)]$.

(iii) $R(x) \equiv x$ is a number, $E(x) \equiv x$ is an even number

$D(x, y) \equiv y$ is divisible by x.

Then $\quad \forall x, y [E(x) \wedge R(y)] \wedge D(x, y) \Rightarrow E(y)]$.

(iv) $T(x) \equiv x$ is a teacher of college, $L(x) \equiv x$ is learned.

$\therefore \quad \forall x [T(x) \Rightarrow L(x)]$.

(v) $S(x) \equiv x$ is student and $W(x) \equiv x$ is wise.

$\therefore \quad \sim \forall x [S(x) \wedge W(x)]$.

(vi) $P(x) \equiv x$ is people, $I(x) \equiv x$ is instrumental music, $H(x, y) \equiv x$ likes to listen y.

$\therefore \quad \exists x [P(x) \wedge H(x, y) \Rightarrow Q(y)]$.

(vii) $P(x) \equiv$ Product of finite number of members

$F(x, y) \equiv x$ is a factor of y.

$\therefore \quad \forall x [P(x) \wedge (x = 0) \Rightarrow \exists y \{F(y, x) \Rightarrow (y = 0)\}]$.

(viii) $F(x) \equiv$ The function f is continuous at x.

$\therefore \quad [\forall \varepsilon > 0 \exists \delta > 0 \forall x \{|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\}] \Rightarrow f(a)$.

Example 113: *Write the following predicate into symbolic form:*

(i) *There are real numbers which are greater than all real numbers.*

(ii) *Some ladies lawyer are such that they are also homely ladies.*

(iii) *Some ladies are lawyer and also homely ladies.*

(iv) *Every student is a member of N.C.C. or N.S.S., but some students are not players.*

(v) *If a and b are non-zero integers and p is a prime number such that $p | ab$ then $p | a$ or $p | b$.*

Solution: (i) Let $R(x) \equiv x$ is real number

$$G(x, y) \equiv x > y.$$

Then the given predicate in symbolic form is

$$\therefore \quad \forall\, x\,[R(x) \Rightarrow \exists\, y\,\{R(y) \wedge G(x, y)\}].$$

(ii) $W(x) \equiv x$ is lady, $L(x) \equiv x$ is lawyer, $H(x) \equiv x$ is homely lady.

Then $\quad \exists\, x\,[W(x) \wedge L(x) \wedge H(x)]$.

(iii) Same as (ii).

(iv) $S(x) \equiv x$ is student. $N(x) \equiv x$ is a member of N.C.C. $M(x) \equiv x$ is a member of N.S.S. $P(x) \equiv x$ is a player.

Then $\quad \forall\, x\,[S(x) \Rightarrow \{N(x) \wedge M(x)\}] \wedge \exists\, y\,[S(y) \wedge \sim P(y)]$.

(v) $I(x) \equiv x$ is a non-zero integer, $P(x) \equiv x$ is prime number, $D(x, y) \equiv x$ is divisible y.

Then $\forall\, a, b, p\,[I(a) \wedge I(b) \wedge P(p) \wedge D(p, ab)] \Rightarrow [D(p, a) \vee D(p, b)]$.

Example 114: *Write the following predicates into symbolic form:*

(i) *All are not mortal.* (ii) *All are immortal.*

(iii) *For all real numbers x, y, z addition is an associative operation.*

Solution: Let x denote any person and $M(x) \equiv x$ is Mortal. Then

(i) $\sim \forall\, x\,[M(x)]$. (ii) $\forall\, x \Rightarrow [\sim M(x)]$.

(iii) Let x, y, z be real numbers. Then $\forall\, x, y, z\,[(x + y) + z = x + (y + z)]$.

Example 115: *Write the following predicate into symbols and also write its negative in symbols:*

"Every rational number is a real number".

Solution: Let $Q(x) \equiv x$ is rational number, $R(x) \equiv x$ is real number, and $P(x) \equiv Q(x) \Rightarrow R(x)$.

Then symbolic form of given predicate is $\forall\, x\, P(x)$.

Negative: $\exists\, x\,[\sim P(x)]$ i.e., $\exists\, x\,[\sim Q(x) \Rightarrow R(x)]$.

Exercise

1. Show that negative of $\exists\, x\,(\sim P(x))$ is $\forall\, x\,(P(x))$.

   [Hint: $\sim \exists\, x\,(\sim P(x)) = \forall x\,(\sim\sim P(x)) = \forall\, x\, P(x)$].

2. Write the empty set ϕ and universal set U by the use of quantifier.

3. Show that $\sim \forall\, x\, P(x) \equiv \exists\, x\,(\sim P(x))$.

4. Define quantifiers, universal quantifiers and existential quantifiers by giving an example.

5. Define existential and universal quantifiers.

6. Which are existential and universal quantifiers? Write three-three statements using each.

7. Explain the following terms and also give examples to explain them:

   (a) Quantifier. (b) Universal quantifier.

   (c) Existential quantifier. (d) Negation of a quantifier.

8. Define Predicate.

9. Translate the following sentences in quantified expressions of Predicate logic

(a) All students need financial acid

(b) Some students need financial aid [U.P.T.U. (B.Tech.) 2009]

10. Show that the following are not equivalent

(a) $\begin{cases} \forall\, x\, (P(x) \to Q(x) \\ \forall\, x\;\; P(x) \to \forall\, x\, Q(x) \end{cases}$ (b) $\begin{cases} \forall\, x \,\forall\, y\; P(x, y) \\ \forall\, y\, \forall\, x\; P(x, y) \end{cases}$

[U.P.T.U. (B.Tech.) 2009]

11. Find out which statements are propositions

(a) Close the door (b) Beware the dogs

(c) $3 + x = 10$ (d) This statement is true

12. Which of the following are statements:

(a) Grass is yellow (b) is the number 5 is prime

(c) Give me pen (d) $\{y : y^2 = 4 = \{2, -2\}$

(e) $a^2 + b^2 \geq 0$ (f) White roses are beautiful

(g) If dogs can bark, then no home guarded by a dog needs to fear intruders

(h) $(7x + 5y = 12 : (x, y) = (1, 1),\ x, y \in N)$

13. Let p be the proposition "x is an even number" and let q be the statement "x is the product of two integers" Translate into symbols each of the following statements

(a) Either x is an even number, or x is a product of two integers

(b) x is an odd number, and x is a product of two integers

(c) Either x is an even number and a product of integers, or x is an odd number and is not a product of integers

(d) x is neither an even number nor a product of integers

14. Negate the following propositions

(a) All the triangles have the sum of their angle equal to 180°

(b) There is an integer between 4 and 8 inclusive

(c) All students live in the hostels

(d) For all x in S, $0 + x = 0$

(e) All differentiable functions are continuous

15. If $p \equiv$ It is about 6:45 PM

$q \equiv$ Sangam Train is about to depart, then find

(a) $p \wedge q$ (b) $p \vee q$ (c) $q \vee (\sim p)$

(d) $p \wedge (\sim q)$ (e) $\sim (p \wedge q)$ (f) $\sim (p \vee q)$

(g) $\sim p \Rightarrow q$ (h) $(\sim p \Rightarrow p)$

16. What are two types of quantifiers?

Answers

2. $X = \phi \equiv \forall\, x\,(x \notin X)$, $X = \cup \equiv \forall\, x\,(x \in X)$

11. (a) No (b) No (c) No (d) Yes

12. (a) Yes (b) No (c) No (d) Yes (e) Yes (f) Yes
(g) Yes (h) Yes

13. (a) $p \vee q$ (b) $p \wedge q$ (c) $p \wedge q \vee (\sim p \wedge \sim q)$ (d) $\phi \sim p \wedge \sim q$

14. (a) There is a triangle whose sum of angles $\neq 180°$
(b) All integers are not between 4 and 8 inclusive
(c) Some students do not leave in hostels
(d) There exists an x in S such that $0 + x \neq 0$
(e) Not all differentiable functions are continuous.

15. (a) It is about 6:45 p.m. and Sangam train is about of depart
(b) Either it is about 6:45 p.m. or Sangam train is about to depart
(c) Either Sangam Train is about to depart or it is not about 6:45 p.m.
(d) It is about to 6:45 p.m. and Sangam train is not about to depart
(e) It is false that it is about 6:45 p.m. and Sangam train is about to depart.
(f) Neither it is about to 6:45 p.m. nor Sangam train is about to depart.
(g) It is not about to 6:45 p.m., then Sangam train is about to depart
(h) If the Sangam train is not about to depart, then it is about to 6:45 p.m.

Objective Type Quesitons

Multiple Choice Questions

1. The proposition $P \wedge P$ is equivalent to
(a) 0 (b) p (c) $-p$ (d) none of these

2. The proposition $\sim (p \wedge q)$ is equivalent to
(a) $\sim p \vee \sim q$ (b) $\sim p \wedge \sim q$ (c) $p \wedge q$ (d) none of these

3. The number of rows in the truth table of a compound statement having n distinct primary or atomic statements is
(a) 2^n (b) n^2 (c) $\lfloor n$ (d) 2^8

4. Which of the following is a tautology
(a) $\sim[p \wedge (\sim p)]$ (b) $p \wedge p$ (c) $\sim p \wedge \sim p$ (d) $p \vee p$

5. Which of the following are tautologies
(a) $((p \vee q) \wedge q) \Leftrightarrow q$ (b) $(p \vee q) \wedge (\sim p) = q$
(c) $(p \vee q) \wedge p \Rightarrow q$ (d) $(p \vee (p \rightarrow q)) \rightarrow p$

6. $p \vee q$ is false when
(a) p is true, q is false (b) p is true, q is true
(c) p is false, q is true (d) p is false, q is false.