Mathematical Induction

What is induction?

- A method of proof
- It does not generate answers: it only can prove them
- Three parts:
  - Base case(s): show it is true for one element
  - Inductive hypothesis: assume it is true for any given element
    - **Must be clearly labeled!!!**
  - Show that if it true for the next highest element

- **Problem 1**
- Use mathematical induction to prove that **$1 + 2 + 3 + \ldots + n = n(n + 1) / 2$** for all positive integers n

Let the statement P (n) be

$1 + 2 + 3 + ... + n = n (n + 1) / 2$

STEP 1: We first show that p (1) is true.

Left Side = 1

Right Side = $1 (1 + 1) / 2 = 1$

Both sides of the statement are equal hence p (1) is true.

STEP 2: We now assume that p (k) is true

$1 + 2 + 3 + ... + k = k (k + 1) / 2$

and show that p (k + 1) is true by adding k + 1 to both sides of the above statement

$1 + 2 + 3 + ... + k + (k + 1) = k (k + 1) / 2 + (k + 1)$

$= (k + 1)(k / 2 + 1)$

$= (k + 1)(k + 2) / 2$

The last statement may be written as

$1 + 2 + 3 + ... + k + (k + 1) = (k + 1)(k + 2) / 2$

Which is the statement p(k + 1).

PROBLEM 2

- Use mathematical induction to prove that $\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2 (n + 1)^2 / 4}$ for all positive integers n.

$1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2 (n + 1)^2 / 4$

STEP 1: We first show that p (1) is true.

Left Side $= 1^3 = 1$

Right Side $= 1^2 (1 + 1)^2 / 4 = 1$

hence p (1) is true.

STEP 2: We now assume that p (k) is true

$1^3 + 2^3 + 3^3 + \ldots + k^3 = k^2 (k + 1)^2 / 4$

add $(k + 1)^3$ to both sides

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = k^2 (k + 1)^2 / 4 + (k + 1)^3$

factor $(k + 1)^2$ on the right side

$= (k + 1)^2 [k^2 / 4 + (k + 1)]$

set to common denominator and group

$= (k + 1)^2 [k^2 + 4 k + 4] / 4$

$= (k + 1)^2 [(k + 2)^2] / 4$

We have started from the statement P(k) and have shown that

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = (k + 1)^2 [(k + 2)^2] / 4$

Which is the statement P(k + 1).

- **Problem 3**
- Use mathematical induction to prove that $\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2 (n + 1)^2 / 4}$ for all positive integers n.

$1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2 (n + 1)^2 / 4$

STEP 1: We first show that p (1) is true.

Left Side = $1^3 = 1$

Right Side = $1^2 (1 + 1)^2 / 4 = 1$

hence p (1) is true.

STEP 2: We now assume that p (k) is true

$1^3 + 2^3 + 3^3 + \ldots + k^3 = k^2 (k + 1)^2 / 4$

add $(k + 1)^3$ to both sides

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = k^2 (k + 1)^2 / 4 + (k + 1)^3$

factor $(k + 1)^2$ on the right side

$= (k + 1)^2 [k^2 / 4 + (k + 1)]$

set to common denominator and group

$= (k + 1)^2 [k^2 + 4 k + 4] / 4$

$= (k + 1)^2 [(k + 2)^2] / 4$

We have started from the statement P(k) and have shown that

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = (k + 1)^2 [(k + 2)^2] / 4$

Which is the statement P(k + 1).

PROBLEM 4

- Prove that for any positive integer number n , $\mathbf{n^3 + 2n}$ is divisible by 3

Statement P(n) is defined by
$n^3 + 2n$ is divisible by 3

$n^3 + 2n$ is divisible by 3

STEP 1: We first show that p (1) is true. Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

3 is divisible by 3

hence p (1) is true.

STEP 2: We now assume that p (k) is true

$k^3 + 2k$ is divisible by 3

is equivalent to

$k^3 + 2k = 3M$, where M is a positive integer.

We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$

Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

Induction example

- Show that the sum of the first *n* odd integers is n^2
  - Example: If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, Show $$\forall n\, P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i - 1 == n^2$$
- Base case: Show that P(1) is true

$$P(1) = \sum_{i=1}^{1} 2(i) - 1 == 1^2$$
$$= 1 == 1$$

Induction example, continued

- Inductive hypothesis: assume true for *k*
  - Thus, we assume that P(*k*) is true, or that

$$\sum_{i=1}^{\boxed{k}} 2i-1 == \boxed{k}^2$$

  - Note: we don't yet know if this is true or not!

- Inductive step: show true for *k*+1
  - We want to show that:

$$\sum_{i=1}^{\boxed{k+1}} 2i-1 == \boxed{(k+1)}^2$$

Induction example, continued

- Recall the inductive hypothesis: $$\sum_{i=1}^{k} 2i-1 == k^2$$
- Proof of inductive step:

$$\sum_{i=1}^{k+1} 2i-1 == (k+1)^2$$

$$2(k+1)-1+\boxed{\sum_{i=1}^{k} 2i-1} == k^2+2k+1$$

$$2(k+1)-1+\boxed{k^2} == k^2+2k+1$$

$$k^2+2k+1 == k^2+2k+1$$

What did we show

- Base case: P(1)
- If P(*k*) was true, then P(*k*+1) is true
  - i.e., P(*k*) → P(*k*+1)

- We know it's true for P(1)
- Because of P(*k*) → P(*k*+1), if it's true for P(1), then it's true for P(2)
- Because of P(*k*) → P(*k*+1), if it's true for P(2), then it's true for P(3)
- Because of P(*k*) → P(*k*+1), if it's true for P(3), then it's true for P(4)
- Because of P(*k*) → P(*k*+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it is true for all possible values of *n*

- In other words, we showed that:

$$\left[\mathrm{P}(1) \wedge \forall k \left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)\right] \rightarrow \forall n\, \mathrm{P}(n)$$

The idea behind inductive proofs

- Show the base case
- Show the inductive hypothesis
- Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
- Show the inductive step

Second induction example

- Show the sum of the first n positive even integers is $n^2 + n$
- Rephrased: $$\forall n \, \mathrm{P}(n) \text{ where } \mathrm{P}(n) = \sum_{i=1}^{n} 2i == n^2 + n$$

- The three parts:
  - Base case
  - Inductive hypothesis
  - Inductive step

Second induction example, continued

- Base case: Show P(1):

$$P(1) = \sum_{i=1}^{1} 2(i) == 1^2 + 1$$
$$= 2 == 2$$

- Inductive hypothesis: Assume

$$P(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

- Inductive step: Show

$$P(k+1) = \sum_{i=1}^{k+1} 2i == (k+1)^2 + (k+1)$$

Second induction example, continued

- Recall our inductive hypothesis: $$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Notes on proofs by induction

- We manipulate the k+1 case to make part of it look like the k case
- We then replace that part with the other side of the k case

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Third induction example

Show $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

- Base case: *n* = 1

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$$

$$1^2 = \frac{6}{6}$$

$$1 = 1$$

- Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction example

- Inductive step: show $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$$

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Show: $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+\boxed{2})}{6}$$

- Base case: $n = 1$:

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+2)}{6}$$

$$1^2 = \frac{7}{6}$$

$$1 \neq \frac{7}{6}$$

- But let's continue anyway…
- Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Inductive step: show $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+2)}{6} = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+2) = (k+1)(k+2)(2k+4)$$

$$2k^3 + 10k^2 + 14k + 6 \neq 2k^3 + 10k^2 + 16k + 8$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Fourth induction example

- Rosen, question 14: show that $n! < n^n$ for all $n > 1$

- Base case: $n = 2$

  $2! < 2^2$

  $2 < 4$

- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

$$(k+1)! = (k+1)\boxed{k!} < (k+1)\boxed{k^k} < (k+1)(k+1)^k = (k+1)^{k+1}$$

Example 6

Prove that

$2.7^n + 3.5^n - 5$ is divisible by 24, for all $n \in \mathbf{N}$.

Let $P(n) : 2.7^n + 3.5^n - 5 = 24d$, when $d \in \mathbf{N}$

For $n = 1$,

$$
\begin{aligned}
\text{L.H.S} &= 2.7^1 + 3.5^1 - 5 \\
&= 2.7 + 3.5 - 5 \\
&= 14 + 15 - 5 \\
&= 24 \\
&= 24 \times 1 \\
&= \text{R.H.S},
\end{aligned}
$$

$\therefore$ $P(n)$ is true for $n = 1$

Assume P(k) is true

$2.7^k + 3.5^k - 5 = 24m$, when $m \in \mathbf{N}$ ...(1)

We will prove that P(k + 1) is true.

$$\text{L.H.S} = 2.7^{k+1} + 3.5^{k+1} - 5$$

$$= 2.7^k . 7^1 + 3.5^k . 5^1 - 5$$

$$= 7. (2.7^k) + 5 . 3.5^k - 5$$

$$\boxed{\begin{aligned} &\textit{From (1): } 2.7^k + 3.5^k - 5 = 24m \\ &\qquad\qquad 2.7^k = 24m - 3.5^k + 5 \end{aligned}}$$

$$= 7 [24m - 3.5^k + 5] + 15.5^k - 5$$

$$= 7 \times 24m - (7 \times 3). 5^k + (7.5) + 15.5^k - 5$$

$$= 7 \times 24m - 21. 5^k + 35 + 15.5^k - 5$$

$$= 7 \times 24m - 21. 5^k + 15.5^k + 35 - 5$$

$$= 7 \times 24m - 6.5^k + 30$$

$= 7 \times 24m - 6\ (5^k - 5)$

$(5^k - 5)$ is a multiple of 4

$= 7 \times 24m - 6\ (4p)$

Here p is a natural number

$= 7 \times 24m - 24p$

$= 24\ (7m - p)$

$= 24 \times r;$

where $r = 7m - p$, is some natural number.

<u>*Rough*</u>

$5^k - 5$

For $k = 1$,

$5^1 - 5 = 0 = 4 \times 0$

For $k = 2$,

$5^2 - 5 = 25 - 5 = 20 = 4 \times 5$

For $k = 3$,

$5^3 - 5 = 125 - 5 = 120 = 4 \times 30$

$\therefore$ P(k + 1) is true whenever P(k) is true.

$\therefore$By the principle of mathematical induction, P(n) is true for n, where n is a natural number

Prove that $1^2 + 2^2 + \ldots + n^2 > \frac{n^3}{3}$, $n \in \mathbf{N}$

Let $P(n) : 1^2 + 2^2 + \ldots + n^2 > \frac{n^3}{3}$, $n \in \mathbf{N}$

For $n = 1$

L.H.S $= 1^2 = 1$

R.H.S $= \frac{1^3}{3} = \frac{1}{3}$

Since $1 > \frac{1}{3}$

L.H.S > R.H.S

$\therefore$ P(n) is true for $n = 1$

Assume P(k) is true

$$P(k) : 1^2 + 2^2 + \dots + k^2 > \frac{k^3}{3} \qquad \dots(1)$$

We will prove that P(k + 1) is true.

$$\text{L.H.S} = 1^2 + 2^2 + 3^2 + \dots + (k + 1)^2$$

$$\text{R.H.S} = \frac{(k+1)^3}{3}$$

L.H.S

$= 1^2 + 2^2 + 3^2 + \dots + (k+1)^2$

$= 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$

$= (1^2 + 2^2 + 3^2 + \dots + k^2) + (k+1)^2$

Using (1): $1^2 + 2^2 + \dots + k^2 > \frac{k^3}{3}$

$> \frac{k^3}{3} + (k+1)^2$

$> \frac{k^3 + 3(k+1)^2}{3}$

$> \frac{1}{3}\ [k^3 + 3(k^2 + 2k + 1)]$

$> \frac{1}{3}\ [k^3 + 3k^2 + 6k + 3]$

$> \frac{1}{3}\ [(k^3 + 1 + 3k^2 + 3k) + (3k+2)]$

$> \frac{1}{3}\ (k^3 + 1 + 3k^2 + 3k) + \frac{1}{3}(3k+2)$

As $\frac{1}{3}(3k+2)$ *is a positive quantity*

$> \frac{1}{3}\ \{k^3 + 1 + 3k^2 + 3k\}$

R.H.S

$= \frac{(k+1)^3}{3}$

Using $(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$= \frac{1}{3}(k^3 + 1^3 + 3k + 3k^2)$

$= \frac{1}{3}(k^3 + 1 + 3k + 3k^2)$

L.H.S > R.H.S

$\therefore$ P(k + 1) is true whenever P(k) is true.

Example 8

Prove the rule of exponents $(ab)^n = a^n b^n$ by using principle of mathematical induction for every natural number.

Let $P(n) : (ab)^n = a^n b^n$.

For $n = 1$,

L.H.S $= (ab)^1 = ab$

R.H.S $= a^1 b^1 = a \times b = ab$

Thus, L.H.S. = R.H.S ,

$\therefore$ P(n) is true for $n = 1$

Assuming P(k) is true

$P(k) : (ab)^k = a^k b^k$ ...(1)

We will prove that P(k + 1) is true.

$R.H.S = a^{k+1} b^{k+1}$

$L.H.S = (ab)^{k+1}$

L.H.S

$= (ab)^{k+1}$

$= (ab)^k (ab)^1$

$= (a^k b^k) (a^1 b^1)$

$= (a^k . a^1) (b^k . b^1)$

$= a^{k+1} . b^{k+1}$

R.H.S

$= a^{k+1} b^{k+1}$

L.H.S = R.H.S

∴ P(k + 1) is true whenever P(k) is true.

$7^{2n} + 2^{3n-3} \cdot 3^{n-1}$ is divisible by 25

Let $P(n): 7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25, $\forall\, n \in N$.

$\therefore\ P(1): 7^2 + 2^0 \cdot 3^0 = 50$ which is divisible by 25.

$\Rightarrow\ P(1)$ is true.

Let $P(k)$ be true. *ie*,

$$7^{2k} + (2^{3k-3})(3^{k-1}) \text{ is divisible by 25.}$$

$$\Rightarrow\quad 7^{2k} + (2^{3k-3})(3^{k-1}) = 25m, \text{ where } m \in Z$$

$$\Rightarrow\quad (2^{3k-3}) \cdot (3^{k-1}) = 25m - 7^{2k} \qquad \ldots\text{(i)}$$

To show $P(k+1)$ is true.

Now, $7^{2(k+1)} + 2^{3(k+1)-3} \cdot 3^{k+1-1}$

$= 7^{2k} \cdot 7^2 + 2^3 \cdot 3 \cdot 2^{3k-3} \cdot 3^{k-1}$

$= 49 \cdot 7^{2k} + 24 \cdot (25m - 7^{2k})$ [from Eq. (i)]

$= 49 \cdot 7^{2k} + 24 \cdot (25m) - 24 \cdot 7^{2k}$

$= 25(7^{2k} + 24m)$, which is divisible by 25.

.'. P (k + 1) is also true, whenever P (k) is true

Hence, by mathematical induction P (n) is true for all n $\in$ N.

Prove by induction : For n ≥ 1, 8 n − 3 n is divisible by 5.

1. Basis of induction :

For $n = 1$ $\quad 8^1 - 3^1 = 5$

$= 5 \cdot 1$

Obviously a multiple of 5.

$\therefore$ P(1) is true.

2. Induction step : Assume that, P(k) in true.
i.e. $8^k - 3^k$ is multiple of 5 say 5 r

i.e. $\quad 8^k - 3^k = 5r$

where r is an integer

Then we have,

$$8^{k+1} - 3^{k+1} = 8^k \cdot 8 - 3^k \cdot 3$$

$$= 8^k \cdot (5 + 3) - 3^k \cdot 3$$

$$= 8^k \cdot 5 + (8^k \cdot 3 - 3^k \cdot 3)$$

$$= 8^k \cdot 5 + 3(8^k - 3^k)$$

Obviously $8^k \cdot 5$ is multiple of 5 and also $8^k - 3^k$ is multiple of 5.

Therefore, $8^{k+1} - 3^{k+1}$ is multiple of 5.

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.