Functions

Unit 1

Discrete Structures and Theory of Logic

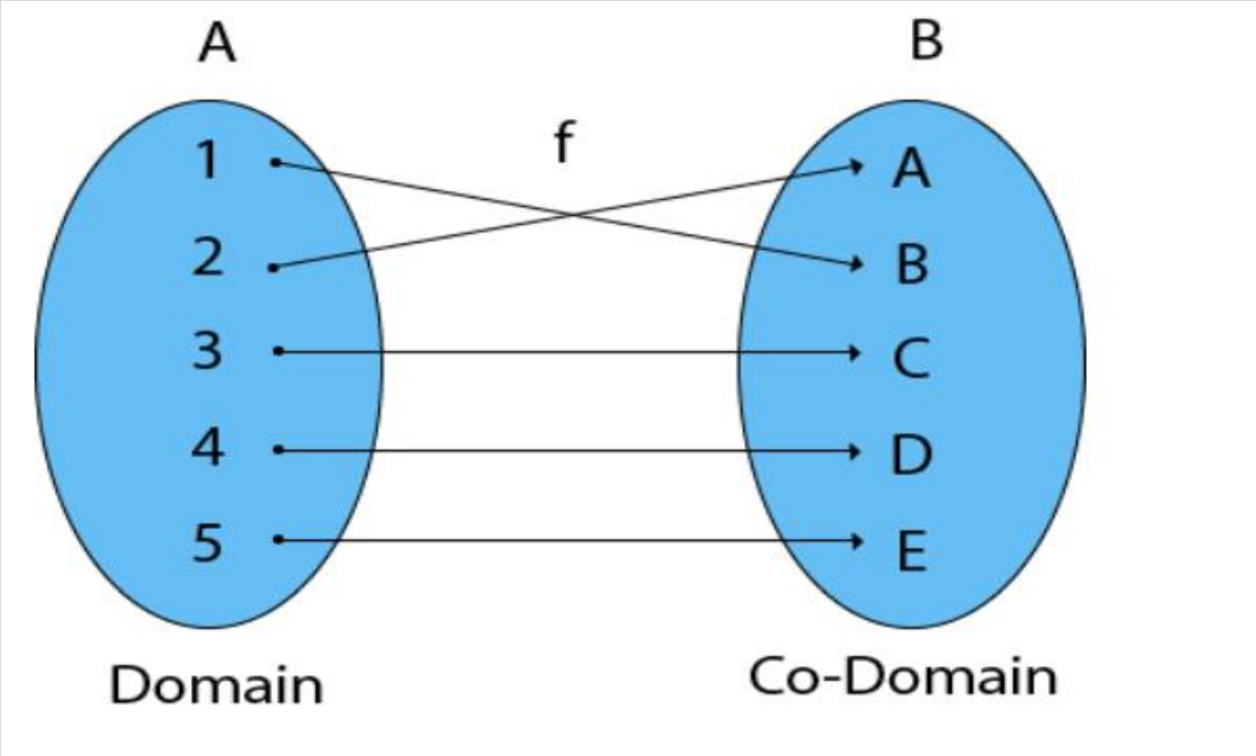
What is Function?

It is a mapping in which every element of set A is uniquely associated at the element with set B.

The set of A is called **Domain** of a function and set of B is called **Co domain**.

Points:

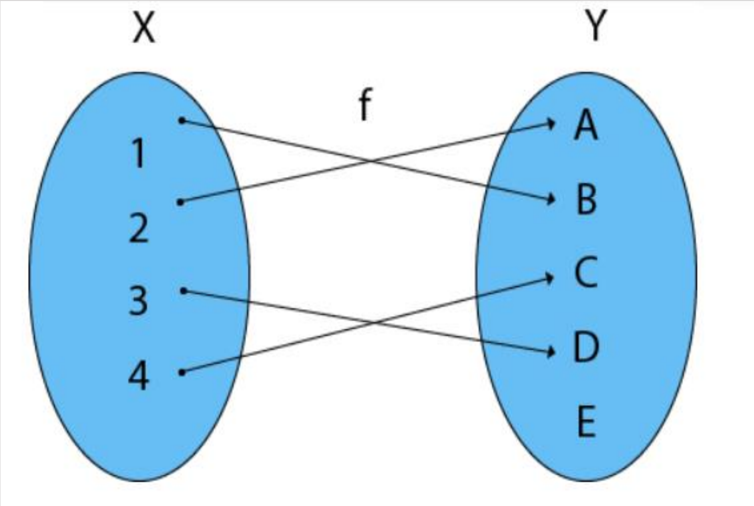
* There may be some elements of set B which are not associated to any element of the set A.
* Each element of set A must be associated to one and only one element of set B.



Domain of a Function: Let f be a function from P to Q. The set P is called the domain of the function f.
Co-Domain of a Function: Let f be a function from P to Q. The set Q is called Co-domain of the function f.
Range of a Function: The range of a function is the set of picture of its domain. In other words, we can say it is a subset of its co-domain. It is denoted as f (domain).

Example: Find the Domain, Co-Domain, and Range of function.

1. Let x = {1, 2, 3, 4}
2. y = {a, b, c, d, e}
3. f = {(1, b), (2, a), (3, d), (4, c) }

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Domain of function: {1, 2, 3, 4}
Range of function: {a, b, c, d}
Co-Domain of function: {a, b, c, d, e}
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Representation of a Function

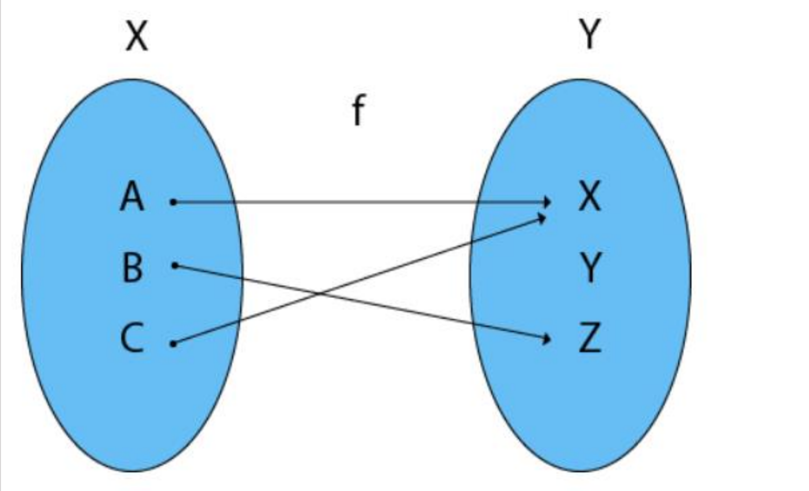
- The two sets P and Q are represented by two circles.

The function f: $P \rightarrow Q$ is represented by a collection of arrows joining the points which represent the elements of P and corresponds elements of Q.

Example:

1.Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ and f: $X \rightarrow Y$ such that

2. $f = \{(a, x), (b, z), (c, x)\}$



- **Example2:** Let X = {x, y, z, k} and Y = {1, 2, 3, 4}. Determine which of the following functions. Give reasons if it is not. Find range if it is a function.

1. f = {(x, 1), (y, 2), (z, 3), (k, 4)
2. g = {(x, 1), (y, 1), (k, 4)
3. h = {(x, 1), (x, 2), (x, 3), (x, 4)
4. l = {(x, 1), (y, 1), (z, 1), (k, 1)}
5. d = {(x, 1), (y, 2), (y, 3), (z, 4), (z, 4)}.

Solution:

1. It is a function. Range (f) = {1, 2, 3, 4}
2. It is not a function because every element of X does not relate with some element of Y i.e., Z is not related with any element of Y.
3. h is not a function because h (x) = {1, 2, 3, 4} i.e., element x has more than one image in set Y.
4. d is not a function because d (y) = {2, 3} i.e., element y has more than image in set Y.

Functions vs Relation

Function

A ***function*** is a set of ordered pairs (x,y) that shows a relationship where there is only one output for every input. In other words, for every value of x, there is only one value for y.

Relation

A ***relation*** is any set of ordered pairs (x,y). A relation has more than one output for at least one input.

A relation that is a function

x	y
0	0
1	1
2	2
3	3

A relation that is not a function

x	y
0	0
1	1
2	2
2	1

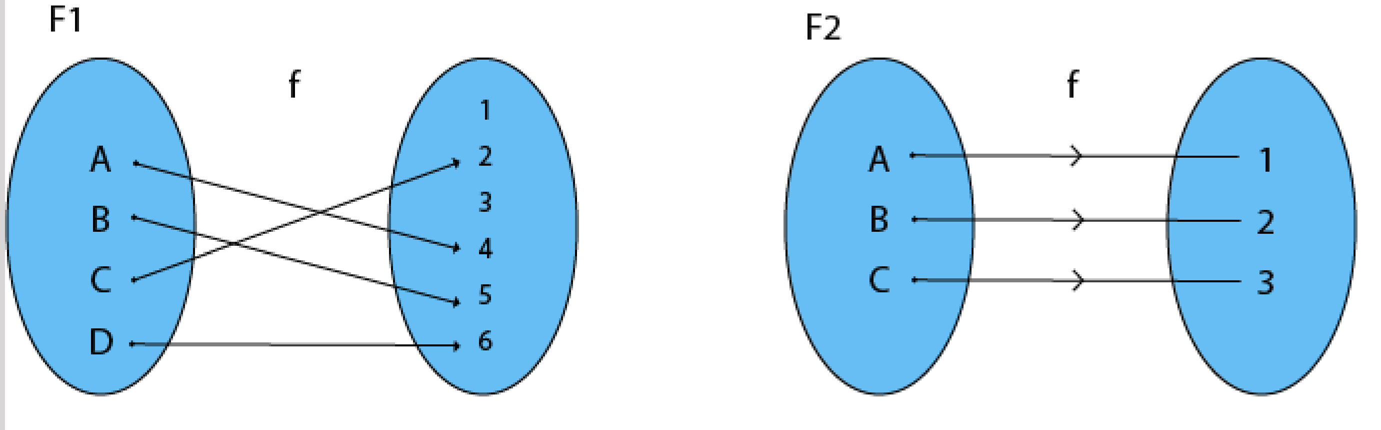
Example A
Determine if the following relation is a function.

x	y
−3.5	−3.6
−1	−1
4	3.6
7.8	7.2

Solution A) The relation is a function because there is only one value of y for every value of x.

Types of Functions

1. Injective (One-to-One) Functions: A function in which one element of Domain Set is connected to one element of Co-Domain Set.

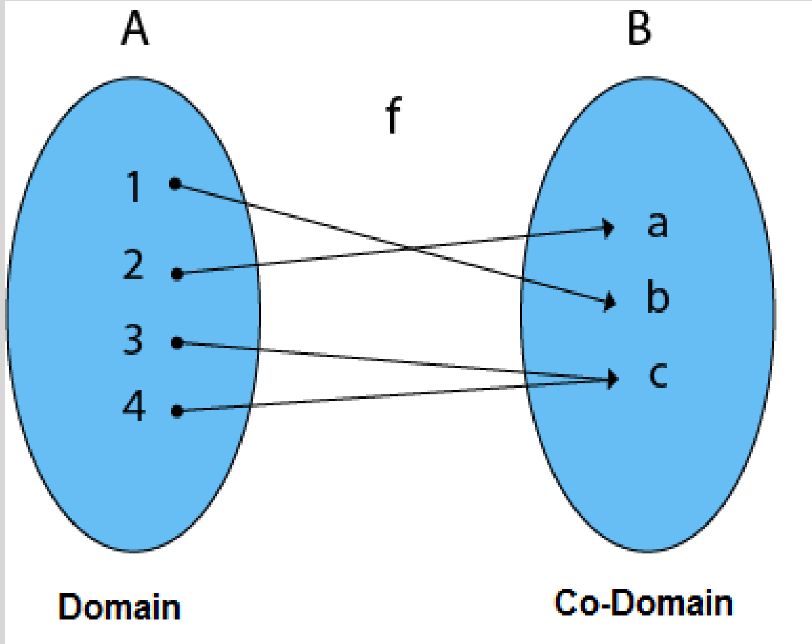


F1 and F2 show one to one Function

2. Surjective (Onto) Functions: A function in which every element of Co-Domain Set has one pre-image.

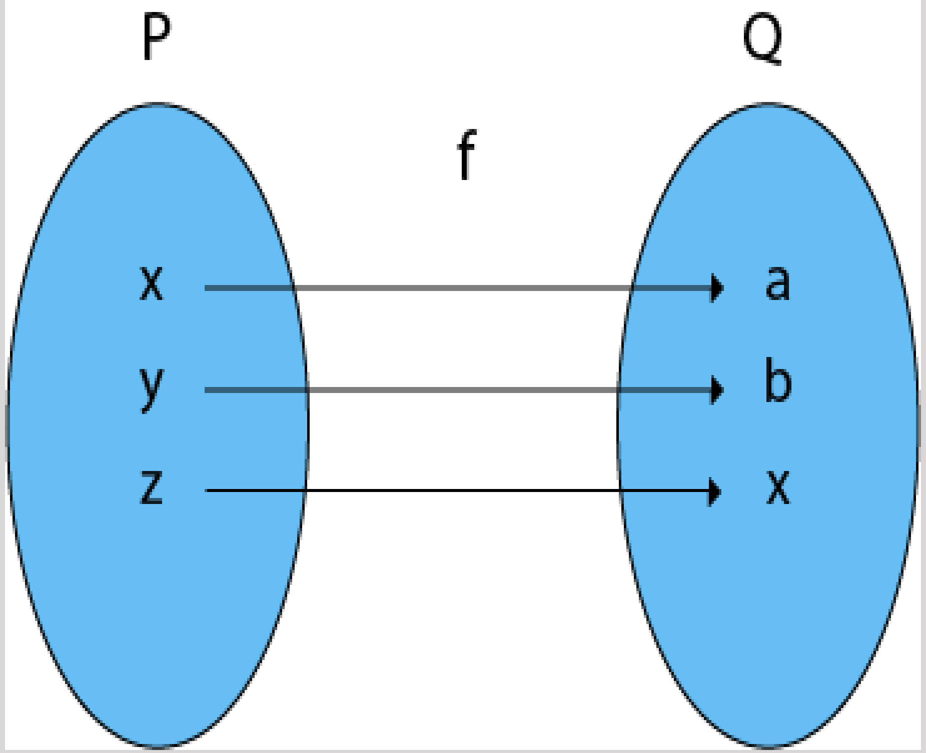
Example: Consider, A = {1, 2, 3, 4}, B = {a, b, c} and f = {(1, b), (2, a), (3, c), (4, c)}.

It is a Surjective Function, as every element of B is the image of some A



Note: In an Onto Function, Range is equal to Co-Domain.

3. Bijective (One-to-One Onto) Functions: A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.



Example:

1.Consider $P = \{x, y, z\}$

$Q = \{a, b, c\}$and $f: P \rightarrow Q$ such that

$f = \{(x, a), (y, b), (z, c)\}$

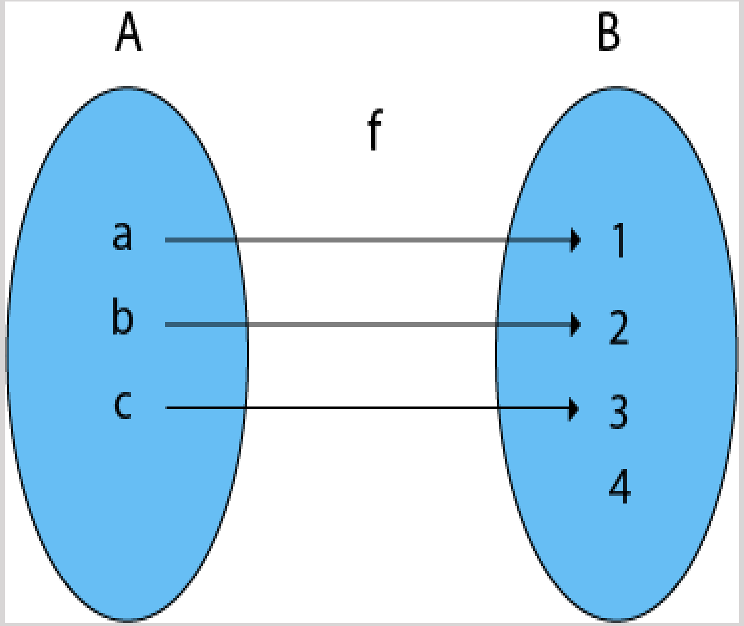
The f is a one-to-one function and also it is onto. So it is a bijective function.

4. Into Functions: A function in which there must be an element of co-domain Y does not have a pre-image in domain X.

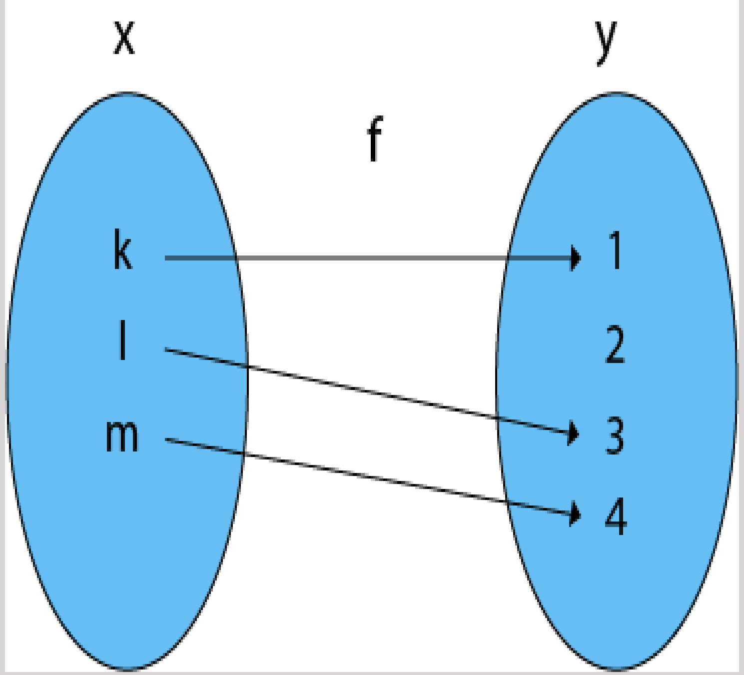
Example:

1. Consider, A = {a, b, c}
2. B = {1, 2, 3, 4} and f: A → B such that
3. f = {(a, 1), (b, 2), (c, 3)}
4. In the function f, the range i.e., {1, 2, 3} ≠ co-domain of Y i.e., {1, 2, 3, 4}

Therefore, it is an into function



5. One-One Into Functions: Let f: X → Y. The function f is called one-one into function if different elements of X have different unique images of Y.



Example:

1. Consider, X = {k, l, m}
2. Y = {1, 2, 3, 4} and f: X → Y such that
3. f = {(k, 1), (l, 3), (m, 4)}

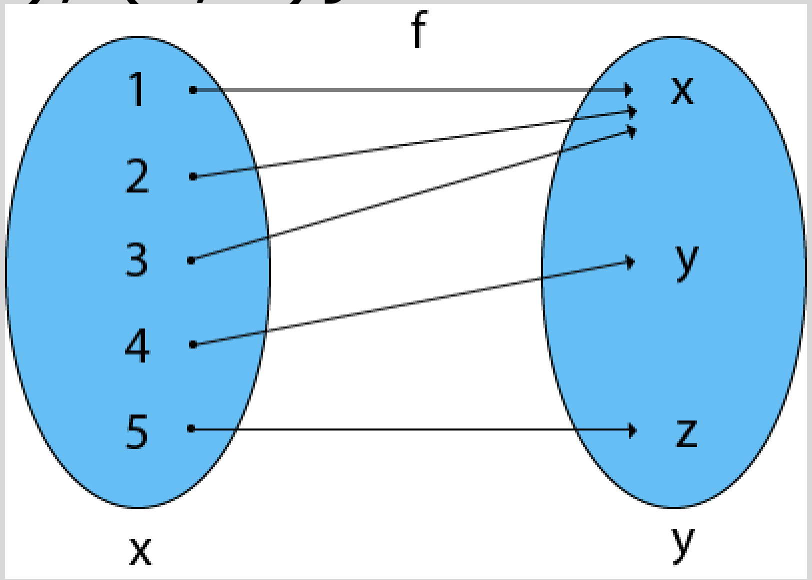
The function f is a one-one into function

6. Many-One Functions: Let f: X → Y. The function f is said to be many-one functions if there exist two or more than two different elements in X having the same image in Y.

Example:

1. Consider X = {1, 2, 3, 4, 5}
2. Y = {x, y, z} and f: X → Y such that
3. f = {(1, x), (2, x), (3, x), (4, y), (5, z)}

The function f is a many-one function

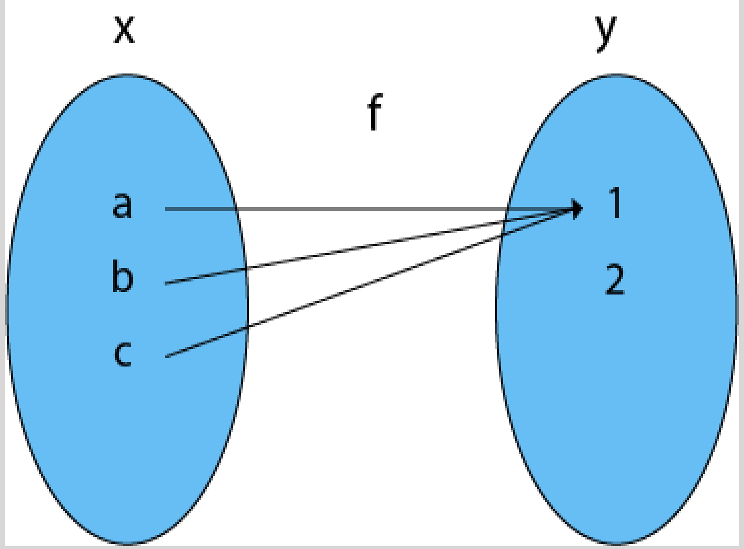


7. Many-One Into Functions: Let f: X → Y. The function f is called the many-one function if and only if is both many one and into function.

Example:

1. Consider X = {a, b, c}
2. Y = {1, 2} and f: X → Y such that
3. f = {(a, 1), (b, 1), (c, 1)}

As the function f is a many-one and into, so it is a many-one into function.

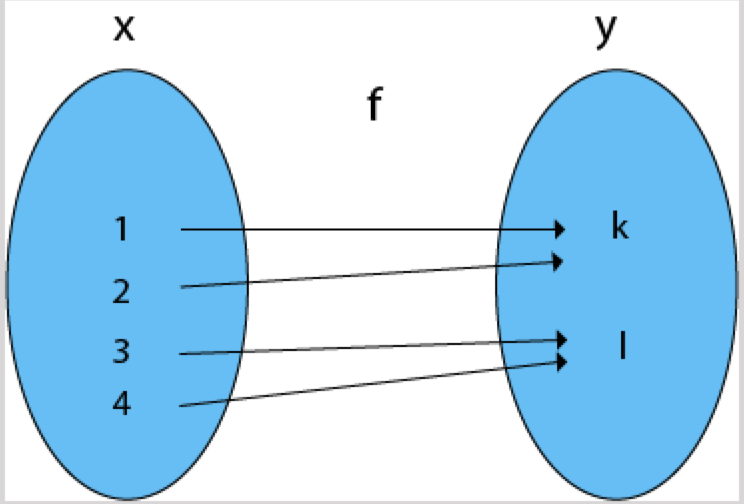


8. Many-One Onto Functions: Let f: X → Y. The function f is called many-one onto function if and only if is both many one and onto.

Example:

1. Consider X = {1, 2, 3, 4}
2. Y = {k, l} and f: X → Y such that
3. f = {(1, k), (2, k), (3, l), (4, l)}

The function f is a many-one (as the two elements have the same image in Y) and it is onto (as every element of Y is the image of some element X). So, it is many-one onto function



Identity Functions

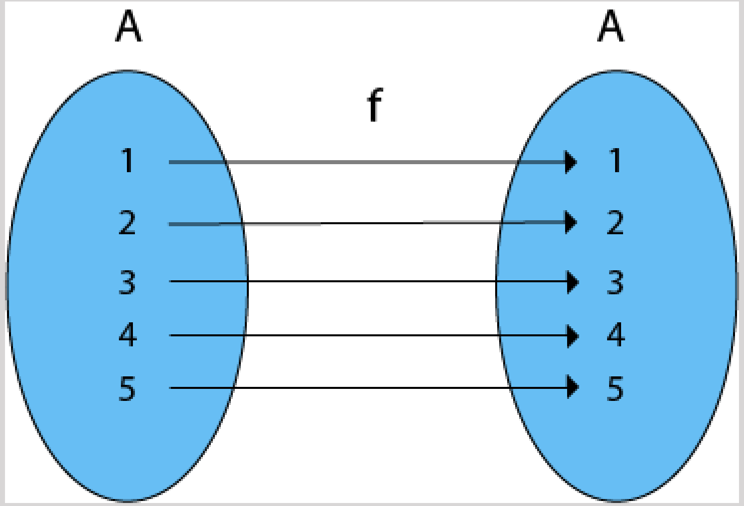
The function f is called the identity function if each element of set A has an image on itself i.e. f (a) = a ∀ a ∈ A.

It is denoted by I.

Example:

1. Consider, A = {1, 2, 3, 4, 5} and f: A → A such that
2. f = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)}.

The function f is an identity function as each element of A is mapped onto itself. The function f is a one-one and onto



Invertible (Inverse) Functions

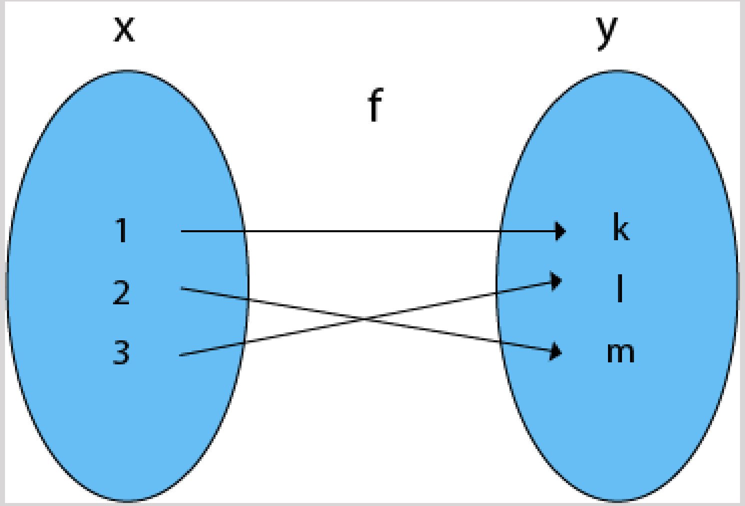
A function f: X → Y is invertible if and only if it is a bijective function.
Consider the bijective (one to one onto) function f: X → Y. As f is a one to one, therefore, each element of X corresponds to a distinct element of Y. As f is onto, there is no element of Y which is not the image of any element of X, i.e., range = co-domain Y.

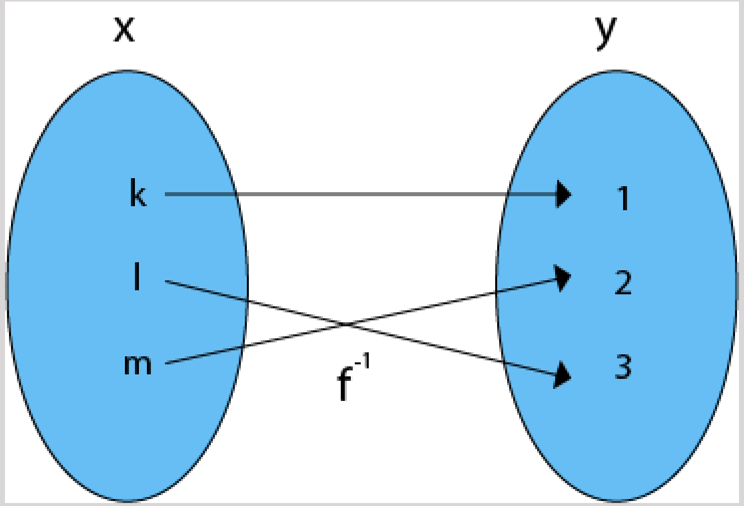
The inverse function for f exists if f^{-1} is a function from Y to X.

Example:

1. Consider, X = {1, 2, 3}
2. Y = {k, l, m} and f: X→Y such that
3. f = {(1, k), (2, m), (3, l)}

The inverse function of f is shown in fig:





Example :1 Consider the function $f : \{1,2,3,4,5,6\} \to \{a,b,c,d\}$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & a & b & b & b & c \end{pmatrix}.$$

Find $f(\{1,2,3\})$, $f^{-1}(\{a,b\})$, and $f^{-1}(d)$.

▼ Solution

$f(\{1,2,3\}) = \{a,b\}$ since a and b are the elements in the codomain to which f sends 1 and 2.

$f^{-1}(\{a,b\}) = \{1,2,3,4,5\}$ since these are exactly the elements that f sends to a and b.

$f^{-1}(d) = \emptyset$ since d is not in the range of f.

Example : 2

Consider the function $f : \{1,2,3,4\} \to \{1,2,3,4\}$ given by

$$f(n) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 4 \end{pmatrix}.$$

a. Find $f(1)$.

b. Find an element n in the domain such that $f(n) = 1$.

c. Find an element n of the domain such that $f(n) = n$.

d. Find an element of the codomain that is not in the range.

Solution

a. $f(1) = 4$, since 4 is the number below 1 in the two-line notation.

b. Such an n is $n = 2$, since $f(2) = 1$. Note that 2 is above a 1 in the notation.

c. $n = 4$ has this property. We say that 4 is a fixed point of f. Not all functions have such a point.

d. Such an element is 2 (in fact, that is the only element in the codomain that is not in the range). In other words, 2 is not the image of any element under f; nothing is sent to 2.

Example : 3

The following functions all have $\{1,2,3,4,5\}$ as both their domain and codomain. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

a. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$. ?

b. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$. ?

c. $f(x) = 6 - x$. ?

d. $f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ (x+1)/2 & \text{if } x \text{ is odd} \end{cases}$. ?

Solution

a. This is neither injective nor surjective. It is not injective because more than one element from the domain has 3 as its image. It is not surjective because there are elements of the codomain (1, 2, 4, and 5) that are not images of anything from the domain.

b. This is a bijection. Every element in the codomain is the image of *exactly* one element of the domain.

c. This is a bijection. Note that we can write this function in two line notation as $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$.

d. In two line notation, this function is $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 3 \end{pmatrix}$. From this we can quickly see it is neither injective (for example, 1 is the image of both 1 and 2) nor surjective (for example, 4 is not the image of anything).

Example : 4

Consider the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ given by the table below:

x	1	2	3	4	5
$f(x)$	3	2	4	1	2

a. Is f injective? Explain.

b. Is f surjective? Explain.

c. Write the function using two-line notation.

▾ Solution

a. f is not injective, since $f(2) = f(5)$; two different inputs have the same output.

b. f is surjective, since every element of the codomain is an element of the range.

c. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 2 \end{pmatrix}$.

Operations on Functions

Functions with overlapping domains can be added, subtracted, multiplied and divided. If f(x)f(x) and g(x)g(x) are two functions, then for all xx in the domain of both functions the sum, difference, product and quotient are defined as follows.

$$(f+g)(x)=f(x)+g(x)$$
$$(f-g)(x)=f(x)-g(x)$$
$$(fg)(x)=f(x)\times g(x)$$
$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)},g(x)\neq 0$$

Operation	Definition	Example: $f(x) = 5x$, $g(x) = x + 2$
Add	$h(x) = f(x) + g(x)$	$h(x) = 5x + (x + 2) = \mathbf{6x + 2}$
Subtract	$h(x) = f(x) - g(x)$	$h(x) = 5x - (x + 2) = \mathbf{4x - 2}$
Multiply	$h(x) = f(x) \cdot g(x)$	$h(x) = 5x(x+2) = \mathbf{5x^2 + 10x}$
Divide	$h(x) = \dfrac{f(x)}{g(x)}$	$h(x) = \dfrac{5x}{x+2}$

Example :

Let $f(x) = 2x + 1$ and $g(x) = x^2 - 4$

Find $(f + g)(x), (f - g)(x), (fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x) \\
&= (2x+1)+\left(x^2-4\right) \\
&= x^2+2x-3
\end{aligned}
$$

$$
\begin{aligned}
(f-g)(x) &= f(x)-g(x) \\
&= (2x+1)-\left(x^2-4\right) \\
&= -x^2+2x+5
\end{aligned}
$$

$$
\begin{aligned}
(fg)(x) &= f(x)\times g(x) \\
&= (2x+1)\left(x^2-4\right) \\
&= 2x^3+x^2-8x-4
\end{aligned}
$$

$$
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x+1}{x^2-4}, x \neq \pm 2
$$

Composition of Functions

The function whose value at x is $f(g(x))$ is called the **composite** of the functions f and g. The operation that combines f and g to produce the composite is called **composition** .

Notation: $(f \circ g)(x)$ or $f(g(x))$

The domain of $f(g(x))$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f .

Example:

If $f(x) = \sqrt{x}$ and $g(x) = 3x - 5$, then find $f(g(4))$.

Substituting 4 for x in the definition of the function g , $g(4) = 3(4) - 5 = 7$. Then, $f(g(4)) = f(7)$

$f(7) = \sqrt{7}$

Example 1:

Let $f(x) = x^2$ and $g(x) = x - 3$. Find $f(g(x))$.

$$\begin{aligned} f(g(x)) &= f(x-3) \\ &= (x-3)^2 \\ &= x^2 - 6x + 9 \end{aligned}$$

Example 2:

Let $f(x) = 2x - 1$ and $g(x) = x + 2$. Find $f(g(x))$.

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= 2(x+2) - 1 \\ &= 2x + 3 \end{aligned}$$

Composition is not commutative. That is, $(f \circ g)(x)$ is usually different from $(g \circ f)(x)$.

Example 3:

Let $f(x) = 3x + 1$ and $g(x) = 2x - 3$.

Find $f(g(x))$ and $g(f(x))$.

$$\begin{aligned} f(g(x)) &= f(2x - 3) \\ &= 3(2x - 3) + 1 \\ &= 6x - 8 \\ g(f(x)) &= f(3x + 1) \\ &= 2(3x + 1) - 3 \\ &= 6x - 1 \end{aligned}$$

Since $6x - 8 \neq 2x - 1$, $f(g(x)) \neq g(f(x))$.

Recursively Defined Functions

A recursive definition has two parts:

1. Definition of the smallest argument (usually f (0) or f (1)).
2. Definition of f (n), given f (n - 1), f (n - 2), etc.

- Here is an example of a recursively defined function:

$$f(0) = 5$$
$$f(n) = f(n-1) + 2$$

We can calculate the values of this function:

This recursively defined function is equivalent to the explicitly defined function $f(n) = 2n + 5$. However, the recursive function is defined only for nonnegative integers.

$f(0)=5$

$f(1)=f(0) + 2 = 5 + 2 = 7$

$f(2)=f(1) + 2 = 7 + 2 = 9$

$f(3)=f(2) + 2 = 9 + 2 = 11$

...

Here is another example of a recursively defined function:

$$\begin{cases} f(0) = 0 \\ f(n) = f(n-1) + 2n - 1 \end{cases}$$

The values of this function are:

$f(0) = 0$
$f(1) = f(0) + (2)(1) - 1 = 0 + 2 - 1 = 1$
$f(2) = f(1) + (2)(2) - 1 = 1 + 4 - 1 = 4$
$f(3) = f(2) + (2)(3) - 1 = 4 + 6 - 1 = 9$
$f(4) = f(3) + (2)(4) - 1 = 9 + 8 - 1 = 16$
...

This recursively defined function is equivalent to the explicitly defined function $f(n) = n^2$. Again, the recursive function is defined only for nonnegative integers.

Here is one more example of a recursively defined function: $\begin{cases} f(0) = 1 \\ f(n) = n \cdot f(n-1) \end{cases}$

The values of this function are:

f (0) = 1
f (1) = 1ƒ*f* (0) = 1ƒ1 = 1
f (2) = 2ƒ*f* (1) = 2ƒ1 = 2
f (3) = 3ƒ*f* (2) = 3ƒ2 = 6
f (4) = 4ƒ*f* (3) = 4ƒ6 = 24
f (5) = 5ƒ*f* (4) = 5ƒ24 = 120
...

This is the recursive definition of the factorial function, $F(n) = n!$.

Not all recursively defined functions have an explicit definition.