4.1 Introduction

- Mathematical induction is a powerful technique in applied mathematics especially in number theory, where many properties of natural numbers are proved by this method.
- In day to day life, we are often required to generalise a particular pattern for the prediction purpose. The generalisation is achieved by using a statement involving a variable as natural number.
- Mathematical induction is very useful technique or tool for the programmers to check whether a program statement is loop invariant or not.
- There are two principles of mathematical induction :
  1) First principle of mathematical induction.
  2) Second principle of mathematical induction.

4.2 First Principle of Mathematical Induction Statement

- Let P(n) be a statement involving a natural number $n \geq n_0$ such that,
  1) If P(n) is true for $n = n_0$ where $n_0 \in N$ and
  2) Assume that P(k) is true for $k \geq n_0$ we prove that P(k+1) is also true,
- Then P(n) is true for all natural numbers $n \geq n_0$.
- Step 1 is called as the **basis of induction.**
- Step 2 is called as the **induction step.**

4.3 Second Principle of Mathematical Induction Statement

AKTU : 2011-12, 2012-13, 2014-15, 2015-16, 2013-14, 2016-17

- Let P(n) be a statement involving a natural number $n \geq n_0$ such that,
  1) If P(n) is true for $n = n_0$ where $n_0 \in N$ and
  2) Assume that P (n) is true for $n_0 < n \leq k$ i.e. $P(n_0+1)(n_0+2)$............P(k) are true.

  we prove that P(k + 1) is true,
- Then P(n) is true for all natural numbers $n \geq n_0$.

Ex.4.3.1 : Prove by mathematical induction for $n \geq 1$.

$$1^2+2^2+3^2+....+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Sol. : Let P(n) the given statement

1. Basis of induction

For $n_0 = 1$ L.H.S. = 1

$$\text{R.H.S.} = \frac{1(2)(3)}{6} = 1 \Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step

Assume that, P(k) is true

i.e. $$1^2+2^2+3^2+...+k^2 = \frac{k(k+1)(2k+1)}{6} \quad ...(1)$$

Then we have, consider

$$1^2+2^2+3^2+...+k^2\ (k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} \quad ... \text{(Using 1)}$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Ex.4.3.2 : Show by induction that , $n \geq 1$

$$1^2+3^2+5^2+..(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Sol. : Let P(n) be the given statement

1. Basis of induction :

For n = 1, L.H.S. = $1^2 = 1$,

$$\text{R.H.S.} = \frac{1(1)(3)}{3} = 1$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true

2. Induction step : Assume that P(k) is true.

i.e. $$1^2+3^2+5^2+\ldots+(2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \ldots (1)$$

Hence

$$[1^2+3^2+5^2+\ldots+(2k-1)^2]+(2k+1)^2 = \frac{k(2k-1)(2k+1)}{3}+(2k+1)^2 \quad \ldots \text{(Using 1)}$$

$$= \frac{(2k+1)}{3}[2k^2-k+3(2k+1)]$$

$$= \frac{(2k+1)}{3}[2k^2+5k+3]$$

$$= \frac{(2k+1)}{3}[2k^2+2k+3k+3]$$

$$= \frac{(2k+1)}{3}[(2k+3)(k+1)]$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

Hence assuming P(k) is true P(k + 1) is also true. Therefore by mathematical induction P(n) in true for all $n \geq 1$.

Ex.4.3.3 : Show that $1^3+2^3+3^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$

Sol. : Let P(n) be the given statement.

1. Basis of induction :

For n = 1,

$$\text{L.H.S.} = 1,$$

$$\text{R.H.S.} = \frac{1(1+1)^2}{4} = 1$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $$1^3+2^3+3^3+\ldots+k^3 = \frac{k^2(k+1)^2}{4} \quad \ldots (1)$$

Then we have

$$(1^3+2^3+3^3+\ldots+k^3)+(k+1)^3 = (1+2+3+\ldots+k)^2+(k+1)^3 \quad \text{(Using 1)}$$

$$= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3$$

$$= (k+1)^2\left[\frac{k^2}{4}+k+1\right]$$

$$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right] = \frac{(k+1)^2\ (k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k+1)^2\ (k+2)^2}{4}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Ex.4.3.4 : Show that $\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\ldots+\frac{n^2}{(2n-1)\ (2n+1)} = \frac{n\ (n+1)}{2(2n+1)}$

Sol. : Let P(n) be the given statement.

1. Basis of induction : For n = 1

We have,

$$\text{L.H.S.} = \frac{1}{1.3} = \frac{1}{3}$$

$$\text{R.H.S.} = \frac{1\ (2)}{1(3)} = \frac{1}{3}$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step : Assume that P(k) is true

i.e. $\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\cdots\frac{k^2}{(2k-1)\ (2k+1)} = \frac{k\ (k+1)}{2\ (2k+1)}$ ... (1)

Then we have,

$$\left[\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\ldots+\frac{k^2}{(2k-1)\ (2k+1)}\right]+\frac{(k+1)^2}{[2\ (k+1)-1][2\ (k+1)+1]}$$

$$= \frac{k\ (k+1)}{2\ (2k+1)}+\frac{(k+1)^2}{(2k+1)\ (2k+3)}$$ ... (Using 1

$$= \frac{(k+1)}{(2k+1)}\left[\frac{k(2k+3)+2\ (k+1)}{2\ (2k+3)}\right] = \frac{(k+1)}{2k+1}\left[\frac{2k^2+5k+2}{2\ (2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k^2+4k+k+2}{2\ (2k+3)}\right] = \frac{(k+1)}{(2k+1)}\left[\frac{2k\ (k+2)+1\ (k+2)}{2\ (2k+3)}\right]$$

$$= \frac{(k+1)\ (k+2)}{2\ (2k+3)} = \frac{(k+1)\ [(k+1)+1]}{2\ [2(k+1)+1]}$$

$\therefore$ P(k + 1) is true.

Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Ex.4.3.5 : Show that a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\ (n+1)} = \frac{n}{n+1}$

b) Show that $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)\ (2n+1)} = \frac{n}{2n+1}$

c) Show that $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-1)\ (3n+1)} = \frac{n}{3n+1}$

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Sol. : Let P(n) be the given statement,

a) 1. Basis of induction :

For n = 1 L.H.S. $= \frac{1}{1.2} = \frac{1}{2}$

R.H.S. $= \frac{1}{1+1} = \frac{1}{2}$

L.H.S. = R.H.S. Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)} = \frac{k}{k+1} \qquad \ldots (1)$$

Then we have

$$\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)} = \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \qquad \ldots \text{(Using 1)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \frac{(k+1)}{(k+1)+1}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all n ≥ 1.

b) Let P(n) : $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

1. Basis of induction : For n = 1

L.H.S. $= \frac{1}{1\cdot 3} = \frac{1}{3}$, R.H.S. $= \frac{1}{3}$

$\frac{1}{3} = \frac{1}{2\cdot 1+1}$

⇒ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true

i.e. $$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \qquad \ldots (1)$$

Then we have,

$$\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2k-1)(2k+1)}\right]+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

$$= \frac{k+1}{2(k+1)+1}$$

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

c) Let P(n) : $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

1. Basis of induction

For n = 1, L.H.S. $= \frac{1}{1.4} = \frac{1}{4}$, R.H.S. $= \frac{1}{4}$

$$\frac{1}{1\cdot 4} = \frac{1}{3\cdot 1+1}$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \ldots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$ ... (1)

Then we have,

$$\left[\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \ldots + \frac{1}{(3k-2)(3k+1)}\right] + \frac{1}{(3k+1)(3k+4)}$$

... (Using 1)

$$= \frac{k}{(3k+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3(k+1)+1)}$$

$$= \frac{k+1}{3(k+1)+1}$$

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Ex.4.3.6 : Prove by induction for

$$n \geq 0.\ 1 + a + a^2 + \ldots + a^n = \frac{1-a^{n+1}}{1-a}$$

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Sol. : Let P(n) be the given statement,

1. Basis of induction

For n = 0, L.H.S. = 1, R.H.S. $= \frac{1-a}{1-a} = 1$

For n = 1, L.H.S. = 1 + a, R.H.S. $= \frac{1-a^2}{1-a} = 1+a$

$\therefore$ For n = 0, 1, L.H.S. = R.H.S.

Hence P(0), P(1) are true.

2. Induction step : Assume that P(k) is true

$$\therefore \quad 1+a+a^2+\ldots+a^k = \frac{1-a^{k+1}}{1-a} \quad \ldots (1)$$

Consider,

$$1+a+a^2+\ldots+a^k+a^{k+1} = \frac{1-a^{k+1}}{1-a} + a^{k+1} \quad \ldots \text{(Using 1)}$$

$$= \frac{1-a^{k+1}+(1-a)a^{k+1}}{1-a}$$

$$= \frac{1-a^{k+1}+a^{k+1}-a^{k+2}}{1-a}$$

$$= \frac{1-a^{k+2}}{1-a}$$

Hence P(k + 1) is true.

Therefore by the mathematical induction P(n) is true for all $n \geq 0$.

Ex.4.3.7 : Prove by mathematical induction that n^2+n is an even number for all $n \geq 1$

Sol. : Let P(n) be the given statement.

1) Basis of induction :

For n = 1, $n^2+n = 2$ which is even

$\therefore$ P(1) is true

2) Induction step :

Assume that P(k) is true

i.e. k^2+k is an even number

i.e. $k^2+k = 2m$; $m \in Z$

Consider $(k+1)^2+(k+1)$

$$= k^2+2k+1+k+1$$

$$= k^2+k+2k+2 = 2m+2(k+1)$$

$$= 2(m+k+1) = \text{even number}$$

$\therefore$ P(k + 1) is true

$\therefore$ By mathematical induction p(n) is true $\forall$ n.

Ex.4.3.8 : Prove by method of induction

$$1^2+4^2+7^2+\ldots+(3n-2)^2 = \frac{n(6n^2-3n-1)}{2}$$

Sol. :

Let P(n) be the given statement.

1) Basis of Induction :

For $n = 1$

$$\text{L.H.S.} = 1^2 = 1,\ \text{R.H.S.} = \frac{1(6-3-1)}{2} = \frac{2}{2} = 1$$

Hence P(1) is true.

2) Induction step :

Assume that p(k) is true.

i.e. $1^2+4^2+7^2+\ldots+(3k-2)^2 = \dfrac{k(6k^2-3k-1)}{2}$

Consider $1^2+4^2+7^2+\ldots+(3k-2)^2+(3k+1)^2$

$$= \frac{k(6k^2-3k-1)}{2}+(3k-1)^2$$

$$= \frac{6k^3-3k^2-k+2(9k^2+6k+1)}{2}$$

$$= \frac{6k^3+15k^2+11k+2}{2}$$

$$= \frac{6k^2(k+1)+9k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(6k^2+9k+2)}{2}$$

$$= \frac{(k+1)+[6(k+1)^2-3(k+1)-1]}{2}$$

Thus P(k + 1) is true.

$\therefore$ By mathematical induction, P(n) is true $\forall$ n.

Ex.4.3.9 : Use mathematical induction to show that $n(n^2-1)$ is divisible by 24, where n is any odd positive number.

Sol. : If $n(n^2-1) = n^3-n$ is divisible by 24.

Then $n^3-n = 24$ (m) where m is any positive integral.

Let P(n) be the given statement,

1. Basis of Induction : For n = 1,

$n(n^2-1) = 0$ which is divisible by 24.

For n = 3, $n(n^2-1) = 24$ which is divisible by 24.

$\therefore$ P(1) and P(3) is true.

2. Induction step :

Assume that P(k) is true.

i.e. $k(k^2-1) = k^3-k$ is divisible by 24.

$\therefore$ $k(k^2-1) = k^3-k = 24\,(m_0)$, $m_0 \in Z$ ... (1)

Consider

$$(k+1)[(k+1)^2-1] = (k+1)^3-(k+1)$$

$$= k^3+3k^2+3k+1-k-1$$

$$= k^3+3k^2+2k$$

$$= (k^3-k)+3k^2+3k \quad \text{... (Using 1)}$$

$$= 24m_0+3k(k+1)$$

(As k(k + 1) is multiple of 8 for k odd positive integer and $k \geq 3$)

$$= 24m_0+3(8m_1)$$

$$= 24(m_0+m_1)$$

$$= 24m_2 \quad (\because m_0+m_1 = m_2)$$

$\therefore$ P(k + 1) is true.

$\therefore$ By mathematical induction P(n) is true for all n odd positive number.

Ex.4.3.10 : Show that $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$.

Sol. : Let P(n) be the given statement.

1. Basis of induction

For $\quad n = 2$

$$2^4 - 4(2^2) = 16 - 16$$

$$= 0 \text{ is divisible by 3 as 0 is divisible by every number}$$

$\therefore$ P(2) is true.

2. Induction step : Assume that P(k) is true

i.e. $k^4 - 4k^2$ is divisible by 3

Then we have,

$$(k+1)^4 - 4(k+1)^2 = k^4 + 4k^3 + 6k^2 + 4k + 1 - 4(k^2 + 2k + 1)$$

$$= (k^4 - 4k^2) + 4(k^3 + 2k) + 6k^2 + 12k - 3$$

$k^4 - 4k^2$ is divisible by 3.

$k^3 + 2k$ is divisible by 3

Also $\quad 6k^2 + 12k - 3 = 3(2k^2 + 4k - 1)$ is divisible by 3.

Hence $(k+1)^4 - 4(k+1)^2$ is divisible by 3.

Hence assuming P(k) is true. P(k + 1) in also true. Therefore P(n) is true for $n \geq 2$.

Ex.4.3.11 : Using mathematical induction, prove that

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

Sol. : Let P(n) be the given statement.

1. Basis of induction

For n = 1 $\quad$ L.H.S = 1, R.H.S. = 1

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) in true.

i.e. $\quad 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2 = (-1)^{k-1} \frac{k(k+1)}{2}$ ... (

Then we have,

$$[1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2] + (-1)^k (k+1)^2$$

$$= (-1)^{k-1} \frac{k(k+1)}{2} + (-1)^k (k+1)^2 \quad \text{... (Using}$$

$$= (-1)^k (k+1)\left[-\frac{k}{2} + (k+1)\right]$$

$$= (-1)^k (k+1)\left[\frac{-k+2k+2}{2}\right] = (-1)^k \frac{(k+1)(k+2)}{2}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) in true for all n ≥ 1.

Ex.4.3.12 : Prove by mathematical induction that for n ≥ 1 :

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1.$$

Sol. : Let P(n) be the given statement.

1. Basis of induction

For $n = 1$, L.H.S $= 1$, R.H.S. $= 1$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true.

i.e. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k!$

$$= (k+1)! - 1 \qquad \ldots (1)$$

Then we have,

$$[1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k!] + (k+1) \cdot (k+1)! \qquad \ldots \text{(Using 1)}$$

$$= [(k+1)! - 1] + (k+1) \cdot (k+1)! = (k+1)! + (k+1) \cdot (k+1)! - 1$$

$$= (k+1)!\,[k+1+1] - 1 = (k+2)\,(k+1)! - 1$$

$$= (k+2)! - 1$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for n ≥ 1.

Ex.4.3.13 : Prove that for any positive integer n the number $n^5 - n$ is divisible by 5.

Sol. : Let P(n) be the given statement.

1. Basis of induction :

For $n = 1$, $1^5 - 1 = 0$ is divisible by 5.

As 0 is divisible by every number.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true.

i.e. $k^5 - k$ is divisible by 5

Then we have

$$(k+1)^5 - (k+1) = (k^5 + {}^5C_1 k^4 + {}^5C_2 k^3 + {}^5C_3 k^2 + {}^5C_4 k + {}^5C_5) - (k+1)$$

$$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$

$$= (k^5 - k) + 5\,[k^4 + 2k^3 + 2k^2 + k]$$

$k^5 - k$ is divisible by 5.

and $5(k^4 + 2k^3 + 2k^2 + k)$ is divisible by 5.

Hence $(k+1)^5 - (k+1)$ is divisible by 5.

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for $n \geq 1$.

Ex.4.3.14 : Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction for $n \geq 1$.

Sol. : Let P(n) be the given statement.

1. Basis of induction :

For n = 1 $\quad 8^1 - 3^1 = 5$

$$= 5 \cdot 1$$

Obviously a multiple of 5.

$\therefore$ P(1) is true.

2. Induction step : Assume that, P(k) in true.
i.e. $8^k - 3^k$ is multiple of 5 say 5 r

i.e. $\quad 8^k - 3^k = 5r$

where r is an integer

Then we have,

$$\begin{aligned} 8^{k+1} - 3^{k+1} &= 8^k \cdot 8 - 3^k \cdot 3 \\ &= 8^k \cdot (5+3) - 3^k \cdot 3 \\ &= 8^k \cdot 5 + (8^k \cdot 3 - 3^k \cdot 3) \\ &= 8^k \cdot 5 + 3(8^k - 3^k) \end{aligned}$$

Obviously $8^k \cdot 5$ is multiple of 5 and also $8^k - 3^k$ is multiple of 5.

Therefore, $8^{k+1} - 3^{k+1}$ is multiple of 5.

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Ex.4.3.15 : Show that the sum of the cubes of three consecutive natural number is divisible by 9.

Sol. : Let n, n + 1, n + 2 be three consecutive natural numbers.

We have to show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Let P(n) be the above statement,

1. Basis of induction : For n = 1

$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$ which is divisible by 9.

$\therefore$ P(1) is true.

2. Induction step : Assume that P(k) is true.
i.e. $k^3 + (k+1)^3 + (k+2)^3$ in divisible by 9.

Then we have,

$$(k+1)^3+(k+2)^3+(k+3)^3 = [(k+1)^3+(k+2)^3]+[k^3+{}^3C_1k^2(3)+{}^3C_2k(3)^2+{}^3C_3(3^3)]$$

$$= (k+1)^3+(k+2)^3+k^3+[9k^2+27k+27]$$

$$= [k^3+(k+1)^3+(k+2)^3]+9[k^2+3k+3]$$

$k^3+(k+1)^3+(k+2)^3$ is divisible by 9 and $9(k^3+3k+3)$ is divisible by 9.

$\Rightarrow (k+1)^3+(k+2)^3+(k+3)^3$ is divisible by 9.

Hence assuming P(k) in true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Ex.4.3.16 : Prove that $n(n^2+5)$ is an integer multiple of 6 for all positive integers n.

Sol. : Let P(n) be the given statement

1) Basis of Induction : For $n = 1$,

$n(n^2+5) = 1(1+5) = 6$ which is of 6 multiple

$\therefore$ P(1) is true

2) Induction step : Assume that P(k) is true

i.e. $k(k^2+5)$ is a multiple of 6

$\therefore \quad k(k^2+5) = k^3+5k = 6a$ where $a \in z$

Consider

$$(k+1)((k+1)^2+5) = (k+1)(k^2+2k+1+5)$$

$$= (k+1)(k^2+5+1+2k)$$

$$= k(k^2+5)+k^2+5+(k+1)(1+2k)$$

$$= k(k^2+5)+k^2+5+k+1+2k^2+2k$$

$$= k(k^2+5)+3k^2+3k+6$$

$$= k(k^2+5)+3(k^2+k+2)$$

$$= 6a+3(k^2+k+2)$$

But k^2+k+2 is a multiple of 2

$$k^2+k+2 = 2b, \quad b \in z$$

$\therefore (k+1)((k+1)^2+5) = 6a+6b = 6(a+b)$ which is a multiple of 6

$\therefore$ P(k + 1) is true

Hence, by mathematical induction, P(n) is true for all n.

Ex.4.3.17 Use mathematical induction to show that $1+2+2^2+\ldots+2^n = 2^{n+1}-1$ for all non negative integers n

AKTU : 2011-12

Then we have,

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = [(k+1)^3 + (k+2)^3] + [k^3 + {}^3C_1 k^2(3) + {}^3C_2 k(3)^2 + {}^3C_3 (3^3)]$$

$$= (k+1)^3 + (k+2)^3 + k^3 + [9k^2 + 27k + 27]$$

$$= [k^3 + (k+1)^3 + (k+2)^3] + 9[k^2 + 3k + 3]$$

$k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9 and $9(k^3 + 3k + 3)$ is divisible by 9.

$\Rightarrow (k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 9.

Hence assuming P(k) in true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Ex.4.3.16 : Prove that $n(n^2+5)$ is an integer multiple of 6 for all positive integers n.

Sol. : Let P(n) be the given statement

1) Basis of Induction : For $n = 1$,

$$n(n^2+5) = 1(1+5) = 6 \text{ which is of 6 multiple}$$

$\therefore$ P(1) is true

2) Induction step : Assume that P(k) is true

i.e. $k(k^2+5)$ is a multiple of 6

$$\therefore \quad k(k^2+5) = k^3+5k = 6a \text{ where } a \in z$$

Consider

$$(k+1)((k+1)^2+5) = (k+1)(k^2+2k+1+5)$$

$$= (k+1)(k^2+5+1+2k)$$

$$= k(k^2+5)+k^2+5+(k+1)(1+2k)$$

$$= k(k^2+5)+k^2+5+k+1+2k^2+2k$$

$$= k(k^2+5)+3k^2+3k+6$$

$$= k(k^2+5)+3(k^2+k+2)$$

$$= 6a+3(k^2+k+2)$$

But k^2+k+2 is a multiple of 2

$$k^2+k+2 = 2b, \quad b \in z$$

$\therefore (k+1)((k+1)^2+5) = 6a + 6b = 6(a+b)$ which is a multiple of 6

$\therefore$ P(k + 1) is true

Hence, by mathematical induction, P(n) is true for all n.

Ex.4.3.17 Use mathematical induction to show that $1+2+2^2+\ldots+2^n = 2^{n+1}-1$ for all non negative integers n

AKTU : 2011-12

Sol. : Let P (n) be $1+2+2^2+...+2^n = 2^{n+1}-1$

(i) Basis of Induction :

For all n = 0 ; $2^0 = 1$ and $2^{0+1}-1 = 1$

and for all n = 1 ; $1+2^1 = 3$ and $2^{1+1}-1 = 3$

Thus P (n) is true for all n = 0 and 1.

(ii) Assume that P (K) is true.

i.e. $1+2+2^2+...+2^K = 2^{K+1}-1$

(iii) Consider, $1+2+2^2+...+2^K+2^{K+1}-1$

$$= 2^{K+1}-1+2^{K+1}$$

$$= 2 \cdot 2^{K+1}-1 = 2^{K+1}-1$$

Hence P (K + 1) is true.

$\therefore$ P (n) is true for all n.

Ex.4.3.18 Prove by mathematical induction that 7 + 77 + 777 + ... + 777 for every $n \in N$ ... 7 = $\frac{7}{10}[10^{n+1}-9n-10]$

AKTU : 2013-14

Sol. : Let P (n) be the given statement.

(i) For n = 1 L.H.S. = 7

R.H.S. = $\frac{7}{81}[10^{1+1}-9-10] = 7$

$\therefore$ P (1) is true.

ii) Assume that P (K) is true

$7 + 77 + 777 + ... + 777 + ... 7 = \frac{7}{81}[10^{K+1}-9K-10]$

(iii) Consider : 7 + 77 + 777 + ... + 777 + ...7

$$= \frac{7}{81}[10^{K+1}-9K-10]+777......77$$

$$= \frac{7}{81}[10^{K+2}-9(K+1)10]$$

$\Rightarrow$ P (K + 1) is true. Hence P (n) is true for all n.

Ex.4.3.19 : Show by induction that any positive integer n greater than or equal to 2 is either a prime or product of primes.

AKTU : 2012-13

Sol. : Let P(n) : Any positive integer $n \geq 2$ is either a prime or product of primes.

1) **Basis of Induction :** For n = 2, we have 2 = 2 which is prime. $\therefore$ P(2) is true.

2) **Induction step :** Assume that P(k) is true for k < n. i.e. for k = 2, 3, ... n − 2, n − 1.

Consider any positive integer n, There are two cases.

i) If n is a prime, then result is trivial

ii) If n is not a prime, then $n = r \cdot s$

where $r, s \in N$. r and $s \geq 2$ and r, s < n

But by induction principle,

$$r = P_1 \cdot P_2 ... P_m \text{ and}$$

$$s = q_1 \cdot q_2 \cdot q_3 ... q_t$$

$\therefore$ where P_i and q_i are primes

$\therefore$ $n = r \cdot s = P_1 P_2 ... P_m \cdot q_1 q_2 ... q_t$

Thus n is a product of primes

Hence the proof.

Ex.4.3.20 : If D_n = set of all positive odd integers = {1, 3, 5, 7, ...} then prove with the help of mathematical induction P(n) : 1 + 3n is divisible by 4.

AKTU : 2015-16

Sol. : Let P(n) be the given statement.

1) **Basis of Induction :** For n = 1, 1 + 3n = 1 + 3 = 4 which is divisible by 4. $\therefore$ P(1) is true,

2) **Induction step :** Assume that P(k) is true where k is an odd integer. i.e. 1 + 3k = 4P where $p \in z$

Consider $1+3(k+2) = 1+3k+6$

$= 4P+6$ which is not divisible by 4

e.g. If n = 15, 1 + 3n = 46 which is not divisible by 4

Hence P(n) is not true for all n.

Thus given example is wrong.

Ex.4.3.21 : By mathematical induction, prove that for any integer n, $(11)^{n+2}+(12)^{2n+1}$ is divisible by 133

Sol. : Let p(n) be the given statement

1) Basis of Induction :

For n = 1, $(11)^{n+2} + (12)^{2n+1} = 11^2 + 12^3 = 3059$ which is divisible by 133

$\therefore$ P(1) is true.

2) Induction step : Assume that P(k) is true

i.e. $(11)^{k+2} + (12)^{2k+1}$ is divisible by 133

$$(11)^{k+2} + (12)^{2k+1} = 133\ a : a \in Z$$

Consider

$$(11)^{k+3} + (12)^{2k+3} = (11)^{k+2}(11) + (12)^{2k+2}(12)^2$$

$$= (11^{k+2} \times 11) + (12^{2k+1} \times 144)$$

$$= (11^{k+2} \times 11) + (12^{2k+1} \times (11 + 133))$$

$$= (11^{k+2} \times 11) + (12^{2k+1} \times 11) + 133(12)^{2k+1}$$

$$= 11 \times (11^{k+2} + 12^{2k+1}) + 133(12)^{2k+1}$$

$$= 11 \times (133a) + 133(12)^{2k+1}$$

$$= 133\,[11a + (12)^{2k+1}]$$ which is divisible by 133

$\therefore$ P(k+1) is true.

Hence by mathematical induction, P(n) is true for all n.

Ex.4.3.22 By principle of mathematical induction, prove that $2.7^n + 3.5^n - 5$ is divisible by 24, for all natural number n.

Sol. : Let P(n) be the given statement

1) Basis of Induction :

For n = 1

$$2.7^n + 3.5^n - 5 = 2 \times 7 + 3 \times 5 - 5$$

$$= 24 \text{ which is divisible by } 24$$

$\therefore$ P(1) is true.

2) Induction step :

Assume that P(k) is true.

i.e. $2.7^k + 3.5^k - 5$ is divisible by 24

i.e. $2.7^k + 3.5^k - 5 = 24\ a\ ;\ a \in Z$

Consider $2.7^{k+1} + 3.5^{k+1} - 5$

$$= 2.7^k.7 + 3.5^k.5 - 5$$

$$= 147^k + 155^k - 5$$

$$= 7(2 \cdot 7^k + 3.5^k - 5) + 30 - 6.5^k = 7(24a) + 30(1 - 5^{k-1})$$

$$= 7a(24) - 30(5^{k-1} - 1) \quad \dots (1)$$

Now we have $1 + x + x^2 + \dots + x^{n-1} = \dfrac{x^n - 1}{x - 1}$

$$\therefore \quad x^n - 1 = (x-1)(1 + x + x^2 + \dots + x^{n-1})$$

$$\therefore \quad 5^k - 1 = (5-1)(1 + 5 + 5^2 + \dots + 5^{k-1})$$

$$\therefore \quad 5^{k-1} - 1 = (5-1)(1 + 5 + 5^2 + \dots + 5^{k-2})$$

$\therefore$ Equation (1) becomes $2.7^{k+1} + 3.5^{k+1} - 5 = 24(7a) - 120(1 + 5 + 5^2 + \dots + 5^{k-2})$

$$= 24[7a - 5(1 + 5 + 5^2 + \dots + 5^{k-2})]$$ which is divisible by 24

$\therefore$ P(k + 1) is true

Hence, by the principle of mathematical induction P(n) is true $\forall$ n.

Ex.4.3.23 : Using mathematical induction prove that

$$3 + 3.5 + 3.5^2 + \dots + 3.5^n = 3\left(\frac{5^{n+1} - 1}{4}\right).$$ **For non-negative number n.**

Sol. : Cancelling 3 from the both sides of given.

Statement, We get .

$$1 + 5 + 5^2 + \dots + 5^n = \frac{5^{n+1} - 1}{5 - 1} \quad \dots (1)$$

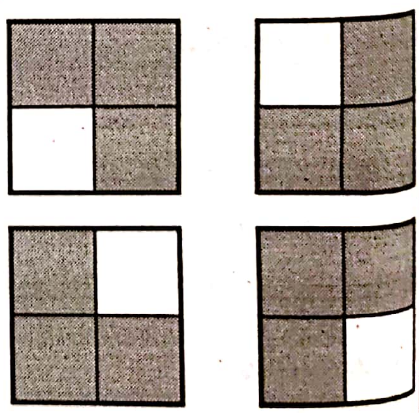
Let P(n) be the above statement.

To prove this refer example 4.3.6 for a = 5.

Ex.4.3.24 : Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be covered by L-shaped pieces, where each piece covers three squares at a time.

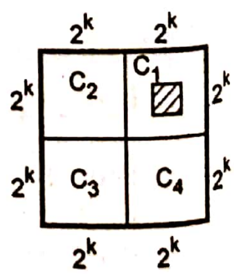
Sol. : Let P(n) be the proposition that any $2^n \times 2^n$ chessboard with one square removed can be covered using L-shaped pieces.

Basis of induction : For n = 1, P(1) implies that any 2×2 chessboard with one square removed can be covered using L shaped pieces. P(1) is true, as seen below.



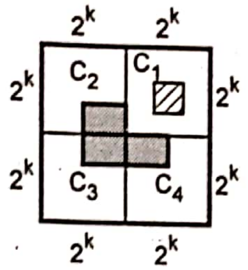
Induction step : Assume that, P(k) is true i.e. any $2^k \times 2^k$ chessboard with one square removed can be covered using L-shaped pieces.

Then, we have to show that P(k + 1) is true. For this consider, a $2^{k+1} \times 2^{k+1}$ chessboard with one square removed. Divide the chessboard into four equal halves of size $2^k \times 2^k$, as shown below.



The square which has been removed, would have been removed from one of the four chessboards, say C_1. Then by induction hypothesis, C_1 can be covered using L-shaped pieces. Now, from each of the remaining chessboards, remove that particular piece (or tile), lying at the centre of the large chessboards.

Then by induction hypothesis, each of these $2^k \times 2^k$ chessboards with a piece (or tile) removed can be covered by the L-shaped pieces. Also the three tiles removed from the centre can be covered by one L-shaped piece. Hence the chessboard of $2^{k+1} \times 2^{k+1}$ can be covered by L-shaped pieces.



Hence proved.

Ex.4.3.25 Suppose we have unlimited stamps of two different denominations, 3 rupees and 5 rupees. We want to show that it is possible to make up exactly any postage of 8 rupees or more using stamps of these two denominations.

Sol. : For k = 8, we have one 5 rupees stamp and one 3 rupees stamp.

For k = 9, replace 5 rupees stamp by two 3 rupees stamp, similarly for k = 10, replace all 3, 3 rupees stamps by, two 5 rupees stamp an so on.

Hence let us assume that, it is possible to make up k rupees stamp using 3 rupees and 5 rupees stamps (for $k \geq 8$).

Now we have to show that it is also possible to make up (k + 1) rupees stamps using 3 rupees and 5 rupees stamps.

We examine two cases :

1) Suppose we make up stamps of k rupees using at least one 5 rupees stamp. Replacing a 5 rupees stamp by two 3 rupees stamp, we can make up k + 1 rupees stamps.

2) Suppose we make up a stamp of k rupees using 3 rupees only. Since $k \geq 8$ we must have at least 3, 3 rupees stamps. Replacing these 3, 3 rupees stamps by two five rupees stamps. We can make up stamps of k + 1 rupees.

Hence proved.

Ex.4.3.26 The king summoned the best mathematicians in the kingdom to the palace to find out how smart they were. The king told them

"I have placed white hats on some of you and black hats on the others. You may look at, but not talk, to, one another. I will leave now and will come back every hour on the hour. Every time I return, I want those of you who have determined that you are wearing white hats to come up and tell me immediately."

As it turned out, at the n^{th} hour every one of the n mathematician who were given white hats informed the king that she knew that she was wearing a white hat. Why ?

Sol. :

1. Basis of induction : For n = 1, there is only one mathematician wearing a white hat. Since the king said that white hats were placed on some one of the mathematician (king never lie). The mathematician who saw that all other mathematicians had on black hats would realize immediately that she was wearing a white hat. Consequently she would inform the king on the first hour. (when the king returned for the first time) that she was wearing a white hat.

Now let n = 2, i.e. two mathematicians wearing white hats.

Consider, one of these two mathematicians. She saw that one of her colleagues was wearing a white hat. She reasoned that if she were wearing a black hat, her colleague would be the only one wearing a white hat. In that case, her colleague would have figured out the situation and informed the king on the first hour. That did not happen shows that she was also wearing a white hat. Conesquently she told the king on the second hour (and so did the other mathematician with a white hat, since all the mathematicians are smart).

2. Induction step : Assume that, if there were k mathematicians wearing white hats, then they would have figured out that they were wearing white hats and informed the king so on the k^{th} hour. Now, suppose that there were k + 1 mathematicians wearing white hats.

Every mathematician wearing a white hat saw that k of her colleagues were wearing white hats. However, that her k colleagures did not inform the king of their findings on the k^{th} hour can only imply that there were more then k. People wearing white hats. Consequently she knew that, she must be wearing a white hat also on the $(k+1)^{th}$ hour. She (together with all other mathematicians wearing white hats) would inform the king their conclusion.

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