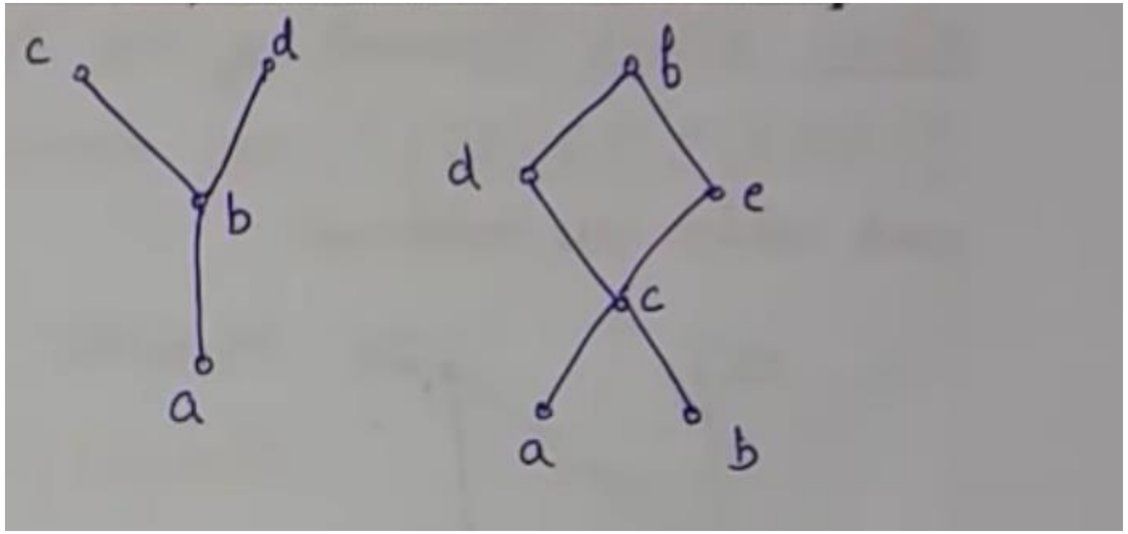
Lattice Unit-3

Discrete Structures and Theory of Logic

Lattice

A Poset in which every pair of elements has both, a least upper bound and a greatest
lower bound is called a lattice.
There are two binary operations defined for lattices –
1.Join – The join of two elements is their least upper bound. It is denoted by $\vee$ .If in a poset for every pair of elements Least Upper bound exists. Then it is known as Join Semi lattice.
2. **Meet –** The meet of two elements is their greatest lower bound. It is denoted by $\wedge$. If in a poset for every pair of element Greatest lower bound exists. Then it is known as Meet Semi lattice.

Example



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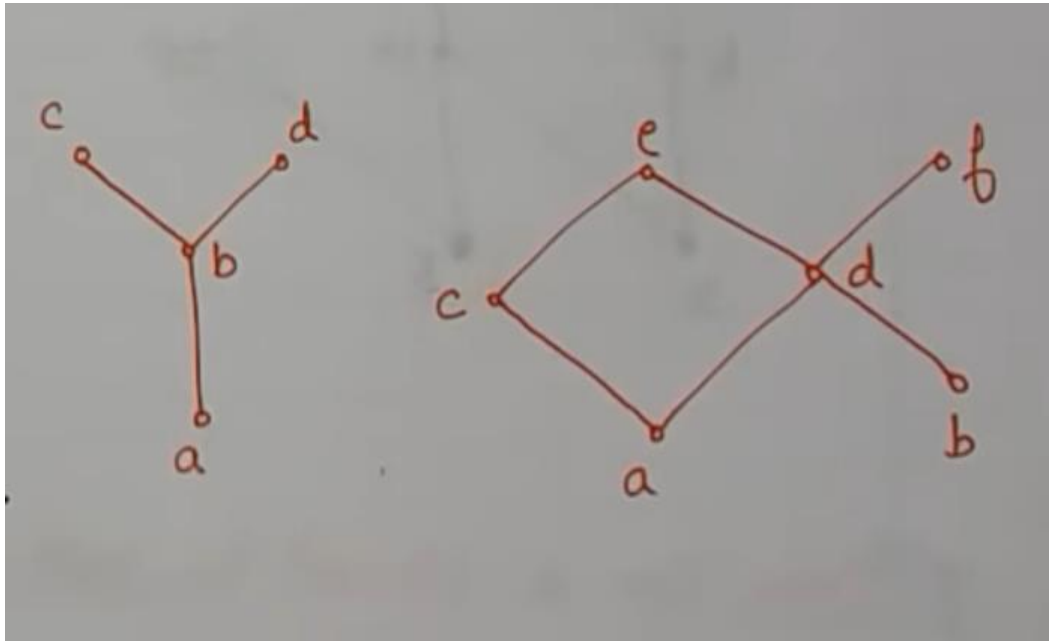
In first Hasse Diagram for (c,d) least upper bound doesn’t exist. Hence it is not join semi lattice.

In second Hasse Diagram for every pair of elements least upper bound exist .Hence it is join semi lattice.



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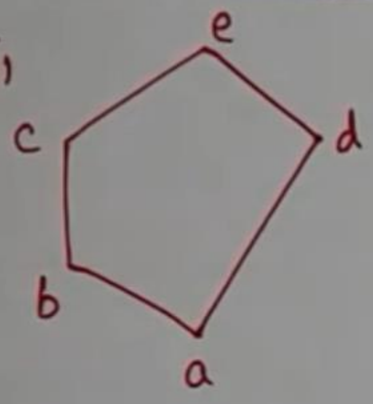
In First Hasse Diagram for every pair of elements greatest lower bound exist .Hence it is meet semi lattice.

In second Hasse Diagram for (a,b) greatest lower bound doesn’t exist. Hence it is not meet semi lattice.

Practice Questions on Lattices:-

(i)



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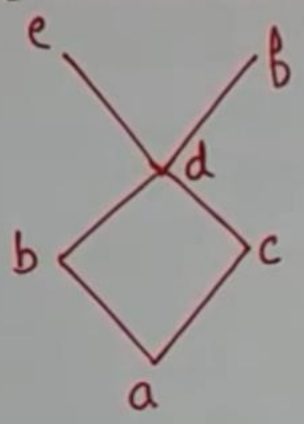

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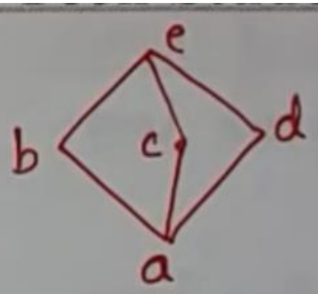

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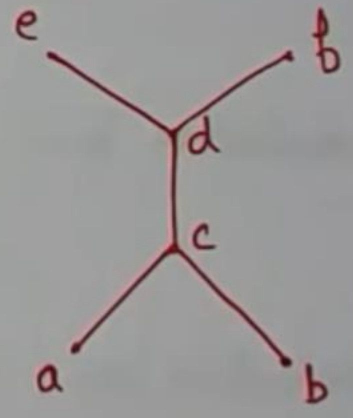

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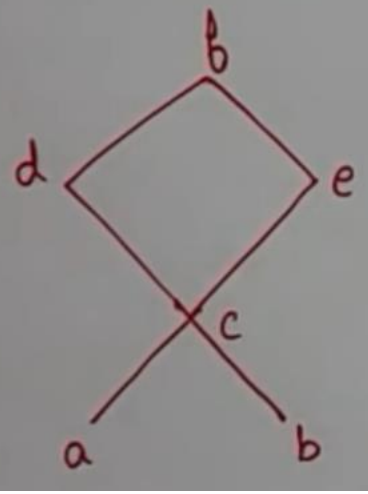
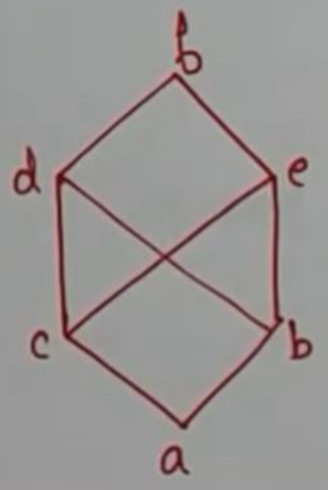


(iv)



Solution

i) It is Lattice. Every pair has join and meet semi lattice.

ii) Not. For (a,b) Greatest lower bound doesn’t exist.

iii)Not. For (e,f) least upper bound doesn’t exist.

iv)Not. For (d,e) pair lower bound are (c,b,a). Since c,b are at same level.It is difficult to find the greatest lower bound.

v) It is Lattice. Every pair has join and meet semi lattice.

vi) Not. For (e,f) least upper bound doesn’t exist. For (a,b) Greatest lower bound doesn’t exist.

Properties of Lattice:

1. Idempotent Properties

   a) $a \vee a = a$
   b) $a \wedge a = a$
2. Commutative Properties

   a) $a \vee b = b \vee a$
   b) $a \wedge b = b \wedge a$

3. Associative Properties

   a) $a \vee (b \vee c) = (a \vee b) \vee c$
   b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption Properties

   a) $a \vee (a \wedge b) = a$
   b) $a \wedge (a \vee b) = a$

Types of Lattice :

1. Bounded Lattice
2. Complete
3. Isomorphic
4. Distributive
5. Complemented

1.Complete Lattice

Definition. A lattice L is said to be *complete* if (i) every subset S of L has a least upper bound (denoted $\sup S$) and (ii) every subset of L has a greatest lower bound (denoted $\inf S$).

Example: Relation (Z,<=)

For any finite set S: (x such that(2<=x<=-2)).
The elements of S={2,1,0,-1,-2}.
For this set both join and meet exists. So it is complete Lattice

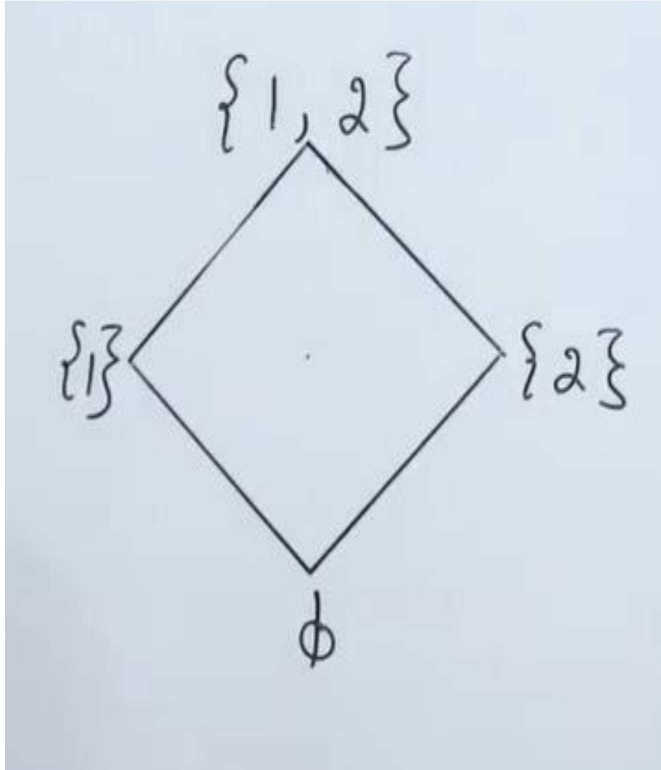
∞
3
2
1
0
-1

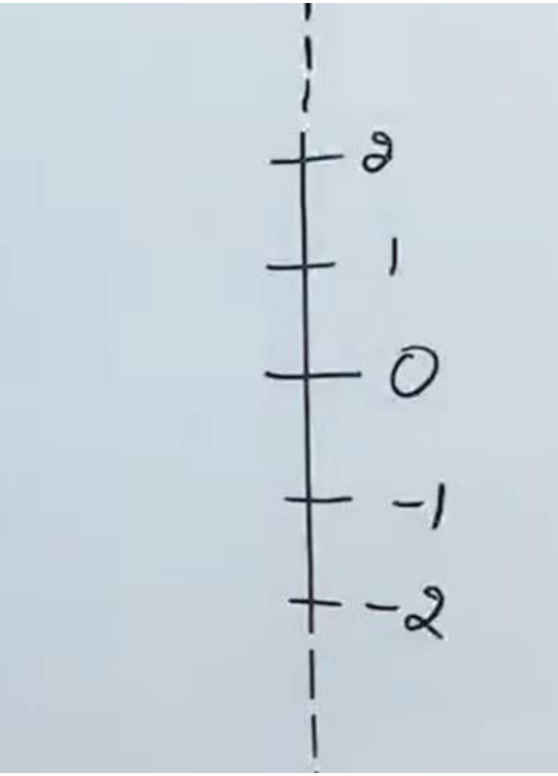
2.Bounded Lattice

- If for a Lattice both the greatest (1)and least(0) element exist is known as a Bounded Lattice.

- Greatest Element: Maximal Element.
- Least Element: Minimal Element.

Example:





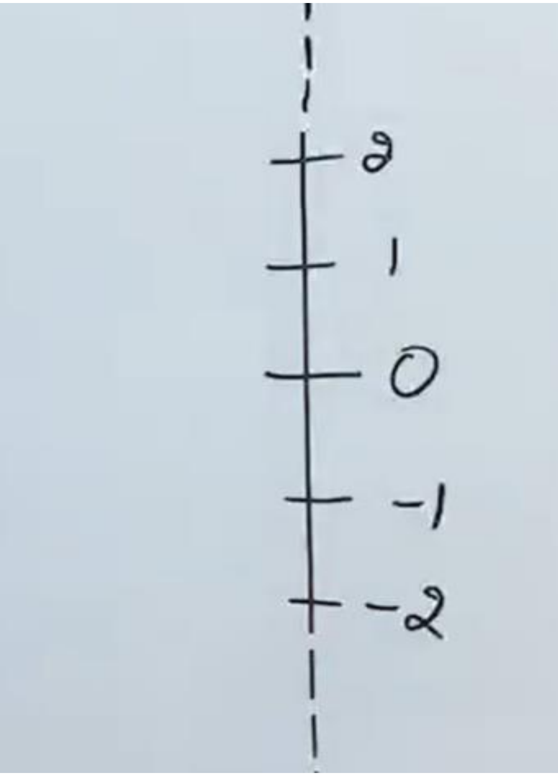
Greatest Element:{1,2}
Least Element: ϕ

For infinite set it is not possible to find maximum and minimum but we can make it bounded

$S=\{Z, <=\} \cup \{-\infty, \infty\}$

Now Greatest Element= ∞

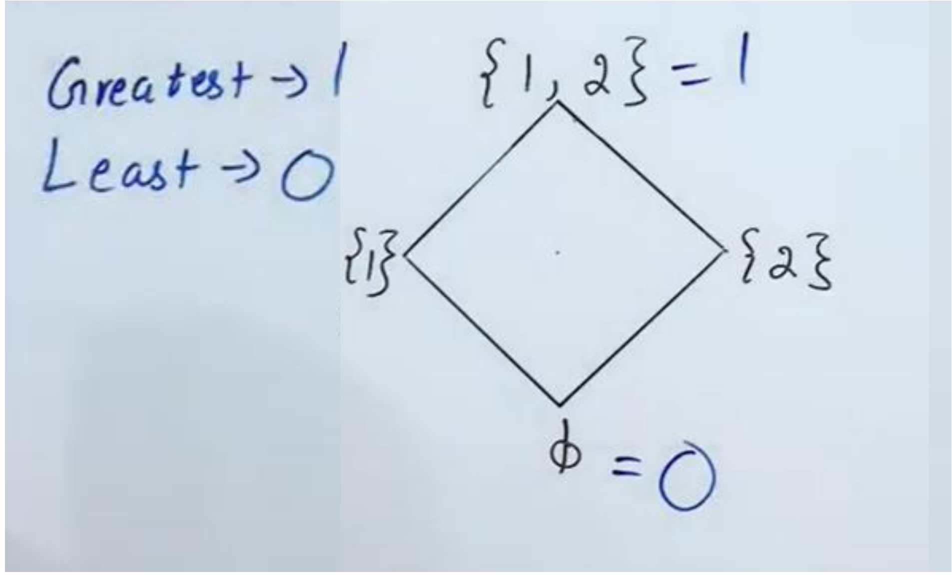
Least Element=- ∞



Properties of Bounded Lattice

If L is a bounded lattice, then for any element $a \in L$, we have the following identities:

1. $a \vee 1 = 1$
2. $a \wedge 1 = a$
3. $a \vee 0 = a$
4. $a \wedge 0 = 0$



3.Isomorphic Lattice

Definition: Two lattices L_1 and L_2 are called isomorphic lattices if there is a bijection (one-one,onto) from L_1 to L_2 f: $L_1 \longrightarrow L_2$, such that

- $f(a \wedge b) = f(a) \wedge f(b)$
- $f(a \vee b) = f(a) \vee f(b)$

Example

Set L={1,2,3,4,5,6}. Take two lattice from this set.

(L1,|) $\longrightarrow$ (L2,<=)

{2,3} ∈ L1

f (a v b) = f (a) v f (b)

L.H.S= f(a v b)=f(2 v 3)

=L.C.M of {2,3}

=f(6)

R.H.S= f (a) v f (b)

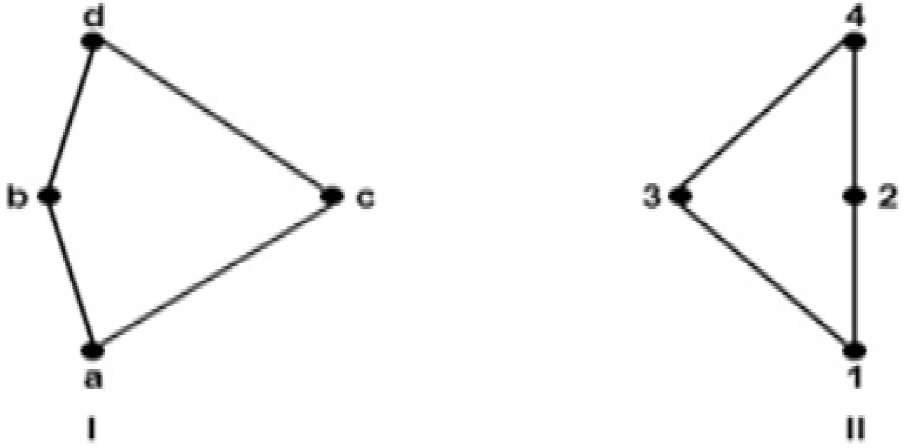
=f(2) v f (3)= either f(2) or f (3)

f(2) ≠ f(6) .hence L.HS ≠ R.H.S.

The two lattice are not isomorphic

Example: Determine whether the lattices shown in fig are isomorphic

$f = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$



$f(b \wedge c) = f(b) \wedge f(c)$

L.H.S$= f(b \wedge c) = f(a) = 1$

R.H.S$= f(b) \wedge f(c) = 2 \wedge 3 = 1$

Both are Isomorphic Lattice

4.Distributive Lattice:

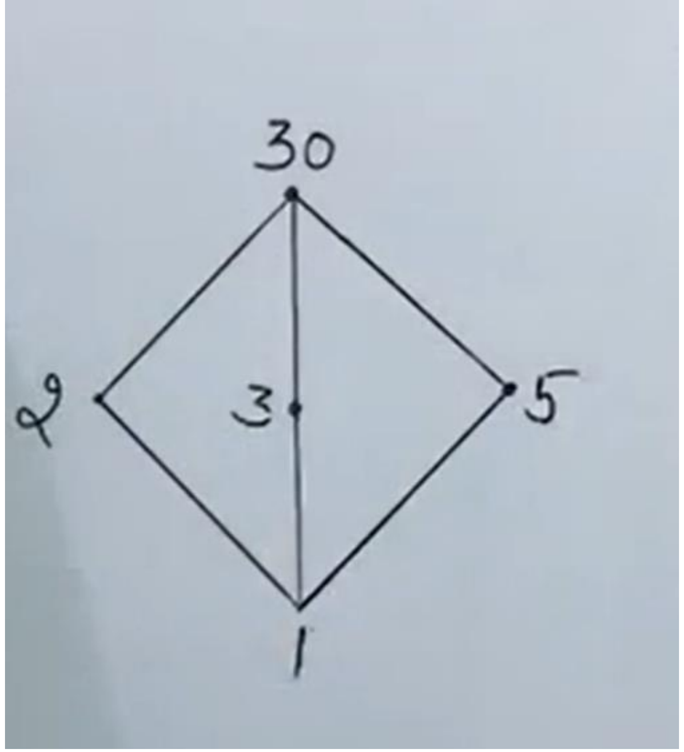
- A lattice L is called distributive lattice if for any elements a, b and c of L,it satisfies following distributive properties:

1.$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

2.$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

- If the lattice L does not satisfies the above properties, it is called a non-distributive lattice.

Example



Take three elements : 2,3,5

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$2 \wedge (3 \vee 5) = (2 \wedge 3) \vee (2 \wedge 5)$

L.HS= $2 \wedge (3 \vee 5) = 2 \wedge 30 = 2$

R.H.S= $(2 \wedge 3) \vee (2 \wedge 5)$
$(2 \wedge 3) = 1$
$(2 \wedge 5) = 1$

$= 1 \vee 1 = 1$

L.H.S $\neq$ R.HS.Hence it is not a distributive Lattice.

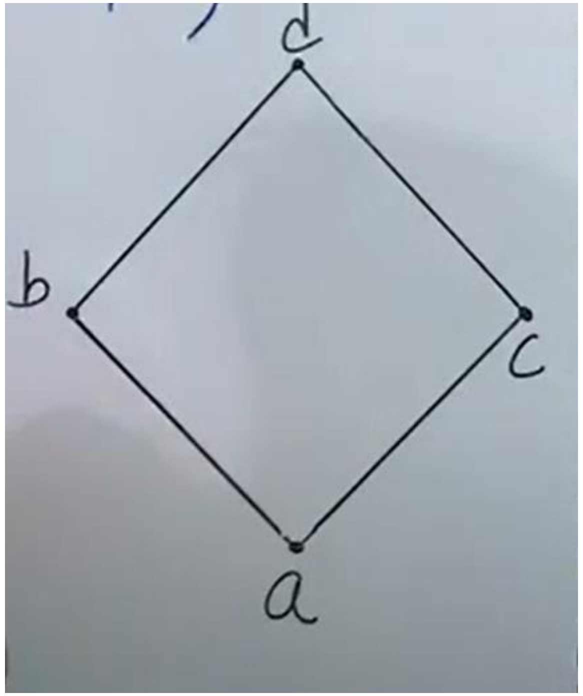
Complement of a Lattice

Let L be a bounded lattice with lower bound o and upper bound I. Let a be an element if L. An element x in L is called a complement of a if

$a \vee x = I$

$a \wedge x = 0$

Example:



1. Check for a and d

a ∨ d join= d = Upper Bound

a ∧ x meet= a= Lower Bound

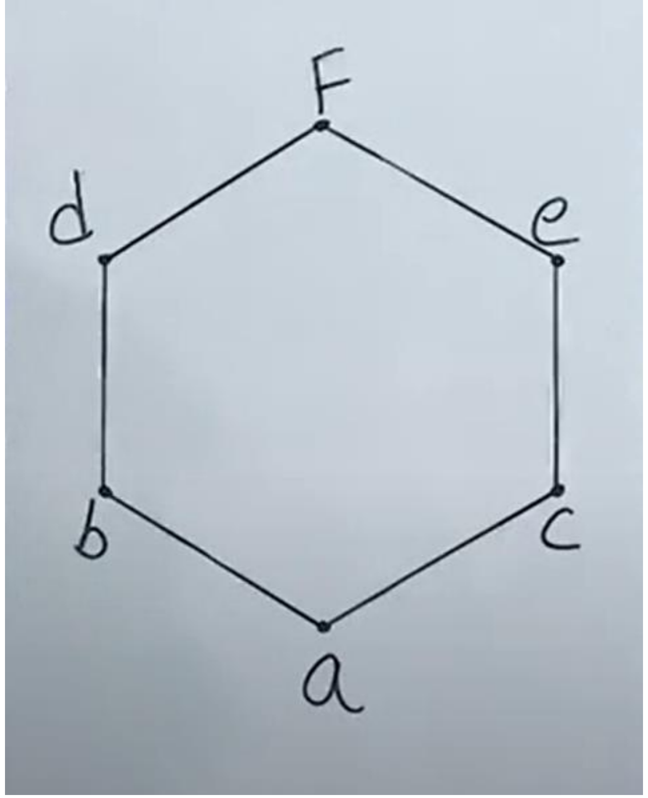
So a,d are complement of each other.

2. Check for b and c

b ∨ c join= d = Upper Bound

b ∧ c meet= a= Lower Bound

So b,c are complement of each other.



1.Check for b and c

b ∨ c join= F = Upper Bound

b ∧ c meet= a= Lower Bound

So b,c are complement of each other.

2. Check for b and e

b ∨ e join= F= Upper Bound

b ∧ e meet= a= Lower Bound

So b,e are complement of each other.

3. Check for b and d

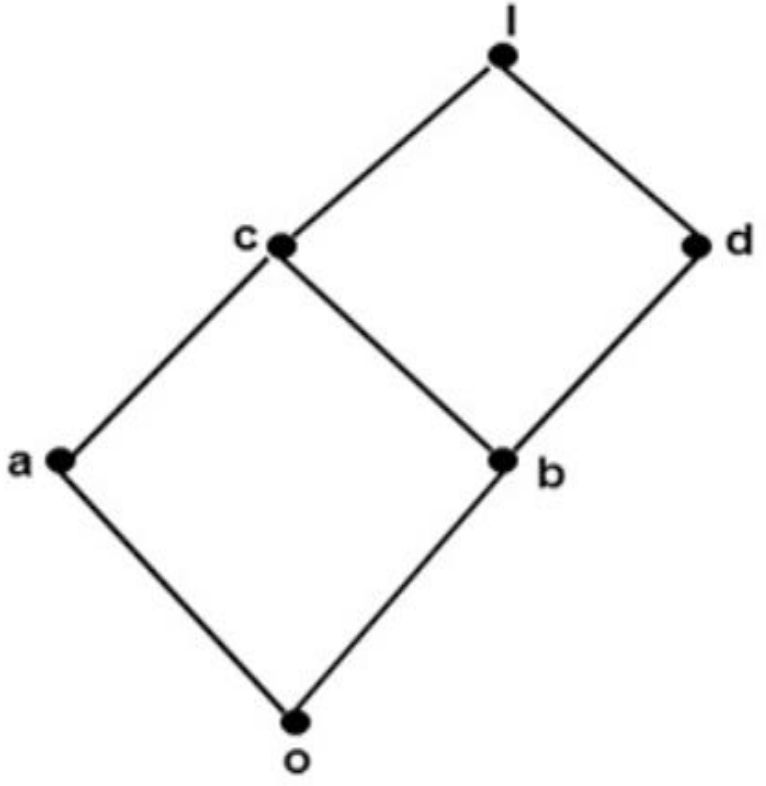
b ∨ d join= d= not Upper Bound

b ∧ d meet= b= not Lower Bound

So b,d are not complement of each other.

Element can have more than one complement.

Example: Determine the complement of a and c in fig:

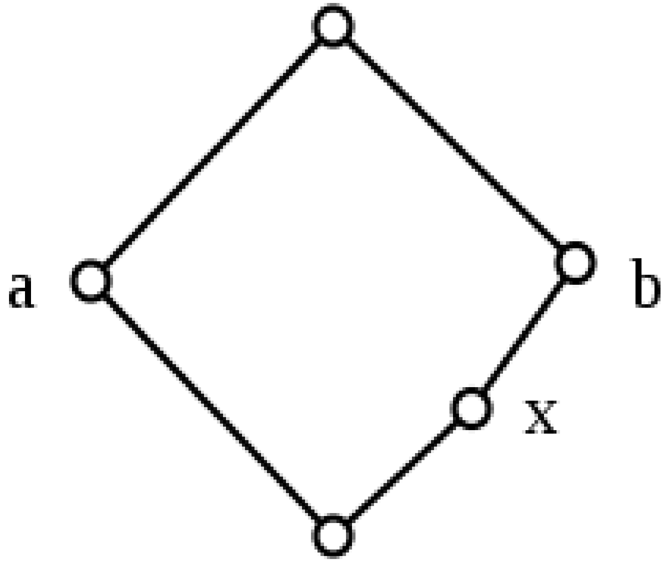


Solution: The complement of a is d. Since, $a \vee d = 1$ and $a \wedge d = 0$

The complement of c does not exist. Since, there does not exist any element c such that $c \vee c'=1$ and $c \wedge c'= 0$.

MODULAR LATTICE

- lattice $(L, \wedge, \vee)$ is called a modular lattice if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$.

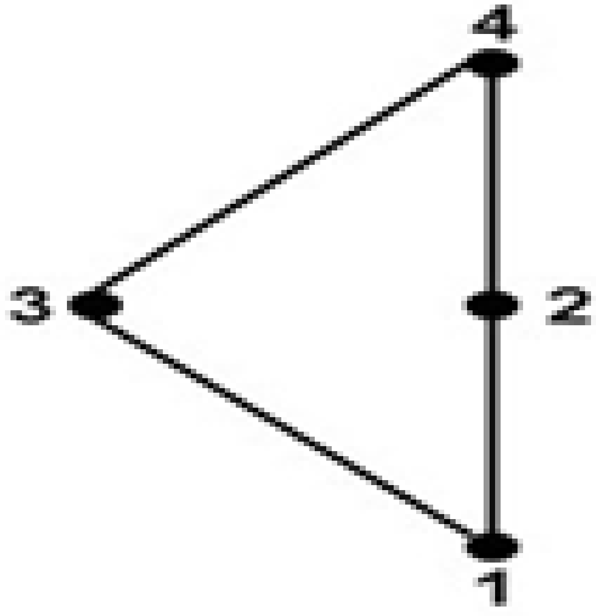


Distributive Lattice:

- A lattice L is called distributive lattice if for any elements a, b and c of L,it satisfies following distributive properties:
  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

  If the lattice L does not satisfy the above properties, it is called a non-distributive lattice.
  Example:
- The power set P (S) of the set S under the operation of intersection and union is a distributive function. Since, both the properties are satisfied for any sets a, b and c of P(S).

The lattice shown in fig II is a distributive. Since, it satisfies the distributive properties for all ordered triples which are taken from 1, 2, 3, and 4.



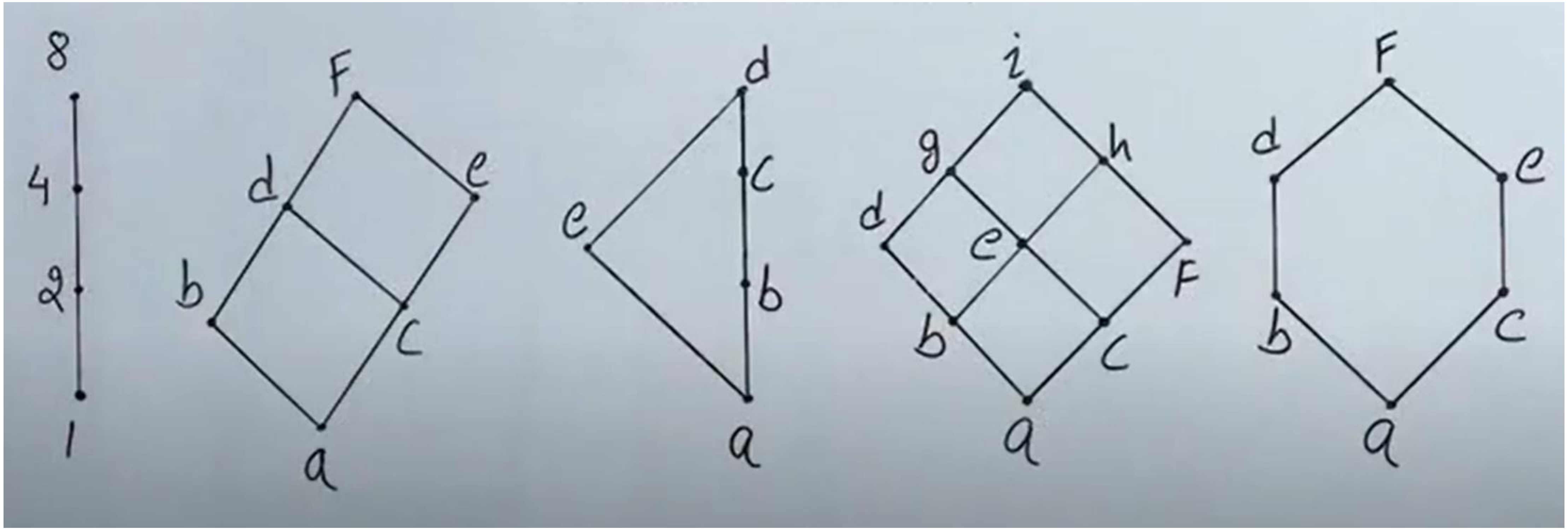
II

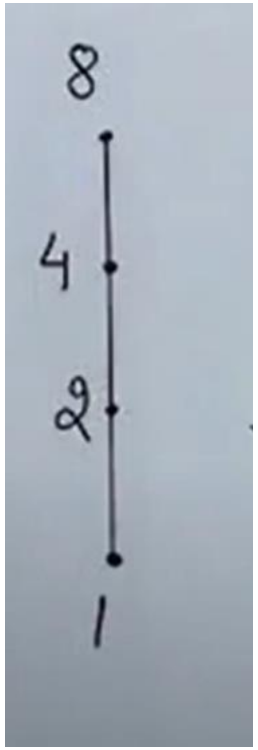
5.Complemented Lattice

A lattice L is said to be complemented if L is bounded and every element in L has a complement.

(atleast one complement)

For the following lattice check whether the lattice is a complemented lattice or not.





$I = 8$

$O = 1$

Check for elements 2,4

$2 \vee 4 = 8 = I$

$2 \wedge 4 = 2 \neq 0.$

Similarly 2,4 will have no complement. Hence it is not complemented Lattice.

1. Not Complemented Lattice. For 2,4 no complement exist

2. Not Complemented Lattice. For d,c no complement exist

3. Yes.It is a Complemented Lattice
4. Not Complemented Lattice. For e no complement exist.
5. Yes.It is a Complemented Lattice