

Ch-1

### Partial Differential Eq<sup>n</sup>

p.d.eqn - An eqn containing two or more partial derivatives of two or more independent variables, K.a. p.d.eqn.

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

Real life p.d.eqn -

1. Laplace eqn -  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$   $U_{xx} + U_{yy} + U_{zz} = 0.$

2. Wave eqn -  $\frac{\partial^2 u}{\partial t^2} = c^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right].$

3. Heat eqn -  $\frac{\partial u}{\partial t} = c^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$

order - Highest ordered derivative is the order of eqn.

Degree - Exponential power of highest ordered derivative.

### Solution of partial differential eqn

Soln is the one which satisfies. Soln contains arbitrary fn & constant; on removing both p.d.eqn is obtained.

### Formation of partial differential Eqn

(i) By elimination of arbitrary constant

e. (i)  $z = ax + by + ab.$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}.$$

d.w.r.t. x and w.r.t. y

$$z = a = p \quad z = b = q.$$

$$z = px + qy + pq.$$

(ii)  $z = ax + a^2 y^2 + b.$

$$\frac{\partial z}{\partial x} = a = p, \quad \frac{\partial z}{\partial y} = 2a^2 y = qp^2.$$

$$(iii) z = (x+a)(y+b)$$

$$\frac{\partial z}{\partial x} = p \quad \frac{\partial z}{\partial y} = q.$$

$$z = xy + xb + ay + ab.$$

$$\frac{\partial z}{\partial x} = y + b \quad \frac{\partial z}{\partial y} = a + x.$$

$$z = (y+b)(x+a)$$

$$\boxed{z = pq}$$

$$(iv) az + b = a^2x + y.$$

$$az = a^2x + y - b$$

$$\frac{\partial z}{\partial x} = a^2 \quad a \frac{\partial z}{\partial y} = 1 \Rightarrow q = \frac{1}{a}.$$

$$p = a \quad q = \frac{1}{a}.$$

$$\boxed{pq = 1}$$

By eliminating arbitrary fn

$$Q. z = f(x^2 - y^2)$$

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)(2x) \quad \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y).$$

$$\frac{p}{q} = \frac{f'(x^2 - y^2)(2x)}{f'(x^2 - y^2)(-2y)}$$

$$\frac{p}{q} = \frac{x}{-y}$$

$$+py = -qx$$

$$py + qx = 0.$$

$$(v) z = \phi(x) \cdot \psi(y)$$

$$\frac{\partial z}{\partial x} = \phi'(x)\psi(y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \phi'(x)\psi'(y)$$

$$\frac{\partial z}{\partial y} = \phi(x)\psi'(y)$$

$$\boxed{z = pq}$$

$$z = \phi(x)\psi(y)$$

(iii)  $z = x + y + F(xy)$

$\frac{\partial z}{\partial x} = 1 + q + F'(xy) \cdot y$

$\frac{\partial z}{\partial y} = 0 + 1 + F'(xy) \cdot x$

$p - 1 = F'(xy) \cdot y$

$q - 1 = F'(xy) \cdot x$

$\frac{p-1}{q-1} = \frac{F'(xy) \cdot y}{F'(xy) \cdot x}$

$\frac{p-1}{q-1} = \frac{y}{x}$

$(p-1)x = (q-1)y$

$px - x = qy - y$

$px - qy = x - y$

(iv)  $z = F(x+it) + g(x+it)$

$\frac{\partial z}{\partial x} = F'(x+it) + g'(x+it)$

$\frac{\partial z}{\partial y} = f'(x+it) \cdot i + g'(x+it) \cdot i$

$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x+it)$

$\frac{\partial^2 z}{\partial y^2} = f''(x+it) \cdot i^2 + g''(x+it) \cdot (i)^2$

$= -f''(x+it) - g''(x+it)$

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = f''(x+it) + g''(x+it) - f''(x+it) - g''(x+it)$

$= 0$

Complete integral :-

The sol<sup>n</sup>  $F(x, y, z, a, b) = 0$  of a first order partial diff. eq<sup>n</sup>, which contains two arbitrary constants is called complete sol<sup>n</sup> or complete integral.

Linear Homogeneous partial Differential Eq<sup>n</sup>

A partial differential eq<sup>n</sup> is said to be linear if it is of first degree in the dependent variable & its partial derivatives and also they are not multiplied together.

Homogeneous eq<sup>n</sup> :- every eq<sup>n</sup> containing dependent variable or its derivati-  
-ve , called Homogeneous eq<sup>n</sup>.

Quasi-linear partial diff. eqn :-  
 If the degree of highest ordered derivative is one.  
 General form of quasi-linear partial diff. eqn of first order.

$$P(x, y, z) \frac{\partial z}{\partial x} + Q(x, y, z) \frac{\partial z}{\partial y} = R(x, y, z). \quad \text{--- (1)}$$

$P, Q \rightarrow$  independent of  $z$ . &  $R$  is linear in  $z$  then eqn (1) is linear differential eqn.

Q. Solve :-

$$t = \sin(xy)$$

$$\frac{\partial^2 z}{\partial y^2} = \sin xy.$$

on integrating.

$$\frac{\partial z}{\partial y} = -\frac{1}{x} \cos(xy) + F(x)$$

The constant is being taken as a fn of  $x$ .

Again integrating,

$$z = -\frac{1}{x^2} \sin(xy) + yF(x) + \phi(x).$$

Q.  $\frac{\partial^2 z}{\partial x^2} + z = 0.$

when  $x=0, z=e^y \quad \frac{\partial z}{\partial x} = 1.$

If  $z$  were a fn of  $x$  alone, the soln would have been -  
 $z = C_1 \cos x + C_2 \sin x$ ;  $C_1, C_2 \rightarrow$  arbitrary constant.

The soln of given eqn is

$$z = F(y) \cos x + \phi(y) \sin x.$$

$$\frac{\partial z}{\partial x} = -F(y) \sin x + \phi(y) \cos x.$$

$$x=0; z=e^y \quad \therefore e^y = F(y)$$

$$x=0 \quad \frac{\partial z}{\partial x} = 1 \quad 1 = \phi(y).$$

Linear partial differential eqn of first order.

A differential eqn involving first order partial derivative p and q, called partial diff. eqn of first order.

i.e. p & q occur in first degree and are not multiplied together.

LAGRANGE'S EQUATION

$$P_p + Q_q = R.$$

1. First write in a form  $P_p + Q_q = R.$

2. Form auxiliary eqn

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

3. Solve eqn by method of grouping or by multiplier.

OR both to get two independent fns  $u=a$ ;  $v=b$ .

4. Then  $\phi(u, v) = 0$  or  $v = f(u)$

(i)  $y z p - x z q = x y.$

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}.$$

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

$$-zdx = ydy.$$

$$\frac{dy}{-xz} = \frac{dz}{xy}$$

$$-ydy = zdz.$$

$$xdx + ydy = 0.$$

$$ydy + zdz = 0.$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

$$\frac{y^2}{2} + \frac{z^2}{2} = c_2.$$

$$\phi(x^2 + y^2, y^2 + z^2) = 0.$$

(ii)  $y^2 p - xyq = x(z-2y)$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$xdx = -ydy.$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$-(z-2y)dy = zdz.$$

$$-zdy + 2ydy = zdz$$

$$2ydy = zdz + zdy.$$

$$y^2 = yz + c_2.$$

$$\phi(x^2 + y^2, y^2 - yz) = 0.$$

linear Partial Diff. eq<sup>n</sup> of points first order questions for most p

A diff. eq<sup>n</sup> involving first order partial derivative p and q, only in called partial diff. eq<sup>n</sup> of first order. (only p & q). If p & q occur only & are not multiplied together then it is called Linear partial Differential eq<sup>n</sup> of first order.

Lagrange's eq<sup>n</sup>

$$\text{P.D.E. of form } P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R \rightarrow f^n \text{ of } z.$$

$P \frac{\partial z}{\partial x}$        $Q \frac{\partial z}{\partial y}$   
 $f^n \text{ of } x$        $f^n \text{ of } y$

standard form of quasi-linear partial differential eq<sup>n</sup> of first order is called Lagrange's Eq<sup>n</sup>.

Working Rule

$$Pp + Qq = R$$

(i) Form auxiliary eq<sup>n</sup>  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ .

(ii) solve this method of grouping OR

method of multipliers OR

to get two independent soln  $u=a$  &  $v=b$  where a, b are arbitrary constant.

(iii) Then  $\phi(u, v) = 0$  or  $v = F(u)$  is general soln of eq<sup>n</sup>  $Pp + Qq = R$ .

Q: solve -

(i)  $yzp - xzq = xy$

given diff. eq<sup>n</sup>  $yzp - xzq = xy$ .

Comparing this with  $Pp + Qq = R$ .

$$P = yz \quad Q = -zx \quad R = xy$$

Lagrange's subsidiary eq<sup>n</sup>

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

taking first & second part, we get

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$xdx + ydy = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$$x^2 + y^2 = c_1.$$

NOW 1st & third

$$\frac{dx}{yz} = \frac{dz}{xy} \Rightarrow \frac{dx}{z} = \frac{dz}{x}$$

$$xdx - zdz = 0.$$

on integrating, we get

$$\frac{x^2}{2} - \frac{z^2}{2} = \frac{c_2}{2}.$$

$$x^2 - z^2 = c_2$$

gen. soln is  $(x^2 + y^2, x^2 - z^2) = 0$ .

$$Q \cdot y^2 P - xyP = x(z-2y)$$

Soln. Comparing with Lagrange's eqn.  
 $P = y^2$   $Q = -xy$   $R = x(z-2y)$

Lagrange's subsidiary eqn are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

taking first & second fraction

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$-x dx = y dy.$$

$$xdx + ydy = 0.$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{C_1}{2}$$

$$x^2 + y^2 = C_1$$

taking 2nd & 3rd fraction

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow \frac{dy}{-y} = \frac{dz}{(z-2y)}$$

$$-z dy + 2y dy = y dz$$

$$2y dy = y dz + z dy$$

on integrating

$$y^2 = yz + C_2.$$

Hence general eqn / soln is

$$\phi(x^2 + y^2, y^2 - yz) = 0.$$

$$(iii) x^2 p + y^2 p = (x+y)z.$$

$$p = x^2 \quad Q = y^2 \quad R = (x+y)z.$$

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}.$$

taking 1 & 2 fractn.

$$\frac{dx}{x^2} = \frac{dy}{y^2}.$$

on integrating

$$-\frac{1}{x} = -\frac{1}{y} - C_1.$$

$$\frac{1}{x} - \frac{1}{y} = C_1.$$

NOW taking again from 1.

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx - dy}{x-y} = \frac{dz}{z}.$$

on integrating, we get

$$\log(x-y) = \log z + \log C_2.$$

$$\frac{x-y}{z} = C_2.$$

Hence general soln is

$$\phi \left[ \frac{1}{x} - \frac{1}{y}, \frac{x-y}{z} \right] = 0.$$

$$(iii) p^2 - q^2 z = z^2 + (x+y)^2 \cdot R = z^2 + (x+y)^2.$$

$$P = z \quad Q = -z$$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

① & ②

$$\frac{dx}{z} = \frac{dy}{-z}$$

$$dx + dy = 0.$$

$$x + y = C_1$$

Now taking ① & ③

$$\frac{dx}{z} = \frac{dz}{z^2 + C_1^2}$$

$$\frac{(z^2 + C_1^2)}{t} dx = zdz$$

$$dx = \frac{zdz}{(z^2 + C_1^2)}$$

$$2dx = \frac{2zdz}{z^2 + C_1^2}$$

On integration.

$$\log(z^2 + C_1^2) = 2x + C_2$$

$$\log(z^2 + (x+y)^2) - 2x = C_2$$

from ① & ② gen. soln is

$$\phi [x+y, \log(x^2 + y^2 + z^2 + 2xy) - 2x] = 0.$$

Q3. Solve the diff. eqn.

$$(i) (mz-ny)p + (nz-lz)q = ly - mx$$

$$P = mz - ny$$

$$Q = nz - lz$$

$$R = ly - mx$$

$$\frac{dx}{mz-ny} = \frac{dy}{nz-lz} = \frac{dz}{ly-mx}$$

Here grouping is not possible now we'll multiply the term which is not here.

using  $x, y, z$  as multipliers, we get. ( $x, y, z$  form me in each fract<sup>n</sup> =  $\frac{adx + dy + dz}{0}$  at a time).

(we added to make denominator zero)  $\rightarrow 0$

$$\therefore adx + dy + dz = 0.$$

which on integration gives -

$$x^2 + y^2 + z^2 = a.$$

Now, again using  $l, m, n$  as multipliers.

each fract<sup>n</sup> =  $\frac{ldx + mdy + ndz}{0}$

which on integrat<sup>n</sup>

$$lx + my + nz = b.$$

from ① & ② general sol<sup>n</sup> is

$$x^2 + y^2 + z^2 = f(lx + my + nz)$$

$$(ii) x^2(y-z)p + y^2(z-x)q = z^2(x-y)$$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using  $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$  as multiplier we get

$$\text{each fract<sup>n</sup>} = \frac{\frac{1}{x^2} dx + \frac{1}{y^2} dy + \frac{1}{z^2} dz}{0}$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0.$$

which on integrat<sup>n</sup>

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1.$$

Again  $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$  as multiplier.

$$\frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

On soln.

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2.$$

$$xyz = c_2.$$

gen. soln

$$\phi\left[\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right] = 0.$$

Q. solve

$$x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$$

$$\text{where } p = \frac{\partial z}{\partial x} \text{ & } q = \frac{\partial z}{\partial y}.$$

soln  $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$ .

using  $x, y, -z$  as multiplier we get

$$\text{each fractn} \Rightarrow \frac{xdx+ydy-zdz}{0}$$

$$xdx+ydy-zdz=0.$$

we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2}.$$

$$x^2 + y^2 - 2z = c_1.$$

Again using  $\frac{1}{x}, \frac{1}{y}, \frac{1}{2}$  as multiplier

$$\frac{\frac{1}{x} dz + \frac{1}{y} dy + \frac{1}{2} dz}{0}$$

on integration

$$\log x + \log y + \log z = \log c_2 \\ xyz = c_2.$$

Hence gen. soln

$$\phi(x^2 + y^2 - 2z, xyz) = 0$$

Q9. Solve following partial eqn

$$x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2) \text{ where } p = \frac{\partial z}{\partial x} \text{ & } q = \frac{\partial z}{\partial y}.$$

Soln: Multiplier  $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$  and  $x, y, z$  as multipliers.

Q10. Solve

$$p + 3q = 5z + \tan(y - 3x).$$

$$p = 1 \quad q = 3 \quad R = 5z + \tan(y - 3x).$$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$$

First & second

$$\frac{dx}{1} = \frac{dy}{3}.$$

$$3dx = dy.$$

$$3x - y = C_1.$$

$$\text{or } C_1 = y - 3x.$$

Second & third or 1st & third

$$\frac{dx}{1} = \frac{dz}{5z + \tan C_1}$$

$$dx(5z + \tan C_1) = dz$$

$$dx = \frac{dz}{5z + \tan C_1} \Rightarrow dx =$$

$$x = \frac{1}{5} \log(5z + \tan C_1) - \frac{C_2}{5}.$$

$$\log[5z + \tan(y - 3x)] - 5x = C_2.$$

General soln

$$\phi[y - 3x, \log(5z + \tan(y - 3x)] - 5x].$$

$$Q \cdot (y^2 + z^2 - x^2)p - 2xyq + 2xz = 0.$$

$$P = (y^2 + z^2 - x^2)$$

$$Q = -2xy$$

$$R = -2xz$$

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

taking 2nd & third

$$\frac{dy}{-2xy} = \frac{dz}{-2xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

on integration

$$\log y = \log z + \log c_1$$

$$\frac{y}{z} = c_1$$

taking first, Now take multipliers x,y,z we have

$$\text{each fraction} = \frac{x dx + y dy + z dz}{-x(x^2 + y^2 + z^2)}$$

$$\frac{x dx + y dy + z dz}{-x(x^2 + y^2 + z^2)}$$

Now take 2nd or 3rd.

$$\frac{x dx + y dy + z dz}{-x(x^2 + y^2 + z^2)} = \frac{dz}{-2xz}$$

$$\frac{2(x dx + y dy + z dz)}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

$$\text{if } x^2 + y^2 + z^2 = t.$$

$$t' = 2x dx + 2y dy + 2z dz.$$

on integration forming  $\frac{f(t)}{f'(t)}$

Hence  $\log(x^2 + y^2 + z^2) = \log z + \log c_2$ .

Q12. solve

$$(ii) \left( \frac{y-z}{yz} \right)p + \left( \frac{z-x}{zx} \right)q = \left( \frac{x-y}{xy} \right).$$

Multiply by  $xyz$  both sides.

$$x(y-z)p + y(z-x)q = z(x-y).$$

$$P = (y-z)x \quad Q = y(z-x) \quad R = z(x-y).$$

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$  as multipliers.

$$= \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz$$

0

$$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0.$$

On integration we get.

$$\log x + \log y + \log z = \log c_1 \\ xyz = c_1.$$

then 1, 1, 1 as multipliers

$$dx + dy + dz = 0.$$

On integration

$$x + y + z = c_2.$$

General soln

$$\phi(xyz, x+y+z) = 0.$$

Q. (ii)  $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x-y)$

$$P = x^2 - y^2 - yz$$

$$Q = x^2 - y^2 - zx$$

$$R = z(x-y)$$

$$\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y^2-zx} = \frac{dz}{z(x-y)}$$

using 1, -1, -1 as multipliers.

$$\text{each fraction} = \frac{dx-dy-dz}{0}$$

$$dx - dy - dz = 0.$$

on integration we get

$$x - y - z = C_1.$$

from 1st & 2nd, we have.

$$\frac{x dx - y dy}{x^3 - xy^2 - xyz - yx^2 + y^3 + yzx} = \frac{dx}{z(x-y)}.$$

$$\frac{x dx - y dy}{(x-y)(x^2-y^2)} = \frac{dz}{z(x-y)}$$

$$\frac{x dx - y dy}{(x^2-y^2)} = \frac{dz}{z}.$$

on integrating

$$\frac{1}{2} \log(x^2-y^2) = \log z + \frac{1}{2} \log C_2.$$

$$\frac{x^2-y^2}{z^2} = C_2.$$

Hence gen. soln

$$\Phi \left[ x - y - z, \frac{x^2-y^2}{z^2} \right].$$

$$(iii) (x^2 + xy)p - (xy + y^2)q = -(x-y)(2x+2y+z).$$

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

on first & second we have

$$\frac{dx}{x} = \frac{dy}{-y}$$

on integrating we have

$$\log x = -\log y + \log C_1.$$

from first & second

$$\frac{dx + dy}{(x-y)(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

$$(2x+2y+z)(dz+dy) + (x+y)dz = 0.$$

$$(x+y+z)d(x+y) + (x+y)(dx+dy) + (x+y)dz = 0.$$

$$(x+y+z)d(x+y) + (x+y)d(x+y+z) = 0.$$

$$d\{(x+y)(x+y+z)\} = 0.$$

Non-Linear Partial Differential Eq<sup>n</sup> of first order.

which involves First Order partial derivatives p and q, with Degree higher than 1 & product is called non-linear partial differential eq<sup>n</sup> of first order.

(i) Eq<sup>n</sup> of the form  $f(p, q) = 0$  i.e;

only p and q & no x, y, z ; a, b are conn. by reltn where a & b conn. by  $f(a, b) = 0$

Q. (i)  $\sqrt{p} + \sqrt{q} = 1$  compl. soln  $z = ax + by + c.$

eq<sup>n</sup> of form  $f(p, q) = 0.$

complete soln is  $x = ax + by + c$

$$\sqrt{a} + \sqrt{b} = 1 \text{ or } b = (1 - \sqrt{a})^2$$

compl. soln in  $z = ax + (1 - \sqrt{a})^2 y + c. \underline{\text{du}}$

$$(ii). pq = p+q.$$

gen. soln  $z = ax + by + c.$

Replace  $p+q$  by  $a^2 b.$

$$ab = a+b$$

or

$$b = \frac{a}{a-1}$$

Now complete soln

$$z = ax + \frac{a}{a-1} y + c.$$

$$\text{Q: } x^2 p^2 + y^2 q^2 = z^2.$$

Here degree is 2 hence non-linear.

The given eqn can be written as.

$$\left( \frac{x}{z} \frac{\partial z}{\partial x} \right)^2 + \left( \frac{y}{z} \frac{\partial z}{\partial y} \right)^2 = 1. \quad \text{--- (1)}$$

$$\text{Let } \frac{\partial x}{z} = \partial X \quad \frac{\partial y}{z} = \partial Y \quad \frac{\partial z}{z} = \partial Z.$$

$$\text{then } x = \log z$$

$$y = \log y \quad \left. \begin{array}{l} \\ \end{array} \right] \text{ on integration}$$

$$z = \log z$$

$$\frac{\partial z}{\partial x} = \frac{x}{z} \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{y}{z} \cdot \frac{\partial z}{\partial y}$$

Now eqn (1) can be written as

$$\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = 1.$$

$$P^2 + Q^2 = 1$$

where;

$$P = \frac{\partial z}{\partial x} \quad \& \quad Q = \frac{\partial z}{\partial y}$$

It is of the form  $f(p, q) = 0$ .

or complete soln if  $z = ax + by + c$ .

where  $a^2 + b^2 = 1$

or

$$b = \sqrt{1-a^2}$$

### Summary

1.  $p, q$  ho  $x, y, z$  na ho.

$$p = \frac{\partial z}{\partial x} = a; q = \frac{\partial z}{\partial y} = b$$

assume complete soln  $z = ax + \underbrace{\phi(a)y + c}_{b = \phi(a)}$

Eq<sup>n</sup> of form  $z = px + qy + f(p, q)$  (CLAIRAUT TYPE)

Q.  $z = px + qy + \sqrt{1+p^2+q^2}$

$$z = px + qy + \sqrt{1+p^2+q^2}$$

$$\text{put } p=a, q=b$$

$$z = ax + by + \sqrt{1+a^2+b^2}$$

Q. find singular integral (free from constant) of  $z = px + qy + pq$ .

$$z = px + qy + f(p, q)$$

complete integral

$$z = ax + by + ab$$

diff. w.r.t.  $a$  &  $b$

$$0 = x + b \quad 0 = y + a$$

$$z = -xy - xy + xy$$

$$z = -xy$$

singular soln of eqn.

Q.  $4xyz = pq + 2px^2y + 2qxy^2$ .

$$x^2 = X \quad y^2 = Y$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = 2x \cdot \frac{\partial z}{\partial X}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = 2y \frac{\partial z}{\partial Y}$$

eqn becomes.

$$4xyz = 4xy \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y} + 4x^3y \frac{\partial z}{\partial X} + 4xy^3 \frac{\partial z}{\partial Y}$$

$$z = x^2 \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \\ = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$$

$$z = px + qy + pq.$$

$$p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y}.$$

$$z = px + qy + f(p, q).$$

$$\text{complete soln } z = ax + by + ab$$

$$z = ax^2 + by^2 + ab.$$

Eqn of form  $f_1(x, p) = f_2(y, p)$ .

$z$  is absent and  $x$ 's term involving  $p$  & can be separated  
those involving  $y$  and  $q$ :

$$f_1(x, p) = f_2(y, p) - a.$$

$$p = F_1(x) \quad q = F_2(y).$$

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} dy = pdx + qdy.$$

$$dz = F_1(x)dx + F_2(y)dy.$$

$$z = \int F_1(x)dx + \int F_2(y)dy + b.$$

Q.  $p^2 - q^2 = x - y.$

$p$  &  $q$  sth  $x$ .

$q$  &  $k$  sth  $y$ .

$p^2 - x = q^2 - y$  which is of the form  $F_1(x, p) = F_2(y, q)$

$$p^2 - x = q^2 - y = a.$$

$$p^2 = x + a \quad ; \quad q^2 = y + a.$$

$$p = \sqrt{x+a} \quad ; \quad q = \sqrt{y+a}.$$

put in soln  $dz = pdx + qdy$ .

$$dz = \sqrt{x+a} dx + \sqrt{y+a} dy$$

$$\text{on integrating; } z = \frac{2}{3} (x+a)^{3/2} + \frac{2}{3} (y+a)^{3/2} + b.$$

Q. Solve :  $yp = 2yx + \log q$ .

Soln.  $p = 2x + \frac{1}{y} \log q$  or  $p - 2x = \frac{1}{y} \log q$ .

which is of the form  $F_1(x, p) = F_2(y, q)$

$$p - 2x = \frac{1}{y} \log q = a$$

then  $p = 2x + a$

$$\text{& } \log q = ay \Rightarrow q = e^{ay}$$

substituting  $p$  &  $q$  in  $dz = pdx + qdy$ ; we get

$$dz = (2x + a)dx + e^{ay}dy.$$

on integrating

$$z = x^2 + tax + \frac{1}{a} e^{ay} + b.$$

### CHARPIT METHOD

general method for finding soln of non-linear partial diff. eqn of first order.

Let given eqn be  $F(x, y, z, p, q) = 0$ . —①

If we can find another relation

$$F(x, y, z, p, q) = 0. —②$$

involving  $x, y, z, p$  &  $q$ , then solve ① & ② for  $p$  &  $q$ , & substi-

- te in  $dz = pdx + qdy$ . —③

The auxiliary eqn are -

$$\frac{dp}{\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z}} = \frac{dz}{-p \frac{\partial F}{\partial p} - q \frac{\partial F}{\partial q}} = \frac{dx}{\frac{-\partial F}{\partial p}} = \frac{dy}{\frac{-\partial F}{\partial q}} = \frac{df}{0}$$

$$\frac{dp}{fx + pfz} = \frac{dq}{fy + qfz} = \frac{dz}{-pf - qfq} = \frac{dx}{-fp} = \frac{dy}{-fq} = \frac{df}{0}$$

$$Q. \quad 2zx - px^2 - 2qxy + pqy = 0$$

$$\frac{\partial F}{\partial x} = 2z - 2px - 2qy$$

$$\frac{\partial F}{\partial y} = -2qx$$

$$\frac{\partial F}{\partial z} = 2x$$

$$\frac{\partial F}{\partial p} = -x^2 + qy$$

$$\frac{\partial F}{\partial q} = -2xy + p$$

$$\frac{dp}{\partial F + p \frac{\partial F}{\partial x}} = \frac{dq}{\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z}} = \frac{dz}{-p \frac{\partial F}{\partial p} - q \frac{\partial F}{\partial q}} = \frac{dx}{-2F} = \frac{dy}{-2F} = \frac{df}{0}$$

$$\frac{dp}{2z - 2py} = \frac{dq}{-2qy + qx^2 - qy} = \frac{dz}{x^2 - qy} = \frac{dy}{2xy - p} = \frac{df}{0}$$

$$dq = 0$$

$$q = a.$$

$$dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a} dx + ady.$$

$$\frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a} dx$$

Integrating

$$\log(z - ay) = \log(x^2 - a) + \log b$$

or

$$z - ay = b(x^2 - a) \text{ complete integration.}$$

Q.  
Soln

$$\text{Solve } (p^2 + q^2)y = qz$$

$$F = (p^2 + q^2)y - qz = 0$$

$$\frac{\partial F}{\partial x} = 0; \frac{\partial F}{\partial y} = p^2 + q^2; \frac{\partial F}{\partial z} = -q; \frac{\partial F}{\partial p} = 2py;$$

$$\frac{\partial f}{\partial q} = 2qy - 2$$

Apply Auxiliary eqn we get,

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{df}{0}$$

from ① & ②

$$p dp + q dq = 0$$

$$\text{Integ. } p^2 + q^2 = a^2$$

$$\text{put in ① } q = \frac{a^2 y}{z}$$

$$\begin{aligned} p &= \sqrt{a^2 - q^2} \\ &= \sqrt{a^2 - \frac{a^4 y^2}{z^2}} = \frac{a}{z} \sqrt{z^2 - a^2 y^2} \end{aligned}$$

$$dz = pdx + qdy = \frac{a}{z} \int z^2 - a^2 y^2 dx + \frac{a^2 y}{z} dy.$$

$$z dz - a^2 y dy = a \int z^2 - a^2 y^2 dz \text{ or } \frac{1}{2} d(z^2 - a^2 y^2) = adx$$

on integrating, we get,

$$\sqrt{z^2 - a^2 y^2} = ax + b$$

$$z^2 = (ax + b)^2 + a^2 y^2.$$

Q. find complete integral of  $px + qy = pq$ .  
 Given Here given eqn is  $F(x, y, z, p, q) = px + qy - pq = 0$ .  $\textcircled{1}$

$\therefore$  charpit's auxiliary eqn are

$$\frac{dp}{\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z}} = \frac{dz}{-p \frac{\partial F}{\partial p} - q \frac{\partial F}{\partial q}} = \frac{dx}{\frac{\partial F}{\partial p}} = \frac{dy}{\frac{\partial F}{\partial q}}$$

$$\frac{dp}{\frac{\partial F}{\partial x}} = p \quad \frac{\partial F}{\partial y} = q \quad \frac{\partial F}{\partial p} = x - q \quad \frac{\partial F}{\partial p} = x - q$$

$$\frac{\partial F}{\partial z} = 0 \quad \frac{\partial F}{\partial z} = 0 \quad \frac{\partial F}{\partial q} = y - p \quad \frac{\partial F}{\partial q} = y - p$$

$$\frac{dp}{p+0} = \frac{dq}{q+0} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{-x+q} = \frac{dy}{-y+p}$$

$$= \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-px+pq-yq+pq} = \frac{dx}{-x+q} = \frac{dy}{-y+p}$$

$$= \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{2pq-px-yq} = \frac{dx}{-x+q} = \frac{dy}{-y+p} \quad \textcircled{2}$$

Taking the last two fraction of (2)

$$\frac{1}{p} dp = \frac{1}{q} dq$$

Integrating,  $\log p = \log q + \log a$

$$\text{or } p = aq \quad \textcircled{3}$$

Substituting the value of  $p$  in  $\textcircled{1}$ , we have

$$aqx + qy - aq^2 = 0 \quad \text{or } aq = ax + y \text{ as } q \neq 0 \quad \textcircled{4}$$

$$\text{from } \textcircled{3} + \textcircled{4} \quad q = (ax + y)/a \quad \text{and } p = ax + y \quad \textcircled{5}$$

Putting  $p, q$  in  $dz = pdx + qdy$ , we get

$$dz = (ax + y)dx + [(ax + y)/a]dy$$

or

$$adz = (ax + y)d(ax + y) = udu \text{ where } u = ax + y.$$

Integrating  $az = (1/2)u^2 + b = \frac{1}{2}(ax+by)^2 + b$

viz complete integral, a & b being arbitrary const.

Q. find a complete soln of integral  $p^2 - q^2 q = y^2 - x^2$ .

$$p^2 - q^2 q - y^2 + x^2 = 0 \quad \text{--- (1)}$$

$$\frac{dp}{2x} = \frac{dq}{-2qy-2y} = \frac{dz}{-p(2p)-q(-q^2)} = \frac{dx}{-2p} = \frac{dy}{y^2} \quad \text{--- (2)}$$

$$\frac{dp}{2x} = \frac{dq}{2y} \text{ taking first & fourth}$$

$$pdq + xdn = 0.$$

On integrating

$$p^2/2 + x^2/2 = a^2/2$$

or

$$p^2 + x^2 = a^2 \quad \text{--- (3)}$$

Solving (1) & (3) for p, q.

$$p = \sqrt{a^2 - x^2}, q = a^2 y^{-2} - 1.$$

$$dz = pdx + q dy$$

$$= \int a^2 - x^2 dx + (a^2 y^{-2} - 1) dy.$$

$$\text{Integrating, } z = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} (x/a) - (a^2/y) - y + b$$

cauchy's method

consider first order partial diff. eqn.

$$a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} = f(x, y) + Ku; \quad u(0, y) = h(y) \quad \text{--- (1)}$$

Let  $u(x, y)$  be the soln of eqn 1.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{--- (2)}$$

from (1) & (2)

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x, y) + Ku} \quad \text{--- (3)}$$

taking (1) & (2)

$$\frac{dx}{a} = \frac{dy}{b} \quad \text{--- (4)}$$

Now eqn becomes

$$bx - ay = c. \quad \text{--- (5)}$$

NOW

$$\frac{dx}{a} = \frac{du}{f(x, y) + Ku} \quad \text{--- (6)}$$

$$\frac{dx}{a} = \frac{du}{f\left(x, \frac{bx-c}{a}\right) + Ku} \quad \text{--- (7)}$$

NOW

$$\frac{du}{dx} - \frac{Ku}{a} = \frac{1}{a} f\left(x, \frac{bx-c}{a}\right) \quad \text{--- (8)}$$

which is a first order linear ordin. diff. eqn.  
 $I.F. = e^{-K/a x}$

soln of (8) is of the form  
 $u = g(x, c) + c_2.$

where  $c_1 = g(c)$ ;  $g$  is an arbitrary const f'n  
from eqn ④

$$u = g(x, c) + g(c)$$

↳ determined by using given conditions.

Q: Use Cauchy's method of characteristics to solve partial diff. eqn.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = xy; \quad u(x, 0) = 0.$$

System of eqn is  $\frac{dx}{1} = \frac{dy}{1} = \frac{du}{x+y}$  — ①

Taking ① & ② we get

$$x - y = c \quad — ②$$

$$x = c + y$$

$c \rightarrow$  constant

Now taking  $\frac{dy}{1} = \frac{du}{x+y}$

$$(x+y)dy = du$$

$$dy = \frac{du}{x+y}$$

$$2y + c$$

$$u(x, y) = y^2 + cy + c_1 \quad — ③$$

from ③  $c_1 = g(c)$  then,

$$u(x, y) = y^2 + cy + g(c)$$

$$= y^2 + y(x-y) + g(x-y) \quad — ④$$

where  $g(x-y)$  is an arbitrary f'n

On applying condition  $u(x, 0) = 0$  on eqn ④ we get

$$0 = g(x)$$

$$g(x-y) = 0$$

consequently, the sol'n is

$$u(x, y) = xy.$$

- Q. use the method of characteristics to solve:-
- $$U_x - U_y = 0 \quad ; \quad U(x, 0) = x.$$
- $$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad ; \quad U(x, 0) = x.$$

The system of eqn

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{0}. \quad \text{--- (1)}$$

taking (1) & (2)

$$\frac{dx}{1} = \frac{dy}{-1}.$$

$$-1dx = 1dy.$$

$$-x = y.$$

$$-y - x = 0.$$

$$-y - x = c. \quad \text{--- (2)}$$

$$\begin{aligned} c &= x + y \\ y &= c + x \end{aligned} \quad \text{--- (3)}$$

Now taking (3) & (3).

$$\frac{dy}{-1} = \frac{du}{0}.$$

$$du = 0.$$

$$U(x, y) = c_1.$$

$$U(x, y) = c_1 = g(c) = g(x+y).$$

$g(x+y)$  is an arbit. f^n.

from given condtn

$$U(x, 0) = x = g(x)$$

$$g(x+y) = (x+y)$$

∴ from eqn (3)

$$U(x, y) = x + y.$$

Partial diff. eqn of 2nd order  
which includes x, y, t but none of higher order

Linear partial diff. eqn with constant coefficients

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2} \partial y} + \dots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}} + C_0 \frac{\partial z}{\partial x} + C_1 \frac{\partial z}{\partial y} + P_0 = f(x, y) \quad \textcircled{1}$$

$A_0, A_1, \dots, A_n, B_0, B_1, \dots, B_{n-1}, C_0, C_1, P_0$  are constants or fn of x and y.

Homogeneous linear partial diff. eqn with constant coefficient.

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = f(x, y) \quad \textcircled{2}$$

$a_0, a_1, a_2, \dots, a_n \rightarrow$  constant, is called homogeneous linear partial differential eqn of nth order with constant coefficients.

D for  $\frac{\partial}{\partial x}$  & D' for  $\frac{\partial}{\partial y}$

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n) z = f(x, y)$$

$$\phi(D, D') z = f(x, y).$$

complete soln

$$z = C \cdot F + P \cdot I$$

Rules for finding C.F.

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0.$$

$$(D^2 + a_1 DD' + a_2 D'^2)z = 0.$$

$$D^2 + a_1 DD' + a_2 D'^2 = 0.$$

is called Auxiliary eqn (A.E.)

Let's its roots be  $m_1$  &  $m_2$ .

Case 1:- Distinct root  $m_1 \neq m_2$ .

$$(D - m_1 D')(D - m_2 D')z = 0. \quad \text{--- (3)}$$

Now sol<sup>n</sup> of  $(D - m_2 D')z = 0$ .

$$\begin{aligned} (D - m_2 D')z = 0. &\Rightarrow \frac{\partial z}{\partial x} - m_2 \frac{\partial z}{\partial y} \\ &\Rightarrow p - m_2 q = 0. \end{aligned}$$

which is of Lagrange's form and auxiliary eqn are -

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}.$$

$$y + m_2 x = a. \quad \text{or} \quad dy + m_2 dx = 0.$$

$$dz = 0 \quad \text{or} \quad z = b.$$

$\therefore z = F_2(y - m_2 x)$  is a sol<sup>n</sup> of  $(D - m_2 D')z = 0$ .

eq<sup>n</sup> (3) will also have sol<sup>n</sup> of

$$(D - m_1 D')z = 0 \quad \text{i.e.} \quad z = F_1(y - m_1 x)$$

Hence sol<sup>n</sup>

$$z = F_1(y - m_1 x) + F_2(y - m_2 x).$$

1.  $m_1, m_2, m_3$  roots

then,

$$\boxed{C.F. = F_1(y - m_1 x) + F_2(y - m_2 x) + F_3(y - m_3 x)}$$

2. Equal roots:-

$$z = \phi(y + mx) + x f(y + mx)$$

$$\text{or } z = F_1(y + mx) + x f_2(y + mx)$$

$$\text{Q. } 4r - 12s + 9t = 0.$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}.$$

$$4 \frac{\partial^2 z}{\partial x^2} - 12 \frac{\partial^2 z}{\partial x \partial y} + 9 \frac{\partial^2 z}{\partial y^2} = 0.$$

$$\left[ 4 \frac{\partial^2}{\partial x^2} - 12 \frac{\partial^2}{\partial x \partial y} + 9 \frac{\partial^2}{\partial y^2} \right] z = 0.$$

$$4D^2 - 12DD' + 9D^2 = 0.$$

$$4m^2 - 12m + 9 = 0.$$

$$(2m-3)^2 = 0.$$

$$m = \frac{3}{2}, \frac{3}{2}.$$

$$C.F. = F_1 \left( y + \frac{3}{2}x \right) + F_2 \left( y + \frac{3}{2}x \right)^2 = 0$$

$$P.I. = 0.$$

$$\text{Q. } \frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 \partial y} + 2 \frac{\partial^3 z}{\partial x \partial y^2} = 0.$$

$$D^3 z - 3D^2 D' z + 2DD' z^2 = 0.$$

$$m^3 - 3m^2 + 2m = 0.$$

$$m(m-1)(m-2) = 0.$$

$$m = 0, 1, 2.$$

$$C.F. = F_1(y) + F_2(y+x) + F_3(y+2x)$$

$$P.I. = 0.$$

$$\text{Q. ii) } r = a^2 t$$

$$\frac{z^2 z'}{D x^2} = a^2 t$$

$$D^2 z = a^2 t \Rightarrow D^2 z - a^2 D'^2 = 0.$$

$$D'^2 = 0.$$

$$m = 0, 0.$$

$$C.F. = F_1(y+0) + 0F_2(y+0)$$

$$C.F. = F_1$$

$$m^2 = a^2$$

$$m^2 - a^2 = 0.$$

$$m = \pm a.$$

$$C.F. = F_1(y+ax) + F_2(y+ax)$$

$$P.I. = 0.$$

$$Z = C.F. + P.I.$$

$$= F_1(y+ax) + F_2(y+ax).$$

$$\text{Q. iii) } D^3 D'^2 + D^2 D'^3 = 0.$$

$$m^3 + m^2 = 0.$$

$$m(m+1) = 0.$$

$$m = 0, 0, -1.$$

$$C.F. = F_1(y+0x) + 0F_2(y+0x) + F_3(y-x)$$

$$P.I. = 0.$$

$$y = C.F. + P.I.$$

But Order is 5 then how 3 roots?

Hence

$$D^2 D'^2 (D + D') = 0$$

$$D^2 = 0.$$

$$m^2 = 0$$

$$m = 0, 0x$$

$$C.F. = F_1(y) + m_1 x F_2(y+x)$$

$$= f_1(y) + x F_2(y).$$

NOTE-

when power of  $D'$  is greater than power of  $D$ ; in this situation,  
formation of auxiliary eqn is

$$\text{put } D' = m \quad D = 1$$

case 1- If roots are distinct like  $m_1, m_2, m_3$ .  
then C.F. is  $F_1(x+m_1y) + F_2(x+m_2y) + F_3(x+m_3y)$ .

case 2- If roots are same like  $m, m, m$ .  
C.F. =  $F_1(x+my) + yF_2(x+my) + y^2F_3(x+my)$ .

$\Rightarrow$  Now back to questn

$$D' = 0.$$

$$m = 0, 0.$$

$$C.F. = F_3(y) + yF_4(y).$$

$$(D + D')$$

$$\Rightarrow (m+1) = 0.$$

$$m = -1.$$

$$C.F. = F_5(y-x).$$

$$C.F. = F_1(y) + xF_2(y) + F_3(x) + yF_4(x) + F_5(y-x).$$

$$C.F.+I = F_1(y) + xF_2(y) + F_3(x) + yF_4(x) + F_5(y-x) + 0.$$

$$(iv) \frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0.$$

$$D^4 z - D^4 z = 0.$$

$$m^4 - 1 = 0. \Rightarrow (m^2 - 1)(m^2 + 1) = 0.$$

$$m = 1, -1, i, -i.$$

$$C.F. = F_1(y+x) + xF_2(y+x) + x^2F_3(y+x) + F_4 \cdot x^3(y+x).$$

$$= F_1(y+x) + xF_2(y+x) + x^2F_3(y+x) + x^3F_4(y+x) + 0.$$

$$C.F. = F_1(y+x) + xF_2(y-x) + F_3(y+ix) + F_4(y-ix) + 0$$

Q. (i)  $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$   
(ii)  $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$ .  
(iii)  $y - 4s + 4t = 0$ .

Q4. Solve  $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$ .

$D^4 z + D'^4 z = 0$ .

$m^4 + 1 = 0$ .

$m^4 - (-1) = 0$ .

$(m^2 - 1)(m^2 + 1) = 0$ .

$m = 1, -1, i, -i$

C.F. =  $F_1(y+x) + F_2(y-x) + F_3(y-iw) + F_4(y+iw)$ .

$m^4 + 1 = 0$ .

$m^4 + 1 + 2m^2 - 2m^2 = 0$ .

$m^4 + 1 + 2m^2 = 2m^2$

$(m^2 + 1)^2 - (m\sqrt{2})^2 = 0$ .

$(m^2 + \sqrt{2}m + 1)(m^2 - \sqrt{2}m + 1) = 0$ .

$m^2 + \sqrt{2}m + 1 = 0$ .

or

$m^2 - \sqrt{2}m + 1 = 0$ .

$m = \frac{-1 \pm i}{\sqrt{2}}, \frac{1 \pm i}{\sqrt{2}}$

$z_1 = \frac{-1+i}{\sqrt{2}}$  and  $z_2 = \frac{1+i}{\sqrt{2}}$ .

$m = z_1, \bar{z}_1, z_2, \bar{z}_2$ .

$z = C.F.P.I.$

$= F_1(y + z_1x) + F_2(y + \bar{z}_1x) + F_3(y + z_2x) + F_4(y + \bar{z}_2x)$ .

- Rules for finding P.I.

\* P.I. of  $F(D, D')z = \phi(x, y)$

is given by :-

$$\frac{1}{F(D, D')} \phi(x, y)$$

case - when  $\phi(x, y)$  is a fn of  $(x, y)$  [Right hand val af  $x, y$  m= no].

To find P.I.

$$F(D, D')z = \phi(ax+by); F(a, b) \neq 0.$$

Hom. fn of degree 'n'.

Step :-

1. Replace  $D$  by  $a$  &  $D'$  by  $b$

$F(D, D')$  to get  $F(a, b)$ .

2. put  $ax+by = u$  now become  $\phi(u)$ .

Now integrate  $\phi(u)$  until n times. (integrate times of power of ord).

$$P.I. = \frac{1}{F(a, b)} \iiint \dots \int \phi(u) du du du \dots du$$

(n times)

3. Replace  $u$  by  $ax+by$  by at least.

\*  $F(a, b) \neq 0$  if  $F(a, b) = 0$  then method will fail.

### Case of Failures :-

1. find eqn of  $F(a, b) = 0$ .

and  $P(D, D')z = \phi(ax+by); F(a, b) = 0$ .

Now

2. Differentiate  $F(D, D')$  partially w.r.t  $D'$  & simultaneously multiply the expression by  $x$ .

3. check  $F'(a, b) \neq 0$  ?

then; P.I. =  $\frac{1}{F(D, D')} \phi(ax+by)$

$$= x \cdot \frac{1}{D} \phi(ax+by)$$

Now if it fails

$$= x^2 \cdot \frac{1}{\frac{2}{D}} \phi(ax+by) \dots$$

If it also fails then  $\frac{1}{\frac{3}{D}} \phi'''(D, D')$

$$\text{Q. (i)} \frac{\partial^3 u}{\partial x^3} - 3 \frac{\partial^3 u}{\partial x^2 \partial y} + 4 \frac{\partial^3 u}{\partial y^3} = e^{x+2y}$$

Now;

C.F. we'll find

Now;

$$P.I. = \frac{1}{D^3 - 3D^2 D' + 4D'^3} e^{x+2y}$$

$$= \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3} \int \int \int e^u du dy dx \quad x+2y = 4$$

$$= \frac{1}{27} e^{x+2y}$$

where  $f_1, f_2, f_3$  are arbitrary fns.

$$\text{Q. } \frac{\partial^2 z}{\partial x^2} + 3 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = x+y.$$

$$D^2 + 3DD' + 2D'^2 = 0.$$

$$m^2 + 3m + 2 = 0.$$

$$m^2 + (2+1)m + 2 = 0.$$

$$m = -1, -2.$$

$$C.F. = F_1(y-x) + F_2(x-2y).$$

Now P.I.

$$= \frac{1}{D^2 + 3D^2 D' + 2D'^2} (x+y).$$

$$= \frac{1}{(1)^4 + 3(1)(1) + 2(1)^2} \iint \sin u du \quad \text{2 bars.}$$

$$= \frac{1}{6} : \frac{u^3}{6}.$$

$$= \frac{u^3}{36} = \frac{(x+y)^3}{36}.$$

$$z = C.F. + P.I.$$

$$= f_1(y-x) + f_2(y-2x) + \frac{(x+y)^3}{36}.$$

Q.  $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y).$

$$D^2 - 2DD' + D'^2$$

$$m^2 - 2m + 1 = 0.$$

$$m^2 - m - m + 1 = 0.$$

$$m(m-1) - 1(m-1) = 0.$$

$$m = 1, 1.$$

$$C.F. = f_1(y+x) + x f_2(y+x)$$

$$P.I. = \sin(2x+3y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2} (\sin(2x+3y)).$$

$$= \frac{1}{(2)^2 - 2(2)(3) + (3)^2} \iint \sin(2x+3y).$$

$$= \frac{1}{(D-D')^2} \sin(2x+3y)$$

$$= \frac{1}{(2-3)^2} \iint \sin u du.$$

$$= -\sin u = -\sin(2x+3y)$$

$$z = C.F. + P.I.$$

$$= f_1(y+x) + x f_2(y+x) - \sin(2x+3y).$$

$$8. \quad r+s-2t = \int 2x+y$$

$$(D^2 + DD' - 2D'^2)z = (2x+y)^{1/2}.$$

$$m^2 + m - 2 = 0.$$

$$(m-1)(m+2) = 0.$$

$$m = 1, -2.$$

$$C.F. = F_1(y+x) + F_2(y-2x).$$

$$P.I. = \frac{1}{D^2 + DD' - 2D'^2} (2x+y)^{1/2}.$$

$$a = 2 \quad b = 1.$$

$$= \frac{1}{D^2 + DD' - 2D'^2} \int \int (u)^{1/2} du du.$$

$$= \frac{1}{(2)^2 + (2)(1) + 2(1)^2} \int \frac{u^{1/2+1}}{1/2+1} du$$

$$= \frac{1}{4+2+2} \left[ \frac{u^{3/2+1}}{3/2+1} \right].$$

$$= \frac{1}{8} \cdot \frac{u^{5/2}}{5/2}$$

$$= \cancel{\frac{2}{5}} \cdot \frac{1}{\cancel{4}} u^{5/2}.$$

$$= \frac{1}{4} \cdot \frac{4}{15} u^{5/2} = \frac{1}{15} (2x+y)^{5/2}.$$

$$Z = C.F. + P.I.$$

$$= F_1(y+x) + F_2(y-2x) + \frac{1}{15} (2x+y)^{5/2}.$$

$$\text{Q. } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x+ty).$$

$$801^n \quad (D^2 + D'^2)z = \cos mx \cos ny + 30(2x+4y).$$

$$m^2 + 1 = 0$$

$$m = \pm L$$

$$c \cdot f = f_1(y + ix) + F_2(y - ix).$$

$$P.I. = \frac{1}{D^2 + D'^2} \left[ f_1(y+in) + f_2(y+in) \right] + \frac{1}{D^2 + D'^2} \left[ \int \int 30(2x+y) \right]$$

$$\textcircled{2} = \frac{1}{D^2 + D^1} \iint 30(u) du du.$$

$$= \frac{1}{(2)^2 + (1)^2} \int \int 30 \sin \theta d\theta du$$

$$= \frac{1}{5} \int 30 u^{\frac{1}{1+1}} du.$$

$$= \frac{1}{5} \int \frac{30u^2}{x} du.$$

$$= \frac{1}{5} \cancel{15} \frac{U^3}{\cancel{3}}$$

$$= u^3 = (2x+y)^3.$$

$$\textcircled{1} \quad \div 8 \times by \ 2.$$

$$= \frac{1}{2} \frac{1}{D^2 + D'^2} R_{\text{WNL}}(m \mathbf{x} + n \mathbf{y}).$$

$$= \frac{1}{2} \left[ \frac{3}{m+n^2} \right] \int \cos u du$$

$$= -\frac{1}{2} (m^2 + n^2) [\cos(mx+ny) + \cos(mn-ny)]$$

$$Z = C + F + P - I$$

$$= -\frac{1}{2(m^2+n^2)} \left( \cos(mx+ny) + \omega_0(mn-ny) + (2x+ny)^3 \right)$$

$$Q. \frac{\partial^3 z}{\partial x^3} - 4 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial x \partial y^2} = 4 \sin(2x+y)$$

$$D^3 - 4D^2 D' + 4DD'^2 = 4 \sin(2x+y).$$

$$m^3 - 4m^2 + 4m = 0.$$

$$m(m^2 - 4m + 4) = 0.$$

$$m = 0, 1, 2.$$

$$C.F. = F_1(y) + F_2(y+2x) + xF_3(y+2x).$$

$$P.I. = \frac{1}{D^3 - 4D^2 D' + 4DD'^2} \iiint (4 \sin(2x+y)) dududu.$$

$$= \frac{1}{(4)^3 - 2(4)^2(1) + 4(2)(1)} \iint 4(-\cos(2x+y)) dudu.$$

$$= \frac{4}{D} \left[ \frac{1}{(D-2D')^2} \sin(2x+y) \right].$$

$$= x \cdot \frac{4}{D} \left[ \frac{1}{2(D-2D')} \sin(2x+y) \right] = 4x^2 \cdot \frac{1}{D} \left[ \frac{1}{2} \sin(2x+y) \right]$$

$$= 2x^2 \frac{1}{D} \sin(2x+y).$$

$$= -2x^2 \frac{\cos(2x+y)}{2} = -x^2 \cos(2x+y).$$

$$z = C.F. + P.I. = F_1(y) + F_2(y+2x) + x \sin(y+2x) - x^2 \cos(2x+y).$$

$$Q. \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y.$$

$$m^2 - m = 0 \quad m = 0, 1.$$

$$C.F. = f_1(y) + f_2(y+x).$$

$$P.I. = \frac{1}{D^2 - DD'} \sin x \cos y.$$

$$= \frac{1}{2(D^2 - DD')} [\sin(x+y) + \sin(x-y)].$$

$$\text{Q. } \frac{\partial^3 z}{\partial x^2 \partial y} - 2 \frac{\partial^3 z}{\partial x^2 y^2} + \frac{\partial^3 z}{\partial y^3} = \frac{1}{x^2}.$$

$$(D^2 D^1 - 2 D D^1)^2 + D^1)^3) z = \frac{1}{x^2}.$$

$$\begin{aligned} m^2 - 2m + 1 \\ = m^2 - (1+1)m + 1 = 0. \\ = m^2 - m - m + 1 = 0. \\ = m(m-1) - 1(m-1) = 0. \\ = (m-1)(m-1) = 0. \\ = m = 1, 1. \end{aligned}$$

$$\begin{aligned} D^1 = m. \\ D^1 (D^2 - 2 D D^1 + D^1)^2 z = \frac{1}{x^2} \\ D = m. \\ (m-1)(m-1) = 0. \\ m = 1, 1. \end{aligned}$$

$$\begin{aligned} c \cdot F = f_2(y+x) + x f_3(y+x) \\ = f_1(x) + f_2(y+x) + x f_3(y+x). \end{aligned}$$

$$\begin{aligned} P \cdot I \cdot &= \frac{1}{D^2 D^1 - 2 D D^1 + D^1)^3} \iint \frac{1}{x^2} \\ &= \frac{1}{D^2 D^1 - 2 D D^1 + D^1)^3} \iint (x^{-2}) \\ &= \frac{1}{D^2 D^1 - 2 D D^1 + D^1)^3} \iint (1x + oy)^{-2} \\ &= \frac{1}{0+0+0} \iint u^2 du dw. \end{aligned}$$

case fail.

power of  $D'$  is more hence we'll multiply 'y' in  $f_3$  case.

diff. w.r.t. to  $D'$  and multiply by 'y'.

$$= y \cdot \frac{1}{\frac{\partial}{\partial b^1} [F(D, D^1)]} \phi(ax+by).$$

$$= y \cdot \frac{1}{3D^1^2 + D^2 - 4DD^1} \left[ \frac{1}{x^2} \right].$$

$$= y \cdot \frac{1}{3(0)^2 + (1)^2 - 4(1)(0)} \iint \frac{1}{u^2} du dw.$$

$$= y \cdot \int -\frac{1}{u} du = y(-\log u) = -y \log u.$$

Hence complete soln is

$$z = C.F. + P.I.$$

$$= f_1(x) + f_2(y+x) + xF_3(y+x) + y\log x \dots$$

where  $f_1, f_2, f_3$  are arbitrary constant.

\* When  $\phi(m, y)$  is of the form  $x^m y^n$ .

$$\text{Q. } \frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$$

$$\text{SOLN } (D^3 - D^{13}) z = x^3 y^3$$

$$m^3 - 1 = 0$$

$$m = 1, \omega, \omega^2$$

where  $\omega$  is one of the <sup>cube</sup> roots of unity.

$$C.F. = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x).$$

$$P.I. = \frac{1}{D^3 - D^{13}} (x^3 y^3) = \frac{1}{D^3 (1 - \frac{D^{13}}{D^3})} (x^3 y^3).$$

$$= \frac{1}{D^3} \left( 1 - \frac{D^{13}}{D^3} \right)^{-1} (x^3 y^3)$$

$$= \frac{1}{D^3} \left( 1 + \frac{D^{13}}{D^3} \right) (x^3 y^3).$$

$$= \frac{1}{D^3} \left[ x^3 y^3 + \frac{1}{D^3} D^{13} (x^3 y^3) \right]$$

$$= \frac{1}{D^3} \left[ x^3 y^3 + \frac{1}{D^3} (x^3) \right]$$

$$= \frac{1}{D^3} (x^3 y^3) + \frac{1}{D^6} (x^3)$$

$$= \frac{x^6 y^3}{6 \cdot 5 \cdot 4}$$

$$= \frac{x^6 y^3}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 0} = \frac{x^6 y^3}{120}$$

$$= \frac{x^6 y^3}{120} + \frac{6 x^9}{10080}$$

$$z = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x) + \frac{x^6 y^3}{120} + \frac{x^9}{10080}.$$

5/8/20  
 Q.  $(D^2 - 6DD' + 9D'^2), z = 12x^2 + 3xy$   
 $m^2 - 6m + 9 = 0$   
 $m = 3, 3$

$C.F. = F_1(y+3x) + xF_2(y+3x)$

$P.I. = \frac{1}{D^2 - 6DD' + 9D'^2} (12x^2) + \frac{1}{D^2 - 6DD' + 9D'^2} (3xy) = P_1 + P_2$

$$\begin{aligned} P_1 &= \frac{1}{D^2 - 6DD' + 9D'^2} (12x^2) \\ &= \frac{1}{D^2 \left[ 1 - \frac{6D'}{D} + \frac{9D'^2}{D^2} \right]} (12x^2) \\ &= \frac{1}{D^2} \left[ 1 - \frac{6D'}{D} + \frac{9D'^2}{D^2} \right]^{-1} (12x^2) \\ &= \frac{1}{D^2} \left[ 1 - \frac{6DD' + 9D'^2}{D} \right]^{-1} (12x^2) \\ &= \frac{1}{D^2} \left[ 1 + \left( \frac{6DD' + 9D'^2}{D} \right) \right] (12x^2) \\ &= \frac{1}{D^2} + \frac{1}{D^2} \\ &= \frac{12}{D^2} \int \int u^2 du du. \end{aligned}$$

$P_2 = \frac{36}{(D - 3D')^2} (xy) = \frac{36}{D^2} \left[ 1 - \frac{3D'}{D} \right]^{-2} (xy).$

$= \frac{36}{D^2} \left[ 1 + \frac{6D'}{D} \right] (xy).$

$= \frac{36}{D^2} \left[ xy + \frac{6}{D} (x) \right] = \frac{36}{D^2} (xy + 3x^2).$

$\text{Ans} = 36 \left( \frac{x^3}{6} y + \frac{x^4}{4} \right) = x^3 y + 9x^4$

$$P.I. = P_1 + P_2 = 6x^3y + 10x^4$$

$$Z = f_1(y+3x) + x f_2(y+3x) + cx^3y + 10x^4.$$

General method for P.I.

$$Q. \quad \frac{D^2Z}{Dx^2} + \frac{D^2Z}{DxDy} - 6\frac{D^2Z}{Dy^2} = y \cos x.$$

$$(D^2 + DD' - 6D^2)Z = y \cos x.$$

$$m^2 + m - 6 = 0.$$

$$(m+2)(m+3) = 0.$$

$$m = 2, -3.$$

$$C.F. = f_1(y+2x) + f_2(y-3x)$$

$$P.I. = \frac{1}{D^2 + DD' - 6D^2} y \cos x = \frac{1}{(D-2D')(D+3D')} y \cos x.$$

$$= \frac{1}{(D-2D')} \int ((+3x) \cos x) dx ; \text{ where } y = c+3x.$$

[uv valid on AEE; log; expo; algebraic; trigono; expo].

$$\left[ \int uv dx = u v_1 - [u'v] + u''v_3 - u'''v_4 + \dots \right]$$

$v_1, v_2, v_3 \dots$  denotes integration

$u', u'', u''' \dots$  denotes differentiation.

$u \rightarrow$  always algebraic fn.

$v \rightarrow$  trigonometric or exponential

$(c+3x) \rightarrow u:$

$\cos x \rightarrow v. \quad u' = 3.$

$v_1 = \sin x$

$$= \frac{1}{D-2D'} \int ((c+3x) \sin x - (3)(\sin x))$$

$$= \frac{1}{D-2D'} \{ c\sin x + 3 \int x \cos nx dx \}$$

$$= \frac{1}{D-2D'} [c\sin x + 3[x\sin x - \int 1 \cdot \sin x dx]]$$

$$= \frac{1}{D-2D'} [(c+3x)\sin x + 3\cos x] \text{ where } c = y-3x.$$

$$= \frac{1}{b-D-2D'} (y\sin x + 3\cos x) \text{ where } b = y-2x.$$

$$= \int (b-2x)\sin x dx + 3\sin x \text{ where } y = b-2x.$$

$$= -b\cos x - 2 \{ x(-\cos x) - \int 1 \cdot (-\cos x) dx \} + 3\sin x.$$

$$= -b\cos x + 2x\cos x - 2\sin x + 3\sin x.$$

$$= -(b-2x)\cos x + \sin x.$$

$$= -y\cos x + \sin x; \text{ where } b = y+2x.$$

$$x = C.P. + P.I.$$

$$= F_1(y+2x) + F_2(y-3x) - y\cos x + \sin x.$$

Q. solve:-  $\frac{D^2Z}{Dx^2} + \frac{D^2Z}{Dx \cdot D} - 2 \frac{D^2Z}{Dy^2} = (y-1)e^x$

$$(D^2 + DD' - 2D')Z = (y-1)e^x.$$

the auxiliary eqn

$$m^2 + m - 2 = 0.$$

$$(m+2)(m-1) \text{ or } (m-1)(m+2) = 0$$

$$\therefore m = -2, 1.$$

$$\therefore F = f_1(y-2x) + f_2(y+x).$$

$$P.I. = (y-1)e^x$$

$$= \frac{1}{D^2 + DD' - 2D'} (y-1)e^x = \frac{1}{D^2 + DD' - 2D'} (y-1)e^x.$$

$$= \frac{1}{(D-D')(D+2D')} \underset{\substack{\downarrow \\ C}}{(y-1)e^x} \underset{\substack{\downarrow \\ x}}{}$$

$$= \frac{1}{(D-D')} \{ (C+2x)e^x \}$$

$$= \frac{1}{(D-D')} \int \left[ \frac{(c+2x-1)e^x}{u} \right] dx$$

$$\int uvdx = (c+2x-1) \cdot e^x - (2)(e^x)$$

$$= \frac{1}{(D-D')} [(c+2x-1)e^x - 2e^x].$$

$$= \frac{1}{(D-D')} [ce^x + 2xe^x - e^x - 2e^x]$$

$$= \frac{1}{(D-D')} [c+2x-1-2]e^x.$$

where  $y = c+2x$

$$= \frac{1}{(D-D')} [(c-1)e^x + 2(x-1)e^x].$$

$$= \frac{1}{D-D'} [(c+2x)e^x - 3e^x]$$

$$= \frac{1}{D-D'} (ye^x - 3e^x); c=y-2x.$$

$$= \int (b-x)e^x dx - 3e^x \quad y=b-x.$$

$$= b e^x - (x-1)x^x - 3e^x.$$

$$= (b-x-2)e^x = (y-2)e^y \text{ wh. } b=y+x.$$

$Z = C.F. + P.I.$

$$= F_1(y+x) + F_2(y-2x) + (y-2)e^y.$$

$F_1$

$F_2$  arbitrary constant

$F_3$

Q. Non-Homogeneous partial diff. eqn with constant coefficient:-  
 $f(D, D') = F(x, y)$  Non-Homo. eqn. ( $D$  is distinct).

$$z = C.F. + P.I.$$

Methods for finding CF.

Resolve to  $D - mD' - a$

Now consider  $(D - mD' - a) z = 0$ .

$$P - mq = az.$$

$$\text{Lagrange's aux. eqn} \quad \frac{dx}{+1} = \frac{dy}{-m} = \frac{dz}{az}$$

then find soln.

$$z = e^{ax} f(y + mx).$$

C.F. of ①

$$(D - m_1 D' - a_1)(D - m_2 D' - a_2) \dots (D - m_n D' - a_n) z = 0$$

$$z = e^{a_1 x} F_1(y + m_1 x) + e^{a_2 x} F_2(y + m_2 x) + \dots + e^{a_n x} f_n(y + m_n x)$$

In case of repeated factor.

$$(D - mD' - a)^3 z = 0.$$

$$\text{we have } z = e^{ax} f_1(y + mx) + e^{ax} \cdot x f_2(y - mx) + x^2 e^{ax} (y + mx)$$

$$\Rightarrow a - 3D' + 2z = 0$$

$$\text{Ans. C.F.} = e^{-2x} F_1(y + 3x).$$

$$(D + 5D' - 6z) = 0$$

$$= C.F. = e^{6x} F_1(y - 5x).$$

\* When have same factors-

$$(D + 3D' - 2)^4 z = 0.$$

$$= e^{2x} F_1(y + 3x) + x e^{2x} f_2(y - 3x) + x^2 e^{2x} f_3(y - 3x) \\ + x^3 e^{2x} f_4(y - 3x).$$

\* When  $F(D, D')$  cannot be factorized into linear factors  
 In such cases, we use a trial method

$$(D - D'^2)z = 0 \quad \text{--- (1)}$$

Let trial soln of (1) be  $z = Ae^{hx+ky}$  --- (2)

from (2)

$$Dz = \frac{\partial z}{\partial x} = Ahe^{hx+ky}$$

$$D'z = \frac{\partial z}{\partial y} = Ak e^{hx+ky}$$

$$D'^2z = \frac{\partial^2 z}{\partial y^2} = AK^2 e^{hx+ky}$$

Putting in (1), we get

$$Ahe^{hx+ky} - AK^2 e^{hx+ky} = 0$$

$$A(h - K^2)e^{hx+ky} = 0$$

$$h = K^2 \quad \text{--- (3)}$$

eqn (2) gives,

$$z = Ae^{K^2x+ky} \quad \text{--- (4)}$$

Now gen. soln.

$$z = \sum A e^{K^2x+ky}$$

Q.  $(D + D' - 1)(D + 2D' - 2)z = 0$ .

$$(D + D' - 1)(D + 2D' - 2)$$

$$z = e^x f_1(y-x) + e^{2x} f_2(y-2x) + 0$$

as P.I. = 0.

Q.

$$\text{solve } e^{-x} + (xe^{-x} + x^2e^{-x})x + \dots - e^{-x}x^2 - \dots$$

$$PD' (D + 2D' + 1)z = 0$$

~~eff~~

$$(D^2 D' + 2DD' + DD')z = 0$$

$$\underbrace{DD'}_{\text{f}_1(y)} \underbrace{(D + 2D' + 1)z}_{\text{f}_2(y-2x)}$$

$$e^{-x} f_1(y) + e^{-x} f_2(y-2x)$$

Q3. Solve  $x + 2s + t + 2p + 2q + z = 0$ .

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + 2 \frac{\partial z}{\partial y} + z = 0.$$

$$(D^2 + DD' + D'^2 + 2D + 2D' + 1)z = 0.$$

$$(D^2 + DD' + D'^2 + 2(D+D') + 1)z = 0$$

$$= (a+b)^2$$

$$(D + D' + 1)^2 z = 0.$$

$$C.F. = e^{-x} f_1(y-x) + x e^{-x} f_2(y-x)$$

$$P.I. = 0.$$

$$z = e^{-x} f_1(y-x) + x e^{-x} f_2(y-x) + 0.$$

(i)  $(D^2 - D'^2 + D + 3D' - 2)z = 0$

(ii)  $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$

(iii)  $\underbrace{(D^2 - D'^2 + 2D - D)}_D + \underbrace{2D' + D'}_{3D} - \cancel{D'^2 - 2D'D} - 2z = 0.$

$$(D^2 - DD' + 2D + DD' - D'^2 + 2D' - D + D' - 2)z = 0.$$

$$(D - D' + 2)(D + D' - 1)z = 0.$$

$$C.F. = e^{-2x} f_1(y+x) + e^x f_2(y-x).$$

$$P.I. = 0.$$

$$z = e^{-2x} f_1(y+x) + e^x f_2(y-x) + 0.$$

OR

$$D^2 - D'^2 + D + 3D' - 2 + \frac{1}{4} - \frac{1}{4}.$$

$$= D^2 + D + \frac{1}{4} - D'^2 + 3D' - 2 - \frac{1}{4}.$$

$$= \left(D + \frac{1}{2}\right)^2 - \left(D'^2 - 3D' - \frac{9}{4}\right)$$

20

general method for Homogeneous diff. eqn to find P.I.

$$P.I. = \frac{1}{F(D, D')} \phi(x, y)$$

$$= \frac{1}{(D-m_1 D')(D-m_2 D') \dots (D-m_n D')} \phi(x, y)$$

$$= \frac{1}{D-m_1 D' - D-m_2 D' - \dots - D-m_n D'} \phi(x, y)$$

Let us evaluate  $\frac{1}{D-m_1 D'} \phi(x, y)$

consider eqn

$$(D-m_1 D')z = \phi(x, y)$$

or

$$p - mq = \phi(x, y)$$

subsidiary eqn are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(x, y)}$$

from first two members

$$y + mx = c$$

last member.

$$z = \int \phi(x, c - mx) dx$$

$$\frac{1}{D-m_1 D'} \phi(x, y) = \int \phi(x, c - mx) dx$$

where  $c$  is replaced by  $y + mx$  after integratn.

Q  $(D^2 + 2DD' + D'^2)z = 2\cos y - xi\sin y$   
 $m^2 + 2m + 1 = 0.$   
 $(m+1)^2 = 0.$   
 $m = -1, -1.$

$c.f. = f_1(y-x) + f_2x(y-x).$

$P.I. = \frac{1}{(D+D')^2} 2\cos y - \frac{1}{(D+D')^2} (xi\sin y) = P_1 - P_2.$

$P_1 = \frac{1}{(D+D')^2} 2\cos y = \frac{2}{(0+1)^2} \int \int \cos u du du \text{ where } y=u.$ 
 $= 2(-\cos y) = -2\cos y.$

$P_2 = \frac{1}{(D+D')^2} xi\sin y = \frac{1}{D+D'} \int x \sin(c+x) dx \text{ ; where } y=c+x.$ 
 $= \frac{1}{D+D'} \left[ x(-\cos(c+x)) - \int 1(-\cos(c+x)) dx \right].$

$= \frac{1}{D+D'} \left[ -x\cos(c+x) + \sin(c+x) \right].$

$= \frac{1}{D+D'} \left[ -x\cos(b+x) + \sin(b+x) \right].$

$= \frac{1}{D+D'} (-x\cos y + \sin y)$

$= \int -x \cos(b+x) dx + \int \sin(b+x) dx \text{ ; } y=b+x.$

$= -x\sin(b+x) - 2\cos(b+x).$

$= -x\sin y - 2\cos y.$

$$Q. \quad z - 4y + 4t + p - 2q = e^{x+y}.$$

The given eqn is.

$$(D^2 - 4D D' + 4D'^2 + D - 2D')z = e^{x+y}.$$

$$(D - 2D')^2 + (D - 2D')z = e^{x+y}.$$

$$(D - 2D')(D - 2D').$$

$$C.F. = F_1(y+2x) + e^{-x}(y+2x).$$

$$P.I. = \frac{1}{D - 2D' + 1} \left[ \frac{2}{1-2} \right] e^{4u} du.$$

$$= -\frac{1}{D - 2D' + 1} e^{x+y} = -e^{x+y} \cdot \frac{1}{D + 1 - 2(D' + 1) + 1} \quad (1).$$

$$= -e^{x+y} \left[ \frac{1}{D - 2D'} \right] (1) = -e^{x+y} \left[ \frac{1}{D - 2D'} (e^{ax+ay}) \right]$$

$$= -xe^{x+y}.$$

$$Q. \quad (D^2 - D')z = xe^{ax+a^2y}.$$

$$P.I. = \frac{1}{(D^2 - D')} (xe^{ax+a^2y})$$

$$= e^{ax+a^2y} \cdot \frac{1}{(D+a)^2 - (D'-a^2)}$$

$$= e^{ax+a^2y} \cdot \frac{1}{D^2 + 2aD - D'} (x)$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD} \left( 1 + \frac{D}{2a} - \frac{D'}{2aD} \right)^{-1} (x).$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD} \left\{ 1 - \left( \frac{D}{2a} - \frac{D'}{2aD} \right) + \dots \right\} x.$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD} \left( x - \frac{1}{2a} \right).$$

$$= e^{ax+a^2y} \left[ \frac{x^2}{4a} - \frac{x}{4a^2} \right].$$

Eq<sup>n</sup> reducible to partial diff. eq<sup>n</sup> with constant coefficient

$$x = e^X$$

$$y = e^Y$$

so that  $X = \log x$

$$Y = \log y$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x}$$

$$= \frac{1}{x} \frac{\partial z}{\partial X}$$

$$\underline{x \frac{\partial z}{\partial x}} = \frac{\partial z}{\partial X} \Rightarrow \therefore x \frac{\partial}{\partial x} = D \left( \equiv \frac{\partial}{\partial X} \right)$$

$$x^2 \frac{\partial^2 z}{\partial x^2} = (D-1)x \frac{\partial z}{\partial X} = (D-1)Dz$$

$$x^3 \frac{\partial^3 z}{\partial x^3} = (D-2)x^2 \frac{\partial^2 z}{\partial X^2} = (D-2)(D-1)Dz \text{ etc.}$$

$$y \frac{\partial z}{\partial y} = D'z ; y^2 \frac{\partial^2 z}{\partial y^2} = (D'-1)D'z$$

$$y^3 \frac{\partial^3 z}{\partial y^3} = (D'-2)(D'-1)D'z$$

$$xy \frac{\partial^2 z}{\partial x \partial y} = DD'z$$