Unit 3: Partial Orderings & Lattice.

POSET Definition

- A relation R on a set S is called a partial order if it is

  - Reflexive
  - Antisymmetric
  - Transitive

r

- A set S together with a partial ordering R is called a partially ordered set or poset and is denoted as "(A,R)".

Introduction

- An equivalence relation is a relation that is reflexive, symmetric, and transitive
- A partial ordering (or partial order) is a relation that is reflexive, *anti*symmetric, and transitive
  - Recall that antisymmetric means that if $(a,b) \in R$, then $(b,a) \notin R$ unless $b = a$
  - Thus, (a,a) is allowed to be in R
  - But since it's reflexive, all possible (a,a) must be in R

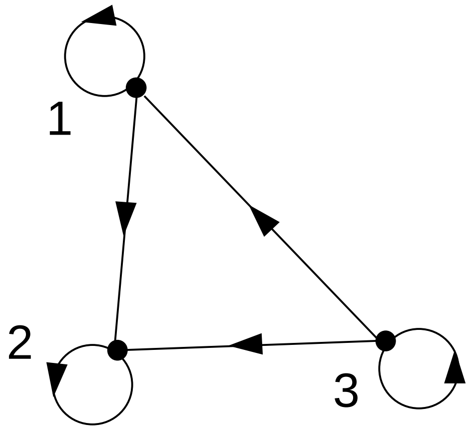
Partially Ordered Set (POSET)

A relation R on a set S is called a *partial ordering* or *partial order* if it is *reflexive*, *antisymmetric*, and *transitive*. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R)

Example (1)

Let S = {1, 2, 3} and

let R = {(1,1), (2,2), (3,3), (1, 2), (3,1), (3,2)}



$\preccurlyeq$

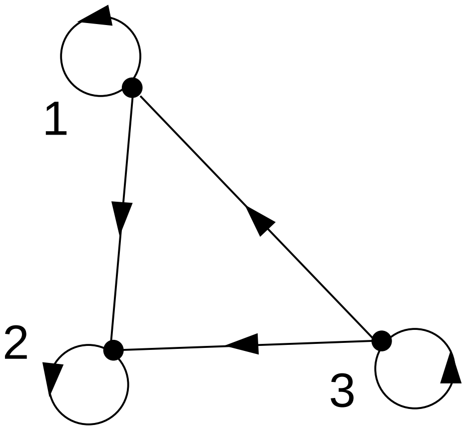
In a poset the notation $a \preccurlyeq b$ denotes that $(a,b) \in R$

This notation is used because the "*less than or equal to*" relation is a paradigm for a partial ordering. (Note that the symbol $\preccurlyeq$ is used to denote the relation in *any* poset, not just the "less than or equals" relation.) The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$

Example

Let S = {1, 2, 3} and

let R = {(1,1), (2,2), (3,3), (1, 2), (3,1), (3,2)}



$2 \preccurlyeq 2$ $\qquad$ $3 \prec 2$

Example (2)

- Show that ≥ is a partial order on the set of integers
  - It is reflexive: $a \geq a$ for all $a \in \mathbf{Z}$
  - It is antisymmetric: if $a \geq b$ then the only way that $b \geq a$ is when $b = a$
  - It is transitive: if $a \geq b$ and $b \geq c$, then $a \geq c$
- Note that ≥ is the partial ordering on the set of integers
- ($\mathbf{Z}$, ≥) is the partially ordered set, or poset

Example (3)

Consider the power set of $\{a, b, c\}$ and the subset relation. $(P(\{a,b,c\}), \subseteq)$

Comparable / Incomparable

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

Example

Consider the power set of $\{a, b, c\}$ and the subset relation. $(P(\{a,b,c\}), \subseteq)$

$$\{a,c\} \not\subseteq \{a,b\} \text{ and } \{a,b\} \not\subseteq \{a,c\}$$

So, $\{a,c\}$ and $\{a,b\}$ are *incomparable*

Totally Ordered, Chains

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called *totally ordered* or *linearly ordered* set, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

- In the poset ($\mathbf{Z}^+, \leq$), are the integers 3 and 9 comparable?
  - Yes, as $3 \leq 9$
- Are 7 and 5 comparable?
  - Yes, as $5 \leq 7$
- As all pairs of elements in $\mathbf{Z}^+$ are comparable, the poset ($\mathbf{Z}^+, \leq$) is a total order
  - a.k.a. totally ordered poset, linear order, or chain

- In the poset ($\mathbf{Z}^+$,|) with "divides" operator |, are the integers 3 and 9 comparable?
  - Yes, as 3 | 9
- Are 7 and 5 comparable?
  - No, as $7 \nmid 5$ and $5 \nmid 7$
- Thus, as there are pairs of elements in $\mathbf{Z}^+$ that are not comparable, the poset ($\mathbf{Z}^+$,|) is a partial order. It is not a chain.

Definition: Let R be a total order on A and suppose $S \subseteq A$. An element s in S is a *least element* of S iff sRb for every b in S.

Similarly for *greatest* element.

Note: this implies that $<a, s>$ is not in R for any a unless $a = s$. (There is nothing smaller than s under the order R).

Well-Ordered Set

$(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and such that every nonempty subset of S has a *least element*.

Example: Consider the ordered pairs of positive integers, $\mathbb{Z}^+ \times \mathbb{Z}^+$ where

$(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$

Well-ordered sets examples

- Example: $(\mathbf{Z},\leq)$
  - Is a total ordered poset (every element is comparable to every other element)
  - It has no least element
  - Thus, it is not a well-ordered set
- Example: $(S,\leq)$ where $S = \{ 1, 2, 3, 4, 5 \}$
  - Is a total ordered poset (every element is comparable to every other element)
  - Has a least element (1)
  - Thus, it is a well-ordered set

Lexicographic Order

This ordering is called *lexicographic* because it is the way that words are ordered in the dictionary.

Given two posets (A_1, R_1) and (A_2, R_2) we construct an *induced* partial order R on $A_1 \times A_2$:

$$< x_1, y_1 > R < x_2, y_2 > \text{ iff}$$

- $x_1 \; R_1 \; x_2$

or

- $x_1 = x_2$ and $y_1 \; R_2 \; y_2$.

Example:

Let $A_1 = A_2 = Z^+$ and $R_1 = R_2 =$ 'divides'.

Then

- $<2, 4>$ R $<2, 8>$ since $x_1 = x_2$ and $y_1 \, R_2 \, y_2$.

- $<2, 4>$ is not related under R to $<2, 6>$ since $x_1 = x_2$ but 4 does not divide 6.

- $<2, 4>$ R $<4, 5>$ since $x_1 \, R_1 \, x_2$. (Note that 4 is not related to 5).

Let Σ be a finite set and suppose R is a partial order relation defined on Σ. Define a relation $\prec$ on Σ^*, the set of all strings over Σ, as follows:

For any positive integers m and n and $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_n$ in Σ^*.

1. If $m \le n$ and $a_i = b_i$ for all i = 1, 2, , . . ., m, then

$$a_1,a_2\ldots a_m \prec b_1b_2\ldots.b_n.$$

2. If for some integer k with $k \le m,\ k \le n,$ and $k \ge 1,\ a_i = b_i$ for all i = 1,2,…,k-1, and $a_k\ R\ b_k$ but $a_k \ne b_k$, then

$$a_1,a_2\ldots a_m \prec b_1b_2\ldots.b_n.$$

3. If ε is the null string and s is any string in Σ^* then $\varepsilon \prec s$.

The Principle of Well-Ordered Induction

Suppose that S is a well-ordered set. Then *P*(x) is true for all $x \in S$, if:

BASIS STEP: *P*(x_0) is true for the least element of S, and

INDUCTION STEP: For every $y \in S$ if *P*(x) is true for all $x \prec y$, then *P*(y) is true.

Hasse Diagrams

Given any partial order relation defined on a finite set, it is possible to draw the directed graph so that all of these properties are satisfied.

This makes it possible to associate a somewhat simpler graph, called a *Hasse diagram*, with a partial order relation defined on a finite set.

Hasse Diagrams (continued)

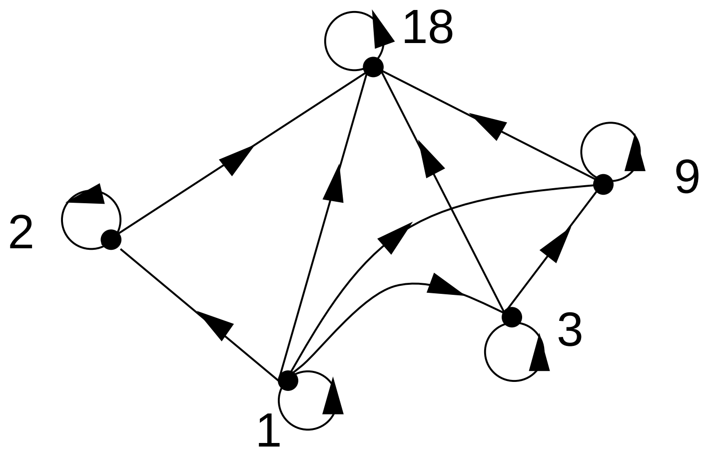
Start with a directed graph of the relation in which all arrows point upward. Then eliminate:

1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

Example

Let A = {1, 2, 3, 9, 19} and consider the “divides” relation on A:

For all $a, b \in A,\ a \mid b \Leftrightarrow b = ka$ for some integer k.

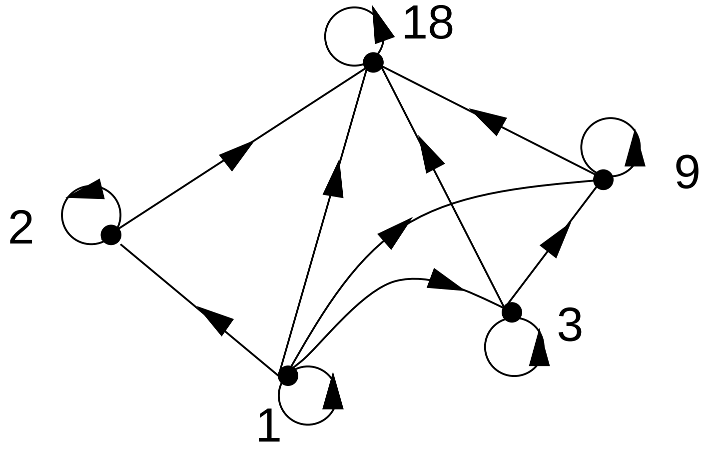


Example

Eliminate the loops at all the vertices.

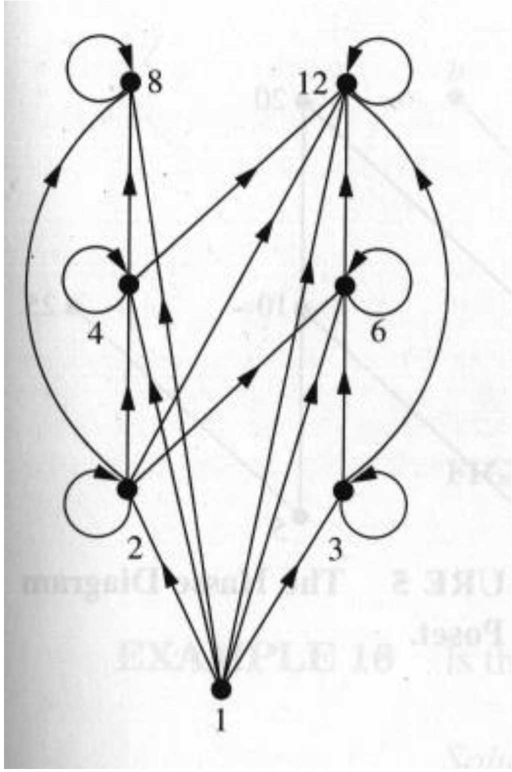
Eliminate all arrows whose existence is implied by the transitive property.

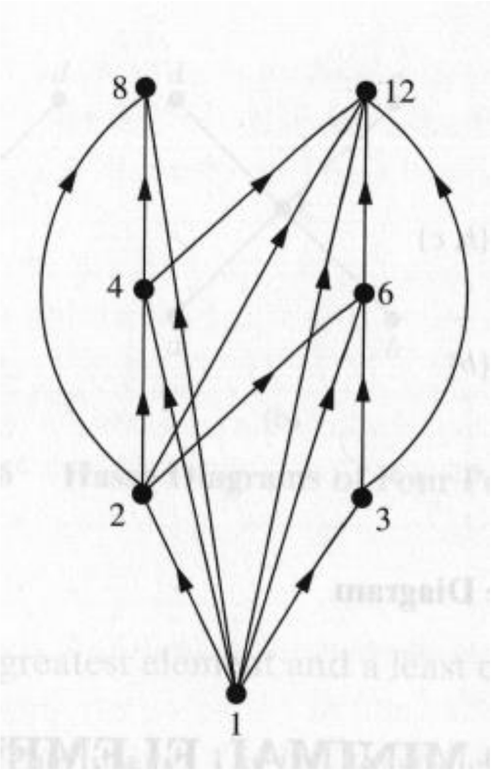
Eliminate the direction indicators on the arrows.

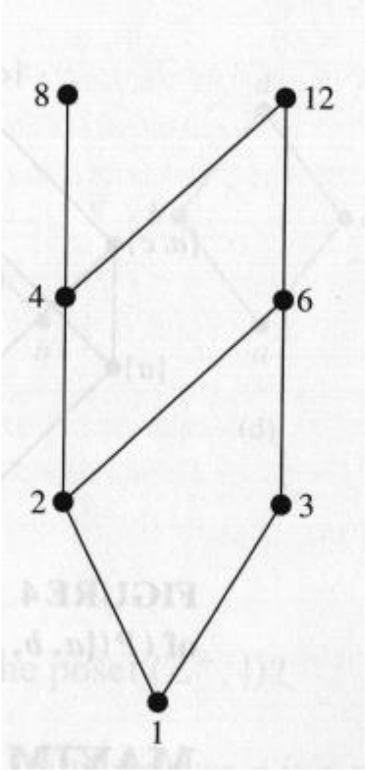


Hasse Diagram

- For the poset ({1,2,3,4,6,8,12}, |)







Construct the Hasse diagram of $(P(\{a, b, c\}), \subseteq)$.

The elements of $P(\{a, b, c\})$ are

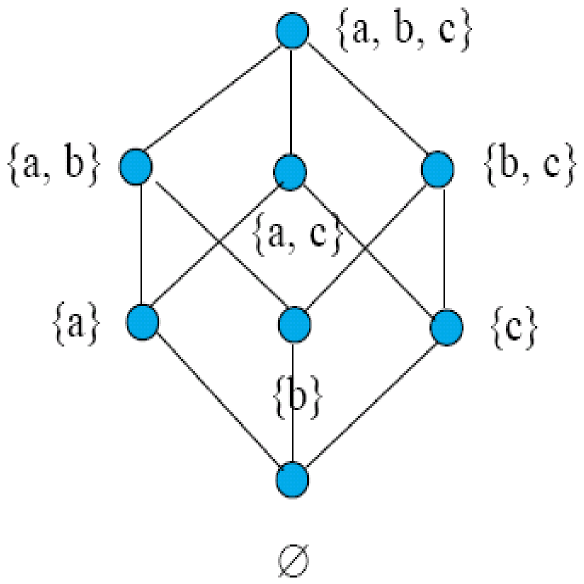
$$\varnothing$$

$$\{a\}, \{b\}, \{c\}$$

$$\{a, b\}, \{a, c\}, \{b, c\}$$

$$\{a, b, c\}$$

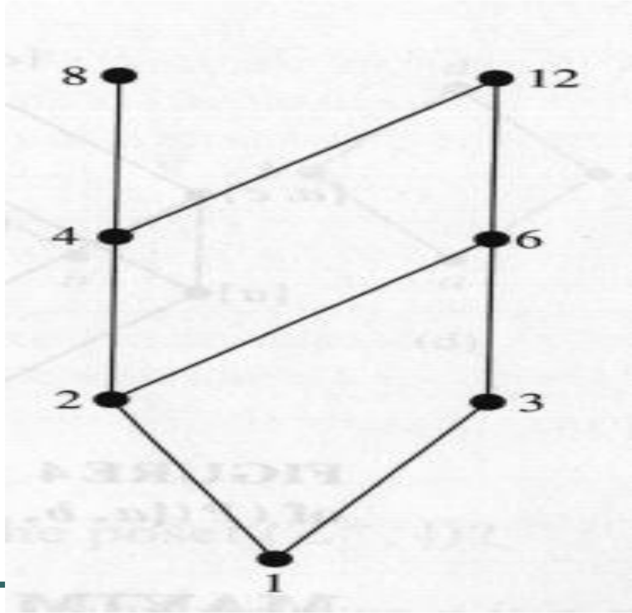
The digraph is



Maximal and Minimal Elements

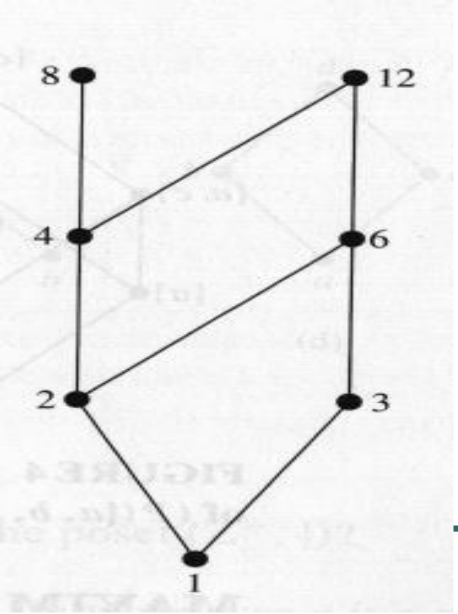
a is a *maximal* in the poset (S, $\preccurlyeq$) if there is no $b \in S$ such that $a \prec b$. Similarly, an element of a poset is called *minimal* if it is not greater than any element of the poset. That is, a is *minimal* if there is no element $b \in S$ such that $b \prec a$.

It is possible to have multiple minimals and maximals.



Greatest Element
Least Element

a is the *greatest element* in the poset (S, $\preccurlyeq$) if $b \preccurlyeq a$ for all $b \in S$. Similarly, an element of a poset is called the *least element* if it is less or equal than all other elements in the poset. That is, a is the *least element* if $a \preccurlyeq b$ for all $b \in S$



Upper bound, Lower bound

Sometimes it is possible to find an element that is greater than all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an *upper bound* of A. Likewise, there may be an element less than all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a *lower bound* of A.

Examples 18, p. 574 in Rosen.

Least Upper Bound, Greatest Lower Bound

The element x is called the *least upper bound* (lub) of the subset A if x is an upper bound that is less than every other upper bound of A.

The element y is called the *greatest lower bound* (glb) of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A.

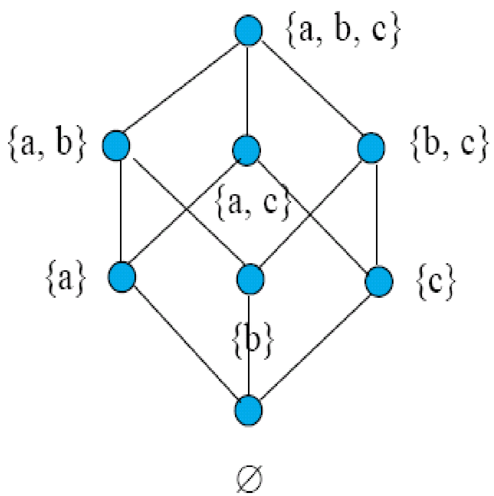
- In the poset $(P(S), \subseteq)$, $\text{lub}(A, B) = A \cup B$. What is the $\text{glb}(A, B)$?

Examples 19 and 20, p. 574 in Rosen.

Lattices

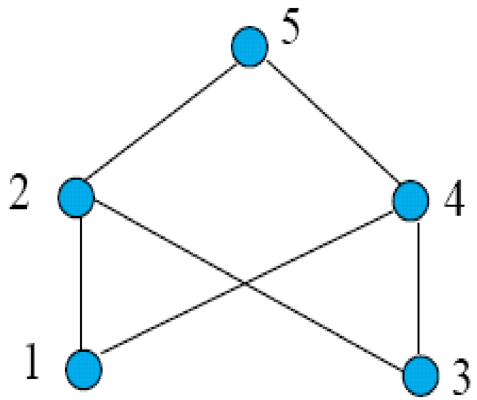
A partially ordered set in which *every pair* of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

$(P(\{a, b, c\}), \subseteq)$



Consider the elements 1 and 3.

- Upper bounds of 1 are 1, 2, 4 and 5.
- Upper bounds of 3 are 3, 2, 4 and 5.
- 2, 4 and 5 are upper bounds for the pair 1 and 3.
- There is no lub since
  - 2 is not related to 4
  - 4 is not related to 2
  - 2 and 4 are both related to 5.
- There is no glb either.



The poset is not a lattice.

Examples 21 and 22, p. 575 in Rosen.