CHAPTER — 2

Application of Partial Differential Equation

Classification of Linear partial differential equation of 2nd Order

The general form of

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

A B C are constant or continuous function x and y, A is (+)ve.

Now Equation is :-

Equation 1) Elliptic if $B^2 - 4AC < 0$

Equation 2) Hyperbolic if $B^2 - 4AC > 0$

Equation 3) Parabolic if $B^2 - 4AC = 0$

1.
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$$

$A = 1$; $B = 1$; $C = 1$

$B^2 - 4AC \Rightarrow (1)^2 - 4(1)(1) \Rightarrow -3$

$-3 < 0$ so it is elliptic

Q 2
$$4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$$

$A = 4$; $B = 4$; $C = 1$

$B^2 - 4AC \Rightarrow 16 - 16 = 0$ so parabolic

Q 3
$$x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 0$$

★

$A = x^2$; $B = 0$; $C = -1$

$B^2 - 4AC \Rightarrow \underline{4x^2} \longrightarrow$ depends on x

here we consider all cases for x

① if $x > 0$ or $x < 0$ implies curve is hyperbolic

② if $x = 0$ curve parabolic

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Q4 $x\dfrac{\partial^2 u}{\partial x^2} + t\dfrac{\partial^2 u}{\partial x \partial t} + \dfrac{\partial^2 u}{\partial t^2} = 0$

$A = x \qquad B = t \qquad C = 1$

$B^2 - 4AC \Rightarrow \boxed{t^2 - 4x}$

if curve is elliptic $t^2 < 4x$
is hyperbolic $t^2 > 4x$
is parabolic $t^2 = 4x$

Q5 Classify the following differential Eq^n as two type in the 2nd Quadrant of X-Y plane.

$\sqrt{y^2+x^2}\, U_{xx} + 2(x-y)\, U_{xy} + \sqrt{y^2+x^2}\, U_{yy} = 0$

$A = \sqrt{y^2+x^2} \qquad B = 2(x-y) \qquad C = \sqrt{y^2+x^2}$

$B^2 - 4AC$

$4(x-y)^2 - 4\left(\sqrt{y^2+x^2}\right)\left(\sqrt{y^2+x^2}\right)$

$4[x^2+y^2-2xy) - 4(y^2+x^2)$

$4[x^2+y^2-2xy-x^2-y^2) \Rightarrow \underline{-8xy}$

1) Hyperoblic as in 2nd quad $\Rightarrow x \to -ve$
$y \to (+)ve$

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Method of Separation of Variable

In this method we assume the solution to be the product of two functions. Each of which involves only of one of the variable.

Ques 1 Use the Method of Separation of Variable to solve the equation.

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad ; \text{ given that } u(x,0) = 6e^{-3x}$$

$u = f(x, t)$

$$u = X(x)\,T(t)$$

Put this solⁿ in eqⁿ (1)

$$\frac{\partial}{\partial x}(XT) = 2\frac{\partial}{\partial t}(XT) + XT$$

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

~~dividing~~ dividing XT both side

$$\frac{T}{XT}\frac{dX}{dx} = \frac{2X}{XT}\frac{dT}{dt} + \frac{XT}{XT}$$

$$\boxed{\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1} = K$$

(2)

(1)

Point to Rem.

$$\frac{1}{X}\frac{dX}{dx} = K \Rightarrow \frac{dX}{X} = K\,dx$$

On integration

$$\log X = Kx + \log C$$

$$\boxed{X = C_1 e^{Kx}}$$

$\log x = \log_e e^{Kx} + \log C$

as $\log_e e = 1$

$\log X = \log C_1 e^{Kx}$

$X = C_1 e^{Kx}$

or

$\frac{1}{X}\frac{dX}{dx} = K$

$\frac{dX}{dx} - KX = 0$

$(D-K)X = 0$

$X = C_1 e^{Kx}$

$$\frac{2}{T}\frac{dT}{dt} + 1 = K$$

$$\frac{2}{T}\frac{dT}{dt} = K-1$$

$$\int \frac{1}{T}\,dT = \int \left(\frac{K-1}{2}\right)dt$$

$$\log T = \frac{(K-1)}{2}\cdot t + \log C_2$$

$$\boxed{T = C_2\, e^{\left(\frac{K-1}{2}\right)t}}$$

$$u = X(x)\,T(t)$$

$$= C_1 e^{Kx}\; C_2 e^{\left(\frac{K-1}{2}\right)t}$$

$$\boxed{u(x,t) = C e^{Kx + \left(\frac{K-1}{2}\right)t}} \quad — (2)$$

from our condition $u(x,0) = 6e^{-3x}$

put $t = 0$ in (2)

$u(x,0) = Ce^{Kx}$

$u(x,0) = 6e^{-3x}$

On comparing $C = 6$, $K = -3$

$$\boxed{u = 6e^{-3x-2t}}$$ Ans

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Ques 2: $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$; $u(0,y) = 8e^{-3y}$

$u = X(x)\,Y(y)$

$\frac{\partial (XT)}{\partial x} = 4\frac{\partial (XY)}{\partial y}$

$Y\frac{dX}{dx} = 4X\frac{dY}{dy}$

$\frac{Y}{XY}\frac{dX}{\partial x} = \frac{4X}{XY}\frac{dY}{dy}$

$$\boxed{\frac{1}{X}\frac{dX}{dx} = \frac{4}{Y}\frac{dY}{dy} = K}$$

eqn 1

$\Rightarrow \frac{1}{X}\frac{dX}{dx} = K$

$\int\frac{dX}{X} = \int K\,dx$

$\log X = Kx + \log C_1$

$$\boxed{X = C_1 e^{Kx}}$$

eqn 2

$\frac{4}{Y}\frac{dY}{dy} = K$

$4\frac{dY}{Y} = \frac{1}{\cancel{4}}K\,dy$

$4\log Y = \cancel{\frac{1}{4}}Ky + \log C_2$

$\log Y^4 = Ky + \log C_2$

$Y^4 = C_2 e^{Ky}$

$Y = \left(C_2 e^{Ky}\right)^{1/4}$

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$$\boxed{\begin{aligned} Y &= C_2^{1/4}\, e^{\frac{Ky}{4}} \\ Y &= C\, e^{\frac{Ky}{4}} \end{aligned}}$$ R|W

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$u = XY$

$u = C_1 e^{Kx} \cdot C e^{\frac{Ky}{4}}$

$u = A\, e^{K^2 x + \frac{Ky}{4}}$

$A = 8\, e^{0 + \frac{y}{4}}$

$\therefore \boxed{K = -12}$

Also $\boxed{-3 = \frac{K}{4} \Rightarrow K = -12}$

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Q3 $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

Soln $u = X(x)\,Y(y)$

$$\frac{\partial^2 (XY)}{\partial x^2} - 2\frac{\partial (XY)}{dx} + \frac{\partial}{\partial y}(XY) = 0$$

$$Y\frac{d^2 X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0$$

$$\frac{Y}{XY}\frac{d^2 X}{dx^2} - \frac{2Y}{XY}\frac{dX}{dx} + \frac{X}{XY}\frac{dY}{dy} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = K$$

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = K$$

$$\frac{d^2X}{dx^2} - 2\frac{dX}{dx} = KX$$

$$(D^2 - 2D - K)X = 0$$

$$(m^2 - 2m - K) = 0$$

$$m = \frac{2 \pm \sqrt{4+4K}}{2} = 1 \pm \sqrt{1+K}$$

$$Y = C_1 e^{(1+\sqrt{1+K})x} + C_2 e^{(1-\sqrt{1+K})x}$$

$$\int \frac{dY}{Y} = \int +K\, dy$$

$$\log Y = +Ky + \log C_3$$

$$\boxed{Y = C_3 e^{+Ky}}$$

$$U = X(x)\,Y(y)$$

$$u = \left[C_1 e^{(1+\sqrt{1+K})x} + C_2 e^{(1-\sqrt{1+K})x}\right] C_3 e^{+Ky}$$

$$u = A e^{(1+\sqrt{1+K})x + Ky} + B e^{(1-\sqrt{1+K})x + Ky}.$$

Q4) $4\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 3u$, ~~$u = 3e^{-x} + e^{-5x}$~~ when $t=0$

$u = 3e^{-x} - e^{-5x}$

Sol^n

$$\boxed{U = X(x)\,T(t)}$$

$$4\frac{\partial}{\partial t}(XT) + \frac{\partial}{\partial x}(XT) = 3(XT)$$

$$4X\frac{dT}{dx} + T\frac{dX}{dx} = 3XT$$

$$\frac{4X}{XT}\frac{dT}{dx} + \frac{T\,dX}{XT\,dx} = \frac{3XT}{XT}$$

$$\frac{4}{T}\frac{dT}{dx} + \frac{1}{X}\frac{dX}{dx} = 3$$

$$\boxed{\frac{4}{T}\frac{dT}{dx} - 3 = -\frac{1}{X}\frac{dX}{dx} = K}$$

① $\frac{4}{T}\frac{dT}{dt} - 3 = K$

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$\frac{4}{T}\frac{dT}{dt} = K+3 \Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{1}{4}(K+3)$

$\int \frac{dT}{T} = \int \frac{1}{4}(K+3)\, dt$

$\log T = \frac{1}{4}\int (K+3)\, dt$

$\log T = \frac{1}{4}(K+3)t + \log C$

$\boxed{T = C_2 e^{\frac{1}{4}(K+3)t}}$

$\frac{dT}{dT} = \left(\frac{K+3}{4}\right)T$

$\frac{dT}{dT} - \left(\frac{K+3}{4}\right)T = 0$

$\left(D - \left(\frac{K+3}{4}\right)\right)T = 0$

$m = \frac{K+3}{4}$

$T = C_2 e^{\left(\frac{K+3}{4}\right)t}$

② $-\frac{1}{X}\frac{dX}{dx} = K$

$\cancel{-\frac{dX}{X} = K dx}$ $\quad \frac{dX}{dx} + KX = 0$

$0 = K dx + \frac{dX}{X}$

$0 = (D+K)X$

$m + K = 0 \Rightarrow m = -K$

$\boxed{X = C_1 e^{-Kx}}$

$u = XT$

$u(x,t) = \underbrace{C_1 C_2}\, e^{-Kx + \frac{1}{4}(K+3)t}$

$u(x,t) = Ce^{-Kx + \frac{1}{4}(K+3)t}$

$t = 0$

$u(x,0) = Ce^{-Kx} = 3e^{-x} - e^{-5x}$.

Let $C = A + B$

$(A+B)e^{-Kx} = 3e^{-x} - e^{-5x}$

$A\cdot e^{-Kx} + Be^{-Kx} = 3e^{-x} - e^{-5x}$

when $A = 3$. $B = -1$

$K = 1$ $K = 5$

$u(x,t) = (A+B)\, e^{-Kx + \left(\frac{K+3}{4}\right) t}$

$u(x,t) = A\, e^{-Kx + \left(\frac{K+3}{4}\right) t} + B\, e^{-Kx + \left(\frac{K+3}{4}\right) t}$

$$\boxed{u(x,t) = 3e^{-x+t} - e^{-5x+2t}}$$

Q5 $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u$

$u(x,0) = 10e^{-x} - 6e^{-4x}$

$u = XT$

$\frac{\partial}{dt}(XT) = \frac{\partial}{\partial x}(XT) - 2XT$

$X\frac{dT}{dt} = T\frac{dX}{dx} - 2XT$

$\frac{X}{XT}\frac{dT}{dt} = \frac{T}{XT}\frac{dX}{dx} - \frac{2XT}{XT}$

$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{dX}{dx} - 2$

$\frac{1}{T}\frac{dT}{dt} + 2 = \frac{1}{X}\frac{dX}{dx} = K$

$\frac{1}{T}\frac{dT}{dt} + 2 = K$

$\frac{1}{T}\frac{dT}{dt} = T(K-2) \Rightarrow \frac{dT}{dt} - T(K-2) = 0$

X ko hamesha akele chorda.

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$[D - (K-2)]\, T = 0$

$[m - (K-2)] = 0$

$m = (K-2)$

$\boxed{T = C_1 e^{(K-2)t}}$

② $\frac{1}{X}\frac{dX}{dx} = K$

$\frac{dX}{dx} = KX$

$\frac{dX}{dx} - KX = 0$

$(D-K)X = 0$

$m - K = 0$

$m = K$

$\boxed{X = C_2 e^{Kx}}$

$~~U T X~~ \quad U = XT$

$u(x,t) = C_1 C_2 e^{(K-2)t + Kx}$

$u(x,0) = C_1 C_2 e^{Kx}$

$u(x,0) = C e^{Kx}$

Let $C = A + B$

$u(x,0) = (A+B)e^{Kx}$

$u(x,t) = Ae^{Kx} + Be^{Kx}$

$u(x,0) = 10e^{-x} - 6e^{-4x}$

$K = -1$
$A = 10$

$K = -4$
$B = -6$

$u(x,t) = 10e^{-(3t+x)} - 6e^{-2(2x+3t)}$

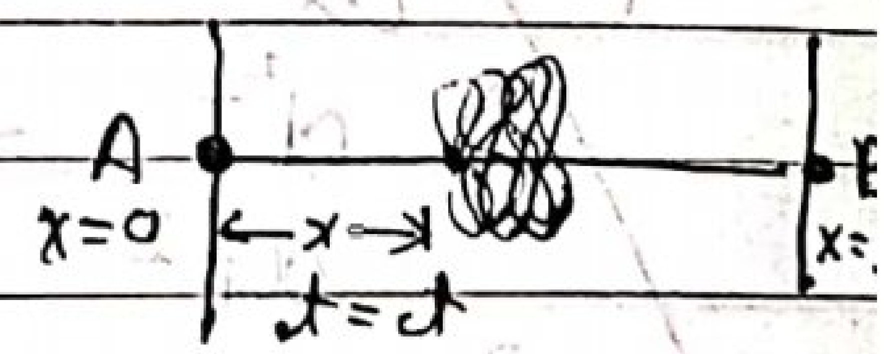
Ans

1-D WAVE EQUATION

VIBRATION OF A STREDCHED STRING

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$ Wave Equation.

where :> y - vertical distribution (wave function)
x - distance
t - time
c = velocity of string

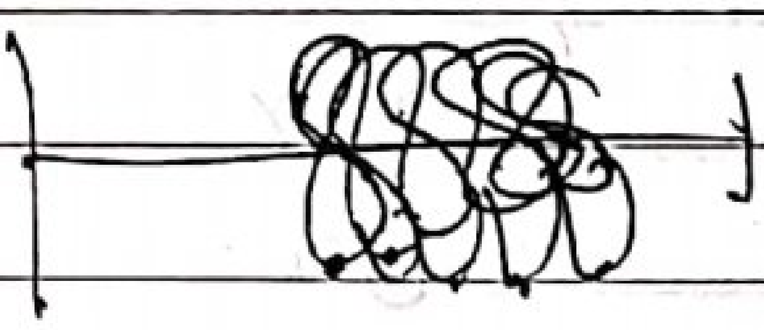


Boundary Condition : (1) $\boxed{y(0, t) = y(l, t) = 0}$

Initial velocity (string released at rest)

$$\boxed{\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0}$$ (2)

dis
time



Initial condition

$$\boxed{y(x, 0) = f(x)}$$ (3)

Solⁿ $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

By method of separation of variable assume solⁿ is
$y(x, t) = X(x)\, T(t)$

$$\frac{\partial^2 (XT)}{dt^2} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$$

$$X \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2}$$

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Q2 A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially in a position given by $y = y_0 \sin^3 \frac{\pi x}{l}$.

If it is released from rest from the position find the displacement $y(x,t) = ?$

Sol^n $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

$y = XT$

$\frac{\partial^2 XT}{\partial t^2} = c^2 \frac{\partial^2 XT}{\partial x^2}$

$X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$

$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2} = -p^2$

$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -p^2$

$\frac{d^2 T}{dt^2} + p^2 c^2 T = 0$

$(D^2 + p^2 c^2) T = 0$

$m^2 + p^2 c^2 = 0$

$\boxed{m = \pm pci}$

$T = C_3 \cos pct + C_4 \sin pct$

$\frac{d^2 X}{dx^2} + p^2 X = 0$

$(D^2 + p^2) X = 0$

$m^2 + p^2 = 0$

$m = \pm pi$

$X = C_1 \cos px + C_2 \sin px$

(1) BC (2) TV (3) IC

$$Y = (c_3 \cos pct + c_4 \sin pct)(c_1 \cos px + c_2 \sin px)$$

$$y(0, t) = y(l, t) = 0$$

$x = 0$

$$Y \Rightarrow (c_3 \cdot \cos cpt + c_4 \sin pct)(c_1) = 0$$

$$\boxed{c_1 = 0}$$

$x = l$

$$Y \Rightarrow (c_3 \cos cpt + c_4 \sin cpt)(c_2 \sin pl) = 0$$

$$c_2 \neq 0 \quad ; \quad pl = n\pi \; ; \; p = \frac{n\pi}{l}$$

$$\boxed{\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0}$$

$$\frac{\partial y}{\partial t} = (c_1 \cos px + c_2 \sin px)(-c_3 cp \sin cpt + c_4 cp \cos cpt)$$

$t = 0$

$$(c_1 \cos px + c_2 \sin px)(c_4 cp \cos cpt) = 0$$

$$\boxed{c_4 = 0}$$

$$y(x, t) = (c_2 \sin px \; c_3 \cos cpt)$$

$$y(x, t) = c_2 c_3 (\sin px)(\cos cpt)$$

$$y(x, t) = b_n (\sin px)(\cos cpt)$$

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$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{cn\pi t}{l}$$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= y_0 \sin^3\frac{\pi x}{l}$$

$\sin 3A = 3\sin A - 4\sin^3 A$

$4\sin^3 A = -\sin 3A + 3\sin A$

$\sin^3 A = -\frac{1}{4}\sin 3A + \frac{3}{4}\sin A$

$$\therefore = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$$b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} = \frac{y_0}{4}\left[\quad\right]$$

$$b_1 = \frac{3y_0}{4}, \quad b_2 = 0, \quad b_3 = -\frac{y_0}{4}$$

$$y(x,t) = b_1 \sin\frac{\pi x}{l}\cos\frac{c\pi t}{l} + b_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi ct}{l} +$$

$$b_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}$$

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}$$

Q3

A tightly stretched string at its ends $x=0$ & $x=l$

at time $t=0$ the string is given in ~~sapko~~ shape.
$F(x) = \mu x(l-x)$ $\qquad \mu \rightarrow$ constant.
and then released. Find the displacement
$y(x,t)$

$Y = XT$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 (XT)}{\partial t^2} = c^2 \frac{\partial^2 XT}{\partial x^2}$$

$$X \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2} \Rightarrow \frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{X}\frac{d^2 X}{dx^2} = -p^2$$

Solve yourself

Direct coming to next step.

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$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{cn\pi t}{l}$$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = \mu x(l-x)$$

$$\boxed{b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx}$$

$$b_n = \frac{2}{l}\int_0^l \mu x(l-x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l \underbrace{(\mu x l - \mu x^2)}_{u} \underbrace{\sin \frac{n\pi x}{l}}_{v}\, dx$$

$$b_n = \frac{2}{l}\left[\cancel{(\mu x l - \mu x^2)} \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + \cancel{(\mu l - 2\mu x)\left(\frac{+\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l}}\right)} + (-2\mu)\left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right) \right]_0^l$$

$$b_n = \left(\frac{2}{l}\right)\left(\frac{-2\mu l^3}{n^3\pi^3}\right)\left[(-1)^n - 1\right]$$

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$$b_n = -\frac{4\mu l}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4\mu l^2}{n^3\pi^3}\left[1-(-1)^n\right] \sin\frac{n\pi x}{l} \cos\frac{cn\pi t}{l}$$

Q4 If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3\frac{\pi x}{l}$ find the displacement $y(x,t)$.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

BC

IC

BC $\Rightarrow$ $y(0,t) = y(l,t) = 0$

$\boxed{IC \Rightarrow \quad y(x,0) = 0}$ $\longrightarrow$ for equilibrium position

IV $\Rightarrow$ $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l}$ $\leftarrow$ at last becoz it is given

B.C $C_1 = 0$, $p = \frac{n\pi}{l}$

$$y(x,t) = C_2 \sin px \,(C_3 \cos cpt + C_4 \sin cpt)$$

$$y(x,0) = 0$$

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$0 = C_2 \sin px \,(C_3)$

$\boxed{C_3 = 0}$

$y(x,t) = \underbrace{C_2 C_4}\, \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$

$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$

$b_n \longrightarrow$ using initial velocity

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi x}{l}\right)\left(\cos \frac{cn\pi t}{l}\right) \cdot \left(\frac{cn\pi}{l}\right)$

$b \sin^3 \frac{\pi x}{l} = \sum_{n=}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l}$

$n=1$
$n=2$
$n=3$

$\sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A$

$\frac{b}{4}\left[3 \sin \frac{n\pi x}{l} - \sin \frac{3\pi x}{l}\right] = b_1 \sin \frac{\pi x}{l} \cdot \frac{c\pi}{l} +$

$b_2 \sin \frac{2\pi x}{l} \cdot \frac{2c\pi}{l} +$

$b_3 \sin \frac{3\pi x}{l} \cdot \frac{3c\pi}{l}$

On comparing

$\frac{3b}{4} = \frac{b_1 c\pi}{l}$

$\therefore b_1 = \frac{3bl}{4c\pi}$

$\therefore b_2 = 0$

$-\frac{b}{4} = b_3 \cdot \frac{3c\pi}{l}$

$\therefore b_3 = -\frac{bl}{12c\pi}$

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$$y(x,t) = b_n \sin\frac{n\pi x}{l} \sin\frac{cn\pi t}{l}$$

$$y(x,t) = \frac{3bl}{4\pi c} \sin\frac{\pi x}{l} \sin\frac{c\pi t}{l} - \frac{bl}{12\pi c} \sin\frac{3\pi x}{l} \sin\frac{3c\pi t}{l}$$

Q5 H/W

A tightly stretched string with fixed ~~at~~ ends $x=0$ & $x=\pi$, is initially at rest in its equilibrium postion. If it is said vibrating by giving to each of its points an initial velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin^3 x$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \Big| \quad Y = X\,T$$

$$X \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2}$$

$$\boxed{\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{X}\frac{d^2 X}{dx^2} = -p^2} \; ; \quad \frac{1}{X}\frac{d^2 X}{dx^2} = -p^2 \Rightarrow (D^2+p^2)X = 0$$

$$\boxed{m = \pm p i}$$

$\rightarrow X = C_1 \cos px + C_2 \sin px$

$(D^2 + p^2c^2)T = 0 \Rightarrow m^2 = -p^2c^2 \Rightarrow m = \pm p c i$

$\rightarrow T = C_3 \cos pct + C_4 \sin pct$

$Y = (C_1 \cos px + C_2 \sin px)(C_3 \cos pct + C_4 \sin pct)$

$\because y(0,t) = y(\pi,t) = 0$

at $x = 0 \Rightarrow 0 = C_1 (C_3 \cos cpt + C_4 \sin cpt) \therefore C_1 = 0$

at $x = \pi \Rightarrow 0 = C_2 \sin p\pi \, (C_3 \cos cpt + C_4 \sin cpt)$

$C_2 \neq 0 \quad \sin p\pi = 0$

$\sin p\pi = \sin n\pi$

$p = n$

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$$\boxed{y(x,t) = (c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)}$$

at equilibrium $y(x,0) = 0$

$\therefore t = 0$

$$\boxed{0 = (c_2 \sin px)(c_3)} \quad \therefore \boxed{c_3 = 0}$$

$$\boxed{y(x,t) = (c_2 \sin px)(c_4 \sin cpt)}$$

$p = n$

$$y(x,t) = c_2 c_4 (\sin nx)(\sin cnt)$$

Main $\rightarrow$ $\boxed{y(x,t) = \sum_{1}^{\infty} b_n (\sin nx)(\sin cnt)}$

$$\frac{\partial y}{\partial t} = b_n \sin nx \cdot \cos cnt \times cn$$

$$\left(\frac{\partial y}{\partial t}\right) = \sum_{1}^{\infty} b_n cn (\sin nx)(\cos cnt)$$

$t = 0$; $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} b_n cn \sin nx$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin nx - 0.04 \sin 3x$$

~~$b_n = \frac{2}{\pi}\int_0^\pi c$ $\quad \sin 3A = 3\sin A - 4\sin^3 A$~~

~~$4\sin^3 A + 4\sin^3 A = 3\sin A$~~

$$\underbrace{0.03 \sin x - 0.04 \sin 3x}_{} = b_1 c \sin x + 2b_2 c \sin 2x + 3b_3 c \sin 3x$$

$\boxed{b_1 c = 0.03}$ $\quad$ $\boxed{2b_2 c = 0}$ $\quad$ $3b_3 c = -0.04$

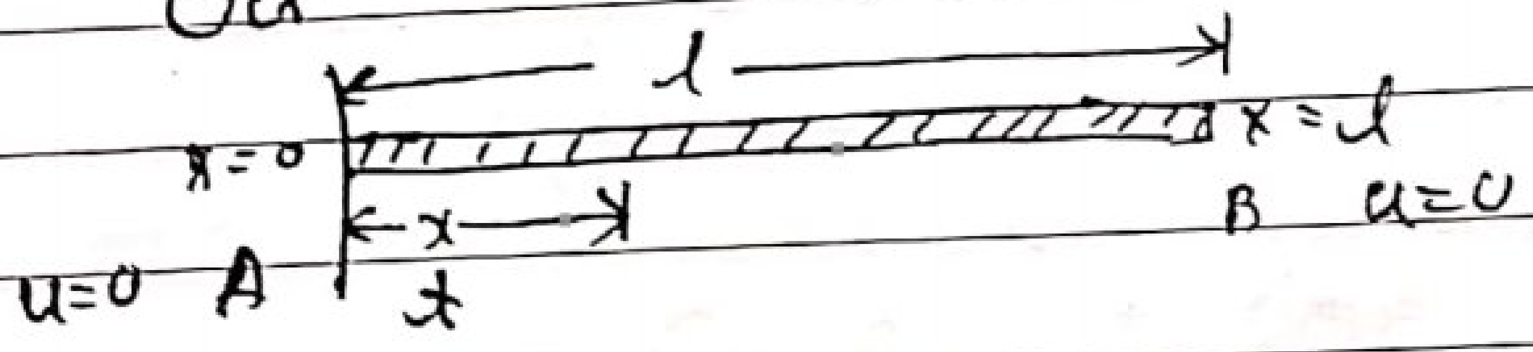
$b_1 = \frac{0.03}{c}$ $\quad$ $b_3 = -\frac{0.04}{3c}$

$$= \left(\frac{0.03}{c} \times \sin x \sin ct\right) - \left(\frac{0.04}{3c} \sin 3x \sin 3ct\right)$$

Solve

One-Dimensional Heat flow Equation.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$



$u \rightarrow$ Temperature distribution (Temp. func)
$c \rightarrow$ Thermal diffusivity (Temprature difference)
$x \rightarrow$ distance ; $t \rightarrow$ time ;

BC ⇒ $u(0,t) = u(l,t) = 0$

I.C ⇒ $u(x,0) = f(x)$

General Solⁿ of heat Eqⁿ.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = XT$$

$$X \frac{dT}{dt} = c^2 T \frac{d^2 X}{dx^2}$$

$$\frac{1}{c^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} = K$$

$$\frac{dT}{dt} + p^2 c^2 T = 0$$

$$(D + p^2 c^2) T = 0$$

$$m + p^2 c^2 = 0$$

$$m = \pm p c i$$

$T = c_3 \cos$

$\boxed{\text{case 1}}$: $K=0$

$$\frac{1}{X}\frac{d^2X}{dx^2}=0$$

$$\boxed{X = C_1 + C_2 x}$$

$$\frac{dT}{dt}=0$$

$$\boxed{T = C_3}$$

$$u(x,t) = (C_1 + C_2 x)C_3$$

Using B.C $\quad u(0,t) = u(l,t) = 0$

at $x=0$

$$u(0,t) = C_1 C_3$$

$$\boxed{0 = C_1 C_3}$$

$C_1 = 0$
$C_3 \neq 0$

at $x = l$

$$u(l,t) = C_2 l C_3$$

$$\boxed{0 = C_2 l C_3}$$

$C_2 = 0$
$C_3 \neq 0$

$C_1 = 0,\ C_2 = 0$ means $u = 0$

which is impossible.

hence this case is neglected.

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$\boxed{\text{case 2}}$ $K = +p^2$

$$\frac{1}{c^2 T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = p^2$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = p^2 \Rightarrow (D^2 - p^2)X = 0 \Rightarrow m^2 = p^2$$

$$\boxed{m = \pm p}$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$\frac{dT}{dt} - p^2c^2T = 0 \Rightarrow (D - p^2c^2)T = 0$$

$$\boxed{m = p^2c^2}$$

$$\boxed{T = c_3 e^{p^2c^2t}}$$

$u = XT$

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{p^2c^2t})$$

$u(0,t) = u(l,t) = 0$

$x = 0$

$$0 = (c_1 + c_2)\, c_3 e^{p^2c^2t}$$

$$\boxed{c_1 + c_2 = 0}$$

$x = l$

$$0 = (c_1 e^{pl} + c_2 e^{-pl})\, c_3 e^{p^2c^2t}$$

$\rightarrow$ $c_1 + c_2 = 0$ $\longrightarrow$ $\boxed{c_2 = -c_1}$

$\rightarrow$ $c_1 e^{pl} + c_2 e^{-pl} = 0$

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$$c_1 e^{pd} - c_1 e^{-pd} = 0$$

$$c_1 (e^{pd} - e^{-pd}) = 0$$

$$\underline{\underline{c_1 = 0}} \quad \& \quad \underline{\underline{c_2 = 0}}$$

hence $u = 0$

$\therefore$ There is no heat which is impossible $\therefore$ case is neglected.

$\boxed{\text{case 3}}$: $k = -p^2$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \qquad\qquad \frac{1}{c^2T}\frac{dT}{dt} = -p^2$$

$$(D^2 + p^2)X = 0 \qquad\qquad (D + p^2c^2)T = 0$$

$$m^2 + p^2 = 0 \qquad\qquad \boxed{m = -p^2c^2}$$

$$\boxed{m = \pm pi}$$

$$X = C_1 \cos px + C_2 \sin px \qquad\qquad T = C_3 e^{-p^2c^2t}$$

$$u(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 e^{-p^2c^2t})$$

$$u(0,t) = u(d,t) = 0$$

$x = 0$

$$\boxed{0 = C_1 C_3 e^{-p^2c^2t}} \rightarrow C_1 = 0, \quad C_3 \neq 0$$

$x = d$

$$0 = (C_2 \sin pd)(C_3 e^{-p^2c^2t})$$

$C_3 \neq 0$

$C_2 \sin pl = 0$

$C_2 \neq 0$

$pl = n\pi$

$p = \frac{n\pi}{l}$

$$u(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} e^{\frac{-c^2 n^2 \pi^2 t}{l^2}}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{-c^2 n^2 \pi^2 t}{l^2}}}$$

Most general

$u(x,0) = f(x)$

$t = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\boxed{b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx}$$

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Q A rod of length l with insulated sides is initially at uniform temperature u_0 its ends are suddenly cooled to $0°c$ and are kept at that temperature. Find the temperature function $u(x,t)$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

B.C $\quad u(0,t) = u(l,t) = 0$

I.C $\quad u(x,0) = u_0 = f(x)$

$$u = XT$$

$$X\frac{dT}{dt} = c^2 T \frac{d^2X}{dx^2}$$

$$\frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -p^2$$

$$\frac{1}{c^2T}\frac{dT}{dt} = -p^2$$

$$\frac{dT}{dt} + p^2c^2T = 0$$

$$(D + p^2c^2)T = 0$$

$$(m + p^2c^2) = 0$$

$$m = -p^2c^2$$

$$\boxed{T = C_3 e^{-p^2c^2 t}}$$

$$\frac{d^2X}{dx^2} + p^2X = 0$$

$$(D^2 + p^2)X = 0$$

$$m^2 + p^2 = 0$$

$$m^2 = -p^2$$

$$m = \pm pi$$

$$\boxed{X = C_1\cos px + C_2\sin px}$$

$$\boxed{u = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-p^2c^2t}}$$

$u(0,t) = u(l,t) = 0$

$x = 0$

$$u = (c_1)\, c_3\, e^{-c^2p^2t} = 0$$

$$\boxed{c_1 = 0}$$

$x = l$

$$u = c_2 \sin pl\, c_3\, e^{-p^2c^2t}$$

$$u = c_2 c_3 \sin pl\, e^{-p^2c^2t}$$

$\sin pl = \sin n\pi$

$$p = \frac{n\pi}{l}$$

$$\boxed{u(x,t) = c_2 c_3 \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2c^2t}{l^2}}}$$

general solⁿ

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

using IC

at $t = 0$

$$u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

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$$b_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\, dx$$

$$b_n = \frac{-2u_0}{l}\left[\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l$$

$$b_n = -\frac{2u_0}{nl}\left[(-1)^n - 1\right]$$

$$b_n = \frac{2u_0}{n\pi}\left[1-(-1)^n\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}\left[1-(-1)^n\right] \sin\frac{\pi x n}{l}\, e^{\frac{-c^2 n^2 \pi^2 t}{l^2}}$$

(7) Find the temperature in a bar of length 2 whose ends are kept at 0 and the lateral surface insulated if the intial temperature is $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$ $t=0$

$BC \Rightarrow u(0,t) = u(2,t) = 0$

$$\frac{\partial u}{dt} = c^2 \frac{\partial^2 u}{dx^2}$$

$IC \Rightarrow u(x,0) = f(x) \Rightarrow \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

$U = XT$

$$\frac{\partial(XT)}{\partial t} = c^2 \frac{\partial^2 XT}{\partial x^2}$$

$$X\frac{dT}{dt} = c^2 T\frac{d^2X}{dx^2}$$

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$$\cancel{\frac{1}{c^2 T}\frac{d^2X}{dx^2} = \frac{1}{X}\frac{d^2X}{}}$$

$$\frac{1}{c^2 T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -p^2$$

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2 c^2 t}$$

Using

at $t = 0$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = b_1 \sin\frac{\pi x}{2} + b_2 \sin\frac{2\pi x}{2} + b_3 \sin\frac{3\pi x}{2} + b_4 \sin\frac{4\pi x}{2} + b_5 \sin\frac{5\pi x}{2}$$

On comparing

$b_1 = 1$; $b_2 = b_3 = b_4 = 0$; $b_5 = 3$

$$\boxed{u(x,t) = \sin\frac{\pi x}{2}\, e^{-\frac{c^2 p^2 t \pi^2}{4}} + 3\sin\frac{5\pi x}{2}\, e^{-\frac{25 c^2 t \pi^2}{4}}}$$

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Q

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0$$

$z(x,0) = 0$
$z(x,\pi) = 0$
$z(0,y) = 4\sin 3y$

$$Z = XY$$

$$Y\frac{dX}{dx} + X\frac{d^2Y}{dy^2} = 0 \Rightarrow \frac{Y\,dX}{XY\,dx} = -\frac{X\,d^2Y}{XY\,dy^2}$$

$$-\frac{1}{X}\frac{dX}{dx} = \frac{1}{Y}\frac{d^2Y}{dy^2} = -p^2$$

$\frac{1}{X}\frac{dX}{dx} = -p^2$

$\frac{dX}{dx} = p^2 X^2$

$(D - p^2)X = 0$

$m = +p^2$

$m = +p^2$

$\boxed{\cancel{X = C_1\cos px + C_2\sin px}}$

$X = C_3 e^{+p^2 x}$

$\frac{1}{Y}\frac{d^2Y}{dy^2} = -p^2$

$\frac{d^2Y}{dy^2} = -p^2 Y$

$(D^2 + p^2)Y = 0$

$m^2 = -p^2$

$m = \pm pi$

$\boxed{Y = C_1\cos py + C_2\sin py}$

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BC $\boxed{z(x,0) = z(x,\pi) = 0}$

$$\boxed{z = C_3 e^{+p^2 x}(C_1 \cos py + C_2 \sin py)}$$

When $y = 0$

$$\boxed{z = C_1 C_3 e^{p^2 x} = 0} \Rightarrow \boxed{C_1 = 0}$$

$y = \pi$

$$z = C_3 e^{+p^2 x}(C_2 \sin p\pi) = 0$$

$$z = C_2 C_3 \sin p\pi \, e^{p^2 x} = 0$$

$C_2 \sin p\pi = 0$

$C_2 \neq 0 \quad \sin p\pi = \sin n\pi$

$p = \frac{n\pi}{\pi} \Rightarrow \boxed{p = n}$

$$Z(x,y) = A \sin p\pi y \, e^{p^2 x}$$

$$Z(x,y) = A \sin n\pi y \, e^{n^2 x}$$

$Z(0,y) \Rightarrow x = 0$

$\Rightarrow A \sin n\pi$

on comparin

$A = 4 \quad n = 3$

$$\boxed{z(x,y) = 4 \sin 3\pi y \, e^{9x}}$$

Ans

Imp CT-1 (5)

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Q $\frac{\partial^2 u}{\partial x^2} = 2u + \frac{\partial u}{\partial y}$

use Method of Separation of Variable subject to conditions $u = 0$. $\frac{\partial u}{\partial x} = e^{-3y} > x = 0$

$u(0,y) = 0$

x ko free
condition x

Soln $u = XY$

$$\underbrace{Y \frac{d^2 X}{dx^2}} = 2XY + X \frac{dY}{dy}$$

$$\cancel{X \frac{d^2 X}{dx^2} = -p^2}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = 2 + \frac{1}{Y}\frac{dY}{dy} = -p^2$$

$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2$

$\frac{d^2X}{dx^2} + p^2X = 0$

$(D^2 + p^2)X = 0$

$m^2 + p^2 = 0$

$m^2 = -p^2$

$\boxed{m = \pm pi}$

$X = C_1 \cos px + C_2 \sin px$

$2 + \frac{1}{Y}\frac{dY}{dy} = -p^2$

$\frac{1}{Y}\frac{dY}{dy} = -p^2 - 2$

$\frac{dY}{dy} = -(p^2+2)Y$

$[D + (p^2+2)]Y = 0$

$m + (p^2+2) = 0$

$m = -(p^2+2)$

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$$Y = C_3 e^{-(p^2+2)}$$

$$\boxed{U = (C_1 \cos px + C_2 \sin px)(C_3 e^{-(p^2+2)y})}$$

$\cancel{u(0,y) = u(x,y) = 0}$

$\cancel{x=0}$

$$\cancel{u \Rightarrow (C_1)(C_3 e^{-(p^2+2)y}) = 0}$$

$$\boxed{\frac{\partial u}{\partial x} = (C_3 e^{-(p^2+2)y})(-C_1 \sin px \cdot p + C_2 p \cos x)}$$

$$\frac{\partial u}{\partial x} = C_3 e^{-(p^2+2)y}(-C_1 p \sin px + C_2 p \cos x)$$

$x=0$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = C_2 C_3 e^{-(p^2+2)y} p$$

$$\underline{p C_2 C_3}\, e^{-(p^2+2)y} = e^{-3y}$$

$$A e^{-(p^2+2)y} = e^{-3y}$$

$A = 1$

$$-(p^2+2) = -3$$

$$-p^2 - 2 = -3$$

$$-p^2 = -3+2$$

$+p^2 = +1$

$p = +1$

$\therefore A = 1$

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$u(0,y) = u(\pi,y) = 0$

$x=0$

$$u = (c_1)\left(c_3 e^{-(p^2+2)y}\right)$$

$$0 = c_1\left(c_3 e^{-(p^2+2)y}\right)$$

$$\boxed{c_1 = 0}$$

$\therefore$ $c_2 c_3 = 1$

$$u(x,y) = e^{-3y} \sin x$$

CT-2

Imp

Ques The ends A and B of a rod of length 20 cm are at temperature $30^\circ C$ and $80^\circ C$ until steady state prevails. Then the temperature of the rod end are changed to $40^\circ C$ and $60^\circ C$ respectively. Find the temperature distribution function $u(x,t)$.

The specific heat density & the thermal conductivity of the material of the rod are such that the combination

$$\frac{k}{\rho\sigma} = c^2 = 1.$$

Temperature Change

$$\boxed{u = u_1(x,t) + u_2(x)}$$

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Initial temperature distribution of the rod

$u_1(x,0) = 30 + \frac{(80-30)}{20} x = 30 + \frac{5}{2} x$ } IC

$u(x,0)$

final temperature distribution in steady state

$u_2 = 40 + \frac{(60-40)}{20} x = 40 + x$

One dimension heat flow

To get u in the intermediate period

$$\boxed{u = \underbrace{u_1(x,t)}_{\text{general temp}} + \underbrace{u_2(x)}_{\text{steady state}}}$$ when both ends are different in initial & final temp

here $u_2(x)$ is the steady state temp distribution

and $u_1(x,t)$ general temp distribution which tend to zero as t (time) increases.

$$\boxed{u = \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t} + 40 + x}$$

A

In steady state

B.C

$u(0,t) = 40$
$u(20,t) = 60$] present state temp.

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using B.C at $x=0$

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t} + 40 + x$$

$$40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2 t}$$

$$0 = \sum_{n=1}^{\infty} a_n e^{-p^2 t}$$

$$\boxed{a_n = 0}$$

$x = 20$ $\quad u(20,t) = 60$

$$60 = 60 + \sum_{n=1}^{\infty} b_n \sin 20p \, e^{\cancel{(20)} -p^2 t}$$

$$0 = \sum_{n=1}^{\infty} b_n \sin 20p \, e^{-p^2 t}$$

$\sin x = \int \cos x$

$\sin 20p = 0 = \sin n\pi$ $\quad -\cos x = \int \sin x$

$$\boxed{p = \frac{n\pi}{20}}$$

Putting the value in equation $\underline{A}$

$$\boxed{u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-\left(\frac{n\pi}{20}\right)^2 t}}$$

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Using Initial condition $u(x,0) = f(x)$

$$30 + \frac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin x\,dx$$

$$\underbrace{\frac{3}{2}x - 10}_{f(x)} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20}$$

$$\therefore\ b_n = \frac{2}{20}\int_0^{20}\left(\frac{3}{2}x - 10\right)\sin\frac{n\pi x}{20}\,dx$$

$$b_n = \frac{1}{10}\int_0^{20}\underbrace{\left(\frac{3}{2}x - 10\right)}_{①}\underbrace{\sin\frac{n\pi x}{20}}_{②}\,dx$$

$$= \frac{1}{10}\left[\left(\frac{3}{2}x - 10\right)\left[-\frac{\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right] - \left[\frac{-3}{2} \times \frac{\sin\frac{n\pi x}{20}}{\frac{n^2\pi^2}{(20)^2}}\right]\right]_0^{20}$$

$$= \frac{1}{10}\left[\left[\frac{3}{2}x - 10\right]\left[-\left(\cos\frac{n\pi x}{20}\right) \times \frac{20}{n\pi}\right]\right]_0^l$$

$$= \frac{1}{10}\left[\left(\frac{3}{2}(20) - 10\right)\left(-\cos\frac{n\pi(20)}{20}\right)\left(\frac{20}{n\pi}\right)\right] -$$

$$\left[(-10)(-) \times \frac{20}{n\pi}\right]$$

$$= \frac{1}{10}\Big[(20)(-1)^n(-1) - 10\Big] \times \frac{2 \times 20}{n\pi}$$

$$\Rightarrow\ b_n = -\frac{20}{n\pi}\left[2(-1)^n + 1\right]$$

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case : <u>perfectly insulated</u>

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<u>Ques</u>

Imp.

The temperature distribution in a bar of length π which is perfectly insulated at ends $x=0$ and $x=\pi$ is governed by the partial differential eq^n $\frac{\partial u}{\partial t} = 1 \cdot \frac{\partial^2 u}{\partial x^2}$ assumming the initial temp. distribution $u(x,0) = fx = \cos 2x$ find the temp distribution at any instant of time

<u>Sol^n</u> $u(x,t) = \sum_{n=0}^{\infty} (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2 t}$ → A

General sol^n of heat equation

★ Since both ends of bar is perfectly insulated means no heat can pass from either sides and boundary conditions are

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0 \qquad \left(\frac{\partial u}{\partial x}\right)_{x=\pi} = 0$$

Using Boundary Condition

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[c_1 \cos px + c_2 \sin px \right) c_3 e^{-p^2 t}$$

$$\frac{\partial u}{\partial x} = \left(-p c_1 \sin px + n c_2 \cos px\right) c_3 e^{-p^2 t}$$

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_{x=0} = p c_2 c_3 e^{-p^2 t}}$$

$$0 = n c_2 c_3 e^{-p^2 t} \Rightarrow \boxed{c_2 = 0}$$

$$\left\{\begin{array}{l} f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow \text{cos series} \\ f(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow \text{sin series} \end{array}\right.$$

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$$\left(\frac{\partial u}{\partial x}\right)_{x=\pi} = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=\pi} = \left(-pC_1 \sin p\pi + pC_2 \cos p\pi\right) C_3 e^{-p^2 t}$$

$$0 = -pC_1 \sin p\pi$$

$\boxed{p = 0}$

$C_1 \neq 0$

$$\sin p\pi = 0 = \sin n\pi$$

$$\boxed{p = n}$$

General equation ; Put C_2 & p in (A).

$$\boxed{u(x,t) = C_1 \cos nx \, C_3 e^{-n^2 t}}$$

The most general solution

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, e^{-n^2 t}}$$

$\left\{\begin{array}{l} \text{jab cos hota hai toh } a_n \\ \text{jab sin hota hai toh } b_n \end{array}\right\}$ Note

frome Case I

$$X = C_1 + C_2 x$$

$$T = C_3$$

$$u(x,t) = (C_1 + C_2 x) C_3$$

$$\frac{\partial u}{\partial x} = C_2 C_3 = \frac{a_0}{2}$$

Using Initial condition

$u(x,0) = f(x) = \cos 2x$

$\cos 2x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

comparing.

$\boxed{a_0 = 0}$

$a_n = 1$
$\boxed{n = 2}$

$\boxed{u(x,t) = e^{-4t} \cos 2x}$ ans.

H/W

Ques Solve $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ under the condition

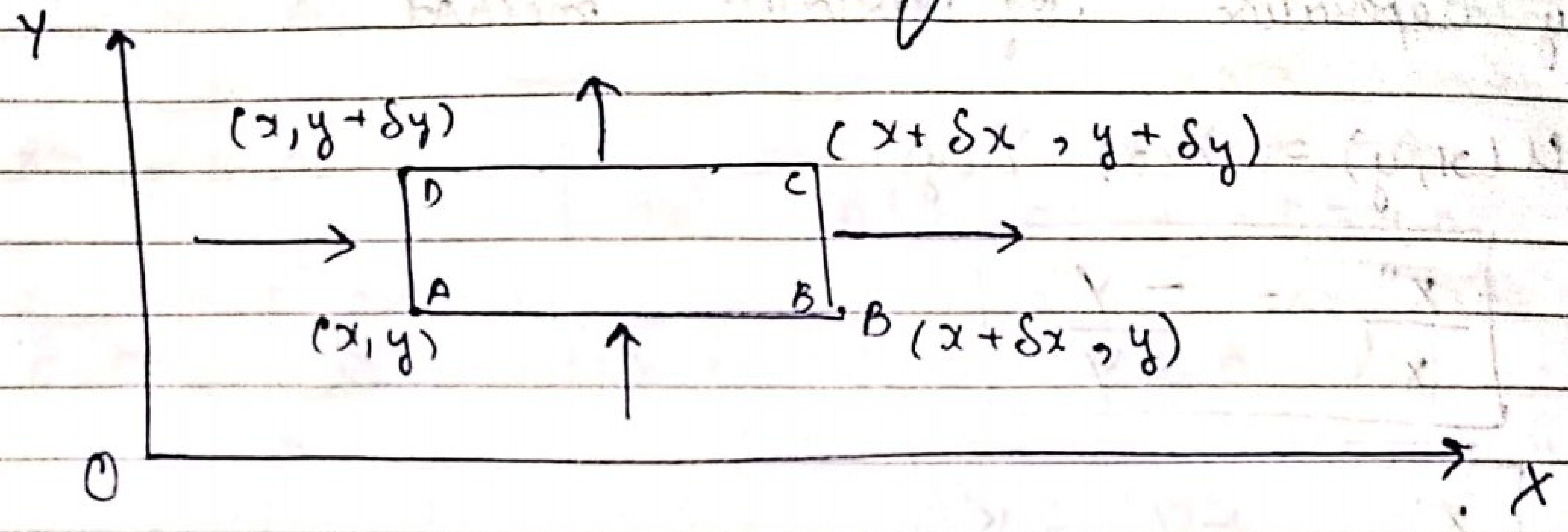
1) $u \neq \infty$ if $t \to \infty$

2) $\frac{\partial u}{\partial x} = 0$ for $x = 0$ or $x = l$

3) $u = lx - x^2$ for $t = 0$ between $x = 0$ & $x = l$

Imp

2-D heat flow



Consider the flow of heat in a metal plate in the XOY plane.
If the temperature at any point is independent of the Z co-ordinate and depends on X, Y and T only then the flow is called two dimensional and the heat flow lies in the plane XOY only and zero along to the normal ~~of~~ to the plane of XOY.

The 2-D heat flow eq^n is

$$\boxed{c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}}$$ Imp.

In steady state condition 2D heat flow equation convert in laplace eq^n.

$$\frac{\partial u}{\partial t}=0 \quad ; \quad \boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$

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Solⁿ of the laplace equation

By seperation of variable method.

$$u(x,y) = X(x)\,Y(y)$$

$$\boxed{\frac{X''}{X} = \frac{-Y''}{Y}}$$

$$\boxed{\frac{X''}{X} = \frac{-Y''}{Y} = K}$$

$\cancel{\frac{X''}{X} = K} =$ case 1 $\qquad K = 0$ $\therefore$

$$\frac{X''}{X} = \frac{-Y''}{Y} = 0. \qquad \boxed{u = (C_1 + xC_2)(C_3 + xC_4)}$$

$$X'' = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow D^2 X = 0 \quad ; \quad m^2 = 0$$

$$\boxed{m = 0, 0}$$

$$\boxed{X = C_1 + xC_2}$$

$$\frac{-Y''}{Y} = 0 \Rightarrow \frac{-Y''}{Y} = 0 \Rightarrow -\frac{\partial^2 u}{\partial y^2} = 0$$

$$-D^2 y = 0 \Rightarrow D^2 y = 0$$

$$m = 0, 0$$

$$\boxed{Y = C_3 + xC_4}$$

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case 2 $K = p^2$

$\therefore u = (c_5 e^{px} + c_6 e^{-px}) \times (c_7 \cos py + c_8 \sin py)$

$$\frac{X''}{X} = -\frac{Y''}{Y} = p^2$$

$\cancel{X'' = p^2 X}$

$\cancel{\frac{\partial^2 u}{\partial x^2} - p^2 \frac{\partial u}{\partial x} = 0}$

$\cancel{(D^2 - p^2 D)\, x = 0}$

$\cancel{D^2 - p^2 D = 0}$

$\cancel{D^2 = p^2 D}$

$\boxed{D = p^2}$

$$\frac{\partial^2 u}{\partial x^2} = p^2 \Rightarrow D^2 - p^2 = 0$$

$$D^2 = p^2 \Rightarrow D = \pm p$$

$$\boxed{X = (c_5 e^{px} + c_6 e^{-px})}$$

$$-\frac{\partial^2 u}{\partial y^2} = p^2 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -p^2$$

$$(D^2 + p^2)\, y = 0$$

$$D^2 = -p^2$$

$$D = pi$$

$$\boxed{Y = c_7 \cos py + c_8 \sin py}$$

case 3 $K = -p^2$

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Of these three solⁿ we have to chose that solⁿ which is consistent with the physical nature of the problem & the given boundary condition

Imp

Ques 1. Use seperation of variable to solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the boundary conditions.

$u(0,y) = u(l,y) = u(x,0) = 0$
and
$\left. u(x,a) = \sin \frac{n\pi x}{l} = f(x) \right]$ (3)

$\underbrace{u(0,y) = u(l,y)}_{B\cdot C\ (1)}, = \underbrace{u(x,0) = 0}_{\text{Initial}\ (2)}$

$B \cdot C \Rightarrow u(0,y) = u(l,y)$

Initial $\Rightarrow u(x,0) = 0$

By Seperation of variable method we know that $\boxed{\frac{X''}{X} = -\frac{Y''}{Y} = K}$

* minus bas Y ko do.

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when $K=0$

$$u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y)$$

Using BC

$x=0$ $\quad u(0,y) = 0 = c_1 (c_3 + c_4 y)$ $\therefore$ $\boxed{c_1 = 0}$

$x=l$ $\quad 0 = c_2 l (c_3 + c_4 y)$ $\therefore$ $\boxed{c_2 = 0}$

$u=0$

here $c_1 = 0$ means there is no heat which is impossible hence case is neglected

---

case 2 $\qquad K = + p^2$

$$u(x,y) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 \cos py + c_4 \sin py\right)$$

Using B.C

$x=0$; $0 = (c_1 + c_2)(c_3 \cos py + c_4 \sin py)$

$\therefore$ $\boxed{c_1 + c_2 = 0}$

$x = l$; $0 = \left(c_1 e^{pl} + c_2 e^{-pl}\right)\left(c_3 \cos py + c_4 \sin py\right)$

$\therefore$ $\boxed{c_1 e^{pl} + c_2 e^{-pl} = 0}$

$\therefore$ $c_1 = c_2 = 0$

which is impossible $\therefore$ case is neglected.

Of these three sol^n we have to chose that sol^n which is consistent with the physical nature of the problem & the given boundary condition.

Imp

Ques 1. Use seperation of variable to solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the boundary conditions.

$u(0,y) = u(l,y) = u(x,0) = 0$
and
$$\left. u(x,a) = \sin\frac{n\pi x}{l} = f(x) \right] \quad (3)$$

$$\underbrace{u(0,y) = u(l,y)}_{B \cdot C \ (1)} = \underbrace{u(x,0) = 0}_{\text{Initial } (2)}$$

$B \cdot C \Rightarrow u(0,y) = u(l,y)$

Initial $\Rightarrow u(x,0) = 0$

By seperation of variable method we know that $\boxed{\frac{X''}{X} = \frac{-Y''}{Y} = K}$

* minus has Y ko lo

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when $K=0$

$u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y)$

Using BC

$x=0$ $\quad u(0,y) = 0 = c_1(c_3 + c_4 y) \therefore \boxed{c_1 = 0}$

$x=l$ $\quad 0 = c_2 l(c_3 + c_4 y) \therefore \boxed{c_2 = 0}$

$u=0$

here $c_1 = 0$ means there is no heat which is impossible hence case is neglected.

---

case 2 $\quad K = +p^2$

$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

Using B.C

$x=0; \quad 0 = (c_1 + c_2)(c_3 \cos py + c_4 \sin py)$

$\therefore \boxed{c_1 + c_2 = 0}$

$x=l; \quad 0 = (c_1 e^{pl} + c_2 e^{-pl})(c_3 \cos py + c_4 \sin py)$

$\therefore \boxed{c_1 e^{pl} + c_2 e^{-pl} = 0}$

$\therefore c_1 = c_2 = 0$

which is impossible $\therefore$ case is neglected.

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Case 3 $\qquad K = -p^2$

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

$x = 0$

$$0 = C_1 (C_3 e^{py} + C_4 e^{-py})$$

$$\therefore \boxed{C_1 = 0}$$

$x = l$

$$0 = (\cancel{C_1 \cos pl}\,(C_2 \sin pl)(C_3 e^{py} + C_4 e^{-py})$$

$C_2 \neq 0$

$\sin pl = \sin n\pi$

$$p = \frac{n\pi}{l}$$

Equation

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

$\therefore \quad C_1 = 0 \quad p = \frac{n\pi}{l}$

$$\boxed{u(x,y) = (C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})}$$

Using IC $\qquad u(x,0) = 0$

$$0 = (C_2 \sin px)(C_3 + C_4)$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

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$c_3 + c_4 = 0 \quad \therefore \quad \boxed{c_4 = -c_3}$

$$u(x,y) = c_2 \sin px \left(c_3 e^{py} - c_3 e^{-py} \right)$$

$$u(x,y) = c_2 c_3 \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)$$

$$\boxed{u(x,y) = \sum b_n \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)}$$

<u>Using</u> $u(x,a) = \sin \frac{n\pi x}{l}$

<u>$y = a$</u>

$$u(x,a) = \sum b_n \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}} \right)$$

$$1 \sin \frac{n\pi x}{l} = u(x,a) = \sum b_n \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}} \right)$$

$$b_n = \frac{1}{2 \times \left[\frac{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}}{2} \right]} \Rightarrow b_n = \frac{1}{2 \sinh \frac{n\pi a}{l}}$$

$$u(x,y) = \frac{1}{2 \sinh \frac{n\pi a}{l}} \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)$$

$$u(x,y) = \frac{1}{\sinh \frac{n\pi a}{l}} \sin \frac{n\pi x}{l} \left(\frac{e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}}{2} \right)$$

$$\boxed{u(x,y) = \frac{\sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l}}{\sinh \frac{n\pi a}{l}}}$$

y differ toh cos aur sin mai y

Ques Imp x differ toh cos aur sin mai x

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Solve the laplace equation in a rectangular X.Y plane with $u(x,0)=0$, $u(x,b)=0$, $u(0,y)=0$ & $u(a,y)=f(y)$ ll y (0<x<a)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

y differ ho raha hai to $K=+p^2$ is valid aur jis ke cos & sin mai y function ho

B.C : ① $u(x,0)=u(x,b)=0$

IC : ② $u(0,y)=0$

: ③ $u(a,y)=f(y)$

at $K=0$

$$u(x,y)=(C_1+C_2x)(C_3+C_4y)$$

$y=0$

$$0=(C_1+C_2x)(C_3)$$

$\therefore$ $\boxed{C_3=0}$

$y=b$

$$0=(C_1+C_2x)(C_3+C_4b)$$

$$0=(C_1+C_2x)(C_4b)$$

$C_4=0$

at $K=p^2$

$$u(x,y)=\left(C_1e^{px}+C_2e^{-px}\right)(C_3\cos py+C_4\sin py)$$

$y=0$

$$0=\left(C_1e^{px}+C_2e^{-px}\right)C_3$$

$$\boxed{C_3=0}$$

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$\underline{y = b}$

$$u(x,y) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 \cos pb + c_4 \sin pb\right) = 0$$

$c_3 \to 0$

$$\boxed{c_3 = 0}$$

$\therefore$

$$\therefore \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_4 \sin pb\right) = 0$$

$$c_4 \neq 0$$

$$\sin pb = \sin n\pi$$

$$\boxed{p = \frac{n\pi}{b}}$$

$\underline{BC}$ $\quad u(x,0) = u(x,b) = 0$

$$\boxed{u(x,y) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_4 \sin py\right)}$$

$\underline{IC}$ $\quad u(0,y) = 0$

$x = 0$

$$u(0,y) = 0 = (c_1 + c_2)(c_4 \sin py)$$

$$\boxed{c_1 + c_2 = 0} \quad \therefore \quad \boxed{c_1 = -c_2}$$

$$u(x,y) = \left(c_1 e^{px} - c_1 e^{px}\right) c_4 \sin py$$

$p = \frac{n\pi}{b}$

$$\boxed{u(x,y) = \left(c_1 e^{\frac{n\pi x}{b}} - c_1 e^{-\frac{n\pi x}{b}}\right) c_4 \sin \frac{n\pi}{b} y}$$

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$$u(x,y) = C_1 C_4 \sin\frac{n\pi}{b} y \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right)$$

$$u(x,y) = b_n \sin\frac{n\pi y}{b} \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right)$$

$$u(a,y) = f(y)$$

$$\underline{x=a}$$

$$u(x,y) = b_n \sin\frac{n\pi y}{b}\, e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}$$

$$u(x,y) = b_n \sin\frac{n\pi y}{b} \left[2\left[\frac{e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}}{2}\right]\right]$$

$$\left[u(a,y) = b_n \sin\frac{n\pi y}{b}\, 2\sinh\frac{n\pi a}{b}\right]$$

$$\cancel{u(x,y) = b_n}$$

$$f(y) = \sum b_n \sin\frac{n\pi y}{b} \cdot 2\sinh\frac{n\pi a}{b}$$

$$\sinh\frac{n\pi a}{b}\; \underline{b_n} = \frac{2}{b}\int_0^b f(y) \sin\frac{n\pi y}{b}\, dy.$$

$$2\sinh\frac{n\pi a}{b}\; b_n = \frac{2}{b}\int_0^b f(y) \sin\frac{n\pi y}{b}\, dy$$

$$\boxed{b_n = \frac{1}{b\sinh\frac{n\pi a}{b}} \int_0^b f(y) \sin\frac{n\pi y}{b}\, dy}$$

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Imp Special case.

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fail the case for $\left.\begin{matrix} K=0 \\ K=p^2 \end{matrix}\right]$

Ques Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$0 \le x \le a$, $0 \le y \le b$

given $\underset{(2)}{u(x,b)} = \underset{(1)}{u(0,y) = u(a,y) = 0}$ & $\underset{(3)}{u(x,0) = x(a-x)}$

Let $K = -p^2$ | $y \to$ change

Solⁿ :

$u = (C_1 \cos px + C_2 \sin px)(C_3 e^{+py} + C_4 e^{-py})$

B.C. $u(0,y) = u(a,y) = 0$

$x = 0$

$0 = C_1 (C_3 e^{py} + C_4 e^{-py})$

$\therefore \boxed{C_1 = 0}$

$x = a$

$0 = (C_2 \sin pa)(C_3 e^{py} + C_4 e^{-py})$

$\therefore pa = n\pi$; $p = \frac{n\pi}{a}$

$$\boxed{u = \sum \left[C_2 \sin \frac{n\pi x}{a} \right] \left[C_3 e^{\frac{n\pi y}{a}} + C_4 e^{-\frac{n\pi y}{a}} \right]}$$

$u(x,b) = 0$

$y = b$

$0 = \sum \left[C_2 \sin \frac{n\pi x}{a} \right] \left[C_3 e^{\frac{n\pi b}{a}} + C_4 e^{-\frac{n\pi b}{a}} \right]$

$0 = \sin \frac{n\pi x}{a} \left[C_2 C_3 e^{\frac{n\pi b}{a}} + C_2 C_4 e^{-\frac{n\pi b}{a}} \right]$

$0 = \sin \frac{n\pi x}{a} \left[A e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}} \right]$

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Let $A e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}} = 0$.

$$\boxed{A e^{\frac{n\pi b}{a}} = -B e^{-\frac{n\pi b}{a}} = -\frac{1}{2} B_n}$$

kuki yeah ~~to parabolic~~ hyperbolic mai convert kardenge.

$u(x,y) = \sin\frac{n\pi x}{a}\left[C_2C_3 e^{\frac{n\pi b}{a}} + C_2C_4 e^{-\frac{n\pi b}{a}}\right]$

$u(x,y) = \sin\frac{n\pi x}{a}\left[-\frac{1}{2}B_n\, e^{-\frac{n\pi b}{a}} / e^{\frac{n\pi y}{a}} + \frac{1}{2}B_n e^{\frac{n\pi b}{a}} / e^{-\frac{n\pi y}{a}}\right]$

$= \frac{1}{2} B_n \sin\frac{n\pi x}{a}\left[e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)}\right]$

$= B_n \sin\frac{n\pi x}{a}\left[\frac{e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)}}{2}\right]$

$$\boxed{u(x,y) = B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi}{a}(b-y)}$$

~~$B_n = \frac{2}{b}\int_0^b$~~

Using IC $\quad u(x,0) = x(a-x)$

$u(x,0) = B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$

$u(x,0) = \sum B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$

$x(a-x) = \sum B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$

$\sin x = \int \cos x$ $\quad (xa - x^2)$

$-\cos x = \int \sin x$

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$$B_n \sinh \frac{n\pi b}{a} = \frac{2}{a}\int_0^a x(a-x)\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[\left[-x(a-x)\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^a - \left[-(a-2x)\frac{a^2}{n^2\pi^2}\sin\frac{n\pi x}{a}\right]_0^a\right.$$

$$\left. + \left[+(-2)\frac{a^3}{n^3\pi^3}\cos\frac{n\pi x}{a}\right]_0^a\right]$$

$$= \frac{2}{a}\left[x(x-a)\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^a + \left[(-2)\frac{a^3}{n^3\pi^3}\cos\frac{n\pi x}{a}\right]_0^a$$

$$= \frac{2}{a}\left[[0-0] + (-2)\frac{a^3}{n^3\pi^3}\left[\cos n\pi - \cos 0\right]\right]$$

$$= \frac{2}{a}\left[(-2)\frac{a^3}{n^3\pi^3}\left[(-1)^n - 1\right]\right]$$

$$= \frac{4}{a}\frac{a^3}{n^3\pi^3}\left[1-(-1)^n\right] \Rightarrow \frac{4a^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

when $n = \text{odd}$ | $n = \text{even}$

$\frac{4a^2}{n^3\pi^3}[2] \Rightarrow \frac{8a^2}{n^3\pi^3}$ | 0

$$u(x,y) = \frac{8a^3}{\pi^3}\sum_{n=1,2\ldots}^{\infty} \frac{\sin\frac{n\pi x}{a}}{n^3 \sinh\frac{n\pi b}{a}} \sinh\frac{n\pi}{a}(b-y)$$

$$\Rightarrow b_n = \frac{8a^2}{n^3\pi^3} \times \frac{1}{\sinh\frac{n\pi b}{a}}$$

Imp

Special Case
infinite length

A rectangular plate with insulated surface is 8 cm wide & so long compared to its width that it may be considered infinite in length without introducing an appropriate error if the temperature along one short edge $y = 0$ is given $u(x,0) = 100 \sin \frac{\pi x}{8}$

where $0 \le x \le 8$ while the two long edges $x = 0$ and $x = 8$ as well as the other short edge is kept at $0^\circ C$ so that steady state temperature at any point of the plate $u(x,y) = 400 e^{-\pi x}$

$$u(x,y) = 100 e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}$$

Let : $k = p^2$ pass

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

boundary condition

$$u(0,y) = u(8,y) = 0$$

$\therefore$ $x = 0$

$$0 = C_1 (C_3 e^{py} + C_4 e^{-py})$$

$$\therefore \quad C_1 = 0$$

$x = 8$

$$0 = (C_2 \sin 8p)(C_3 e^{py} + C_4 e^{-py})$$

$$8p = n\pi \Rightarrow \boxed{p = \frac{n\pi}{8}}$$

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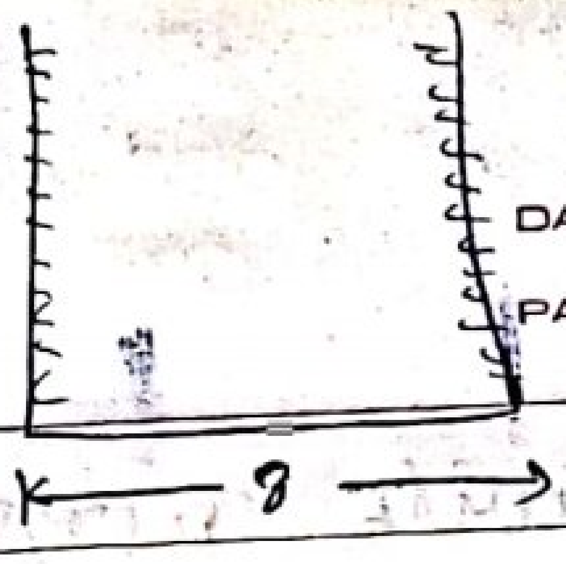

infinite length

$$e^{\infty} = \infty$$

$$e^{-\infty} = 0$$

$u(x,y) = 0$ ★

$$\boxed{\lim_{y \to \infty} u(x,y) = 0}$$ for infinite length

$$\therefore u(x,y) = \left[C_2 \sin \frac{n\pi x}{8} \right] \left[C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}} \right]$$

now $(u(x,y) = 0)$

$$\lim_{y\to\infty} u(x,y) = \cancel{u(x,} 0 = \left[C_2 \sin \frac{n\pi x}{8} \right] \lim_{y \to \infty} \left[C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}} \right.$$

$$e^{-\infty} = 0$$

for this condition only if

$$\therefore \left[C_3 = 0 \right] \text{ because } \left[e^{\infty} = \infty \right]$$

$$u(x,y) = 0 = C_2 C_4 \sin \frac{n\pi x}{8} \cdot e^{-\frac{n\pi y}{8}}$$

$$u(x,y) = b_n \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}}$$

$$u(x,y) = b_n \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}}$$

$u(x,0)$

~~$u(x,y)$~~ =

Let $y = 0$

$$\boxed{u(x,0) = b_n \sin \frac{n\pi x}{8}} \quad \text{— (1)}$$

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On comparing with $u(x,0) = 100 \sin \frac{\pi x}{8}$

$b_n = 100$ $\quad$ $n = 1$

$$u(x,y) = b_n \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}}$$

$$u(x,y) = 100 \sin \frac{\pi x}{8} e^{-\frac{\pi y}{8}}$$ Ans

Transmission Lines

1. Consider the flow of electricity in an insulated cable.
2. Let V be the potential and I the current at time t at a point P of the cable at a distance x from given point.
3. Let R, L, C, G be respectively the resistance, inductance, capacitance and leakance to the ground per unit length of the cable.

4. Each assumed to be constant.

Equation of Telephone

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} + (RC + LG)\frac{\partial V}{\partial t} + RGV \quad -①$$

$$\frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2} + (LG + RC)\frac{\partial I}{\partial t} + RGI \quad -②$$

Note 1 : if $L = G = 0$

then equation 1 & 2

$$\boxed{\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}} \qquad \boxed{\frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t}}$$

which are known as telegraph equation

which are similar to 1-D heat flow equation

Note 2 : if $R = G = 0$

then equation 1 & 2

$$\boxed{\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}} \qquad \boxed{\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}}$$

known as radio equation

Similar to wave equation.

Neglecting ya negligible
then eqⁿ $\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 e}{\partial t^2}$

Imp

Note 3 : If R and G are negligible then the transmission line becomes

$$\boxed{\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t}} \qquad \boxed{\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}}$$

Note 4 If $L = C = 0$ the equation becomes

$$\boxed{\frac{\partial^2 V}{\partial x^2} = RGV} \qquad \boxed{\frac{\partial^2 I}{\partial x^2} = RGI}$$

known as summarine cable

Ques find the current 'i' & voltage 'e' in a line of length l, t seconds after the ends are suddenly grounded given that

$$i(x,0) = i_0 \quad , \quad e(x,0) = e_0 \sin \frac{\pi x}{l}$$

Also R & G are negligible

Since R & G are negligible then lines become

$$\boxed{\frac{\partial e}{\partial x} = -L \frac{\partial i}{\partial t}} \quad \text{--- (1)} \qquad \boxed{\frac{\partial i}{\partial x} = -C \frac{\partial e}{\partial t}} \quad \text{--- (2)}$$

for eliminating i differentiating (1) with respect to 'x' & (2) w.r.t 't'

$$\boxed{\frac{\partial^2 e}{\partial x^2} = -L \frac{\partial^2 i}{\partial x \partial t}} \quad \text{— (3)}$$

(1) BC
(2) IV
(3) IC

$$\boxed{\frac{\partial^2 i}{\partial x \partial t} = -C \frac{\partial^2 e}{\partial t^2}} \quad \text{— (4)}$$

$$\boxed{\frac{\partial^2 e}{\partial x^2} = LC \frac{\partial^2 e}{\partial t^2}} \quad \text{from } 3 \,\&\, 4$$

→ similiar to wave

since the ends are grounded.

BC $\quad e(0,t) = e(l,t) = 0$

IC $\quad e(x,0) = e_0 \sin \frac{\pi x}{l}$ → Initial condition hamesha at last.

here given $\boxed{i = i_0 \text{ when } t = 0}$

$i = i_0$ then differentiating w.r.t to x

$$\frac{\partial i}{\partial x} = 0 \qquad \left[\frac{\partial i}{\partial x} = -C \frac{\partial e}{\partial t}\right]$$

$$\Longrightarrow \left(\frac{\partial e}{\partial t} = 0\right)_{t=0} \qquad \text{IV}$$

By Method of seperation of varable

$$e = X(x)\, T(t)$$

$$X = C_1 \cos px + C_2 \sin px$$

$$T =$$

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$$\frac{\partial^2 e}{\partial x^2} = LC \frac{\partial^2 e}{\partial t^2}$$

$$e = XT$$

$$\frac{\partial^2 XT}{\partial x^2} = LC \frac{\partial^2 XT}{\partial t}$$

$$\frac{T}{XT} \frac{d^2 X}{dx^2} = \frac{LCX}{XT} \frac{d^2 T}{dt}$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{LC}{T} \frac{d^2 T}{dt} = -p^2$$

$$\frac{d^2 X}{dx^2} = -p^2 X \Rightarrow (D^2 + p^2) X = 0$$

$$m^2 = -p^2$$

$$m = \pm p i$$

$$\boxed{X = C_1 \cos px + C_2 \sin px}$$

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$$e(x,t) = (C_1 \cos px + C_2 \sin px)\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$$

BC $e(0,t) = 0$

$x = 0$

$$0 = C_1\left[C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right]$$

$$\therefore \boxed{C_1 = 0}$$

$$e(l,t) = 0$$

$$0 = (C_2 \sin pl)\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$$

$$\boxed{p = \frac{n\pi}{l}}$$

$$e(x,t) = (C_2 \sin px)\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$$

IV : $\left(\frac{\partial e}{\partial t} = 0\right)_{t=0}$

$$\frac{\partial e}{\partial t}_{t=0} = (C_2 \cos px)\left(-\frac{p}{\sqrt{LC}} C_3 \sin \frac{pt}{\sqrt{LC}} + \frac{p}{\sqrt{LC}} C_4 \cos \frac{pt}{\sqrt{LC}}\right)$$

$$0 = (C_2 \cos px)\left[\frac{p}{\sqrt{LC}} C_4\right]$$

$$\therefore \boxed{C_4 = 0}$$

$$\boxed{e(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}}$$

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sin → b_n
cos → a_n

x → function of → cos

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$$e(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi t}{l\sqrt{LC}}$$

IC :

$$e(x,0) = e_0 \sin\frac{\pi x}{l}$$

$$e_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\underline{\underline{b_n = e_0}} \quad ; \quad \underline{\underline{n = 1}}$$

$$e(x,t) = e_0 \sin\frac{\pi x}{l} \cos\frac{\pi t}{l\sqrt{LC}} \quad \text{— (A)}$$

$$\frac{\partial e}{\partial x} = -L\frac{\partial i}{\partial t}$$

differentiate A w.r.t x

$$\frac{\partial e}{\partial x} = \frac{e_0 \pi}{l} \cos\frac{\pi x}{l} \cos\frac{\pi t}{l\sqrt{LC}}$$

$$-L\frac{di}{dt} = \left(\frac{e_0\pi}{l} \cos\frac{\pi x}{l} \cdot \cos\frac{\pi t}{l\sqrt{LC}}\right).$$

$$i = \int \frac{-1}{L} \frac{e_0\pi}{l} \left(\cos\frac{\pi x}{l} \cos\frac{\pi t}{l\sqrt{LC}}\right) dt$$

$$i = -\frac{e_0 \cancel{\pi}}{L \cancel{l}} \frac{\cos \pi x}{l} \frac{\frac{\sin \pi t}{l\sqrt{LC}}}{\frac{\cancel{\pi}}{l\sqrt{LC}}} + C$$

$$i = -\sqrt{\frac{C}{L}}\, e_0 \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}} + C$$

put $t = 0$

$$i_0 = 0 + C$$

$\therefore$
$$\boxed{i = -\sqrt{\frac{C}{L}}\, e_0 \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}} + i_0}$$

Q Negleding R and G find the EMF ~~v(t)~~ v of x (t) $v(x,t)$ in a line of length (l) t seconds after the ends are suddenly grounded given that $i(x,0) = i_0$ and $v(x,0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l}$

when R and G are negilible

$\therefore$ $\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t}$ — (1) $\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}$ — (2)

when R & G are negilible then

$$\boxed{\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 e}{\partial t^2}}$$

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suddenly grounded

BC $v(0,t) = v(l,t) = 0$ (2)

IC $v(x,0) = e_1 \sin \frac{x\pi}{l} + e_5 \sin 5\frac{\pi x}{l}$ (1)

$i(x,0) = i_c \quad \therefore \frac{\partial i}{\partial x} = 0$

$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \Rightarrow \boxed{0 = \frac{\partial v}{\partial t}}$

$\left(\frac{\partial v}{\partial t}\right)_{t=0} = 0$ — (3)

$e(x,t) = (C_1 \cos px + C_2 \sin px)\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$

Boundary condition

(1)

$0 = e(0,t) = C_1\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$

$\therefore \boxed{C_1 = 0}$

$0 = e(l,t) = C_2 \sin pl\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$

$\therefore \boxed{p = \frac{n\pi}{l}}$

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$\therefore \quad e(x,t) = (C_2 \sin px)\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$

where $p = \frac{n\pi}{l}$

Using $\left(\frac{\partial v}{\partial t}\right)_{t=0} = 0$

$\frac{\partial v}{\partial t} = (C_2 \sin px)\left(-C_3 p \sin \frac{pt}{\sqrt{LC}} + C_4 p \cos \frac{pt}{\sqrt{LC}}\right)$

$t=0$

$0 = (C_2 \sin px)\left(C_4 p\right) \Rightarrow 0 = C_2 C_4 p \sin px$

$\therefore \quad \boxed{C_4 = 0}$

$e(x,t) = C_2 C_3 \sin px \cos \frac{pt}{\sqrt{LC}}$

$$\left[e(x,t) = \sum_1^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}} \right]$$

IC : $v(x,0)$

$t=0$

$v(x,0) = \sum_1^{\infty} b_n \sin \frac{n\pi x}{l}$ — (1)

$v(x,0) = e_1 \sin \frac{x\pi}{l} + e_5 \sin \frac{5\pi x}{l}$ — (2)

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Comparing 1 & 2

$$b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} +$$

$$b_4 \sin\frac{4\pi x}{l} + b_5 \sin\frac{5\pi x}{l} = e_1 \sin\frac{\pi x}{l} +$$

$$e_5 \sin\frac{5\pi x}{l}$$

$\therefore$ $b_1 = e_1$ & $b_5 = e_5$

$b_2 = b_3 = b_4 = 0$

$$\therefore \boxed{e(x,t) = e_1 \cdot \sin\frac{\pi x}{l} \cos\frac{\pi t}{l\sqrt{LC}} + e_5 \sin\frac{5\pi x}{l} \cos\frac{5\pi t}{l\sqrt{LC}}}$$