# Shor's algorithm

## A very short introduction

Jinghong Yang

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# Problem to Solve

$N = pq$, where $p, q$ are prime.

- You know $N$
- Want to find $p$ and $q$.

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# How to factor $N$

Suppose we somehow find a number $M$ such that

$$\gcd(M, N) \neq 1$$
$$\gcd(M, N) \neq N$$

Then $\gcd(M, N)$ is just a non-trivial factor of $N$.

We can find $\gcd(M, N)$ using Euclid's algorithm.

# Review: Congruence

$$a \equiv b \mod N$$

Basic properties:

If $a \equiv b \mod N$ and $c \equiv d \mod N$, then

- $a \pm c \equiv b \pm d \mod N$
- $ac \equiv bd \mod N$

# A bit of number theory

Find order $r$ given $a$, $N$:

$$a^r \equiv 1 \mod N$$

Define function $f_{a,N}$:

$$f_{a,N}(j) := a^j \mod N$$

Clearly, we have

$$\begin{aligned} f_{a,N}(j+r) &= a^{j+r} \mod N \\ &= (a^j \mod N) * (a^r \mod N) \\ &= (a^j \mod N) * 1 \\ &= f_{a,N}(j) \end{aligned} \tag{1}$$

# A bit of number theory

$$
\begin{aligned}
a^r &\equiv 1 \mod N \\
a^r - 1 &\equiv 0 \mod N \\
\left(a^{r/2} - 1\right)\left(a^{r/2} + 1\right) &\equiv 0 \mod N
\end{aligned} \tag{2}
$$

Unless $a^{r/2} \equiv -1 \mod N$, we have

$$
\gcd(a^{r/2} - 1, N) = p, \quad \gcd(a^{r/2} + 1, N) = q \tag{3}
$$

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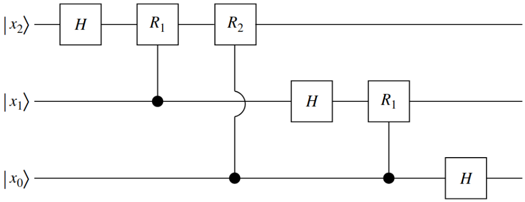
# Quantum Fourier Transform

$n$: the number of qubits. Definition:

$$U_{FT}|x\rangle = \sum_y \frac{1}{2^{n/2}} e^{2i\pi xy/2^n} |y\rangle \tag{4}$$




(b)

Figure 5.9 (a) The box $U_{\mathrm{FT}}$. (b) A circuit constructing $U_{\mathrm{FT}}$ in the case $n = 3$.

Figure: Quantum Fourier Transform illustration from Bellac (2006).

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_d = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2^d} \end{pmatrix} \tag{5}$$

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Suppose we get a quantum computer, with n+m qubits.
Want to factor $N$ ($2^n > N^2$). Given random number $a$.
Initial state:

$$|\Phi\rangle = \frac{1}{2^{n/2}} \left( \sum_{x=0}^{2^n-1} |x\rangle_n \right) \otimes |0 \cdots 0\rangle_m \tag{6}$$

This step is easy; for the first n qubits, cast them in $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
Next, we transform the state into

$$|\psi\rangle = \frac{1}{2^{n/2}} \left( \sum_{x=0}^{2^n-1} |x\rangle_n \right) \otimes |a^x \pmod{N}\rangle_m \tag{7}$$

Then, measure the output register (last m qubits). Suppose the result is $f_0$. The solution to $f_0 = f_{aN}(x) = a^x \mod N$ will have the form $x = x_0 + kr$.
Thus, after the measurement, the first n qubits are:

$$|\psi_0\rangle = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} |x_0 + kr\rangle_n \tag{8}$$

where $K \approx 2^n/r$.
Now, do a Quantum Fourier Transform $U_{FT}$ on $|\psi_0\rangle$.

$$\langle y| U_{FT} |\psi_0\rangle = \frac{1}{2^{n/2}} \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} e^{2i\pi y(x_0+kr)/2^n} \tag{9}$$

After the Quantum Fourier Transform, suppose you do a measurement, and you get $y$.

We will show later that if there's an integer $j$ such that $y$ is close to $\frac{j2^n}{r}$, you can obtain $r$.
For now, let's look at the probability of getting each $y$.

$$\begin{aligned} p(y) =& \frac{1}{2^n K} \left| \sum_{k=0}^{K-1} e^{2i\pi y(x_0+kr)/2^n} \right|^2 \\ =& \frac{1}{2^n K} \left| \sum_{k=0}^{K-1} e^{2i\pi kry/2^n} \right|^2 \end{aligned} \tag{10}$$

$$\sum_{k=0}^{K-1} e^{2i\pi kry/2^n} = \frac{1 - e^{2i\pi Kry/2^n}}{1 - e^{2i\pi ry/2^n}} = e^{\frac{i\pi r(K-1)}{2^n}} \frac{\sin\left(\pi yKr/2^n\right)}{\sin\left(\pi yr/2^n\right)} \qquad (11)$$

Thus,

$$p(y) = \frac{1}{2^n K} \frac{\sin^2\left(\pi yKr/2^n\right)}{\sin^2\left(\pi yr/2^n\right)} \qquad (12)$$

Suppose $\frac{2^n}{r}$ happens to be an integer, which means it is exactly equal to $K$. Then

$$p(y) = \frac{1}{2^n K} \frac{\sin^2(\pi y)}{\sin^2(\pi y/K)} = \begin{cases} \frac{1}{r} & \text{if } y = jK \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

This means $y/j = K = 2^n/r$, which means

$$\frac{j}{r} = \frac{y}{2^n} \tag{14}$$

Otherwise, we write

$$y = j\frac{2^n}{r} + \delta_y$$

$$\begin{aligned} P(y) =& \frac{1}{2^n K} \frac{\sin^2\left(\pi\left(j\frac{2^n}{r} + \delta_y\right) Kr/2^n\right)}{\sin^2\left(\pi\left(j\frac{2^n}{r} + \delta_y\right) r/2^n\right)} \\ =& \frac{1}{2^n K} \frac{\sin^2\left(\pi\delta_y Kr/2^n\right)}{\sin^2\left(\pi\delta_y r/2^n\right)} \end{aligned} \tag{15}$$

We know $\frac{2}{\pi}x \leq \sin x \leq x$ if $0 \leq x \leq \frac{\pi}{2}$. Thus, when we have $|\delta_y| \leq \frac{1}{2}$, we will get [1]

$$p(y) \geq \frac{4}{\pi^2}\frac{K}{2^n} \approx \frac{4}{\pi^2}\frac{1}{r} \approx 0.405\frac{1}{r} \tag{16}$$

This means we have roughly 40% chance of having $y$ close to $j2^n/r$.

[1] Recall $K \approx \frac{2^n}{r}$

Now, we have $\left|y - j\frac{2^n}{r}\right| \leq \frac{1}{2}$. Thus,

$$\left|\frac{y}{2^n} - \frac{j}{r}\right| \leq \frac{1}{2 * 2^n} \tag{17}$$

So $j/r$ must lie in a region of size $1/2^n$ around $\frac{y}{2^n}$.
Obviously, for fractions, unless $\frac{a}{b} = \frac{c}{d}$, we always have $\left|\frac{a}{b} - \frac{c}{d}\right| \geq \frac{1}{bd}$.
Suppose fractions $\frac{j_1}{r_1}$ and $\frac{j_2}{r_2}$ both lie in this region, and they are not equal, then [2]

$$\left|\frac{j_1}{r_1} - \frac{j_2}{r_2}\right| \geq \frac{1}{r_1 r_2} \geq \frac{1}{N^2} \geq \frac{1}{2^n} \tag{18}$$

Thus, the value $\frac{j}{r}$ is unique.

[2] Recall that $2^n > N^2$; also, by number theory, we have $r < N$.

M. L. Bellac. *A short introduction to quantum information and quantum computation*. Cambridge University Press, 1 edition, 2006. ISBN 0-521-86056-3,978-0-521-86056-7,978-0-511-22009-8,0-511-22009-X.

P. W. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5): 1484–1509, Oct 1997. ISSN 1095-7111. doi: 10.1137/s0097539795293172. URL http://dx.doi.org/10.1137/S0097539795293172.

# Questions