Shor's algorithm

A very short introduction

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Problem to Solve

$N = pq$, where p, q are prime.

- You know N
- Want to find p and q.

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How to factor N

Suppose we somehow find a number M such that

$$\gcd(M, N) \neq 1$$
$$\gcd(M, N) \neq N$$

Then $\gcd(M, N)$ is just a non-trivial factor of N.

We can find $\gcd(M, N)$ using Euclid's algorithm.

Review: Congruence

$$a \equiv b \mod N$$

Basic properties:

If $a \equiv b \mod N$ and $c \equiv d \mod N$, then

- $a \pm c \equiv b \pm d \mod N$
- $ac \equiv bd \mod N$

A bit of number theory

Find order r given a, N:

$$a^r \equiv 1 \mod N$$

Define function $f_{a,N}$:

$$f_{a,N}(j) := a^j \mod N$$

Clearly, we have

$$\begin{aligned} f_{a,N}(j+r) &= a^{j+r} \mod N \\ &= (a^j \mod N) * (a^r \mod N) \\ &= (a^j \mod N) * 1 \\ &= f_{a,N}(j) \end{aligned} \tag{1}$$

A bit of number theory

$$\begin{aligned} a^r &\equiv 1 \mod N \\ a^r - 1 &\equiv 0 \mod N \\ \left(a^{r/2} - 1\right)\left(a^{r/2} + 1\right) &\equiv 0 \mod N \end{aligned} \tag{2}$$

Unless $a^{r/2} \equiv -1 \mod N$, we have

$$\gcd(a^{r/2} - 1, N) = p, \quad \gcd(a^{r/2} + 1, N) = q \tag{3}$$

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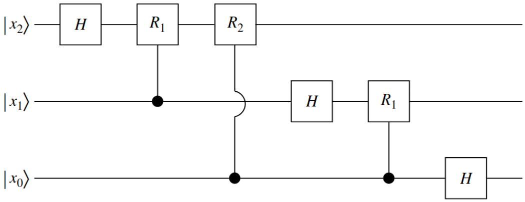
Quantum Fourier Transform

Notation reminder: $|7\rangle = |111\rangle$, $|6\rangle = |110\rangle$ so on.
n: the number of qubits. Definition:

$$U_{FT} |x\rangle = \sum_y \frac{1}{2^{n/2}} e^{2i\pi xy/2^n} |y\rangle \tag{4}$$



(b)

Figure 5.9 (a) The box U_{FT}. (b) A circuit constructing U_{FT} in the case $n = 3$.

Figure: Quantum Fourier Transform illustration from Bellac (2006).

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_d = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2^d} \end{pmatrix} \tag{5}$$

Some notations

$$|x\rangle = |x_{n-1} \cdots x_1 x_0\rangle \tag{6}$$

$$\begin{aligned} x &= x_0 + 2x_1 + \cdots + 2^{n-1}x_{n-1} \\ y &= y_0 + 2y_1 + \cdots + 2^{n-1}y_{n-1} \end{aligned} \tag{7}$$

Denote

$$.x_p x_{p-1} \cdots x_1 x_0 = \frac{x_p}{2} + \frac{x_{p-1}}{2^2} + \cdots + \frac{x_0}{2^p} \tag{8}$$

E.g. $n = 3$,

$$\begin{aligned} \exp\left(2\pi i \frac{xy}{8}\right) &= \exp\left(2\pi i \frac{1}{8} \left(x_0 + 2x_1 + 4x_2\right)\left(y_0 + 2y_1 + 4y_2\right)\right) \\ &= \exp\left(2\pi i \left[y_0 \left(\frac{x_2}{2} + \frac{x_1}{4} + \frac{x_0}{8}\right) + y_1 \left(\frac{x_1}{x} + \frac{x_0}{4}\right) + y_2 \frac{x_0}{2}\right]\right) \\ &= \exp(2\pi i \left(y_0.x_2x_1x_0 + y_1.x_1x_0 + y_2.x_0\right)) \end{aligned} \tag{9}$$

$$
\begin{aligned}
U_{FT}\left|x\right\rangle =&\frac{1}{2^{n/2}}\sum_{y} e^{2\pi i y_{n-1}.x_0}\cdots e^{2\pi i y_0.x_{n-1}\cdots x_1 x_0}\left|y_{n-1}\cdots y_1 y_0\right\rangle \\
=&\frac{1}{2^{n/2}}\left(\sum_{y_{n-1}} e^{2\pi i y_{n-1}.x_0}\left|y_{n-1}\right\rangle\right)\cdots\left(\sum_{y_1} e^{2\pi i y_1.x_{n-2}\cdots x_0}\left|y_1\right\rangle\right) \\
&\left(\sum_{y_0} e^{2\pi i y_0.x_{n-1}\cdots x_0}\left|y_0\right\rangle\right) \\
=&\frac{1}{2^{n/2}}\left(\left|0\right\rangle + e^{2\pi i.x_0}\left|1\right\rangle\right)\otimes\cdots\otimes\left(\left|0\right\rangle + e^{2\pi i.x_{n-2}\cdots x_0}\left|1\right\rangle\right) \\
&\otimes\left(\left|0\right\rangle + e^{2\pi i.x_{n-1}\cdots x_0}\left|1\right\rangle\right)
\end{aligned}
\tag{10}
$$

When $n = 3$,

$$
\frac{1}{\sqrt{8}}\left(\left|0\right\rangle_2 + e^{2\pi i.x_0}\left|1\right\rangle_2\right)\left(\left|0\right\rangle_1 + e^{2\pi i.x_1x_0}\left|1\right\rangle_1\right)\left(\left|0\right\rangle_0 + e^{2\pi i.x_2x_1x_0}\left|1\right\rangle_0\right)
\tag{11}
$$

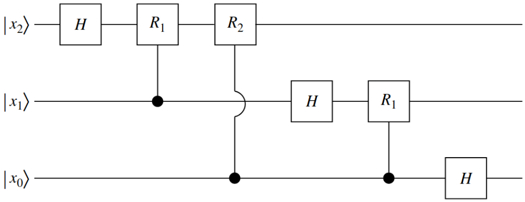


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Figure 5.9 (a) The box U_{FT}. (b) A circuit constructing U_{FT} in the case $n = 3$.

Figure: Quantum Fourier Transform illustration from Bellac (2006).

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_d = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2^d} \end{pmatrix} \tag{12}$$

What do we get from the circuit:

$$\frac{1}{\sqrt{8}}\left(|0\rangle_2 + e^{2\pi i.x_2x_1x_0}|1\rangle_2\right)\left(|0\rangle_1 + e^{2\pi i.x_1x_0}|1\rangle_1\right)\left(|0\rangle_0 + e^{2\pi i.x_0}|1\rangle_0\right) \tag{13}$$

What do we want:

$$\frac{1}{\sqrt{8}}\left(|0\rangle_2 + e^{2\pi i.x_0}|1\rangle_2\right)\left(|0\rangle_1 + e^{2\pi i.x_1x_0}|1\rangle_1\right)\left(|0\rangle_0 + e^{2\pi i.x_2x_1x_0}|1\rangle_0\right) \tag{14}$$

Solution, interpret the output as its bit reversal

$$|y\rangle = |y_0y_1\cdots y_{n-1}\rangle \tag{15}$$

instead of

$$|y_{n-1}\cdots y_1y_0\rangle$$

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Suppose we get a quantum computer, with n+m qubits.
Want to factor N ($2^n > N^2$). Given random number a.
Initial state:

$$|\Phi\rangle = \frac{1}{2^{n/2}} \left(\sum_{x=0}^{2^n-1} |x\rangle_n \right) \otimes |0 \cdots 0\rangle_m \tag{16}$$

This step is easy; for the first n qubits, cast them in $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
Next, we transform the state into

$$|\psi\rangle = \frac{1}{2^{n/2}} \left(\sum_{x=0}^{2^n-1} |x\rangle_n \right) \otimes |a^x \pmod N\rangle_m \tag{17}$$

Then, measure the output register (last m qubits). Suppose the result is f_0. The solution to $f_0 = f_{aN}(x) = a^x \mod N$ will have the form $x = x_0 + kr$.

Thus, after the measurement, the first n qubits are:

$$|\psi_0\rangle = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} |x_0 + kr\rangle_n \tag{18}$$

where $K \approx 2^n/r$.

Now, do a Quantum Fourier Transform U_{FT} on $|\psi_0\rangle$.

$$\langle y| U_{FT} |\psi_0\rangle = \frac{1}{2^{n/2}} \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} e^{2i\pi y(x_0+kr)/2^n} \tag{19}$$

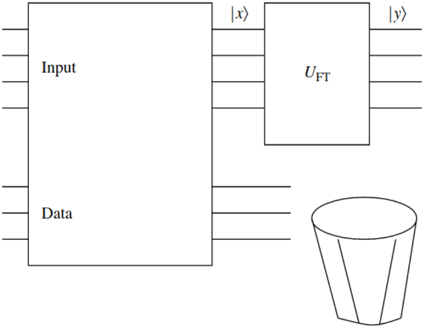

Figure 5.10 Schematic depiction of calculation of the period. The qubits of the output register are discarded.

Figure: Figure from Bellac (2006)

After the Quantum Fourier Transform, suppose you do a measurement, and you get y.

We will show later that if there's an integer j such that y is close to $\frac{j2^n}{r}$, you can obtain r.
For now, let's look at the probability of getting each y.

$$\begin{aligned} p(y) &= \frac{1}{2^n K} \left| \sum_{k=0}^{K-1} e^{2i\pi y(x_0+kr)/2^n} \right|^2 \\ &= \frac{1}{2^n K} \left| \sum_{k=0}^{K-1} e^{2i\pi kry/2^n} \right|^2 \end{aligned} \tag{20}$$

$$\sum_{k=0}^{K-1} e^{2i\pi kry/2^n} = \frac{1 - e^{2i\pi Kry/2^n}}{1 - e^{2i\pi ry/2^n}} = e^{\frac{i\pi r(K-1)}{2^n}} \frac{\sin\left(\pi yKr/2^n\right)}{\sin\left(\pi yr/2^n\right)} \tag{21}$$

Thus,

$$p(y) = \frac{1}{2^n K} \frac{\sin^2\left(\pi yKr/2^n\right)}{\sin^2\left(\pi yr/2^n\right)} \tag{22}$$

Suppose $\frac{2^n}{r}$ happens to be an integer, which means it is exactly equal to K. Then

$$p(y) = \frac{1}{2^n K} \frac{\sin^2(\pi y)}{\sin^2(\pi y / K)} = \begin{cases} \frac{1}{r} & \text{if } y = jK \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

This means $y/j = K = 2^n/r$, which means

$$\frac{j}{r} = \frac{y}{2^n} \tag{24}$$

Otherwise, we write

$$y = j\frac{2^n}{r} + \delta_y$$

$$\begin{aligned} P(y) =& \frac{1}{2^n K} \frac{\sin^2\left(\pi\left(j\frac{2^n}{r} + \delta_y\right) Kr/2^n\right)}{\sin^2\left(\pi\left(j\frac{2^n}{r} + \delta_y\right) r/2^n\right)} \\ =& \frac{1}{2^n K} \frac{\sin^2\left(\pi\delta_y Kr/2^n\right)}{\sin^2\left(\pi\delta_y r/2^n\right)} \end{aligned} \tag{25}$$

We know $\frac{2}{\pi}x \leq \sin x \leq x$ if $0 \leq x \leq \frac{\pi}{2}$. Thus, when we have $|\delta_y| \leq \frac{1}{2}$, we will get [1]

$$p(y) \geq \frac{4}{\pi^2}\frac{K}{2^n} \approx \frac{4}{\pi^2}\frac{1}{r} \approx 0.405\frac{1}{r} \tag{26}$$

This means we have roughly 40% chance of having y close to $j2^n/r$.

[1]Recall $K \approx \frac{2^n}{r}$

Now, we have $\left|y - j\frac{2^n}{r}\right| \leq \frac{1}{2}$. Thus,

$$\left|\frac{y}{2^n} - \frac{j}{r}\right| \leq \frac{1}{2 * 2^n} \tag{27}$$

So j/r must lie in a region of size $1/2^n$ around $\frac{y}{2^n}$.
Obviously, for fractions, unless $\frac{a}{b} = \frac{c}{d}$, we always have $\left|\frac{a}{b} - \frac{c}{d}\right| \geq \frac{1}{bd}$.
Suppose fractions $\frac{j_1}{r_1}$ and $\frac{j_2}{r_2}$ both lie in this region, and they are not equal, then [2]

$$\left|\frac{j_1}{r_1} - \frac{j_2}{r_2}\right| \geq \frac{1}{r_1 r_2} \geq \frac{1}{N^2} \geq \frac{1}{2^n} \tag{28}$$

Thus, the value $\frac{j}{r}$ is unique.

[2] Recall that $2^n > N^2$; also, by number theory, we have $r \leq N$.

M. L. Bellac. *A short introduction to quantum information and quantum computation*. Cambridge University Press, 1 edition, 2006. ISBN 0-521-86056-3,978-0-521-86056-7,978-0-511-22009-8,0-511-22009-X.

P. W. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5): 1484–1509, Oct 1997. ISSN 1095-7111. doi: 10.1137/s0097539795293172. URL http://dx.doi.org/10.1137/S0097539795293172.

Questions