HOMOTOPY LIMITS: AN ∞-CATEGORICAL PERSPECTIVE

YOYO JIANG

Abstract. In many cases, we would like to work with a weaker notion of equivalence than isomorphism in a category, such as homotopy equivalences of topological spaces, and infinity categories given a natural way to make categorical constructions that respect these equivalences. In this paper, we begin by introducing quasi-categories and describe some of their properties before discussing some categorical constructions with an emphasis on (co)limits. We relate ∞-categorical (co)limits to classical homotopy (co)limits, and describe examples in the setting of topological spaces.

Contents

1. Introduction

In a first course in topology, we learn that the correct notion of "sameness" between topological spaces is that of homeomorphisms: bijections that preserve topological structure. When we begin studying algebraic topology, however, we often find it more convenient to work with spaces up to homotopy equivalence or even weak homotopy equivalences, since these are the equivalences that our invariants (homotopy groups, (co)homology, etc.) are able to distinguish. Hence, we might want to work in a place (a category) that does not differentiate between homotopy equivalent spaces. However, if we try to make our favourite constructions, we face some problems.

Example 1.1. One of the best ways to build our favourite topological spaces is by gluing together simpler spaces, which is formalized using colimits. For instance, we could build the 2-sphere S^2 by taking the pushout of the diagram

$$D^2 \hookleftarrow S^1 \hookrightarrow D^2 \tag{1}$$

which takes two disks and glues them along their boundaries. If we stop distinguishing between homotopy equivalent spaces, we can replace the diagram with

$$* \leftarrow S^1 \rightarrow * \tag{2}$$

since disks are contractible. The pushout of this new diagram is obtained by gluing together two points, which is a point. Clearly, this does not give us the sphere that we are after. In general, we find that we have lost access to many of our favourite spaces (for instance, CW complexes), since gluing together contractible spaces no longer gives us anything interesting.

A naive idea for a better category to work in is the homotopy category of topological spaces, where homotopic maps are identified and isomorphisms are indeed given by homotopy equivalences. Unfortunately, it turns out that straight-up quotienting by homotopies destroys too much of the categorical structure; for instance, the homotopy category does not have most limits and colimits. Heuristically, the problem is that we need to not only remember *when* two maps are homotopic, but also *how* they are homotopic. In other words, we need to remember "higher dimensional" information encoded by homotopies. A good place to do this is in an ∞-category. Specifically, we will describe a model of an $(\infty, 1)$-category, which has n-morphisms for all $n \in \mathbb{N}$ such that any m-morphism for $m > 1$ is invertible. This should bring to mind the prototypical example of topological spaces, in which our higher morphisms are given by homotopies, which are invertible by reversing the parametrization. It turns out that many of our favourite categorical constructions can be generalized to $(\infty, 1)$-categories, and surprisingly many of the theorems that hold for 1-categorical constructions still hold in the ∞-categorical setting.

Our goal in this expository document is twofold. First, we wish to introduce ∞-categories in a way that is more categorical in approach than simplicial. Although we will be focusing our discussion on quasi-categories, which model ∞-categories using simplicial sets, we try to present proofs using the ∞-cosmos framework developed by Riehl and Verity (see [RV22]), which are more model agnostic and often 2-categorical in nature. We will try our best to avoid discussion of fibrations and simplicial combinatorics by black boxing key results when necessary. In Section **??**, we introduce some necessary facts about simplicial sets including a formula for adjunctions and slice and join constructions. In Section 3, we introduce quasi-categories using the example of spaces and motivate the definition by drawing analogies to categories. We also introduce some properties about quasi-categories that make them an ∞-cosmos, which will be used in later proofs. In Section 5, we introduce adjunctions and limits using the ∞-cosmos framework, and we describe their universal properties in Section 6.

Secondly, we hope to motivate ∞-categories by introducing examples that "arise in nature" for which the 1-categorical language is not sufficient. The prototypical example is the ∞-category of spaces, where higher structure need to be taken into account when making homotopy invariant constructions as we saw above. In Section 4, we will introduce ∞-categories that arise from simplicially enriched categories and model categories. In the last two sections we show that familiar classical constructions have the expected universal property when considered ∞-categorically, and we give explicit examples of ∞-categorical (co)limits in spaces.

1.1. **Conventions.** We assume all topological spaces are compacted generated and weakly Hausdorff, and we let Top denote the cartesian closed category of cgwh spaces where internal homs have the compact open topology.

We let Δ denote the simplex category whose objects are non-empty finite ordinals $[n]$ and order preserving maps, and we let $\mathsf{sSet} := \mathsf{Set}^{\Delta^{\mathrm{op}}}$ denote the cartesian closed category of simplical sets.

We let $\Delta^n := \Delta(-, [n])$ be the n-simplex, and we let $\Lambda^n_k := \partial\Delta^n \setminus d^k\Delta^{n-1}$ be the (n, k)-horn constructed by discarding the k-th face from the boundary of the n-simplex.

We let $\mathbb{N} := \mathbb{Z}_{\geq 0}$.

2. Useful Facts about Simplicial Sets

Before we begin the discussion of ∞-categories, we recall some useful facts about simplicial sets, which are used to define the most popular model for $(\infty, 1)$-categories. Our approach to defining adjunctions involving sSet follows [Rie], which also gives a more detailed introduction to simplicial sets.

2.1. **A Formula for Adjunctions.** In this subsection, we describe a formula for constructing adjunctions involving the category of simplicial sets. First, we recall the density theorem, which states that every presheaf can be canonically represented as a colimit of representables. See [Rie16, Section 6.5] for a more detailed discussion.

Proposition 2.1 (density)**.** Let $\mathcal{C}$ be a category and $F \in \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ be a presheaf on $\mathcal{C}$. Then, F is the colimit of the diagram

$$\int F \xrightarrow{\Pi} \mathcal{C} \overset{よ}{\hookrightarrow} \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}} \tag{3}$$

where $\int F$ denotes the category of elements on F and Π denotes the canonical projection functor.

Proof. See [Rie16, Theorem 6.5.7]. □

Specializing to the case where $\mathcal{C} = \Delta$, we get:

Example 2.2. Every simplicial set is canonically a colimit of n-simplices. Given a simplicial set X, we call the category of elements $\int X$ the *category of simplices* of X. By the Yoneda lemma, we can equivalently describe $\int X$ as the comma category $よ \downarrow X$, whose objects are n-simplices $\Delta^n \to X$ for all $n \in \mathbb{Z}_{\geq 0}$. We can thus write the colimit (3) as

$$X \cong \operatorname*{colim}_{\Delta^n \to X} \Delta^n.$$

Equivalently, we can write the colimit as the coequalizer

$$\begin{tikzcd}
\Delta^m \arrow[r, "\theta"] \arrow[d, "\iota_{\sigma,\theta}"'] & \Delta^n \arrow[d, "\iota_\sigma"] & \\
\coprod\limits_{\substack{\sigma\colon \Delta^n \to X \\ \theta\colon \Delta^m \to \Delta^n}} \Delta^m \arrow[r, shift left] \arrow[r, shift right] & \coprod\limits_{\sigma\colon \Delta^n \to X} \Delta^n \arrow[r, two heads] & X \\
\Delta^m \arrow[u, "\iota_{\sigma,\theta}"] \arrow[ur, "\iota_{\sigma\theta}"'] & &
\end{tikzcd} \tag{4}$$

where the two maps being coequalized restrict to the top and bottom diagrams respectively on the copy of Δ^m indexed by σ and θ. We can simplify this coequalizer in two ways. First, it suffices to vary θ over the coface and codegeneracy maps since these generate Δ. Secondly, every degenerate σ is identified through the coequalizer to some non-degenerate simplex, so we can range over all non-degenerate σ's and take θ to only be coface maps. These simplifications allow us to explicitly write down a finite colimit for in small cases, as we will do in the next example.

Example 2.3. Consider the horn Λ^2_1, whose has three 0-simplices $\{0, 1, 2\}$ and two non-degenerate 1-simplices. The simplified colimit formula describes Λ^2_1 as a quotient of

$$\Delta^0_{(0)} \amalg \Delta^0_{(1)} \amalg \Delta^0_{(2)} \amalg \Delta^1_{(10)} \amalg \Delta^1_{(21)} \tag{5}$$

where we index each $\sigma\colon \Delta^1 \to X$ by the subscript $(d^0\sigma\ d^1\sigma)$. The quotienting object consists of four copies of Δ^1, indexed by σ varying over the two non-degenerate 1-simplices and θ varying over the two coface maps $d^0, d^1\colon \Delta^0 \rightrightarrows \Delta^1$. To give an example, when σ is given by the 1-simplex (10) and $\theta = d^0$, the correspond summand Δ^0 maps to $\Delta^1_{(10)}$ through d^1 in the top arrow of (4) and to $\Delta^0_(1)$ in the bottom arrow. Hence, the summand has the effect of identifying $\Delta^0_{(1)}$ with the codomain of $\Delta^1_{(10)}$ in the coproduct (5). Similarly, the other three summands of the quotienting coproduct identify the domains and codomains of the copies of Δ^1 in (5) with the corresponding 0-simplices to obtain the correctly shaped inner horn.

Heuristically, the n-simplices form a kind of "basis" for the category of simplicial sets from which all simplicial sets are assembled. Just like how linear maps are determined by extending the mapping on basis vectors, colimit preserving functors out of sSet are given by extending the mapping on n-simplices:

Proposition 2.4 (nerve-realization adjunction)**.** Let $F\colon \Delta \to \mathcal{E}$ be a functor into a cocomplete category. Then, there exists an extension $|\cdot|_F\colon \mathsf{sSet} \to \mathcal{E}$ along the Yoneda embedding

$$\begin{tikzcd} \Delta \arrow[rr, "F"] \arrow[dr, "よ"'] & & \mathcal{E} \\ & \mathsf{sSet} \arrow[ur, dashed, "|\cdot|_F"'] & \end{tikzcd}$$

given by taking a simplicial set $X \in \mathsf{sSet}$ to

$$|X|_F := \operatorname*{colim}_{\Delta^n \to X} F[n] \cong \int^{[n]\in\Delta} X_n \cdot F[n].$$

Furthermore, $|\cdot|_F$ has a right adjoint $N_F\colon \mathcal{E} \to \mathsf{sSet}$ defined as the composite

$$N_F := \mathcal{E} \xhookrightarrow{よ} \mathsf{Set}^{\mathcal{E}^{\mathsf{op}}} \xrightarrow{F^*} \mathsf{Set}^{\Delta^{\mathsf{op}}}$$

which sends an object $e \in \mathcal{E}$ to the simplicial set whose n-simplices are

$$N_F(e)_n = \mathcal{E}(F[n], e).$$

We will generally refer to functors of the form $|\cdot|_F$ as *realization* and functors of the form N_F as *nerve.*

Proof. Since $\mathcal{E}$ is cocomplete, the pointwise Kan extension formula is well defined, and since the Yoneda embedding is fully faithful, the Kan extension restricts to F on Δ. Finally, we can calculate that

$$\begin{aligned}
\mathcal{E}(|X|_F, e) &\cong \mathcal{E}\left(\operatorname*{colim}_{\Delta^n \to X} F[n], e\right) \\
&\cong \lim_{\Delta^n \to X} \mathcal{E}(F[n], e) \\
&\cong \lim_{\Delta^n \to X} N_F(e)_n \\
&\cong \lim_{\Delta^n \to X} \mathsf{sSet}\big(\Delta^n, N_F(e)\big) && \text{(Yoneda lemma)} \\
&\cong \mathsf{sSet}\left(\operatorname*{colim}_{\Delta^n \to X} \Delta^n, N_F(e)\right) \\
&\cong \mathsf{sSet}(X, N_F(e)) && \text{(density)}
\end{aligned}$$

naturally in all $X \in \mathsf{sSet}$ and $e \in \mathcal{E}$ which checks that $|\cdot|_F \dashv N_F$ indeed defines an adjunction. $\square$

Example 2.5 (geometric realization)**.** Define a functor $F\colon \Delta \to \mathsf{Top}$ by sending each $[n]$ to the standard n-simplex

$$F[n] := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 \right\} \subseteq \mathbb{R}^{n+1}$$

and sending the coface and codegeneracy maps to the corresponding topological coface and codegeneracy maps. The extension $|\cdot| := |\cdot|_F$ is called *geometric realization*, which sends a simplicial set X to the colimit

$$|X| := \operatorname*{colim}_{\Delta^n \to X} |\Delta^n| \cong \int^{[n]\in\Delta} X_n \cdot |\Delta^n|.$$

The right adjoint N_F sends a topological space E to the simplicial set whose n-cells are

$$N_F(E)_n := \mathsf{Top}(|\Delta^n|, E),$$

which are exactly the singular n-cells of E. We will thus call the functor $S_\bullet := N_F$ the *singular complex functor.*

Example 2.6. Let $\iota : \Delta \to \mathsf{Cat}$ be the inclusion functor. The nerve functor $N := N_\iota$ sends a small category $\mathcal{C}$, to the simplicial set whose n-cells are functors $[n] \to \mathcal{C}$. Such a functor is determined by the image of the atomic arrows, and exactly picks out a string of n composable arrows in $\mathcal{C}$. The degeneracy map $s_i \colon N\mathcal{C}_n \to n\mathcal{C}_{n+1}$ inserts the identity morphism at the i-th place; the outer face maps $d_0, d_n \colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ leaves out the outermost morphisms; the inner face maps $d_i \colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ for $0 < i < n$ composes the i-th and $i+1$-th morphisms.

One can also think about n-simplices in the nerve as commutative diagrams of the shape of the n-simplex. For instance, a 2-simplex in $N\mathcal{C}$ is a pair of composable morphisms

$$x \xrightarrow{f} y \xrightarrow{g} z,$$

in $\mathcal{C}$, which can be equivalently described as a commutative triangle

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & \arrow[u, phantom, "\#"] & z. \end{tikzcd}$$

The face maps, as we might expect, pick out the morphisms opposite the chosen vertex: d_0 picks out g which opposes the 0-th vertex, d_2 picks out f which opposes the second vertex, and d_1 picks out the composite $g \circ f$ which opposes the first vertex. For an example of degeneracy maps, given a 1-simplex in $N\mathcal{C}$ which is a morphism

$$x \xrightarrow{f} y,$$

the map s_0 repeats the 0-th vertex x by inserting id_x to form the commutative triangle

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, equal] \arrow[rr, "f"'] & \arrow[u, phantom, "\#"] & y, \end{tikzcd}$$

while the map s_1 repeats the first vertex y by inserting id_y.

We will now describe a left adjoint to ι. The existence of a left adjoint is guaranteed by Proposition 2.4, but we will give a simpler description than the Kan extension formula. Given a simplicial set X, the objects of $\tau_1 X$ is defined to be the vertices $\mathrm{ob}\,\tau_1 := X_0$. The morphisms are freely generated by the edges X_1 subject to relations given by X_2 which we describe below; the domain and codomain of the generating morphisms are given by the the face maps $d_1, d_0 \colon X_1 \to X_0$; the identity morphisms for each object is given by the degeneracy map $s_0 \colon X_0 \to X_1$. The relation on the morphisms are as follows: consider the free graph on X_0 generated by arrows in X_1; for each 2-cell $\sigma \in X_2$

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\sigma\Uparrow"] & z \end{tikzcd}$$

bounded by $f := d_2\sigma$, $g := d_0\sigma$ and $h := d_1\sigma$, impose the relation that $h = gf$. Associativity of composition is true for the free graph on X_0 generated by X_1 and thus true for the quotient $\mathrm{mor}\,\tau_1 X$. Unitality is given by the relation imposed by degenerate 2-cells coming from $s_0, s_1 \colon X_1 \to X_2$. We have thus shown that $\tau_1 X$ is indeed a category.

To show that $\tau_1 \dashv N$ is an adjunction, given a simplicial set X and a category $\mathcal{C}$. notice that a functor $F \colon \tau_1 X \to \mathcal{C}$ is given by an assignment on objects $F \colon X_0 \to \mathrm{ob}\,\mathcal{C}$ along with an assignment on morphisms given on generating arrows $F \colon X_1 \to \mathrm{mor}\,\mathcal{C}$ such that domain and codomain is respected and that for any 2-cell

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\sigma\Uparrow"] & z \end{tikzcd}$$

in X, we have $F(g \circ f) = Fh$. We will use this to construct a map of simplicial sets $\phi\colon X \to N\mathcal{C}$. The data of ϕ is given at the level of n-cells by a set of functions

$$\phi_n\colon X_n \to (N\mathcal{C})_n$$

for all $n \in \mathbb{N}$ that respect face and degeneracy maps. For $n = 0$, we take the assignment of F on objects. For $n = 1$, we take the assignment of F on generating morphisms. The naturality for face and degeneracy maps between $n = 0$ and $n = 1$ are guaranteed by functoriality. For $n = 2$, we define a function

$$\phi_2\colon X_2 \to \operatorname{mor}\mathcal{C} \underset{\operatorname{ob}\mathcal{C}}{\times} \operatorname{mor}\mathcal{C}$$

by sending a 2-cell below left

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[urr, phantom, "\sigma\Uparrow"] & & z \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & Fy \arrow[dr, "Fg"] & \\ Fx \arrow[ur, "Ff"] \arrow[rr, "Fh"'] \arrow[urr, phantom, "\#"] & & Fz \end{tikzcd}$$

to the commutative square above right, which is well defined by our assumption that $gf = h$. Mapping on higher cells are similarly well-defined by using the relationship imposed by the 2-cells. It is clear that this construction defines a bijective correspondence between maps $\tau_1 X \to \mathcal{C}$ and $X \to N\mathcal{C}$, so we are done.

Remark 2.7. Using the explicit description, we can check that τ_1 preserves products. In particular, this makes $\tau_1 \dashv N$ an sSet-enriched adjunction (where hom simplicial sets in Cat are taken to be nerves of hom categories).

2.2. **Joins of Simplicial Sets.**

In this subsection, we will very briefly introduce join and slice simplicial sets, which is a generalization of their 1-categorical counterparts, which are a useful tool for constructing new ∞-categories "analytically".

Definition 2.8. Let X, Y be simplicial sets. Their *join* is the simplicial set $X \star Y$ whose n-simplices is the set

$$(X \star Y)_n := X_n \amalg \left(\coprod_{p+1+q=n} X_p \times Y_q \right) \amalg Y_n. \tag{6}$$

The face and degeneracy maps are given on the first and third summands are by the corresponding maps on X and Y, and on the second summand for a pair $(\sigma, \tau) \in X_p \times Y_q$ by

$$d_i(\sigma,\tau) := \begin{cases} (d_i\sigma, \tau), & \text{if } i \le p, \\ (\sigma, d_{i-1-p}\tau), & \text{if } i > p. \end{cases} \qquad s_i(\sigma,\tau) := \begin{cases} (s_i\sigma, \tau), & \text{if } i \le p, \\ (\sigma, s_{i-1-p}\tau), & \text{if } i > p. \end{cases}$$

We have canonical simlpicial subset inclusions $X \hookrightarrow X \star Y$ and $Y \hookrightarrow X \star Y$ by including at the n-cell level into the first and third summands respectively.

Example 2.9. Let $\mathcal{C}, \mathcal{D}$ be categories. Their *categorical join* is the category $\mathcal{C} \star \mathcal{D}$ whose

- objects are $\operatorname{ob}(\mathcal{C} \star \mathcal{D}) := \operatorname{ob}\mathcal{C} \amalg \operatorname{ob}\mathcal{D}$;
- morphisms are

$$\mathcal{C} \star \mathcal{D}(a,b) := \begin{cases} \mathcal{C}(a,b), & \text{if } a, b \in \mathcal{C}; \\ \mathcal{D}(a,b), & \text{if } a, b \in \mathcal{D}; \\ *, & \text{if } a \in \mathcal{C} \text{ and } b \in \mathcal{D}; \\ \varnothing, & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{C}. \end{cases}$$

Then, the nerve of the join is (naturally) isomorphic to the join of the nerve

$$N(\mathcal{C} \star \mathcal{D}) \cong N\mathcal{C} \star N\mathcal{D},$$

which justifies the choice of notation. One can either check this fact by explicitly writing down the simplices or using an alternate description of the join as a Day convolution on augmented simplicial sets (see [RV22, Appendix D.2]).

Example 2.10. Using Example 2.9, we may obtain the join of simplices $\Delta^n \star \Delta^m$ as the nerve of the join $[n] \star [m]$, which we can easily check to be $[n+1+m]$. Hence, we have

$$\Delta^n \star \Delta^m \cong \Delta^{n+1+m}.$$

The join defines a bifunctor $- \star -\colon \mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet}$, which we can see either from direct construction or using the Day convolution perspective. Rephrasing Definition 2.8, we have the following characterization of the n-simplices in $X \star Y$:

Lemma 2.11. For each n-simplex $\rho\colon \Delta^n \to X \star Y$, exactly one of the following holds:

- The morphism ρ factors through the simplicial subset $X \hookrightarrow X \star Y$;
- The morphism ρ factors through the simplicial subset $Y \hookrightarrow X \star Y$;
- There exist unique $p, q \in \mathbb{N}$ such that $p + 1 + q = n$ and σ factors as a composition
$$\Delta^n \cong \Delta^p \star \Delta^q \xrightarrow{\sigma\star\tau} X \star Y$$
for some unique $\sigma \in X_p$ and $\tau \in Y_q$.

Proof. The three cases correspond to the three summands of (6). In the case of the middle summand, we note that $\sigma \star \tau$ corresponds exactly to the pair

$$(\sigma, \tau) \in X_p \times Y_q \subseteq (X \star Y)_n,$$

as under the Yoneda identification the map picks out the image of the unique non-degenerate n-simplex of Δ^n, which is given by the pair of unique non-degenerate simplices in $\Delta^p_p \times \Delta^q_q$ that map to (σ, τ) under $\sigma \star \tau$. $\square$

Example 2.12. We consider $X \star \Delta^0$, which should be thought about as the cone under X (the geometric realization $|X \star \Delta^0|$ is indeed the topological cone over $|X|$). Its non-degenerate n-simplices are

- the cone point $\Delta^0 \hookrightarrow X \star \Delta^0$,
- the non-degenerate n-simplices of X and
- for each non-degenerate $(n-1)$-simplex of X, an n-simplex whose final vertex is the cone point and the opposing face is the chosen $(n-1)$-simplex.

Remark 2.13. The cone point in $X \star \Delta^0$ is *terminal* in the following sense: for all $n \in \mathbb{Z}_{\geq 1}$ and all spheres σ in $X \star \Delta^0$ whose final vertex is the cone point (which we denote by $\top$), the lifting problem

$$\begin{tikzcd}
\Delta^0 \arrow[r, "\{n\}"'] \arrow[rr, bend left, "\top"] & \partial\Delta^n \arrow[r, "\sigma"] \arrow[d, hook] & X \star \Delta^0 \\
& \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

has a solution. To see this, consider the n-cell given by $\overline{\sigma} := d_n\sigma \star \top$ such that

$$\begin{tikzcd}
d^n\Delta^n \amalg \Delta^0 \arrow[r, hook] \arrow[dr, hook] & \partial\Delta^n \arrow[r, "\sigma"] & X \star \Delta^0 \\
& \Delta^n \arrow[ur, dashed, "\overline{\sigma}"'] &
\end{tikzcd}$$

commutes, which we take to be the candidate for the solution to the lifting problem. It suffices to check that $\overline{\sigma}$ agrees with σ on $\partial\Delta^n$, which we can check by checking that they agree on $d^k\Delta^n$ for all $0 \leq k \leq n$. For $k = n$, this is true by construction. For all $k < n$, we notice that $\top$ is the final vertex in both $d_k\sigma$ and $d_k\overline{\sigma}$ and that

$$d_n d_k \sigma = d_n d_k \overline{\sigma}$$

are $(n-2)$-simplices in X that agree by construction. Hence, $d_k\sigma$ and $d_k\overline{\sigma}$ are both given by the unique $(n-1)$-simplex $d_n d_k \sigma \star \top$ from $d_n d_k \sigma$ to $\top$, and thus agree.

Definition 2.14. Let $f\colon K \to X$ be a morphism of simplicial sets. We define the *slice simplicial set of X over f* to be the simplicial set $X_{/f}$ whose n-simplices are given by

$$\left(X_{/f}\right)_n := \{\sigma\colon \Delta^n \star K \to X : \sigma|_K = f\}$$

and whose face and degeneracy maps are given on the left component of the join. Explicitly, given a non-decreasing functor $\alpha\colon [m] \to [n]$, the associated function $\alpha^*\colon \left(X_{/f}\right)_n \to \left(X_{/f}\right)_m$ sends $\sigma\colon \Delta^n \star K \to X$ to the composite

$$\Delta^m \star K \xrightarrow{\alpha \star K} \Delta^n \star K \xrightarrow{\sigma} X.$$

Example 2.15. Let $\mathcal{J}, \mathcal{C}$ be small categories and $F\colon \mathcal{J} \to \mathcal{C}$ be a functor. Then, we observe that $(N\mathcal{C})_{/NF} \cong N(\mathcal{C}_{/F})$: the n-simplices of $N\mathcal{C}_{/NF}$ are given by maps

$$\Delta^n \star N\mathcal{J} \to N\mathcal{C}$$

that restrict to NF on $N\mathcal{J}$. This is the same as functors

$$[n] \star \mathcal{J} \to \mathcal{C}$$

that restrict to F on $\mathcal{C}$, which give the n-simplices of $N(\mathcal{C}_{/F})$.

Example 2.16. Specializing F in the above example to an object $c\colon 1 \to \mathcal{C}$, we get a canonical isomorphism $(N\mathcal{C})_{/c} \cong N(\mathcal{C}_{/c})$.

Proposition 2.17. Let $f\colon K \to X$ be a morphism of simplicial sets. Then, for any simplicial set Y, there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{sSet}}(Y, X_{/f}) \cong \mathrm{Hom}_{K/\mathsf{sSet}}(Y \star K, X)$$

Proof. This is true by definition for the simplices Δ^n, and since any simplicial set is canonically a colimit of the simplices, it suffices to show that both sides preserve colimits in the Y variable. This is true on the right side as the functor $- \star K\colon \mathsf{sSet} \to {}^{K/}\mathsf{sSet}$ preserves small colimits ([RV22, Lemma D.2.7]). ☐

Corollary 2.18. The join and slice functors define an adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, shift left=1.5, "-\star K"] \arrow[r, phantom, "\perp"] & {}^{K/}\mathsf{sSet} \arrow[l, shift left=1.5, "-/-"] \end{tikzcd}.$$

3. The ∞-Cosmos of Quasi-Categories

3.1. **The ∞-Category of Topological Spaces.** Before we define what an ∞-category is, we consider the prototypical example of topological spaces. This is a special case of a homotopy coherent nerve, which we will discuss in 4.1, but we hope that the familiar setting of spaces will serve well for motivation.

Informally speaking, an ∞-category is an object that has objects and morphisms (just like in a 1-category), but also 2-morphisms (i.e. morphisms between morphisms), 3-morphisms, and so on. We will describe the ∞-category of topological spaces, which we will denote by $\mathcal{S}$, in this informal manner by describing the morphisms at each level.

As in the 1-category of topological spaces, the objects of $\mathcal{S}$ are topological spaces, and the morphisms of $\mathcal{S}$ are continuous maps. The 2-morphisms are more interesting: these are given by homotopies between maps. Specifically, 2-cells in $\mathcal{S}$ are given by

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\alpha\Uparrow"] & Z \end{tikzcd}$$

where f, g and h are continous maps, and $\alpha\colon X \times I \to Z$ is a homotopy from h to gf such that $\alpha(-,0) = h$ and $\alpha(-,1) = gf$. (Note that instead of describing 2-morphisms as homotopies between two parallel continuous maps, we have opted for a more "triangular" shape. This is due to the fact that our preferred model of ∞-categories — quasi-categories — are simplicial in nature rather than globular. We can always recover homotopies between parallel maps by setting either f or g to be the identity map.)

Our 3-morphisms should be morphisms between 2-morphisms, i.e. homotopies between homotopies. Suppose we have the following continous maps

$$\begin{tikzcd} & X_1 \arrow[rr, "f_{31}"] \arrow[dr, "f_{21}"] & & X_3 \\ X_0 \arrow[ur, "f_{10}"] \arrow[urrr, "f_{30}"] \arrow[rr, "f_{20}"'] & & X_2 \arrow[ur, "f_{32}"'] & \end{tikzcd} \tag{7}$$

that do *not* necessarily commute. A 3-cell in $\mathcal{S}$ with this 1-skeleton is given by 2-cells (omitting labels for 1-cells)

$$\begin{tikzcd}[column sep=small] & X_1 \arrow[dr] & \\ X_0 \arrow[ur] \arrow[rr] & \arrow[u, phantom, "\alpha^1_{20}\Uparrow"] & X_2 \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & X_1 \arrow[dr] & \\ X_0 \arrow[ur] \arrow[rr] & \arrow[u, phantom, "\alpha^1_{30}\Uparrow"] & X_3 \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & X_2 \arrow[dr] & \\ X_1 \arrow[ur] \arrow[rr] & \arrow[u, phantom, "\alpha^2_{31}\Uparrow"] & X_3 \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & X_2 \arrow[dr] & \\ X_0 \arrow[ur] \arrow[rr] & \arrow[u, phantom, "\alpha^2_{30}\Uparrow"] & X_3 \end{tikzcd}$$

filling in the faces of the tetrahedron (7), along with a double homotopy

$$\beta\colon I^2 \to \mathsf{Top}(X_0, X_3)$$

of the form

$$\begin{tikzcd} f_{30} \arrow[r, "\alpha^1_{30}"] \arrow[d, "\alpha^2_{30}"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & f_{31}f_{10} \arrow[d, "\alpha^2_{31}*f_{10}"] \\ f_{32}f_{20} \arrow[r, "f_{32}*\alpha^1_{20}"'] & f_{32}f_{21}f_{10} \end{tikzcd}$$

filling in the "body" of the tetrahedron. Note that this is a picture, not a diagram! The vertices given by the f's are not objects in $\mathcal{S}$ but rather points in the space $\mathsf{Top}(X_0, X_3)$; the edges are not morphisms but paths in the mapping space, i.e. homotopies. We denote $*$ for the concatenation of homotopies, and we abuse notation to denote the constant homotopy at a map $f\colon X \to Y$ by f itself. (As in the 2-cell case, we can recover globular 3-morphisms by setting certain edges and faces to be identity.)

We encourage the reader to take a guess on what higher cells should look like (an explicit description of 4-cells in $\mathcal{S}$ is given in [Rav23, Section 7]), but we will now begin a more formal discussion of ∞-categories by introducing a popular model: quasi-categories.

Definition 3.1. A *quasi-category* is a simplicial set A satisfying the inner horn lifting property: for all $n > 0$ and all $0 < k < n$, the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & A \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a solution. A map of simplicial sets between quasi-categories is called a (∞-)functor.

Remark 3.2. More generally, we call maps $f\colon A \to B$ of simplicial sets *inner fibrations* if they have the right lifting property against inner horn inclusions, i.e. lifting problems of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

have solutions. A simplicial set A is a quasi-category if and only if the unique map to the terminal simplicial set is an inner fibration.

An important subset of inner fibrations are *isofibrations*, which are inner fibrations that also satisfy right lifting properties against the (nerve of the) two endpoint inclusion functors $\mathbb{1} \hookrightarrow \mathbb{I}$, where $\mathbb{I}$ is the walking isomorphism category. We will denote isofibrations by double headed arrows "$\twoheadrightarrow$".

Although isofibrations have no homotopy-theoretic significance (it turns out that all functors between quasi-categories are equivalent to isofibrations), they allow us to work with ∞-categories while considering strictly commuting diagrams (a bit like how fibrations in model categories allow us to compute homotopy-theoretic constructions strictly).

Remark 3.3. At a first glance, the definition for quasi-categories does not look like the familiar axioms for 1-categories, so let us unpack the definition a bit more at low dimensions. A quasi-category A comes with the data of n-cells A_n for all $n \in \mathbb{N}$. We will say that the 0-cells of A are its *objects* and the 1-cells its *morphisms*. The face maps $d_0, d_1\colon A_1 \to A_0$ pick out the codomain and domain of a morphism respectively, and the degeneracy map $s_0\colon A_0 \to A_1$ picks out the identity morphisms. Composition is a bit less obvious: a pair of composable morphisms

$$x \xrightarrow{f} y \xrightarrow{g} z$$

in A corresponds to a map $g.f\colon \Lambda^2_1 \to A$, and by inner horn lifting, we can find some lift

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "g.f"] \arrow[d, hook] & A \\ \Delta^2 \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

that fills in the 2-cell

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\alpha\Uparrow"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & {} & z \end{tikzcd}$$

We consider the morphism $h := d_1\alpha$ to be *a* composite of g with f, and we say that the 2-cell α witness the composition relationship. Note that there is no axiom saying that the horn filler has to be *unique*, and in fact, enforcing uniqueness of horn filler will give us an ordinary category (Example 3.4). However, it turns out that composition is "unique up to contractible choice", meaning that given any pair of composable morphisms (g, f), the space of composites

$$\begin{tikzcd} \mathrm{comp}(g,f) \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}(\Delta^2, A) \arrow[d] \\ \{(g,f)\} \arrow[r, hook] & \mathrm{Hom}(\Lambda^2_1, A) \end{tikzcd} \tag{8}$$

is contractible. (A strengthening of this fact can be found as [Cis19, Corollary 3.2.9], which says that A is a quasi-category *if and only if* the vertical map on the right of (8) is a trivial fibration, which one can think of encoding the uniqueness of composition "in families".) This justifies the heuristic that a quasi-category can be thought of as a category with weak composition.

It remains to describe the analogues of the associativity and unitality axioms. Unitality is easy: given any morphism $f\colon x \to y$, we wish to find a filler for

$$\begin{tikzcd}
& y \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[rr, "f"'] & & y
\end{tikzcd}$$

witnessing f as a composite of f with id_y, and such a filler is given by the degenerate 2-cell $s_1 f$. Similarly, $s_0 f$ witness the unitality for composing f with id_x. Associativity is a bit more complicated. Given a triple of composable morphisms

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w,$$

we can find composites

$$\begin{tikzcd}
& y \arrow[rr, dashed, "l"] \arrow[dr, dashed, "g"] \arrow[d, phantom, "\alpha\Uparrow"] & \arrow[d, phantom, "\beta\Downarrow"] & w \\
x \arrow[ur, "f"] \arrow[rr, dashed, "k"'] & {} & z \arrow[ur, "h"'] &
\end{tikzcd}$$

using fillers for Λ^2_1. We wish to establish a relationship between a composite of $l.f$ with a composite of $h.k$. We first find a filler

$$\begin{tikzcd}
& y \arrow[dr, "l"] \arrow[d, phantom, "\gamma\Uparrow"] & \\
x \arrow[ur, "f"] \arrow[rr, dashed, "m"'] & {} & w
\end{tikzcd}$$

for the back face of the tetrahedron. Notice that we have filled in three out of the four faces of the 3-cell whose edges are

$$\begin{tikzcd}
& y \arrow[rr, "l"] \arrow[dr, "g"] & & w \\
x \arrow[ur, "f"] \arrow[urrr, "m"] \arrow[rr, "k"'] & & z \arrow[ur, "h"'] &
\end{tikzcd}$$

meaning that we can now use inner horn filling for $\Lambda^3_1 \hookrightarrow \Delta^3$. This allows us to fill in the bottom face with some 2-cell

$$\begin{tikzcd}
& z \arrow[dr, "h"] \arrow[d, phantom, "\delta\Uparrow"] & \\
x \arrow[ur, "k"] \arrow[rr, "m"'] & {} & w
\end{tikzcd}$$

exhibiting m as a composite of h with k, as well as fill in the body of the 3-cell, which heuristically tells us that the two ways of obtaining the composite m from f, g, and h — by composing g with f first or by composing h with g first — agree, which is our new notion of "weak associativity".

Our description of the analogues of the 1-category axioms for quasi-categories is justified by the fact that 1-categories are examples of quasi-categories:

Example 3.4. Let $\mathcal{C}$ be a small category. We will show that the nerve of $\mathcal{C}$ is a quasi-category. Consider a lifting problem

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & N\mathcal{C} \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

When $n \geq 4$, the diagram automatically has a unique lift: the horn Λ^n_k contains all the 2-cells of Δ^n, so a map $\Lambda^n_k \to N\mathcal{C}$ picks out a diagram in which all faces commute, and hence the entire diagram commutes. When $n = 2$ and $k = 1$, a map $\Delta^2_1 \to N\mathcal{C}$ corresponds to a pair of composable arrows

$$x \xrightarrow{f} y \xrightarrow{g} z,$$

which we can uniquely fill to a commutative triangle using the composite $g \circ f$

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & \# & z. \end{tikzcd}$$

which gives us a unique lift to Δ^2. When $n = 3$ and $k = 1$, a map $\Delta^3_1 \to N\mathcal{C}$ corresponds to a diagram

$$\begin{tikzcd} & x \arrow[dr, "\ell"] \arrow[dd, "g"] & \\ w \arrow[ur, "f"] \arrow[rr, "m"] \arrow[dr, "k"'] & & z \\ & y \arrow[ur, "h"'] & \end{tikzcd}$$

in which all but the bottom face commutes, i.e. we have the relations

$$gf = k, \quad hg = \ell, \quad \ell f = m.$$

To lift this to a map from Δ^3, it suffices to check that the bottom face also commutes, which is true since $hk = hgf = \ell f = m$. A similar argument yields unique fillers for Λ^3_2.

We thus have a fully faithful embedding $\mathsf{Cat} \hookrightarrow \mathsf{qCat}$, and we will often conflate a 1-category with its nerve. Furthermore, it is straightforward to check that the essential image of the embedding is exactly the quasi-categories that satisfy unique lifting with respect to inner horns.

Example 3.5 (opposite category)**.** The opposite category of a 1-category $\mathcal{C}$ is defined to be the category $\mathcal{C}^{\mathsf{op}}$ where

- the objects are the same as $\mathcal{C}$
- for $X, Y \in \mathcal{C}$, the morphisms in $\mathcal{C}^{\mathsf{op}}$ are given by

$$\mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

We generalize this idea to define the *opposite* of any simplicial set S by taking S^{op} to be the simplicial set where

- the n-cells are the same as S:

$$S^{\mathsf{op}}_n := S_n$$

- the face and degeneracy maps have "the opposite ordering", i.e.

$$(d_i : S^{\mathsf{op}}_n \to S^{\mathsf{op}}_{n-1}) := (d_{n-i} : S_n \to S_{n-1})$$

$$(s_i : S^{\mathsf{op}}_n \to S^{\mathsf{op}}_{n+1}) := (s_{n-i} : S_n \to S_{n+1}).$$

It is easy to see that a simplicial set A is a quasi-category if and only if A^{op} is, as A lifts againsts (n, k)-horns if and only if A^{op} lifts against $(n, n-k)$-horns.

Example 3.6. Any Kan complex is a quasi-category, since Kan complexes satisfy lifting properties with respect to all horn inclusions and in particular the inner horn inclusions.

Example 3.7. Given a topological space X, the singular complex $S_\bullet X$ is a Kan complex and thus a quasi-category. Recall that the n-cells of $S_\bullet X$ are given by continuous maps $|\Delta^n| \to X$. Hence, objects of $S_\bullet X$ are given by points in X; a morphism $p : x \to y$ for points $x, y \in X$ is given by a map $p : |\Delta^1| \to X$ such that $p(0) = x$ and $p(1) = y$, in other words, a path from x to y. Now, given paths

$$x \xrightarrow{p} y \xrightarrow{q} z,$$

a composite is given by a singular 2-cell $\alpha\colon |\Delta^2| \to X$, depicted by

$$\begin{tikzcd} & y \arrow[dr, "q"] & \\ x \arrow[ur, "p"] \arrow[rr, dashed, "r"'] & \arrow[u, phantom, "\alpha\Uparrow"] & z \end{tikzcd}$$

Notice that this is both a diagram in $S_\bullet X$ and a picture in the space X. Under a fixed homeomorphism $|\Delta^2| \cong I^2$, we see that α encodes the data of a homotopy r to the standard concatenation $p * q$ (i.e. the path sendings $t \in [0,1]$ to $p(2t)$ when $t \leq 1/2$ and $q(2(t-1/2))$ when $t \geq 1/2$). There are many choices for such a composite: we could take the identity homotopy from $p * q$ to itself, but we could also take any other reparametrization of the interval, for instance, fixing any $a \in (0,1)$ and defining a path $r_a\colon I \to X$ sending $t \in I$ to

$$r_a(t) := \begin{cases} p\left(\frac{t}{a}\right), & t \in [0,a] \\ q\left(\frac{1-t}{1-a}\right), & t \in [a,1]. \end{cases}$$

In the quasi-categorical setting, no one choice of reparametrization for concatenation is preferred over the others, and we instead consider the (contractible) space of all possible reparametrizations. Although this approach forces us to keep track of more information, it is in some ways more natural than simply quotienting out by the homotopy relationship which destroys a lot of structure. We may also recover the quotient by taking the homotopy category, which we describe now.

Definition 3.8 (homotopy relation in a quasi-category)**.** Fix a parallel pair of morphisms f, g in a quasi-category A. We call a 2-cell such as below left a *left homotopy*, and we say that $f \sim_l g$ are *left homotopic* if such a 2-cell exists;

$$\begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & \arrow[u, phantom, "\Uparrow"] & y \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, equal] \arrow[rr, "g"'] & \arrow[u, phantom, "\Uparrow"] & y \end{tikzcd} \tag{9}$$

similarly, we call a 2-cell such as above right a *right homotopy*, and we say that $f \sim_r g$ are *right homotopic* if such a 2-cell exists.

Lemma 3.9. Parallel morphisms are left homotopic if and only if they are right homotopic, in which case we say they are homotopic. Furthermore, the relationship induced by homotopies is an equivalence relation on morphisms that is stable under composition.

Proof. We show the equivalence of left and right homotopies by constructing a 3-cell with edges

$$\begin{tikzcd} & x \arrow[dr, "f"] \arrow[dd, "f"] & \\ x \arrow[ur, equal] \arrow[rr, "g"] \arrow[dr, "f"'] & & y \\ & y \arrow[ur, equal] & \end{tikzcd}$$

where the left and right faces can be filled with degenerate 2-cells, so by inner horn fillers, filling the bottom face with a left homotopy $f \sim_l g$ is possible if and only if we can fill the back face with a right homotopy $f \sim_r g$.

The reflexivity of both homotopy relations are witnessed by degenerate 2-cells on morphisms. To show symmetry for the left homotopy relation, suppose we have a left homotopy $f \sim_l g$. We construct a 3-cell whose edges are given by

$$\begin{tikzcd}
& y \arrow[dd, equal] \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[rr, "f"] \arrow[dr, "g"'] & & y \\
& y \arrow[ur, equal] &
\end{tikzcd}$$

by filling in the left face with the given homotopy, the right face with the degenerate 2-cell id_y, the back face with id_f, and using horn filling for Λ^3_1 to find a filler for the bottom face, which gives us a left homotopy $g \sim_l f$. A similar argument gives us symmetry for the right homotopy relation. Finally, to show transitivity, given left homotopies $f \sim_l g$ and $g \sim_l h$, construct a 3-cell whose edges are given by

$$\begin{tikzcd}
& x \arrow[dd, equal] \arrow[dr, equal] & \\
y \arrow[ur, "f"] \arrow[rr, "h"] \arrow[dr, "g"'] & & y \\
& y \arrow[ur, equal] &
\end{tikzcd}$$

by filling in the right face with a degenerate cell and the left and bottom faces with the given homotopies. Using horn lifting for Λ^3_2, we obtain the desired homotopy $f \sim_l h$ in the back face.

Finally, to show that homotopy is stable under composition, let $f, g\colon x \to y$ be homotopic 1-cells and $h\colon y \to z$. Given (a choice of) composites

$$\begin{tikzcd}
& y \arrow[dr, "h"] \arrow[d, phantom, "\alpha\Uparrow"] & \\
x \arrow[ur, "f"] \arrow[rr, "h\circ f"'] & {} & z
\end{tikzcd}
\qquad
\begin{tikzcd}
& y \arrow[dr, "h"] \arrow[d, phantom, "\beta\Uparrow"] & \\
x \arrow[ur, "g"] \arrow[rr, "h\circ g"'] & {} & z
\end{tikzcd}$$

construct a 3-cell whose edges are

$$\begin{tikzcd}
& x \arrow[dd, "f"] \arrow[dr, "h\circ f"] & \\
x \arrow[ur, equal] \arrow[rr, "h\circ g"] \arrow[dr, "g"'] & & z \\
& y \arrow[ur, "h"'] &
\end{tikzcd}$$

by filling in the left face with the given homotopy $f \sim g$, filling in the right and bottom faces with α and β respectively, and using inner horn filler for Λ^3_2. The filled in back face gives us our desired homotopy $h \circ f \sim h \circ g$. A similar argument for left homotopies grants us stability under precomposition. $\square$

Definition 3.10 (homotopy category of a quasi-category)**.** Let A be a quasi-category. Its *homotopy category* is the category $\mathsf{h}A$ whose

- objects are 0-cells of A, $\mathrm{ob}(\mathsf{h}A) := A_0$;
- morphisms are homotopy classes of 1-cells of A;
- identity morphism at $a \in A_0$ is represented by the degenerate 1-cell $s_0 a \in A_1$;
- composition of a composable pair of arrows (g, f) is represented by any 1-cell h such that a 2-cell of the form

$$\begin{tikzcd}
& b \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow"] & \\
a \arrow[ur, "f"] \arrow[rr, "h"'] & {} & c
\end{tikzcd}$$

exists.

To justify this definition, we check that:

Lemma 3.11. The homotopy category $\mathsf{h}A$ is well defined and naturally isomorphic to $\tau_1 A$.

Proof. Given a composable pair of arrows (g, f), inner horn filler for Λ^2_1 guarantees the existence of a composite. Furthermore, we can check that any two composites h, h' are necessarily homotopic by considering a 3-cell with edges

$$\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[dd, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h'"] \arrow[dr, "h"'] & & z \arrow[dl, equal] \\
 & z &
\end{tikzcd}$$

filling in the left and back faces with the 2-cells witnessing composition, the right face with a degenerate 2-cell, and using inner horn filler to obtain the desired homotopy $h \sim h'$ in the bottom face. In addition to the stability of homotopy from Lemma 3.9, we see that composition is well-defined. Unitality and associativity are given by the "weak" analogues described in Remark 3.3.

To show that $\mathsf{h}A \cong \tau_1 A$, it suffices to show that h defines another left adjoint to nerve. Let $\mathcal{C}$ be a category and $F\colon \mathsf{h}A \to \mathcal{C}$ be a functor. We would like to define a map of simplicial sets $A \to N\mathcal{C}$. On 0-cells, we take the assignment of F on objects, and on 1-cells, we take the assignment of F on morphisms. By 2-coskeletonality of $N\mathcal{C}$, it suffices for us to check that for any 2-cell in A of the form

$$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\sigma\Uparrow"] & & z
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & Fy \arrow[dr, "Fg"] & \\
Fx \arrow[ur, "Ff"] \arrow[rr, "Fh"'] \arrow[rr, phantom, shift left=3, "\#"] & & Fz
\end{tikzcd}$$

the diagram on the right in $\mathcal{C}$ commutes, which holds by definition. Conversely, we may construct a functor $G : \mathsf{h}A \to \mathcal{C}$ from the data of a map $A \to N\mathcal{C}$ by taking the mapping on objects and morphisms from the mapping on 0- and 1-cells respectively, where functoriality is guaranteed by the mapping on 2-cells. It is clear that the two procedures define a bijective correspondence. $\square$

Example 3.12. Let $\mathcal{C}$ be a category. Two morphisms $f, g\colon x \to y$ are homotopic if and only if the triangle

$$\begin{tikzcd}
 & y \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[rr, "g"'] & & y
\end{tikzcd}$$

commutes, i.e. $f = g$. Hence, the homotopy category of $\mathcal{C}$ is just $\mathcal{C}$ itself.

Example 3.13. Let X be a topological space. Two paths $p, q\colon x \to y$ in $S_\bullet X$ are homotopic in the sense of Definition 3.8 if and only if they are homotopic in the topological sense. Hence, the homotopy category $\mathsf{h}S_\bullet X$ is the category whose objects are points in X and whose morphisms are paths in X up to homotopy, i.e. the fundamental groupoid $\Pi_1 X$.

Example 3.14. The homotopy category of the quasi-category of topological spaces is the "naive homotopy category", in which objects are topological spaces and maps are homotopy classes of continuous maps.

Example 3.15 (full subcategory)**.** Let A be a quasi-category, and $B_0 \subseteq A_0$ be a subset of objects. We say that the *full subcategory of A spanned by the objects B_0* is the pullback

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d] \\ \mathsf{h}B \arrow[r, hook] & \mathsf{h}A \end{tikzcd}$$

in simplicial sets where $\mathsf{h}B \subseteq \mathsf{h}A$ is defined to be the full (1-)subcategory of $\mathsf{h}A$ spanned by the objects B_0. Inner horn lifting for B is checked by using the universal property of the pullback.

Definition 3.16 (isomorphism in a quasi-category)**.** A morphism $f\colon x \to y$ in a quasi-category A is an *isomorphism* if its image in the homotopy category $\mathsf{h}A$ is. Explicitly, f is an isomorphism if and only if there exists morphisms $g, g'\colon y \to x$ and 2-cells

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow"] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & {} & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "\Uparrow"] & \\ y \arrow[ur, "g'"] \arrow[rr, equal] & {} & y \end{tikzcd}$$

witnessing g, g' as homotopy inverses to f.

Remark 3.17. Although by the definition above, the two homotopy inverses g and g' are not necessarily equal, we can always choose them to be so by finding a homotopy $g \sim g'$ using inner horn lifting for the back face of the 3-cell below left

$$\begin{tikzcd} & x \arrow[dr, equal] \arrow[dd, "f"] & \\ y \arrow[ur, "g'"] \arrow[rr, "g"] \arrow[dr, equal] & & x \\ & y \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr, equal] \arrow[dd, "g'"] & \\ y \arrow[ur, equal] \arrow[rr, equal] \arrow[dr, "g"'] & & y \\ & x \arrow[ur, "f"'] & \end{tikzcd}$$

then using inner horn lifting for the bottom face of the 3-cell above right to get our desired homotopy $fg \sim \mathrm{id}_y$.

Example 3.18. An isomorphism in $\mathcal{S}$ is a homotopy equivalence.

Example 3.19. Using the outer horn lifting properties for the 2-cell, we see that any morphism in a Kan complex is an isomorphism.

In fact, all ∞-groupoids (quasi-categories in which every morphism is invertible) are Kan complexes (this fact is sometimes referred to as the *homotopy hypothesis*):

Theorem 3.20 (Joyal)**.** A quasi-category is a Kan complex if and only if its homotopy category is a groupoid.

Proof. See [RV22, Corollary 1.1.15]. □

Every quasi-category A contains a canonical *maximal sub-∞-groupoid*, given by taking the pullback

$$\begin{tikzcd} A^{\simeq} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d] \\ (\mathsf{h}A)^{\simeq} \arrow[r, hook] & \mathsf{h}A \end{tikzcd}$$

inside simplicial sets, where $\mathsf{h}A^{\simeq}$ denotes the maximal subgroupoid of the 1-category $\mathsf{h}A$. In particular, a morphism $\mathbb{2} \to A$ is an isomorphism if and only if it factors through $A^{\simeq}$. By the pullback universal property, any functor from an ∞-groupoid into A factors through $A^{\simeq}$, which gives us a functor

$$(-)^{\simeq}\colon \mathsf{qCat} \to \mathsf{Kan} \tag{10}$$

right adjoint to the inclusion

$$\mathsf{Kan} \hookrightarrow \mathsf{qCat}.$$

The fact that ∞-groupoids are Kan complexes allow us to lift isomorphisms to *homotopy coherent isomorphisms*, which are represented by functors $\mathbb{I} \to A$ where $\mathbb{I}$ is the nerve of the walking isomorphism category:

Corollary 3.21. An arrow f in a quasi-category A is an isomorphism if and only if it extends to a homotopy coherent isomorphism, i.e. a solution to the lifting problem

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A. \\ \mathbb{I} \arrow[ur, dashed] & \end{tikzcd}$$

exists.

Proof. The converse direction is easily shown by composing the functor $\mathbb{I} \to A$ with the quotienting map $A \to \mathsf{h}A$ for a homotopy inverse to f. To check the forward direction, we know that the arrow f is an isomorphism, so it necessarily lands in the subgroupoid

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A^{\simeq} \subseteq A \\ \mathbb{I} \arrow[ur, dashed] & \end{tikzcd}$$

which is a Kan complex by assumption. It then suffices to show that the inclusion $\mathbb{2} \to \mathbb{I}$ is anodyne (meaning that it lifts against Kan fibrations), which we can check by inductively constructing the inclusion as a sequential composite of outer horn inclusions. We denote the two objects in $\mathbb{I}$ by 0 and 1, and we denote the two non-identity morphisms as $0 \xrightarrow{f} 1$ and $1 \xrightarrow{g} 0$. The nerve of $\mathbb{I}$ has exactly two non-degenerate n-cells for each $n \in \mathbb{N}$ given by alternating composites of f and g's of length n, either $f \dots fgf$ and $g \dots gfg$ when n is odd or $gf \dots fgf$ and $fg \dots gfg$ when n is even. Denote $\sigma_n \in \mathbb{I}_n$ for the non-degenerate n-cell that starts at the vertex 0 (i.e. whose rightmost letter is f) and $\tau_n \in \mathbb{I}_n$ for the non-degenerate n-cell that starts at the vertex 1 (i.e. whose rightmost letter is g), and observe that the outer faces of σ_n are given by

$$d_0\sigma_n = \tau_{n-1} \quad \text{and} \quad d_n\sigma_n = \sigma_{n-1}$$

and the inner faces are degenerate.

Let $X_n \subseteq \mathbb{I}$ be the subsimplicial set that contains all k-cells of $\mathbb{I}$ for $k < n$ and all n-cells of $\mathbb{I}$ except σ_n. We can easily check that

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d, hook] & X_n \arrow[d, hook] \\ \Delta^n \arrow[r, "\sigma_n"'] \arrow[ur, phantom, "\ulcorner"] & X_{n+1} \end{tikzcd}$$

is a pushout diagram, which we may do levelwise: by n-skeletality it suffices to check for all k-cells when $k \leq n$; on k-cells when $k < n$, this is the identity pushout diagram; on n-cells, this is a pushout diagram by construction. $\square$

We are interested in studying categorical constructions in quasi-categories. Just like how it is often useful in 1-category theory to study the 2-category of 1-categories, we will formally study ∞-category theory by considering the "$(\infty, 2)$-category of $(\infty, 1)$-categories". An axiomatization of this notion is that of ∞-cosmoi due to Riehl and Verity. We will not be explicitly defining ∞-cosmoi in this exposition since we will only focus on the example of quasi-categories, but our constructions and proofs will be heavily influenced by the cosmological approach following [RV22], and most can be generalized to arbitrary ∞-cosmoi or at least ∞-cosmoi of $(\infty, 1)$-categories.

We first introduce diagram ∞-categories. In ordinary category theory, we usually index diagrams by categories, but we often also implicitly index diagrams by graphs. The generalization is to index diagrams in the quasi-categorical setting by simplicial sets. Given a simplicial set J and a quasi-category A, we will call a map of simplicial sets $d\colon J \to A$ a *diagram*. Since sSet is cartesian closed, we naturally have simplicial set structure on the collection of diagrams from J to A, which we denote by

$$A^J := \mathsf{sSet}(J, A).$$

Using simplicial combinatorics, one can prove that these are in fact quasi-categories.

Theorem 3.22 (Joyal)**.** If J is a simplicial set and A is a quasi-category, then A^J is a quasi-category. Moreover, a 1-simplex in A^J is an isomorphism if and only if its components at each $j \in J$ is an isomorphism in A.

Proof. We omit the proof of this difficult theorem; one reference for the proof is [RV22, Corollary 1.1.22]. □

In particular, given two quasi-categories A, B, the collection of functors

$$\mathsf{Fun}(A, B) := \mathsf{sSet}(A, B)$$

forms a quasi-category, and qCat is cartesian closed. This describes the category of quasi-categories as an "$(\infty, 2)$-category", which heuristically means that it has n-morphisms for all $n \in \mathbb{Z}_{\geq 0}$ which are invertible for $n > 2$: the $(m+1)$-morphisms in qCat are given by m-morphisms in functor quasi-categories, which are invertible for $m > 1$. Since taking homotopy category preserves products (Remark 2.7), we can perform base change to obtain a 2-category of quasi-categories by taking the homotopy categories of functor categories. It turns out, somewhat surprisingly, that many ∞-categorical constructions in quasi-categories (and more generally in any ∞-cosmos) can be characterized in this 2-category.

Definition 3.23. The *homotopy 2-category of quasi-categories* is the 2-category $\mathfrak{h}\mathsf{qCat} := \mathsf{h}_*\mathsf{qCat}$ obtained by performing base change on qCat along $\mathsf{h}\colon \mathsf{qCat} \to \mathsf{Cat}$. Explicitly, it is the 2-category whose

- objects are quasi-categories,
- 1-cells are ∞-functors (maps of simplicial sets) and
- 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ are homotopy classes of 1-cells in $\mathsf{Fun}(A, B)$ from f to g, which are represented by functors $A \times \Delta^1 \to B$.

We will refer to the 2-cells in $\mathfrak{h}\mathsf{qCat}$ as ∞*-natural transformations* and invertible 2-cells as ∞*-natural isomorphisms*.

Remark 3.24 (∞-cosmoi). Definition 3.23 can be generalized to arbitrary ∞-cosmoi, which are quasi-categorically enriched categories $\mathcal{K}$ equipped with a specified subcategory of isofibrations that satisfy certain completeness axioms ([RV22, Definition 1.2.1]). We refer to objects in an ∞-cosmos as ∞*-categories*. Given an ∞-cosmos $\mathcal{K}$, we can base change along $\mathsf{h}\colon \mathsf{qCat} \to \mathsf{Cat}$ to obtain a 2-category $\mathfrak{h}\mathcal{K}$, which we refer to as the *homotopy 2-category of* $\mathcal{K}$. Examples of ∞-cosmoi include: (see [RV22, Chapter 1.2] for details)

- ordinary categories Cat;
- Kan complexes Kan, which model $(\infty, 0)$-categories or ∞-groupoids (3.20);
- models of $(\infty, 1)$-categories such as quasi-categories, complete Segal spaces, and Segal categories;
- models of (∞, n)-categories such as θ_n-spaces and iterated complete Segal spaces.

Many of the results in the rest of the document can be easily generalized to arbitrary ∞-cosmos, as we will often work 2-categorically in $\mathfrak{hqCat}$. Some analytic results about quasi-categories can even be transferred to other models $(\infty, 1)$-categories through model independence results due to Riehl and Verity (see [RV22, Part III]), but these are much beyond the scope of this document.

Definition 3.25. An *equivalence of quasi-categories* is an equivalence in the homotopy 2-category $\mathfrak{hqCat}$. Explicitly, the data of an equivalence consists of a pair of functors $f\colon A \leftrightarrows B\colon g$ and ∞-natural isomorphisms $\alpha\colon \mathrm{id}_A \cong gf$ and $\beta\colon fg \cong \mathrm{id}_B$.

Since isomorphisms in quasi-categories lift to homotopy coherent isomorphisms (Corollary 3.21), we have the following:

Corollary 3.26. A functor $f\colon A \to B$ between quasi-categories defines an equivalence if and only if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval, meaning that there exist maps $g\colon B \to A$,

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, two heads, "\mathsf{ev}_0"'] \arrow[d, two heads, "\mathsf{ev}_1"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, two heads, "\mathsf{ev}_0"'] \arrow[d, two heads, "\mathsf{ev}_1"] \\ & B. \end{tikzcd}$$

Remark 3.27. The alternate description of equivalences makes it easy to check that taking homotopy category preserves equivalences: the composites

$$\mathsf{h}A \xrightarrow{\mathsf{h}\alpha} \mathsf{h}(A^{\mathbb{I}}) \to (\mathsf{h}A)^{\mathbb{I}} \quad \text{and} \quad \mathsf{h}B \xrightarrow{\mathsf{h}\beta} \mathsf{h}(B^{\mathbb{I}}) \to (\mathsf{h}B)^{\mathbb{I}}$$

exhibit $\mathsf{h}g$ as an inverse to $\mathsf{h}f$.

4. More Examples of Quasi-Categories

Before discussing ∞-category theory, let us see a few more complicated examples of quasi-categories.

4.1. **The Homotopy Coherent Nerve.**

We saw in Section 3.1 that the homotopical data in Top can be encoded by considering *spaces* rather than sets of continous maps, i.e. in a topologically enriched category. However, although topological categories do present one model for $(\infty, 1)$-categories, they are not the easiest to work with for ∞-categorical constructions, since often it is the more natural to consider weak composition (for instance, in the case of the concatenation of two paths). In this section, we will describe a general construction that produces a quasi-category from a topological category. More specifically, we will first perform a base change to produce a Kan-complex enriched category, then describe the homotopy coherent nerve of such a category using the formula for realization-nerve adjunction on simplicial sets (Proposition 2.4).

Corollary 4.1. Let $\mathcal{C}$ be a topological category. Since $S_\bullet\colon \mathsf{Top} \to \mathsf{Kan}$ is a right adjoint, it preserves products and can be given a strong monoidal structure. Hence, we may regard $\mathcal{C}$ as a Kan-complex enriched category using base change, with Hom-complexes given by

$$\mathcal{C}(c,d)_\bullet := S_\bullet\, \mathcal{C}(c,d)$$

for all $c, d \in \mathcal{C}$. Explicitly, given any $n \in \mathbb{N}$, an n-cell of $\mathcal{C}(c,d)$ is given by a map $\Delta^n \to \mathcal{C}(c,d)_\bullet$ in simplicial sets, which is equivalently given by a map $|\Delta^n| \to \mathcal{C}(c,d)$ in topological spaces under transposition, i.e. a singular n-simplex in the mapping space $\mathcal{C}(c,d)$. In particular, we may regard Top as a Kan-complex enriched category.

Now, we will use Proposition 2.4 to come up with an adjunction that goes between simplicial sets and simplicially enriched categories. To do this, we define a cosimplicial object in $\mathsf{sSet}\text{-}\mathsf{Cat}$:

Definition 4.2. The *homotopy coherent n-simplex* $\mathbb{C}[n]$ is the simplicially enriched category whose set of objects is $\{0, 1, \ldots, n\}$, and for $0 \leq j, k \leq n$ the mapping simplicial set $\mathbb{C}[n](j, k)$ has r-cells given by sequences of nested subsets

$$\{j, k\} \subseteq T^0 \subseteq \ldots \subseteq T^r \subseteq [j, k].$$

The face and degeneracy maps are given by omitting and duplicating T^i. In other words, the mapping simplicial set is the nerve

$$\mathbb{C}[n](j, k) := N\left({}^{\{j,k\}}/\mathcal{P}[j, k]\right).$$

where $\mathcal{P}[j, k]$ denotes the power set of $[j, k]$. In particular, an r-arrow $T^0 \subseteq \ldots \subseteq T^r$ is non-degenerate if and only if each inclusion is proper. We define a functor $\mathbb{C}\colon \Delta \to \mathsf{sSet\text{-}Cat}$ by defining the coface and codegeneracy maps to be the usual ones on objects, which naturally extends to a mapping on hom-simplicial sets.

Example 4.3. Let's write down a few small examples to get an intuition for what the mapping-simplicial sets are in the homotopy coherent simplices. When $n = 0$, we have the terminal simplicial category $\{0\}$. When $n = 1$, we have two objects $\{0, 1\}$, and the hom-simplicial set from 0 to 1 has a single 0-arrow, giving us the discrete simplicial category $[1]$. When $n = 2$, we have three objects $\{0, 1, 2\}$. The hom's between adjacent numbers still have a single 0-arrow, but the hom between 0 and 2 is more interesting: it is the nerve of $\{0, 2\} \subseteq \{0, 1, 2\}$, i.e. Δ^1. In general, we visually represent the 0-arrow given by a subset

$$\{j, k\} \subseteq \{j, j_1, \ldots, j_i, k\} \subseteq [j, k]$$

as a string of arrows from j to k with vertices at each j_i

$$j \to j_1 \to \ldots \to j_i \to \ldots \to k.$$

Hence, we can visualize $\mathbb{C}[2]$ as a diagram

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

where the 2-cell represents the 1-arrow $\{0, 2\} \to \{0, 1, 2\}$. When $n = 3$, the hom from 0 to 3 is the nerve of the poset

$$\begin{array}{ccc} \{0,3\} & \subseteq & \{0,2,3\} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \{0,1,3\} & \subseteq & \{0,1,2,3\} \end{array}$$

which is the simplicial square $\Delta^1 \times \Delta^1$. In general, we can check that for all $n \in \mathbb{N}$ and $0 \leq j, k \leq n$, the mapping simplicial set can be described as

$$\mathbb{C}[n](j, k) \cong \begin{cases} \varnothing, & j > k; \\ \Delta^0, & j = k; \\ (\Delta^1)^{k-j-1}, & j < k. \end{cases}$$

By Kan extension on the homotopy coherent simplices, we get the following:

Corollary 4.4. There is an adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{sSet\text{-}Cat} \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd} \tag{11}$$

given by left Kan extension of $\mathbb{C}$ along the Yoneda embedding. We call $\mathfrak{C}$ the *homotopy coherent realization* and $\mathfrak{N}$ the *homotopy coherent nerve*. Explicitly, $\mathcal{N}$ sends a simplicially enriched category $\mathcal{M}$ to the simplicial set whose n-simplices are

$$(\mathfrak{N}\mathcal{M})_n := \mathsf{sSet\text{-}Cat}(\mathfrak{C}\Delta^n, \mathcal{M}).$$

In particular, taking the homotopy coherent nerve of a category enriched in Kan complexes gives us a quasi-category:

Theorem 4.5 ([CP86])**.** The adjunction (11) restricts to

$$\begin{tikzcd} \mathsf{qCat} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{Kan\text{-}Cat} \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd}$$

between quasi-categories and Kan-complex enriched categories.

Remark 4.6. In fact, the adjunction (11) is a Quillen adjunction between the Joyal model structure and the Bergner model structure (due to Joyal). Heuristically, this says that quasi-categories and Kan-complex enriched categories are equivalent as models for $(\infty, 1)$-categories.

Example 4.7. We can now make the description of the ∞-category of topological spaces in Section 3.1 precise: we obtain $\mathcal{S}$ by the cartesian closed category of topological spaces, base change to a Kan-complex enrichment, and take the homotopy coherent nerve. Given $X, Y \in \mathsf{Top}$, the mapping Kan complex is defined to be

$$\mathsf{Map}_{\mathcal{S}}(X, Y) := S_\bullet\,\mathsf{Top}(X, Y),$$

and its n-cells are given by maps

$$\Delta^n \to S_\bullet\,\mathsf{Top}(X, Y) \quad \leftrightsquigarrow \quad |\Delta^n| \to \mathsf{Top}(X, Y).$$

An n-cell in the homotopy coherenet nerve of Top is thus given by a simplicial functors $\mathfrak{C}\Delta^n \to \mathsf{Top}$, the data of which is determined by a map of simplicial sets

$$\mathfrak{C}\Delta^n(0, n) \cong (\Delta^1)^{n-1} \to S_\bullet\,\mathsf{Top}(X, Y)$$

for some topological spaces X, Y, which is what we described in Section 3.1 for small n.

Remark 4.8. Although we defined the ∞-category of spaces $\mathcal{S}$ using topological spaces, it is more standard and often more convenient to describe $\mathcal{S}$ as the homotopy coherent nerve of the cartesian closed category Kan. To do this, one needs to check that the mapping simplicial set between Kan complexes satisfy the Kan condition. The Quillen adjunction between the Kan-Quillen model structure on sSet and the Quillen model structure on Top given by geometric realization $\dashv$ singular complex lifts to an equivalence of quasi-categories between the two different approaches to defining $\mathcal{S}$.

Example 4.9. The category of quasi-categories qCat is naturally an $(\infty, 2)$-category using the self-enrichment. We can obtain an $(\infty, 1)$-category of quasi-categories by forgetting all non-invertible 2-morphisms. Specifically, we perform base change along the maximal sub-∞-groupoid functor (10) which produces a Kan-enriched category $\mathsf{qCat}^{\simeq}$ where objects A, B are quasi-categories and hom simplicial sets are given by $\mathrm{Hom}(A, B) := \mathsf{Fun}(A, B)^{\simeq}$. The quasi-category of quasi-categories is the homotopy coherent nerve of $\mathsf{qCat}^{\simeq}$, which we simply denote by qCat when there is no ambiguity.

Example 4.10. Under the Dold-Kan correspondence, non-negatively graded chain complexes can be equivalently regarded as simplicial abelian groups, which are automatically Kan complexes. This allows us to base change from categories enriched in chain complexes (which are called *dg categories*) to Kan-complex enriched ones, which then lets us produce a quasi-category by taking homotopy coherent nerve. A standard example is the quasi-category $\mathsf{Ch}_R^{\geq 0}$ of non-negatively graded chain complexes over R, which is called the *derived category* of R as it turns out that it is the ∞-categorical localization of the 1-category of non-negatively graded chain complexes at quasi-isomorphisms. See [Lur25, Subsection 00ND] for the details of this construction.

4.2. **The Underlying Quasi-Category of a Model Category.** Up until now, our method for writing down a quasi-category consisted of explicitly writing down n-cells at every level. It may come as a surprise, then, that the data of a quasi-category can be equivalently described by an ordinary categtory together with a specified collection of weak equivalences. In particular, model structures on ordinary categories are a convenient way to present ∞-categorical data, since the structure of fibrations and cofibrations allow us to better control localization. We will also see later that model structures give us a way to write down functors between ∞-categories more easily by just writing down a functor at the point set level. In this section, we will describe several equivalent methods to obtain the underlying quasi-category of a category with weak equivalences.

Definition 4.11. A *category with weak equivalences* is a category $\mathcal{C}$ with a subcategory $W \subseteq \mathcal{C}$ which

- contains all isomorphisms in $\mathcal{C}$;
- satisfies two-out-of-three, given composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{C}$, if two of $\{f, g, g \circ f\}$ are in W, then so is the third.

The most conceptual way of obtaining the underlying ∞-category of a category with weak equivalences is through ∞-categorical localization.

4.2.1. *Localization.* In commutative algebra, localization is a way of formally adjoining inverses to certain elements in a commutative ring. This can be generalized to categories. We will first recall the definition of localization for an ordinary category before presenting the definition in the ∞-categorical setting.

Definition 4.12. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories, and let W be a collection of morphisms in $\mathcal{C}$. We say that F exhibits $\mathcal{D}$ as a 1-categorical localization of $\mathcal{C}$ with respect to W if for all categories $\mathcal{E}$, the functor induced by precomposition

$$\mathsf{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{-\circ F} \mathsf{Fun}(\mathcal{C}, \mathcal{E})$$

is fully faithful, and functors in the essential image are those who send morphisms in W to isomorphisms.

Remark 4.13. (1) The universal property of localization as stated above is 2-categorical, in the sense that we require precomposition with F to determine an equivalence of categories between $\mathsf{Fun}(\mathcal{D}, \mathcal{E})$ and the full subcategory $\underset{W\mapsto\cong}{\mathsf{Fun}}(\mathcal{C}, \mathcal{E})$ of $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$ spanned by the functors sending W to isomorphisms. If we ask for precomposition to determine a bijection on objects instead, the universal property determines a category uniquely up to isomorphism, which we call the *strict localization* of $\mathcal{C}$ with respect to W. Any strict localization is in particular a 1-categorical localization.

(2) By the Yoneda lemma, any two strict localizations of $(\mathcal{C}, W)$ are isomorphic. Two weak localizations may not be isomorphic but are necessarily equivalent: given two localizations $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{E}$ for the same collection of morphisms W, use the essential surjectivity of

$$\mathsf{Fun}(\mathcal{D}, \mathcal{E}) \underset{\sim}{\xrightarrow{F^*}} \underset{W\mapsto\cong}{\mathsf{Fun}}(\mathcal{C}, \mathcal{E})$$

to find some functor $K\colon \mathcal{D} \to \mathcal{E}$ such that $KF \cong G$. Similarly, using the essential surjectivity of G^*, find some $H\colon \mathcal{E} \to \mathcal{D}$ such that $HG \cong F$. Now, by fully faithfulness we have

$$HKF \cong HG \cong F \implies HK \cong \mathrm{id}_{\mathcal{E}}$$

and

$$KHG \cong KF \cong G \implies KH \cong \mathrm{id}_{\mathcal{D}}$$

so K and H define an equivalence between $\mathcal{D}$ and $\mathcal{E}$.

We can always formally construct a 1-categorical localization as follows:

Proposition 4.14. Let $\mathcal{C}$ be a category and W be a collection of morphisms in $\mathcal{C}$. We can construct a category $\mathcal{C}[W^{-1}]$ by adjoining a new morphism $w^{-1}\colon y \to x$ for each morphism $w\colon x \to y$ in W, and imposing the relations that $w^{-1} \circ w = \mathrm{id}_x$ and $w \circ w^{-1} = \mathrm{id}_y$. The canonical map $\mathcal{C} \to \mathcal{C}[W^{-1}]$ exhibits $\mathcal{C}[W^{-1}]$ as a strict localization. We will sometimes denote $\mathsf{Ho}\mathcal{C} := \mathcal{C}[W^{-1}]$ when the class of weak equivalences is clear from context.

Remark 4.15. If $\mathcal{C}$ is small, then the strict localization $\mathcal{C}[W^{-1}]$ is also small. However, if $\mathcal{C}$ is only locally small, then $\mathcal{C}[W^{-1}]$ is not necessarily locally small. This gives us evidence that the strict localization is not often the most well behaved, and we often would like to look for other models that are equivalent but not necessarily isomorphic to $\mathcal{C}[W^{-1}]$. One example where such a model exists is in the case when $\mathcal{C}$ admits a model structure.

Proposition 4.16 (Quillen)**.** Let $\mathcal{M}$ be a model category with functorial factorization, and denote the full subcategory of fibrant-cofibrant objects by $\mathcal{M}_{\mathsf{cf}}$. For any two objects $x, y \in \mathcal{M}_{\mathsf{cf}}$, there is an equivalence relationship on the hom set $\mathcal{M}(x, y)$ which is closed under composition (two equivalent maps are said to be homotopic). Construct a category $\mathsf{h}\mathcal{M}_{\mathsf{cf}}$ whose objects are that of $\mathcal{M}_{\mathsf{cf}}$ and whose hom sets are homotopy classes of maps

$$[x, y] := \mathcal{M}(x, y)_{/\sim}.$$

The composite functor

$$\mathcal{M} \xrightarrow{RQ} \mathcal{M}_{\mathsf{cf}} \to \mathsf{h}\mathcal{M}_{\mathsf{cf}}$$

of the quotient functor with the fibrant-cofibrant replacement functor exhibits $\mathsf{h}\mathcal{M}_{\mathsf{cf}}$ as a 1-categorical localization of $\mathcal{M}$ with respect to weak equivalences.

Proof. See [BGH22], Chapter 2, sections 3.3 and 3.4. □

We mention a few familiar examples:

Example 4.17. (1) The 1-categorical localization of topological spaces by weak homotopy equivalences is given by the category of CW-complexes with homotopy classes of maps (using the Quillen model structure and Whitehead's theorem).
(2) The 1-categorical localization of chain complexes by quasi-isomorphisms is given by the category of projective chain complexes with chain homotopy classes of maps.

We now consider ∞-categorical localization.

Definition 4.18. Let A, B be a quasi-categories, let $f\colon A \to B$ be a functor, and let W be a collection of morphisms in A. We say that f exhibits B as a *localization of A with respect to W* if for all quasi-categories E, the precomposition functor

$$\mathsf{Fun}(B, E) \xrightarrow{\circ f} \mathsf{Fun}(A, E) \tag{12}$$

is an equivalence onto the full subcategory $\underset{W\mapsto\cong}{\mathsf{Fun}}(A, E) \subseteq \mathsf{Fun}(A, E)$ spanned by functors that map edges in W to isomorphisms, which we define by taking the pullback

$$\begin{tikzcd}
\underset{W\mapsto\cong}{\mathsf{Fun}}(A,E) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(A,E) \arrow[d] \\
\underset{W\mapsto\cong}{\mathsf{hFun}}(A,E) \arrow[r, hook] & \mathsf{hFun}(A,E)
\end{tikzcd}$$

in the category of simplicial sets.

Remark 4.19. The ∞-categorical localization is unique up to equivalence: since the homotopy category functor preserves equivalences (Remark 3.27), the functor (12) yields an equivalence of 1-categories

$$\mathsf{hFun}(B, E) \xrightarrow{\circ f} \underset{W\mapsto\cong}{\mathsf{hFun}}(A, E).$$

A similar argument as in Remark 4.13 gives us the desired equivalence between two localizations.

The uniqueness of localization justifies the following definition:

Definition 4.20. Let $(\mathcal{C}, W)$ be a category with weak equivalences. Suppose there exists some quasi-category A with a functor $f\colon \mathcal{C} \to A$ exhibiting A as the localization of $\mathcal{C}$ with respect to W. Then, we say that A is the *underlying quasi-category* of $\mathcal{C}$ and denote it by $\mathsf{uq}\mathcal{C}$.

The following theorem gives us evidence that we have the "correct" definition of equivalences between quasi-categories:

Theorem 4.21 (Dwyer-Kan)**.** Quillen equivalences induce equivalences of underlying quasi-categories.

Proof. Quillen equivalences are proved to induce weak equivalences of hammock localizations in [DK80b, Proposition 5.2] in the Bergner model structure on simplicial categories, which becomes an equivalence of quasi-categories upon taking fibrant replacement and homotopy coherent nerve (which is a right Quillen functor from the Bergner model structure to the Joyal model structure). See [Maz16] for a summary of the history of theorems relating homotopical structures on model categories to structures on the underlying quasi-categories. □

Proposition 4.22. Let $\mathcal{C}$ be a small category with a collection of morphisms $W \subseteq \mathcal{C}$. Suppose $f\colon \mathcal{C} \to B$ exhibits B as an ∞-categorical localization of $\mathcal{C}$ with respect to W. Then, the composite functor

$$\mathcal{C} \xrightarrow{f} B \xrightarrow{\pi} \mathsf{h}B$$

exhibits the homotopy category $\mathsf{h}B$ as a 1-categorical localization of $\mathcal{C}$ with respect to W.

Proof. Given a category $\mathcal{E}$, we would like to show that

$$\mathsf{Fun}(\mathsf{h}B, \mathcal{E}) \xrightarrow{\pi^*} \mathsf{Fun}(B, \mathcal{E}) \xrightarrow{f^*} \underset{W \mapsto \cong}{\mathsf{Fun}}(\mathcal{C}, \mathcal{E})$$

is an equivalence. By the universal property of localization, f^* is an equivalence, and by the homotopy-nerve adjunction, so is π^*. □

Remark 4.23. In fact, the existence of the ∞-categorical localization is also guaranteed by a (∞-categorical) colimit construction similar to the category of fractions in Proposition 4.14 (see [Lur25, Subsection 01MZ] for details). However, similarly to the category of fractions, this construction is often not the most helpful characterization of the localization. Since our goal is to characterize the underlying quasi-category of model categories, we will describe a specific construction for the localization of model categories in the following section, which originally appears in [DK80a].

Definition 4.24 (Hammock Localization)**.** Let $(\mathcal{C}, W)$ be a category with weak equivalences. Define its hammock localization $\mathcal{L}^H(\mathcal{C}, W)$ to be the simplicial category whose objects are those of $\mathcal{C}$ and whose mapping simplicial set between $X, Y \in \mathcal{C}$ has k-simplices given by "reduced hammocks of width k" from X to Y, which are commutative diagrams

$$\begin{tikzcd}
 & A_{0,1} \arrow[r, no head] \arrow[d, "\wr"] & A_{0,2} \arrow[r, no head] \arrow[d, "\wr"] & \cdots \arrow[r, no head] & A_{0,n-1} \arrow[d, "\wr"'] & \\
 & A_{1,1} \arrow[r, no head] \arrow[d, "\wr"] & A_{0,2} \arrow[r, no head] \arrow[d, "\wr"] & \cdots \arrow[r, no head] & A_{1,n-1} \arrow[d, "\wr"'] & \\
X \arrow[uur, no head] \arrow[ur, no head] \arrow[dr, no head] & \vdots \arrow[d, "\wr"] & \vdots \arrow[d, "\wr"] & & \vdots \arrow[d, "\wr"'] & Y \arrow[uul, no head] \arrow[ul, no head] \arrow[dl, no head] \\
 & A_{k,1} \arrow[r, no head] & A_{k,2} \arrow[r, no head] & \cdots \arrow[r, no head] & A_{k,n-1} &
\end{tikzcd}$$

where the length of the hammock is $n \geq 1$ and the morphisms in the hammock are such that

i) all vertical maps are in W;
ii) horizontal maps in the same columns go in the same direction and can go left if they are in W;
iii) maps in adjacent columns go in different directions.

The face and degeneracy maps are given by omitting and duplicating rows. Composition is defined by horizontally pasting hammocks and reducing by

i) composing adjacent columns that point in the same direction and
ii) omitting any columns that contain only identity maps.

We have a canonical simplicial functor $\mathcal{C} \to \mathcal{L}^H(\mathcal{C}, W)$ by sending a morphism to a length 1, width 0 hammock.

Proposition 4.25 (Hinich)**.** Let (C, W) be a category with weak equivalences. Let R denote fibrant replacement in the Bergner model structure. The canonical map

$$C \to \mathfrak{N}R\mathcal{L}^H(C, W)$$

exhibits $\mathfrak{N}R\mathcal{L}^H(C, W)$ as a localization of C with respect to W.

Proof. By [Hin16, Proposition 1.2.1], there is a weak equivalence

$$(N\mathcal{C}, N\mathcal{W}) \to \mathfrak{N}R\mathcal{L}^H(C, W)^\natural$$

in the model category of marked simplicial sets, where $\natural$ denotes the natural marking in which exactly the isomoprhisms are marked. By [Hin16, Proposition 1.1.3], the ∞-categorical localization is represented by the fibrant replacement map. Composing the two weak equivalences, we see that $\mathfrak{N}R\mathcal{L}^H(C, W)$ is weakly equivalent to the ∞-categorical localization in the model category of marked simplicial sets, which exactly means that they are equivalent as quasi-categories. $\square$

The upshot is that in a simplicial model category, there is a nicer model of the ∞-categorical localization by the following result:

Proposition 4.26 ([DK80b])**.** Let $\mathcal{M}$ be a simplicial model category. Then, the simplicial categories of fibrant-cofibrant objects $\mathcal{M}_{\mathsf{cf}}$ is DK-equivalent to $\mathcal{L}^H(\mathcal{M}_0, W)$, where $\mathcal{M}_0$ denotes the underlying model category of $\mathcal{M}$. Hence, the quasi-categories $\mathfrak{N}\mathcal{M}_{\mathsf{cf}}$ and $\mathfrak{N}R\mathcal{L}^H(\mathcal{M}_0, W)$ are equivalent and the canonical map $N\mathcal{M}_0 \to \mathfrak{N}\mathcal{M}_{\mathsf{cf}}$ is a localization map.

Example 4.27. The archetypical example of a simplical model category is the Kan-Quillen model structure on simplicial sets. The fibrant-cofibrant objects are Kan complexes, so the underlying ∞-category is the ∞-category of spaces (Remark 4.8).

5. Basic ∞-Category Theory

Our goal in this section is to introduce basic ∞-category theory by working in the 2-category of quasi-categories. We begin with adjunctions, which have a familiar 2-categorical definition:

Definition 5.1. An *adjunction* between quasi-categories is one in the homotopy 2-category $\mathfrak{h}\mathsf{qCat}$. Specifically, an adjunction consists of the the data of

- two quasi-categories $A, B \in \mathsf{qCat}$;
- a pair of functors $f \colon B \to A$ and $u \colon A \to B$;
- a pair of ∞-natural transformations $\eta \colon \mathrm{id}_B \Rightarrow uf$ and $\epsilon \colon fu \Rightarrow \mathrm{id}_A$ satisfying the triangle identities:

$$
\begin{tikzcd}
 & B \arrow[rr, equal] \arrow[dr, "f"] & & B \\
A \arrow[ur, "u"] \arrow[rr, equal] & \arrow[u, phantom, "\epsilon\Downarrow"] & A \arrow[ur, "u"'] \arrow[u, phantom, "\eta\Downarrow"] &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "u"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\overset{\mathrm{id}_u}{\Rightarrow}"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[rr, equal] \arrow[dr, "u"] & & A \\
B \arrow[ur, "f"] \arrow[rr, equal] & \arrow[u, phantom, "\eta\Uparrow"] & B \arrow[ur, "f"'] \arrow[u, phantom, "\epsilon\Uparrow"] &
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\overset{\mathrm{id}_f}{\Rightarrow}"] & A
\end{tikzcd}
$$

which in equations say that $\epsilon u \circ u\eta = \mathrm{id}_u$ and $\epsilon f \circ f\eta = \mathrm{id}_f$.

Example 5.2. Adjunctions between nerves of 1-categories are just adjunctions between the underlying 1-categories.

Example 5.3. Simplicially enriched adjunctions between Kan-enriched categories induce adjunctions between the corresponding homotopy coherent nerves. However, these are too "strict" in some sense, and do not account for all adjunctions between the nerves.

The following theorem provide us with our first non-strict ∞-categorical adjunctions between quasi-categories that underlie model categories.

Theorem 5.4 ([Maz16])**.** Quillen adjunctions between model categories induce a canonical adjunction between their underlying quasi-categories.

Remark 5.5. An analytic definition of adjunctions involving cartesian and cocartesian fibrations is used instead of our categorical definition in [Maz16]. A description of this definition and its equivalence can be found in [RV22, Appendix F.5].

Although the 2-categorical definition of adjunctions might not be the useful when constructing examples, it allow us to prove facts about ∞-categories by doing 2-category theory, and the proofs are often similar to the 1-categorical versions.

Proposition 5.6. Adjunctions are unique up to isomorphism: given $f\colon B \to A$ and $u, u'\colon A \to B$ such that $f \dashv u$ and $f \dashv u'$, $u \cong u'$ are isomorphic. Conversely, given an adjunction $f \dashv u$ and an isomorphism $u \dashv u'$, we have $f \dashv u'$.

Proof. In the first direction, denote the unit maps for u and u' by η and η' and the counit maps by ϵ and ϵ'. The composite

$$\begin{tikzcd}
& B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\epsilon\Downarrow"] & & B \\
A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u'"'] \arrow[u, phantom, "\eta'\Downarrow"] &
\end{tikzcd}$$

is an isomorphism with inverse

$$\begin{tikzcd}
& B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\epsilon'\Downarrow"] & & B \\
A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u'"'] \arrow[u, phantom, "\eta\Downarrow"] &
\end{tikzcd}$$

which we see by using the triangle identities for both adjunctions. In the converse direction, let $f \dashv u$ be an adjunction with unit and counit maps η and ϵ, and let $\alpha\colon u \Rightarrow u'$ be an isomorphism with inverse $\beta\colon u' \Rightarrow u$. Define the new unit and counit maps for $f \dashv u'$ by

$$\epsilon' := \begin{tikzcd}
& B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
A \arrow[ur, "u"'] \arrow[ur, bend left=60, "u'"] \arrow[ur, phantom, bend left=30, "\beta\Downarrow"] \arrow[rr, equal] & {} & A
\end{tikzcd}
\qquad
\eta' := \begin{tikzcd}
& A \arrow[dr, "u"'] \arrow[dr, bend left=60, "u'"] \arrow[dr, phantom, bend left=30, "\alpha\Downarrow"] \arrow[d, phantom, "\eta\Uparrow"] & \\
B \arrow[ur, "f"] \arrow[rr, equal] & {} & B
\end{tikzcd}$$

and it is straightforward to check that they satisfy the triangle identities using the triangle identities for $f \dashv u$. ☐

Proposition 5.7. Adjunctions compose: given adjunctions as below left

$$\begin{tikzcd}
C \arrow[r, shift left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left, "u"] \arrow[r, shift left, "f'"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "u'"]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C \arrow[r, shift left, "f'f"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "uu'"]
\end{tikzcd}$$

the composites $f'f \dashv uu'$ form an adjunction.

Proof. Denote the unit and counit maps for $f \dashv u$ by η and ϵ, and those for $f' \dashv u'$ by η' and ϵ'. Define the unit and counit maps for the composite $f'f \dashv uu'$ by

$$\begin{tikzcd}
C \arrow[rrrr, equal] \arrow[dr, "f"'] & & \arrow[d, phantom, "\eta\Uparrow"] & & C \\
& B \arrow[rr, equal] \arrow[dr, "f'"'] & {} \arrow[d, phantom, "\eta'\Uparrow"] & B \arrow[ur, "u"'] & \\
& & A \arrow[ur, "u'"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & C \arrow[dr, "f"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \\
& B \arrow[ur, "u"] \arrow[rr, equal] & {} \arrow[d, phantom, "\epsilon'\Downarrow"] & B \arrow[dr, "f'"] & \\
A \arrow[ur, "u'"] \arrow[rrrr, equal] & & {} & & A
\end{tikzcd}$$

and the triangle identities follow from stacking those for $f \dashv u$ and $f' \dashv u'$. $\square$

Proposition 5.8. Any equivalence $A \underset{g}{\overset{f}{\underset{\leftarrow}{\overset{\rightarrow}{\simeq}}}} B$ of quasi-categories can be promoted to an adjunction $f \dashv g$.

Proof. Let $\alpha\colon \mathrm{id}_A \cong gf$ and $\beta\colon fg \cong \mathrm{id}_B$ be the structure ∞-natural isomorphisms for the equivalence. We wish to construct unit and counit natural isomorphisms that satisfy the triangle identities. The failure of α and β to satisfy the triangle identities is encoded in the natural isomorphism

$$\begin{tikzcd}
& B \\
A \arrow[ur, bend left=60, "f"] \arrow[ur, bend right=30, "f"'] \arrow[ur, phantom, "\phi\Uparrow\cong"] &
\end{tikzcd}
:=
\begin{tikzcd}
& B \arrow[rr, equal] \arrow[dr, "g"] \arrow[d, phantom, "\alpha\Uparrow"] & \arrow[d, phantom, "\beta\Uparrow"] & B \\
A \arrow[ur, "f"] \arrow[rr, equal] & {} & A \arrow[ur, "f'"'] &
\end{tikzcd}$$

which we use to define the unit map

$$\begin{tikzcd}
& B \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\eta"] & \\
A \arrow[ur, "f"] \arrow[rr, equal] & {} & A
\end{tikzcd}
:=
\begin{tikzcd}
& B \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\alpha"] & \\
A \arrow[ur, bend left=60, "f"] \arrow[ur, "f"'] \arrow[ur, phantom, bend left=30, "\psi\Uparrow"] \arrow[rr, equal] & {} & A
\end{tikzcd}$$

taking $\psi := \phi^{-1}$. Keep the counit map the same by taking $\epsilon := \beta$. One of the triangle identities is satisfied by construction. For the other triangle identity, since the composite

$$\begin{tikzcd}
& B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\epsilon\cong\Downarrow"] & \arrow[d, phantom, "\eta\cong\Downarrow"] & B \\
A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u"'] &
\end{tikzcd}$$

is an isomorphism, to check that it is the identity natural transformation it suffices to check that it is idempotent, which follows by the first triangle identity. $\square$

Now that we have the notion of an adjunction, we have one way to generalize the concept of limits:

Definition 5.9. An ∞-category A *admits all limits indexed by a simplicial set* J if the diagonal map $A \xrightarrow{\Delta} A^J$ admits a right adjoint

$$\begin{tikzcd}
A \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend left, "\lim"]
\end{tikzcd}.$$

Example 5.10. A terminal object is a limit over the empty diagram, and is hence given by an element $t\colon 1 \cong A^{\varnothing} \to A$ and an adjunction

$$\begin{tikzcd}
A \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "t"]
\end{tikzcd}.$$

The functorial definition of limits is very nice: using the fact that adjunctions compose, we immediately get:

Corollary 5.11. Right adjoints preserve functorial limits.

Unfortunately, this easy definition does not cover all cases. Often times, we will have categories that admit certain limits but not others with the same indexing category. We need a definition that deals with colimits on a case by case basis. Let's try to draw some inspiration from 1-category theory.

Definition 5.12. Let $\mathcal{C}, \mathcal{D}$ be categories, and let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor. An object $\lim F$ together with a cone $\lambda\colon \Delta \lim F \Rightarrow F$ is a *limit* for F if one of the following equivalent statements hold:

(1) for all $c \in \mathcal{C}$, given a set of maps $\mu_d\colon c \to Fd$ natural in $d \in D$, there exists a unique map $\overline{\mu}\colon c \to \lim F$ such that
$$\begin{tikzcd} c \arrow[d, dashed, "\exists!\overline{\mu}"'] \arrow[dr, "\mu_d"] & \\ \lim F \arrow[r, "\lambda_d"'] & Fd \end{tikzcd}$$
commutes for all $d \in \mathcal{D}$;

(2) for all $c \in \mathcal{C}$, given any cone $\mu\colon \Delta c \Rightarrow F$, there exists a unique map $\overline{\mu}\colon c \to \lim F$ such that $\lambda\Delta\overline{\mu} = \mu$;

(3) the natural transformation
$$\mathcal{C}(c, \lim F) \to \mathcal{C}^{\mathcal{D}}(\Delta c, F)$$
corresponding to λ under Yoneda is a natural isomorphism;

(4) the cone λ is terminal in the category of cones over F.

Definition (2) is easy to encode 2-categorically: represent the limit cone λ diagrammatically as
$$\begin{tikzcd} & \mathcal{C} \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\lim F"] \arrow[r, "F"'] \arrow[r, phantom, shift left=3, "\lambda\Downarrow"] & \mathcal{C}^{\mathcal{D}} \end{tikzcd}$$
and the universal property of the limit then says that given any object $c \in \mathcal{C}$ and cone $\mu\colon \Delta c \Rightarrow F$, there exists a unique factorization $\overline{\mu}\colon \operatorname{colim} F \to c$, depicted 2-categorically as:
$$\begin{tikzcd} 1 \arrow[r, "c"] \arrow[d] \arrow[dr, phantom, "\mu\Downarrow"] & \mathcal{C} \arrow[d, "\Delta"] \\ 1 \arrow[r, "F"'] & \mathcal{C}^{\mathcal{D}} \end{tikzcd} = \begin{tikzcd} 1 \arrow[r, "c"] \arrow[d] \arrow[r, phantom, shift right=3, "\exists!\overline{\mu}\Downarrow"] & \mathcal{C} \arrow[d, "\Delta"] \\ 1 \arrow[ur] \arrow[r, "F"'] \arrow[r, phantom, shift left=3, "\lambda\Downarrow"] & \mathcal{C}^{\mathcal{D}} \end{tikzcd}$$
In the ∞-categorical setting, it is not enough to just consider the global elements (i.e. objects), but it turns out that we obtain the correct definition by considering generalized elements. We will encode this generalization using the following 2-categorical notion:

Definition 5.13. Given a cospan $C \xrightarrow{g} A \xleftarrow{f} B$, a functor $r\colon C \to B$ and a 2-cell
$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\rho\Downarrow"] & A \end{tikzcd} \tag{13}$$
defines an *absolute right lifting of g through f* if for any 2-cell as below left
$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\chi\Downarrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[r, phantom, shift right=3, "\exists!\overline{\chi}\Downarrow"] & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\rho\Downarrow"] & A \end{tikzcd}$$
there exists a unique factorization as above right. In this case, we say that the 2-cell (13) is an *absolute right lifting diagram.*

The adjective "absolute" refers to the following stability property:

Lemma 5.14. Absolute right lifting diagrams are stable under restriction of domain: let

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\rho\Downarrow"'] & A \end{tikzcd}$$

be an absolute right lifting diagram. For any $c\colon X \to C$, the whiskered diagram

$$\begin{tikzcd} & & B \arrow[d, "f"] \\ X \arrow[r, "c"'] & C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\rho\Downarrow"'] & A \end{tikzcd}$$

is an absolute right lifting diagram, i.e. the pair $(rc, \rho c)$ defines an absolute right lifting of gc through f.

Proof. Given a 2-cell

$$\begin{tikzcd} Z \arrow[rr, "b"] \arrow[d, "x"'] \arrow[drr, phantom, "\chi\Downarrow"] & & B \arrow[d, "f"] \\ X \arrow[r, "c"'] & C \arrow[r, "g"'] & A \end{tikzcd}$$

we use the absolute lifting property of λ to obtain a factorization

$$\begin{tikzcd} Z \arrow[r, "b"] \arrow[d, "x"'] & B \arrow[dd, "f"] \\ X \arrow[d, "c"'] \arrow[r, phantom, "\chi\Downarrow"] & {} \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} Z \arrow[r, "b"] \arrow[d, "x"'] \arrow[dr, phantom, "\exists!\bar{\chi}\Downarrow"] & B \arrow[dd, "f"] \\ X \arrow[d, "c"'] & {} \\ C \arrow[uur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\rho\Downarrow"] & A \end{tikzcd}$$

exhibiting $(rc, \rho c)$ as an absolute right lifting diagram. □

Example 5.15. Given a pair of functors $f\colon B \to A$ and $u\colon A \to B$, we claim that a natural transformation $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ defines a counit for an adjunction if and only if

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, "\epsilon\Downarrow"'] & A \end{tikzcd}$$

is an absolute right lifting diagram. To show this, we obtain the unit η for the adjunction using absolute lifting for id_f

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "="] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} = \begin{tikzcd} B \arrow[r, equal] \arrow[d, "f"'] \arrow[r, phantom, "\eta\Downarrow"', near start] & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, "\epsilon\Downarrow", near end] & A \end{tikzcd}$$

which satisfies one of the triangle identities by definition. For the other triangle identity, the universal property of the absolute right lifting diagram tells us that pasting with ϵ to the right is injective, hence the equality

$$\begin{tikzcd} & B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\epsilon\Downarrow"] & {} \arrow[d, phantom, "\eta\Downarrow"] & B \arrow[dr, "f"] \arrow[d, phantom, "\epsilon\Downarrow"] & \\ A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \end{tikzcd} = \begin{tikzcd} & B \arrow[dr, "f"] \arrow[d, phantom, "\epsilon\Downarrow"] & \\ A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \end{tikzcd}$$

gives us the second triangle identity.

The absolute lifting diagram encoding of the counit recovers the "transposition" axiomatization of adjunctions: the factorization

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\chi\Downarrow"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "f"'] \arrow[r, phantom, shift right=3, "\exists!\overline{\chi}\Downarrow"] & B \arrow[d, "f"] \\ A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, shift right=3, "\epsilon\Downarrow"] & A \end{tikzcd}$$

induces a bijection between generalized arrows

$$\chi\colon fb \to a \quad \leftrightsquigarrow \quad \overline{\chi}\colon b \to ua.$$

In particular, taking $\mathcal{K} = \mathsf{Cat}$ and $X = 1$ recovers the usual definition of adjunction.

Using an easy argument that is exactly the same as showing the uniqueness of limits in 1-categories, we show the uniqueness of absolute liftings:

Proposition 5.16. Absolute right lifting diagrams are unique up to isomorphism: given two absolute right lifting diagrams

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\rho\Downarrow"] & A \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r'"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\rho'\Downarrow"] & A \end{tikzcd}$$

for the same cospan $B \xrightarrow{f} A \xleftarrow{g} C$, there exists some unique 2-cell isomorphism $\phi\colon r' \cong r$ such that

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, bend left, "r'"] \arrow[ur, "r"'] \arrow[ur, phantom, shift left=2, "\phi\Downarrow\cong"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\rho\Downarrow"] & A \end{tikzcd} \quad = \quad \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r'"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\rho'\Downarrow"] & A. \end{tikzcd}$$

Proof. Obtain ϕ by factorizing ρ' through ρ using absolute right lifting:

$$\begin{tikzcd} C \arrow[r, "r'"] \arrow[d, equal] \arrow[dr, phantom, "\rho'\Downarrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} C \arrow[r, "r'"] \arrow[d, equal] \arrow[r, phantom, shift right=3, "\exists!\phi\Downarrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] \arrow[ur, "r"] \arrow[ur, phantom, shift right=3, "\rho\Downarrow"] & A \end{tikzcd}$$

Similarly obtain a 2-cell $\psi\colon r \Rightarrow r'$ using absolute right lifting for ρ'. The composite $\phi\psi\colon r \Rightarrow r$ satisfies that $\rho \cdot f\phi\psi = \rho$, so by uniqueness we have $\phi\psi = \mathrm{id}_r$. Similarly check that $\psi\phi = \mathrm{id}_{r'}$, so ψ defines an inverse to ϕ. ☐

Proposition 5.17. Absolute right lifting diagrams are invariant under isomorphisms: given an absolute right lifting diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\rho\Downarrow"] & A \end{tikzcd}$$

and natural isomorphisms

$$\begin{tikzcd} C \arrow[r, bend left, "r'"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\theta\Downarrow\cong"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\phi\Downarrow\cong"] & A \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\psi\Downarrow\cong"] & A, \end{tikzcd}$$

the pasted composite

$$\begin{tikzcd} & B \arrow[d, "f'"] \\ C \arrow[ur, "r'"] \arrow[r, "g'"'] \arrow[ur, phantom, "\rho'\Downarrow" near end, shift right=3] & A \end{tikzcd} \quad := \quad \begin{tikzcd} & B \arrow[d, "f"'] \arrow[d, bend left=60, "f'"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}", shift left=3] \\ C \arrow[ur, bend left=50, "r'"] \arrow[ur, "r"'] \arrow[ur, phantom, "\theta\Downarrow", shift left=3] \arrow[r, "g"] \arrow[ur, phantom, "\rho\Downarrow", shift right=3] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\psi\Downarrow", shift right=3] & A \end{tikzcd}$$

is an absolute lifting diagram.

Proof. Given a 2-cell

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f'"] \\ C \arrow[r, "g'"'] & A, \end{tikzcd}$$

we compose with inverses to obtain a 2-cell

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f'"', bend right] \arrow[d, bend left, "f"] \arrow[d, phantom, "\overset{\phi^{-1}}{\Rightarrow}"] \\ C \arrow[r, "g'"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\psi^{-1}\Downarrow", shift right=3] & A, \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\exists!\bar{\chi}\Downarrow" near start] & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, "\rho\Downarrow" near end, shift right=3] & A \end{tikzcd}$$

which factors through the absolute lifting diagram ρ as above right. Our desired lift through ρ' is given by pasting $\overline{\chi}$ with the isomorphism θ^{-1}. $\square$

Definition 5.18. Let A be an ∞-category, J be an simplicial set, and $d\colon J \to A$ be a diagram. A *limit* of d is an absolute right lifting of the diagram d through the constant diagram functor $\Delta\colon A \to A^J$:

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[ur, phantom, "\lambda\Downarrow" near end, shift right=3] & A^J \end{tikzcd}$$

We call the 2-cell $\lambda\colon \Delta \lim D \Rightarrow d$ the *limit cone*.

Example 5.19. Let $\mathcal{C}, \mathcal{D}$ be a 1-categories and $F\colon \mathcal{D} \to \mathcal{C}$ be a functor. A quasi-categorical limit of F is just a 1-categorical limit for F: we would need to check absolute lifting for arbitrary quasi-categories

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\chi"] & \mathcal{C} \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[r, "F"'] & \mathcal{C}^{\mathcal{D}} \end{tikzcd}$$

but by the adjunction $\mathsf{h} \dashv N$, the generalized arrow χ factors through $\mathsf{h}X$. Hence, we only need to check absolute lifting for 1-categories, which we claim is equivalent to the ordinary limit condition (which say that the absolute lifting condition is true for $\mathbb{1}$). Given a category $\mathcal{E}$ and a natural transformation

$$\begin{tikzcd} \mathcal{E} \arrow[r, "G"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\chi"] & \mathcal{C} \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[r, "F"'] & \mathcal{C}^{\mathcal{D}} \end{tikzcd}$$

we would like to factor it uniquely through the 1-categorical limit cone, which we will denote by $\lambda\colon \Delta \lim F \Rightarrow F$. In other words, we would like to show that

$$(\mathcal{C}^{\mathcal{D}})^{\mathcal{E}}(\Delta \circ G, \Delta F) \cong \mathcal{C}^{\mathcal{E}}(G, \Delta \lim F).$$

Writing natural transformations as ends, we have

$$\begin{aligned}(\mathcal{C}^{\mathcal{D}})^{\mathcal{E}}(\Delta \circ G, \Delta F) &\cong \int_{d\in\mathcal{D}}\int_{e\in\mathcal{E}} \mathcal{C}(Ge, Fd)\\ &\cong \int_{e\in\mathcal{E}}\int_{d\in\mathcal{D}} \mathcal{C}(Ge, Fd)\\ &\cong \int_{e\in\mathcal{E}} \mathcal{C}\left(Ge, \int_{d\in\mathcal{D}} Fd\right)\\ &\cong \int_{e\in\mathcal{E}} \mathcal{C}(Ge, \lim F)\\ &\cong \mathcal{C}^{\mathcal{E}}(G, \Delta \lim F)\end{aligned}$$

as desired.

Example 5.20. Let $\mathcal{C}$ be a 1-category and J be a simplical set. We have a natural identification

$$\mathsf{Fun}(J, N\mathcal{C}) \cong N\,\mathsf{Fun}(\tau_1 J, \mathcal{C})$$

of quasi-categories. Hence, quasi-categorical limits in nerves of 1-categories can always be computed as 1-categorical limits after replacing the indexing simplicial set by a graph.

Stability of absolute lifting diagrams tells us that functorial limits are limits:

Corollary 5.21. Suppose A admits all limits of shape J. Then, a limit for a diagram $d\colon J \to A$ is given by evaluating $\lim\colon A^J \to A$ at d, i.e. the composite

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"'] & A^J \arrow[ur, "\lim"] \arrow[r, equal] \arrow[ur, phantom, shift left=3, "\epsilon\Downarrow"] & A^J \end{tikzcd}$$

is absolute right lifting.

From the uniqueness and isomorphism invariance of absolute liftings (Propositions 5.16 and 5.17), we get the following:

Corollary 5.22. Limits are unique up to isomorphism: any two limits (ℓ, λ), (ℓ', λ') of a diagram $d\colon J \to A$ are isomorphic in A through an isomorphism $\alpha\colon \ell \to \ell'$ such that $\lambda' \circ \Delta\alpha = \lambda$ which is unique up to homotopy.

Remark 5.23. In fact, the isomorphism relating (ℓ, λ) and (ℓ', λ') is unique not just up to homotopy but up to contractible choice, meaning that the space of such isomorphisms is contractible (see Corollary 6.33 and Proposition 6.34).

Corollary 5.24. Two naturally isomorphic diagrams $d, d'\colon J \to A$ have isomorphic limits. Any object ℓ that is isomorphic to $\lim d$ is a limit for d.

Since all of our definitions are 2-categorical, we are able to prove the following standard result using 2-categorical techniques (in an arbitrary ∞-cosmos):

Proposition 5.25. Right adjoints preserve limits.

Proof. Let $\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "u"] \end{tikzcd}$ be an adjunction and $d\colon J \to A$ be a diagram. Fix a limit $\lim d \in A$ with limit cone $\lambda\colon \Delta \lim d \Rightarrow d$. We first recall the strategy for proving this statement for 1-categories: given a cone $\mu\colon \Delta b \Rightarrow ud$ for some $b \in B$, we take the transpose $\mu^\dagger\colon \Delta fb \Rightarrow d$, factor it as $\overline{\mu^\dagger}\colon fb \to \lim d$, which we then transpose again to obtain the desired factorization. The 2-categorical strategy is the same, but we need to write out the transposes as pasting diagrams using the unit and counit 2-cells. Our goal is to show that

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\lim d"] \arrow[r, "d"'] \arrow[r, phantom, "\lambda\Downarrow", shift left=3ex] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd} \tag{14}$$

is a absolute right lifting diagram. Start with a 2-cell of the form

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[d, "!"'] & \arrow[d, phantom, "\chi\Downarrow"] & B \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"'] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd}$$

and take the transpose (below left)

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[d, "!"'] & \arrow[d, phantom, "\chi\Downarrow"] & B \arrow[r, "f"] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"'] & A^J \arrow[r, "u^J"] \arrow[rr, equal, bend right=50, ""{name=E}] & B^J \arrow[r, "f^J"] \arrow[d, phantom, "\epsilon^J\Downarrow"] & A^J \\ & & {} & \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "!"'] \arrow[dr, phantom, "\exists!\phi\Downarrow"] & B \arrow[r, "f"] & A \arrow[d, "\Delta"] \\ 1 \arrow[urr, "\lim d"] \arrow[rr, "d"'] & \arrow[r, phantom, "\lambda\Downarrow"] & A^J \end{tikzcd}$$

which we can factor using our hypothesis (above right). The (unique) candidate for our desired factorization is the 2-cell

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "!"'] \arrow[dr, phantom, "\exists!\phi\Downarrow"] & B \arrow[r, "f"'] \arrow[rr, equal, bend left=50] & A \arrow[u, phantom, "\eta\Downarrow"] \arrow[r, "u"'] & B \\ 1 \arrow[urr, bend right, "\lim d"'] & & & \end{tikzcd}$$

as upon pasting with (14), we get

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "!"'] \arrow[dr, phantom, "\exists!\phi\Downarrow"] & B \arrow[r, "f"] \arrow[rrr, equal, bend left=50, "\eta\Downarrow"] & A \arrow[rr, "u"'] \arrow[d, "\Delta"] & & B \arrow[d, "\Delta"] \\ 1 \arrow[urr, "\lim d"] \arrow[rr, "d"'] & \arrow[r, phantom, "\lambda\Downarrow"] & A^J \arrow[rr, "u^J"'] & & B^J \end{tikzcd} = \begin{tikzcd} X \arrow[rr, "b"] \arrow[d, "!"'] & \arrow[d, phantom, "\chi\Downarrow"] & B \arrow[r, "f"'] \arrow[d, "\Delta"] \arrow[rr, equal, bend left=50, "\eta\Downarrow"] & A \arrow[r, "u"'] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"'] & A^J \arrow[r, "u^J"] \arrow[rr, equal, bend right=50, "\epsilon^J\Downarrow"'] & B^J \arrow[r, "f^J"] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd}$$

where using the enriched functoriality of the simplicial cotensor $\mathsf{sSet} \times \mathsf{qCat} \to \mathsf{qCat}$ we may rewrite the above right as

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[d, "!"'] & \arrow[d, phantom, "\chi\Downarrow"] & B \arrow[d, "\Delta"] & & \\ 1 \arrow[r, "d"'] & A^J \arrow[r, "u^J"] \arrow[rr, equal, bend right=50, "\epsilon^J\Downarrow"'] & B^J \arrow[r, "f^J"] \arrow[rr, equal, bend left=50, "\eta^J\Downarrow"] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd}$$

which is equal to χ by the triangle equality. $\square$

6. Universal Properties of Adjunctions and Limits

6.1. **Naturality and Discrete Fibrations.** Now that we have defined limits and colimits in an ∞-category, we would like to show that they satisfy an expected universal property, just like the 1-categorical limit (Definition 5.12, (3)). Unfortunately, naturality is more difficult to encode in a ∞-categorical setting, as we need to consider functorial mappings for higher morphisms.

Example 6.1. Let $A \in \mathsf{qCat}$ be a quasicategory and $a, b \in A_0$ be objects. We can define the hom-space as the pullback

$$\begin{tikzcd} \mathrm{Hom}_A(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, "(\mathrm{cod},\mathrm{dom})"] \\ \mathbb{1} \arrow[r, "(b,a)"'] & A \times A \end{tikzcd}$$

of simplicial sets. We will soon introduce techniques to show that $\mathrm{Hom}_A(a, b)$ is actually a space (i.e. a Kan complex), but let's first focus on a different question. We know from our experience with 1-category theory that one of the most important functors is the representable functor, which we use to define universal properties, prove the Yoneda lemma, etc. We definitely would like to define something similar for ∞-categories, for instance, a functor $\mathrm{Hom}_A(-, a) : A^{\mathsf{op}} \to \mathsf{Kan}$ that at the object level maps a to $\mathrm{Hom}_A(x, a)$. However, manually defining such a functor requires us to specify mappings at every cell level in a compatible way, which are not obvious, especially since we no longer have strict composition. We will sidestep this issue by considering certain kinds of fibrations, which encode the data of a functor in a much simpler way. Let's go back to 1-category land for some inspiration.

Definition 6.2. Let $\mathcal{C} \in \mathsf{Cat}$ be a small category and $F : \mathcal{C} \to \mathsf{Set}$ be a functor. The category of elements of F is a category $\mathrm{el}\, F$ whose

- objects are pairs (c, x), where $c \in \mathcal{C}$ is an object and $x \in Fc$ is an element;
- morphisms $(c, x) \to (d, y)$ are given by morphisms $f : c \to d$ in $\mathcal{C}$ such that $Ff(x) = y$.

It turns out that together with the canonical projection $\Pi :\ \mathrm{el}\, F \to \mathcal{C}$, the category of elements encodes all information about F. The categories over $\mathcal{C}$ that arise as categories of elements of functors, which we call discrete fibrations, can be characterized by a lifting property as follows:

Definition 6.3. A discrete fibration is a functor $p : \mathcal{E} \to \mathcal{C}$ such that for all objects $e \in \mathcal{E}$ and all morphisms $g : c \to pe$, there exists a unique lift $h : d \to e$ in $\mathcal{E}$ such that $Fh = g$. A discrete opfibration is a functor $p : \mathcal{E} \to \mathcal{C}$ such that $p^{\mathsf{op}} : \mathcal{E}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ is a discrete fibration.

Example 6.4. Let $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ be a functor. The projection $\Pi :\ \mathrm{el}\, F \to \mathcal{C}$ is a discrete fibration: given a pair (c, x) in $\mathrm{el}\, F$ and a morphism $g : b \to c$ in $\mathcal{C}$, we have a unique lift to $(b, Fg(x)) \to (c, x)$ in $\mathrm{el}\, F$ (note the contravariance).

As the name suggests, fibers of discrete fibrations are discrete:

Lemma 6.5. Let $p : \mathcal{E} \to \mathcal{C}$ be a discrete fibration. Then, for all objects $c \in \mathcal{C}$, the fiber

$$\begin{tikzcd} \mathcal{E}_c \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, two heads, "p"] \\ \{c\} \arrow[r, hook] & \mathcal{C} \end{tikzcd}$$

is a discrete category, i.e. a set.

Proof. The morphisms of $\mathcal{E}_c$ are morphisms $f : e \to e'$ in $\mathcal{E}$ such that $pf = \mathrm{id}_c$. Since $\mathrm{id}_{e'} : e' \to e'$ is another lift of f, by the discrete fibration assumption we necessarily have $f = \mathrm{id}_{e'}$. Hence, all morphisms in $\mathcal{E}_c$ are identity morphisms. $\square$

As we mentioned earlier, taking categories of elements is a lossless way to encode the data of a presheaf as a discrete fibration, which is captured in the following:

Proposition 6.6. There is an equivalence of categories

$$\mathsf{Cat}^{\mathsf{DFib}}_{/\mathcal{C}} \underset{\mathsf{Un}}{\overset{\mathsf{St}}{\rightleftarrows}} \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Set})$$

between the category of discrete fibrations, regarded as a full subcategory of the slice category $\mathsf{Cat}_{/\mathcal{C}}$, and the category of presheaves on $\mathcal{C}$. This is called the *straightening-unstraightening* equivalence, where St is called the straightening functor and Un is called the unstraightening functor.

Proof. Define St by sending a discrete fibration $p\colon \mathcal{E} \to \mathcal{C}$ to the presheaf $\mathsf{St}(p)\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ that sends an object $c \in \mathcal{C}$ to the set $\mathcal{E}_c$. Given any morphisms $f\colon c \to d$ in $\mathcal{C}$ and an element $x \in \mathcal{E}_d$, we get a unique lift

$$\begin{tikzcd} \exists y \arrow[r, dashed, "\exists!"] \arrow[d, mapsto] & x & \in & \mathcal{E} \arrow[d, two heads, "p"] \\ c \arrow[r, "f"'] & d & \in & \mathcal{C} \end{tikzcd}$$

which we define to be $\mathsf{St}(p)f(x) := y$. Functoriality is guaranteed by the uniqueness of lifts. The action of St on morphisms is given by sending a fibered functor

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "r"] \arrow[dr, two heads, "p"'] & & \mathcal{F} \arrow[dl, two heads, "q"] \\ & \mathcal{C} & \end{tikzcd}$$

to the natural transformation that sends $x \in \mathcal{E}_c$ to $rx \in \mathcal{F}_c$, where naturality is again guaranteed by uniqueness of lifts. The unstraightening functor Un is defined by sending a presheaf to its category of elements. It is straightforward to check that the two procedures define inverses to each other. $\square$

When working with quasi-categories, it is often much easier to write down the corresponding fibration than the functor itself. We cite the following ∞-categorical analogue of Proposition 6.6 without proof to motivate our discussion of universal properties in the following sections using fibered equivalences rather than isomorphisms of functors:

Theorem 6.7 ([Lur09], Theorem 2.2.1.2)**.** Let A be a small quasi-category. Then, there is an equivalence of quasi-categories

$$\mathsf{qCat}^{\mathsf{LFib}}_{/A} \underset{\mathsf{Un}}{\overset{\mathsf{St}}{\rightleftarrows}} \mathsf{Fun}(A^{\mathsf{op}}, \mathcal{S})$$

where the left hand side denotes the subcategory of the slice quasi-category over A spanned by left fibrations (maps of quasi-categories with horn fillers for Λ^n_k for all $0 \le k < n$).

Example 6.8. Let's return to our motivating example from 6.1, first in a 1-categorical setting. Given a category $\mathcal{C}$, consider the representable functor

$$\mathcal{C}(-, c)\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set},$$

whose category of elements has

- objects $(b \in \mathcal{C}, f\colon b \to c)$;
- morphisms $(b, f) \to (b', f')$ given by maps $g\colon b \to b'$ such that $f'g = f$.

Notice that the category of elements is exactly the slice category $\mathcal{C}/c$, with the defining discrete fibration being the forgetful functor $\mathcal{C}/c \to \mathcal{C}$. By Proposition 6.6, we can encode naturality statements in terms of fibered equivalences. For instance, a functor $F\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is representable if and only if there exists a fibered equivalence $\mathcal{C}/c \simeq_{\mathcal{C}} \mathrm{el}\, F$. As an application, given a functor $K\colon \mathcal{J} \to \mathcal{C}$, the defining universal property of the limit $\lim K$ can be encoded as a fibered equivalence

$$\mathcal{C}/\lim K \simeq_{\mathcal{C}} \mathrm{el}\left(\mathcal{C}^{\mathcal{J}}(\Delta, K)\right)$$

where the category of elements on the right hand side is the category of cones.

6.2. **Comma Categories.** The framework in which we will discuss universal properties in an ∞-cosmos is that of comma categories.

Lemma 6.9. The map $(\mathrm{cod}, \mathrm{dom})\colon A^{\mathbb{2}} \to A \times A$ is an isofibration.

Proof. We first check that the map is an inner fibration. Changing to simplicial set notation, we recognize this map as $A^{\Delta^1} \to A^{\partial\Delta^1}$, induced by precomposing the boundary inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$. Given $0 < k < n \in \mathbb{N}$, we can transpose a lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & A^{\Delta^1} \arrow[d] \\ \Delta^n \arrow[r, "\beta"'] \arrow[ur, dashed] & A^{\partial\Delta^1} \end{tikzcd}$$

as follows: the given data is equivalent to

$$\begin{tikzcd} \Lambda^n_k \times \partial\Delta^1 \arrow[r, hook] \arrow[d, hook] & \Delta^n \times \partial\Delta^1 \arrow[d] \arrow[ddr, bend left, "\beta"] & \\ \Lambda^n_k \times \Delta^1 \arrow[r] \arrow[drr, bend right, "\alpha"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists!"] & \\ & & A \end{tikzcd}$$

and the desired lift transposes to a map

$$\Delta^n \times \Delta^1 \to A$$

that agrees with α and β on their respective domains. In other words, the inner horn lifting problem transposes to a lifting problem

$$\begin{tikzcd} (\Delta^n \times \partial\Delta^1) \cup (\Lambda^n_k \times \Delta^1) \arrow[r] \arrow[d, hook] & A \\ \Delta^n \times \Delta^1 \arrow[ur, dashed] & \end{tikzcd}$$

which is a special case of a "Leibniz product" of an inner horn inclusion with a monomorphism, shown combinatorially to be inner anodyne in [RV22, Corollary D.3.11].

To show that $(\mathrm{cod}, \mathrm{dom})$ lifts against the endpoint inclusions $\mathbb{1} \hookrightarrow \mathbb{I}$, we make use of the description of homotopy coherent isomorphisms in Corollary 3.21. Given a lifting problem

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, hook] & A^{\mathbb{2}} \arrow[d, "(\mathrm{cod},\mathrm{dom})"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & A \times A \end{tikzcd}$$

let $f\colon a \to b$ denote the morphism in A picked out by $\mathbb{1} \to A^{\mathbb{2}}$, and let $g\colon a \xrightarrow{\cong} c$ and $h\colon b \xrightarrow{\cong} d$ be the isomorphisms in A picked out by $\mathbb{I} \to A \times A$ (taking $\mathbb{1} \hookrightarrow \mathbb{I}$ to be, without loss of generality, the domain inclusion). Using inner horn lifting for 2-cells twice, we obtain some composite

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, dashed, "\exists k" description] & b \arrow[d, "h"] \arrow[dl, phantom, "\Uparrow" near start] \\ c \arrow[u, "g^{-1}"] \arrow[r, dashed, "\exists \ell"'] \arrow[ur, phantom, "\Uparrow" near start] & d \end{tikzcd}$$

which we use in the (3,2)-horn filler for

$$\begin{tikzcd} & c \arrow[dd, "g^{-1}" near start] \arrow[dr, "\ell"] & \\ a \arrow[ur, "g"] \arrow[rr, "k"' near start] \arrow[dr, equal] & & d, \\ & a \arrow[ur, "k"'] & \end{tikzcd}$$

the back face of which can be pasted to obtain a square

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "k" description] & b \arrow[d, "h"] \arrow[dl, phantom, "\Uparrow" near start] \\ c \arrow[r, "\ell"'] \arrow[ur, phantom, "\Leftarrow" near start] & d \end{tikzcd}$$

giving ourselves a map $\Delta^1 \times \Delta^1 \to A$ that transposes to a lift

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, hook] & A^{\mathbb{2}} \arrow[dd] \\ \mathbb{2} \arrow[ur, dashed] \arrow[d, hook] & \\ \mathbb{I} \arrow[r] & A \times A \end{tikzcd}$$

Since isomorphisms in functor categories can be checked objectwise (Theorem 3.22), we obtain our desired lift. □

Remark 6.10. Lemma 6.9 is a special case of the stability of isofibrations under "Leibniz exponentials" (see [RV22, Proposition 1.1.20]), which is axiomatized as a part of the definition for an ∞-cosmos.

Definition 6.11. Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a diagram of ∞-categories. The *comma category* is constructed as the pullback

$$\begin{tikzcd} \mathrm{Hom}_A(f,g) \arrow[r, "\phi"] \arrow[d, two heads, "(p_1,p_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"] \\ C \times B \arrow[r, "f\times g"'] & A \times A \end{tikzcd}$$

in the ∞-cosmos $\mathcal{K}$. It is sometimes also denoted as $f \downarrow g$. This construction comes with a canonical natural transformation

$$\begin{tikzcd} & \mathrm{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

which we call the comma cone.

The comma category of quasi-categories unfortunately does not satisfy the 2-categorical universal property of a comma object as we might expect. Instead, it satisfies a weaker kind of universal property, introduced by Riehl and Verity in [RV15], which uniquely characterizes it up to equivalence. We will only cite it here, and the interested reader is encouraged to read [RV22, Chapter 3] for more details.

Proposition 6.12. The canonical functor

$$\mathsf{hFun}(X, f \downarrow g) \to \mathsf{hFun}(X, f) \downarrow \mathsf{hFun}(X, g) \tag{15}$$

is *smothering*, meaning that it is surjective on objects, full and conservative. Equivalently, this says that the comma cone satisfies the following weak universal property in $\mathfrak{h}\mathcal{K}$:

i) 1-cell induction: given $B \xleftarrow{b} X \xrightarrow{c} C$ and a natural transformation

$$\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, phantom, "\overset{\alpha}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad = \quad \begin{tikzcd} & X \arrow[d, dashed, "\exists\lceil\alpha\rceil"] \arrow[ddl, bend right, "c"'] \arrow[ddr, bend left, "b"] & \\ & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

there exists a functor $\lceil\alpha\rceil \colon X \to \mathrm{Hom}_A(f,g)$ such that $p_0\lceil\alpha\rceil = b$, $p_1\lceil\alpha\rceil = c$ and $\phi\lceil\alpha\rceil = \alpha$ as depicted above right.

ii) 2-cell induction: given $a, a' \colon X \to \mathrm{Hom}_A(f,g)$ and natural transformations $\tau_0 \colon p_0 a \Rightarrow p_0 a'$ and $\tau_1 \colon p_1 a \Rightarrow p_1 a'$ such that

$$\begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_1}{\Leftarrow}"] & & \\ f\downarrow g \arrow[dr, "p_1"'] & {} & f\downarrow g \arrow[dl, "p_1"] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ & C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\ & & A & \end{tikzcd} \quad = \quad \begin{tikzcd} & & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\ & f\downarrow g \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & {} & f\downarrow g \arrow[dl, "p_0"] \\ C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] & \\ & A & & \end{tikzcd} \tag{16}$$

then there exists a natural transformation $\tau \colon a \Rightarrow a'$ such that $p_0\tau = \tau_0$ and $p_1\tau = \tau_1$.

iii) 2-cell conservativity: A natural transformation

$$\begin{tikzcd} X \arrow[r, bend left, "a'"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\Downarrow\tau"] & f\downarrow g \end{tikzcd}$$

is an isomorphism if and only if $p_0\tau$ and $p_1\tau$ are.

Proof. [RV22, Proposition 3.4.6] □

Example 6.13. Let $\mathcal{C} \xrightarrow{G} \mathcal{A} \xleftarrow{F} \mathcal{B}$ be a cospan of 1-categories. The comma category $F \downarrow G$ is the category in which

- objects are triples $(b \in \mathcal{B}, c \in \mathcal{C}, \alpha \colon Fb \to Gc \in \mathcal{A})$

- a morphism $(b,c,\alpha) \to (b',c',\alpha')$ is given by a pair of maps $\beta\colon b \to b'$ and $\gamma\colon c \to c'$ such that the square

$$\begin{tikzcd} Fb \arrow[r, "\alpha"] \arrow[d, "F\beta"'] & Gc \arrow[d, "G\gamma"] \\ Fb' \arrow[r, "\alpha'"'] & Gc' \end{tikzcd}$$

commutes.

Example 6.14. We continue to let $\mathcal{K} = \mathsf{Cat}$ and specialize to a few familiar examples.

- Consider the cospan $1 \xrightarrow{c} \mathcal{C} \xleftarrow{d} \mathcal{C}$ given by two objects in $\mathcal{C}$. The comma category $c \downarrow d$ has objects the maps $c \to d$ and no non-identity morphisms, so it is just the hom-set $\mathrm{Hom}_{\mathcal{C}}(c,d)$. This justifies the hom notation for comma categories.
- Consider the cospan $1 \xrightarrow{c} \mathcal{C} \xleftarrow{\mathrm{id}_c} \mathcal{C}$. The comma category $c \downarrow \mathcal{C}$ is isomorphic to the slice category ${}^{c/}\mathcal{C}$. Similarly, the comma category $\mathcal{C} \downarrow c$ is isomorphic to $\mathcal{C}_{/c}$.
- Consider the cospan $1 \xrightarrow{c} \mathcal{C} \xleftarrow{F} \mathcal{D}$. The comma category $c \downarrow F$ is the generalized slice category ${}^{c/}F$, where objects are maps $a \to Fd$ and morphisms are commutative triangles

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[d, phantom, "\#"] & \\ Fd \arrow[rr, "Fg"'] & {} & Fd'. \end{tikzcd}$$

- Consider the cospan $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^{\mathcal{J}} \xleftarrow{F} 1$. The comma category $\Delta \downarrow F$ is the category of cones over F: objects are cones $\mu\colon \Delta c \Rightarrow F$ and morphisms are commutative triangles

$$\begin{tikzcd} \Delta c \arrow[rr, "\Delta f"] \arrow[dr, "\mu"'] & \arrow[d, phantom, "\#"] & \Delta c' \arrow[dl, "\mu'"] \\ & F. & \end{tikzcd}$$

- Consider the cospan $\mathcal{C} \overset{よ}{\hookrightarrow} \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}} \xleftarrow{F} 1$. The comma category $よ \downarrow F$ is the category of elements $\int_{\mathcal{C}} F$.

Example 6.15. Let A be a quasi-category, and let $a, b\colon 1 \to A$ be objects. The comma category $\mathrm{Hom}_A(a,b)$ is defined as the pullback

$$\begin{tikzcd} \mathrm{Hom}_A(a,b) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "{(\mathrm{cod},\mathrm{dom})}"] \\ 1 \arrow[r, "{(b,a)}"'] & A \times A \end{tikzcd}$$

which we already know is a quasi-category. It then suffices to check that $\mathrm{Hom}_A(a,b)$ is an ∞-groupoid in order to check that it is a Kan complex. By 2-cell conservativity, a morphism $f\colon \mathbb{2} \to \mathrm{Hom}_A(a,b)$ is an isomorphism if and only if the whiskered composites $p_0 f, p_1 f\colon \mathbb{2} \rightrightarrows 1$ are isomorphisms, which is always the case since 1 is 2-terminal.

As one might expect, taking homotopy coherent nerves preserve mapping spaces (up to homotopy):

Theorem 6.16 (Lurie)**.** Let $\mathcal{M}$ be a Kan-complex enriched category. For every pair of objects $x, y \in \mathcal{M}$, there is a canonical map

$$\mathcal{M}(x,y) \to \mathrm{Hom}_{\mathfrak{N}(\mathcal{M})}(x,y)$$

that is a homotopy equivalence of Kan complexes.

Proof. This was originally proved in [Lur09] using a theorem of Joyal; a direct proof is given in [HK20]. ☐

The weak universal property of the comma ∞-category $f \downarrow g$ can be used to show that they "represent 2-cells between f and g" in the following way:

Proposition 6.17. Whiskering with the comma cone ϕ induces a bijection between natural transformations depicted below left

$$\left\{ \begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \right\} \cong \left\{ \begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[dd, "\lceil \alpha \rceil"] & \\ C & & B \\ & \mathrm{Hom}_A(f,g) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd} \right\}_{/\cong}$$

and fibered isomorphism classes of maps as displayed above right, where fibered isomorphisms are given by invertible 2-cells

$$\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, bend right, "\lceil \alpha' \rceil"'] \arrow[dd, bend left, "\lceil \alpha \rceil"] \arrow[dd, phantom, "\overset{\gamma}{\cong}"] & \\ C & & B \\ & \mathrm{Hom}_A(f,g) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd}$$

such that $p_0\gamma = \mathrm{id}_b$ and $p_1\gamma = \mathrm{id}_c$.

Proof. The fiber of any smothering functor is a connected groupoid: fullness and surjectivity on objects are used to lift the identity morphism in the codomain to a morphism between any two objects in the fiber, and conservativity is used to check that such a lift is an isomorphism. The action of the smothering functor (15) defines a bijection between objects of the codomain and their fibers, which are respectively the left and right sides of the bijection in the statement of the result. ☐

Proposition 6.18. (uniqueness of comma categories) Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a diagram of ∞-categories, and suppose we have a 2-cell

$$\begin{tikzcd} & E \arrow[dl, two heads, "e_1"'] \arrow[dr, two heads, "e_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

which induces a map $e := (\lceil \epsilon \rceil \colon E \to f \downarrow g)$ to the comma category by 1-cell induction. Then, e is a fibered equivalence $E \simeq_{C \times B} f \downarrow g$ if and only if the 2-cell ϵ satisfies the weak universal property described in Proposition 6.12.

Proof. First, assuming e is a fibered equivalence, we would like to show that the composite

$$\mathsf{hFun}(X, E) \underset{\sim}{\xrightarrow{e_*}} \mathsf{hFun}(X, f \downarrow g) \to \mathsf{hFun}(X, f) \downarrow \mathsf{hFun}(X, g) \tag{17}$$

is smothering. Since e_* is an equivalence which is in particular full and conservative, so is the composite, and it suffices to show that the composite is surjective on objects. Given an object

$\alpha \in \mathsf{hFun}(X, f) \downarrow \mathsf{hFun}(X, g)$, which encodes the data of a 2-cell

$$\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] & \\ C \arrow[dr, "g"'] & \arrow[l, phantom, ""] \Leftarrow \alpha & B \arrow[dl, "f"] \\ & A, & \end{tikzcd}$$

we use 1-cell induction for $f \downarrow g$ to obtain a map $\lceil \alpha \rceil \colon X \to f \downarrow g$. Then, we claim that the map

$$X \xrightarrow{\lceil \alpha \rceil} f \downarrow g \xrightarrow[\sim]{e^{-1}} E$$

maps to α under the composite (17): since the equivalence is fibered, the composite $ee^{-1}\lceil \alpha \rceil$ is fibered isomorphic to $\lceil \alpha \rceil$ and hence pastes with the universal arrow ϕ to become α.

Conversely, suppose ϵ satisfies the weak universal property. Using 1-cell induction for ϵ on the universal arrow ϕ, we obtain a map $e' := (\lceil \phi \rceil \colon f \downarrow g \to E)$. By Proposition 6.17 (used here for the universal property of E), to check that $e'e \cong \mathrm{id}_E$ over $C \times B$ it suffices to check that they compose with ϵ to the same 2-ccell, which is true by construction. The same argument using Proposition 6.17 for $f \downarrow g$ shows that $ee' \cong \mathrm{id}_{f\downarrow g}$ over $C \times B$. ☐

Using the comma categories that we just described, we can rephrase part (3) of Definition 5.12 as saying that the canonical functor

$$\mathcal{C} \downarrow \lim F \to \Delta \downarrow F$$

induced by

$$\begin{tikzcd} \mathcal{C} \arrow[r, equal] \arrow[d, equal] & \mathcal{C} \arrow[d, "\Delta"'] \arrow[r, phantom, "\lambda\Downarrow"] & 1 \arrow[l, "\lim F"'] \arrow[d, equal] \\ \mathcal{C} \arrow[r, "\Delta"'] & \mathcal{C}^{\mathcal{J}} & 1 \arrow[l, "F"] \end{tikzcd}$$

defines a fibered equivalence over $\mathcal{C}$. We will generalize this kind of universal property to the quasi-categorical setting.

Definition 6.19. A map between isofibrations of quasi-categories

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

is a *fibered equivalence* if the underlying functor f is an equivalence.

Proposition 6.20. Fibered equivalences between isofibrations are stable under pullback: given a fibered equivalence

$$\begin{tikzcd} E \arrow[rr, "\sim", "f"'] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

and a functor $g\colon A \to B$ of quasi-categories, the pullback $A \times_B f$ is an fibered equivalence.

$$\begin{tikzcd}
 & A \times_B F \arrow[r] \arrow[ddl, bend right, two heads] \arrow[d, phantom, "\lrcorner"] & F \arrow[dd, bend left, two heads, "q"] \\
A \times_B E \arrow[ur, "\sim"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[ur, "\sim", "f"'] \arrow[d, two heads, "p"'] & \\
A \arrow[r, "g"'] & B &
\end{tikzcd}$$

Proof. See [RV22, Proposition 3.3.4], taking r and q to be identity. The proof uses some nice properties of isofibrations as alluded in Remark 3.2 that we do not go into in this exposition. ☐

Specializing f to be a global element, we get the following:

Corollary 6.21. Fibered equivalences are *fiberwise equivalences*: given a fibered equivalence

$$\begin{tikzcd}
E \arrow[rr, "\sim", "f"'] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\
 & B &
\end{tikzcd}$$

and given any $b\colon 1 \to B$, the induced map $f_b\colon E_b \to F_b$ is an equivalence.

Fibered equivalences to fiberwise equivalences are as natural isomorphisms to objectwise isomorphisms: when p, q are (co)cartesian fibrations, the analogies are precise under the straightening-unstraightening correspondence. Just like how not all objectwise isomorphisms are necessarily natural, not all fiberwise equivalences can be promoted to fibered equivalences. However, it turns out that in quasi-categories (more generally, in $(\infty, 1)$-categories), fibered equivalences can be detected fiberwise. This is exactly the unstraightened version of the fact that natural isomorphisms can be detected objectwise (Theorem 3.22).

Proposition 6.22. A functor between left fibrations

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\
 & B &
\end{tikzcd}$$

is a fibered equivalence if and only if it is a fiberwise equivalence.

Proof. A more general statement is proven as [RV22, Proposition 12.2.11] for cartesian functors between cocartesian fibrations. It is straightforward to check from definition that left fibrations are cocartesian fibrations and all functors between them are cartesian. ☐

Finally, we cite a recognition theorem from [RV22] that specializes to familiar universal properties for adjunctions and limits:

Theorem 6.23 ([RV22], Theorem 3.5.8)**.** Given functor $r\colon C \to B$, $f\colon B \to A$ and $g\colon C \to A$, there is a bijection between natural transformations displayed below left and fibered isomorphism classes of maps of spans displayed below right

$$\left\{ \begin{tikzcd}
 & B \arrow[d, "f"] \\
C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A
\end{tikzcd} \right\} \cong \left\{ \begin{tikzcd}
 & B \downarrow r \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, "y"] & \\
C & & B \\
 & f \downarrow g \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] &
\end{tikzcd} \right\}_{/\cong}$$

constructed by sending ρ to the fibered isomorphism class of the functor obtained by 1-cell induction on the pasted diagram $\rho\phi$, depicted below:

$$\begin{tikzcd} & B \downarrow r \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ C \arrow[rr, "r"] \arrow[dr, "g"'] & \overset{\rho}{\Leftarrow} & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & B \downarrow r \arrow[d, dashed, "\exists y"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\ & f \downarrow g \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

Moreover, ρ displays r as an absolute right lifting of g through f if and only if the corresponding map $y\colon B \downarrow r \to f \downarrow g$ is an equivalence, in which case we say that $f \downarrow g$ is *right representable.*

Using the absolute lifting diagram encoding of an adjunction in Example 5.15 and applying Theorem 6.23 to the cospan $A \xrightarrow{\mathrm{id}_A} A \xleftarrow{f} B$ we immediately obtain the familiar universal property of adjunctions:

Corollary 6.24. A pair of functors $f\colon B \rightleftarrows A\colon u$ defines an adjunction $f \dashv u$ if and only if there exists a fibered equivalence

$$f \downarrow B \simeq_{B\times A} A \downarrow u. \tag{18}$$

Remark 6.25. By 6.21, the fibered equivalence (18) is also a fiberwise equivalence, meaning that for all $a\colon 1 \to A$ and all $b\colon 1 \to B$, the induced map

$$\mathrm{Hom}_B(fa, b) \to \mathrm{Hom}_A(a, ub)$$

is an equivalence. Naturality in a and b is encoded in the "fibered-ness" of the equivalence, and in fact comes for free by Proposition 6.22.

Similarly, applying Theorem 6.23 to the cospan $1 \xrightarrow{d} A^J \xleftarrow{\Delta} A$, we obtain the universal property for limits:

Corollary 6.26. A cone $\lambda\colon \Delta\ell \Rightarrow d$ defines a limit cone for d if and only if the induced functor

$$\begin{tikzcd} A \downarrow \ell \arrow[rr, "\lceil\lambda\rceil"] \arrow[dr] & & \Delta \downarrow d \arrow[dl] \\ & A & \end{tikzcd}$$

is a fibered equivalence, which occurs if and only if for all global elements $a\colon 1 \to A$, the pullback of $\lceil\lambda\rceil$ along a yields equivalences of mapping spaces

$$\mathrm{Hom}_A(a, \ell) \xrightarrow{\sim} \mathrm{Hom}_{A^J}(\Delta a, d).$$

6.3. **The ∞-Category of Cones.** Finally, we will generalize part (4) of Definition 5.12 to quasi-categories, which begins with a representability theorem for comma categories.

Theorem 6.27. The comma category $f \downarrow g$ associated to a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ is right representable if and only if the codomain projection functor admits a right adjoint right inverse

$$\begin{tikzcd} & f \downarrow g \arrow[dl, bend right, "p_1"'] \arrow[dr, "p_0"] & \\ C \arrow[ur, dashed, bend right, "i"'] \arrow[ur, phantom, "\perp"] & & B \end{tikzcd}$$

in which $r := p_0 i$ defines the representing functor and the natural transformation $\phi i\colon fr \Rightarrow g$ defines an absolute right lifting of g through f.

Proof. See [RV22, Theorem 3.5.12]. □

Specializing to when f is $\Delta\colon A \to A^J$ and g is $d\colon 1 \to A^J$, we obtain an alternate characterization of limits in terms of terminal cones:

Corollary 6.28. A functor $d\colon J \to A$ admits a limit if and only if the category of cones $\Delta \downarrow F$ admits a terminal element, which pastes with the universal arrow to become the limit cone.

6.4. **A lifting property to characterize limits.** In this subsection, we will give one more characterization of limit cones. For 1-categories, an alternative description of the category of cones can be given using slice categories. Given a functor $F\colon \mathcal{J} \to \mathcal{C}$ between categories, the category of cones over F can be described as the slice category $\mathcal{C}_{/F}$, which is isomorphic as categories over $\mathcal{C}$ to the comma category $\Delta \downarrow F$. We will generalize this to the quasi-category setting using slice simplicial sets (Definition 2.14). We first note that slices of quasi-categories are quasi-categories:

Proposition 6.29 (Joyal)**.** Let J be a simplicial set, let A be a quasi-category, and let $d\colon J \to A$ be a diagram. Then, $A_{/d}$ is a quasi-category.

Proposition 6.30 ([RV22, Proposition D.6.4])**.** Let J be a simplicial set, A be a quasi-category, and $d\colon J \to A$ be a diagram. Then, there is a canonical equivalence

$$\begin{tikzcd} A_{/d} \arrow[rr, "\sim"] \arrow[dr, two heads, "\mathsf{res}"'] & & \Delta \downarrow d \arrow[dl, two heads, "\mathsf{res}"] \\ & A. & \end{tikzcd}$$

In particular, $A_{/d}$ satisfies the weak universal property of the comma category (Proposition 6.12), and its elements are represented by cones

$$\begin{tikzcd} & 1 \arrow[dl, equal] \arrow[dr, "a"] & \\ 1 \arrow[dr, "d"'] \arrow[rr, phantom, "\overset{\mu}{\Leftarrow}"] & & A \arrow[dl, "\Delta"] \\ & A^J. & \end{tikzcd}$$

Specializing to the case where d is a global element and the constant diagram functor is identity on A, we get:

Corollary 6.31. Let A be a quasi-category and let $a\colon 1 \to A$ be an object. Then, there is a canonical equivalence

$$\begin{tikzcd} A_{/a} \arrow[rr, "\sim"] \arrow[dr, two heads, "\mathsf{res}"'] & & A \downarrow a \arrow[dl, two heads, "\mathsf{res}"] \\ & A. & \end{tikzcd}$$

To summarize, we have the following equivalent characterizations for a terminal object in A:

Proposition 6.32. Let A be a quasi-category. An object $t\colon 1 \to A$ is terminal in A if one of the following equivalent condition holds:

(i) There exists an adjunction

$$\begin{tikzcd} A \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1. \arrow[l, bend left, "t"] \end{tikzcd}$$

(ii) The object t is a limit over the empty diagram $\varnothing \to A$.

(iii) The domain projection functor

$$A \downarrow t \xrightarrow[\sim]{p_0} A$$

is a trivial fibration.

(iv) For all objects $a \in A$, the mapping space $\mathrm{Hom}_A(a, t)$ is contractible.

(v) The projection functor

$$A_{/t} \xrightarrow{\sim} A$$

is a trivial fibration.

(vi) Any sphere in A whose final vertex is t admits a filler: for all $n \in \mathbb{Z}_{\geq 1}$, the lifting problem

$$\begin{tikzcd} \Delta^0 \arrow[r, "\{n\}"'] \arrow[rr, bend left, "t"] & \partial\Delta^n \arrow[r] \arrow[d, hook] & A \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{19}$$

has a solution.

Proof. From the absolute lifting diagram characterization of adjunctions (Example 5.15), we get (i) if and only if (ii). By the universal property of the adjunction described in Corollary 6.24, t is a right adjoint to ! if and only if

$$\begin{tikzcd} ! \downarrow 1 \arrow[rr, "\sim"] \arrow[dr, two heads] & & A \downarrow t \arrow[dl, two heads] \\ & A & \end{tikzcd}$$

is a fibered equivalence, but $! \downarrow 1 = A$ by definition, so we get (i) if and only if (iii). Using the fact that fibered equivalences between between left fibrations are detected fiberwise (Proposition 6.22; one could convince oneself that $A \downarrow a \twoheadrightarrow A$ is a left fibration using its straightening-unstraightening correspondence to the representable functor, a non-trivial fact which is proven in [RV22, p. 5.5.14]), we get (iii) if and only if (iv). By the fibered equivalence of the two models of slice categories (Corollary 6.31), we get (iii) if and only if (v).

We will conclude by showing that (v) if and only if (vi). By definition, a map of simplicial sets is a trivial fibration if and only if it satisfies left lifting against all sphere inclusions, i.e. for all $m \in \mathbb{Z}_{\geq 0}$, the lifting problem

$$\begin{tikzcd} \partial\Delta^m \arrow[r, "\sigma"] \arrow[d, hook] & A_{/t} \arrow[d, two heads, "\wr"] \\ \Delta^m \arrow[r, "\tau"'] \arrow[ur, dashed] & A \end{tikzcd} \tag{20}$$

It suffices to show that this lifting problem is equivalent to the lifting problem (19). By the join-slice adjunction (Corollary 2.18), the data of σ is equivalent to a map

$$\tilde{\sigma} \colon \partial\Delta^m \star \Delta^0 \to A$$

such that $\tilde{\sigma}|_{\Delta^0} = t$. It is straightforward to check that $\partial\Delta^m \star \Delta^0 = \Lambda^{m+1}_{m+1}$, and that the n-simplex τ provides the data for the missing face in the outer horn. Hence, the given data in the lifting problem (20) is exactly a map

$$\begin{tikzcd} \Delta^0 \arrow[r, "\{n\}"'] \arrow[rr, bend left, "t"] & \partial\Delta^n \arrow[r] & A \end{tikzcd}$$

for $n = m + 1$ and the solution to (20) transposes to define a solution to (19). ☐

Using the analytical description of terminal objects in quasi-categories, we show that terminal objects are unique up to contractible choice:

Corollary 6.33. Let A be a quasi-category and $T \subseteq A$ be the full subcategory of terminal objects. Then, T is either empty or contractible.

Proof. If A does not admit any terminal objects, then T is empty. Suppose A admits a terminal object. To check that T is contractible, it suffices to check that the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & T \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a solution for any $n \in \mathbb{N}$. Since every object in T is terminal, by the lifting property characterization of the terminal object in Proposition 6.32 we conclude the desire result. ☐

Finally, we summarize equivalent characterizations of a limit:

Proposition 6.34. Let A be a quasi-category, J be a simplicial set and $d\colon J \to A$ be a diagram. A cone $\lambda\colon \Delta\ell \Rightarrow d$ is a limit cone for d if one of the following equivalent conditions hold:

(i) The diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "l"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3ex, "\lambda\Downarrow"] & A^J \end{tikzcd}$$

is absolute right lifting.

(ii) The functor $A \downarrow \ell \to \Delta \downarrow d$ induced by the composite diagram

$$\begin{tikzcd} & A \downarrow \ell \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ 1 \arrow[rr, "\ell" description] \arrow[dr, "d"'] & \overset{\lambda}{\Leftarrow} & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd}$$

is a fibered equivalence over A.

(iii) The functor described in (ii) induces a fiberwise equivalence: for all $a\colon 1 \to A$,

$$\mathrm{Hom}_A(a, \ell) \to \mathrm{Hom}_{A^J}(\Delta a, d)$$

is an equivalence of Kan complexes (by which we mean an isomorphism in the ∞-category of Kan complexes).

(iv) The cone $\lceil\lambda\rceil\colon 1 \to \Delta \downarrow d$ obtained by 1-cell induction is terminal.

(v) The cone $\lceil\lambda\rceil\colon 1 \to A_{/d}$ obtained by 1-cell induction is terminal.

An analytical result about quasi-categories by Joyal and Lurie can be used to show that limits are automatically functorial:

Proposition 6.35. Let A be a quasi-category and J be a simplicial set. If A admits limits for all $d\colon \mathbb{1} \to A^J$, then there exists a limit functor

$$\lim\colon A^J \to A$$

that is right adjoint to the contant diagram functor $\Delta\colon A \to A^J$.

Proof. See [RV22, Corollary 12.2.10]. ☐

7. Limits in Homotopy Coherent Nerves

So far, we have discussed the motivation behind the definition of (co)limits, and we have shown that they satisfy the expected universal property. However, we have not yet presented any non-trivial examples. In the ∞-categorical setting, showing that an object satisfies the universal property of the (co)limit requires checking that certain hom-spaces are equivalent, which is much less straightforward than checking that certain hom-sets are bijective in the 1-categorical setting. Our goal in this section will be to identify explicit models for ∞-categorical (co)limits in quasi-categories that arise as the homotopy coherent nerves of Kan-complex enriched categories, following [RV20].

We first recall the notion of weighted (co)limits, the "correct" notion of (co)limits in the enriched setting. We will consider the general case where we enrich over any closed symmetric monoidal category $\mathcal{V}$, but in all applications we will take $\mathcal{V} = \mathsf{sSet}$.

Definition 7.1. Let $\mathcal{D}, \mathcal{M}$ be $\mathcal{V}$-categories, and $F \colon \mathcal{D} \to \mathcal{M}$ be a $\mathcal{V}$-functor. Let $W \colon \mathcal{D} \to \mathcal{V}$ be a $\mathcal{V}$-functor, which we call the weight. A W-weighted limit over F is an object $\lim^W F$ in $\mathcal{M}$ along with isomorphisms

$$\mathcal{M}(m, \lim{}^W F) \cong \mathcal{V}^{\mathcal{D}}(W, \mathcal{M}(m, F))$$

natural in $m \in \mathcal{M}$. We say that a simplicial natural transformation $W \to \mathcal{M}(m, F)$ is a W-cone over F with apex m.

Remark 7.2. Suppose $\mathcal{M}$ is cotensored over $\mathcal{V}$, in the sense that for all $v \in \mathcal{V}$ and all $m, n \in \mathcal{M}$, there exists some $m^v \in \mathcal{M}$ such that

$$\mathcal{V}(v, \mathcal{M}(n, m)) \cong \mathcal{M}(n, m^v)$$

naturally. Then, we can write the universal property of the weighted limit to write

$$\begin{aligned}
\mathcal{M}(m, \lim{}^W F) &\cong \mathcal{V}^{\mathcal{D}}(W, \mathcal{M}(m, F) \\
&\cong \int_{d \in \mathcal{D}} \mathcal{V}(Wd, \mathcal{M}(m, Fd)) \\
&\cong \int_{d \in \mathcal{D}} \mathcal{M}(m, Fd^{Wd}) \\
&\cong \mathcal{M}\left(m, \int_{d \in \mathcal{D}} Fd^{Wd}\right)
\end{aligned}$$

using enriched ends and coends. By the (enriched) Yoneda lemma, we get the formula

$$\lim{}^W F \cong \int_{d \in \mathcal{D}} Fd^{Wd}. \tag{21}$$

Dually, we have the formula

$$\operatorname{colim}{}^W F \cong \int^{d \in \mathcal{D}} Wd \otimes Fd \tag{22}$$

for weighted colimits when $\mathcal{M}$ is tensored over $\mathcal{V}$, meaning that for all $v \in \mathcal{V}$ and all $m, n \in \mathcal{M}$, there exists some $v \otimes m \in \mathcal{M}$ such that

$$\mathcal{V}(v, \mathcal{M}(m, n)) \cong \mathcal{M}(v \otimes m, n).$$

The weight of the (co)limit describes the "shape" of the cones over the diagram. By "fattening up" the weight, we are able to describe some sort of homotopy coherent cones that are homotopically well behaved. Our goal is to describe a particular weight $W_X \colon \mathfrak{C}[X] \to \mathsf{sSet}$ for all simplicial sets X such that W_X-weighted limit cones for functors $F \colon \mathfrak{C}[X] \to \mathcal{M}$ correspond to ∞-categorical limit cones for the transposed functor $F \colon X \to N_{hc}(\mathcal{M})$. To do this, we first describe an alternate way to present a simplicial weight $W \colon \mathcal{M} \to \mathsf{sSet}$ as a simplicial category, which we call its collage.

Definition 7.3. Given a weight $W: \mathcal{D} \to \mathsf{sSet}$, the collage of W is a simplicial category $\mathrm{coll}(W)$ that contains $\mathcal{D}$ as a full subcategory along with precisely one extra object $\perp$ whose endomorphism space is the point. For all $d \in \mathcal{D}$, the simplicial set $\mathsf{Map}_{\mathrm{coll}(W)}(d, \perp)$ is taken to be empty, and we define

$$\mathsf{Map}_{\mathrm{coll}(W)}(d, \perp) := Wd.$$

The composition operations

$$Wd \times \mathsf{Map}_{\mathcal{M}}(d, d') \to Wd'$$

are given by the transpose of the action of the functor W.

The collage construction gives us an equivalent way to describe simplicial weights, by the following proposition:

Proposition 7.4. The collage construction defines a fully faithful functor

$$\mathsf{sSet}^{\mathcal{D}} \xhookrightarrow{\mathrm{coll}} {}^{\mathbb{1}+\mathcal{D}/}\mathsf{sSet\text{-}Cat}$$

whose essential image is comprised of the simplicial functors $F: \mathbb{1} + \mathcal{M} \to \mathcal{C}$ that are bijective on objects, fully faithful on $\mathbb{1} + \mathcal{M}$ and have the property that any arrow with codomain $F\perp$ is the identity. Furthermore, the collage functor coll admits a (1-categorical) right adjoint

$${}^{\mathbb{1}+\mathcal{D}/}\mathsf{sSet\text{-}Cat} \xrightarrow{\mathrm{wgt}} \mathsf{sSet}^{\mathcal{D}}$$

which sends a simplicial functor $F: \mathbb{1} + \mathcal{D} \to \mathcal{C}$ to $\mathsf{Map}_{\mathcal{C}}(F\perp, F-)$.

Proof. See [RV20, Proposition 5.2.3] $\square$

Corollary 7.5. Functors $\mathrm{coll}(W) \to \mathcal{M}$ that restrict to m on $\mathbb{1}$ and F on $\mathcal{D}$ correspond to W-cones over F with apex m, i.e. natural transformations $W \to \mathcal{M}(m, F)$.

Proof. Use the adjunction in Proposition 7.4 and notice that $\mathrm{wgt}\,\mathcal{M}$, considering the composite $\mathbb{1} + \mathcal{D} \to \mathrm{coll}(W) \to \mathcal{M}$ as a category under $\mathbb{1} + \mathcal{D}$, is exactly the functor $\mathcal{M}(m, F): \mathcal{D} \to \mathsf{sSet}$. $\square$

Now, given a simplicial set X, the shape of cones for functors out of X in the quasi-category setting is given by functors out of the simplicial join $\Delta^0 \star X$. Taking the homotopy coherent realization of the simplicial set inclusion $\Delta^0 + X \hookrightarrow \Delta^0 \star X$, we get a inclusion of simplicial categories $\mathbb{1} + \mathfrak{C}X \hookrightarrow \mathfrak{C}[\Delta^0 \star X]$. By the counit isomorphism of the adjunction in Proposition 7.4, this simplicial category under $\mathbb{1} + \mathfrak{C}X$ is isomorphic to the collage of its corresponding weight, which we denote by $W_X: \mathfrak{C}X \to \mathsf{sSet}$.

Definition 7.6. We call the weight W_X the weight for the pseudo limit of a homotopy coherent diagram of shape X, and we call a W_X-weighted limit of such a diagram $F: \mathfrak{C}X \to \mathcal{M}$ the pseudo limit of the diagram. Explicitly, the functor W_X sends a vertex $x \in X$ to the Kan complex $\mathfrak{C}[\Delta^0 \star X](\perp, x)$, where $\perp$ denotes the cone point Δ^0 in $\Delta^0 \star X$.

It turns out that the pseudo limit gives us one model of the ∞-categorical limit, in the following sense:

Theorem 7.7 ([RV20], Theorem 6.1.4)**.** Let $\mathcal{M}$ be a Kan-complex enriched category, let X be a simplicial set, and let $F: \mathfrak{C}[X] \to \mathcal{M}$ be a homotopy coherent diagram. If F admits a pseudo limit in $\mathcal{M}$, then the W_X-weighted limit cone

$$\Lambda: \mathfrak{C}[\Delta^0 \star X] \to \mathcal{M}$$

transposes to define a limit cone over the transposed diagram

$$F: X \to N_{hc}(\mathcal{M}).$$

7.1. **Examples in Topological Spaces.** To get some intuition of what ∞-categorical (co)limits look like, we will apply Theorem 7.7 in the case when $\mathcal{M}$ is the familiar category of topological spaces. We will denote Top for the category of cgwh (compactly generated weakly Hausdorff) spaces, which is cartesian closed. We will use $\mathsf{Map}(-,-)$ to denote the internal hom of Top (the set of continuous maps with the compact open topology). In Example 4.7, we saw that base change using the singular complex map $S_\bullet\colon \mathsf{Top} \to \mathsf{Kan}$ realizes Top as a Kan-complex enriched category, where the enriched hom, which we denote by $\mathsf{Top}(-,-)$, is the simplicial set

$$\mathsf{Top}(X,Y) := S_\bullet \mathsf{Map}(X,Y)$$

for $X, Y \in \mathsf{Top}$. Our goal will be to use the formulas (21) and (22) to explicitly write down our desired pseudo (co)limits, for which we need simplicial (co)tensors in Top.

Proposition 7.8. The simplicial category Top admits tensors and cotensors: given a simplicial set K and a topological space X, the tensor is given by

$$K \otimes X \cong |K| \times X,$$

and the cotensor is given by

$$X^K \cong \mathsf{Map}(|K|, X).$$

Proof. First, we notice that the singular complex $\vdash$ geometric realization adjunction can be promoted to an enriched adjunction of simplicial categories, i.e. for all simplicial sets K and topological spaces Z, we have

$$\mathsf{sSet}(K, S_\bullet X) \cong \mathsf{Top}(|K|, X).$$

We check this isomorphism at the level of simplices: for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathsf{sSet}(K, S_\bullet X)_n &\cong \underline{\mathsf{sSet}}(K \times \Delta^n, S_\bullet X) \\
&\cong \underline{\mathsf{Map}}(|K \times \Delta^n|, X) \\
&\cong \underline{\mathsf{Map}}(|K| \times |\Delta^n|, X) \\
&\cong \underline{\mathsf{Map}}(|\Delta^n|, \mathsf{Map}(|K|, X)) \\
&\cong \underline{\mathsf{sSet}}(\Delta^n, S_\bullet \mathsf{Map}(|K|, X)) \\
&\cong S_n \mathsf{Map}(|K|, X) \\
&\cong \mathsf{Top}(|K|, X)_n
\end{aligned}$$

where crucially we used the fact that geometric realization preserves products (note that we use underline to denote the underlying set of the enriched homs). Now, for any topological spaces X, Y and simplicial sets K, we have

$$\begin{aligned}
\mathsf{sSet}(K, \mathsf{Top}(X,Y)) &= \mathsf{sSet}(K, S_\bullet \mathsf{Map}(X,Y)) \\
&\cong \mathsf{Top}(|K|, \mathsf{Map}(X,Y)) \\
&\cong \mathsf{Top}(|K| \times X, Y)) \\
&\cong \mathsf{Top}(X, \mathsf{Map}(|K|, Y))
\end{aligned}$$

where the last two lines exhibit the universal properties of our candidates for tensors and cotensors respectively. $\square$

Now, we are finally ready to see some examples of ∞-categorical (co)limits in spaces. We will start with a few small examples of colimits since they are easier to visualize.

Example 7.9 (coproduct)**.** Take $\mathcal{D} = \mathfrak{C}[\Delta^0 \amalg \Delta^0] = \mathbb{1} \amalg \mathbb{1}$. A functor $\mathcal{D} \to \mathsf{Top}$ simply encodes the data of two topololgical spaces X, Y. The collage of the weight $W := W_{\Delta^0 \amalg \Delta^0}$ is the geometric realization of Λ^2_2, which we may depict as

$$\begin{tikzcd} 0 \arrow[dr] & & 1 \arrow[dl] \\ & \top & \end{tikzcd} .$$

The corresponding weight is simply the trivial one (that sends all objects in $\mathcal{D}$ to $*$), so the pseudo colimit is simply the ordinary colimit.

Example 7.10 (mapping cylinder)**.** Take $\mathcal{D} = \mathfrak{C}[\Delta^1]$. A functor $F\colon \mathfrak{C}[\Delta^1] \to \mathsf{Top}$ encodes the data of a continuous map $f\colon X \to Y$ between topological spaces. If we took the 1-categorical colimit of this functor, we would simply get Y: a cocone under F is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

which is determined by the map g. The ∞-categorical colimit is more interesting. A map from the ∞-categorical colimit corresponds to a homotopy *coherent* cocone under F

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

as a W_{Δ^1}-weighted cocone is given by a functor from $\mathfrak{C}[\Delta^1 \star \Delta^1] = \mathfrak{C}[\Delta^2]$ into Top. Denote $W := W_{\Delta^1}$ for simplicity. Using formula (22), the pseudo colimit can be written as

$$\operatorname{colim}^W F \cong \int^{i \in \mathfrak{C}[\Delta^1]} |Wi| \times Fi.$$

Let's first compute what $W\colon \mathfrak{C}[\Delta^1]^{\mathsf{op}} \to \mathsf{sSet}$ is. Recall that the collage of W is the category $\mathfrak{C}[\Delta^2]$, which we can depict as

$$\begin{tikzcd} 0 \arrow[rr, "10"] \arrow[dr, "\top 0"'] & \arrow[d, phantom, "\Rightarrow"] & 1 \arrow[dl, "\top 1"] \\ & \top & \end{tikzcd}$$

labelling arrows by "codomain domain". Note that morphisms in homotopy coherent realizations are given by strings of composable arrows, which we will notate by using "." to separate atomic arrows in the string. Using Proposition 7.4, we see that the weight maps the object 0 to

$$\mathfrak{C}[\Delta^2](0, \top) \cong \{\top 0 \Rightarrow \top 1.10\} \cong \Delta^1,$$

the object 1 to

$$\mathfrak{C}[\Delta^2](1, \top) \cong \{\top 1\} \cong \Delta^0,$$

and the morphism $0 \to 1$ to precomposition by 10, which in this case gives us the boundary map d^0. Now, we can write the pseudo colimit as the pushout

$$\begin{tikzcd} {|W_1| \times F_0} \arrow[r] \arrow[d] & {|W_1| \times F_1} \arrow[d] \\ {|W_0| \times F_0} \arrow[r] & \operatorname{colim}^W F \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} = \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_1"'] & Y \arrow[d] \\ {|\Delta^1| \times X} \arrow[r] & \operatorname{colim}^W F \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

which is the familiar mapping cylinder construction from topology.

Example 7.11 (double mapping cylinder)**.** Take $\mathcal{D} := \mathfrak{C}[\Lambda^2_1] \cong \mathfrak{C}[\Delta^1 \amalg_{\Delta^0} \Delta^1]$, and a functor $F \colon \mathcal{D} \to \mathsf{Top}$ given by continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

The collage of the weight $W := W_{\Lambda^2_1}$ is the rigidification of the simplicial set

$$(\Delta^1 \amalg_{\Delta^0} \Delta^1) \star \Delta^0 \cong (\Delta^1 \star \Delta^0) \amalg_{(\Delta^0 \star \Delta^0)} (\Delta^1 \star \Delta^0) \cong \Delta^2 \amalg_{\Delta^1} \Delta^2$$

(since joins preserve connected colimits) which looks like

$$\begin{tikzcd}
 & 1 \arrow[dr] \arrow[dd] & \\
0 \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Rightarrow \qquad \Rightarrow"] & & 2 \arrow[dl] \\
 & \top &
\end{tikzcd}$$

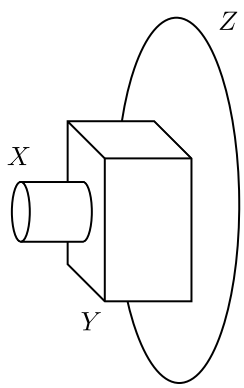
so our weight W has the following mappings:

$$\begin{tikzcd}
0 \arrow[d] & \mapsto & \Lambda^2_1 \\
1 \arrow[d] & \mapsto & \Delta^1 \arrow[u, "d^0"'] \\
2 & \mapsto & \Delta^0. \arrow[u, "d^0"']
\end{tikzcd}$$

The pseudo colimit is given by the colimit under the diagram

$$\begin{tikzcd}
 & {|\Delta^1| \times X} \arrow[dl, "d^0"'] \arrow[dr, "{|\Delta^1| \times f}"] & & Y \arrow[dl, "d^0"'] \arrow[dr, "g"] & \\
{|\Lambda^2_1| \times X} & & {|\Delta^1| \times Y} & & Z
\end{tikzcd}$$

that looks something like



which we may recognize as the classical double mapping cylinder.

Example 7.12 (mapping telescope)**.** We generalize the double mapping cylinder in Example 7.11 to arbitrary sequence of continuous maps

$$X_0 \xrightarrow{f_{1,0}} X_1 \xrightarrow{f_{2,1}} X_2 \to \ldots \to X_n \to \ldots$$

that may or may not stabilize (i.e. $f_{i,i-1}$ is the identity map for all $i \geq k$ for some $k \in \mathbb{N}$). The indexing simplicial set is

$$S := \Delta^1 \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} \ldots = \operatorname{coeq}\left(\coprod_{n \in \mathbb{N}} \Delta^0_{(n)} \rightrightarrows \coprod_{k \in \mathbb{N}} \Delta^1_{(k)} \right)$$

where the two maps in the coequalizer are given on $\Delta^0_{(n)}$ by

$$\begin{tikzcd} \Delta^0_{(n)} \arrow[r, "d^0"] \arrow[dr, "d^1"'] & \Delta^1_{(n)} \\ & \Delta^1_{(n+1)}. \end{tikzcd}$$

The collage of the weight $W := W_S$ is the rigidification of

$$\operatorname{coeq}\left(\coprod_{n \in \mathbb{N}} \Delta^1_{(n)} \rightrightarrows \coprod_{k \in \mathbb{N}} \Delta^2_{(k)} \right)$$

which looks like

$$\begin{tikzcd} 0 \arrow[r] \arrow[drr] & 1 \arrow[r] \arrow[dr] & 2 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[dl] & \ldots \\ & & \top & \ldots & \end{tikzcd}$$

with right-pointing 2-arrows in each of the triangular cells bound by n, $n+1$ and $\top$. The weight W_S maps each $n \in \mathbb{N}$ to

$$\mathsf{Map}(n, \top) = \{n \to n+1 \to n+2 \to \ldots\} \cong S$$

and maps $n \to n+1$ to inclusion maps

$$\{n+1 \to n+2 \to \ldots\} \hookrightarrow \{n \to n+1 \to n+2 \to \ldots\} \tag{23}$$

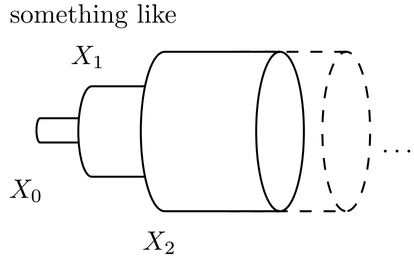
i.e. shift maps $S \hookrightarrow S$. We can realize its geometric realization as the half open interval $|\mathsf{Map}(n, \top)| \cong [n, \infty)$ so that the inclusion maps (23) are mapped to the inclusions

$$[n+1, \infty) \subseteq [n, \infty).$$

The pseudo colimit is thus the topological space

$$\left(\coprod_{n \in \mathbb{N}} [n, \infty) \times X_n \right) / \sim$$

where the equivalence relation $\sim$ identifies $[n+1, \infty) \times X_n$ with its image under $\mathrm{id} \times f_{n+1,n}$ in $[n+1, \infty) \times X_{n+1}$, which looks something like



where we have an infinitely expanding "telescope".

Example 7.13 (homotopy pushout)**.** Take $\mathcal{D} = \mathfrak{C}[\Lambda^2_0]$, the cospan category, which we will denote by $1 \leftarrow 0 \rightarrow 2$. A functor $\mathcal{D} \to \mathsf{Top}$ is some cospan of spaces $Y \xleftarrow{f} X \xrightarrow{g} Z$. The cone shape for the pseudo colimit is the simplicial set $\Lambda^2_0 \star \Delta^0 \cong \Delta^1 \times \Delta^1$, the realization of which we depict as follows: (omitting labels for arrows, which are always "codomain domain"):

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=2, "\Nearrow"] \arrow[dr, phantom, shift right=2, "\Swarrow"] & 2 \arrow[d] \\ 1 \arrow[r] & \top. \end{tikzcd}$$

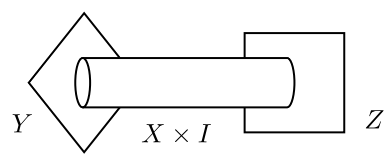
The corresponding weight $W\colon \mathfrak{C}[\Lambda^2_0] \to \mathsf{sSet}$ sends the object 0 to

$$\mathfrak{C}[\Delta^1 \times \Delta^1](0, \top) = \left\{ \begin{tikzcd} \top 0 \arrow[r] \arrow[d] & \top 1.10 \\ \top 2.20 & \end{tikzcd} \right\} \cong \Lambda^2_0,$$

the object 1 to $\{\top 1\} \cong \Delta^0$ and the object 2 to $\{\top 2\} \cong \Delta^0$; the morphisms $0 \to 1$ and $0 \to 2$ are sent to the two endpoint inclusions $\Delta^0 \to \Lambda^2_0$. Using the coend formula, we get that the pseudo colimit for the cospan is the colimit under the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "\iota_1"] & & X \arrow[dl, "\iota_2"'] \arrow[dr, "g"] & \\ Y & & {|\Lambda^2_0| \times X} & & Z \end{tikzcd}$$

looking something like



which we may recognize as the homotopy pushout.

Example 7.14 (mapping cone)**.** Specializing Example 7.13 to when $Z = *$, the pseudo colimit is obtained by gluing one end of the cylinder $I \times X$ (letting $I := [0,1]$ denote the topological interval) to Y and identifying the other end to a point. This construction gives us the mapping cone, or the homotopy cofiber.

Example 7.15 (suspension)**.** Specializing Example 7.13 even more to when $Y = Z = *$, the pseudo colimit is obtained by identifying both ends of the cylinder $I \times X$, which gives us the suspension of X.

Now let's see some examples of limits.

Example 7.16 (product)**.** Similar to Example 7.9, the homotopy product is just the ordinary product.

Example 7.17. Take $\mathcal{D} = \mathfrak{C}[\Delta^1]$ again. The cone shape is once again Δ^2, but we will depict it as

$$\begin{tikzcd} & \bot \arrow[dl, "0\bot"'] \arrow[dr, "1\bot"] \arrow[d, phantom, "\Leftarrow"] & \\ 0 \arrow[rr, "10"'] & {} & 1 \end{tikzcd}$$

since we now want cones and not cocones. The weight for the pseudo limit is a functor $W\colon \mathfrak{C}[\Delta^1] \to \mathsf{sSet}$ sending the object 0 to $\{0\bot\} \cong \Delta^0$, the object 1 to $\{1\bot \Rightarrow 10.0\bot\} \cong \Delta^1$, and the morphism

$0 \to 1$ to the codomain inclusion map $d^0 \colon \Delta^0 \to \Delta^1$. The psuedo limit is the pullback

$$\begin{tikzcd} \lim^W F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_1^{W_1} \arrow[d] \\ F_0^{W_0} \arrow[r] & F_1^{W_0} \end{tikzcd} \quad = \quad \begin{tikzcd} \lim^W F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "\pi_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

which we can describe as the subspace of $X \times Y^I$ consisting of the points

$$\{(x, p \colon I \to Y) : p(1) = f(x)\}.$$

In other words, this is the space of paths in Y that end in the image of X.

Example 7.18 (homotopy pullback)**.** Take $\mathcal{D} = \mathfrak{C}[\Lambda^2_2]$, which we denote by $0 \to 2 \leftarrow 1$. We depict the cone shape $\Delta^0 \star \Lambda^2_2 \cong \Delta^1 \times \Delta^1$ as follows:

$$\begin{tikzcd} \perp \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\Nearrow", shift left=2] \arrow[dr, phantom, "\Swarrow", shift right=2] & 1 \arrow[d] \\ 0 \arrow[r] & 2 \end{tikzcd}$$

Similar to the case for the homotopy pushout, the weight for this pseudo limit is the functor $W \colon \mathfrak{C}[\Lambda^2_2] \to \mathsf{sSet}$ sending the objects 0 and 1 to Δ^0, the object 2 to Λ^0_2, and the two morphisms $0 \to 2$ and $1 \to 2$ to the two endpoint inclusions. Given a span $X \xrightarrow{f} Y \xleftarrow{g} Z$, the pseudo limit is the limit over the diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y^I \arrow[dl, "p_0"] \arrow[dr, "p_1"'] & & Z \arrow[dl, "g"] \\ & Y & & Y & \end{tikzcd}$$

This is the subspace of $X \times Y^I \times Z$ consisting of triples (x, α, z) such that the path α starts at x and ends at z.

Example 7.19 (loop space)**.** Specializing Example 7.18 to the case where $X = Z = *$ and $f = g = \{y\}$ for a point $y \in Y$, we see that the pseudo limit is the subspace of paths that start and end at y, in other words, the loop space $\Omega(Y, y)$.

8. Classical Perspectives on Homotopy Limits

We now have a more concrete description of (co)limits in Kan-complex enriched categories, and we have seen some examples of familiar constructions in the category of topological spaces. To give ourselves more computational tools, we will turn to look at the classical topic of homotopy (co)limits, which turn out to give us models for ∞-categorical (co)limits in nice settings.

8.1. **Derived Functors.**

In this section, we will briefly recall the theory of derived functors on homotopical categories, following [Rie14], which we will use to define (traditional) homotopy limits.

Definition 8.1. A *homotopical category* is a category $\mathcal{M}$ with a distinguished subcategory $\mathcal{W} \subseteq \mathcal{M}$ that contains all object and satsifies the 2-of-6 property: given composable morphisms

$$w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z,$$

if hg and gf are in $\mathcal{W}$, then so are f, g, h, hgf. A *homotopical functor* between homotopical categories is one that preserves weak equivalences.

Definition 8.2. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor. A total right derived functor for F is a left Kan extension

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Uparrow"] & \mathcal{N} \arrow[d, "\delta"] \\ \mathsf{Ho}\mathcal{M} \arrow[r, dashed, "\boldsymbol{L}F"'] & \mathsf{Ho}\mathcal{N} \end{tikzcd}$$

(note the opposing convention in handedness). Dually, a total left derived functor is a right Kan extension.

A total derived functor can be equivalently described as a homotopical functor from $\mathcal{M} \to \mathsf{Ho}\mathcal{N}$. In some situations, we can lift this to an approximation $\mathcal{M} \to \mathcal{N}$, which we will simply call a *derived functor*.

Definition 8.3. A right derived functor for $F\colon \mathcal{M} \to \mathcal{N}$ is a homotopical functor $\mathbb{R}F\colon \mathcal{M} \to \mathcal{N}$ with a natural transformation $\lambda\colon F \Rightarrow \mathbb{R}F$ such that

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\delta\lambda\Downarrow"] & \mathcal{N} \arrow[d, "\delta"] \\ \mathsf{Ho}\mathcal{M} \arrow[r, "\delta\mathbb{R}F"'] & \mathsf{Ho}\mathcal{N} \end{tikzcd}$$

is a total right derived functor. As before, one can easily dualize to define right derived functors.

Remark 8.4. Let $\mathbb{R}F, \mathbb{R}'F$ be two right derived functors for $F\colon \mathcal{M} \to \mathcal{N}$. By the uniqueness of Kan extensions, we know that the total derived functors $\delta\mathbb{R}F$ and $\delta\mathbb{R}'F$ are naturally isomorphic. If $\mathcal{N}$ is saturated (meaning that weak equivalences are exactly the maps that become isomorphisms in the homotopy category — in particular all model categories are saturated), then we know that for all $m \in \mathcal{M}$, there exists some weak equivalence $\mathbb{R}Fm \xrightarrow{\sim} \mathbb{R}'Fm$. We do not necessarily know, however, that $\mathbb{R}F$ and $\mathbb{R}'F$ themselves are isomorphic, and we will often see examples where we have different point set level models for the same derived functors.

Example 8.5. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a homotopical functor between homotopical categories. We claim that F is a left and right derived functor for itself with the identity natural transformation. Indeed, let $\tilde{F}\colon \mathsf{Ho}\mathcal{M} \to \mathsf{Ho}\mathcal{N}$ be the factorization

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[d, "\delta"] \\ \mathsf{Ho}\mathcal{M} \arrow[r, dashed, "\exists!\tilde{F}"'] & \mathsf{Ho}\mathcal{N} \end{tikzcd}$$

arising from the universal property of $\mathsf{Ho}\mathcal{M}$. We have

$$\begin{aligned} \mathsf{Ho}\mathcal{N}^{\mathsf{Ho}\mathcal{M}}(\mathbb{R}F, G) &\cong \mathsf{Ho}\mathcal{N}^{\mathcal{M}}(\delta F, G\gamma) && \text{(Kan extension)} \\ &= \mathsf{Ho}\mathcal{N}^{\mathcal{M}}(\tilde{F}\gamma, G\gamma) \\ &\cong \mathsf{Ho}\mathcal{N}^{\mathsf{Ho}\mathcal{M}}(\tilde{F}, G) && \text{(universal property)} \end{aligned}$$

naturally in $G\colon \mathsf{Ho}\mathcal{M} \to \mathsf{Ho}\mathcal{N}$, so $\tilde{F} \cong \boldsymbol{L}F$ is a total right derived functor, and F is a right derived functor for itself. The same argument applies for the dual case.

Theorem 8.6 ([DHKS04])**.** Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between homotopical categories, and let $R\colon \mathcal{M} \to \mathcal{M}$ be an endofunctor equipped with a natural weak equivalence $q\colon 1 \xRightarrow{\sim} R$ which we call a *right deformation*. Let $\mathcal{M}_R \subseteq \mathcal{M}$ be the full subcategory containing the image of R, and suppose F is homotopical on $\mathcal{M}_R$. Then, $\mathbb{R}F = FR$ is a right derived functor of F.

Example 8.7. Suppose $\mathcal{M}$ admits a model structure. By Ken Brown's lemma (see [Rie14, Lemma 2.2.6]), a right Quillen functor $F\colon \mathcal{M} \to \mathcal{N}$ preserves weak equivalences between fibrant objects. Hence, fibrant replacement is a right deformation for F, making FR a right derived functor of F.

Now that we have a general formalism for computing derived functors, we define homotopy limits as simply the derived functor of the limit functor

Definition 8.8. Let $\mathcal{M}$ be a homotopical category, and let $\mathcal{D}$ be a small category. We consider the functor category $\mathcal{M}^{\mathcal{D}}$ as a homotopical category, where the weak equivalences are given pointwise. Then, any right derived functor of $\lim\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$ is called a *homotopy limit* functor, denoted by holim, and any left derived functor of $\operatorname{colim}\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$ is called a *homotopy colimit* functor, denoted by hocolim.

Example 8.9. If the functor category admits a model structure (e.g. projective, injective, Reedy, etc.), then we may compute homotopy (co)limits by (co)fibrantly replacing. This is convenient when we have a good description of what cofibrant and fibrant objects are in the functor model structure. For instance, this allows us to deduce that pushouts of cofibrant objects along a cofibration are homotopy pushouts (using the Reedy model structure on the span category.

Example 8.10. (Bousfield-Kan formula) The flexibility of the deformation approach to derived functors is that it works for any homotopical categories, not just ones admitting a model structure. This is helpful when computing homotopy (co)limits, since there isn't always an obvious model structure on functor categories. One deformation that always works in the setting of a simplicial model category is the classical Bousfield-Kan formula. See

8.2. **Homotopy (Co)limits in a Simplicial Model Category.** In Section 7, we described a model for ∞-categorical (co)limits in homotopy coherent nerves of Kan-complex enriched categories. In this section, we will show that the pseudo (co)limits we defined model homotopy (co)limits in the derived functor sense, in the setting of simplicial model categories where we have access to both the structure of hom-spaces as well as the structure of fibrations and cofibrations.

Theorem 8.11 ([Gam10])**.** Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{D}$ be a small simplicial category. The weighted limit bifunctor

$$(\mathsf{sSet}^{\mathcal{D}})^{\mathsf{op}} \times \mathcal{M}^{\mathcal{D}} \xrightarrow{\lim^{-}(-)} \mathcal{M}$$

is right Quillen when we endow the domain with either the (projective, projective) or (injective, injective) model structure.

Dually, the weighted colimit bifunctor

$$\mathsf{sSet}^{\mathcal{D}^{\mathsf{op}}} \times \mathcal{M}^{\mathcal{D}} \xrightarrow{\operatorname{colim}^{-}(-)} \mathcal{M}$$

is left Quillen when we endow the domain with the (projective, injective) or (injective, projective) model structure.

We will present all the results in this section for the case of limits and leave the reader to dualize arguments for the case of colimits.

Corollary 8.12. Let $W\colon \mathcal{D} \to \mathsf{sSet}$ be a weight that is projectively cofibrant and pointwise contractible, i.e. weakly equivalent to $*$. A model for the homotopy limit functor is given by

$$\mathcal{M}^{\mathcal{D}} \xrightarrow{R_*} \mathcal{M}_f^{\mathcal{D}} \xrightarrow{\lim^W} \mathcal{M}$$

by first taking pointwise fibrant replacement of the diagram, then taking limit weighted by W.

Proof. The functor

$$* \times \mathrm{id}\colon \mathcal{M}^{\mathcal{D}} \to (\mathsf{sSet}^{\mathcal{D}})^{\mathsf{op}} \times \mathcal{M}^{\mathcal{D}}$$

sending a functor F to the pair $(*, F)$, where $*$ denotes the constant diagram at $*$, preserves weak equivalences, fibrations and cofibrations by the definition of the product model structure. The limit functor is the composite

$$\mathcal{M}^{\mathcal{D}} \xrightarrow{*\times \mathrm{id}} (\mathsf{sSet}^{\mathcal{D}})^{\mathsf{op}} \times \mathcal{M}^{\mathcal{D}} \xrightarrow{\lim^{-}(-)} \mathcal{M}$$

and by the pseudofunctoriality of taking derived functors, we obtain our desired derived functor as a composite of the derived functors for $* \times \mathrm{id}$ and lim. Since $* \times \mathrm{id}$ is homotopical, it is a derived functor for itself (Example 8.5). By Theorem 8.11, a derived functor for the weighted limit is given by fibrant replacement in the (projective, projective) model structure. Fibrant replacement in $\mathcal{M}^{\mathcal{D}}_{\mathsf{proj}}$ is given simply by pointwise fibrant replacement of the objects. Fibrant replacement in $(\mathsf{sSet}^{\mathcal{D}}_{\mathsf{proj}})^{\mathsf{op}}$ is given by cofibrant replacement in $\mathsf{sSet}^{\mathcal{D}}_{\mathsf{proj}}$, and for conical limits weighted by $*$, such a replacement is given by W. $\square$

Proposition 8.13. Let $\mathcal{M}$ be a simplicial model category and let X be a simplicial set. Let $\mathcal{D} := \mathfrak{C}[X]$ be the homotopy coherent realization of X and $F\colon \mathcal{D} \to \mathcal{M}$ be a simplicial functor. Let $RQ\colon \mathcal{M} \to \mathcal{M}_{cf}$ be a bifibrant replacement functor. Then, we have a model of the homotopy limit given by

$$\operatorname{holim} F \simeq \lim{}^{W_X} RQF.$$

Proof. Since homotopy limit is invariant under weak equivalences, we have

$$\operatorname{holim} F \simeq \operatorname{holim} RQF.$$

We can then assume, without loss of generality, that F lands in the bifibrant subcategory $\mathcal{M}_{cf}$, and the statement we would like to check reduces to

$$\operatorname{holim} F \simeq \lim{}^{W_X} F.$$

By Corollary 8.12, this reduces to checking that W_X is projectively cofibrant and pointwise contractible, which is the content of Corollary 8.21. $\square$

Since homotopy (co)limits always exist in simplicial model categories (for instance, using the Bousfield-Kan formulas), we have:

Corollary 8.14. Underlying quasi-categories of simplicial model categories are complete and cocomplete.

By Dugger's theorem (see [Dug01]), every model category is Quillen equivalent to a simplicial one. We can thus generalize Proposition 8.13 to all model categories by checking that Quillen equivalences preserve homotopy limits:

Lemma 8.15. Let $G\colon \mathcal{M} \to \mathcal{N}$ be the left or right adjoint in an Quillen equivalence with inverse K, and let $F\colon \mathcal{D} \to \mathcal{M}$ be a functor. Then, we have a weak equivalence

$$G \operatorname{holim} F \simeq \operatorname{holim} GF$$

natural in F.

Proof. Using the fact that Quillen equivalences induce equivalences on homotopy categories, we can write down the following diagram of homotopy categories:

$$\begin{tikzcd} \mathsf{Ho}(\mathcal{N}^{\mathcal{D}}) \arrow[r, shift right, "\mathsf{Ho}(K_*)"'] \arrow[r, phantom, "\simeq"] & \mathsf{Ho}(\mathcal{M}^{\mathcal{D}}) \arrow[l, shift right, "\mathsf{Ho}(G_*)"'] \arrow[r, shift right, "\operatorname{holim}"'] \arrow[r, phantom, "\perp"] & \mathsf{Ho}\mathcal{M} \arrow[l, shift right, "\mathsf{Ho}\Delta"'] \arrow[r, shift right, "\mathsf{Ho}G"'] \arrow[r, phantom, "\simeq"] & \mathsf{Ho}\mathcal{N} \arrow[l, shift right, "\mathsf{Ho}K"'] \end{tikzcd}$$

The top composite at the point set level sends an object $n \in \mathcal{N}$ to ΔGKn, which admits a counit or unit map to or from (depending on whether G is the left or right adjoint) Δn, which upon taking homotopy categories becomes a natural isomorphism to the constant diagram functor $\mathsf{Ho}\Delta$. Hence, as equivalences preserve adjunctions, the bottom composite, which sends GF to $G \operatorname{holim} KGF$, is naturally isomorphic to the homotopy limit functor on $\mathcal{N}$. Combining this with the weak equivalence $KGF \simeq F$ using the other (co)unit map, we have a natural weak equivalence

$$G \operatorname{holim} F \simeq G \operatorname{holim} KGF \simeq \operatorname{holim} GF$$

as desired. $\square$

Corollary 8.16. Let $\mathcal{M}$ be a model category, $\mathcal{D}$ be a category, and $F\colon \mathcal{D} \to \mathcal{M}$ be a functor. Then, a homotopy limit $\operatorname{holim} F$ is a limit of F in the underlying quasi-category $\mathsf{uq}\mathcal{M}$.

Alternatively, we lift the Quillen equivalence from Dugger's theorem to an equivalence of underlying quasi-categories using Theorem 4.21 and the fact that equivalences preserve (co)limits to immediately get:

Corollary 8.17. Underlying quasi-categories of model categories are complete and cocomplete.

We turn back to finishing the proof for Propositon 8.13. To show that the weight for the pseudo limit is projectively cofibrant, we will introduce some definitions from [RV20].

Definition 8.18. Let $\mathcal{D}$ be a small simplicial category, and fix a pair of objects $[n] \in \Delta$ and $d \in \mathcal{D}$. The projective n-cell associated with d is the simplicial natural transformation

$$\partial\Delta^n \times \mathcal{D}(d,-) \hookrightarrow \Delta^n \times \mathcal{D}(d,-). \tag{24}$$

We say that a monomorphism $V \to W$ in $\mathsf{sSet}^{\mathcal{D}}$ is a projective cell complex if it may be expressed as a countable composite of pushouts of coproducts of projective cells. A weight $W\colon \mathcal{D} \to \mathsf{sSet}$ is said to be a flexible weight if the inclusion $\varnothing \to W$ is a projective cell complex.

Proposition 8.19. Projective cell complexes are projective cofibrations.

Proof. Since cofibrations are stable under pushouts, coproducts and countable composition, it suffices to show that the projective n-cells (24) are cofibrations in the projective model structure. Fix a trivial fibration $Y \xrightarrow{\sim} X$ in $\mathsf{sSet}^{\mathcal{D}}_{\mathsf{proj}}$, i.e. a natural transformation that is pointwise a trivial fibrations. We would like to show that any lifting problem

$$\begin{tikzcd} \partial\Delta^n \times \mathcal{D}(d,-) \arrow[r] \arrow[d, hook] & Y \arrow[d, two heads, "\wr"] \\ \Delta^n \times \mathcal{D}(d,-) \arrow[r] \arrow[ur, dashed, "\exists ?"] & X \end{tikzcd} \tag{25}$$

has a solution. To do this, we construct a simplicially enriched adjunction

$$- \times \mathcal{D}(d,-) : \mathsf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{sSet}^{\mathcal{D}} : \mathsf{ev}_d \tag{26}$$

as follows: for any simplicial set K and a functor $Z\colon \mathcal{D} \to \mathsf{sSet}$, we have the following chain of isomorphisms of simplicial sets

$$\begin{aligned} \mathsf{sSet}^{\mathcal{D}}\Big(K \times \mathcal{D}(d,-), Z\Big) &= \int_{x\in\mathcal{D}} \mathsf{sSet}\Big(K \times \mathcal{D}(d,x), Zx\Big) \\ &\cong \int_{x\in\mathcal{D}} \mathsf{sSet}\Big(K, \mathsf{sSet}\big(\mathcal{D}(d,x), Zx\big)\Big) \\ &\cong \mathsf{sSet}\left(K, \int_{x\in\mathcal{D}} \mathsf{sSet}\big(\mathcal{D}(d,x), Zx\big)\right) \\ &\cong \mathsf{sSet}\Big(K, \mathsf{sSet}^{\mathcal{D}}\big(\mathcal{D}(d,-), Z\big)\Big) \\ &\cong \mathsf{sSet}(K, Zd) \end{aligned}$$

each of which is natural in K and Z. Going back to our lifting problem (25) and forgetting the enriched adjunction (26) to an adjunction of the underlying categories, we may transpose the lifting problem to an equivalent one

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & Yd \arrow[d, two heads, "\wr"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists ?"] & Xd \end{tikzcd}$$

inside sSet, which admits a solution. $\square$

Remark 8.20. In fact, a corollary of Quillen's small object argument (see for instance, [Rie14, Corollary 12.2.4]) tells us that *all* projective cofibrations are given by projective cell complexes, and hence cofibrant objects in $\mathsf{sSet}^{\mathcal{D}}_{\mathsf{proj}}$ are exactly the flexible weights.

Corollary 8.21. Let X be a simplicial set, and let $\mathcal{D} := \mathfrak{C}X$. The weight for the pseudo limit $W_X : \mathfrak{C}X \to \mathsf{sSet}$ (Definition 7.6) is projectively cofibrant and pointwise contractible.

Proof. By [RV20, Lemma 5.2.9], W_X is flexible and hence projectively cofibrant by Proposition 8.19. To see that W_X is also pointwise contractible, recall that for any vertex $x \in X$, we have

$$W_X(x) = \mathfrak{C}[\Delta^0 \star X](\perp, x)$$

by definition. Using the identification of the mapping spaces in a homotopy coherent nerve (Theorem 6.16), we get that

$$\mathfrak{C}[\Delta^0 \star X](\perp, x) \simeq \mathrm{Hom}_{\Delta^0 \star X}(\perp, x).$$

Finally, the fact that $\perp$ is initial in $\Delta^0 \star X$ (Remark 2.13) tells us that

$$\mathrm{Hom}_{\Delta^0 \star X}(\perp, x) \simeq *$$

is contractible using characterization (iv) in Proposition 6.32, the proof of which does not use the fact that A is a quasi-category. $\square$

Acknowledgements

This expository thesis started at the 2024 UChicago REU under the mentorship of Peter May, whom I would like to thank not only for his mathematical guidance but also for his commitment to running this wonderful program. I would also like to thank members of the topology and category theory community at Johns Hopkins: Maru Sarazola for introducing me to and making me fall in love with category theory and math in general; Anish Chedalavada for being an amazing DRP mentor and for both mathematical and emotional support during the summer of 2024; Rok Gregoric for helping me get over the fear of Higher Algebra (and many other things); Daniel Caranza and Tim Campion for teaching me stuff about ∞-categories that came up in this document (and stuff that didn't); and everyone else for giving me a wonderful mathematical experience as an undergraduate at Hopkins. Finally, I would like to thank my advisor Emily Riehl for her endless patience and kindness, for teaching me most of the things in this document, and for being a constant source of inspiration both within and without mathematics.

References

[BGH22] Andrew J. Blumberg, Teena Gerhardt, and Michael A. Hill, eds. *Stable categories and structured ring spectra*. Vol. 69. Mathematical Sciences Research Institute Publications. Cambridge University Press, Cambridge, 2022, pp. xi+426.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*. Vol. 180. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2019, pp. xviii+430. DOI: 10.1017/9781108588737.

[CP86] Jean-Marc Cordier and Timothy Porter. "Vogt's theorem on categories of homotopy coherent diagrams". In: *Math. Proc. Cambridge Philos. Soc.* 100.1 (1986), pp. 65–90. DOI: 10.1017/S0305004100065877.

[DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy limit functors on model categories and homotopical categories*. Vol. 113. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2004, pp. viii+181. DOI: 10.1090/surv/113.

[DK80a] W. G. Dwyer and D. M. Kan. "Calculating simplicial localizations". In: *J. Pure Appl. Algebra* 18.1 (1980), pp. 17–35. DOI: 10.1016/0022-4049(80)90113-9.

[DK80b] W. G. Dwyer and D. M. Kan. "Function complexes in homotopical algebra". In: *Topology* 19.4 (1980), pp. 427–440. DOI: 10.1016/0040-9383(80)90025-7.

[Dug01] Daniel Dugger. "Replacing model categories with simplicial ones". In: *Trans. Amer. Math. Soc.* 353.12 (2001), pp. 5003–5027. DOI: 10.1090/S0002-9947-01-02661-7.

[Gam10] Nicola Gambino. "Weighted limits in simplicial homotopy theory". In: *J. Pure Appl. Algebra* 214.7 (2010), pp. 1193–1199. DOI: 10.1016/j.jpaa.2009.10.006.

[Hin16] Vladimir Hinich. "Dwyer-Kan localization revisited". In: *Homology Homotopy Appl.* 18.1 (2016), pp. 27–48. DOI: 10.4310/HHA.2016.v18.n1.a3.

[HK20] Fabian Hebestreit and Achim Krause. *Mapping spaces in homotopy coherent nerves*. 2020. eprint: arXiv:2011.09345.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. DOI: 10.1515/9781400830558.

[Lur25] Jacob Lurie. *Kerodon*. https://kerodon.net. 2025.

[Maz16] Aaron Mazel-Gee. "Quillen adjunctions induce adjunctions of quasicategories". In: *New York J. Math.* 22 (2016), pp. 57–93.

[Rav23] Douglas C. Ravenel. "What is an ∞-category?" In: *Grad. J. Math.* 8.2 (2023), pp. 1–14.

[Rie] Emily Riehl. *A Leisurely Introduction to Simplicial Sets*. URL: https://math.jhu.edu/~eriehl/ssets.pdf.

[Rie14] E. Riehl. *Categorical Homotopy Theory*. Cambridge University Press, 2014.

[Rie16] E. Riehl. *Category Theory in Context*. Dover Publications, 2016.

[RV15] Emily Riehl and Dominic Verity. "The 2-category theory of quasi-categories". In: *Adv. Math.* 280 (2015), pp. 549–642. DOI: 10.1016/j.aim.2015.04.021.

[RV20] Emily Riehl and Dominic Verity. "Recognizing quasi-categorical limits and colimits in homotopy coherent nerves". In: *Appl. Categ. Structures* 28.4 (2020), pp. 669–716. DOI: 10.1007/s10485-020-09594-x.

[RV22] Emily Riehl and Dominic Verity. *Elements of ∞-category theory*. Vol. 194. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2022, pp. xix+759. DOI: 10.1017/9781108936880.