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Newton's Iteration

Introduction:

Roots of functions are input values that result in the function evaluating to zero. Graphically, roots can be located where the graph of a function intersects with the x-axis. How can calculators and computers precisely calculate roots of functions? Newton's Iteration is one possible way. Also known as Newton's Method and the Newton-Raphson Method, it is a numerical analysis method of closely approximating real roots of smooth or differentiable functions. It can be used to calculate roots of functions such as $f(x) = x^5 - x - 7$. This can be applied to calculate roots of numbers, such as $\sqrt{90}$ or $\sqrt[3]{75}$, and it can also be applied to precisely approximate division and reciprocals (Weisstein).

Concept:

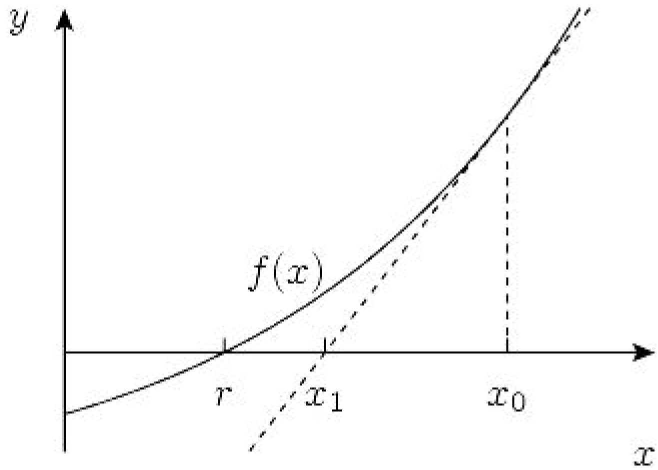
Newton's Iteration estimates roots of functions using its derivative. A derivative represents the slope of the tangent line at a particular point along the function. It begins with an initial given estimate of the root, x_0. Find the point at which the line tangent to the function at x_0 intersects with the x-axis. This becomes the new estimate, x_1 for the next iteration. This process is repeated until the estimate converges (Salvy, Weisstein).

Recurrence Equation:

Below is the equation that describes the Newton Iteration algorithm. x_i represents the estimate for the root at each iteration.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This equation can be derived as follows:



(source)

The slope of the tangent line at $(x_0, f(x_0))$, $f'(x_0)$, can be expressed using the following equation: $f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1}$, obtained by inserting the points $(x_0, f(x_0))$ and $(x_1, 0)$ into the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Solve the slope equation for x_1:

$$(x_0 - x_1) f'(x_0) = f(x_0) - 0$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_1)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If we derive the equation for x_2, we will end up with $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ and so on for $x_3, x_4 \ldots$. The sequence can then be summarized using the Newton Iteration equation (Weisstein).

Division and Reciprocals:

One of the most common applications of this algorithm is calculating divisions and reciprocals. Divisions can be expressed as $\frac{a}{b}$ or $a \times \frac{1}{b}$ (and a reciprocal of an integer b is $\frac{1}{b}$). The key to division and reciprocals is calculating the $\frac{1}{b}$, thus we will express this as $\frac{1}{b} = x$.

$$b - \frac{1}{x} = 0$$

$$f(x) = b - x^{-1}$$

$$f'(x) = x^{-2}$$

Substitute *f(x)* and *f'(x)* into the recurrence equation and simplify:

$$x_{n+1} = x_n - \frac{b - x_n^{-1}}{x_n^{-2}}$$

$$x_{n+1} = x_n(2 - bx_n)$$

Example 1:

Use Newton's Iteration to estimate $\sqrt[5]{1000}$. Start with $x_0 = 3.5$ and estimate the root up to x_4.

$$\sqrt[5]{1000} = 1000^{\frac{1}{5}} = x$$

$$x - 1000^{\frac{1}{5}} = 0$$

$$f(x) = x^5 - 1000$$

$$f'(x) = 5x^4$$

$x_1 = 3.5 - \frac{f(3.5)}{f'(3.5)} = 4.132778009$

$x_2 = 4.132778009 - \frac{f(4.132778009)}{f'(4.132778009)} = 3.991807286$

$x_3 = 3.991807286 - \frac{f(3.991807286)}{f'(3.991807286)} = 3.9811209306$

$x_4 = 3.9811209306 - \frac{f(3.9811209306)}{f'(3.9811209306)} = 3.981071707$

Root calculated on a TI-84 calculator = 3.981071706

Example 2

Use Newton's Iteration to estimate $\frac{1}{115}$. Start with $x_0 = .001$ and estimate to x_5

$x_1 = .001(2 - 115 \cdot .001) = .001885$

$x_2 = .001885(2 - 115 \cdot .001885) = .0033613791$

$x_3 = .0033613791(2 - 115 \cdot .0033613791) = .0054233882$

$x_4 = .0054233882(2 - 115 \cdot .0054233882) = .0074642653$

$x_5 = .0074642653(2 - 115 \cdot .0074642653) = .0085212761$

Quotient calculated on a T1-84 calculator = .0086956522

Quadratic Convergence

The rate of convergence represents how quickly a converging sequence approaches its limit. Newton's Iteration's rate of convergence is quadratic. Quadratic convergence means that generally, Newton's Iteration will converge faster compared to other rates of convergence, such as linear or superlinear. It also means that the number of correct digits in an estimate roughly doubles with each iteration of the algorithm (Peng).

Estimating x_0 using Divide and Conquer:

The result is not always guaranteed to converge. One problem that happens is if the initial guess provided is too far from the root(s), the estimates may never converge. This can be handled by choosing a guess that is between the x values in which the function goes from negative to positive (or vice versa). The intermediate value theorem states that if a function *f(x)* is continuous on [*a,b*] and *f(a)* and *f(b)* have different signs (positive / negative), then there must be a root between a and b. Thus, you can start with a large boundary, [A, B], halve it and search for a sign change in each half. If there is a sign change in the first half, [A, $\frac{A+B}{2}$], then the boundaries can be adjusted and you can repeat until the sign does not change anymore in either half (Knill).

Code Implementation using Divide and Conquer:

Time Complexity:

$O(M(n))$ where *M(n)* represents the time complexity of the multiplication algorithm that is being used by the calculator / computer (Salvy).

Works Cited

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