ML_AS10 (P1 & P3)   111652042 顏友君

1. Consider a forward SDE

$$dx_t = f(x_t,t)\,dt + g(x_t,t)dW_t,$$

show that the corresponding probability flow ODE is written as

$$dx_t = \left[f(x_t,t) - \frac{1}{2}\frac{\partial}{\partial x}g^2(x_t,t) - \frac{g^2(x_t,t)}{2}\frac{\partial}{\partial x}\log p(x_t,t)\right]dt.$$

Consider the forward Ito SDE:

$$dx_t = f(x_t,t)\,dt + g(x_t,t)\,dW_t,$$

where $f(x_t,t)$ is the drift coefficient, $g(x_t,t)$ is the diffusion coefficient and W_t is a standard Wiener process.

We want to find a deterministic ODE whose marginal density $p(x_t,t)$ at every time t matches that of the SDE. This ODE is called the probability flow ODE.

The time evolution of the marginal density $p(x_t,t)$ corresponding to the SDE is governed by the Fokker-Planck equation:

$$\frac{\partial p(x_t,t)}{\partial t} = -\frac{\partial}{\partial x}[f(x,t)\,p(x_t,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[g^2(x,t)\,p(x_t,t)] \quad \text{— ①}$$

For a deterministic flow $dx_t = v(x_t,t)\,dt$, the marginal density evolves according to the continuity eq.:

$$\frac{\partial p(x_t,t)}{\partial t} = -\frac{\partial}{\partial x}[v(x,t)\,p(x_t,t)] \quad \text{— ②}$$

By ① and ②, we equate their right-hand sides:

$$-\frac{\partial}{\partial x}(vp) = -\frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2 p)$$

$$\Rightarrow \frac{\partial}{\partial x}(fp - vp) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2 p)$$

Assuming the boundary terms vanish (i.e. p and $g^2 p \to 0$ at $\to \infty$), integrate once over x:

$$fp - vp = \frac{1}{2}\frac{\partial}{\partial x}(g^2 p)$$

Then, we solve for $v(x,t)$:

$$v(x,t) = f(x,t) - \frac{1}{p(x_t,t)}\frac{1}{2}\frac{\partial}{\partial x}[g^2(x,t)\,p(x_t,t)]$$

$$= f(x,t) - \frac{1}{2}\frac{1}{p(x_t,t)}\left[\left(\frac{\partial}{\partial x}g^2(x,t)\right)p(x_t,t) + g^2(x,t)\frac{\partial}{\partial x}p(x_t,t)\right] \quad \left(\frac{1}{p}\frac{\partial}{\partial x}p = \frac{\partial}{\partial x}\log p\right)$$

$$= f(x,t) - \frac{1}{2}\left[\frac{\partial}{\partial x}g^2(x,t) + g^2(x,t)\frac{\partial}{\partial x}\log p(x_t,t)\right] = \frac{dx_t}{dt}$$

Hence, we show that the corresponding probability flow ODE is

$$dx_t = \left[f(x_t,t) - \frac{1}{2}\frac{\partial}{\partial x}g^2(x_t,t) - \frac{g^2(x_t,t)}{2}\frac{\partial}{\partial x}\log p(x_t,t)\right]dt \quad \square$$

3. Unanswered Question：

在 Reverse SDE 中，模型是透過一步步去除噪聲，來恢復出真實的資料樣本，

但在現實世界中，像是天氣預測或金融市場中，資料中的噪聲往往不是

論文假設的高斯噪聲，而是未知或依據資料改變的非高斯噪聲，

我想問如果正向過程的噪聲並非高斯分佈，甚至依賴資料本身，

那反向 SDE 還能穩定且可靠的重建出原始訊號嗎？