6.046/18.410 Problem Set 1

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1 Problem 1-1: 6.006 Review

1.1 Part (a)

1.1.1 Suppose $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$

This statement is FALSE.

Consider $f(n) = n^2$ and $g(n) = n^2 + n$. Obviously we have $f(n) = \Theta(g(n))$. But $2^{g(n)} = 2^n 2^{f(n)}$, so it is impossible to find $C_1 \geqslant 0$ and $n_0 \in \mathbb{N}$ such that $\forall n > n_0, C_1 \cdot 2^{g(n)} \leqslant 2^{f(n)}$.

In this case, $2^{g(n)}$ is NOT an asymptotic lower bound of $2^{f(n)}$. So the original statement is false.

1.1.2 For any constants $a, b > 0$, $af(n) + bg(n) = \Theta(\max(f(n), g(n)))$

This statement is TRUE.

Let $C_1 = \min(a, b)$ and $C_2 = a + b$. Clearly we have $C_1, C_2 > 0$ and

$$0 \leqslant C_1 \max(f(n), g(n)) \leqslant af(n) + bg(n) \leqslant C_2 \max(f(n), g(n))$$

which means that $af(n) + bg(n) = \Theta(\max(f(n), g(n)))$.

1.1.3 Suppose $f(n) = o(1)$, then $f(n)g(n) = o(1)$

This statement is FALSE.

Consider $f(n) = n^{-1}$ and $g(n) = n$. We have $f(n) = o(1)$, but $f(n)g(n) = 1 \neq o(1)$.

1.1.4 Rank functions by order of growth

The ordering (growth rate from large to small, i.e. $g_k = \Omega(g_{k+1})$ for $k = 1, 2, \cdots, 11$) is given below. The proof can be found after this list.

$$\begin{aligned}
g_1 &= n! \\
g_2 &= 4^n \\
g_3 &= 2^n \\
g_4 &= 3^{\log^2 n} \\
g_5 &= (\log n)^{\log n} = g_6 = n^{\log \log n} \\
g_7 &= n^{10} \\
g_8 &= n^3 \\
g_9 &= n \log n \\
g_{10} &= \sum_{k=1}^{n} \log k \\
g_{11} &= \log \log n \\
g_{12} &= 100000^{100000000}
\end{aligned}$$

$g_5 = (\log n)^{\log n}$ and $g_6 = n^{\log\log n}$ belong to the same equivalent class. In fact, they are equal. $g_9 = n\log n$ and $g_{10} = \sum_{k=1}^{n} \log k$ belong to the same equivalent class. Each of the remaining functions is partitioned into its own equivalent class.
Proof: To compare the first 7 functions in the list, we can take log first.

$$
\begin{aligned}
f_1 &= \log g_1 \approx n\log n - n \text{ [Stirling approx.]}\\
f_2 &= \log g_2 = n\log 4\\
f_3 &= \log g_3 = n\log 2\\
f_4 &= \log g_4 = \log^2 n \log 3\\
f_5 &= \log g_5 = \log n \log\log n\\
f_6 &= \log g_6 = \log n \log\log n\\
f_7 &= \log g_7 = 10\log n
\end{aligned}
$$

Then it is clear that $g_1 = \Omega(g_2), \cdots, g_6 = \Omega(g_7)$. Moreover, we notice that $g_5 = g_6$. The rest of the functions can be compared without taking log. The only slightly tricky one is $g_{10} = \sum_{k=1}^{n} \log k$. Notice that $\log n \leqslant g_{10} \leqslant n\log n$, then the comparisons can be made.

Finally, I will prove $g_9 = \Theta(g_{10})$. Since $y = \log x$ is concave, an integral can be used as the lower bound of the sum:

$$g_{10} \geqslant (\ln 2)^{-1} \int_1^n \ln x dx = n\log n - \frac{n-1}{\ln 2}$$

On the other hand $g_{10} \leqslant n\log n$. So $g_{10} = \Theta(n\log n)$, and then $g_9 = \Theta(g_{10})$.

1.2 Recurrences

1.2.1 $T(n) = 10T(n/3) + n^2$

$T(n) = \Theta(n^{\log_3 10})$.
Proof: Use the master theorem (case 1). $a = 10$, $b = 3$, $\log_b a = \log_3 10 > 2$. So $n^2 = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$. Therefore, $T(n) = \Theta(n^{\log_3 10})$.

1.2.2 $T(n) = 9T(n/3) + n^2 \log n$

$T(n) = \Theta(n^2 \log^2 n)$.
Proof: Use the master theorem (case 2). $a = 9$, $b = 3$, $\log_b a = 2$. So $n^2 \log n = \Theta(n^{\log_b a} \log n)$. Therefore, $T(n) = \Theta(n^2 \log^2 n)$.

1.2.3 $T(n) = T(\sqrt{n}) + \log n$

$T(n) = \Theta(\log n)$.
Proof:

$$
\begin{aligned}
T(n) &= T(\sqrt{n}) + \log n\\
&= T(n^{1/4}) + \log n + \frac{1}{2}\log n\\
&= \cdots\\
&= \Theta(1) + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\log n\\
&= \Theta(\log n)
\end{aligned}
$$

1.2.4 $T(n) = T(n/4) + T(n/2) + n$

$T(n) = \Theta(n)$.

Proof: The recurrence is linear. So the guess is $T(n) = \Theta(n)$. To prove this, use the substitution method. Suppose $C_1 n \leqslant T(n) \leqslant C_2 n$, where C_1 and C_2 are positive. Then for the lower bound we have

$$\begin{aligned} T(n) &= T(n/4) + T(n/2) + n \\ &\geqslant \left(\frac{1}{4}C_1 + \frac{1}{2}C_1 + 1\right) n \\ &= \left(\frac{3}{4}C_1 + 1\right) n \\ [desired] &\geqslant C_1 n \end{aligned}$$

Obviously, if $C_1 \leqslant 4$, the desired inequality holds. On the other hand, for the upper bound,

$$\begin{aligned} T(n) &= T(n/4) + T(n/2) + n \\ &\leqslant \left(\frac{1}{4}C_2 + \frac{1}{2}C_2 + 1\right) n \\ &= \left(\frac{3}{4}C_2 + 1\right) n \\ [desired] &\leqslant C_2 n \end{aligned}$$

Obviously, if $C_2 \geqslant 4$, the desired inequality holds. This completes the prove that $T(n) = \Theta(n)$.

1.2.5 $T(n) = T(2n/3) + T(n/3) + n \log n$

$T(n) = \Theta(n \log^2 n)$.

Proof: Use the recursion tree method. The tree is shown below (only the coefficients are written out). We notice that $2/3 + 1/3 = 1$, thus the coefficients in each level sum up to unity. This feature guarantees that there are $\Theta(n)$ leaves in the tree, eaching contributing constant runtime. Moreover, each level contributes $\Theta(n \log n)$ since the sum of coefficients in a level is unity. And there are $\Theta(\log n)$ levels (bounded between $\log_{3/2} n$ and $\log_3 n$). In conclusion, we have

$$T(n) = \Theta(n) + \Theta(n \log n)\Theta(\log n) = \Theta(n \log^2 n)$$