6.046/18.410 Problem Set 4

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1 Learn to Fuel Wisely

1.1 Part (a)

Description: First run APSP on the graph $G = (V, E)$ with edge weights $l(u, v)$. Construct a new unweighed undirected graph $G' = (V, E')$, using all original vertices but newly defined edges, as follows: check $\delta(u, v)$ for all pairs. If $\delta(u, v) \leqslant K$, create an edge (u, v). By this construction, all the island pairs that can be reached within one fill are connected. Finally run APSP on the new graph G'. The longest of all pairs shortest paths in G' gives the value t we want.
Correctness: We prove that in order to get from any vertex u to any distinct vertex v, the smallest number of refills is the length of the shortest path $\delta'(u, v)$ in G'. Then the correctness of the algorithm follows.

Given a shortest path p'_{uv} from u to v in G', we can use the following refilling strategy: start by filling at u, then until reaching v, we always go to the subsequent vertex in p'_{uv} and refill there. This can always be done since, by the construction of G', we can reach from a vertex in p'_{uv} to its subsequent vertex in one fill.

2 Lazy Random Homework Solving

2.1 Part (a)

Proof: By induction on k. When $k = 1$, suppose that there are r_1 friends working on this problem. $r_1 \geqslant r$. The assignment becomes invalid when all these r_1 friends are assigned to TA Nirvan (N) or Kelly (K). The possibility is $P[1, \text{invalid}] = 2 \times 2^{-r_1} \leqslant 2^{1-r} = k2^{1-r}$. So the statement is true for the base case.

When $k > 1$, suppose the statement holds for $k - 1$. Therefore, $P[k - 1, \text{invalid}] \leqslant (1 - k)2^{1-r}$. In other words, $P[k - 1, \text{valid}] \geqslant 1 - (k - 1)2^{1-r}$. When we add the k-th problem, we have

$$P[k, \text{valid}] = P[k - 1, \text{valid}]P[1, \text{valid}|\text{assignment valid for previous } k - 1 \text{ problems}]$$

where the second factor is a conditional probability: the probability for the assignment to be valid for the one-problem case (the k-th problem), under the condition that the assignment is valid for the $k - 1$-problem case (the privious $k - 1$ problems). If we use the unconditional probability, the equality becomes an inequality because the unconditional probability is always no larger.

$$P[k, \text{valid}] \geqslant P[k - 1, \text{valid}]P[1, \text{valid}]$$

The inequality can be further relaxed,

$$\begin{aligned} P[k, \text{valid}] &\geqslant (1 - (k - 1)2^{1-r})(1 - 2^{1-r}) \\ &= 1 - k2^{1-r} + (k - 1)2^{2-2r} \\ &\geqslant 1 - k2^{1-r} \end{aligned}$$

which means $P[k, \text{invalid}] \leqslant k2^{1-r}$. Therefore, by induction, the statements is true for all k. Larry fails to choose a valid assignment with probability at most $k2^{1-r}$.