Lecture 3 - Axioms of Consumer Preference and the Theory of Choice

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A brief outline

Topics for this Lecture Note

1. Consumer preference theory

    (a) Notion of utility function

    (b) Axioms of consumer preference

    (c) Monotone transformations

2. Theory of choice

    (a) Solving the consumer's problem
    - Ingredients
    - Characteristics of the solution
    - Interior vs corner solutions

    (b) Constrained maximization for consumer

    (c) Interpretation of the Lagrange multiplier

Where we are headed

Theory

1. Consumer preference theory

2. Theory of choice

    (a) Constrained utility maximization

    (b) Constrained expenditure minimization

3. Individual demand functions

    - Consumer demand holding the consumer's budget constant
    - Consumer demand holding the consumer's utility constant
    - Linking these two demand functions

4. Market demand

    - Market demand for a single good as a function of its price (partial equilibrium)
    - Market demand for all goods as a function of their relative prices (general equilibrium)

Applications

1. Dead weight loss of Christmas
2. How should charitable goods be distributed: Free distribution or cost sharing?
3. Giffen goods: Do they exist? Are they relevant?

1 Consumer Preference Theory

A consumer's utility from consumption of a given bundle "A" is determined by a personal *utility function.*

1.1 Cardinal and ordinal utility

- *Cardinal utility function*

  According to this approach $U(A)$ is a cardinal number, that is:

  U : *consumption bundle* $\longrightarrow$ R^1 measured in "utils"

- *Ordinal utility function*

  More general than cardinal utility function

  U provides a "ranking" or "preference ordering" over bundles.

$$U : (A, B) \longrightarrow \begin{cases} A \ ^P \ B \\ B \ ^P \ A \\ A \ ^I \ B \end{cases}$$

  Used in demand/consumer theory

Cardinal vs Ordinal Utility Functions

The problem with cardinal utility functions comes from the difficulty in finding the appropriate measurement index (metric).

- Is 1 util for person 1 equivalent to 1 util for person 2? Or if we increase a person's utility from 1 to 2, is she twice as happy?
- By being unit-free *ordinal* (but not cardinal) utility functions avoid these problems.
- What's important about a utility function is that it allows us to model how people make personal choices, that is, how they choose among competing alternatives. We do not need to know how many "utils" people experience from each choice to obtain this information; we just need to know how they rank choices.

- It's much harder—and not always meaningful, but not always meaningless—to make/model interpersonal comparisons of utility than to study *intrapersonal* choices, that is, the choices that an individual selects among different options (often, for simplicity, different bundles of goods). It turns out that we don't need to make interpersonal comparisons of utility to make reach positive and even some strong normative conclusions. This is part of the power of economic theory: it goes an amazingly long distance on a very small (but not weak) set of assumptions.

1.2 The Axioms of Consumer Preference Theory

Next up on our agenda: the axioms of consumer theory. These axioms were developed for three purposes:

1. To allow for mathematical representation of utility functions
2. To portray rational behavior (rational in this case means 'optimizing')
3. To derive "well-behaved" demand curves

When we get through the entire corpus of consumer theory, you may be surprised to realize that all of the results and predictions we will have derived rest entirely, and with no further assumptions, on the small set of axioms below.

1.2.1 Axiom 1: Preferences are Complete ("Completeness")

For any two bundles A and B, a consumer can establish a preference ordering. That is, for any comparison of bundles, she will choose one and only one of the following:

1. $A \,^P\, B$
2. $B \,^P\, A$
3. $A \,^I\, B$

Without this property, preferences are undefined.

1.2.2 Axiom 2: Preferences are Transitive ("Transitivity")

- For any consumer if $A \,^P\, B$ and $B \,^P\, C$ then it must be that $A \,^P\, C$.
- This axiom says that consumers are consistent in their preferences.

1.2.3 Axiom 3: Preferences are Continuous ("Continuity")

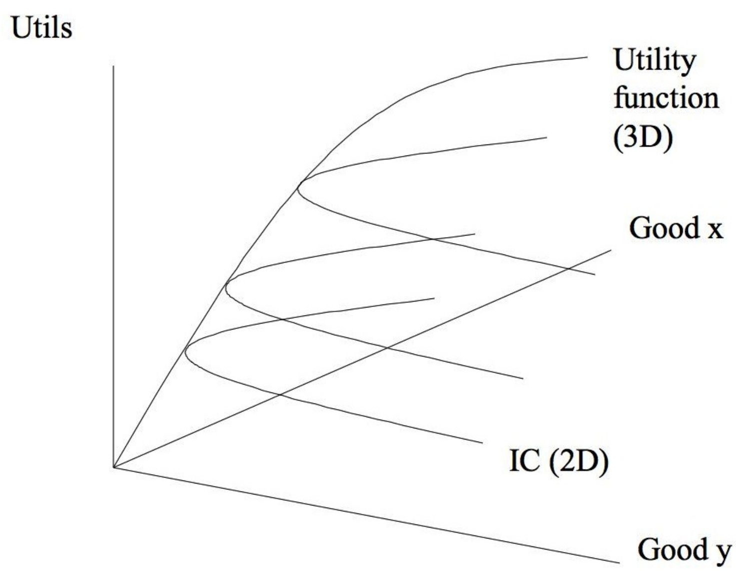
- If $A \,^P\, B$ and C lies within an ε radius of B then $A \,^P\, C$.
- We need continuity to derive well-behaved demand curves.

Given Axioms 1- 3 are obeyed we can always define a utility function.

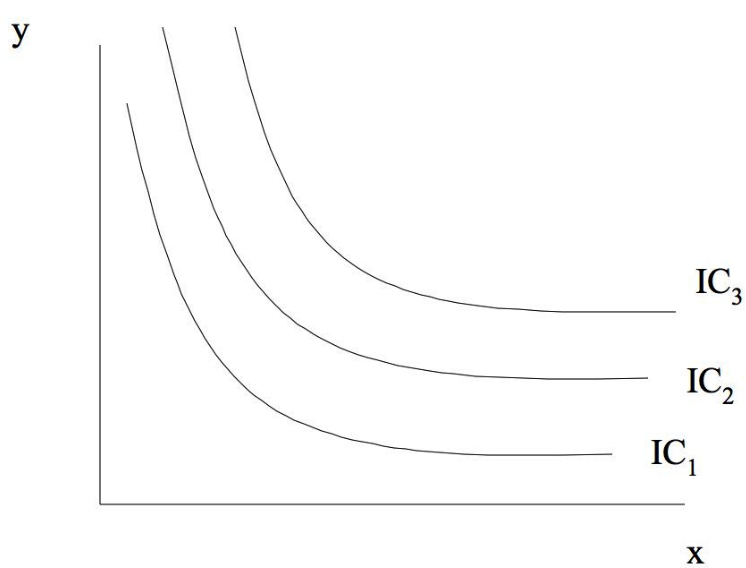
- Any utility function that satisfies Axioms 1- 3 *cannot have indifference curves that cross.*

Indifference Curves

- Define a level of utility say $U(x) = \overline{U}$ then the indifference curve for $\overline{U}$, $IC(\overline{U})$ is the locus of consumption bundles that generate utility level $\overline{U}$ for utility function $U(x)$.
- An *Indifference Curve Map* is a sequence of indifference curves defined over every possible bundle and every utility level: $\{IC(0), IC(\varepsilon), IC(2\varepsilon), ...\}$ with $\varepsilon =$ *epsilon*



- Indifference curves are level sets of this utility function.



- $$\left.\begin{array}{l} IC_3 \longrightarrow \text{ Utility level } U_3 \\ IC_2 \longrightarrow \text{ Utility level } U_2 \\ IC_1 \longrightarrow \text{ Utility level } U_1 \end{array}\right\} U_3 > U_2 > U_1$$

- This is called an Indifference Curve Map. Properties:
  - Every consumption bundle lies on some indifference curve (by the completeness axiom)
  - Indifference curves are smooth due to the continuity axiom. We don't have sharp discontinuous preference reversals.

Axioms 4 and 5 are introduced to reflect observed behavior

- They simplify problems greatly, but they are not *necessary* for a theory of rational choice.

1.2.4 Axiom 4: Non-Satiation (Never Get Enough)

- Given two bundles, A and B, composed of two goods, X and Y, where X_A = amount of X in A, similarly X_B, and Y_A = amount of Y in A, similarly Y_B. If $X_A = X_B$ and $Y_A > Y_B$ (assuming utility is increasing in both arguments) then $A \ ^P \ B$, regardless of the levels of X_A, X_B, Y_A, Y_B
- This implies that:
  1. The consumer always places positive value on more consumption (if a good is a bad, e.g., pollution, just reverse the sign so the gad is a good, e.g., less pollution).
  2. Indifference curve map stretches out endlessly (there is no upper limit to utility)
- While this axiom is not necessary to construct a well-defined utility function, it is helpful.

1.2.5 Axiom 5: Diminishing Marginal Rate of Substitution

- This axiom is also unnecessary to construct a well-defined utility function, but we believe it generally captures a fundamental feature of human preferences.
- This axiom also makes the mathematics of consumer theory much simpler; it means we will stumble across relatively few corner solutions where the consumer maximizes utility by spending the entire budget on one good.
- In order to define this axiom we need to introduce the concept of Marginal Rate of Substitution and some further preliminary explanations.

- *Definition:* MRS measures willingness to trade one bundle for another. Example:

  Bundle A = (6 hours of sleep, 50 points on the problem set)

  Bundle B = (5 hours of sleep, 60 points on the problem set)

  If A and B lie on the same indifference curve

  A student is willing to give up 1 more hour of sleep for 10 more points on the problem set.

  Her willingness to substitute sleep for grade points at the margin (i.e. for 1 fewer hours of sleep) is:

  $\frac{10}{-1} = -10$

$$MRS \text{ (sleep for points)} = |-10| = 10$$

- MRS is measured along an indifference curve and it *may* vary along the same indifference curve. If so, we must define the MRS relative to some bundle (starting point).

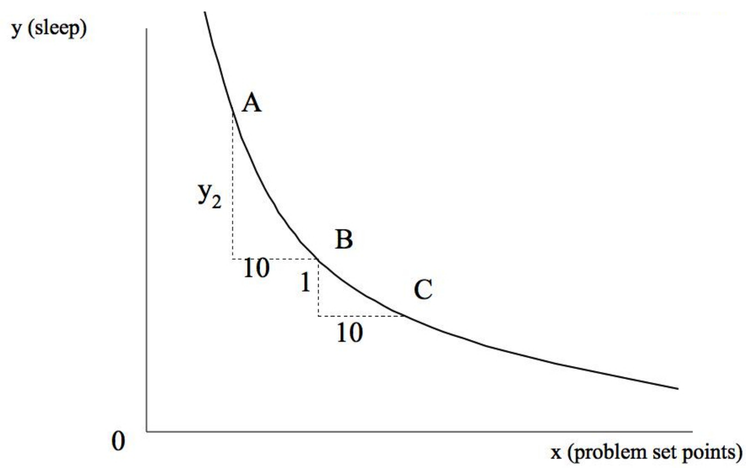
- By definition, utility is constant along an indifference curve: $dU = 0$. Therefore:

$$\begin{aligned}
0 &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \\
0 &= MU_x dx + MU_y dy \\
-\frac{dy}{dx} &= \frac{MU_x}{MU_y} = \text{MRS of x for y}
\end{aligned}$$

- Note the slightly confusing terminology. MRS of x for y is the marginal utility of x divided by the marginal utility of y (holding total utility constant), which is equal to $-dy/dx$.

- MRS must always be evaluated at some particular point (consumption bundle) on the indifference curve. So one should really write MRS(x', y') where (x', y') is a particular consumption bundle.

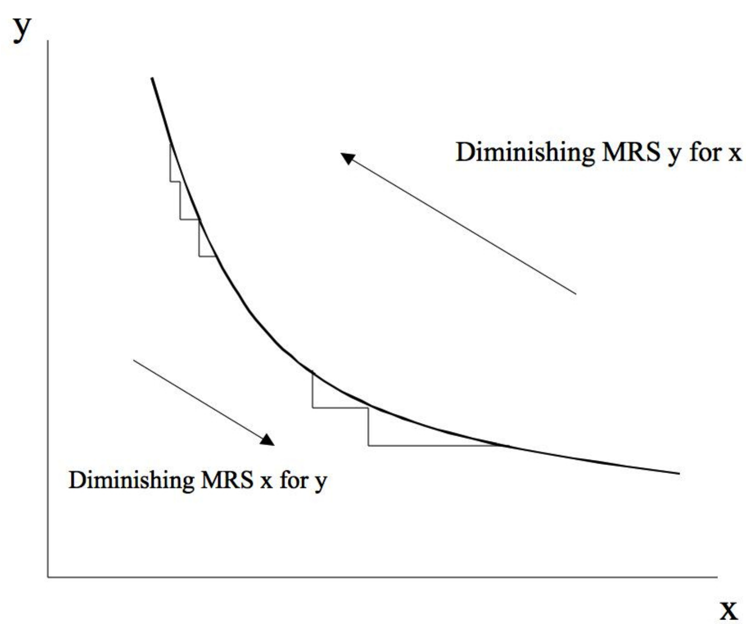
- We are ready to explain what is meant by Diminishing Marginal Rate of Substitution.



- MRS of x for y decreases as we go down the indifference curve.

- This indifference curve exhibits diminishing MRS: the rate at which (at the margin) a consumer is willing to trade x for y – that is, the amount of y she demands in exchange for one unit of x – diminishes as the level of x consumed goes up.

- That is, the slope of the indifference curve between points B and C is less than the slope of the curve between points A and B. Mathematically, this means that MRS of x for y (the ratio MU_x/MU_y) is decreasing in x.



- Diminishing MRS implies that consumers prefer diversity in consumption.

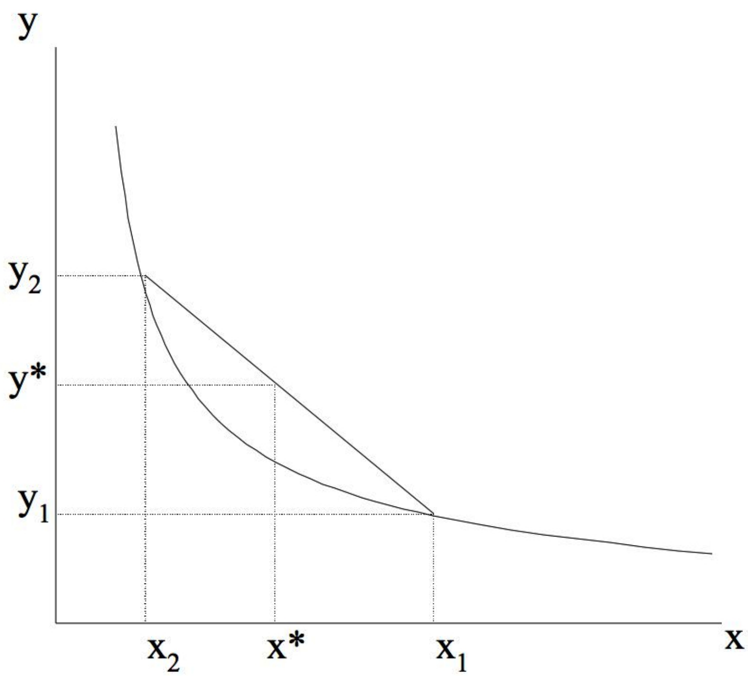
- A formal definition of diminishing MRS can be given through the mathematical notion of *convexity*.

- *Definition.* A function $U(x, y)$ is convex if for any arguments (x_1, y_1) and (x_2, y_2) where $(x_1, y_1) \neq (x_2, y_2)$:

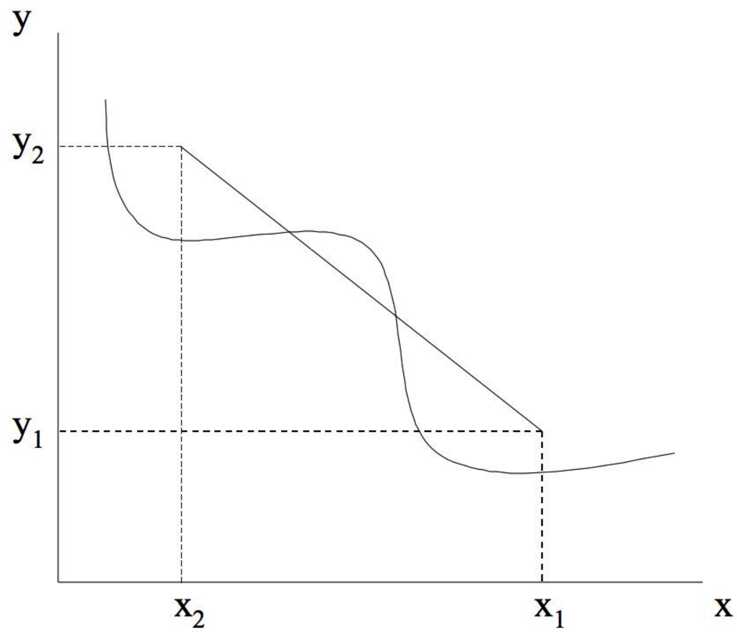
$$U(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \geq \alpha U(x_1, y_1) + (1-\alpha)U(x_2, y_2),$$

where $\alpha \in (0, 1)$.

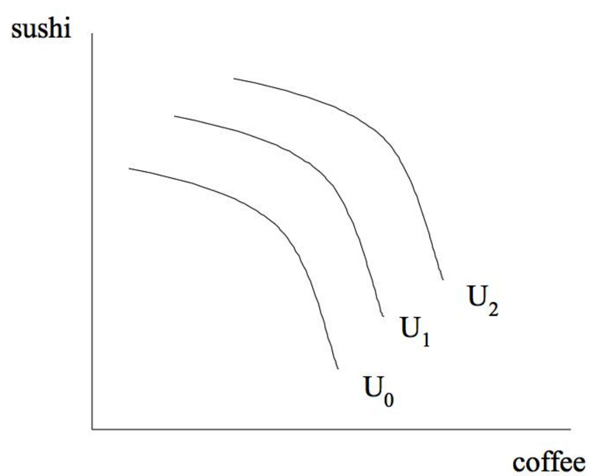
- Suppose the two bundles, (x_1, y_1) and (x_2, y_2) are on the same indifference curve. The diminishing MRS property states that the convex combination of these two bundles is on a higher indifference curve than either of the two initial bundles.



- where $x^* = \alpha x_1 + (1-\alpha)x_2$ and $y^* = \alpha y_1 + (1-\alpha)y_2$. This is verified for every $\alpha \in (0, 1)$.
- A utility function $U(\cdot)$ exhibits diminishing MRS iff the indifference curves of $U(\cdot)$ are convex.
- The following is an example of a non-convex curve:



- In this graph not every point on the line connecting two points above the curve is also above the curve, therefore the curve is not convex.
- Q: Suppose you love drinking coffee and eating sushi, but you strongly dislike consuming them together. Thus, the more you consume of one (at a sitting), the less you want of the other. How do we draw these preferences?

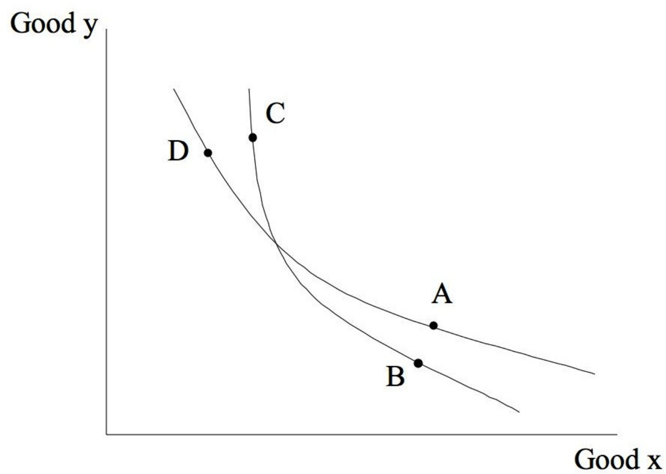


- [Note that we assume here that both goods are consumed at once. If you could preserve the (fresh) coffee or the (fresh) sushi and then consume it later, you would likely have convex preferences over sushi and coffee.]

- If your indifference curves were concave as above, and holding the budget set constant, you should not diversify consumption.

1.2.6 Non-crossing of indifference curves

- Using the transitivity and non-satiation axioms, we can demonstrate that indifference curves cannot intersect.
- Proof: say two indifference curves intersect:



- According to these indifference curves:

  $A \,^{P}\, B$ (by non-satiation)

  $B \,^{I}\, C$

  $C \,^{P}\, D$ (by non-satiation)

  $D \,^{I}\, A$

- By the above mentioned axioms, $A \,^{P}\, D$ and $A \,^{I}\, D$, which is a contradiction.

1.3 Cardinal vs Ordinal Utility

- A utility function of the form $U(x, y) = f(x, y)$ is cardinal in the sense that it reads off "utils" as a function of consumption.
- But we don't know what utils are or how to measure them. Nor do we assume that 10 utils is twice as good as 5 utils. That would be a *cardinal* assumption.
- Our notion of consumer preferences focuses on the consumer's ranking (or ordering) over bundles of goods, rather than the consumer's "level" of utility for a given bundle. That is, we want to know if $A \,^{P}\, B$ but not by how much.

- *We do, however, care that the MRS along an indifference curve is well defined*, i.e. we do want to know precisely how people trade off among goods in indifferent (equally preferred) bundles.

- In consumer theory, we choose to use ordinal not cardinal utility functions. As we show below, ordinal utility functions preserve the MRS property on which we are focused but dispense with cardinal properties (e.g., counting utils).

- Q: How do we preserve properties of utility that we care about and believe in (a. ordering is unique; and b. MRS exists) without imposing cardinal properties? We can weaken our definition of utility functions by stating that a utility function is only defined up to a "*monotonic transformation*." That is, if utility function $g'()$ is a monotone transformation of utility function $g()$, we will say they are identical for purposes of consumer theory.

- *Definition:* Monotonic Transformation

  Let I be an interval on the real line (R^1) then: $g : I \longrightarrow R^1$ is a monotonic transformation if g is a strictly increasing function on I.

  If $g(x)$ is differentiable then $g'(x) > 0 \; \forall x$

  Informally: A monotone transformation of a variable is a *rank-preserving* transformation. [Note: not all rank-preserving transformations are differentiable.]

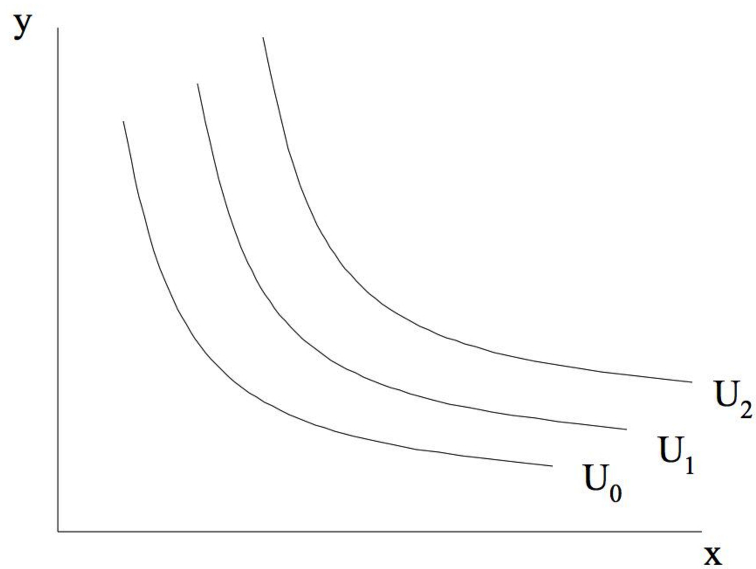
- *Examples* of monotone and non-monotone transformations Let y be defined on R^1:

  1. $x = y + 1$ [monotone]
  2. $x = 2y$ [monotone]
  3. $x = \exp(y)$ [monotone]
  4. $x = abs(y)$ [non-monotone]
  5. $x = y^2$ [monotone if $y \geq 0$]
  6. $x = \ln(y)$ [monotone if $y > 0$]
  7. $x = y^3$ [monotone if $y \geq 0$]
  8. $x = -\frac{1}{y}$ [monotone if $y \geq 0$]
  9. $x = \max(y^2, y^3)$ [monotone if $y \geq 0$]
  10. $x = 2y - y^2$ [non-monotone]

- If $U_2(.)$ is a monotone transformation of $U_1(.)$, i.e. $U_2(.) = f(U_1(.))$ where $f(.)$ is monotone in U_1 as defined earlier, then:

  1. U_1 and U_2 exhibit identical preference rankings
  2. MRS of $U_1(\overline{U})$ and $U_2(\overline{U})$ are the same
  3. U_1 and U_2 are equivalent for consumer theory

- Example: $U(x,y) = x^\alpha y^\beta$ (Cobb-Douglas)



- What is the MRS along an indifference curve U_0?

$$
\begin{aligned}
U_0 &= x_0^\alpha y_0^\beta \\
dU_0 &= \alpha x_0^{\alpha-1} y_0^\beta dx + \beta x_0^\alpha y_0^{\beta-1} dy \\
\left.\frac{dy}{dx}\right|_{U=U_0} &= -\frac{\alpha x_0^{\alpha-1} y_0^\beta}{\beta x_0^\alpha y_0^{\beta-1}} = \frac{\alpha}{\beta}\frac{y_0}{x_0} = \frac{\partial U/\partial x}{\partial U/\partial y}
\end{aligned}
$$

- Consider now a monotonic transformation of U:

$$
\begin{aligned}
U^1(x,y) &= x^\alpha y^\beta \\
U^2(x,y) &= \ln(U^1(x,y)) \\
U^2 &= \alpha \ln x + \beta \ln y
\end{aligned}
$$

- What is the MRS of U^2 along an indifference curve?

$$
\begin{aligned}
U_0^2 &= \ln U_0 = \alpha \ln x_0 + \beta \ln y_0 \\
dU_0^2 &= \frac{\alpha}{x_0} dx + \frac{\beta}{y_0} dy = 0 \\
\left.\frac{dy}{dx}\right|_{U^2=U_0^2} &= \frac{\alpha}{\beta}\frac{y_0}{x_0} = \frac{\partial U/\partial x}{\partial U/\partial y}
\end{aligned}
$$

which is the same as we derived for U^1.

Optional: General demonstration of this result

- How do we know that monotonic transformations always preserve the MRS of a utility function?
- Let $U = f(x, y)$ be a utility function
- Let $g(U)$ be a monotonic transformation of $U = f(x, y)$ and differentiable
- The MRS of $g(U)$ along an indifference curve where $U_0 = f(x_0, y_0)$ and $g(U_0) = g(f(x_0, y_0))$
- By totally differentiating this equality we can obtain the MRS.
- $$\begin{aligned} dg(U_0) &= g'(f(x_0, y_0))f_x(x_0, y_0)dx + g'(f(x_0, y_0))f_y(x_0, y_0)dy \\ -\left.\frac{dy}{dx}\right|_{g(U)=g(U_0)} &= \frac{g'(f(x_0, y_0))f_x(x_0, y_0)}{g'(f(x_0, y_0))f_y(x_0, y_0)} = \frac{f_x(x_0, y_0)}{f_y(x_0, y_0)} = \frac{\partial U/\partial x}{\partial U/\partial y} \end{aligned}$$

  which is the MRS of the original function $U(x, y)$.