Problem Set 6

This problem set is due at **11:59 pm** on **Wednesday, October 28, 2015**.

Problem 6- 1: Shipping To Save The World

Disaster has struck! Several cities around the world have been devastated by natural disasters due to global warming. Luckily, the city of Boston made preparations in advance to prevent such disasters, so we are safe at MIT.

In order to help the cities that have been affected by the disaster, you decide to start a non-profit, distributed shipping system which sends supplies to these cities. You research the disaster and find that there are a total of n cities $C = \{c_1, c_2, \ldots, c_n\}$ that are affected and in need of supplies. Your goal is to send supplies to the cities and save the world.

You buy a total of $m < n$ ships $s_1, s_2, \ldots, s_m$ located in different places around the world. Each ship s_j can transport supplies from its origin to a small subset of the cities $Y_j \subseteq C$ where $|Y_j| \leq 5$. Together, all the ships cover every city, possibly with overlaps. Each city c_i, needs a total of f_i supplies. You decide to send the supplies over the course of a week (7 days $D = \{d_1, \ldots, d_7\}$). Because of financial reasons, on each day d_i, you can only send a total of v_i supplies. Finally, since you don't want the ship captains to get exhausted from working continuously every day of the week, each ship s_j, will only be working on $X_j \subseteq D$ days.

An example instance of this problem for $m = 3$ and $n = 5$ would look something like this:

- You have ships s_1, s_2 and s_3.
- There are affected cities c_1, c_2, c_3, c_4 and c_5 where $f_i = 10$ for all $i \in [1 \ldots 5]$.
- Ship s_1 travels to cities $Y_1 = \{c_1, c_2, c_3\}$ and operates on days $X_1 = \{d_1, d_2, d_3\}$.
- Ship s_2 travels to cities $Y_2 = \{c_2, c_3, c_5\}$ and operates on days $X_2 = \{d_3, d_4\}$.
- Ship s_3 travels to cities $Y_3 = \{c_4, c_5\}$ and operates on days $X_3 = \{d_5, d_6, d_7\}$.
- On each day d_i, you can supply a total amount of $v_i = \{10 \text{ if } i \text{ even, else } 5\}$ supplies.

(a) Define a shipping plan to be a valid assignment for each day, d_i, consisting of the following specifications.

- How many supplies does each ship transport?
- How does each ship distribute its supplies among the cities that it visits?

(i) Design and analyze an efficient algorithm to decide whether there exists a shipping plan such that each city c_i receives exactly f_i supplies.
(ii) Assuming such a shipping plan exists, give an efficient algorithm to determine it.

The following is a valid shipping plan for the example instance.

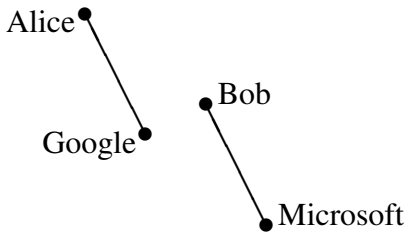
- On day d_1, s_1 delivers 5 supplies to c_1.
- On day d_2, s_1 delivers 5 supplies to c_1 and 5 supplies to c_2.
- On day d_3, s_2 delivers 5 supplies to c_2.
- On day d_4, s_2 delivers 10 supplies to c_3.
- On day d_5, s_3 delivers 5 supplies to c_4.
- On day d_6, s_3 delivers 5 supplies to c_4 and 5 supplies to c_5.
- On day d_7, s_3 delivers 5 supplies to c_5.

(b) You figure out a great shipping plan for part (a). Unfortunately, you hear that one of your ships failed an international quality assurance test and cannot be operational. In your original plan, this ship was scheduled to transport a total of k supplies. Now you must re-evaluate your plan and check if it is still possible to supply the needs of all the cities without this ship. Give an efficient algorithm to decide whether it is still possible to find such a shipping plan and analyze its runtime as a function of k.

Problem 6- 2: Career Fair.

The career fair has come to MIT again and all the sudents have arrived at the Johnson center, resumés at the ready. Unfortunately, career day isn't a student holiday this year, so everyone's in a rush to get to their classes. Each employer at the career fair can hire at most one student, and in order to minimize the time spent at the career fair, the students have each agreed to speak with exactly one employer. Help the students choose which of the employers they should speak to.

There are n students and m employers. Student i takes time $t_{i,j}$ to reach employer j, where $1 \leq i \leq n$ and $1 \leq j \leq m$. At the start of the career fair, the students simultaneously begin moving toward the employers. Each employer can hire at most one student, and each student must speak with exactly one employer. Design an algorithm that chooses an employer for each student such that the last student reaches her employer as quickly as possible.



In the example above, the times are proportional to the (euclidean) distance between the points, but in general this may not be true. The optimal solution, shown above, sends Alice to Google and Bob to Microsoft. Note that pairing Bob with Google and Alice with Microsoft is suboptimal.