Problem Set 4

This problem set is due at **11:59 pm** on **Wednesday, October 14th, 2015**. The exercises are **optional**, and should not be submitted.

Exercise 4-1: Do Exercise 15-1 in CLRS on page 404.

Exercise 4-2: Do Exercise 15-9 in CLRS on page 410.

Exercise 4-3: Do Problem 15-11 in CLRS on page 411.

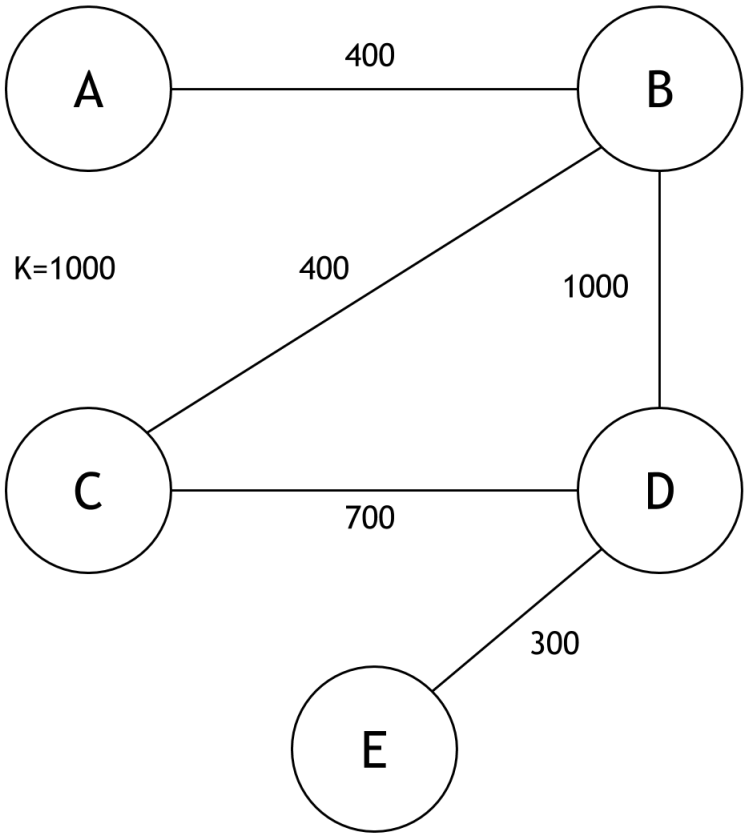
Exercise 4-4: Do Exercise 25.2-2 in CLRS on page 699.

Exercise 4-5: Do Exercise 25.2-4 in CLRS on page 699.

Problem 4-1: Learn to Fuel Wisely

Kiki lives in an archipelago containing N islands. Some pairs of islands, u and v, are connected by a bridge of $l(u, v)$ miles in length. Each island can be reached from any other island via bridges.

(a) Kiki plans to travel around in a car that consumes 1 gallon of gas per mile, and the gas tank has the maximum capacity of K gallons. At the beginning of the journey, the tank is empty. On each island, Kiki can either fill up the entire tank, or not refill it at all. Assume that the lengths of all bridges are less than or equal to K. Find the smallest number of refills t such that for all pairs of islands x and y, Kiki can go from x to y using at most t refills (including the first fill). Describe and analyze your algorithm.



Example: In the above diagram, $t = 2$ since we can get from any island to another using at most two refills. Notice that the path with the least number of refills to get from A to E is $A \to B \to C \to D \to E$, which requires two refills at A and C. There is another path $A \to B \to D \to E$ which has less total length and less number of bridges, but requires a greater number of refills.

(b) Kiki discovers a notebook which belongs to her grandfather. In the notebook, her grandfather records the lengths of shortest paths between all pairs of N islands. However, a new island, L, is being constructed with bridges connected to some of the existing N islands. Kiki would like to keep her grandfathers notebook up to date by updating the lengths of shortest paths between all pairs of $N + 1$ islands. Describe and analyze an algorithm that she can use.

Problem 4-2: Lazy Random Homework Solving

Lazy Larry is a student with n friends and k different problems to do for homework. Luckily for Lazy Larry, at least r of his friends are still working on each problem. Notice that each friend may be working on more than one problem.

Even though he is lazy, Larry is a very persuasive leader. He plans to arrange it so that, by sending his friends, he can get feedback on every problem from both awesome TAs Neo

Nirvan and Kool Kelly during their office hours. Unfortunately, each of his friends only has time to attend one of the office hours and the TAs only give 1 office hour each, but luckily each friend can get feedback on all problems he or she is working on from the office hours he does attend.

We will call an assignment of friends to office hours *valid* if, for each problem at least one person gets feedback from Neo Nirvan and at least one person gets feedback from Kool Kelly.

(a) Suppose Lazy Larry assigns his friends independently and uniformly at random to the two TAs. Show that Larry fails to choose a valid assignment with probability at most $k2^{1-r}$.

(b) Part (a) implies that a valid assignment of Larry's friends is guaranteed to be possible provided the number of problems k is less than a certain value (which depends on r). What is this value? Explain and justify your answer.

(c) Suppose now that $k \leq 2^{r-2}$. Help Larry find an algorithm which will produce a valid assignment in expected linear time (notice that the size of the input for this problem is $O(kn)$ because one needs to list all of the students working on each problem).

(d) After everyone helps Larry finish his problem set, Larry throws them a party. Unfortunately, current fashion trends call for everyone to wear identical red and grey beaver shaped boots. They are all too similar to be distinguished but are of slightly different sizes, and Larry lazily dumped them all in a closet, so now he needs your help to sort them out.

Lazy Larry can choose a guest to try on one of the pairs of boots and find whether the boots fit them (which occurs only if it belongs to that guest), or the boots are too small or too big. However, Larry cannot directly compare boots to each other, nor can he directly compare his friend's foot sizes.

Design and analyze a randomized algorithm that could help Lazy Larry get out of this unfortunate mess and return the boots to their rightful owners. Your algorithm should require Larry to have a guest try on a pair of boots at most $O(n \log n)$ total times in expectation. [Hint: pivot]