6.046/18.410 Problem Set 4

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1 Learn to Fuel Wisely

2 Lazy Random Homework Solving

2.1 Part (a)

Proof: By induction on k. When $k = 1$, suppose that there are r_1 friends working on this problem. $r_1 \geqslant r$. The assignment becomes invalid when all these r_1 friends are assigned to TA Nirvan (N) or Kelly (K). The possibility is $P[1, \text{invalid}] = 2 \times 2^{-r_1} \leqslant 2^{1-r} = k2^{1-r}$. So the statement is true for the base case.

When $k > 1$, suppose the statement holds for $k - 1$. Therefore, $P[k-1, \text{invalid}] \leqslant (1-k)2^{1-r}$. In other words, $P[k-1, \text{valid}] \geqslant 1 - (k-1)2^{1-r}$. When we add the k-th problem, we have

$$P[k, \text{valid}] = P[k-1, \text{valid}]P[1, \text{valid}|\text{assignment valid for previous } k-1 \text{ problems}]$$

where the second factor is a conditional probability: the probability for the assignment to be valid for the one-problem case (the k-th problem), under the condition that the assignment is valid for the $k-1$-problem case (the privious $k-1$ problems). If we use the unconditional probability, the equality becomes an inequality because the unconditional probability is always no larger.

$$P[k, \text{valid}] \geqslant P[k-1, \text{valid}]P[1, \text{valid}]$$

The inequality can be further relaxed,

$$\begin{aligned} P[k, \text{valid}] &\geqslant (1-(k-1)2^{1-r})(1-2^{1-r}) \\ &= 1 - k2^{1-r} + (k-1)2^{2-2r} \\ &\geqslant 1 - k2^{1-r} \end{aligned}$$

which means $P[k, \text{invalid}] \leqslant k2^{1-r}$. Therefore, by induction, the statements is true for all k. Larry fails to choose a valid assignment with probability at most $k2^{1-r}$.