8.511 Problem Set 7

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1 Graphene in a Magnetic Field

1.1 Part (a)

Letting $|H - EI| = 0$, we have

$$E^2 = (\hbar v)^2 (k_x - ik_y)(k_x + ik_y) = (\hbar v|\mathbf{k}|)^2$$

Therefore, we recover the linear spectrum

$$E_{\mathbf{k}}^{\pm} = \pm \hbar v |\mathbf{k}|$$

1.2 Part (b)

$$\begin{aligned}
\pi_x &= -i\partial_x + \frac{e}{\hbar c} A_x = -i\partial_x \\
\pi_y &= -i\partial_y + \frac{e}{\hbar c} A_y = -i\partial_y + \frac{eB}{\hbar c} x
\end{aligned}$$

Therefore, the Schrödinger equation is

$$\hbar v \begin{pmatrix} 0 & -i\partial_x - \partial_y - i\frac{eB}{\hbar c}x \\ -i\partial_x + \partial_y + i\frac{eB}{\hbar c}x & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = E \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Acting by H on both sides, we have

$$(\hbar v)^2 \begin{pmatrix} (-i\partial_x - \partial_y - i\frac{eB}{\hbar c}x)(-i\partial_x + \partial_y + i\frac{eB}{\hbar c}x) & 0 \\ 0 & (-i\partial_x + \partial_y + i\frac{eB}{\hbar c}x)(-i\partial_x - \partial_y - i\frac{eB}{\hbar c}x) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = E^2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

which simplifies to

$$(\hbar v)^2 \begin{pmatrix} -\partial_{xx} + (i\partial_y - \frac{eB}{\hbar c}x)^2 + \frac{eB}{\hbar c}[\partial_x, x] & 0 \\ 0 & -\partial_{xx} + (i\partial_y - \frac{eB}{\hbar c}x)^2 - \frac{eB}{\hbar c}[\partial_x, x] \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = E^2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Since $[\partial_x, x] = 1$, the equation for ϕ_2 is thus

$$(\hbar v)^2 \left(-\partial_{xx} + \left(i\partial_y - \frac{eB}{\hbar c}x \right)^2 - \frac{eB}{\hbar c} \right) \phi_2 = E^2 \phi_2$$

1.3 Part (c)

The y dependence only enters by a phase $e^{ik_y y}$. Replacing $\partial_y \to ik_y$,

$$(\hbar v)^2 \left(-\partial_{xx} + \left(k_y + \frac{eB}{\hbar c} x \right)^2 - \frac{eB}{\hbar c} \right) f(x) = E^2 f(x)$$

This is just a harmonic oscillator equation,

$$\left(-\frac{\hbar^2}{2m} \partial_{xx} + \frac{1}{2} m\omega^2 \left(x + \frac{\hbar c}{eB} k_y \right)^2 \right) f(x) = E' f(x)$$

where

$$\omega = \frac{eB}{mc}$$
$$E' = \frac{E^2}{2mv^2} + \frac{1}{2}\hbar\omega$$

The eigenvalues are

$$E'_n = \left(n + \frac{1}{2} \right) \hbar\omega \qquad (n = 0, 1, 2 \cdots)$$

which means

$$E_n^2 = 2nmv^2 \hbar\omega = \frac{2n\hbar eBv^2}{c} \qquad (n = 0, 1, 2 \cdots)$$

So we get the Landau levels (LL hereafter)

$$E_n^{\pm} = \pm v \sqrt{\frac{2e\hbar}{c} Bn} \qquad (n = 0, 1, 2 \cdots)$$

Plugging in numerical values,

$$E_1^{+} \approx 9.18 \times 10^{-2} \text{ eV} \sim 10^3 \text{ K}$$

This gap is much larger than LL spacing for a free 2DEG under the same magnetic field,

$$\Delta E_{free} = \frac{\hbar eB}{mc} \approx 1.16 \times 10^{-3} \text{ eV} \sim 13.4 \text{ K}$$

1.4 Part (d)

The degeneracy comes from three factors. (1) There are two inequivalent Dirac cones. (2) Spin degeneracy. Zeeman effect is on the order of $\mu_B B \sim 10^{-4}$ eV for $B \sim 10$ T, which is much smaller than LL spacing. So we think of spin degeneracy as not being lifted. (3) $E_n^{\pm}$ does not depend on k_y.

Since the sample is of finite size, k_y takes discrete values $k_y = \frac{2\pi N'}{L_y}$. Moreover, k_y is bounded, for the localization in x direction must stay within the sample.

$$\frac{\hbar c}{eB}(k_{y,\max} - k_{y,\min}) = \frac{hc}{eBL_y} N' = L_x$$

Therefore,

$$N' = \frac{eBL_x L_y}{hc} = \frac{AB}{\phi_0}$$

where A is the size of the sample. Considering two inequivalent Dirac cones and two spins, the degeneracy is

$$N = 4N' = \frac{4AB}{\phi_0}$$

1.5 Part (e)

Suppose n LLs are filled at field intensity B. Since LL degeneracy is linear in B, cnB electrons are accommodated, where c is a porprotional constant. This means that

$$\int_0^{E_F} g(\varepsilon)d\varepsilon = cnB$$

The fact $g(\varepsilon) \propto \varepsilon$ implies that $E_F \propto \sqrt{nB}$. Since $E_F = E_n$, we conclude that LLs scale with $\sqrt{B}$.

2 Shubnikov-de Haas Oscillation

2.1 Part (a)

In SI units, Shubnikov-de Haas relation gives

$$\Delta\left(\frac{1}{B}\right) = \frac{2\pi e}{\hbar S_F}$$

where S_F is the extremal cross-sectional area of the Fermi surface. In free electron model, the Fermi surface is a sphere whose radius is k_F. To calculate k_F, we have

$$\frac{1}{V}\frac{\frac{4}{3}\pi k_F^3 \times 2}{\left(\frac{2\pi}{L}\right)^3} = n$$

Therefore, $k_F = \sqrt[3]{3\pi^2 n}$, where n is the spatial density of electrons. Since Na has bcc lattice and is of valence 1,

$$n = \frac{2}{a^3} = \frac{2}{(429.06\text{ pm})^3} = 2.532 \times 10^{28}\text{ m}^{-3}$$

Therefore, $k = 9.08 \times 10^9\text{ m}^{-1}$, and the period is

$$\Delta\left(\frac{1}{B}\right) = \frac{2e}{\hbar k_F^2} = 3.68 \times 10^{-5}\text{ T}^{-1}$$

2.2 Part (b)

$B = 10$ T corresponds to

$$n = \frac{\frac{1}{B}}{\Delta\left(\frac{1}{B}\right)} \approx 2716$$

Therefore, the area enclosed by the real space orbital is

$$A = \frac{\Phi}{B} = \frac{n\Phi_0}{B} = \frac{nh}{eB} = 1.12 \times 10^{-12}\text{ m}^2$$