Electron lifetime (detailed calculation). Yijun Jiang 1

(a). $\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} = \dfrac{\hbar^2|\vec{k}+\vec{q}|^2}{2m} - \dfrac{\hbar^2 k^2}{2m}$

$= \dfrac{\hbar^2}{m}\vec{k}\cdot\vec{q} = \dfrac{\hbar^2}{m} k_F q_{\parallel} \underset{\text{should}}{=} \hbar\omega(q) = \hbar V_s q$

Therefore, $\dfrac{\hbar k_F}{m} q_{\parallel} = V_F q_{\parallel} = V_s q$. so $q_{\parallel} = \dfrac{V_s}{V_F} q \ll q$

This means that $q \simeq q_{\perp}$

Moreover, $E = \hbar\omega(q) \ll \hbar\omega_D = \hbar k_F V_s$, so $q \ll k_F$

Therefore, we get the following hierarchy:

$q_{\parallel} \ll q_{\perp} \ll k_F$

$|g(q)| = \dfrac{1}{V}\dfrac{4\pi e^2}{k_{TF}^2}\sqrt{\dfrac{N\hbar}{2M}}\dfrac{1}{V_s\sqrt{\omega(q)}}$

V: volume. $N = n_0 V$. $k_{TF}^2 = 4\pi e^2 \cdot 2N(0)$

$N(0) = \dfrac{1}{2\pi^2}\dfrac{2M}{\hbar^2}k_F$

Therefore,

$|g(q)| = \dfrac{1}{V}\dfrac{\pi^2\hbar^2}{2mk_F}\sqrt{\dfrac{N\hbar}{2M}}\dfrac{1}{V_s}\sqrt{\omega(q)}$

Fermi's Golden Rule:

$$\frac{\hbar}{\tau_{\vec{k}}} = 2\pi\sum_q |g(q)|^2 \left\{ \begin{array}{l} (1-f(\varepsilon_{\vec{k}+\vec{q}}))(n(\omega)+1)\,\delta(\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} + \omega(q)) + \\ (1-f(\varepsilon_{\vec{k}+\vec{q}}))\,n(\omega)\,\delta(\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} - \omega(q)) \end{array} \right\}$$

$$\xrightarrow{T=0} 2\pi \sum_{\substack{q \\ |\vec{k}+\vec{q}|>k_F}} \frac{1}{V^2}\frac{\pi^4}{k_F^2}\left(\frac{\hbar^2}{2m}\right)^2 \frac{n_0 V\hbar}{2M}\frac{V_s\cdot q}{V_s^2}\cdot\delta(\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} + \omega(q))$$

Use the following simplifications:

$3\pi^2 n = 3\pi^2 n_0 = k_F$

$\delta(\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} + \omega(q)) = \delta(\hbar V_F q_{\parallel} - \hbar V_s q_{\perp}) = \dfrac{1}{\hbar V_F}\delta(q_{\parallel} - \dfrac{V_s}{V_F}q_{\perp})$

$\sum_q \longrightarrow \dfrac{(2\pi)^3}{V}\int d^3\vec{q} = \dfrac{(2\pi)^3}{V}\int d^2 q_{\perp}\int dq_{\parallel}$

$|q| \leq \dfrac{E}{\hbar V_s}$

$$\frac{\hbar}{\tau_E} = \frac{\pi\cdot(2\pi)^3}{3}\,\frac{1}{M}\left(\frac{\hbar^2}{2m}\right)^2\frac{k_F}{v_F}\,\frac{1}{v_s}\int_{|q|\le\frac{E}{\hbar v_s}} q_\perp\, d^2q_\perp\, dq_\parallel\,\delta\!\left(q_\parallel-\frac{v_s}{v_F}q_\perp\right)$$

$$\frac{1}{\tau_E} = \frac{\pi\cdot(2\pi)^3}{6}\,\frac{1}{M}\cdot\frac{\hbar^2}{2m}\,\frac{1}{v_s}\int_0^{\frac{E}{\hbar v_s}} q_\perp\cdot 2\pi q_\perp\, dq_\perp$$

Use $v_s^2 = \frac{m}{3M}v_F^2 = \frac{1}{3mM}\hbar^2 k_F^2$,

$$mM = \frac{1}{3}\,\frac{\hbar^2 k_F^2}{v_s^2}$$

$$\frac{1}{\tau_E} = \frac{\pi(2\pi)^3}{12}\cdot\cancel{3}\cdot\frac{v_s^{\cancel{2}}}{\cancel{\hbar^2}k_F^2}\cdot\cancel{\hbar^2}\cdot\cancel{\frac{1}{v_s}}\cdot 2\pi\cdot\cancel{\frac{1}{3}}\left(\frac{E}{v_s}\right)^3\frac{1}{\hbar^3}$$

$$= \frac{4\pi^5}{3}\,\frac{E^3}{v_s^2 k_F^2\hbar^3}$$

Since $\omega_D = v_s k_F$,

$$\frac{1}{\tau_E} \sim \frac{1}{\hbar}\,\frac{E^3}{(\hbar\omega_D)^2}$$

(b) Add $1-\cos\theta \approx \frac{1}{2}\theta^2 = \frac{1}{2}\left(\frac{q_\perp}{k_F}\right)^2$ to the integral.

$$\frac{1}{\tau_E} = 4\pi^5\,\frac{v_s}{k_F^2}\int_0^{\frac{E}{\hbar v_s}}\frac{1}{2k_F^2}\,q_\perp^4\, dq_\perp$$

$$= \frac{2\pi^5}{5}\,\frac{v_s}{k_F^4}\,\frac{E^5}{\hbar^5 v_s^5}$$

$$= \frac{2\pi^5}{5}\,\frac{E^5}{v_s^4 k_F^4\hbar^5}$$

$$\sim \frac{1}{\hbar}\,\frac{E^5}{(\hbar\omega_D)^4}$$