Lecture 4 - Theory of Choice and Individual Demand

Sari Pekkala Kerr, MIT Department of Economics

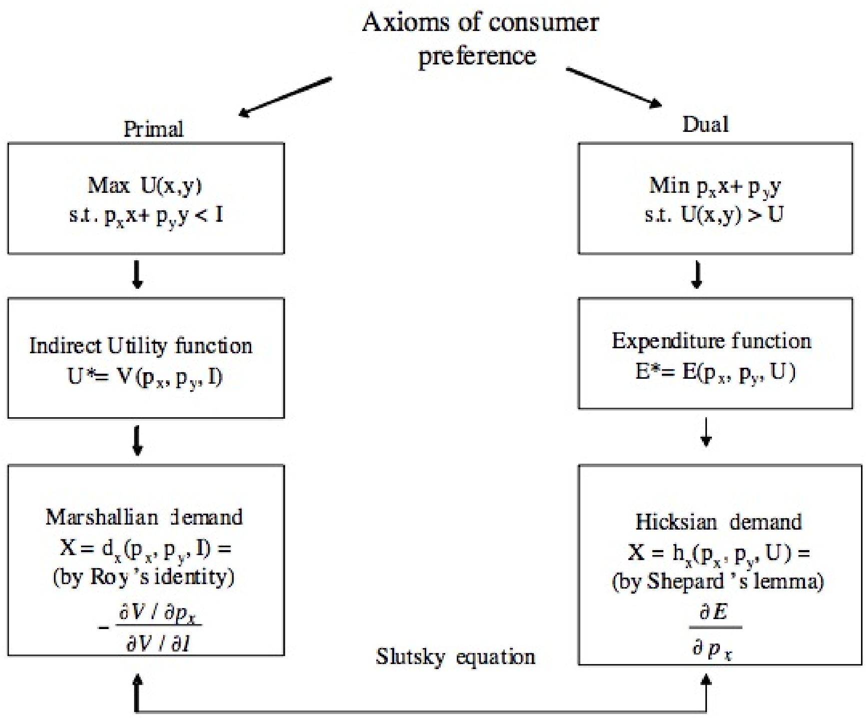
Compiled on 9/12/2015

1 Theory of consumer choice

Agenda for next several lectures

1. Utility maximization
2. Indirect utility function
3. Application: Waldfogel paper on gift giving
4. Expenditure function
5. Relationship between Expenditure function and Indirect utility function
6. Demand functions
7. Application: Cohen and Dupas paper on insecticide treated bed nets (ITNs)
8. Income and substitution effects
9. Normal and inferior goods
10. Compensated and uncompensated demand (Hicksian, Marshallian)
11. Application: Jensen and Miller paper on Giffen goods
12. Application: Miller and Urdinola paper on coffee price fluctuations and child survival in Colombia

Roadmap:



2 Utility maximization subject to budget constraint

Ingredients

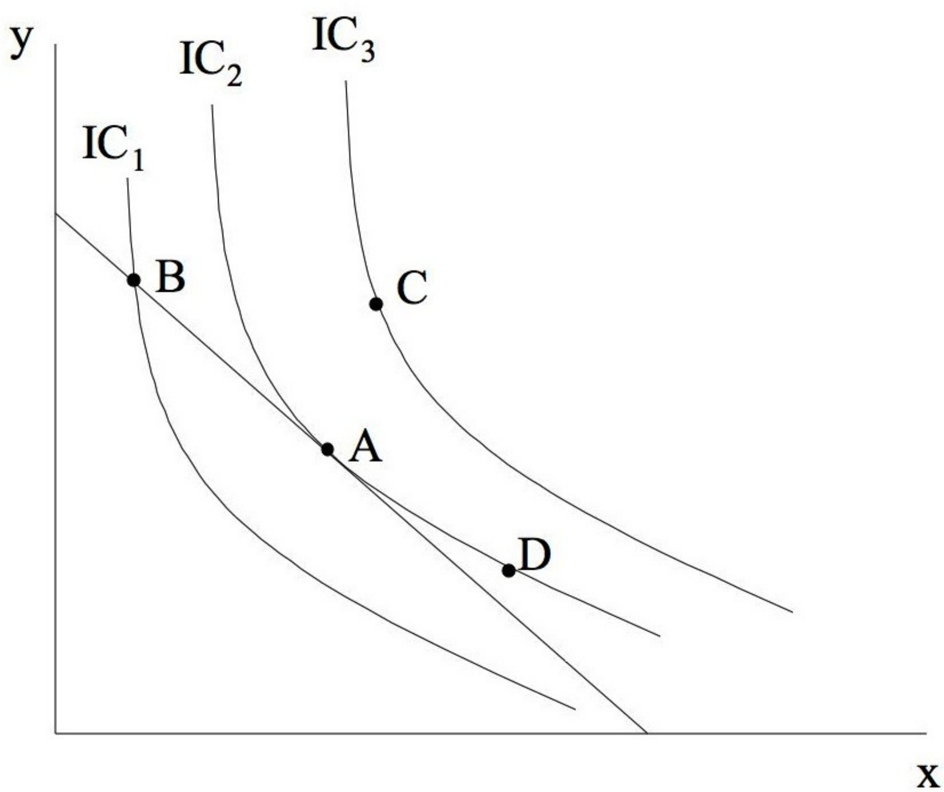
- Utility function (preferences)
- Budget constraint
- Price vector

Consumer's problem

- Maximize utility subject to budget constraint.
- Characteristics of solution:
  - Budget exhaustion (non-satiation)
  - For most solutions: psychic trade-off = monetary payoff
  - Psychic trade-off is MRS

- Monetary trade-off is the price ratio

- From a visual point of view utility maximization corresponds to point A in the diagram below
  - The slope of the budget set is equal to $-\frac{p_x}{p_y}$
  - The slope of each indifference curves is given by the MRS

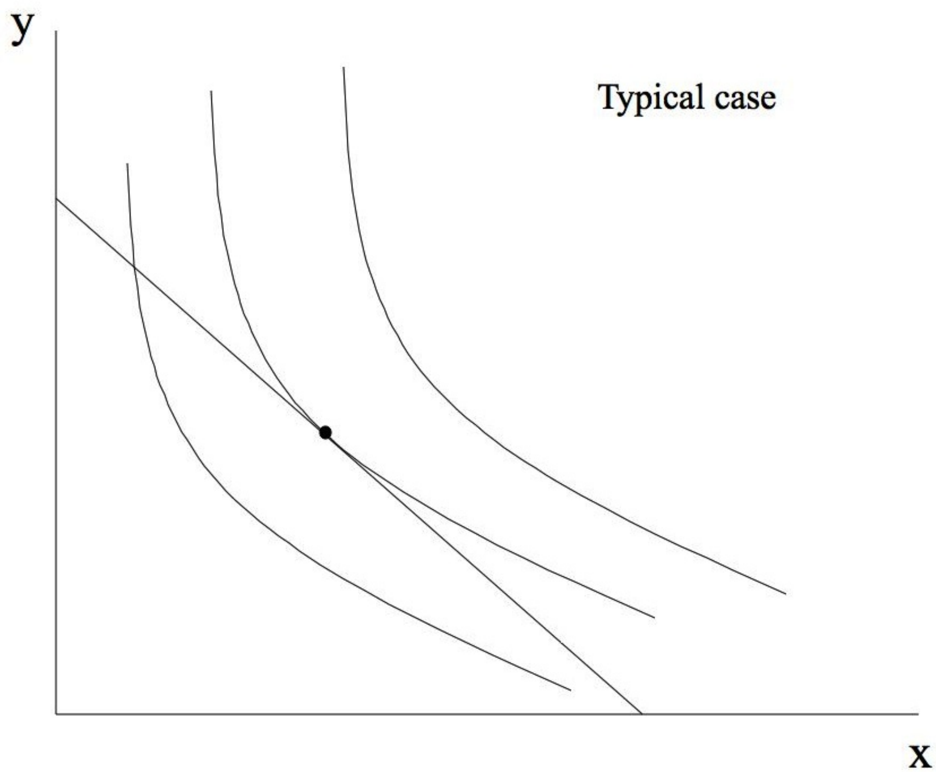


- We can see that $A\ ^P\ B$, $A\ ^I\ D$, $C\ ^P\ A$. Why should one choose A?

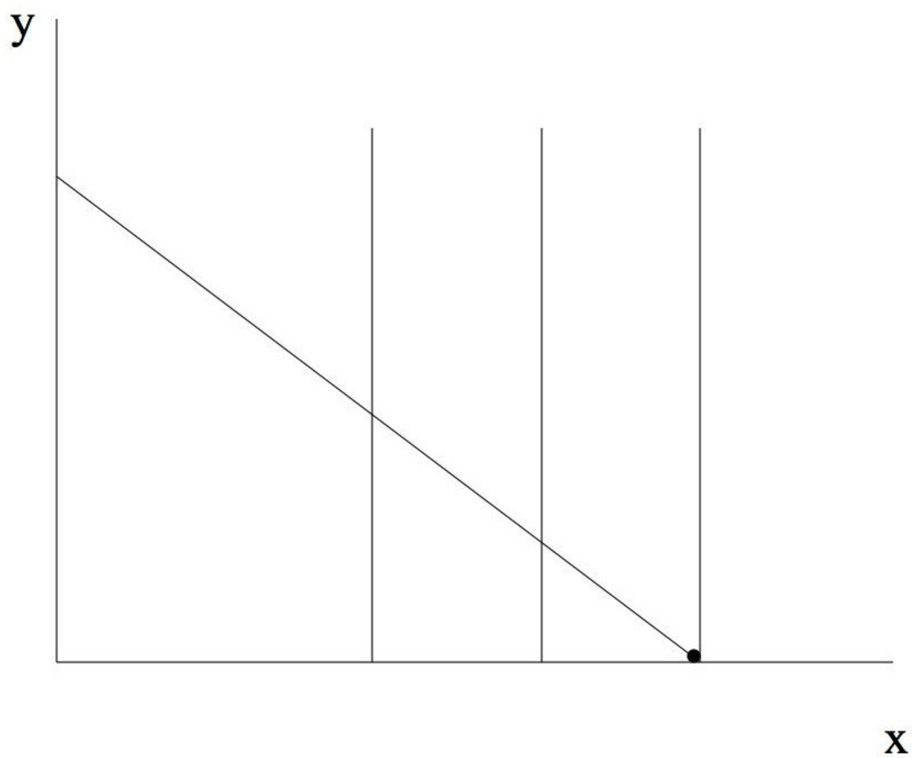
2.1 Interior and corner solutions

There are two types of solution to this problem, interior solutions and corner solutions

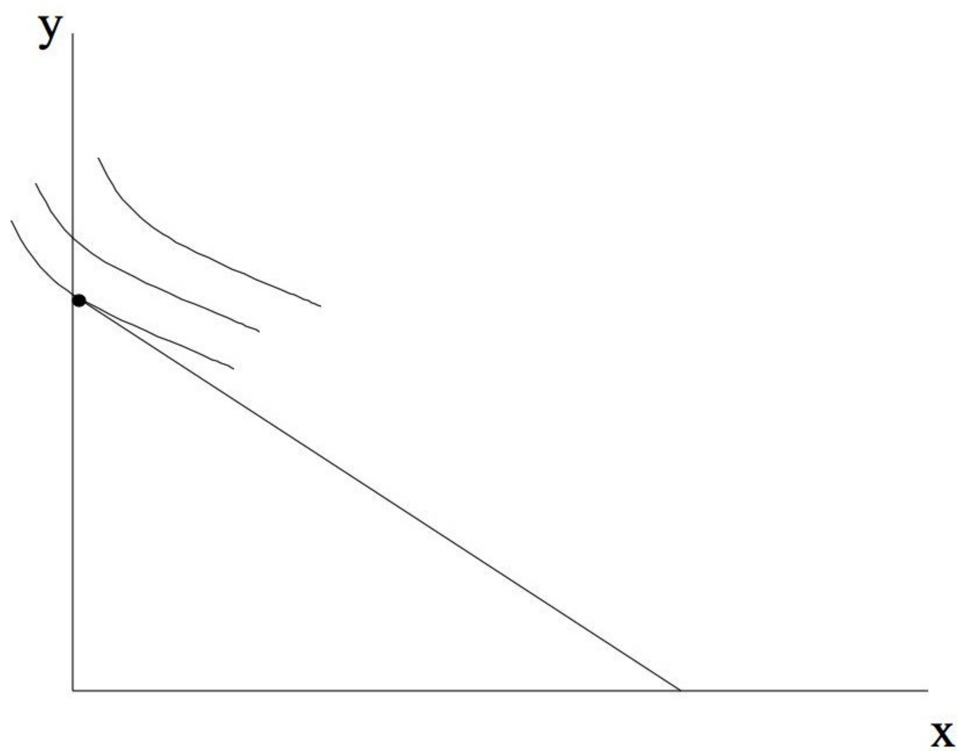
- The figure below depicts an interior solution



- The next figure depicts a corner solution. In this specific example the shape of the indifference curves means that the consumer is indifferent to the consumption of good y. Utility increases only with consumption of x. Thus, the consumer purchases x exclusively.

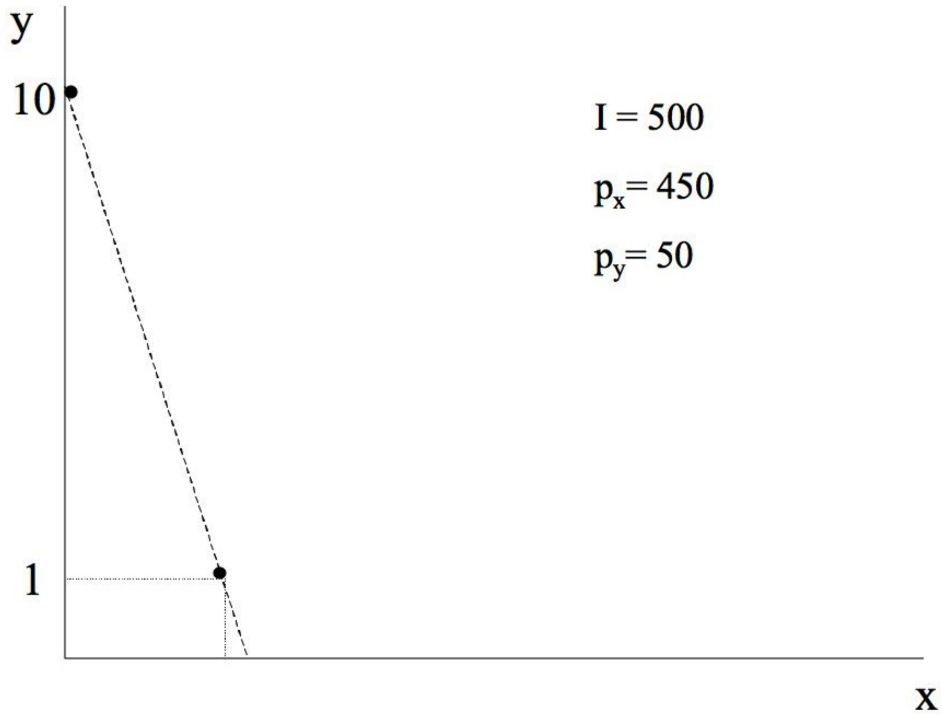


- In the following figure, the consumer's preference for y is sufficiently strong relative to x that the the psychic trade-off is always lower than the monetary trade-off. (This must be the z case for many products that we don't buy.)



- What this means is that the corners (more precisely, the axes), serve as constraints. The consumer would prefer to choose a bundle with negative quantities of x and positive quantities of y. That's not feasible. But to solve the problem using the Lagrangian method, we'd need to specifically impose these non-negativity constraints to prevent a non-sensical solution.

- Another type of "corner" solution can result from indivisibilities the bundle (often called integer constraints).



- Given the budget and set of prices, only two bundles are feasible—unless the consumer could purchase non-integer quantities of good x. We normally abstract from indivisibility.

- Going back to the general case, how do we know a solution exists for consumer, i. e. how do we know the consumer can choose? The axiom of completeness guarantees this. Every bundle is on some indifference curve and can therefore be ranked: $A\ ^I\ B$, $A \succ B$, $B \succ A$.

2.2 Mathematical solution to the Consumer's Problem

- Mathematics

$$
\begin{aligned}
& \max_{x,y} U(x,y) \\
s.t. \ \ p_x x + p_y y &\leq I \\
L &= U(x,y) + \lambda(I - p_x x - p_y y) \\
1. \qquad \frac{\partial L}{\partial x} &= U_x - \lambda p_x = 0 \\
2. \qquad \frac{\partial L}{\partial y} &= U_y - \lambda p_y = 0 \\
3. \qquad \frac{\partial L}{\partial \lambda} &= I - p_x x - p_y y = 0
\end{aligned}
$$

- Rearranging (1) and (2):

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

  This means that the psychic trade-off is equal to the monetary trade-off between the two goods.

- Equation (3) states that budget is exhausted (non-satiation).

- Also notice that:

$$
\begin{aligned}
\frac{U_x}{p_x} &= \lambda \\
\frac{U_y}{p_y} &= \lambda
\end{aligned}
$$

- What is the meaning of λ?

2.3 Interpretation of λ, the Lagrange multiplier

- At the solution of the Consumer's problem (more specifically, an interior solution), the following conditions will hold:

$$\frac{\partial U/\partial x_1}{p_1} = \frac{\partial U/\partial x_2}{p_2} = ... = \frac{\partial U/\partial x_n}{p_n} = \lambda$$

  This expression says that at the utility-maximizing point, the next dollar spent on each good yields the same marginal utility.

- So what is $\frac{dU^*}{dI}$? Return to Lagrangian:

$$\begin{aligned}
L &= U(x,y) + \lambda(I - p_x x - p_y y) \\
\frac{\partial L}{\partial x} &= U_x - \lambda p_x = 0 \\
\frac{\partial L}{\partial y} &= U_y - \lambda p_y = 0 \\
\frac{\partial L}{\partial \lambda} &= I - p_x x - p_y y = 0 \\
\frac{dL}{dI} &= \left(U_x \frac{\partial x^*}{\partial I} - \lambda p_x \frac{\partial x^*}{\partial I}\right) + \left(U_y \frac{\partial y^*}{\partial I} - \lambda p_y \frac{\partial y^*}{\partial I}\right) + \lambda
\end{aligned}$$

  By substituting $\lambda = \left.\frac{U_x}{p_x}\right|_{x=x^*}$ and $\lambda = \left.\frac{U_y}{p_y}\right|_{y=y^*}$, we see that both expressions in parenthesis are zero.

- We conclude that:

$$\frac{dL}{dI} = \frac{\partial L}{\partial I} = \lambda$$

  λ equals the "shadow price" of the budget constraint, i.e. it expresses the quantity of utils that could be obtained with the next dollar of consumption. *Note that this expression only holds when $x = x^*$ and $y = y^*$. If x and y were not at their optimal values, then the total derivative of L with respect to I would also include additional cross-partial terms. These cross-partials are zero at $x = x^*$ and $y = y^*$.*

- What does the "shadow price" mean? It's essentially the "utility value" of relaxing the budget constraint by one unit (e.g., one dollar).

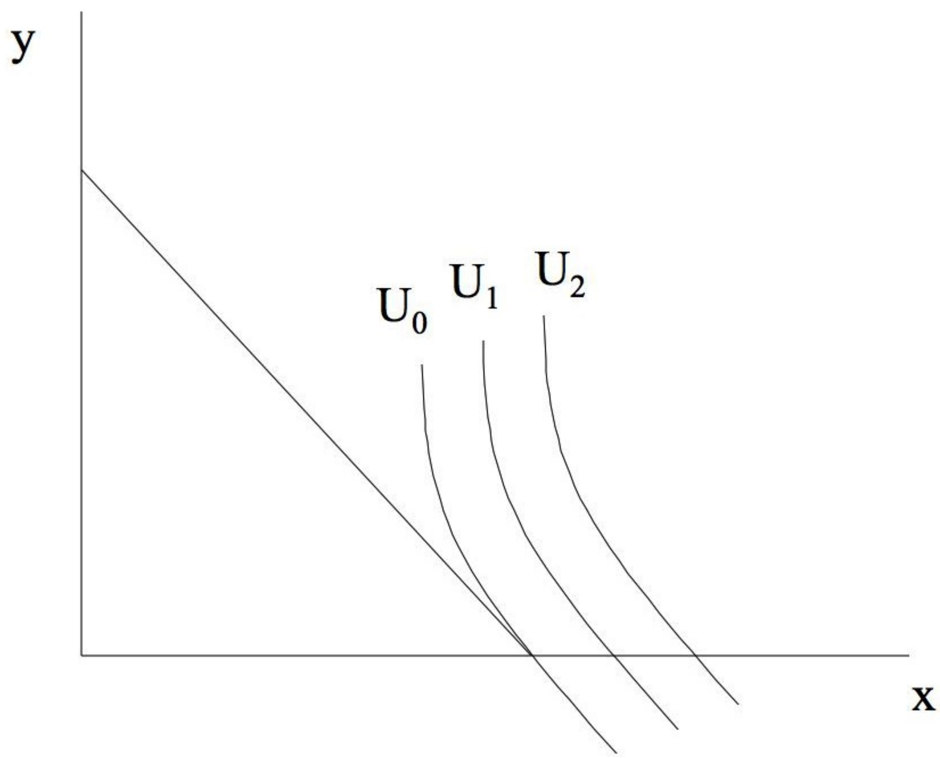
- Note that this shadow price is not uniquely defined since it corresponds to the marginal utility of income in "utils"–an ordinal value. Thus, the shadow price is defined only up to a monotonic transformation.

- We could also have determined that $dL/dI = \lambda$ without calculations by applying the envelope theorem. Note that the envelope theorem for constrained problems says that $\frac{dU^*}{dI} = \frac{\partial L}{\partial I} = \lambda$. At the utility maximizing solution to this problem, x^* and y^* are already optimized and so an infinitesimal change in I does not alter these choices. Hence, the effect of I on U depends only on its direct effect on the budget constraint and does not depend on its indirect effect (due to re-optimization) on the choices of x and y. This "envelope" result is only true in a small neighborhood around the solution to the original problem.

2.4 Corner solutions

- When at a corner solution, consumer buys zero of some good and spends the entire budget on other goods.

- What problem does this create for the Lagrangian?



- The problem above is that a point of tangency doesn't exist for positive values of y. Hence we also need to impose "non-negativity constraints": $x \geq 0$, $y \geq 0$. This will not be important for problems in this class, but it's easy to add these constraints to the maximization problem.

2.5 An Example Problem

- Consider the following example problem:

$$U(x, y) = \frac{1}{4} \ln x + \frac{3}{4} \ln y$$

- Notice that this utility function satisfies all axioms:

  1. Completeness, transitivity, continuity [these are pretty obvious]
  2. Non-satiation: $U_x = \frac{1}{4x} > 0$ for all $x > 0$. $U_y = \frac{3}{4y} > 0$ for all $y > 0$. In other words, utility rises continually with greater consumption of either good, though the rate at which it rises declines (diminishing marginal utility of consumption).
  3. Diminishing marginal rate of substitution:
     - Along an indifference curve of this utility function: $\bar{U} = \frac{1}{4}\ln x_0 + \frac{3}{4}\ln y_0$.
     - Totally differentiate: $0 = \frac{1}{4x_0}dx + \frac{3}{4y_0}dy$.
     - Which provides the marginal rate of substitution $-\frac{dy}{dx}|_{\bar{U}} = \frac{U_x}{U_y} = \frac{4y_0}{12x_0}$.
     - The marginal rate of substitution of x for y is increasing in the amount of y consumed and decreasing in the amount of x consumed; holding utility constant, the more y the consumer has, the more y he would give up for one additional unit of x.

- Example values*:* $p_x = 1$, $p_y = 2$, $I = 12$. Write the Lagrangian for this utility function given prices and income:

$$
\begin{aligned}
& & \max_{x,y} U(x,y) \\
s.t.\ \ p_x x + p_y y &\leq& I \\
L &=& \frac{1}{4}\ln x + \frac{3}{4}\ln y + \lambda(12 - x - 2y) \\
1. \qquad \frac{\partial L}{\partial x} &=& \frac{1}{4x} - \lambda = 0 \\
2. \qquad \frac{\partial L}{\partial y} &=& \frac{3}{4y} - 2\lambda = 0 \\
3. \qquad \frac{\partial L}{\partial \lambda} &=& 12 - x - 2y = 0
\end{aligned}
$$

- Rearranging (1) and (2), we have

$$
\begin{aligned}
\frac{U_x}{U_y} &=& \frac{p_x}{p_y} \\
\frac{1/4x}{3/4y} &=& \frac{1}{2}
\end{aligned}
$$

- The interpretation of this expression is that the MRS (psychic trade-off) is equal to the market trade-off (price-ratio).

- What's $\frac{dL}{dI}$? As before, this is equal to λ, which from (1) and (2) is equal to:

$$\lambda = \frac{1}{4x^*} = \frac{3}{8y^*}.$$

The next dollar of income could buy one additional x, which has marginal utility $\frac{1}{4x^*}$ or it could buy $\frac{1}{2}$ additional y, which provides marginal utility $\frac{3}{4y^*}$ (so, the marginal utility increment is $\frac{1}{2} \cdot \frac{3}{4y^*}$).

- It's important that $dL/dI = \lambda$ is defined in terms of the optimally chosen x^*, y^* . Unless we are at these optima, the envelope theorem does not apply. In that case, dL/dI would also depend on the cross-partial terms: $\left(U_x \frac{\partial x}{\partial I} - \lambda p_x \frac{\partial x}{\partial I}\right) + \left(U_y \frac{\partial y}{\partial I} - \lambda p_y \frac{\partial y}{\partial I}\right)$.

- Incidentally, you should be able to solve for the prices and budget given, $x^* = 3, y^* = 4.5$.

- Having solved that, you can verify that $\frac{1}{4x^*} = \frac{3}{8y^*} = \lambda$. That is, at prices $p_x = 1$ and $p_y = 2$ and consumption choices $x^* = 3, y^* = 4.5$, the marginal utility of a dollar spent on either good x or good y is identical.

2.6 Lagrangian with Non-negativity Constraints [Optional]

$$\begin{aligned}
& \max U(x,y) \\
s.t.\ p_x x + p_y y &\leq I \\
y &\geq 0 \\
L &= U(x,y) + \lambda(I - p_x x - p_y y) + \mu\,(y - 0) \\
\frac{\partial L}{\partial x} &= U_x - \lambda p_x = 0 \\
\partial L &= U_y - \lambda p_y + \mu = 0 \\
\mu y &= 0
\end{aligned}$$

- Final equation above implies that $\mu = 0$, $y = 0$, or both. (This is called a "complementary slackness" condition; either the constraint is slack, implying $\mu = 0$, or the constraint is binding, implying that $y = 0$, and so in either case, we have that the product $\mu y = 0$.)

- We then have three cases.

1. $y = 0,\ \mu \neq 0$ (since $\mu \geq 0$ then it must be the case that $\mu > 0$)

$$\begin{aligned} U_y - \lambda p_y + \mu &= 0 \longrightarrow U_y - \lambda p_y < 0 \\ \frac{U_y}{p_y} &< \lambda \\ \frac{U_x}{p_x} &= \lambda \end{aligned}$$

   Combining the last two expressions:

$$\frac{U_x}{U_y} > \frac{p_x}{p_y}$$

   This consumer would like to consume even more x and less y, but she cannot.

2. $y \neq 0,\ \mu = 0$

$$\begin{aligned} U_y - \lambda p_y + \mu &= 0 \longrightarrow U_y - \lambda p_y = 0 \\ \frac{U_y}{p_y} &= \frac{U_x}{p_x} = \lambda \end{aligned}$$

   Standard FOC, here the non-negativity constraint is not binding.

3. $y = 0,\ \mu = 0$

   Same FOC as before:

$$\frac{p_x}{p_y} = \frac{U_x}{U_y}$$

   Here the non-negativity constraint is satisfied with equality so it doesn't distort consumption.

3 Indirect Utility Function

- For any:
  - Budget constraint
  - Utility function
  - Set of prices

We obtain a set of optimally chosen quantities:

$$\begin{aligned} x_1^* &= x_1(p_1, p_2, ..., p_n, I) \\ &... \\ x_n^* &= x_n(p_1, p_2, ..., p_n, I) \end{aligned}$$

So when we say

$$\max U(x_1, ..., x_n) \ s.t. \ PX \leq I$$

we get as a result:

$$U(x_1^*(p_1, ..., p_n, I), ..., x_n^*(p_1, ..., p_n, I)) \ \equiv V(p_1, ..., p_n, I).$$

We call $V(\cdot)$ the "Indirect Utility Function." This is the value of maximized utility under given prices and income.

- So remember the distinction:
  - Direct utility: utility from consumption of $(x_1, ..., x_n)$
  - Indirect utility: utility obtained when facing $(p_1, ..., p_n, I)$

- Example

$$\begin{aligned} \max U(x, y) &= x^{.5}y^{.5} \\ s.t. \ p_x x + p_y y &\leq I \\ L &= x^{.5}y^{.5} + \lambda(I - p_x x - p_y y) \\ \frac{\partial L}{\partial x} &= .5x^{-.5}y^{.5} - \lambda p_x = 0 \\ \frac{\partial L}{\partial y} &= .5x^{.5}y^{-.5} - \lambda p_y = 0 \\ \frac{\partial L}{\partial \lambda} &= I - p_x x - p_y y = 0 \end{aligned}$$

- We obtain the following:

$$\lambda = \frac{.5x^{-.5}y^{.5}}{p_x} = \frac{.5x^{.5}y^{-.5}}{p_y},$$

which simplifies to:

$$x = \frac{p_y y}{p_x}.$$

- Substituting into the budget constraint gives us

$$\begin{aligned} I - p_x \frac{p_y y}{p_x} - p_y y &= 0 \\ p_y y &= \frac{1}{2} I, \quad p_x x = \frac{1}{2} I \end{aligned}$$

$$x^* = \frac{I}{2p_x}, \quad y^* = \frac{I}{2p_y}$$

Half of the budget goes to each good.

- Thus, for a consumer with $U(x, y) = x^{0.5} y^{0.5}$, budget I, and facing prices p_x and p_y will choose x^* and y^* and obtain utility:

$$U(x^*, y^*) = \left(\frac{I}{2p_x} \right)^{.5} \left(\frac{I}{2p_y} \right)^{.5}.$$

Thus, the indirect utility for this consumer is

$$V(p_x, p_y, I) = U(x^*(p_x, p_y, I), y^*(p_x, p_y, I)) = \left(\frac{I}{2p_x} \right)^{.5} \left(\frac{I}{2p_y} \right)^{.5}$$

- Why bother calculating the indirect utility function? It saves us time. Instead of recalculating the utility level for every set of prices and budget constraints, we can plug in prices and income to get consumer utility. This comes in handy when working with individual demand functions. Demand functions give the quantity of goods purchased by a given consumer as a function of prices and income (or utility).

4 The Carte Blanche Principle

- One immediate implication of consumer theory is that consumers make optimal choices for themselves given prices, constraints, and income. [Generally, the only constraint is that they can't spend more than their income, but we'll see examples where there are additional constraints.]

- This observation gives rise to the Carte Blanche principle: consumers are always weakly better off receiving a cash transfer than an in-kind transfer of identical monetary value. [Weakly better off in that they may be indifferent between the two.]

- With cash, consumers have Carte Blanche to purchase whatever bundle or goods are services they can afford – including the good or service that alternatively could have been transferred to them in-kind.

- Prominent examples of in-kind transfers given to U.S. citizens include Food Stamps, housing vouchers, health insurance (Medicaid), subsidized educational loans, child care services, job training, etc. [An exhaustive list would be long indeed.]

- Economic theory suggests that, relative to the equivalent cash transfer, these in-kind transfers serve as *constraints* on consumer choice. If consumers are rational, constraints on choice cannot be beneficial.

- For example, consider a consumer who has income $I = 100$ and faces the choice of two goods, food and housing, at prices p_f, p_h, each priced at 1 per unit. The consumer's problem is

$$\begin{aligned} & \max_{f,h} U(f,h) \\ s.t.\ f + h & \leq 100 \end{aligned}$$

- The government decides to provide a housing subsidy of 50. This means that the consumer can now purchase up to 150 units of housing but no more than 100 units of food. The consumer's problem is:

$$\begin{aligned} & \max_{f,h} U(f,h) \\ s.t.\ f + h & \leq 150 \\ h & \geq 50. \end{aligned}$$

- Alternatively, if the government had provided 50 dollars in cash instead, the problem would be:

$$\begin{aligned} & \max_{f,h} U(f,h) \\ s.t.\ f + h & \leq 150. \end{aligned}$$

- The government's transfer therefore has two components:

  1. An expansion of the budget set from I to $I' = I + 50$.
  2. The imposition of the constraint that $h \geq 50$.

- The canonical economist's question is: why do both (1) and (2) when you can just do (1) and potentially improve consumer welfare at no additional cost to the government?

(Of course, I don't expect you to accept this argument as gospel truth. But it's a good default position—better, perhaps, than the alternative default that it's better for the government to dictate choices to consumers than to allow them to make them for themselves.)

5 A Simple Example: The Deadweight Loss of Christmas

- Joel Waldfogel's 1993 *American Economic Review* paper provides a stylized (and controversial) example of the application of the Carte Blanche principle.
- Waldfogel observes that gift-giving is equivalent to an in-kind transfer and hence should be less efficient for consumer welfare than simply giving cash.
- In January, 1993, he asked the following questions of approximately 150 Yale undergraduates about holiday gifts received in 1992:
  1. What were the gifts worth in cash value (purchase price)?
  2. How much would the students be willing to pay for these gifts if they didn't already have them?
  3. How much would the students we willing to accept in cash in lieu of the gifts? (Usually higher than willingness to pay – an economic anomaly.)
- For each gift, Waldfogel calculated the gift's "yield," $Y_j = V_j/P_j$, where P_j is the purchase price and V_j is the student's willingness to pay.
- As theory (and intuition) would predict, the yield was, on average, well below one-hundred percent. Waldfogel concludes that in-kind gift giving "destroys" economic value relative to the cost-equivalent cash gift.
- Figure I of Waldfogel illustrates the idea transparently:

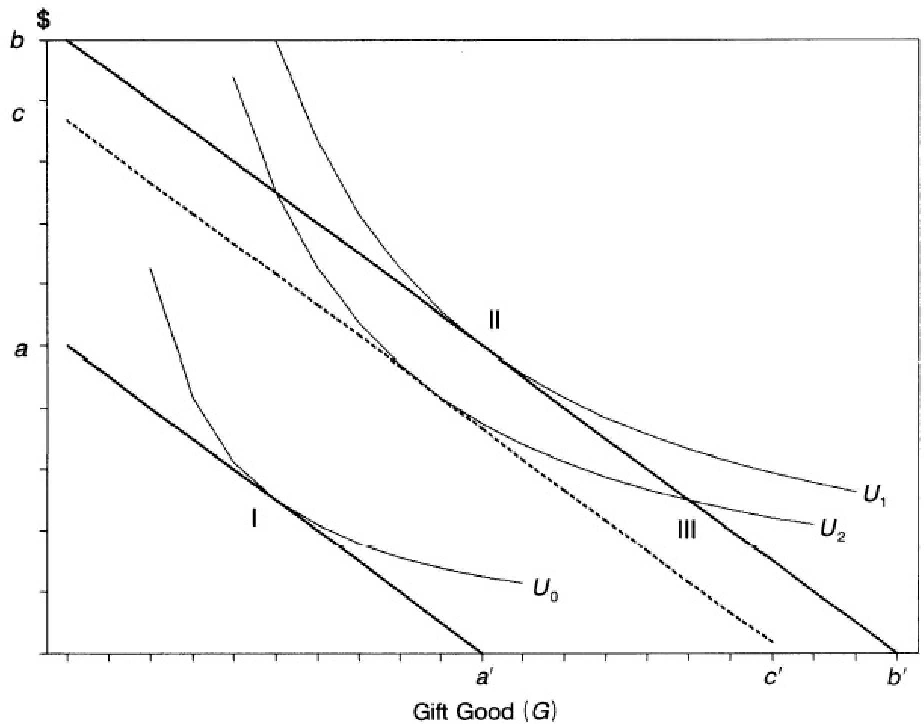


FIGURE 1. GIFT-GIVING AND DEADWEIGHT LOSS

- The budget line aa' is the original budget set.
- The line bb' is the budget set for an in-kind transfer.
- U_1 is the highest feasible indifference curve achievable for this consumer with budget set bb'. This is achieved with consumption bundle II.
- The intersection of U_2 and bb', labeled III, is the consumption bundle with the in-kind gift. The amount of G is selected by the gift-giver rather than the recipient.
- Although III lies on bb', it is not on the highest achievable indifference curve achievable with budget set bb'.
- Line cc' is the actual budget set the consumer would require to attain utility U_2 if his choice set were not constrained by the gift giver.
- The "deadweight loss" of gift-giving relative to the equivalent cash transfer in this example is equal to $(b' - c') \times p_g$.

Several interesting observations from the article:

1. Value "destruction" is greater for distant relatives, e.g., grandparents.

2. Value "preservation" is near-perfect for friends

3. Groups that tend to "destroy" the most value are the most likely to give cash instead

It's useful to interpret the basic regression result given on the top of page 1332:

$$\begin{array}{rcccl} \ln(value_i) & = & -0.314 & + & 0.964 \ln(price_i) \\ & & (0.44) & & (0.08) \end{array}.$$

- The things in parentheses are standard errors. Since 0.964 is much larger than 2×0.08, the relationship between value and price is statistically significant.

- The derivative of value with respect to price is (recall that $\partial/\partial x$ of $\ln x$ is $\partial x/x$):

$$\frac{\partial \ln(value_i)}{\partial \ln(price_i)} = \frac{\partial value_i}{value_i} \cdot \frac{price_i}{\partial price_i} = 0.964.$$

  That is, a 1 percent rise in price translates into a 0.964 percent rise in value.

- But, there is a major difference between the value and price. Rewriting the equation and exponentiating:

$$\begin{aligned} \ln(\text{value}_i) &= \ln(\exp(-0.314)) + 0.964 \ln(\text{price}_i) \\ &= \ln\left(\exp(-0.314) \times \text{price}_i^{0.964}\right) \end{aligned}$$

$$\begin{aligned} \text{value}_i &= \exp(-0.314) \times \text{price}_i^{0.964} \\ &= 0.73 \times \text{price}_i^{0.964} \end{aligned}$$

- So, for a \$100 gift, the approximate recipient valuation is about \$62.

- You can see why it's handy to use natural logarithms to express these relationships. They readily allow for proportional effects. The regression equation above says that the value of a gift is *approximately* equal to 96% of its price minus approximately 31 percent (logarithmic transformations are non-linear, which is why we emphasize the word approximate).

- The Waldfogel article generated a surprising amount of controversy, even among economists, most of whom probably subscribe to the Carte Blanche principle. But to many non-economist readers, this article seems to exemplify the well-worn gripe about economists, "They know the price of everything and the value of nothing."

- *What is Waldfogel missing?*