04/12 Thurs

f function $(a,b) \to R$

Assume f has derivatives all order at $c \in (a,b)$

$$T(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

$$= \sum_{k \geq 0} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$R_n(x) = f(x) - T_{n-1}(x)$$

$$= f(x) - \sum_{k \leq n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

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→ idea is to make $R_n(x)$ smaller as $n \to \infty$

ex) $R_1(x) = f(x) - f(c)$

$R_2(x) = f(x) - f(c) - f'(c)(x-c)$

<u>Thm</u> <Taylor's Thm>

Assume that f has derivatives up to order n on $(a,b) \to f^{(n)}(x)$ exists $\forall x \in (a,b)$

Then, $\forall x_0 \in (a,b)$, there is a y_0 between x_0 & c s.t. $R_n(x_0) = \dfrac{f^{(n)}(y_0)}{n!}(x_0-c)^n$

ex) $n=1 \to R_1(x_0) = f(x_0) - f(c)$

But the Taylor's Thm says $\exists y_0$ between x_0 and c, so, it's equal to

$$\frac{f'(y_0)}{1!}(x_0-c)^1 = f'(y_0)(x_0-c)$$

so, it can be think of it as upgraded version of M.V.T.

pf) call M to be the # s.t.

$$f(x_0) = T_{n-1}(x_0) + \frac{M(x_0-c)^n}{n!}$$

$$= \sum_{k \le n-1} \frac{f^{(k)}(c)}{k!}(x_0-c)^k + \frac{M(x_0-c)^n}{n!}$$

Then, $R_n(x_0)$ is by def. $f(x_0) - T_{n-1}(x_0) = \dfrac{M(x_0-c)^n}{n!}$

But the problem reduces to show that $\exists y_0$ between x_0 and c s.t. $f^{(n)}(y_0) = M$.

Define $g(x) = T_{n-1}(x) + \frac{M(x-c)^n}{n!}$: This is a degree n polynomial.

$$g(c) = f(c)$$
$$g'(c) = T'_{n-1}(c) = f'(c)$$
$$g''(c) = f''(c)$$
$$\vdots$$
$$g^{(n-1)}(c) = f^{(n-1)}(c)$$
$$g^{(n)}(c) = M$$

→ ☆ cuz $T_{n-1}(x)$ is a degree of $n-1$.

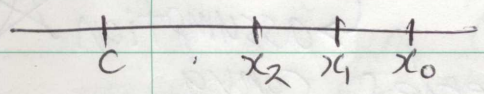
Now if $h(x) = f(x) - g(x)$,

then $h(c) = 0$, $h'(c) = 0$, $h''(c) = 0$, ----

---, $h^{(n-1)}(c) = 0$.

Notice that $h(x_0) = f(x_0) - g(x_0) = 0$,

as M is defined to be $f(x_0) = \underbrace{T_{n-1}(x_0) + \frac{M(x_0-c)^n}{n!}}_{g(x_0)}$



So, $h(c) = h(x_0) = 0$, and Rolle's Thm,

$\exists x_1$ s.t. $h'(x_1) = 0$ between c and x_0

s.t. $h'(c) = h'(x_1) = 0$.

Also, again by Rolle's Thm, $\exists x_2$ s.t. $h''(x_2) = 0$

between c and x_1 s.t. $h''(c) = h''(x_2) = 0$.

And, we keep doing this, we get x_{n-1} s.t.

$h^{(n-1)}(c) = h^{(n-1)}(x_{n-1}) = 0$.

And, again by Rolle's Thm, $\exists x_n$ between c and x_{n-1}

s.t. $h^{(n)}(x_n) = 0$.

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And I know that

$$h(x) = f(x) - \underbrace{T_{n-1}(x)}_{\text{polynomial degree } n} - \frac{M(x-c)^n}{n!}$$

constant

So, $h^{(n)}(x) = f^{(n)}(x) - M$

So, take $y_0 = x_n$, then y_0 is between c and x_0, and also $f^{(n)}(y_0) = M$

i.e. $R_n(x_0) = \frac{f^{(n)}(y_0)}{n!}(x_0-c)^n$

$\boxed{h^{(n)}(x_n) = 0}$ QED

Let's estimate $R_n(x_0) = \frac{f^{(n)}(y_0)}{n!}(x_0-c)^n$

and $\frac{(x_0-c)^n}{n!} \to 0$ as $n \to \infty$.

But, how about $f^{(n)}(y_0)$ ~~[illegible]~~ convg to as $n \to \infty$?

Thm) Let's assume that $\exists c > 0$ s.t. $\forall n$, $\forall x \in (a,b)$: $|f^{(n)}(x)| < c$, then $\forall x$: $R_n(x) \xrightarrow{n\to\infty} 0$

assumption

Actually, $\sup_{(a,b)} |R_n(x)| \to 0$

In this case, the Taylor series convg uniformly to f on (a,b).

So, $|R_n(x_0)| = \left|\frac{f^{(n)}(y_0)}{n!}(x_0-c)^n\right| \le c\frac{|x_0-c|^n}{n!} \xrightarrow{n\to\infty} 0$

and if $-\infty < a, b < +\infty$, $\sup_{(a,b)} |R_n(x_0)| \xrightarrow{n\to\infty} 0$

hence Taylor series uniformly convg to f on (a,b).

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ex) $f(x) = \sin x$.

Taylor series f at $c=0$.

$$T(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

The Thm we just proved shows that $T(x) = f(x)$.

☆ It's sufficient to check that $f(x) = \sin x$ satisfies the assumption of the Thm.

Notice that $f^{(n)}(x)$ is either $\sin x$, $\cos x$, $-\cos x$, $-\sin x$, and c can be 2 !!!

ex) $f(x) = e^x$

$$T(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

If you fix a bounded interval (a, b) then $f^{(n)}(x) = e^x$ for $\forall n$, so,

$$|f^{(n)}(x)| = e^x < c \text{ for } \forall x \in (a,b), \text{ and } \forall n$$

So, the Thm says $T(x)$ convg uniformly to e^x on (a, b).

$T(x) = e^x$ for $\forall x \in \mathbb{R}$ since a and b are arbitrary. ☆

Q. If $f_n \to f$ pointwise & uniform convg and say all f_n have property A, then does f satisfy property A? (say A could be continuity, differentiable, integrable, etc)

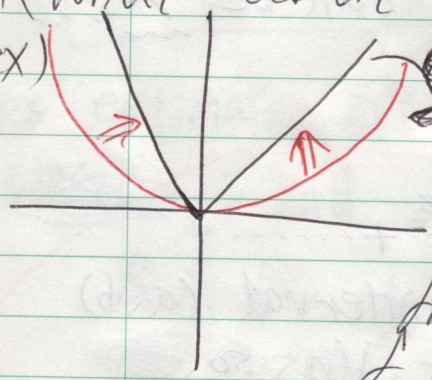
☆ $f_n \to f$ pointwise convg and f_n is continuous, does <u>not</u> mean f is continuous...

↳ ex) $f_n(x) = x^n$ on $[0,1]$.

But, if $f_n \to f$ uniformly convg and f_n is continuous, <u>means</u> f is continuous...

Then, what about differentiability?

ex)



$f(x) = |x|$

So, in this case, even $f_n \to f$ uniformly and f_n is differentiable, f does <u>not</u> have to be differentiable

Even $f_n \to f$ uniformly and f_n & f are differentiable, f_n' may <u>not</u> convg f'!

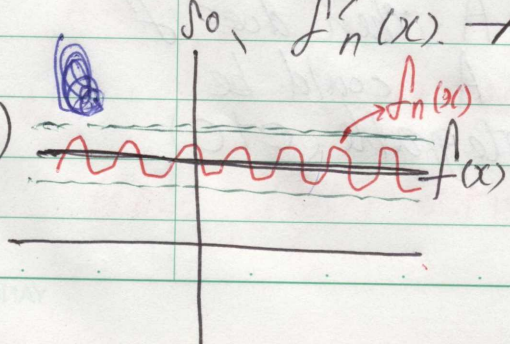
ex) $f_n(x) = \dfrac{\sin nx}{n}$ $\qquad$ $\sup\left|\dfrac{\sin nx}{n}\right| = \dfrac{1}{n}$

$f_n \to f(x) = 0$ (and also uniformly convg)

But, $f_n'(x) = \cos nx$ and $f'(x) = 0$,

So, $f_n'(x) \not\to f'(x)$ as $n \to \infty$.

ex)



(constant)

⊛ function f is integrable if only finite points are not the same

↳ Mid 2 practice ⊛

Now, Integrability.

$f_n \to f$ pointwisely,

f_n is integrable $\not\longrightarrow$ f is integrable.

ex)

f_n is integrable

$f_n(x) = 1$ except at $r_1, \cdots, r_n$

where $f_n(r_1) = f_n(r_2) = \cdots = f_n(r_n) = 0$.

where $\mathbb{Q} \cap [0,1] = \{r_1, r_2, \cdots\}$.

$f_n \to f$ pointwise

If x is irrational, $f_n(x) = 1 \xrightarrow{n\to\infty} f(x) = 1$

If $x \in \mathbb{Q}$, then n is large.

$f_n(x) = 0 \xrightarrow{n\to\infty} f(x) = 0$.