04/03 Tue

Review of Fundamental Thm of calculus

Derivative

$f'(x_0)$ exists if $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists.

Integrability

f is integrable over $[a,b]$ if $U(f) = L(f)$.

where $U(f,p)$ is upper sum w.r.t. sups $\longrightarrow U(f)$

$L(f,p)$ is lower sum w.r.t. infs $\xrightarrow[\text{over } P]{\text{limit}} L(f)$

☆ Criterion $\Rightarrow$ $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. if mesh$(p) < \delta$, then $U(f,p) - L(f,p) < \varepsilon$.

Under appropriate assumptions,

$\int f' = f$

$(\int f)' = f$

The 'F.T.C'

FTC really is two thms...

Thm 1) If f' is integrable, then $\int_a^b f' = f(b) - f(a)$

Sketch of pf) For a partition $p = \{a = t_0 < t_1 < \cdots < t_n = b\}$, we have $f(b) - f(a) = \sum_{k=0}^{n-1} f(t_{k+1}) - f(t_k)$

M.V.T $\leftarrow$ $= \sum_{k=0}^{n-1} f'(c_k)(t_{k+1} - t_k)$, where $c_k \in [t_k, t_{k+1}]$

so, $L(f') \le f(b) - f(a) \le U(f')$

So, $L(f') = f(b) - f(a) = U(f') = \int_a^b f'$, since f' is integrable.

Thm 2) Assume f is integrable on $[a, b]$, if f is continuous at x_0, then $F(x_0) = \int_a^{x_0} f(t)dt$ is differentiable at x_0 and $F'(x_0) = f(x_0)$

Sketch of pf) $F(x) - F(x_0) = \int_a^x f(t)dt - \int_a^{x_0} f(t)dt$

$= \int_{x_0}^{x} f(t)dt$ [here, if $x < x_0$, $\int_{x_0}^{x} f(t)dt$ means $-\int_x^{x_0} f(t)dt$]

The assumption that f is continuous at x_0 implies that when x is near x_0, $f(t)$ is near $f(x_0)$ for $\forall\, t \in [x, x_0]$, and so $\int_{x_0}^{x} f(t)dt \approx f(x_0)(x - x_0)$

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{1}{x - x_0} \int_{x_0}^{x} f(t) - f(x_0)\, dt \right|$$

$$\leq \frac{1}{|x - x_0|} \varepsilon |x - x_0| = \varepsilon \quad \text{when } |x - x_0| < \delta$$

Sequence of functions $\longrightarrow$ ch 24

We already discussed seq of real #, and the notion of a seq $\{x_n\}_{n \in \mathbb{N}}$ converging. Sequences are nice, for example, to find a solution of $f(x) = x$ has? We consider the sequence x_0, $x_1 = f(x_0)$, $x_2 = f(x_1)$, ……

Under some conditions on f (ex、$|f'|<1$)、one can show $\{x_n\}$ convg to some L and $f(L)=L$. Now, we can do sth similar for differential equations, e.g., $f'=f$ form sequence $f_0=1$、$f_1=1+x$、$f_2=1+x+\frac{x^2}{2!}$、----

We need to define the notion of convergence for a seq of functions.

Definition) Suppose $\{f_n\}_{n\in\mathbb{N}}$, $f_n: S\to\mathbb{R}$, we say $\{f_n\}$ converges pointwise to a function $f: S\to\mathbb{R}$ if $\underbrace{\lim_{n\to\infty} f_n(x)}_{\text{limit of seq of real number (for fixed x)}}=f(x)$ for $\forall x\in S$.

ex) $f_n(x)=\frac{1}{n}x$

$$\lim_{n\to\infty} f_n(x)=0$$

Seq of functions

ex) $f(x)=x^n$ on $[0,1]$ → differentiable

$$\lim_{n\to\infty} x^n=\begin{cases}0, & 0\le x<1\\ 1, & x=1\end{cases}$$

→ after pointwise convg not even continuous

☆Note each $f_n=x^n$ is differentiable, but $\lim_{n\to\infty} f_n$ is not even continuous.

Def) $\{f_n\}$ is convg uniformly on S to a function f if $\underbrace{\sup_{x\in S} |f_n(x) - f(x)|}_{\text{seq of real \#}} \xrightarrow{n\to\infty} 0$

In other words, for all $\varepsilon > 0$, $\exists$ a N s.t. for $\forall$ $n > N$ and all $x \in S$, $|f_n(x) - f(x)| < \varepsilon$

ex) $f_n(x) = \frac{x}{n}$ on R, $f_n \to f \equiv 0$

$f_n \longrightarrow f$ not convg uniformly.

ex) $f_n(x) = \frac{x}{n}$ on $\frac{x}{n}$ on $[0,1]$, $f_n \to f \equiv 0$

$f_n \longrightarrow f$ uniformly convg.

Seq of functions

ex) $f_n(x) = x^n$, $f_n \to f = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

a) on $[0,1]$ uniformly convg? $\lim_{n\to\infty} \sup_{[0,1]} |f_n - f| = 1$ No

b) on $[0,1)$: // ? $\lim_{n\to\infty} \sup_{[0,1)} |f_n - f|$

$= \lim_{n\to\infty} \sup_{[0,1)} |f_n - 0| = 1$ No

c) on $[0,\frac{1}{2}]$ // ? $\lim \sup_{[0,\frac{1}{2}]} |f_n - f| = \lim \sup_{[0,\frac{1}{2}]} |f_n - 0| = \lim_{n\to\infty} \frac{1}{2^n} = 0$

Yes

Note Uniform convg implies pointwise convg

Thm) If $\{f_n\}$ is a seq of continuous functions and $f_n \to f$ uniformly, then f is continuous.

pf) Given $\varepsilon > 0$, $\exists N$ s.t. $n > N$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for $\forall x \in S$.

Then, for $x_0 \in S$, $|f(x) - f(x_0)| =$

$$= |f(x) - f_{N+1}(x) + f_{N+1}(x) - f_{N+1}(x_0) + f_{N+1}(x_0) - f(x_0)| \le |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

when $|x - x_0| < \delta$ with $|x - x_0| < \delta$, s.t. $|f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\varepsilon}{3}$

as f_n is continuous

Thm) Let $\{f_n\}$ be a seq of continuous functions on $[a, b]$, and suppose $f_n \to f$ uniformly.

Then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n = \int_a^b f$

pf) $\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \le \int_a^b |f_n - f|$

For large n, s.t. $\sup_{[a,b]} |f_n(x) - f(x)| < \varepsilon$

$\int_a^b |f_n - f| < \varepsilon (b - a)$

QED