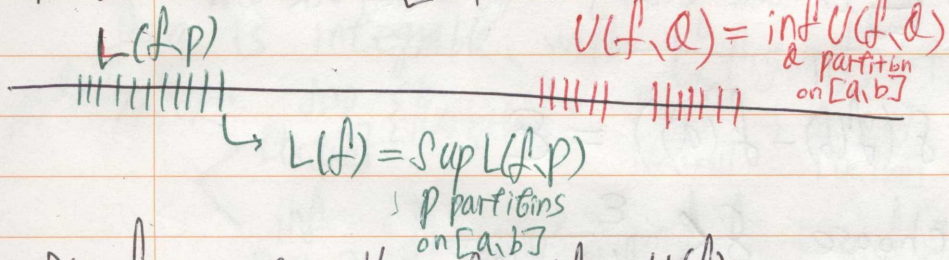
03/15 Thurs

f bounded on $[a,b]$



So, f is integrable if $L(f) = U(f)$.

Criterion for integrability:

f is integrable $\Leftrightarrow \forall \varepsilon > 0, \exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$

or

// $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. if $\text{mesh}(P) < \delta$, $U(f,P) - L(f,P) < \varepsilon$.

where $\text{mesh}(P) = \max\{t_{k+1} - t_k, k = 0, \cdots, n-1\}$.

$$\lim_{\text{mesh}(P) \to 0} (U(f,P) - L(f,P)) = 0$$

$$U(f,P) - L(f,P) = \sum_{k=0}^{n-1} \underbrace{\left(\sup_{[t_k,t_{k+1}]} f - \inf_{[t_k,t_{k+1}]} f\right)}_{\text{depends on } f} \underbrace{(t_{k+1}-t_k)}_{\text{determined by mesh}} < \varepsilon$$

☆ The smaller the mesh, the bigger n is. ☆

Thm) If f is a monotonic function on $[a,b]$, then f is integrable.

pf) Say f is increasing.

Then, $U(f,P) - L(f,P) = \sum_{k=0}^{n-1} (f(t_{k+1}) - f(t_k))(t_{k+1}-t_k)$

as $\sup_{[t_k,t_{k+1}]} f = f(t_{k+1})$ and $\inf_{[t_k,t_{k+1}]} f = f(t_k)$.

$$< \delta \underbrace{\sum_{k=0}^{n-1} [f(t_{k+1}) - f(t_k)]}_{= \delta(f(b)-f(a)) = \varepsilon} \quad \text{if } t_{k+1}-t_k < \delta.$$

Here, we choose $\delta \leq \dfrac{\varepsilon}{f(b)-f(a)}$

Therefore, by the criterion, f is integrable.

QED.

Thm) If f is continuous on $[a,b]$, then f is integrable.

pf) $U(f,P) - L(f,P) = \sum_{k=0}^{n-1} \left(\sup_{[t_k,t_{k+1}]} f - \inf_{[t_k,t_{k+1}]} f\right)(t_{k+1}-t_k)$

Here, if they are continuous, meaning no gap and $\sup f$ and $\inf f$ should not be that difference.

And here, thus, we need uniformly continuity. ☆

Choose this δ for $\dfrac{\varepsilon}{b-a}$ and consider partition P with mesh$(P) < \delta$. We claim that $\sup_{[t_k,t_{k+1}]} f - \inf_{[t_k,t_{k+1}]} f < \dfrac{\varepsilon}{b-a}$.

by uniform continuity ☆

By continuity of f, $\exists\, x, y \in [t_k, t_{k+1}]$ s.t.

$\sup\limits_{[t_k, t_{k+1}]} f = f(x)$ and $\inf\limits_{[t_k, t_{k+1}]} f = f(y)$,

$|x-y| \le t_{k+1} - t_k < \delta$ (by uniformly-continuous)

since $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Thus,

$$U(f,P) - L(f,P) = \sum_{k=0}^{n-1} \left(\sup_{[t_k, t_{k+1}]} f - \inf_{[t_k, t_{k+1}]} f\right)(t_{k+1} - t_k)$$

$$< \sum_{k=0}^{n-1} \frac{\varepsilon}{b-a} \cdot (t_{k+1} - t_k) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

$\Rightarrow f$ is integrable.

QED

Thm) Suppose f and g are integrable on $[a,b]$, then $f+g$ is integrable, with $\int_a^b f+g = \int_a^b f + \int_a^b g$.

pf) Trick: $\sup\limits_{[t_k, t_{k+1}]} \{f(x)+g(x)\} \le \sup\limits_{[t_k, t_{k+1}]} f + \sup\limits_{[t_k, t_{k+1}]} g$

$\inf\limits_{[t_k, t_{k+1}]} \{f(x)+g(x)\} \ge \inf\limits_{[t_k, t_{k+1}]} f + \inf\limits_{[t_k, t_{k+1}]} g$

This inequalities imply:

$$U(f+g, P) \le U(f,P) + U(g,P)$$

$$L(f+g, P) \ge L(f,P) + L(g,P)$$

so, $U(f+g,P) - L(f+g,P) \le (U(f,P) - L(f,P)) + (U(g,P) - L(g,P)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

Since f & g are integrable, choose δ s.t. $\text{mesh}(P) < \delta$,

hence $U(f+g,P) - L(f+g,P) < \varepsilon$.

QED

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Since $f+g$ is integrable, $L(f+g) = U(f+g)$.

$$U(f+g) \le U(f+g, P) \le U(f,P) + U(g,P)$$

and $L(f,P) + L(g,P) \le L(f+g, P) \le L(f+g)$

and $L(f+g) = \int_a^b f+g) = U(f+g)$

As mesh $P \to 0$, $L(f,P) + L(g,P) \to L(f) + L(g)$

and $U(f,P) + U(g,P) \to U(f) + U(g)$

But, since f & g are integrable, $L(f) = U(f)$ and $L(g) = U(g)$. By Squeeze Thm,

$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

QED

Cor) If f & g are integrable on $[a,b]$ and $f \le g$, then $\int_a^b f \le \int_a^b g$.

pf) $h = g - f$ is integrable by Thm, and $h \ge 0$.

Hence by def, $\int_a^b h = U(h) \ge 0$

as $U(h) = \inf_P \sum_{k=0}^{n-1} \left(\sup_{[t_k, t_{k+1}]} h\right)(t_{k+1} - t_k) \ge 0$.

≥ 0 ≥ 0

So, $0 \le \int_a^b h = \int_a^b (g-f) = \int_a^b g - \int_a^b f$.

Therefore, $\int_a^b g \ge \int_a^b f$.

QED

Thm) Intermediate value Thm for integrals.

If f is continuous on $[a,b]$, then $\exists x_0 \in [a,b]$ s.t. $f(x_0) = \frac{1}{b-a} \int_a^b f$ = avg value of f over $[a,b]$.

pf) Say $m = \min_{[a,b]} f$, $M = \max_{[a,b]} f$. I.V.T. for continuous

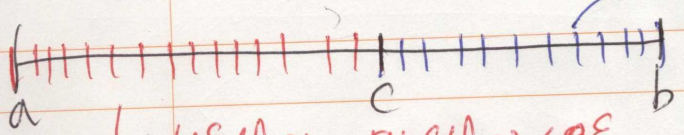
f implies that any $c \in [m, M]$, $\exists x_0 \in [a,b]$ s.t.
$f(x_0) = c$.

Since
$$\begin{cases} f \le M \\ \frac{1}{b-a}\int_a^b f \le \frac{1}{b-a}\int_a^b M = \frac{M(b-a)}{(b-a)} = M \\ f \ge m \\ \frac{1}{b-a}\int_a^b f \ge \frac{1}{b-a}\int_a^b m = \frac{m(b-a)}{(b-a)} = m \end{cases}$$

I.V.T. $\Rightarrow \exists x_0 \in [a,b]$ s.t. $f(x_0) = \frac{1}{b-a}\int_a^b f$

Thm) If f is integrable on $[a,c]$ and $[c,b]$, the f is integrable on $[a,b]$, and
$\int_a^b f = \int_a^c f + \int_c^b f$

$U_c^b(f, P_2) - L_c^b(f, P_2) < \frac{\varepsilon}{2}$



$U_a^c(f, P_1) - L_a^c(f, P_1) < \frac{\varepsilon}{2}$

Define $P = P_1 \cup P_2$, there is a partition of $[a,b]$.
$U_a^b(f, P) = U_a^c(f, P_1) + U_c^b(f, P_2)$
and $L_a^b(f, P) = L_a^c(f, P_1) + L_c^b(f, P_2)$ → (?) How...?
Then, $U_a^b(f, P) - L_a^b(f, P) < \varepsilon$

Therefore, f is integrable on $[a,b]$.

Thm) If g is nonnegative and integrable, s.t. $\int_a^b g = 0$.
Assume that g is continuous.
Then, $g \equiv 0$ on $[a,b]$.

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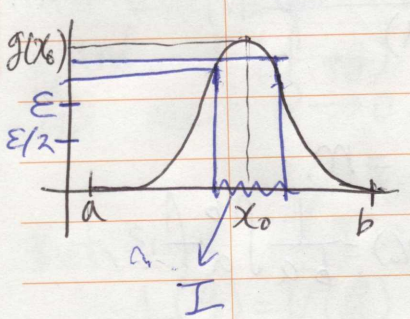
pf) say $g(x_0) > \varepsilon$ for $x_0 \in [a,b]$. Then $\exists$ an interval I around x_0 s.t. $\forall x \in I$, $g(x) > \frac{\varepsilon}{2}$ since g is continuous.



so, $\int_a^b g = \int_I g + \underbrace{\int_{\text{complement of } I} g}_{\geq 0 \text{ since } g \geq 0}$

$\geq \int_I g > \frac{\varepsilon}{2} \cdot \text{length}(I)$

> 0

contradict...