03/06 Tue

Def) f is differentiable at a (f is defined a neighborhood of a) if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ = exists.

Denote the value to be $f'(a)$.

① For all sequences $\{x_n\}_{n\in\mathbb{N}}$ s.t. $\lim_{n\to\infty} x_n = a$

$\lim_{n\to\infty} \frac{f(x_n)-f(a)}{x_n - a}$ exists (and all are equal)

② $\delta-\varepsilon$ formulation:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta$ implies

$\left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < \varepsilon$.

Thm) Assume that f and g are differentiable at $x=a$

Then a) $f \pm g$ is differentiable at a and $(f\pm g)'(a) = f'(a) \pm g'(a)$

b) fg is differentiable at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

c) Assume $g(a) \neq 0$. Then $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$

pf of (b) Need to show $\lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a}$ exists

Take any sequence $\{x_n\}_{n\in\mathbb{N}}$ s.t. $\lim x_n = a$.

consider $\lim_{n\to\infty} \frac{f(x_n)g(x_n)-f(a)g(a)}{x_n - a}$

$$\frac{f(x_n)g(x_n) - f(a)g(a)}{x_n-a} = \frac{f(x_n)g(x_n) - f(a)g(x_n) + f(a)g(x_n) - f(a)g(a)}{x_n-a}$$

$$= \frac{f(x_n) - f(a)}{x_n-a} g(x_n) + \frac{g(x_n)-g(a)}{x_n-a} f(a)$$

(g is differentiable so, g is continuous at a)

$$= f'(a)g(a) + g'(a)f(a) \quad \text{at } a$$

Thm) (Chain Rule) Suppose f is differentiable at a, and g is differentiable at $f(a)$. Then $g\circ f$ is differentiable at a, with

$$(g\circ f)'(a) = g'(f(a))f'(a).$$

Idea of pf) $\lim_{x\to a} \frac{(g\circ f)(x)-(g\circ f)(a)}{x-a} = \lim_{x\to a} \frac{(g\circ f)(x)-(g\circ f)(a)}{f(x)-f(a)} \cdot \frac{f(x)-f(a)}{x-a}$

The problem is when $f(x)=f(a)$ when $\lim x=a$, as $f(x)-f(a)$ denominator $=0$, then we cannot go further using this equal sign.

Thm) Say f is a function s.t. f attains max (or min) at x_0. Assume that $f:(a,b)\to\mathbb{R}$ f is differentiable at x_0, then $f'(x_0)=0$.

ex) If we don't assume that f is differentiable at x_0, then the Thm is false.

↳ $f(x)=|x|$, then $x_0=0$ but $f'(x_0)$ does not exist

pf) $f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$. Say x_0 is the max.

i) Consider $\{x_n\}_{n\in\mathbb{N}}$ that convg from the $\boxed{\text{left}}$ $(x_n<x_0)$ then $\frac{f(x_n)-f(x_0)}{x_n-x_0}\geq 0$ as $\underbrace{f(x_n)-f(x_0)}_{\text{max}}\leq 0$ and $x_n-x_0<0$

ii) Now, consider $\{x_n\}$ convg from the right $(x_n>x_0)$, then, $\frac{f(x_n)-f(x_0)}{x_n-x_0}\leq 0$

(As f is differentiable at x_0, f is continuous at x_0)
for the first sequence, we obtain the limit $f'(x_0) = \lim \frac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0$, and for the second equation, we have $f'(x_0) \leq 0$. therefore $f'(x_0) = 0$

QED

☆ Rolle's Thm ☆

Say $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable in (a, b). Assume $f(a) = f(b)$. Then, $\exists\, c \in (a, b)$ s.t. $f'(c) = 0$.

Since $f : [a, b] \to \mathbb{R}$ is continuous and $[a, b]$ is a closed, bounded interval, f must achieve max and min. Unless the function f is constant, either x_{min} or x_{max} is in (a, b), not on boundary. By previous Thm, either $f'(x_{max}) = 0$ or $f'(x_{min}) = 0$.

QED

☆ Mean Value Thm ☆ (General ver. of Rolle's Thm)

suppose $f : [a, b]$ is continuous and f is differentiable in (a, b). Then $\exists\, c \in (a, b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$.

pf) consider the limit func: $L(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$
consider the difference $g(x) = f(x) - L(x)$ → To rotate axis, and prove in similar way as last thm.

$\Longrightarrow$ C.T.

$g(a) = f(a) - L(a) = f(a) - f(a) = 0$ and

$g(b) = \cancel{f(b) - f(a) + f(a)} = f(b) - L(b) = f(b) - (f(b) - f(a) + f(a)) = 0$.

So, g satisfies the assumption of Rolle's Thm.

Hence, $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$, so

$f'(c) - L'(c) = 0$, where $L'(c)$ is the slope of the linear $= \frac{f(b)-f(a)}{b-a}$.

Thus, $\exists\, c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

⟨Corollary⟩

Suppose $f: [a,b] \to \mathbb{R}$ and f is differentiable s.t. $f'(x) = 0$ for $\forall x$. Then, f is a constant function.

pf) Say $f(x_1) \neq f(x_2)$ for some $x_1 \neq x_2 \in (a,b)$.

Then, by M.V.T., $\exists\, c$ between x_1 & x_2, s.t.

$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \neq 0$. Contradicts!!!

QED.

⟨Corollary⟩

Assume $f: (a,b) \to \mathbb{R}$ is differentiable.

a) $f'' > 0$, then f is strictly inc

b) $f'' < 0$, then f is strictly dec.

$\boxed{\text{pf)}}$ a) Say $x_1 < x_2$ in (a,b). Then, by M.V.T.,

$\exists\, c \in (x_1, x_2)$ s.t. $0 < f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$

Since $x_1 < x_2$, $f(x_1) < f(x_2)$, therefore f is strictly inc.

QED

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ex) Show that $x \geq \sin x$ whenever $x \geq 0$.

$\Rightarrow$ Let $f(x) = x - \sin x$. ($f : \mathbb{R} \to \mathbb{R}$)

$f'(x) = 1 - \cos x \geq 0$ for $\forall x$. so f is inc. ☆

If $x \geq 0$, $f(x) \geq f(0) = 0$

so, $x \geq \sin x$ for $x \geq 0$.

Thm) Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and f^{-1} exists. Assume that $f'(x_0) \neq 0$ then f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$

⌞ <Idea>