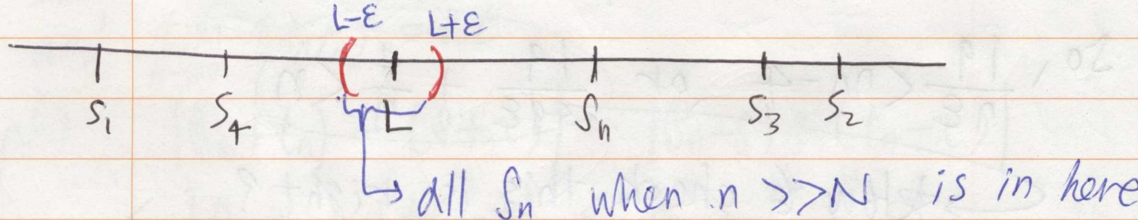
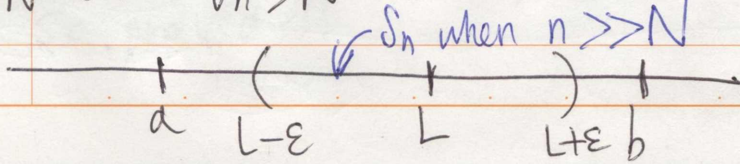
01/30 Tue

Def) Given a sequence $\{s_n\}_{n \in \mathbb{N}}$, we say $\{s_n\}$ converges to L, $\lim_{n\to\infty} s_n = L$ if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$ $|s_n - L| < \varepsilon$.



N depends on ε

Corollary) If $\lim_{n\to\infty} s_n = L$ and you have $a < L < b$, then $\exists N$ s.t. $\forall n > N$ $a < s_n < b$.

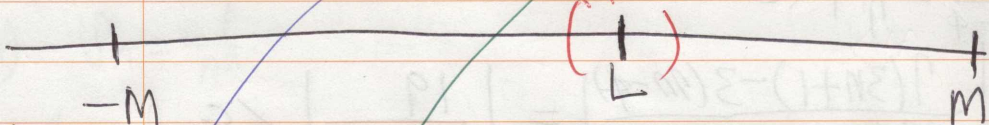


Corollary) If $\lim_{n\to\infty} s_n = L$, then $\{s_n\}_{n\in N}$ is bounded

ex) $\exists M>0$ s.t. $|s_n| \le M$ for $\forall n \in N$.

$|s_n| < \max\{L-1, L+1\}$ when $n > N$.

s_n is here when $n \gg N$



pf) Since $\lim_{n\to\infty} s_n = L$, $\exists N$ s.t. $\forall n > N$
$s_n \in (L-1, L+1)$ so, if we define $M = \max\{|s_1|, \dots, |s_N|, |L+1|, |L-1|\}$, then
$|s_n| < M+1$ $\forall n \in N$.

M

arbitrary

same

($\le$) include equal sign makes more sense!

this is also fine

included to show s_n is actually bounded near $|L|$.

★ ex) $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$

$$\left|s_n - \frac{3}{7}\right| = \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| = \left|\frac{21n+7-21n+12}{7(7n-4)}\right| = \frac{19}{7(7n-4)} < \varepsilon$$

So, $\frac{19}{7\varepsilon} < 7n-4$, or $\frac{19}{49\varepsilon} + \frac{4}{7} < n$

→ ★How to check this is right?

⇒ when $\varepsilon \downarrow$, then $n \uparrow$.

Let $N = \frac{19}{49\varepsilon} + \frac{4}{7}$, then $\forall n > N$, then $\left|s_n - \frac{3}{7}\right| < \varepsilon$.

By def, $\lim_{n\to\infty} s_n = \frac{3}{7}$. $\boxed{\text{QED}}$

ex) $\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} = 4.$

pf) $|S_n - 4| = \left|\frac{4n^3+3n}{n^3-6} - 4\right| = \left|\frac{3n+24}{n^3-6}\right| =$

$= \left|\frac{3n+24}{n^3-6}\right| < \varepsilon.$ assume $n > 2$.

For $n > 2$ (sufficiently large n), $\frac{3n+24}{n^3-6} < ? < \varepsilon$

You don't need to solve for N in a strict sense.

This can solve for N easily.

For example, it can be $\frac{4n}{n^3-n}$ given $n > 24$

So, $\frac{4}{n^2-1} < \varepsilon$ or $\frac{4}{\varepsilon}+1 < n^2$ or $n > \sqrt{\frac{4}{\varepsilon}+1}$

If we define $N = \max\{24, \sqrt{\frac{4}{\varepsilon}+1}\}$

Thm) Given $\{S_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ s.t. $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} t_n = t$. Then, $\lim_{n\to\infty}(S_n + t_n) = S + t$ and $\lim_{n\to\infty}(S_n t_n) = St$

pf) $|(S_n + t_n) - (S+t)| = |(S_n - S) + (t_n - t)|$

$\leq |S_n - S| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

And it is because $\lim_{n\to\infty} S_n = S$, $\lim_{n\to\infty} t_n = t$,

$\exists N_1$ s.t. $|S_n - S| < \frac{\varepsilon}{2}$ and $\exists N_2$ s.t. $|S_n - S| < \frac{\varepsilon}{2}$

If we define $N = \max\{N_1, N_2\}$, then $\forall n > N$.

So, proved!!!

b) $|s_n t_n - st| = |(s_n - s)t_n + st_n - st|$

$= |(s_n - s)t_n + (t_n - t)s| \le |(s_n - s)t_n| + |s(t_n - t)|$

$= |(s_n - s)||t_n| + |s||t_n - t| \le M|s_n - s| + |s||t_n - t|$

Since $\lim_{n\to\infty} t_n = t$, $\exists M$ s.t. $|t_n| < M$

Since $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, $\exists N_1$ s.t. $|s_n - s| < \frac{\varepsilon}{2M}$ and $\exists N_2$ s.t. $|t_n - t| < \frac{\varepsilon}{2|s|}$ for $n > n_1$ and $n > n_2$.

Let $N = \max\{n_1, n_2\}$

Then $n > N$ and $|s_n t_n - st| < \varepsilon$.

Thm) Suppose $\lim_{n\to\infty} s_n = s$ and assume that $s_n \neq 0$ and $s \neq 0$, Then, $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s_n}$

pf) $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s \cdot s_n}\right| = \frac{|s_n - s|}{|s||s_n|}$ → give bound for this!!

need to show $\exists \delta > 0$ s.t. $|s_n| > \delta \ \forall n \in N$

To find δ, consider $(s - \frac{|s|}{2}, s + \frac{|s|}{2}) \not\ni 0$

$\lim_{n\to\infty} s_n = s$, it implies $n > N$, $s_n \in (s - \frac{|s|}{2}, s + \frac{|s|}{2})$

$\delta = \min\{|s_1|, \dots, |s_N|, \frac{|s|}{2}, \frac{3|s|}{2}\} > 0$ $\quad \delta = \inf\{|s_n|, n \in N\}$

then $|s_n| > \delta \ \forall n \in N$

Not the same

Inf is not correct as it can be 0.

finite set so, min = inf here and $s_n \neq 0$

We just showed that if $\lim s_n = s \to$

If you chose N s.t. $|s_n - s| < \varepsilon|s|\delta$ for $n > N$.

→ in general