03/20 Tue

f bounded on $[a,b]$ is integrable.

$$U(f) = L(f)$$

$$\inf_{\text{all } p} \{U(f,p)\} \qquad \sup_{\text{all } p} \{L(f,p)\}$$

⟨Criterion of integrability⟩

① $\forall \varepsilon > 0$, $\exists$ partition p of $[a,b]$ s.t. $U(f,p) - L(f,p) < \varepsilon$

② $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\text{mesh}(p) < \delta \rightarrow U(f,p) - L(f,p) < \varepsilon$.

Thm) If f is integrable on $[a,b]$, then the $|f|$ is also integrable on $[a,b]$. Moreover, $\left| \int_a^b f \right| \leq \int_a^b |f|$

↳ analogous to triangle inequality.

pf) If we already showed that $|f|$ is integrable, then since $-|f| \le f \le |f|$. Hence, $\int_a^b -|f| \le \int_a^b f \le \int_a^b |f|$.

So, $\left|\int_a^b f\right| \le \int_a^b |f|$.

Need to show that $|f|$ is integrable on $[a,b]$.

claim $\sup_{[t_k,t_{k+1}]} |f| - \inf_{[t_k,t_{k+1}]} |f| \le \sup_{[t_k,t_{k+1}]} f - \inf_{[t_k,t_{k+1}]} f$

↳ i) if $f>0$, for $\forall$ values, they are equal

ii) if $f<0$, for $\forall$ values, they are equal.

ii) if f have different signs, $\inf|f|=0$, and will have inequality.

So, $U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$, since

$\sum_k (t_{k+1}-t_k)(\sup|f| - \inf|f|) \le \sum_k (t_{k+1}-t_k)(\sup f - \inf f)$

since f is integrable, we have $\forall \varepsilon>0, \exists \delta>0$ s.t. $mesh(P)<\delta \rightarrow U(f,P)-L(f,P)<\varepsilon$.

From the inequality $U(|f|,P)-L(|f|,P) \le U(f,P)-L(f,P) < \varepsilon$, $|f|$ is integrable

QED

Thm) (Fundamental Thm of Calculus I)

Given f is continuous on $[a,b]$, and differentiable on (a,b). Assume that f' is integrable.

Then, $\int_a^b f' = f(b)-f(a)$.

pf) Start with a partition $P=\{a=t_0<t_1<\cdots<t_n=b\}$.

$f(b)-f(a) = (f(b)-f(t_{n-1})) + (f(t_{n-1})-f(t_{n-2})) + \cdots + (f(t_1)-f(a))$

M.V.T applied to f on $[t_k,t_{k+1}]$ given $\exists x_k \in (t_k,t_{k+1})$

s.t. $f(t_{k+1})-f(t_k) = f'(x_k)(t_{k+1}-t_k)$

$\overrightarrow{c.t}$

So, $f(b)-f(a) = f'(x_{n-1})(t_n - t_{n-1}) + f'(x_{n-2})(t_{n-1} - t_{n-2})$

$+ \cdots + f'(x_0)(t_1 - t_0)$

$\leq \sup_{[t_{n-1}, t_n]} f' (t_n - t_{n-1}) + \sup_{[t_{n-2}, t_{n-1}]} f' (t_{n-1} - t_{n-2})$

$+ \cdots + \sup_{[t_0, t_1]} f' (t_1 - t_0)$

$= U(f', P)$

Similarly, I have $L(f', P) \leq f(b) - f(a) \leq U(f', P)$ for $\forall P$

$L(f') \leq f(b) - f(a) \leq U(f')$. Since f' is integrable

$L(f') = U(f') = \int_a^b f' = f(b) - f(a)$ (?) why...?

Thm) Let f be integrable on $[a,b]$. Define $F(x) = \int_a^x f(t)dt$. Then a) F is continuous on $[a,b]$

b) If f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

pf) a) Start with $\varepsilon > 0$ and x_0, we need to find $\delta > 0$ s.t. $\forall x \in (x_0 - \delta, x_0 + \delta)$, $F(x) \in (F(x_0) - \varepsilon, F(x_0) + \varepsilon)$. Consider $F(x) - F(x_0) = \int_a^x f - \int_a^{x_0} f$.

But

$a \quad x_0 \quad x$ or $a \quad x \quad x_0$

so, we put absolute value to not worry about sign.

$|F(x) - F(x_0)| = \left| \int_a^x f - \int_a^{x_0} f \right| = \left| \int_{x_0}^x f \right| \leq \int_{x_0}^x |f|$.

Assume $|f| < M$, then $\int_{x_0}^x |f| \leq M|x - x_0|$

choose $\delta = \frac{\varepsilon}{M}$ and $|F(x) - F(x_0)| \leq \int_{x_0}^x |f| \leq M \underbrace{|x - x_0|}_{< \delta}$

$< M \cdot \frac{\varepsilon}{M} = \varepsilon$.

DATE

b) Assume that f is continuous at x_0.

$F'(x_0) = \lim_{x \to x_0} \frac{F(x)-F(x_0)}{x-x_0}$, which is equal to $f(x_0)$

It's sufficient to show $\lim_{x \to x_0} \left(\frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right) = 0$.

$\frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\int_{x_0}^{x} f(t)\,dt$.

Notice that $f(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x} f(x_0)$

Hence, $\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)$

$= \frac{1}{x-x_0}\int_{x_0}^{x} \underline{(f(t)-f(x_0))}\,dt$

↳ small when x is near x_0, since f is continuous at x_0.

★ continuity of f at x_0.

Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. whenever $|t-x_0| < \delta$, $|f(t)-f(x_0)| < \varepsilon$.

So, $\left| \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right| \leq \left| \frac{1}{x-x_0} \right| \int_{x_0}^{x} |f(t)-f(x_0)|\,dt$

$< \frac{1}{|x-x_0|}\int_{x_0}^{x} \varepsilon = \varepsilon$, when $|x-x_0| < \delta$

Triangular inequality

By def, $F'(x_0)$ exists and equal to $f(x_0)$