04/10 Tue

$f, g : S \to R$

↳ $\sup_{x\in S} |f(x) - g(x)| \Rightarrow$ Distance in domain S of f and g

Def) $f_n \to f$ convg uniformly on S if

$\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = 0$ ....

Uniform convg $\to$ pointwise convg

($\forall x \in S$, $f_n(x) \xrightarrow{n\to\infty} f(x)$)

$\sum_{k=0}^{\infty} g_k$ where $g_k : S \to R$ is a func.

partial sum $S_n(x) = \sum_{k\le n} g_k(x)$

Does $\{S_n\}$ have a limit function?

- Start with pointwise convg for $\forall x \in S$ $S_n(x) \xrightarrow{n\to\infty} S(x)$?
- Analyze whethere $\{S_n\}_{n\in N}$ convg uniformly in S

Thm) If $\{f_n\} \to f$ uniformly and each f_n is continuous, then f is also continuous....

<power series> $\sum_{k\ge 0} a_k x^k$

pointwise convg: For what $x \in R$ s.t. $\sum_{k\ge 0} a_k x^k$ convg?

Root test

check $\limsup_{k\to\infty} \sqrt[k]{|a_k x^k|} = \lim_{k\to\infty} \sqrt[k]{|a_k|}\, |x|$

$= |x| \cdot \limsup_{k\to\infty} \sqrt[k]{|a_k|}$

Root test implies:

if $|x| \limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1$, $\Leftrightarrow |x| < \dfrac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$

$\| $

R (radius of convg of the power series)

then $\sum a_k x^k$ convg.

↳ The series pointwise convg in $(-R, R)$

↳ The series does not convg or divg.

You can define a function $g(x)$ on $(-R, R)$

s.t. $g(x) = \sum_{k\geq 0} a_k x^k$.

In general, $\sum_{k\geq 0} a_k x^k$ does not convg uniformly in $(-R, R)$

ex) $\sum_{k\geq 0} x^k$

$R = 1$, so $\sum x^k$ convg pointwise in $(-1, 1)$.

$\| \quad \frac{1}{1-x}$ in $(-1, 1)$.

$S_n(x) = 1 + x + x^2 + \cdots + x^n$ & $S(x) = \frac{1}{1-x}$

$\sup_{(-1,1)} |S_n(x) - S(x)| = \infty$ for $\forall n$ because

$|S_n(x)| \leq n$ when $x \in (-1, 1)$ where

$S(x)$ is unbounded in $(-1, 1)$.

So, $S_n \not\to S$ uniformly in $(-1, 1)$.

radius of convg...

Thm) If $0 < R_1 < R$, then $\sum a_k x^k$ convg uniformly on $[-R_1, R_1]$

pf) Use weirstrass M test.

In $[-R_1, R_1]$, $|a_k x^k| \le |a_k| R_1^k$.

Notice that $\sum_{k\ge 0} |a_k| R_1^k$ is convg (by root test together with $R_1 < R$)

weirstrass M test $\Rightarrow \sum_{k\ge 0} a_k x^k$ convg uniformly in $[-R_1, R_1]$

Cor) The limit function $\sum_{k\ge 0} a_k x^k$ is continuous in $(-R, R)$.

pf) $\sum_{k\ge 0} a_k x^k$ is continuous in $[-R_1, R_1]$ for $\forall R_1 < R$.

so, $\sum_{k\ge 0} a_k x^k$ is continuous in $(-R, R)$.

ex) $\sum_{k\ge 0} \frac{x^k}{k!}$

$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{\frac{1}{k!}}}$ and $\sqrt[k]{k!} \xrightarrow{k\to\infty} \infty$

or, use the ratio test!!

Does the series convg uniformly in $\mathbb{R}$?

$\sum \frac{x^k}{k!} = e^x$

$\sup_{x\in\mathbb{R}} |e^x - s_n(x)| = \infty$, so the series does not convg uniformly in $\mathbb{R}$. However, $\forall R_1 < R = \infty$, the series convg uniformly in $[-R_1, R_1]$

Taylor series

f is defined in (a, b), and $c \in (a, b)$.

Assume that f has derivatives of all order at c. (ex) $f^{(n)}(c)$ is defined $\forall n$)

From these # $f^{(n)}(c)$, we construct

$$T(x) = \sum_{n \geq 0} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Q. How does $T(x)$ and f relate to each other?

For what x does $T(x)$ convg? $T(c) = f(c)$

In general, $T(x)$ may not be convg at any other $x \neq c$.

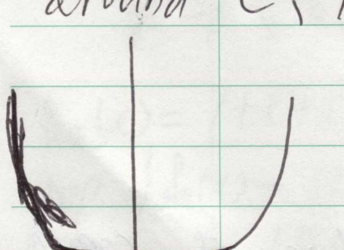
For example, there are smooth (have all derivatives) function f s.t. $f^{(k)}(c) = (k!)^2$

Taylor series at c: $\sum_{k \geq 0} \frac{(k!)^2}{k!}(x-c)^k = \sum_{k \geq 0} k!(x-c)^k$

only convg at $x = c$

Thm) $\exists$ smooth f s.t. $f^{(k)}(c) = a_k$ $\forall k$.

Even if $T(x)$ convg in an open interval around c, $T(x) \neq f(x)$.



For these types of func,

$f^{(k)}(c) = 0$ $\forall k$

$T(x) = f(c)$ $\forall x$, but $f(x) \neq f(c)$ is <u>not</u> the constant func.

Need more conditions to say $T(x)$ agree with f in an open interval around C.

$$R_n(x) = f(x) - \sum_{k \le n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$\hookrightarrow$ n-th remainder

Idea) we want to give an upper bound for $|R_n(x)|$

$|R_0(x)| = |f(x) - f(c)|$

if we assume f' exists, it will be $|f'(y)(x-c)|$ where $\exists\, y$ between x and C.

~~then~~ And, if we assume f' is bounded by M, then, $|f'(y)(x-c)| \le M(x-c)$

Similar statements for $|R_n(x)|$ assuming $f^{(n+1)}$ exists and bounded.