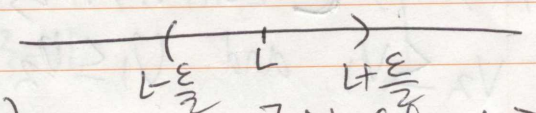
02.06 Tue

def) A sequence $\{S_n\}_{n\in\mathbb{N}}$ is cauchy if $\forall \varepsilon>0, \exists N$ s.t. $\forall m,n>N, |S_n-S_m|<\varepsilon$.

Thm) A cauchy sequence in $\mathbb{R}$ is equivalent to a convergence sequence

pf) $\Leftarrow$: Say $\lim S_n = L$

so, by def when $n>N$



$S_n \in (L-\frac{\varepsilon}{2}, L+\frac{\varepsilon}{2})$. $\forall \varepsilon>0, \exists N$ s.t. $n>N$

$|S_n-L|<\frac{\varepsilon}{2}$. $\Longrightarrow$

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$\Longrightarrow |S_m - S_n| \leq |S_m - L| + |L - S_n|$ by Triangular inequality

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

So, $\{S_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

<u>proved</u>

$\Longrightarrow$: Suppose $\{S_n\}$ is a cauchy sequence.

(observation) Suppose $A \subset \mathbb{R}$ s.t. $|a_1 - a_2| < \varepsilon$ for $\forall\, a_1, a_2 \in A$. Then $\sup A - \inf A \leq \varepsilon$

So. Given $\delta > 0$, $\exists\, a_1 \in A$ s.t. $a_1 > \sup A - \delta$

and $a_2 \in A$ s.t. $a_2 < \inf A + \delta$

So, $(\sup A - \delta) - (\inf A + \delta) < a_1 - a_2 < \varepsilon$

So, $\sup A - \inf A < \varepsilon + 2\delta \;\; \forall \delta$

It implies $\sup A - \inf A \leq \varepsilon$ (as it is for $\forall \delta$).

Consider $A = \{S_n \mid n > N\}$. cauchy implies that A satisfies the condition of the observation.

If I define $U_N = \inf\{S_n \mid n > N\}$ and $V_N = \sup\{S_n \mid n > N\}$

Then, $V_N - U_N \leq \varepsilon$. Notice if $N_1 > N_2$,

$\{S_n \mid n > N_1\} \subset \{S_n \mid n > N_2\}$ (sup gets smaller)

So, $V_2 \leq V_1$ and $U_1 \leq U_2$

So, $U_1 \leq U_2 \leq U_3 \leq \cdots \leq V_3 \leq V_2 \leq V_1$

Here, $\{U_n\}_{n\in\mathbb{N}}$ is an increasing sequence and bounded above. So, $U = \lim U_n$ $(= \lim\inf S_n)$ ☆

$\sup U$

And, $V = \lim V_n$

$\inf V$

increasing & bounded above

Actually, $U = V$, as because

$\lim(V_n - U_n) = 0$ ☆ by using $V_N - U_N \le \varepsilon$.

We now show that $\lim S_n = U = V$.

$U_1 \le S_2 \le V_1 = \sup\{S_n \mid n > 1\}$

$\inf\{S_n \mid n > 1\}$

So, $U_2 \le S_3 \le V_2$

$\vdots$

$U_{n-1} \le S_n \le V_{n-1}$

converge to U converge to V.

And as $U = V$, squeeze lemma implies $\lim S_n = U = V$.

Ex) $S_n = (-1)^n$.

$U_1 = \inf\{S_n \mid n > 1\} = -1$

$V_1 = \sup\{S_n \mid n > 1\} = 1$

$U_2 = \inf\{S_n \mid n > 2\} = -1$

$V_2 = \sup\{S_n \mid n > 2\} = 1$

$\vdots$

$U_1 = U_2 = \cdots U_n = -1 < 1 = V_n = \cdots = V_1$

$\Rightarrow$ so, this is not cauchy.

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ex) $S_n = \frac{(-1)^n}{n} \Rightarrow \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \cdots\}$

$u_1 = -\frac{1}{3} = \inf\{S_n \mid n > 1\}$

$v_1 = \frac{1}{2}$

$u_2 = -\frac{1}{3}$

$v_2 = \frac{1}{4}$

$\vdots$

$\Rightarrow$ cauchy & converging

so, $u_1 \le u_2 \le u_3 \le \cdots \le v_3 \le v_2 \le v_1$

$u = 0$ and $v = 0$

$\to \limsup S_n$

$\to \liminf S_n$

Given a sequence $\{S_n\}$ (bounded sequence), ~~no matter~~ whether or not $\{S_n\}$ is convergent, we can always define $\limsup S_n$ and $\liminf S_n$

(Collorary) If $\limsup S_n = \liminf S_n$, then $\{S_n\}$ converges.

Q. How to show a sequence is not converging?

Def) Given $\{S_n\}_{n \in \mathbb{N}}$, a subsequence $\{t_k = S_{n_k}\}_{k \in \mathbb{N}}$ satisfies $n_1 < n_2 < n_3 < \cdots$.

ex) $S_1 \; S_2 \; S_3 \; S_4 \; S_5 \; S_6 \; S_7 \cdots$

$S_2 = t_1$, $S_3 = t_2$, $S_6 = t_3$

This is a subsequence of $\{S_n\}$.

Thm) If $\lim S_n = L$, then for any subsequence $\{S_{n_k}\}_{k\in\mathbb{N}}$ $\lim_{k\to\infty} S_{n_k} = L$

pf) for $\forall \varepsilon > 0$, $\exists N$ s.t. $n > N$, $|S_n - L| < \varepsilon$.
And n_k is at least k, hence if $k > N$, then
$n_k > N$, so, $|S_{n_k} - L| < \varepsilon$.
So, by def, $\lim_{k\to\infty} S_{n_k} = L$.

Ex) $S_n = \sin\frac{n\pi}{3}$
So, $S_1 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $S_2 = \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and $S_3 = 0$
and $S_4 = -\frac{\sqrt{3}}{2}$ and $S_5 = -\frac{\sqrt{3}}{2}$ and $S_6 = 0$

How check they are converging?
$\Rightarrow$ $\{S_{3n}\}_{n\in\mathbb{N}}$ has limit 0 and
$\{S_{6n+1}\}_{n\in\mathbb{N}}$ has limit $\frac{\sqrt{3}}{2}$
So, not converging....

Given a sequence $\{S_n\}$, $S = \{t \in \mathbb{R} \mid t$ is the limit of some subsequences $\{S_{n_k}\}$ of $\{S_n\}\}$

Ex) In the case $\lim S_n = L$, $S = \{L\}$.
But previous example has $S = \{\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\}$.

Thm) Suppose $\{S_n\}$ is a bounded sequence.
Then, a) $S \neq \emptyset$ (Bolzano-Weierstrass thm)
b) $\max S = \lim\sup S_n$
$\min S = \lim\inf S_n$

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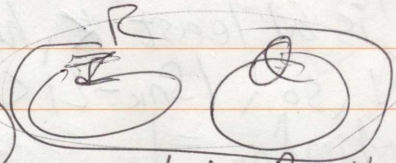
?

☆HW1 $\Rightarrow N \overset{\text{bijection}}{\approx} N \times N \approx \mathbb{Q}$

Ex) I will construct a sequence $\{S_n\}$ s.t. $\{S_n \mid n \in N\} = \mathbb{Q}$.

$\Rightarrow$ From a bijection of $N \to \mathbb{Q}$, I get this sequence $S = \mathbb{R}$

subsequential limit



Lemma) Given $\{S_n\}_{n \in N}$ and $t \in \mathbb{R}$, there exists a subsequences $\{S_{n_k}\}_{k \in N}$ converging to t.

$\Longleftrightarrow$ $\forall \varepsilon > 0$, $(t-\varepsilon, t+\varepsilon)$ contains infinitely many elements of $\{S_n\}_{n \in N}$.

subsequential limit for $S_n = \mathbb{Q}$ can actually approach to I, so it is $\mathbb{Q} \cup I \Rightarrow \mathbb{R}$!!!