02/20 Tue

Series $\sum_{n=1}^{\infty} a_n$

def) $\sum_{n=1}^{\infty} a_n$ converges if $\left\{ S_k = \sum_{n=1}^{k} a_n \right\}_{k \in \mathbb{N}}$ converges.

def) $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Cauchy's criterion

$\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow$ $\{S_k\}_{k \in \mathbb{N}}$ converges

$\Leftrightarrow$ $\{S_k\}_{k \in \mathbb{N}}$ is cauchy

$\Leftrightarrow$ $\forall \varepsilon > 0, \exists N$ s.t. $m > n > N$

$$|S_m - S_n| = \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon$$

Corollary: If $\sum a_n$ converges, then replacing

$m = n+1 > n > N$: $|a_{n+1}| < \varepsilon$.

Hence, $\lim_{n \to \infty} |a_n| = 0$. (ex) $\lim a_n = 0$)

☆ But, $\lim a_n = 0$ does <u>not</u> guarantee $\sum a_n$ conv.

ex) $\sum \frac{1}{n}$ diverges to ∞.

☆ Absolute convg $\not\Longleftrightarrow$ Convg

ex) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$

converges, but not absolutely converges.

Comparison Test

Given $\sum_{n=1}^{\infty} a_n$. Assume that $|a_n| \le M_n$ for $\forall n$.
and also $\sum_{n=1}^{\infty} M_n$ converges.
Thus $\sum_{n=1}^{\infty} a_n$ Converges

ex) $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(1) why add up 1? since the def says work for $\forall n$...

(2) $\frac{1}{n^2} \le \frac{1}{n(n-1)}$ is not for $\forall n$...

Notice $\frac{1}{n^2} < \frac{1}{n(n-1)}$ for $n \ge 2$

we know that $\sum_{n \ge 2} \frac{1}{n(n-1)}$ converges

$\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots = 1 - \frac{1}{n}\right)$ when $n \to \infty$, $1 - \frac{1}{n} \to 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2} \le 1 + \sum_{n \ge 2} \frac{1}{n(n-1)} = 2$

Root test

Given $\sum_{n=1}^{\infty} a_n$

① will converge if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

② will diverge if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

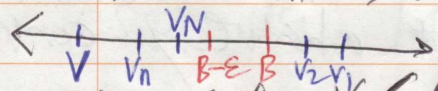
③ does not give info if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le 1 \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

YANGJI

Given a sequence $\{b_n\}_{n\in\mathbb{N}}$, what is the implication of $\lim_{n\to\infty} \sup b_n < B$?

$$V = \lim_{n\to\infty} \sup b_n = \lim_{n\to\infty} V_n \text{, where } V_n = \sup\{b_k : k > n\}$$

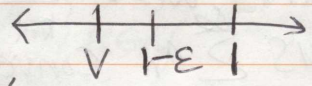
Remember $\{V_n\}_n$ is decreasing



Given $\varepsilon > 0$, s.t. $V < B-\varepsilon$, $\exists N$ s.t. $V_n < B-\varepsilon$.
Therefore, $b_k < B-\varepsilon$ $\forall k > N$.

pf of ratio test ①: call $V = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

pick $\varepsilon > 0$ s.t. $V < 1-\varepsilon$



$\exists N$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < 1-\varepsilon$, for $\forall n > N$.

So, $|a_{n+1}| < |a_n|(1-\varepsilon)$.

So, ex) $|a_{N+3}| < (1-\varepsilon)|a_{N+2}| < (1-\varepsilon)^2 |a_{N+1}|$

$\vdots$

$$|a_n| < (1-\varepsilon)^{n-(N+1)} |a_{N+1}|$$

So, we can see that $n > N$ implies

$$|a_n| < |a_{N+1}| (1-\varepsilon)^{n-(N+1)}$$

$\vdots$

$$|a_{N+1}| \le |a_{N+1}|$$

$\vdots$

$$|a_1| \le |a_1|$$

By comparison test, since RHS is

$$|a_1| + \cdots + |a_N| + \underbrace{\sum_{k=0}^{\infty} |a_{N+1}| \underbrace{(1-\varepsilon)^k}_{\text{geometric series}}}_{\text{geometric convg}}$$ converges

and $0 < 1-\varepsilon < 1$

So, I have $\sum_{n=1}^{\infty} |a_n|$ convg, so $\sum a_n$ convg.

QED

Given $\{b_n\}_{n\in\mathbb{N}}$, what does it mean by

$u = \liminf b_n > B$, where $u_n = \inf\{b_k : k > n\}$

Given $\varepsilon > 0$ s.t. $B + \varepsilon < u$, $\exists N$ s.t. $u_N > B + \varepsilon$.

Hence $\exists N$ s.t. $b_k > B + \varepsilon$, for $\forall k > N$.

Pf of ratio test ②: $u = \liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$

Hence $\exists N$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| > 1$, $n > N$.

So, $|a_{n+1}| > |a_n|$ for $n > N$.

This implies $\lim a_n$ can never be 0.

Therefore $\sum a_n$ diverges.

Root Test

Given $\sum_{n=1}^{\infty} a_n$,

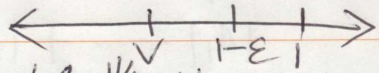
① will converge if $\limsup |a_n|^{1/n} < 1$

② will diverge if $\liminf |a_n|^{1/n} > 1$

③ gives no info if $\liminf |a_n|^{1/n} \le 1 \le \limsup |a_n|^{1/n}$

YANGJI

pf ① root $\Rightarrow$ $V = \limsup \underbrace{|a_n|^{1/n}}_{=b_n} < 1$



Given $\varepsilon > 0$ s.t. $1-\varepsilon > V$, $\exists N$ s.t. $|a_n|^{1/n} < 1-\varepsilon$ for $\forall n > N$.

So, $|a_n| < (1-\varepsilon)^n$ for $\forall n > N$.

By comparison test, $\sum |a_n|$ is dominated by

$$|a_1| + \cdots + |a_n| + \sum_{n>N} (1-\varepsilon)^n.$$

geometric series

We have $\sum_{n=1}^{\infty} |a_n|$ converges

therefore $\sum_{n=1}^{\infty} a_n$ converges.

pf ② root $\Rightarrow$ $\liminf |a_n|^{1/n} > 1$ implies $\exists N$ s.t. $|a_n|^{1/n} > 1$ for $n > N$.

So, it implies $|a_n| > 1$ for $n > N$.

Therefore $\sum a_n$ diverges since $\lim a_n$ cannot be 0.

ex) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Ratio test: $\limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \left|\frac{n+1}{3n}\right| = \lim \left|\frac{n+1}{3n}\right|$

$= \boxed{\frac{1}{3}} < 1$.

Ratio test implies the series conv.

Root test: $\limsup |a_n|^{1/n} = \limsup \left|\frac{n}{3^n}\right|^{1/n} = \frac{1}{3} \limsup n^{\frac{1}{n}} =$

$= \frac{1}{3}(1) = \frac{1}{3} < 1$

Root test implies the series conv.

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$\boxed{\text{Note}}$ In general, root test is stronger than ratio test. when ratio test does not give info, root tese may give some info. But not vice versa.

ex) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

$=\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\cdots$

So, ratio test $\frac{a_{n+1}}{a_n} \Rightarrow 2 \quad \frac{1}{8} \quad 2 \quad \frac{1}{8} \quad 2 \quad \frac{1}{8} \cdots$

So, $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 2$

$\liminf \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{8}$

No info (R: ratio test is really sensitive, with based on the next value.)

However, the root test gives some info.

$\Rightarrow |a_n|^{1/n} \Rightarrow \left(\frac{1}{2}\right)^1 \quad (1)^1 \quad \left(\frac{1}{8}\right)^{1/3} \quad \left(\frac{1}{4}\right)^{1/4} \cdots$

$\longrightarrow \frac{1}{2}$

Therefore, root test implies $\sum$ convg.