03/13 Tue

Given f bounded on $[a,b]$, $P = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$

For each $[t_k, t_{k+1}]$, define $\sup_{[t_k, t_{k+1}]} f$, $\inf_{[t_k, t_{k+1}]} f$

$$U(f,P) = \sum_{k=0}^{n-1} \sup_{[t_k, t_{k+1}]} f \,(t_{k+1} - t_k)$$

$$L(f,P) = \sum_{k=0}^{n-1} \inf_{[t_k, t_{k+1}]} f \,(t_{k+1} - t_k)$$

always $L(f) \le U(f)$

(?) $b-a$ is inf and $L(f,P)$ is inf...?

Upshot: Given any P and Q partitions of $[a,b]$, we have $L(f,P) \le U(f,Q)$

Define $L(f) = \sup_{\text{all partitions } P \text{ of } [a,b]} L(f,P)$ $\le B$ $\le B(b-a)$

independent of P

independence of Q

$U(f) = \inf_{\text{all partitions } Q \text{ of } [a,b]} U(f,Q)$

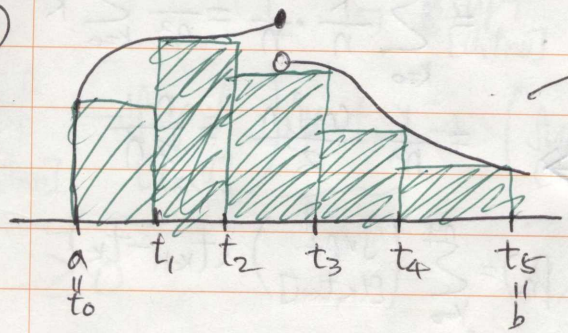
always between $-B(b-a)$ & $B(b-a)$

$U(f)$ & $L(f)$ are finite → because $|f| \le B$

↳ $\forall x \in [a,b] : -B \le f(x) \le B$

f is integrable if $U(f) = L(f)$ and denote
$\int_a^b f = U(f) = L(f)$

ex)



(?) If it's not continuous it cannot be integrable...?

No... continuity and integrable are not related... cuz the more chops make uncontinuity meaningless...

Ex) $f: [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Given P, $L(f,P) = \sum_{k=0}^{n-1} \left(\inf_{[t_k, t_{k+1}]} f \right) (t_{k+1} - t_k) = 0(t_{k+1} - t_k) = 0$

$$U(f,P) = \sum_{k=0}^{n-1} \left(\sup_{[t_k, t_{k+1}]} f \right) (t_{k+1} - t_k) = b - a$$

$$L(f) = \sup_{\text{all partition } P} L(f,P) = 0$$

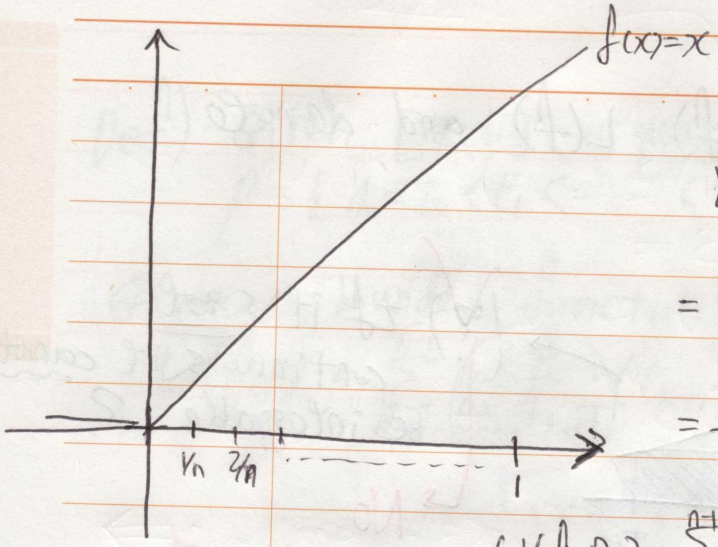
$$U(f) = \inf_{\text{all partition } P} U(f,P) = b - a$$

Thus, not integrable...

ex) $f(x) = x$ in $[0,1]$. We'll show that there are partitions $\{P_n\}$ of $[a,b]$ s.t. $L(f,P_n) \xrightarrow{n \to \infty}$ value $\xleftarrow{n \to \infty} U(f,P_n)$

$P_n = \{ t_0 = 0 < t_1 = \frac{1}{n} < t_2 = \frac{2}{n} < \cdots < t_n = 1 \}$

$\Longrightarrow$ c.t.



$$L(f,P_n)=\sum_{k=0}^{n-1}\left(\inf_{[t_k,t_{k+1}]} f\right)(t_{k+1}-t_k)$$

$$=\sum_{k=0}^{n-1}\frac{k}{n}\cdot\frac{1}{n}=\frac{1}{n^2}\sum_{k=0}^{n-1}k$$

$$=\frac{1}{n^2}\,\frac{(n-1)n}{2}=\frac{n-1}{2n}$$

$$U(f,P_n)=\sum_{k=0}^{n-1}\left(\sup_{[t_k,t_{k+1}]} f\right)(t_{k+1}-t_k)$$

$$=\sum_{k=0}^{n-1}\left(\frac{k+1}{n}\right)\left(\frac{1}{n}\right)=\frac{1}{n^2}\sum_{k=0}^{n-1}(k+1)$$

$$=\frac{1}{n^2}\cdot\frac{n(n+1)}{2}=\frac{n+1}{2n}$$

Why we bring partition $n\to\infty$? to make it more accurate...?

Yes

$$\lim_{n\to\infty} L(f,P_n)=\frac{1}{2}=\lim_{n\to\infty} U(f,P_n)$$

So, $\frac{1}{2}\le L(f)$ and $U(f)\le\frac{1}{2}$

and $L(f)\le U(f)$, so $U(f)=L(f)=\frac{1}{2}$.

By def, f is integrable on $[0,1]$ with $\int_a^b f=\frac{1}{2}$

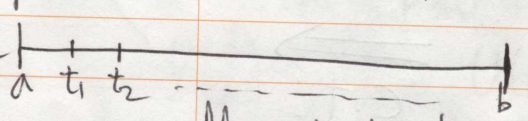
Def) We say partition Q is more refined than partition P if $P\subset Q$

$P=\{t_0,t_1,\dots,t_k,\dots,t_n\}$

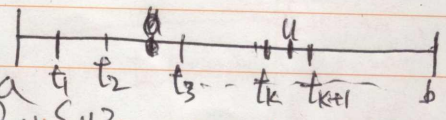
Lemma) If $P\subset Q$, then $L(f,P)\le L(f,Q)\le \boxed{U(f,Q)\le U(f,P)}$ ①

Proof for ① $\Rightarrow$ Let's prove $U(f,P)\ge U(f,Q)$.

Let P be



and Q be



It's sufficient to show when $Q=P\cup\{u\}$

cuz we can recursively add one point to keep proving ☆

Say $u \in (t_k, t_{k+1})$

$$U(f,P) - U(f,Q) = \sup_{[t_k, t_{k+1}]} f\,(t_{k+1} - t_k) - \Big(\sup_{[t_k, u]} f\,(u - t_k) + \sup_{[u, t_{k+1}]} f\,(t_{k+1} - u)\Big)$$

$$= \underbrace{\Big(\sup_{[t_k, t_{k+1}]} f - \sup_{[t_k, u]} f\Big)}_{\geq 0} \underbrace{(u - t_k)}_{\geq 0} + \underbrace{\Big(\sup_{[t_k, t_{k+1}]} f - \sup_{[u, t_{k+1}]} f\Big)}_{\geq 0} \underbrace{(t_{k+1} - u)}_{\geq 0}$$

So, $U(f,P) \geq U(f,Q)$

(cor) $\forall P, Q$ partition, $L(f,P) \leq U(f,Q)$

pf) consider the partition $R = P \cup Q$. R is more refined for both P and Q.

From the lemma, $U(f,Q) \geq U(f,R)$ and $L(f,P) \leq L(f,R)$. And $L(f,R) \leq U(f,R)$.

Thus, $L(f,P) \leq U(f,Q)$

Thm) (Criterion of integrability)

f is integrable iff $\forall \varepsilon > 0$, $\exists P$ partition s.t. $U(f,P) - L(f,P) < \varepsilon$.

pf) $\Rightarrow$: Suppose f is integrable, then $U(f) = L(f)$.

$\left(U(f) = \inf_{\text{all partitions } P} U(f,P)\right)$

$\exists P_1$ s.t. $U(f,P_1) \leq U(f) + \frac{\varepsilon}{2}$

$\exists P_2$ s.t. $L(f,P_2) \geq L(f) - \frac{\varepsilon}{2}$

$\left(L(f) = \sup_{\text{all partition } P} L(f,P)\right)$

YANGJI

Subtracting these inequalities, we get

$$U(f, P_1) - L(f, P_2) \le \left(U(f) + \frac{\varepsilon}{2}\right) - \left(L(f) - \frac{\varepsilon}{2}\right)$$
$$= \varepsilon$$

choose $P = P_1 \cup P_2$, so, $U(f, P_1) \ge U(f, P)$
and $L(f, P_2) \le L(f, P)$

Hence, $U(f, P) - L(f, P) \le U(f, P_1) - L(f, P_2) \le \varepsilon$

$\Leftarrow$: Suppose $\forall \varepsilon > 0, \exists P : U(f, P) - L(f, P) < \varepsilon$.

Notice $U(f) \le U(f, P)$ and $L(f) \ge L(f, P)$

And, $0 \le U(f) - L(f) \le U(f, P) - L(f, P) < \varepsilon$, for $\forall \varepsilon > 0$

And, $U(f) = L(f)$. So, f is integrable. ☆

QED

<δ-ε formulation>

Def: Given a partition $P = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$.

Then, mesh $(P) = \max_{k=0,\dots,n-1} \{t_{k+1} - t_k\}$

↳ ☆ observe that the smaller the mesh is, the more segments P has.

Thm) (Criterion of integrability)

f is integrable on $[a, b]$ iff for $\forall \varepsilon > 0, \exists \delta > 0$ s.t. mesh $(P) < \delta$ we have $U(f, P) - L(f, P) < \varepsilon$ for all P of $[a, b]$

$$U(f,P) - L(f,P) \overset{\text{by def}}{=\!=} \sum_{k=0}^{n-1} \left(\sup_{[t_k, t_{k+1}]} f - \inf_{[t_k, t_{k+1}]} f \right) (t_{k+1} - t_k)$$

☆ mesh(P) smaller $\longrightarrow$ n bigger

There's a trade-off between the smallness of each summand and the # of summations.

(P) $\longrightarrow$ $\sup f \approx \inf$ and $t_{k+1} \approx t_k$ as $n \uparrow$ ☆