01.18 Thurs

set of integers

$N \subset Z \subset Q$

↳ $\{p/q \mid p, q \in Z, q \neq 0\}$

$a+b=b+a$ (commutatirity)

$(a+b)+c=a+(b+c)$ (associativity)

$a \times b = b \times a$ (commutativity)

$(a \times b) \times c = a \times (b \times c)$ (associativity)

$a \times (b+c) = a \times b + a \times c$ (distributivity)

$a+0=a$
$a \times 1 = a$) identity

* claim) there is no $x \in Q$ s.t. $x \cdot x = 2$

↳ so, we want to extend Q and get another set

☆ But $\sqrt{2}x^3 + \sqrt[3]{\frac{7+\sqrt{4}}{3}}x + 1 = 0$ has an algebraic solution, as $\sqrt{2}$, $\sqrt[3]{\frac{7+\sqrt{4}}{3}}$ is algebraic!!!

Def) A number x is an algebraic # if it is a solution of a polynomial with integer coefficient

↳ ex) x_0 is a solution of $C_n x^n + C_{n-1}x^{n-1} + \cdots + C_0 = 0$ where $C_n, \dots, C_0 \in Z$, $C_n \neq 0$.

↳ ex) $x_0 = \frac{p}{q} \longrightarrow qx_0 - p = 0 \longrightarrow qx - p = 0$, $q, p \in Z$

↳ ex) $\sqrt{2}$ is an algebraic #, as $x^2 - 2 = 0$

ex) $x_0 = \sqrt[3]{\frac{4+\sqrt{7}}{3}}$ is an algebraic #, since $9x_0^6 - 24x_0^3 + 16 - 7 = 0$.

But how do we know an algebraic # x_0 is rational?

Thm) Suppose $x_0 = \frac{p}{q}$ is rational sol of $c_n x^n + \cdots + c_1 x + c_0 = 0$ where $c_n, \cdots, c_0 \in \mathbb{Z}$, $c_n \neq 0$. Then, p divides c_0 and q divides c_n.

Ex) Assume $\sqrt{2} = \frac{p}{q}$ is rational. Since $\sqrt{2}$ is a solution of $x^2 - 2 = 0$. Then p divides -2 and q divides 1.

Thus, $\sqrt{2}$ must be either ± 1, ± 2

This is a contradiction since they do not satisfy the equation. So, $\sqrt{2}$ is not rational.

pf) Assume that p and q have no common factors.

$c_n \left(\frac{p}{q}\right)^n + \cdots + c_1 \left(\frac{p}{q}\right) + c_0 = 0$. And, multiply q^n both sides, then, $c_n p^n + \cdots + c_1 p q^{n-1} + c_0 q^n = 0$

As $c_n p^n, \cdots, c_1 p q^{n-1}$ is divisible by p, $c_0 q^n$ is divisible by p as $c_0 q^n = -c_n p^n - \cdots - c_1 p q^{n-1}$.

And, this implies c_0 [illegible] is divisible by p as q^n is not.

Now, do the similar thing, c_n is divisible by q.

ex) Find all rational solutions of the equation
$5x^4 - 7x^3 + 2x - 6 = 0$

Sol) Rational Zero Thm says if $x_0 = \frac{p}{q}$ is a rational sol, then p divides -6 and q divides 5. So $p \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$ and $q \in \{\pm 1, \pm 5\}$. The rest is writing down all possible p/q and sub to the equation.

Corollary) Say x_0 is a rational solution of
$x^n + C_{n-1}x^{n-1} + \cdots + C_0 = 0$, where $C_{n-1}, \cdots, C_0 \in \mathbb{Z}$
Then x_0 has to be an integer.

$\mathbb{Q} \subset \{\text{algebraic } \#\}$

1) If x_0, x_1 is algebraic, $x_0 + x_1$ and $x_0 x_1$ are algebraic
2) If x_0 is algebraic, $\frac{1}{x_0}$ is algebraic.

Def) A field $\mathbb{F}$ is a set with $+, \times, -, /, 0, 1$ and there arithmetic satisfy the standard properties of $+, \times, -, /$.

ex) $\mathbb{Q}$ is a field
$\mathbb{Z}$ is not a field
$\{\text{Algebraic } \#s\}$ is a field.

'completion axiom' ☆

Def) $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

geometric distance from x to 0.

properties $\Rightarrow$ $|xy| = |x||y|$
$|x+y| \leq |x| + |y|$
$|x-y| \leq |x| + |-y| = |x| + |y|$

$|x-y|$: geometric distance between x and y

$|x|$: geo dist between x and 0.