# 01. 23 Tue

$\mathbb{Q} \Rightarrow$ field $\Rightarrow$ have $(+, -, \times, /, 0, 1)$

ex) $(x^2+5)(x-\frac{1}{x}) + \frac{1}{x^4 - \frac{4}{7}x^2 + \frac{1}{x^2+1}} = 0$

$\hookrightarrow \frac{p(x)}{q(x)}$ with coeff of these $p$ and $q$ are in $\mathbb{Q}$.

$\rightarrow$ You cannot construct $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

or $\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$

with these properties.

☆ $\mathbb{Q} \longrightarrow$ completion $\longrightarrow \mathbb{R}$ $\longrightarrow$ Cauchy sequence

$\hookrightarrow$ whenever you have a sequence in $\mathbb{Q}$ that are supposed to be converged, it will introduce an element that is the limit of the sequence.

$\hookrightarrow$ ex) $1 \quad 1.4 \quad 1.41 \quad 1.414 \cdots \longrightarrow \sqrt{2}$

☆ $S \subset \mathbb{R}$ is a subset of $\mathbb{R}$

Def) $M$ is the maximum of $S$ if

- $M \in S$
- $\forall s \in S, s \leq M$

ex) $S = \{-1, 1, 5, 7, -10\} \Rightarrow \max S = 7$

ex) $N = \{1, 2, 3, \cdots\} \Rightarrow \max N$ does not exist

ex) $S = \{x \in R \mid x < 1\} \Rightarrow \max S$ does not exist because $1 \notin S$.

def) $m$ is $\min S$ if
- $m \in S$
- $\forall s \in S,\ s \geq m$.

☆ observation) Not all subsets in $R$ have maximum or minimum.

def) Given $S \subset R$, $M$ is an upper bound of $S$ if $\forall s \in S,\ s \leq M$.

ex) $S = \{1, -1, 5, 7, -10\} \Rightarrow$ any $\# \geq 7$ (not unique)

def) $m \in R$ is a lower bound of $S$ if $\forall s \in S,\ s \geq m$

ex) $S = \{n^{(-1)^n} \mid n \in N\} = \{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \cdots\}$

$\Rightarrow 0$ is a lower bound for $S$

There is no upper bound.

Is $0$ is the biggest lower bound for $S$? $\Rightarrow$ Yes

$\sup S$

$\|$

def) Given $S \subset R$, $M$ the least upper bound of $S$ if
- $M$ is an upper bound of $S$
- $\forall \varepsilon > 0$, $M - \varepsilon$ is not an upper bound of $S$.

↳☆ $\exists s \in S$ s.t. $s > M - \varepsilon$

Inf S
‖

Def) Given $S \subset \mathbb{R}$, $m$ is the greatest lower bound of $S$ if
- $m$ is a lower bound of $S$
- $\forall \varepsilon > 0$, $m+\varepsilon$ is not a lower bound of $S$
  ↳ ex) $\exists s \in S$ s.t. $s < m+\varepsilon$.

Completion Axiom ⇒ Given $S \subset \mathbb{R}$ and $S$ is bounded above (ex) $S$ has an upper bound), then $\sup S$ exists.

☆ $\mathbb{R}$ satisfies the completion axiom from the way $\mathbb{R}$ is constructed.
However, $\mathbb{Q}$ does not satisfy the completion axiom.
⇒ ex) $S \subset \mathbb{Q}$.
$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$
There is no lowest rational upper bound

Thm) If $S \subset \mathbb{R}$ is bounded below, then $\inf S$ exists

pf) Consider the set $S' = \{-s \mid s \in S\}$
↳ pg. 24
If $m$ is a lower bound for $S$, then $-m$ is an upper bound for $S'$.
Since $\sup S'$ exists (by completion axiom),
$-\sup S'$ is the greatest lower bound of $S$.
Thus $\inf S$ exists and is equal to $-\sup S'$.

↳ ☆ $\operatorname{Inf} S = -(\operatorname{Sup}(S'))$

Thm) <Denseness of $\mathbb{Q}$ in $\mathbb{R}$>
Given 2 real # $a, b \in \mathbb{R}$, $a<b$, $\exists q \in \mathbb{Q}$ s.t. $a<q<b$.

↳pf) we can assume that $0<a<b$
(if $a<0<b$, then $q=0$.
if $a<b<0$, then consider the case $0<-b<-a$

want to find $m, n \in \mathbb{N}$, $a<\frac{m}{n}<b$, so $an<m<bn$.
Two steps ⇒
① $\exists n \in \mathbb{N}$ s.t. $n(b-a)>1$
② Since $nb-na>1$, there is a number $m \in \mathbb{N}$ between $na$ and $nb$.

For ①, using Archemedean property (given $a, b>0$, $\exists n \in \mathbb{N}$ s.t. $na>b$), to $b-a$ and $1$.
So, prove Archemedean propt.
Suppose $na<b$ $\forall n \in \mathbb{N}$. consider $S=\{na \mid n \in \mathbb{N}\} \subset \mathbb{R}$. $S$ is bounded above by $b$. So completion axiom implies that $M=\sup S$ exists.
Hence $M \geq na$ $\forall n \in \mathbb{N}$ and since $a>0$, $M-a$ is not an upper bound for $S$.
Hence, $\exists n_0 \in \mathbb{N}$ s.t. $M-a<n_0 a$, so $M<(n_0+1)a$ and it contradicts. as $(n+1)a \in S$
So, ① is proved.
For ②,