02/13 Tue

$$\sum_{n=1}^{\infty} a_n$$

Def) The partial sums of $\sum_{n=1}^{\infty} a_n$ are $\{S_k\}$:

$$S_k = \sum_{n=1}^{k} a_n = a_1 + \cdots + a_k$$

Def) We say $\sum_{n=1}^{\infty} a_n$ is convergent if $\{S_k\}_{k\in\mathbb{N}}$ is convergent.

If $\lim_{k\to\infty} S_k = \pm\infty$, then we say $\sum_{n=1}^{\infty} a_n$ diverges to $\pm\infty$

Else, we just say $\sum_{n=1}^{\infty} a_n$ diverges

ex) $\sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow S_k = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}$

$$= \frac{1}{2}\left(\frac{1-\frac{1}{2^k}}{1-\frac{1}{2}}\right) = 1 - \frac{1}{2^k}$$

ex) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \rightarrow S_k = \frac{1}{1\cdot 2} + \cdots + \frac{1}{k(k+1)}$

$$= \frac{k}{k+1}$$

ex) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

ex) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ → growing faster....

→ alternate series...

ex) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \frac{(-1)^1}{1} + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \cdots = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$

↳ converge

$\sum_{n=1}^{\infty} \frac{1}{n}$

↳ diverge → cauchy criterion.

Thm) $\sum_{n=1}^{\infty} a_n$ is converged $\Longleftrightarrow$ $\{S_k\}$ is cauchy sequence

↳ $|S_m - S_n| < \varepsilon \Leftrightarrow |a_{n+1} + \cdots + a_m| < \varepsilon$, if $m > n$.

Def) We say $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Thm) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges. (converse does <u>not</u> hold)

ex) $\sum \frac{(-1)^n}{n}$ => converge & but not absolute converge

pf) We need to show that $\sum a_n$ satisfies the Cauchy-criterion.

$\forall \varepsilon > 0$, $\exists N$ s.t. $m > n > N$ implies $|a_{n+1} + \cdots + a_m| < \varepsilon$.

Using triangular inequality, $|a_{n+1} + \cdots + a_m| \leq$
$\underbrace{|a_{n+1}| + \cdots + |a_m|}$

Denote $t_k = \sum_{n=1}^{k} |a_n|$, then $\|$ implies $t_m - t_n$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, Cauchy-criterion says:

$\forall \varepsilon > 0$, $\exists N$ s.t. $m > n > N$: $|t_m - t_n| < \varepsilon$.

So when $m > n > N$: $|a_{n+1} + \cdots + a_m| \leq |t_m - t_n| < \varepsilon$.

Therefore, $\{S_k\}$ is a Cauchy sequence and $\sum_{n=1}^{\infty} a_n$ converges

Thm) Comparison Test:

Suppose $\exists \{M_n\}_n$ s.t. $|a_n| \leq M_n$, and moreover $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges.

pf) We will show this by Cauchy Criterion.

$\forall \varepsilon > 0$, $\exists N$ s.t. $m > n > N$: $|a_{n+1} + \cdots + a_m| < \varepsilon$.

Notice $|a_{n+1} + \cdots + a_m| \le |a_{n+1}| + \cdots + |a_m| \le M_{n+1} + \cdots + M_m$

Since $\sum M_n$ converged by assumption, by cauchy criterion, $\exists N$ s.t. $m > n > N$: $M_{n+1} + \cdots + M_m < \varepsilon$

Cauchy-criterion implies $\sum a_n$ converges.

ex) If you take $M_n = |a_n|$, we'll recover the theorem absolute convergence $\Rightarrow$ convergence.

ex) $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$\Rightarrow$ for $n > 1$, $\frac{1}{n^2} < M_n = \frac{1}{(n-1)n}$.

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$ (as M_n cannot start from 1)

Since $\sum M_n$ converges (by earlier example), $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Ex) $\sum_{n=1}^{\infty} \frac{n^2+2n}{2^n}$

$n^2 + 2n < 2^{\frac{n}{2}} = (\sqrt{2})^n$ for small n.

And, $\frac{n^2+2n}{2^n} < \frac{2^{\frac{n}{2}}}{2^n} = \left(\frac{1}{\sqrt{2}}\right)^n = M_n$.

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What matters is $n^2+2n < 2^{n/2}$ when n is large.
So to say is true, try out the first few n until it is ~~the~~ true and use induction.

Ratio test → use $\left|\frac{a_{n+1}}{a_n}\right|$

Root test → use $|a_n|^{1/n}$