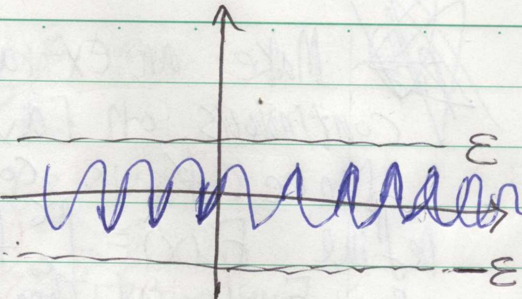
04/19 Thurs

* $f_n \to f$ uniformly
  - continuous (o)
  - integrable (o)
  - differentiable (x) → ex)



Thm) Given f_n differentiable, and assume that $f_n' \to g$ uniformly.
Also, assume that $\exists x_0$ s.t. $\lim_{n\to\infty} f_n(x_0) = c$
Then, $\{f_n\}$ convg uniformly to some func f and $f' = g$.

ex) $f_n(x) = x + n$
Notice that f_n is differentiable
$$f_n'(x) = 1 \longrightarrow g(x) \equiv 1.$$
But $\{f_n\}$ does not convg (even pointwise).

In the case with extra assumption f_n' is continuous, the proof is easier. (using FTOC)

$f(x) = \sum_{n\geq 0} a_n x^n$. $R = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$

The series converges pointwise in $(-R, R)$ and also converges uniformly in $[-R_1, R_1]$ for $\forall R_1 < R$

$S_k(x) = \sum_{n\leq k} a_n x^n$ is a polynomial (nice because it is — continuous, differentiable, smooth [infinitely differentiable], integrable, ---)

f has to be continuous in $(-R, R)$

Is f differentiable? what is f'?

$$S_k'(x) = (a_0 + a_1 x + \cdots + a_k x^k)' = a_1 + 2a_2 x + \cdots + k a_k x^{k-1}$$

Analyzing the sequence $\{S_k'\}$ is the same as analyzing the power series $\sum_{n\geq 1} n a_n x^{n-1}$

$$g(x) = \sum_{n\geq 1} n a_n x^{n-1}$$

$$\tilde{R} = \frac{1}{\limsup_{n\to\infty} \sqrt[n-1]{n|a_n|}} \qquad \left(R = \frac{1}{\limsup \sqrt[n]{|a_n|}} \right)$$

But we want it to be n ---- ☆

So, $x g(x) = \sum_{n\geq 1} n a_n x^n$.

The radius of convg of the power series is

$$R' = \frac{1}{\limsup \sqrt[n]{n|a_n|}}$$

☆ $g(x)$ convg iff $x g(x)$ convg.

So, $\tilde{R} = R'$.

It reduces to find what R' is.

$$\limsup_{n\to\infty} \sqrt[n]{n|a_n|} \overset{?}{=} \lim_{\text{sup}} \sqrt[n]{|a_n|}$$

pf) If the $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then using $\lim_{n\to\infty} \sqrt[n]{n} = 1$

we get $\lim_{n\to\infty} \sqrt[n]{n|a_n|} = (\lim \sqrt[n]{n})(\lim \sqrt[n]{|a_n|}) = \lim \sqrt[n]{|a_n|}$

Hence, $\limsup \sqrt[n]{n|a_n|} = \limsup \sqrt[n]{|a_n|}$

$\sqrt[n_k]{n_k} \to 1$

Notice that $\{\sqrt[n_k]{n_k |a_{n_k}|}\}$ convg $\Leftrightarrow \{\sqrt[n_k]{|a_{n_k}|}\}$ convg

Actually, they convg to the same value.

Therefore, the set of subsequential limits of the 2 ~~other~~ sequences $\{\sqrt[n]{|a_n|}\}$ and $\{\sqrt[n]{n|a_n|}\}$ are the same. Hence, $\limsup \sqrt[n]{n|a_n|} = \limsup \sqrt[n]{|a_n|}$

From the limsup relations, we get $R' = R$.
$\tilde{R} = R' = R$. So, $\sum_{n\geq 1} n a_n x^{n-1}$ has the same radius of convg on $\sum_{n\geq 0} a_n x^n$.

$$S_k'(x) = a_1 + 2a_2 x + \cdots\cdots + k a_k x^{k-1}$$

$$\Big\downarrow \text{uniformly}$$

$$g(x) = \sum_{n\geq 1} n a_n x^{n-1} \quad \text{in } [-R_1, R_1] \ \forall R_1 < R.$$

Also, $S_k(0) = a_0 \quad \forall k$

Therefore, by the Thm, we obtain that $\{S_k\}$ conv uniformly to f and also $f' = g$ in $[-R_1, R_1]$

Hence, f' exists in $(-R, R)$ and $f' = g$ in $(-R, R)$.

$$\frac{d}{dx} \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} \frac{d}{dx}(a_n x^n) \quad \text{in } (-R, R)$$

ex) $\sum_{n\geq 0} \frac{x^n}{n!}$. $\quad R = \infty$

From what we showed, this func is differentiable and the derivative is $\sum_{n\geq 1} \frac{n x^{n-1}}{n!} = \sum_{n\geq 1} \frac{x^{n-1}}{(n-1)!}$ which is the same as the original series.

↳ ex) $f'(x) = f(x)$ in the case.

ex) $f_1(x) = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}(-1)^n = x - \frac{x^3}{5} + \frac{x^5}{5!} - \cdots$

$f_2(x) = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

Then, $R_1 = R_2 = \infty$

$f_1'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = f_2(x)$

$f_2'(x) = -(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) = -f_1(x)$.

Def) (S,d) is a metric space (the set of points)

given $x, y \in S$ we get $d(x,y)$.

d satisfies the following properties:

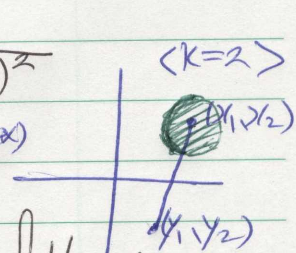
- $d(x,y) \geq 0$ and is 0 if $x=y$
- $d(x,y) = d(y,x)$
- Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$

ex) $S = R^k = \{(x_1, \ldots, x_k) \mid \text{each } x_i \in R\}$

Say $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in R^k

$d_{std}(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_k-y_k)^2}$

$B_\varepsilon(x) = \{y \in R^k \text{ s.t. } d_{std}(x,y) < \varepsilon\} \to$ ex)

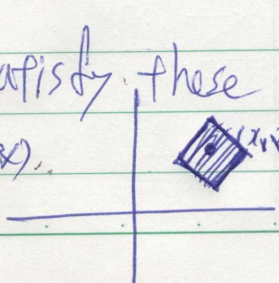


Define a differene metric on R^k as follows:

$d_{taxi\text{-}cab}(x,y) = \sum_{i=1}^{k} |x_i - y_i|$

(※ Taxicap also satisfy these properties)

$B_\varepsilon^{Taxicab}(x) = \{y \mid d_{taxi\text{-}cab}(x,y) < \varepsilon\} \to$ ex).



These 2 metrics are different, but the notions of convg are the same,

i.e.) $\{x^{(n)}\}$ convg to x w.r.t. d_{std}

$\iff$ $\{x^{(n)}\}$ convg to x w.r.t. $d_{taxi-cab}$

Topology $\Rightarrow$ when you only care above convg issue instead of the specific metric.

These 2 metrics d_{std} and $d_{taxi-cab}$ give the same topology in R^k.

Ex) $S = \{$ Bounded function $f : [0,1] \to R \}$

$d(f,g) = \sup_{[0,1]} |f-g|$

d is a metric on S.

$f_n \to f$ w.r.t. the metric d is the same as

$\sup_{[0,1]} |f_n - f| \xrightarrow{n\to\infty} 0$ ($f_n \to f$ uniformly on $[0,1]$).