04/24 TUE

(S, d) metric space

$d : S \times S \to R$

Def) $d(x, y)$: distance between x and y

a) $d(x, y) \geq 0$ (equal when $x = y$)

b) $d(x, y) = d(y, x)$

c) $d(x, z) \leq d(x, y) + d(y, z)$

ex) $S=\mathbb{R}$, d_{std} $\Big\}$ → Not the same metric space
$S=\mathbb{Q}$, d_{std}

ex) $S=\mathbb{R}^k$, $d_{std}(x,y) = \sqrt{\sum_{i=1}^{k}(x_i-y_i)^2}$

$d_{taxicab}(x,y) = \sum_{i=1}^{k}|x_i-y_i|$

Def) Suppose $\{x_n\}_{n\in\mathbb{N}} \subset (S,d)$, we say $x_n \xrightarrow{n\to\infty} x_0$ if $\lim_{n\to\infty} d(x_n,x_0)=0$

ex) In the case $S=\mathbb{R}^k$, we'll show that: $\{x_n\} \xrightarrow{n\to\infty} x_0$ in d_{std} $\iff$ $\{x_n\} \xrightarrow{n\to\infty} x_0$ in $d_{taxicab}$

pf) Say $x,y\in\mathbb{R}^k$, $d_{std}(x,y) \le d_{taxicab}(x,y)$

If $x_n \xrightarrow{n\to\infty} x_0$ in d_{std}, $\frac{1}{k} d_{taxicab}(x,y) \le d_{taxicab}(x_n,x_0) \le \underbrace{k\cdot d_{std}(x_n,x_0)}_{\to 0}$

so, $x_n \to x_0$ in $d_{taxicab}$.

If $x_n \xrightarrow{n\to\infty} x_0$ in $d_{taxicab}$, then $d_{std}(x_n,x_0) \le \underbrace{d_{taxicab}(x_n,x_0)}_{\to 0}$

so, $x_n \xrightarrow{n\to\infty} x_0$ in d_{std}.

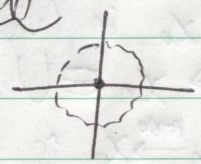
Thm) If $x_n \xrightarrow{n\to\infty} x_0$ and also $x_n \xrightarrow{n\to\infty} y_0$, then $x_0=y_0$.

pf) $d(x_0,y_0) \le d(x_0,x_n) + d(x_n,y_0) \xrightarrow{n\to\infty} 0$

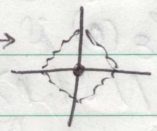
so, $d(x_0,y_0)=0$, so by def, $x_0=y_0$.

$B_r(x_0) = \{x\in S \mid d(x_0,x) < r\}$ → open

For $(\mathbb{R}^2, d_{std})$ ⇒ : $B_r(0)$

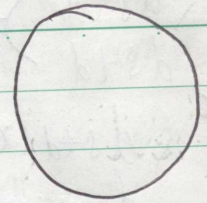


$(\mathbb{R}^2, d_{taxicab})$ ⇒ : $B_r(0)$ →



Def) $U\subset S$ is open if $\forall x_0\in U$, $\exists r>0$ s.t. $B_r(x_0)\subset U$.

ex) $\Rightarrow U$ (open)

 $\Rightarrow U$ (not open)

One can think an given set as a union of open balls. $\left(U = \bigcup_{x_0 \in U} B_{r_{x_0}}(x_0)\right)$

↳ uncountable

☆ properties ☆

a) If $U_1, \dots, U_n$ are open in S, then $U_1 \cap \dots \cap U_n$ is also open in S.

pf) Take $x_0 \in U_1 \cap \dots \cap U_n$. $B_r(x_0) \subset U_1 \cap \dots \cap U_n$, where $r = \min\{r_1, \dots, r_n\}$ where $B_{r_i}(x_0) \subset U_i$.

(If U_i is <u>not</u> finite, then r might become 0, and not hold anymore)

↳ ex) $U_n = B_{\frac{1}{2}+\frac{1}{n}}(0)$

$\bigcup_{n \in \mathbb{N}} U_n = \{x \in S \mid d(x,0) \le \frac{1}{2}\}$ is not open.

↳ $d(x,0) < \frac{1}{2} + \frac{1}{n}$ for $\forall n$

$\parallel$

$d(x,0) \le \frac{1}{2}$

b) If $\{U_\alpha\}_{\alpha \in A}$ is any collection of open sets, then $\bigcup_{\alpha \in A} U_\alpha$ is open.

Thm) $x_n \xrightarrow{n\to\infty} x_0$ in $(S,d) \Leftrightarrow \forall$ open set $U \ni x_0$, $\exists N$ s.t. $n > N \to x_n \in U$.

pf) $\Leftarrow$: $x_n \xrightarrow{n\to\infty} x_0$ is the same as $d(x_n, x_0) \xrightarrow{n\to\infty} 0$

$U = B_\varepsilon(0)$ is an open.

So, $\exists N$ s.t. $n > N$ implies $x_n \in U$ (i.e. $d(x_n, x_0) < \varepsilon$),

hence, $x_n \xrightarrow{n\to\infty} x_0$.

$\Rightarrow$: since U is open and $x_0 \in U$, $\exists r>0$ s.t. $B_r(x_0) \subset U$.
since $x_n \xrightarrow{n\to\infty} x_0$, $\exists N$ s.t. $n>N$ implies $x_n \in B_r(x_0)$
so, $x_n \in U$, $\forall n > N$.

QED

ex) $\left.\begin{array}{l} R^k, d_{std} \\ R^k, d_{taxicab} \end{array}\right) \Rightarrow$ Convg in d_{std} $\Vert$ convg in $d_{taxicab}$

$\boxed{\{\text{open sets in } (R^k, d_{std})\} = \{\text{open sets in } (R^k, d_{taxicab})\}}$

a ball can be writen as $\bigcup$ of diamond
< a diamond can be written as $\bigcup$ of open balls.

Def) A seq $\{x_n\}_{n\in N} \subset S$ is cauchy if $\forall \varepsilon>0, \exists N>0$ s.t. $d(x_m, x_n) < \varepsilon$ when $m, n > N$

Def) (S, d) is complete if every cauchy sequence convg in S

ex) (R, d_{std}) is complete
< (Q, d_{std}) is not complete

ex) $S = \{f : [0,1] \to R, \text{bounded}\}$
$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$
IS S complete?

$\Rightarrow$ start with a cauchy seq $\{f_n\}_{n\in N}$
$\sup_{x\in[0,1]} |f_n(x) - f_m(x)| \xrightarrow{m,n\to\infty} 0$ (1)
For each x, (1) implies $\{f_n(x)\}_{n\in N}$ is a cauchy seq of real #.
$\Rightarrow f_n(x) \xrightarrow{n\to\infty}$ sth we call $f(x)$
check that $d(f_n, f) \xrightarrow{n\to\infty} 0$ S is complete.

ex) $S = \{f : [0,1] \to R, \text{continuous}\}$ with same d.
$\{f_n\}$ is cauchy $\Leftrightarrow$ $\{f_n\}$ is uniformly cauchy
in the matrix d

S is complete since $\{f_n\}$ uniformly cauchy & continuous
$\Rightarrow f_n \to f$ and f is continuous
(ex) a cauchy seq in S convg in S.) S is complete

$(S_1, d_1) \xrightarrow{f} (S_2, d_2)$

Def) f is continuous if whenever $x_n \to x_0$ in (S_1, d_1), the outputs $f(x_n) \to f(x_0)$ in (S_2, d_2)

ex) $S_1 = \{ g : [a,b] \to \mathbb{R}, \text{ integrable} \}$

$d_1(g_1, g_2) = \sup_{[a,b]} |g_1 - g_2|$

$F : (S_1, d_1) \to (\mathbb{R}, d_{std})$

$F(g) = \int_a^b g$

Is F continuous? $\Leftrightarrow$ If $f_n \to f$ in d_1 (i.e. uniformly on $[a,b]$), then $\int_a^b f_n \to \int_a^b f$.