04/17 Tue

Q. $f_n \to f$ uniformly, Suppose f_n is continuous, differentiable or integrable.

Does f satisfy the same property?

continuous (o)
differentiable (x)
integrable (o)

Thm) Suppose $f_n \to f$ uniformly in S (i.e. $[a,b]$, (a,b), $(a,b]$, ...)

If f_n is continuous in S, then f is continuous in S.

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pf) Take $x_0 \in S$ and given $\varepsilon > 0$. You want to find $\delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow \underbrace{|f(x) - f(x_0)|}_{\text{SS}} < \varepsilon$

$|f_n(x) - f_n(x_0)|$ when n is large

$$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$$
$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

Since $f_n \to f$ uniformly, $\exists N$ s.t. $n > N$:
$$\sup_{x \in S} |f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

Let's choose $n = N+1$, then $|f(x) - f_{N+1}(x)| < \frac{\varepsilon}{3}$ and $|f_{N+1}(x_0) - f(x_0)| < \frac{\varepsilon}{3}$.

Since f_{N+1} is continuous, $\exists \delta > 0$ s.t.
$$|x - x_0| < \delta \Rightarrow |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\varepsilon}{3}.$$
Therefore, when $|x - x_0| < \delta$, we have
$$|f(x) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$
Hence, f is continuous at x_0 $\forall x_0 \in S$, and therefore f is continuous in S.

QED

Thm) Suppose f_n is integrable on $[a, b]$ and $f_n \to f$ uniformly on $[a, b]$.

Then, a) f is also integrable on $[a, b]$

b) If you define $F_n(x) = \int_a^x f_n$ and $F(x) = \int_a^x f$, then $F_n \to F$ uniformly on $[a, b]$.

ⓐ

pf) Want to show $\exists P$ partition of $[a,b]$ s.t.

$U(f,P) - L(f,P) < \varepsilon$ or $\sum_{i=0}^{n-1} \left(\sup_{[t_i,t_{i+1}]} f - \inf_{[t_i,t_{i+1}]} f \right)(t_{i+1}-t_i)$

$< \varepsilon$.

Strategy: relate $U(f,P) - L(f,P)$ with $U(f_n,P) - L(f_n,P)$.

Suppose $\sup_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon$.

i.e. $-\varepsilon < f_n(x) - f(x) < \varepsilon \quad \forall x \in [a,b]$

$\iff f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon \quad \forall x \in [a,b]$

prove

$\forall x \in [t_i, t_{i+1}]$, $f(x) \le f_n(x) + \varepsilon \le \sup_{[t_i,t_{i+1}]} f_n + \varepsilon$

or, $f(x) \le \sup_{[t_i,t_{i+1}]} f_n + \varepsilon$

So, $\sup_{[t_i,t_{i+1}]} f \le \sup_{[t_i,t_{i+1}]} f_n + \varepsilon$

↳ lowest upper bound ☆

$\inf_{[t_i,t_{i+1}]} f \ge \inf f_n - \varepsilon$

So, $\sup_{[t_i,t_{i+1}]} f - \inf_{[t_i,t_{i+1}]} f \le \sup_{[t_i,t_{i+1}]} f_n - \inf_{[t_i,t_{i+1}]} f_n + 2\varepsilon$

So, $\cancel{\cancel{\gamma}} \le \sum_{i=0}^{n-1} \left(\sup_{[t_i,t_{i+1}]} f_n - \inf_{[t_i,t_{i+1}]} f_n + 2\varepsilon \right)(t_{i+1}-t_i)$

$= U(f_n,P) - L(f_n,P) + 2\varepsilon(b-a)$

Given $\varepsilon > 0$, choose N s.t. $n > N$: $\sup_{[a,b]} |f - f_n| < \varepsilon$.

Take $n = N+1$, then

$U(f,P) - L(f,P) \le U(f_{N+1},P) - L(f_{N+1},P) + 2\varepsilon(b-a)$

↳ fixed

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Since f_{N+1} is integrable, $\exists P$ s.t.

$$U(f_{N+1}, P) - L(f_{N+1}, P) < \varepsilon$$

Therefore, $U(f, P) - L(f, P) < \varepsilon \underbrace{(2b-2a+1)}_{\Rightarrow \text{fixed}}$

Hence, f is integrable on $[a,b]$

I can change $\sup|f-f_n| < \varepsilon$, to get ε only

(b) $\begin{cases} F_n(x) = \int_a^x f_n \\ F(x) = \int_a^x f \end{cases}$ (F exists since f is integrable)

$$|F_n(x) - F(x)| = \left|\int_a^x (f_n - f)\right| \le \int_a^x |f_n - f| \le \int_a^x \sup_{[a,b]} |f_n - f|$$

Notice, $\forall x$ $|f_n(x) - f(x)| \le \sup_{[a,b]} |f_n - f|$

Hence, $|F_n(x) - F(x)| \le (x-a) \sup_{[a,b]} |f_n - f| \le (b-a) \sup_{[a,b]} |f_n - f|$

So, $\sup_{[a,b]} |F_n - F| \le (b-a) \sup_{[a,b]} |f_n - f| \approx 0$ as $n \to \infty$.

Hence, $\sup_{[a,b]} |F_n - F| \xrightarrow{n\to\infty} 0$, and thus $F_n \to F$ uniformly in $[a,b]$.

cannot be negative....

<Differentiable>

$f_n \to f \not\Rightarrow f_n' \to f'$

Thm) Suppose f_n is differentiable on $[a,b]$.

Moreover $\{f_n'\}$ convg uniformly to g on $[a,b]$.

Assume that $\exists x_0 \in [a,b]$ s.t.

$\lim_{n\to\infty} f_n(x_0) = C \in \mathbb{R}$.

Then, $f_n \to f$ uniformly on $[a,b]$ s.t. $f' = g$.

pf) Make an extra assumption: each f_n' is continuous on $[a,b]$.

Since f_n' is continuous, f_n' is integrable.

Define $F_n(x) = \int_{x_0}^{x} f_n'$ → by assumption, convg uniformly to g

By Fundamental thm of Calculus, $F_n(x) = f_n(x) - f_n(x_0)$

Since f_n' is continuous & $f_n' \to g$ uniformly, I have g is also continuous.

In particular, g is integrable and I can define

$$G(x) = \int_{x_0}^{x} g$$

From previous thm, $F_n(x) \to G$ uniformly on $[a,b]$.

$$\underbrace{F_n(x)}_{= f_n(x) - f_n(x_0)}$$

$f_n(x_0)$ → convg to C as $n \to \infty$

Since $\lim_{n\to\infty} f_n(x_0) = C$, I obtain that

$$f_n(x) - C \to G \text{ uniformly in } [a,b]$$

Define $f(x) = G(x) + C$. Hence $f_n \to f$ uniformly in $[a,b]$.

Since g is continuous, FTOC implies that

$$f' = G' = g$$

QED.

$\sum_{n\geq 0} a_n x^n$ and $R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$

The series convg pointwise in $(-R, R)$, convg uniformly in $[-R_1, R_1]$, $\forall\ 0 < R_1 < R$.

So, the series is a continuous func in $(-R_1, R_1)$

Is the series differentiable?

$$f_n(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{n\geq 0} a_n x^n$$

$$f_n'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} = \sum_{n\geq 1} n a_n x^{n-1}$$