03/01 Thurs

Thm ⟨Intermediate Value Thm⟩
Given a continuous function $f:[a,b]\to\mathbb{R}$, then any value c between $f(a)$ and $f(b)$, $\exists\, x\in[a,b]$ s.t. $f(x)=c$.

Thm) Suppose f is strictly increasing $f:[a,b]\to\mathbb{R}$
Then, Image of $f=[f(a),f(b)]$
$\Longleftrightarrow$ f is continuous

pf) $\Leftarrow$: Use I.V.T

$\Rightarrow$: Assume f is not continuous at x_0.
ex) $\exists$ a seq $\{x_n\}_{n\in\mathbb{N}}$ s.t. $\lim x_n = x_0$ but $\{f(x_n)\}_{n\in\mathbb{N}}$ does not converge to $f(x_0)$

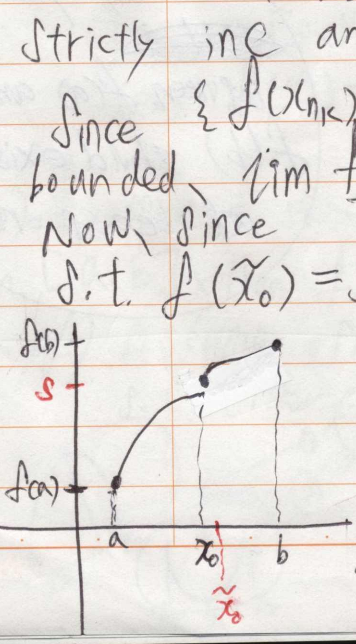
From $\{x_n\}_{n\in\mathbb{N}}$, pick a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that is either inc/dec

Hence, since f is strictly inc, $\{f(x_{n_k})\}_{k\in\mathbb{N}}$ is either strictly inc/strictly dec.

Assume without loss of generality, that $\{x_{n_k}\}$ is strictly inc and thus $\{f(x_{n_k})\}_{k\in\mathbb{N}}$ is strictly inc.

Since $\{f(x_{n_k})\}_{k\in\mathbb{N}}$ is strictly inc and bounded, $\lim f(x_{n_k}) = S$ exists.

Now, since Image $f=[f(a),f(b)]$, $\exists\,\tilde{x}_0\in[a,b]$ s.t. $f(\tilde{x}_0)=S$. ~~Notice $\tilde{x}_0$ ... x_0~~



(original assumption, $\lim\limits_{k\to\infty} f(x_{n_k}) \neq f(x_0)$
$S \overset{!!}{=} f(\tilde{x}_0)$)

Notice $f(\tilde{x}_0) > f(x_{n_k})$ implies $\tilde{x}_0 > x_{n_k}$ (since f is $\nearrow$)

Since $x_{n_k}\to x_0$, then $\tilde{x}_0 \geq x_0$.

On the other hand,

~~But this contradicts that~~ $f(\tilde{x}_0) \le f(x_0)$ → implies $\tilde{x}_0 \le x_0$, as f is strictly inc.

($f(x_{n_k}) < f(x_0)$ since $x_{n_k} < x_0$ implies

$f(\tilde{x}_0) \le f(x_0)$ since $\lim_{k\to\infty} f(x_{n_k}) = f(\tilde{x}_0)$)

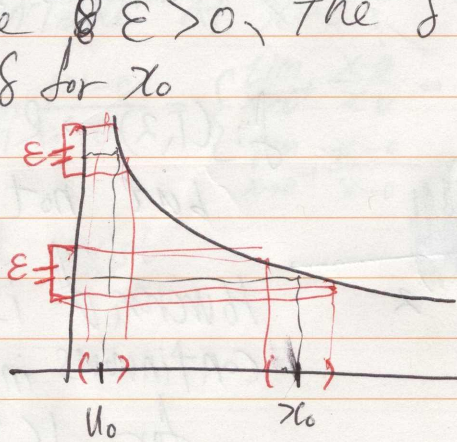
So $\tilde{x}_0 = x_0$, and hence $\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$ → contradict to the assumption.

↳ My goal is to show $x_0 = \tilde{x}_0$!!!

Thm) If $f : [a, b] \to [c, d]$ is continuous and have the inverse f^{-1}. Then, f^{-1} must be continuous

pf) f has inverse f^{-1}, meaning f is a bijection. Since f is a 1-1 and continuous, then f is either strictly inc/dec. If f is strictly inc, then f^{-1} is also strictly inc. Furthermore, image of f^{-1} is $[a, b]$ from bijection of f. Applying the previous theorem to f^{-1}, we obtain f^{-1} is also continuous.

☆ δ depends on x_0 and ε.
For the same $\varepsilon > 0$, the δ for u_0 is much smaller than δ for x_0

↳ 

Def) f is uniformly continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $|x-y| < \delta$, then $|f(x)-f(y)| < \varepsilon$.

(equivalent is: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. δ works for all x_0 ☆ in the def of continuity of f at x_0)

ex) $f(x) = \frac{1}{x}$, $x \in (0, \infty)$ is not uniformly continuous.

δ depends on ε only !! ☆

ex) $f(x) = \frac{1}{x}$.

a) when interval is $[1, 2]$.

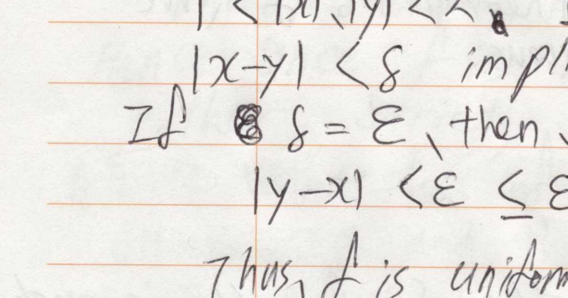
Given $\varepsilon > 0$, we want to solve for δ.

$|f(x) - f(y)| < \varepsilon$, or, $\left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$, or, $\left|\frac{y-x}{xy}\right| < \varepsilon$, or, $|y-x| < \varepsilon |x||y|$. since $x, y \in [1, 2]$, $1 < |x|, |y| < 2$. I need to define δ so, $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

If $\delta = \varepsilon$, then, $|x-y| < \delta = \varepsilon$ implies $|y-x| < \varepsilon \leq \varepsilon |x||y|$, implies $|f(x) - f(y)| < \varepsilon$.

Thus, f is uniformly continuous on $[1, 2]$.

ex)



$f: (1, 2) \to R$ is continuous, but not uniformly continuous.

However, f is uniformly continuous in $(1, 2-\ell)$ for $1 < \ell < 2$.

→ Then in any smaller set $(c,d) \subset [a,b]$, f should be uniformly continuous.

Thm) $f: [a,b] \to \mathbb{R}$ continuous, then f must be uniformly continuous.

Thm) $f: (a,b) \to \mathbb{R}$, and f is uniformly continuous $\iff$ f can be continuously extended to the endpoints a & b.

☆ f is continuously extended to a & b means you can define $f(a)$ & $f(b)$ s.t. $f: [a,b] \to \mathbb{R}$ is continuous.

Def) $f: S \to \mathbb{R}$, we say f is differentiable at $a \in S$, if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists. Notation

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \text{ if exists.}$$

ex) $f(x) = |x|$. Is f differentiable at $x=1$?

$$\lim_{x \to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1} \frac{|x|-1}{x-1}$$

$$= \lim_{x \to 1} \frac{x-1}{x-1} = 1$$

Is f differentiable at $x=0$?

$$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 1} \frac{|x|-0}{x-0} = \begin{cases} \lim_{x \to 0^+} \frac{x-0}{x-0} = 1 \\ \lim_{x \to 0^-} \frac{-x-0}{x-0} = -1 \end{cases} \ne$$

Hence, limit does not exist.

Thm) If f is differentiable at a, then f is continuous at a.

pf) pick a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a.
Since f is differentiable at $x=a$,

$$\lim_{n\to\infty} \frac{f(x_n)-f(a)}{x_n-a} = f'(a).$$

Since $\lim_{n\to\infty}(x_n-a)=0$, this implies $\lim_{n\to\infty}(f(x_n)-f(a)) = 0$. Therefore, f is continuous at $x=a$.