04/05 Thurs

$\{f_n\}: S \to \mathbb{R}$

def) $f_n \to f$ pointwise
if $\forall x \in S: \lim_{n\to\infty} f_n(x) = f(x)$

def) $f_n \to f$ uniformly
if $\sup_{x\in S} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$

ex) $\forall \varepsilon > 0, \exists N$ s.t. when $n > N$, we have
$|f_n(x) - f(x)| < \varepsilon, \forall x \in S$.

observation: Uniform convg $\longrightarrow$ pointwise convg

ex) $f_n(x) = x^n$ on $S = [0,1]$

Q. Does $\{f_n\} \to f$ pointwise?
For $x \in S = [0,1]$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

So, $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

Q. Does $\{f_n\} \to f$ uniformly?

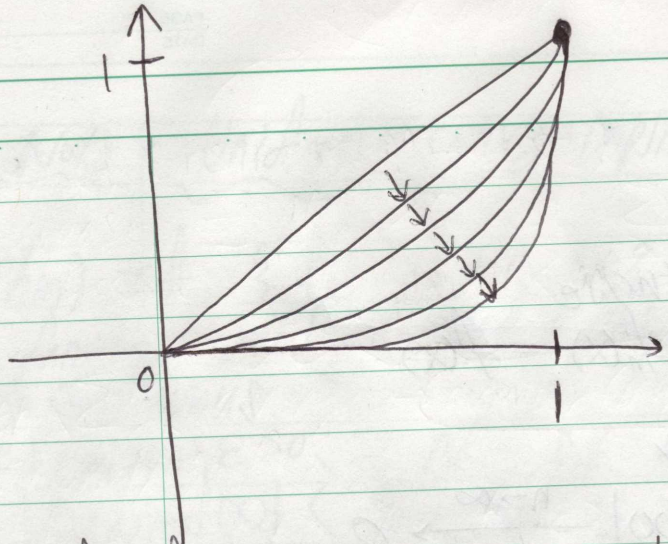
$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup(\{x^n - 0 \mid 0 < x < 1\} \cup \underbrace{\{1^n - 1 = 0\}}_{\text{when } x=1})$

$= \sup([0,1) \cup \{0\}) = 1$

So $\{f_n\}$ X convg to f uniformly on S.

Consider $S = [0, \frac{1}{2}]$
In this case, $\sup_{x\in[0,\frac{1}{2}]} |f_n(x) - f(x)| = \sup_{x\in[0,\frac{1}{2}]} x^n = (\frac{1}{2})^n \xrightarrow{n\to\infty} 0$

$f_n \to f$ uniformly on $[0, \frac{1}{2}]$.



Thm) If f_n is continuous and $f_n \to f$ uniformly, then f is also continuous.

Def) $\{f_n\}$ is a Cauchy seq of functions if
$\forall \varepsilon>0, \exists N$ s.t. $m,n>N$ $\underbrace{\sup_{x\in S}|f_m(x)-f_n(x)|}_{\text{This is the measurement of how close } f_m \,\&\, f_n \text{ is.}}<\varepsilon$.

Thm) If $\{f_n\}_{n\in\mathbb{N}}$ is a cauchy seq of functions defined on S, then $\{f_n\}_{n\in\mathbb{N}}$ uniformly convg.

pf) Let's first check that $\forall x\in S$ $\lim_{n\to\infty} f_n(x)$ exists

From the def of cauchy seq of func, we have $\{f_n(x)\}_{n\in\mathbb{N}}$ is a cauchy seq. Hence $\underbrace{\lim_{n\to\infty} f_n(x)}_{=f(x)}$ exists.

What is left is to check that $f_n \to f$ uniformly in S.

Given $\varepsilon>0$, $\exists N$ s.t. $m,n>N$ $|f_m(x)-f_n(x)|<\frac{\varepsilon}{2}$.

Since $\lim_{n\to\infty} f_n(x)=f(x)$, we obtain $|f_m(x)-f(x)|\le\frac{\varepsilon}{2}<\varepsilon$, $\forall x\in S$ $m>N$.

Therefore, $\{f_n\}\to f$ uniformly.

observation: $\{f_n\}$ convg uniformly $\Longleftrightarrow$ $\{f_n\}$ is a Cauchy seq of func.

we're gonna apply to the case $\sum_{k=0}^{\infty} g_k$ (series).

Def) $\sum_{k=0}^{\infty} g_k$ convg pointwise uniformly (in S) if partial sums $\{S_n = \sum_{k\le n} g_k\}$ convg pointwise uniformly. (in S)

ex) $\sum_{k=0}^{\infty} x^k$, where $g_k(x) = x^k$

What is S for this series?

$S_n(x) = \sum_{k\le n} x^k$ is defined in $\mathbb{R}$

Notice $\{S_n(x)\}_{n\in\mathbb{N}}$ is not convg for $\forall x \in \mathbb{R}$.

For example, $x=1$, $S_n(1) = n+1 \xrightarrow{n\to\infty} \infty$.

Here, we'll see that the series $\sum x^k$ convg pointwise in $(-1,1)$.

Does $\sum x^k$ convg uniformly in $(-1,1)$?

$\sum_{k=0}^{\infty} g_k$ uniformly convg $\overset{def}{\Longleftrightarrow}$ $\{S_n = \sum_{k\le n} g_k\}$ uniformly convg

$\Longleftrightarrow$ $\{S_n\}$ is a cauchy seq of func.

$$\boxed{m \ge n,\ S_m - S_n = \sum_{n+1\le k\le m} g_k}$$

Thm) $\sum g_k$ convg uniformly iff $\forall \varepsilon > 0$, $\exists N$ s.t. $m \ge n > N$ $\sup_{x\in S} \Big|\underbrace{\sum_{n+1\le k\le m} g_k}_{S_m - S_n}\Big| < \varepsilon$

☆ comparison Test

Thm) (Weirstrass M-test)

Suppose $\exists M_n > 0$ s.t. $\sum_{n\geq 0} M_n < +\infty$.

Assume $\sup_{x\in S} |g_n(x)| < M_n$ for $\forall n$.

Then $\sum g_k$ conv uniformly in S.

number (magnitude of the fn on S)

pf) Need to show '$\forall \varepsilon > 0$, $\exists N$ s.t. $m \geq n > N$, $\sup_{x\in S} \left| \sum_{n+1\leq k\leq m} g_k \right| < \varepsilon$'

By Triangular inequality, it is less than / equal to $\sum_{n+1\leq k\leq m} |g_k|$.

$$\sup_{x\in S} \left| \sum_{n+1\leq k\leq m} g_k \right| \leq \sup_{x\in S} \left(\sum_{n+1\leq k\leq m} |g_k| \right) \leq \sum_{n+1\leq k\leq m} \sup_{x\in S} |g_k|$$

$$< \sum_{n+1\leq k\leq m} M_k < \varepsilon$$

☆ since $\sum M_k < \infty$, $m, n > N$ it implies this

Thus, series $\sum g_k$ conv uniformly in S.

power series $\sum_{k\geq 0} a_k x^k$

Q. For which domain $S \subset \mathbb{R}$, does the series conv pointwise?

Q. Does the series conv uniformly in S, or in some smaller subset of S?

ex) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ in $S = \mathbb{R}$

Does the series conv uniformly in $\mathbb{R}$?

Main tool to answer this is Root Test for pointwise conv, and Weirstrass M-test for uniformly conv.

☆ stronger than ratio test