04/26 Thurs

(?) if and only if ?

$(S, d) \rightsquigarrow$ open balls $B_r(x)$ $\rightsquigarrow$ open sets : U is open in S if $\forall x \in U, \exists r$ s.t. $B_r(x) \subset U$, where $U = \bigcup_{x \in U} B_{r_x}(x)$

ex) {open sets in (R^k, d_{std})} = {open sets in $(R^k, d_{taxicab})$}

So, conv in d_{std} = conv in $d_{taxicab}$

ex) (R^k, d_{exotic}) where $d_{exotic}(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$

$x_n \to x_0$ in d_{exotic}? By def: $d(x_n, x_0) \xrightarrow{n\to\infty} 0$ → since 1 when $x_n = x_0$, so in this d_{exotic},

$x_n \xrightarrow{n\to\infty} x_0$ iff $x_n = x_0$ $n > N$.

(?) should exclude $y = x$..?

$B_r(x) = \begin{cases} \{x\}, & r < 1 \\ R^k, & r \geq 1 \end{cases}$

$= \{s \in R^k \mid d_{exo}(x, s) < r\}$

So, any set U is open.

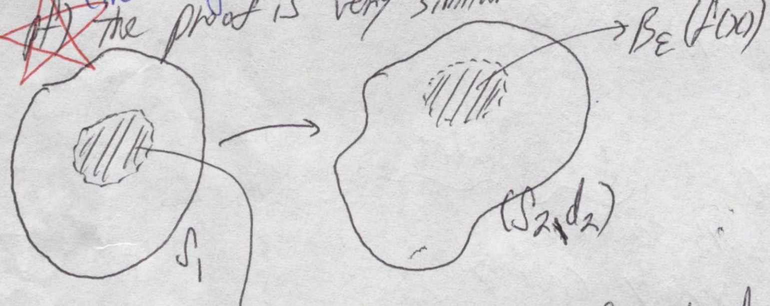
So, {open sets in (R^k, d_{exotic})} $\supseteq$ {open sets in (R^k, d_{std})}

Def) $f : (S_1, d_1) \to (S_2, d_2)$ is continuous if $\forall x \in S_1$ and any sequence $x_n \to x$ w.r.t. d_1 we have $f(x_n) \to f(x)$ w.r.t. d_2.

Thm) f is continuous iff $\forall x, \forall \varepsilon, \exists \delta > 0$ s.t. if $d_1(x, s) < \delta$, then $d_2(f(x), f(s)) < \varepsilon$.

(more general way of proving continuity!!!)

Pf) The proof is very similar....



Since $d_1(x, s) < \delta \Rightarrow d_2(f(x), f(s)) < \varepsilon$ we have $B_\delta(x) \subset$ preimage of ball $B_\varepsilon(f(x))$. So, x is an interior of the pre-image. Actually, any point s in the pre-image is an interior. So, the preimage $f^{-1}(B_\varepsilon(f(x)))$ is an open set.

$f^{-1}(B_\varepsilon(f(x))) = \{s \in S_1 \text{ s.t. } f(s) \in B_\varepsilon(f(x))\}$

Thm) f is continuous $\Leftrightarrow$ for every open set $V \subset S_2$, the preimage $f^{-1}(V)$ is an open set in S_1.

closed set

Def) A set $E \subset S$ is closed if the complement $S \setminus E$ is open.

ex) $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is an open set. So, $[a, b]$ is closed.

ex) $\mathbb{Q} \subset (\mathbb{R}, d_{std})$

$\mathbb{R} \setminus \mathbb{Q} = I$ is not open (HW 13) $\longrightarrow$ $s \in \mathbb{R} \setminus \mathbb{Q}$, any ball $B_r(s)$ must contain a rational #.

ex) $(\mathbb{R}, d_{exotic})$

In this metric, any set is closed.

<properties>

a) If $E_1, \cdots, E_n$ are closed sets, then $\bigcup_{i=1}^{n} E_i$ is also closed.

b) If $\{E_\alpha\}_{\alpha \in A}$ is any collection of closed sets, then $\bigcap_{\alpha \in A} E_\alpha$ is also closed.

pf) Let $U_i = S \setminus E_i$ is open by def.

$\bigcap U_i$ = complement of E_i and $\bigcup U_i$ = complement of $\bigcap E_i$.

By properties of open sets, $\bigcap U_i$ is open, $\bigcup U_\alpha$ is open.

So, the complements are closed sets.

☆ In general, we don't have $\bigcup_{\alpha \in A} E_\alpha$ is closed.

$A = \bigcup_{p \in A} \{p\}$ If A is not closed, then $\bigcup_{p \in A} \{p\}$ is not closed.

Def) A point $x \in S$ is a limit point of E if $\exists \{x_n\} \subset E$ that conv to x.

↳☆ In general, a limit point of E does not have to be in E.

↳

Thm) A set $E \subset S$ is closed iff E contains all of its limit points

pf) $\Rightarrow$:) Suppose E is closed. If $x \in S$ is a limit point of E, so $\exists$ a seq

$x_n \xrightarrow{n \to \infty} x$ and $x_n \in E$.

If $x \notin E$, then $x \in \underbrace{S \setminus E}_{\text{open since } E \text{ is closed}}$.

Hence, $\exists r$ s.t. $B_r(x) < S \setminus E$. But since $x_n \to x$, $\underbrace{x_n}_{\in E} \in B_r(x)$, $n > N$

which is a contradiction.

So, E must contain all of its limit points.

$\Leftarrow$:) Suppose E contains all of limit points. We need to show. We need to show E is closed ($= S \setminus E$ is open). Assume that $S \setminus E$ is not open. ($\exists x \in S \setminus E$ s.t. $\forall r > 0$, $B_r(x) \not\subset S \setminus E$). So, $\exists x_r \in B_r(x)$ and also in E. So, we get a seq $\{x_r\}$ in E that conv to x, so x is a limit point of E. By assumption, $x \in E$ and $x \notin E$ which contradicts.

[a,b] has the property:
1) any sequence in [a,b] muse have a subsequence that conv to a point in [a,b]
(Bolzano-weirsterass Thm)

☆ $f: [a,b] \to R$ is coneinuous then f is bunded and has max & min.