02/22 Thurs

**Comparison Test** $\sum_{n=1}^{\infty} a_n$

Assume that $|a_n| \le M_n$ $n \in \mathbb{N}$

If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is convergent.

The proof of root/ratio test is just the comparison test: given $\limsup_{n\to\infty} (\ ) < 1$, the series $\sum_{n=1}^{\infty} |a_n|$ is dominated by a geometric series for $n > N$.

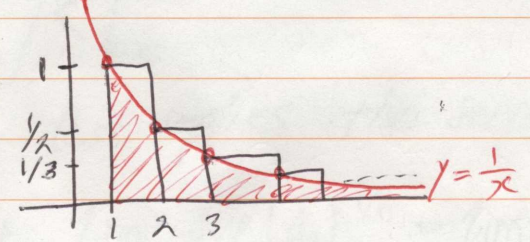
Given $\sum a_n$, assume that $\exists B_n > 0$ s.t. $|a_n| \ge B_n$ for $\forall n$ (or, $\forall n > N$). Then $\sum_{n=1}^{\infty} |a_n|$ divg.

↳ absolute value since $a_n$ might not be positive.

(or, given $\sum a_n$, assume that $\exists B_n$ s.t. $a_n \ge B_n$ ....)

**Integral Test:** ex) $\sum_{n=1}^{\infty} \frac{1}{n}$ divg.

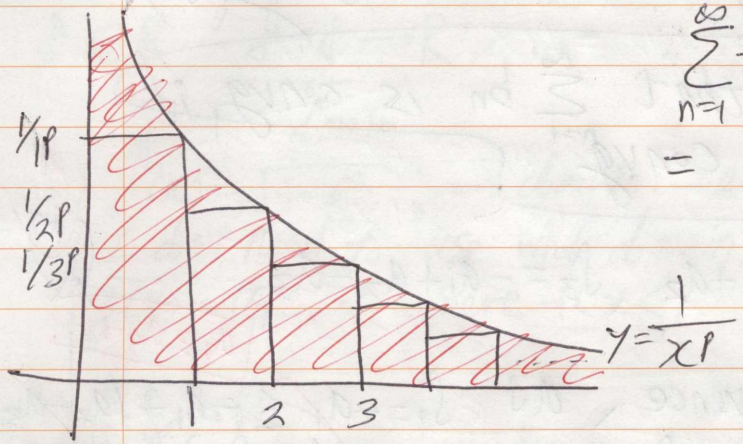
↳




$$\sum_{n=1}^{\infty} \frac{1}{n} \ge \text{Area} = \int_1^{\infty} \frac{1}{x}\,dx = \ln x \Big|_1^{\infty} = +\infty.$$

$\therefore$ Divg.

prove $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convg for $p>1$.




$$\sum_{n=1}^{\infty} \frac{1}{n^p} \le \text{Area}$$

$$= \int_0^{\infty} \frac{1}{x^p} dx = \int_0^{\infty} x^{-p} dx$$

$$= \frac{1}{-p+1} x^{-p+1} \Big|_0^{\infty}$$

$$= \frac{-1}{p-1} \frac{1}{x^{p-1}} \Big|_0^{\infty}$$

$$= 0 + \infty = \infty$$

So, I need to fix it. We don't care about first rectangle as it's just a constant

so $\sum_{n=1}^{\infty} \frac{1}{n^p} \le 1 + \int_1^{\infty} \frac{1}{x^p} dx = 1 + \frac{x^{-p+1}}{-p+1} \Big|_1^{\infty}$

$= 1 + \frac{1}{p-1}$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $(p>1)$

Thm) $a_n = f(n)$ where $f$ is decreasing function, and assume $f \ge 0$. Then, $\sum_{n=1}^{\infty} f(n)$ is convergent iff $\int_1^{\infty} f(x)dx$ is finite.

Thm. Alternating Series Thm

Say $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$ and assume that $\lim a_n = 0$. Then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ convg.

ex) $a_n = \frac{1}{n}$.

Since $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is convg.

$\boxed{\text{Definition}}$

proof) Remember that $\sum_{n=1}^{\infty} b_n$ is convg, if $S_k = \sum_{n=1}^{k} b_n$ is convg.

$S_1 = -a_1$, $S_2 = -a_1 + a_2$, $S_3 = -a_1 + a_2 - a_3$, .....

$\boxed{a_3 \geq a_4 - a_5}$

For odd subsequence, as $S_1 = -a_1 \leq -a_1 + \underbrace{(a_2 - a_3)}_{\geq 0} = S_3$

and so on, $\underset{-a_1}{S_1} \leq \underset{-a_1+a_2-a_3}{S_3} \leq \underset{-a_1+a_2-a_3+a_4-a_5}{S_5} \leq \cdots (\leq S_2)$

For even subsequence, as $S_2 = -a_1 + a_2 \geq -a_1 + a_2 + \underbrace{(-a_3 + a_4)}_{\leq 0} = S_4$

and so on, $\underset{-a_1+a_2}{S_2} \geq \underset{-a_1+a_2-a_3+a_4}{S_4} \geq \underset{-a_1+a_2-a_3+a_4-a_5+a_6}{S_6} \geq \cdots (\geq S_1)$

And the subsequences $\lim_{k\to\infty} S_{2k+1} = c$ and $\lim_{k\to\infty} S_{2k} = d$, by monotonic sequence Thm.
(monotonic bunded seq is convg)

What is left is to show $c = d$. Notice that

$S_{2k+1} - S_{2k} = -a_1 + a_2 - \cdots - a_{2k+1} - (-a_1 + a_2 + \cdots + a_{2k})$
$= -a_{2k+1}$

Since $\lim a_n = 0$, we have $\lim_{k\to\infty} (S_{2k+1} - S_{2k}) = 0$

$$c - d = \lim S_{2k+1} - \lim 2k$$

Therefore $c = d$ and $\lim S_n = c = d$.

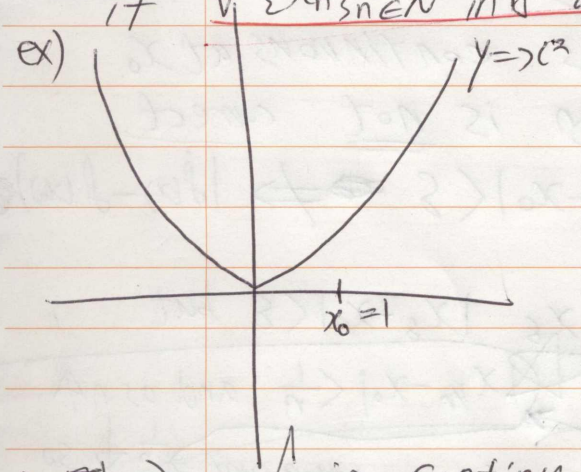
# Functions

$S \subset \mathbb{R}$. $f: \underbrace{S}_{\text{domain}} \to \mathbb{R}$

ex) $f(x) = \sqrt{4-x^2}$, $\Rightarrow$ domain: $-2 < x < 2$.
but I don't need to use whole domain. For example,
$g(x) = \sqrt{4-x^2}$, where $-1 \le x \le 1$.

Def) $f: S \to \mathbb{R}$
Let $x_0 \in S$. We say $f$ is continuous at $x_0$
if $\forall \{x_n\}_{n\in\mathbb{N}}$ in $S$ and $\lim x_n = x_0$, $\lim f(x_n) = f(x_0)$
(a) (b)

ex)




Why is $f(x)=x^2$ continuous at $x_0$?
Given $\{x_n\}_{n\in\mathbb{N}}$ s.t. $\lim x_n = x_0$,
then $\lim_{n\to\infty} f(x_n) = \lim x_n^2$
$= (\lim x_n)(\lim x_n) = x_0^2 = f(x_0)$

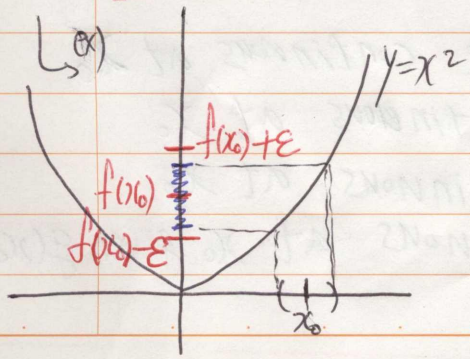
(d)

Thm) $f$ is continuous at $x_0$ iff $\forall \varepsilon > 0, \exists \delta > 0$
s.t. whenever $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$
for $x \in S$.
(c)

(b)

(c) - yin

ex)




YANGJI

pf) $\Leftarrow$ direction: Assume $\delta-\varepsilon$ formulation.

Given $\{x_n\}_{n\in\mathbb{N}}$ s.t. $\lim x_n = x_0$. ~~We need~~ to show that $\lim f(x_n) = f(x_0)$

Then, I will prove ⓓ is true

Take $\varepsilon>0$, by $\delta-\varepsilon$ formulation, $\exists\, \delta_{(\varepsilon)}>0$ s.t. $|x-x_0|<\delta$, $|f(x)-f(x_0)|<\varepsilon$

By $\lim x_n = x_0$, $\exists N_{(\delta)}$ s.t. $n>N$ $|x_n-x_0|<\delta$; hence $n>N$ implies $|f(x_n)-f(x_0)|<\varepsilon$

(3) We try to do: If ⓒ is true, given ⓐ, we try to prove ⓑ

This implies $\lim f(x_n) = f(x_0)$.

$\forall \varepsilon>0$, $\exists \delta>0$ s.t. $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$

$\Rightarrow$ direction: Assume $f$ is continuous at $x_0$.

Assume $\delta-\varepsilon$ formulation is <u>not</u> correct.

ex) $\exists\, \varepsilon>0$, $\forall \delta>0$, $|x-x_0|<\delta \not\Rightarrow |f(x)-f(x_0)|<\varepsilon$.

ex) $\exists\, \varepsilon>0$ s.t. $\forall \delta>0$ : $\exists\, x_\delta$ $|x_\delta - x_0|<\delta$ but $|f(x)-f(x_0)|\geq\varepsilon$.

$|x_{1/n}-x_0|<\frac{1}{n}$ and as $n\uparrow$ $\frac{1}{n}\downarrow$, so it conv g

for $\delta=\frac{1}{n}$. Notice $\{x_{1/n}\}_{n\in\mathbb{N}}$ is convergent to $x_0$, but $\{f(x_{1/n})\}$ doesn't conv g to $f(x_0)$.

This is contradiction to continuity of $f$ at $x_0$.

(12)

Thm) Suppose $f$ and $g$ are continuous at $x_0$

Then, a) $f+g$ is continuous at $x_0$

b) $fg$ is continuous at $x_0$

c) $f/g$ is continuous at $x_0$ given $g(x_0)\neq 0$.

Given $\{x_n\}_{n\in\mathbb{N}}$ $\lim x_n = x_0$

$$\lim (f+g)(x_n) = \lim(f(x_n)+g(x_n)) =$$

$$\lim f(x_n) + \lim g(x_n) = f(x_0)+g(x_0) = (f+g)(x_0)$$