03/08 Thurs

Thm) Suppose $f:(a,b)\to \mathbb{R}$ that has inverse f^{-1}. Assume that f is continuous. Given $x_0\in(a,b)$, suppose f is differentiable at x_0 and $f'(x_0)\neq 0$. then f^{-1} is differentiable at $y_0=f(x_0)$ with

$$(f^{-1})'(y_0)=\frac{1}{f'(x_0)}$$

pf) I need to show $\displaystyle\lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}=\frac{1}{f'(x_0)}$

Choose $\{y_n\}_{n\in\mathbb{N}}\xrightarrow{n\to\infty} y_0$. Need $\displaystyle\lim_{n\to\infty}\frac{f^{-1}(y_n)-f^{-1}(y_0)}{y_n-y_0}=\frac{1}{f'(x_0)}$

So, as $\begin{cases} f^{-1}(y_n) = x_n \\ f(x_n) = y_n \end{cases}$, $\lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)}$.

I can get

Since f is continuous, f^{-1} is also continuous,

thus, $f^{-1}(y_n) \xrightarrow{n\to\infty} f^{-1}(y_0)$

Since $\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0)$ and $f'(x_0) \neq 0$,

so, $\lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}$.

QED

ex) $f(x) = x^3 : R \to R$

$\Rightarrow$ So, $f^{-1}(x) = \sqrt[3]{x}$

f^{-1} fails to be differentiable at $x_0 = 0$,

because $f'(x_0) = 0$...

Thm) Suppose $f : (a, b) \to R$ that is differentiable and $\exists M > 0$ s.t. $|f'(x)| < M \ \forall x \in (a, b)$. Then, f is uniformly continuous in (a, b).

ex) $(a, b) = (-\infty, \infty)$ and $f(x) = x^2$.

$\Rightarrow$ Not uniformly continuous in $(-\infty, \infty)$

But, if a, b are finite, then f is uniformly continuous in (a, b).

pf) Given $\varepsilon>0$, we need to show $\exists\, \delta>0$ s.t. $|x-y|<\delta$ implies $|f(x)-f(y)|<\varepsilon$.

Since f is differentiable, M.V.T. implies $\exists\, c\in(a,b)$ between x and y s.t. $\frac{f(x)-f(y)}{x-y}=f'(c)$.

So, $|f(x)-f(y)| = |f'(c)|\,|(x-y)| < M\,|x-y|$

Take $\delta=\frac{\varepsilon}{M}$, then, $|f(x)-f(y)| < M\cdot\frac{\varepsilon}{M}=\varepsilon$ whenever $|x-y|<\delta$.

By def, f is uniformly continuous. ☆

ex) $f(x)=\sin x$. Is f uniformly continuous in $(-\infty,\infty)$?

$f'(x)=\cos x$, and $|f'(x)|=|\cos x|\le 1$.
So, by Thm, f is uniformly continuous.

☆ Thm) L'Hopital's Rule ☆ but when $\lim|g(x)|=\infty$, as $\lim\frac{f}{g}=L$, $\lim|f(x)|=\infty$ as well.

I want to compute $\lim_{x\to s}\frac{f(x)}{g(x)}$ (s can be $\pm\infty$).

Here, f and g are defined near s, but not necessarily defined at s.

Assume f and g are differentiable, g' is nonzero near s, so that $\lim_{x\to s}\frac{f'(x)}{g'(x)}=L$.

Also, $\lim_{x\to s}f(x)=\lim_{x\to s}g(x)=0$ or $\lim_{x\to s}|f(x)|=\lim_{x\to s}|g(x)|=\infty$.

Then, $\lim_{x\to s}\frac{f(x)}{g(x)}=L$ ☆ NO $\leftarrow$ necessary ex) $\lim_{x\to 0} \frac{\sin x}{x}$

$\Rightarrow$ Since $\lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \cos x = 1$

L'Hopital's rule says $\lim \frac{\sin x}{x} = 1$

ex) $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ $\frac{\infty}{\infty}$ form

So, $\lim_{x\to 0^+} \frac{(\ln x)'}{(\frac{1}{x})'} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$

ex) $\lim_{x\to\infty} \frac{e^x}{x^2}$ $\frac{\infty}{\infty}$ form

So, $\lim_{x\to\infty} \frac{(e^x)'}{(x^2)'} = \lim \frac{e^x}{2x}$ still $\frac{\infty}{\infty}$ form

So, $\lim_{x\to\infty} \frac{(e^x)'}{(2x)'} = \lim_{x\to 0} \frac{e^x}{2} = \infty$

so, set $S=0$, and as M.V.T. requires f&g continuous at end points, we say $\lim f = \lim g = f = g = 0$

☆ we don't prove L'hopital for every S, so we just think of one case ($\frac{0}{0}$ form)

(?) why need them?

pf of special case of L'Hopital)

Assume that f and g are continuous, so that $\lim_{x\to S} f(x) = \lim_{x\to S} g(S) = f(S) = g(S) = 0$

Lemma) (Generalized M.V.T.)

If f&g are differentiable in (a,b), then $\exists c \in (a,b)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

pf) $f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$

consider the function $h(x) = f(x)(g(b)-g(a)) - g(x)(f(b)-f(a))$. So, $h'(x) = f'(x)(g(b)-g(a)) - g'(x)(f(b)-f(a))$.

$\Longrightarrow$ c.t.

$\Rightarrow$ C.t. Notice that $h(a) = h(b)$

Rolle's Thm $\Rightarrow \exists c \in (a,b)$ s.t. $h'(c) = 0$

$$h(a) = f(a)(g(b)-g(a)) - g(a)(f(b)-f(a)) = h(b)$$

QED.

~~Continue~~ Go back to special case of L'hopital

$$\Rightarrow \lim_{x\to s} \frac{f(x)}{g(x)} = \lim_{x\to s} \frac{f(x)-f(s)}{g(x)-g(s)}.$$

$$\exists c \in (x,s) \quad \text{s.t.} \quad \frac{f'(c)}{g'(c)} = \frac{f(x)-f(s)}{g(x)-g(s)}$$

Since $x \to s$, we have $c \to s$ (as $x \le c \le s$).

Therefore $\lim_{x\to s} \frac{f(x)-f(s)}{g(x)-g(s)} = \lim_{c\to s} \frac{f'(c)}{g'(c)} = L.$

(as we originally assumed $L := \lim_{x\to s} \frac{f'(x)}{g'(x)}$)

(f & g are continuous on $[a,b]$, and differentiable on (a,b))

In our class, f is not integrable (Riemann)

ex) $f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ and $\int_0^1 f(x)\,dx$

More advanced class, f is Lebecgue integrable.

In continuous case, Riemman = Lebecgue.

Def) Given $[a,b]$, a ~~[illegible]~~ partition P,

$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$

Given a bounded function f, we define the following:

$$L(f,P) = \sum_{k=0}^{n-1} \inf_{[t_k, t_{k+1}]} f \cdot (t_{k+1} - t_k)$$

$= \inf\{f(x) \mid x \in [t_k, t_{k+1}]\}$.

$$U(f,P) = \sum_{k=0}^{n-1} \sup_{[t_k, t_{k+1}]} f \ (t_{k+1} - t_k)$$