02.08 Thurs

Given a bounded sequence $\{s_n\}_{n \in \mathbb{N}}$, define → so u_n & v_n can be real

$u_n = \inf\{s_k \mid k > n\}$ and $v_n = \sup\{s_k \mid k > n\}$

$$u_1 \leq u_2 \leq \cdots \leq \underset{\liminf u_n = \lim u_n}{\overset{\parallel}{u}} \leq \underset{\lim v_n = \limsup s_n}{\overset{\parallel}{v}} \leq \cdots \leq v_2 \leq v_1$$

Define $S = \{t \in \mathbb{R} \mid t$ is the limit of some subsequences of $\{s_n\}\}$

ex) the set of subsequential limits of $\{s_n\}$

Thm Given a bounded sequence, then

① $S \neq \emptyset$ (Bolzano-Weierstrass Thm)

② $\max S = \limsup s_n$

③ $\min S = \liminf s_n$

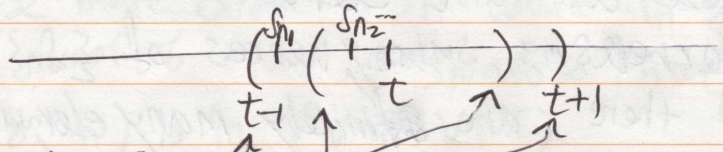
Lemma) $t \in S \iff \forall \varepsilon > 0$, $(t-\varepsilon, t+\varepsilon)$ contains infinitely many elements of $\{s_n\}$

In fact if $t \in S$, you can choose a subsequence $\{s_n\}$ that is monotonic and $\lim_{k\to\infty} s_{n_k} = t$.

pf) $\Longrightarrow$: If $\lim_{k\to\infty} s_{n_k} = t$, then $\forall \varepsilon > 0$ $(t-\varepsilon, t+\varepsilon)$ must contain s_{n_k} when $k > K$.

Therefore, $(t-\varepsilon, t+\varepsilon)$ contains infinitely many elements of $\{s_n\}$

$\Longleftarrow$:



when $\varepsilon = 1$

when ε s.t. $\varepsilon < |t - s_{n_1}|$

⋮

So as $n_1 < n_2 < n_3 < \cdots$,

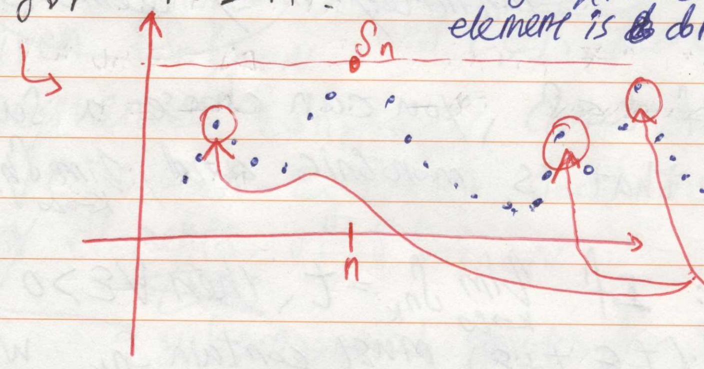
$s_{n_1}, s_{n_2}, s_{n_3}, \cdots \longrightarrow t$.

Keep on choosing the smaller interval, each time pick $n_{k+1} > n_k$, and you get $\{s_{n_k}\} \to t$.

pf for B-W Thm)

ⓐ $S \neq \emptyset$. If we can show that there is a monotonic subsequence $\{S_{n_k}\}$, then $\{S_{n_k}\}$ has to converge and therefore $S \neq \emptyset$.

we define S_n is dominant if $S_n \geq S_m$ for $m > n$. → ☆ decreasing sequence, then every element is dominant. ☆



There are two cases

⇒ 1) If there are infinitely many dominant elements. → property of elements

These dominant elements will form a decreasing subsequences of $\{S_n\}$

If there are finitely many elements, lets say starting from N, no S_n is dominant $n > N$.

(S_n is not dominant ⇒ $\exists\, m > n$ where $S_m > S_n$)

Start with S_{N+1}, since S_{N+1} is not dominant $\exists m$ where $m > N+1$ and $S_{N+1} < S_m$.

Keep on applying. We get an increasing subsequence

(b) First we want to show given $t \in S$

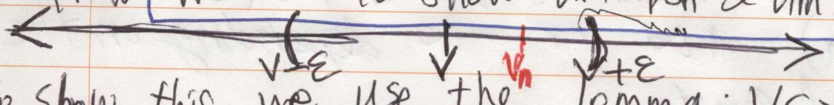
$\liminf s_n \le t \le \limsup s_n$

Let's say $t = \lim s_{n_k}$.

$\underbrace{u_{n_k - 1}}_{\to\, u = \liminf s_n} \le s_{n_k} \le \underbrace{v_{n_k - 1}}_{\to\, v = \limsup s_n} = \sup\{s_l \mid l > n_k - 1\}$

Since $\lim s_{n_k} = t$, we obtain $u \le t \le v$.

So, now $\boxed{\text{we need to show } \limsup s_n \text{ \& } \liminf s_n \in S}$.



To show this, we use the lemma: $\forall \varepsilon > 0$, $(v-\varepsilon, v+\varepsilon)$ has infinitely many elements in $\{s_n\}$.

Since by def, $v = \lim v_n$. So, $(v-\varepsilon, v+\varepsilon)$ contains all v_n, $n > N$.

If $v_n = \sup\{s_k \mid k > n\}$ is an element in $\{s_n\}$ then we move to v_{n+1}

These must exist $k > n$ where $v_n - \delta < s_k \le v_n$

Here we want to choose δ small so that $(v_n - \delta, v_n)$ is in $(v-\varepsilon, v+\varepsilon)$.

Keep on this iteration, we find an infinite # of elements of $\{s_n\}$ that are in $(v-\varepsilon, v+\varepsilon)$.

By lemma, $v \in S$. Since $v \ge t$ $t \in S$, v has to be $\max S$.

~~Def~~) $\lim s_n = +\infty$ if $\forall M > 0$, $\exists N$ s.t. $n > N$ implies $s_n > M$.

Define similarly for $\lim s_n = -\infty$.

Thm) Starting with $\{S_n\}$, $\{t_n\}$ and $\lim S_n = +\infty$

and $\{t_n\}$ is bounded below, then

$\lim (S_n + t_n) = +\infty$.

In particular, if $\lim t_n$ exist and $\neq -\infty$

then $\lim (S_n + t_n) = +\infty$.

Assume $\lim S_n = +\infty$. If $\lim t_n \neq 0$, then

$\lim (S_n t_n) = +\infty$ or $-\infty$ depending on the sign of $\lim t_n$.

when is

$\lim \sup S_n = +\infty$? => when sequence is not bounded above.

☆ But it cannot $\lim \sup S_n = -\infty$.

$\lim \inf S_n = -\infty$? → when seq is not bounde below.