02/01 Thurs

Def) A sequence $\{S_n\}_{n\in N}$ has $\lim_{n\to\infty} S_n = L$ if $\forall \varepsilon > 0, \exists N$ s.t $\forall n > N$ $|S_n - L| < \varepsilon$.

Thm) Suppose $\lim_{n\to\infty} S_n = S$. Assume that $S_n \neq 0$, $S \neq 0$. Then, the sequence $\left\{\frac{1}{S_n}\right\}_{n\in N}$ satisfies

$$\lim_{n\to\infty} \frac{1}{S_n} = \frac{1}{S}.$$

pf) $\varepsilon > 0$, $\left|\frac{1}{S_n} - \frac{1}{S}\right| = \left|\frac{S - S_n}{S S_n}\right| = \frac{|S - S_n|}{|S|\,|S_n|} < \varepsilon$,

arbitrary small

fixed

bounded

for $n > N$.

If you can show $\frac{1}{|S_n|} < M$, $\forall n \in N$

..... Last lecture.....

We proved ① $\{S_n\}$, $\{t_n\}$ where $\lim S_n = S$ and $\lim t_n = t$. Then, $\lim (S_n + t_n) = S + t$ and $\lim (S_n t_n) = S t$

② If $t_n \neq 0$, $t \neq 0$, then $\lim \frac{S_n}{t_n} = \frac{S}{t}$

☆ Squeeze Thm.

Suppose $a_n \le S_n \le b_n$, and assume that $\lim a_n = \lim b_n = L$. Then $\lim S_n = L$.

pf) $a_n - L \le S_n - L \le b_n - L$. If $a_n \ge L$, then $|S_n - L| \le |b_n - L|$. If $b_n < L$, then $|S_n - L| \le |a_n - L|$.

$\Longrightarrow$ c.e

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If $a_n < L < b_n$, then, $|S_n - L| \le \max\{|a_n - L|, |b_n - L|\}$

In any case, we have $|S_n - L| \le \max\{|a_n - L|, |b_n - L|\}$

Since $\lim a_n = L$, $\forall \varepsilon > 0$, $\exists N_1$, s.t. $|a_n - L| < \varepsilon$

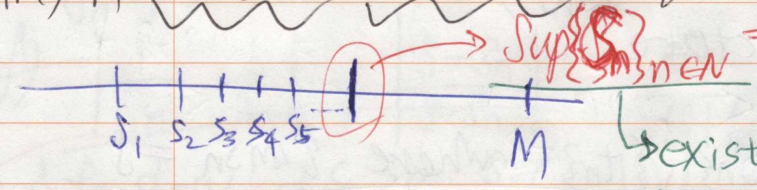
and $\lim b_n = L$, $\forall \varepsilon > 0$, $\exists N_2$, s.t. $|b_n - L| < \varepsilon$

If $N = \max\{N_1, N_2\}$, then $|S_n - L| < \varepsilon$. Therefore by def, $\lim S_n = L$.

☆ What if you don't know L, thus cannot prove by def... ??? Then, how to show $\lim S_n$ exist???

Monotone Sequence $\Rightarrow$ is either an increasing sequences or a decreasing sequence.

Thm) A bounded monotone sequence is converged.



exists because the set is bounded above by M + completeness axiom

pf) Assume that $\{S_n\}_{n\in N}$ is monotonically increasing

We claim that $\lim S_n = S$. (Define $S = \sup\{S_n \mid n \in N\}$)

Given $\varepsilon > 0$, $|S_n - S| = S - S_n < \varepsilon$, as $S_n \le S$.

So, $S_n > S - \varepsilon$. By def of sup,

$S - \varepsilon$ is not an upper bound of $\{S_n \mid n \in N\}$.

Hence, $\exists N$ s.t. $S_N > S - \varepsilon$

Since $\{S_n\}$ is ↗, $\forall n > N$ we have $S_n > S_N > S - \varepsilon$

So, $|S_n - S| < \varepsilon$. By def of limit, $\lim S_n = S$.

as $\underbrace{S - S_n}_{= |S_n - S|} < \varepsilon$.

ex) Given $0 < a < 1$. Show that $\lim a^n = 0$

pf) Let $S_n = a^n$. Notice that S_n is $\searrow$

because $S_{n+1} = a a^n = a \cdot S_n < S_n$ as $0 < a < 1$ and $S_n > 0$.

Also, the sequence $\{S_n\}$ is bounded below by 0.

(?) How to prove?

Thm implies that $\lim S_n$ exists

This is weaker statement than $\lim a^n = 0$

and as $0 < a < 1$, we can say

① $0^n < a^n < 1^n$, and use Squeeze Thm.

S_n is positive so, bounded below 0.

② ~~And as~~ Let $L = \lim S_n$, then $L = \lim S_{n+1}$.

By the identity $S_{n+1} = a \cdot S_n$, we muse have

$L = \lim S_{n+1} = \lim (a S_n) = a \lim S_n = aL$.

Since $a \neq 0$ or 1, $L = 0$.

ex) Given $\{S_n\}_{n \in \mathbb{N}}$. And, $S_1 = 5$, $S_{n+1} = \dfrac{S_n^2 + 5}{2 S_n}$

Show that $\lim S_n$ exists.

$\Rightarrow$ $S_1 = 5$, $S_2 = 3$, $S_3 = \frac{7}{3}$

we guess from the first three terms that $\{S_n\} \searrow$.

Notice that $S_n > 0$. Since $\{S_n\} \searrow$ and $S_n > 0$, $\lim S_n$ exists

wish $S_n \geq S_{n+1}$, or, $S_n \geq \dfrac{S_n^2 + 5}{2 S_n}$, or, $2 S_n^2 \geq S_n^2 + 5$, or, $S_n^2 \geq 5$,

or, $S_n \geq \sqrt{5}$. Now, I need to prove $S_n \geq \sqrt{5}$ to check.

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Let's see $s_{n+1} \geq \sqrt{5}$, or, $\frac{s_n^2+5}{2s_n} \geq \sqrt{5}$

and as $s_n > 0$, $s_n^2+5 \geq 2 \cdot s_n \sqrt{5}$, or,

$(s_n - \sqrt{5})^2 \geq 0$. So, now I know

$s_n \geq s_{n+1}$ if $s_n \geq \sqrt{5}$ (?) why...?

And $L = \lim s_n$ and $L = \lim s_{n+1} = \lim \frac{s_n^2+5}{2s_n}$

and $L \neq 0$, it will be $\frac{L^2+5}{2L}$. solve for L, I

get $L = \sqrt{5}$.

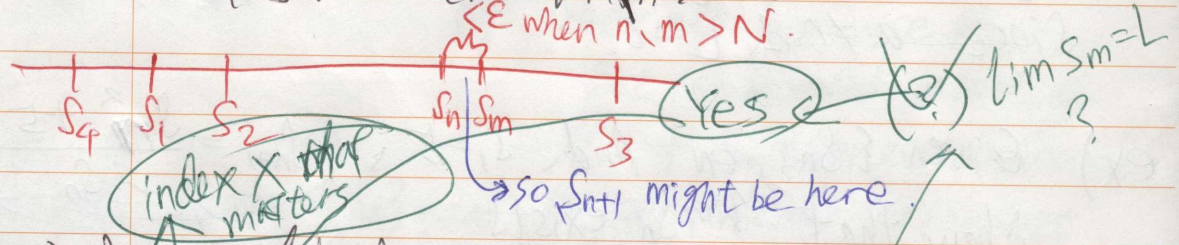
we have $s_n \geq s_{n+1} \Leftrightarrow s_n \geq \sqrt{5}$ and

I proved $s_{n+1} \geq \sqrt{5}$ (index X matters, so $s_n \geq \sqrt{5}$)

Q. How do we show $\lim s_n$ exist without knowing the value $\lim s_n$?

Def) A sequence $\{s_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall m, n > N$

$< \varepsilon$ when $n, m > N$.



Thm) Suppose $\lim s_n$ exists. Then $\{s_n\}$ is a cauchy sequence

pf) Let $L = \lim s_n$. Given $\varepsilon > 0$, $\exists N$ s.t. $n > N$ where $|s_n - L| < \frac{\varepsilon}{2}$. By Triangular inequality we have $|s_m - s_n| = |(s_m - L) + (L - s_n)| \leq |s_m - L| + |L - s_n|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. By def, $\{s_n\}_{n \in \mathbb{N}}$ is a cauchy sequence.

Thm) If $\{S_n\}_{n\in\mathbb{N}}$ is a cauchy-sequence, then $\{S_n\}$ is bounded.

pf) Let $\varepsilon=1$. Then, by def, $\exists N$ s.t. $\forall m, n > N$

$|S_m - S_n| < \varepsilon = 1$ $\quad$ <1 for $m,n>N$. $\quad$ S_m S_n

In particular, $\forall m > N$, $|S_m - S_{N+1}| < 1$.

Hence, $|S_m| = |S_m - S_{N+1} + S_{N+1}| \le |S_m - S_{N+1}| + |S_{N+1}|$

$< |S_{N+1}| + 1$

(?) why?

Hence, $|S_n| \le \max\{|S_1|, \cdots, |S_N|, |S_{N+1}|+1\}$.

for $\forall n$,

Take care of $|S_m|$, $m > N$.

So, $|S_n|$ is bounded.

as $N+1 = n$

we let