02/27 Tue

Def: $f: S \to R$ (i.e.) $S \subset R$, $S = [a,b]$, $S = (a,b)$, $(a,b]$, $[a,b)$, $R \setminus \{0\}$, or etc)

we say f is continuous at $x_0 \in S$ if $\forall \{x_n\}_{n \in N} \subset S$ s.t. $\lim_{n \to \infty} x_n = x_0$, then $\lim f(x_n) = f(x_0)$

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

f is not continuous at 0.

$\lim f(x_n) = 0 \neq f(x_0 = 0) = 1$ for $\forall \{x_n\}$ s.t. $\lim x_n = 0$.

<observation>

Although the def requires to check all sequences $\{x_n\}$ s.t. $\lim x_n = x_0$, in reality, it's sufficient to check for sequence $\{x_n\} \to x_0$ from the left and from the right.

☆ $(\delta - \varepsilon)$ formulation of continuity $\Rightarrow$

depends on ε and x_0

f is continuous at x_0 if $\forall \varepsilon > 0$, $\exists \delta$ s.t.

$|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

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Thm) Given f & g are continuous at x_0, then

a) $f+g$
b) $f-g$
c) $f \cdot g$
d) f/g

are also continuous at x_0 provided $g(x_0) \neq 0$.

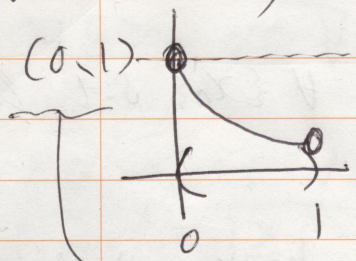
☆ Yeah! cuz f is not defined at 0.

Thm) Suppose $g \circ f$ is defined and f is continuous at x_0 and g is continuous at $f(x_0)$. Then $g \circ f$ is continuous at x_0.

(?) $f: [0,1] \to R$ where $f(x) = \frac{1}{x}$ not work as f is not contin ...?

Thm) Suppose $f: [a,b] \to R$ is continuous. Then
① f is bounded (i.e. $\exists M$ s.t. $|f(x)| < M$ for $\forall x \in [a,b]$) and ② f has max and min.

ex)



$\Rightarrow f$ has no maximum in $(0,1)$
(f is not defined at 1)

pf) Assume f is not bounded. i.e.) $\exists \{x_n\} \subset [a,b]$ s.t. $f(x_n) \to +\infty$ (or $-\infty$).
Bolzano-Weierstrass Thm says: Since $\{x_n\}_{n \in N}$ is bounded, $\exists$ subsequence $\{x_{n_k}\}_{k \in N}$ s.t. $\lim_{k \to \infty} x_{n_k} = x_0$
The closedness of $[a,b]$ requires $x_0 \in [a,b]$.
Now, $\underbrace{\lim_{k \to \infty} f(x_{n_k})}_{= +\infty} = f(x_0)$, but $f(x_0) \neq +\infty$.

Contradiction

(?) It doesn't have to be...?
By the Thm, when limit exists $\exists$ subsequence has the same lim

Now, let's prove whether f has max & min.

Since f is bounded, $\sup f = \sup\{f(x) \mid x \in [a,b]\}$ exists. If $\exists x_0 \in [a,b]$ s.t. $f(x_0) = \sup f$, then x_0 is the maximal point. Otherwise, $\exists \{x_n\} \in [a,b]$ s.t. $\lim f(x_n) = \sup f$. Now, $\underbrace{\lim f(x_{n_k})}_{\sup f} = f(x_0)$

Hence f must have maximum.

☆ <Intermediate Value Thm>

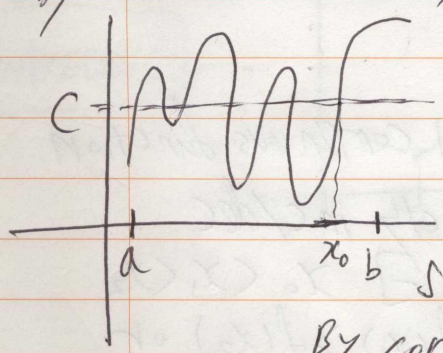
Given $f: [a,b] \to \mathbb{R}$ s.t. f is continuous. Let c be a value between $f(a)$ and $f(b)$. Then $\exists x_0 \in [a,b]$ s.t. $f(x_0) = c$.

pf) Assume $f(a) < f(b)$

$S = \{x \in [a,b] \mid f(x) < c\}$. First of all, $S \neq \phi$ as $a \in S$. S is bounded, meaning $\sup S$ is defined. Call $x_0 = \sup S$. Notice $x_0 < b$.

Indeed $f(x_0) < f(b)$. $\exists \{x_n\} \subset S$ s.t. $\lim x_n = x_0$, by continuity of f, $\lim f(x_n) = f(x_0) \le c < f(b)$

$\uparrow c$



I just proved that $f(x_0) \le c$.

Now, to show $f(x_0) \ge c$, pick a sequence in (x_0, b) s.t. $\lim u_n = x_0$

By continuity of f: $\lim f(x_n) = f(x_0)$

$\ge c$

since $(x_0, b) \cap S = \phi$. Hence $f(x_0) \ge c$.

Therefore, $f(x_0) = c$.

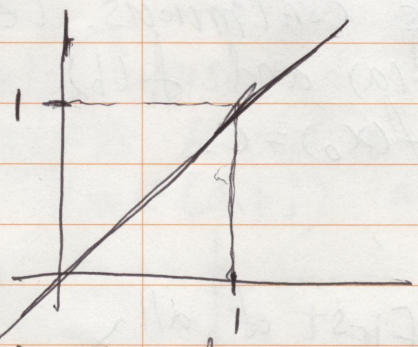
ex) Show that $f(x) = x^3 - 2x - 5$ has a zero between $(-2, 3)$

pf) f is continuous in $\mathbb{R}$.

$f(-2) = -9$ & $f(3) = 16$

Intermediate Thm implies $\exists\, x_0$ between -2 and 3 s.t. $f(x_0) = 0$

ex) Suppose $f: [0,1] \to [0,1]$ is continuous. Show that $\exists\, x_0 \in [0,1]$ s.t. $f(x_0) = x_0$



pf) Consider $g(x) = f(x) - x$.

$g(0) = f(0) - 0 = f(0) \geq 0$

$g(1) = f(1) - 1 \leq 0$. So, by Intermediate value thm

$\exists\, x_0 \in [0,1]$ s.t. $g(x_0) = 0$

(ex) $f(x_0) = x_0$)

Thm) Suppose f is a 1-1 (injective) continuous function on $[a,b]$, then f is either stictly inc/dec.

pf) Assume otherwise. Then $\exists\, x_0 < x_1 < x_2$ s.t. either $f(x_0) < f(x_1)$, $f(x_1) > f(x_2)$ or

① $f(x_0) > f(x_1)$, $f(x_1) < f(x_2)$.

(For ①, by intermediate value Thm, $\exists$ $x_0 \le u \le x_1$ s.t. $f(u)=C$ and $x_1 \le v \le x_2$ s.t. $f(v)=C$.)

Applying I.V.T. to f on $[x_0, x_1]$ and $[x_1, x_2]$ given C between $f(x_0)$ and $f(x_1)$, between $f(x_2)$ and $f(x_1)$.

$\exists$ $u \in [x_0, x_1]$ and $v \in [x_1, x_2]$ s.t. $f(u)=f(v)=C$

So, contradiction of 1-1.

Same for ②.

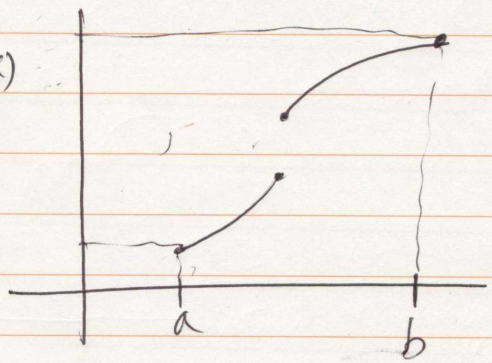
<I.V.T> Another version.

Thm) If f is strictly increasing and continuous $f:[a,b] \to R$. Then, the image of f is $[f(a), f(b)]$

Thm) If f is strictly inc $f:[a,b] \to R$ and $\text{Img } f = [c, d]$. Then f is continuous.

$c = f(a)$, $d = f(b)$

Range f

$\{f(x) \mid x \in [a,b]\}$

ex)