01/25 Thurs

$S \subset \mathbb{R}$

def) $\sup S$ is the least upper bound of S

$\{m \in \mathbb{R} \mid m \geq s \ \forall s \in S\}$

$\inf S$ is the greatest lower bound of S

completeness axiom $\Rightarrow$ $\mathbb{R}$ satisfies $\forall S \subset \mathbb{R}$ s.t.

S is bounded above, then $\sup S$ exists.

ex) $S = (a, b)$

$\sup S = b$, $\inf S = a$

ex) $S = \{r \in \mathbb{Q} \mid r^2 < 2\} \subset \mathbb{R}$

$\sup S = \sqrt{2}$, $\inf S = -\sqrt{2}$

No max & min

$\mathbb{Q}$ does not satisfy completeness axiom because $S \subset \mathbb{Q}$ but $\sup S$ is not in $\mathbb{Q}$.

Thm) If $a < b$ real, then $\exists q \in \mathbb{Q}$ s.t. $a < q < b$.

pf) Assume $0 < a < b$. We need to find $a < \frac{m}{n} < b$ such that $m, n \in \mathbb{N}$.

so, $na < m < nb$. $\Longrightarrow$

Archemedian property of $\mathbb{R}$ helps us find $n \in \mathbb{N}$ s.t. $n(b-a) > 1$, $\rightarrow$ $nb > na + 1 \geq \sup S + 1$

we want to show $\exists\, m \in \mathbb{N}$ s.t. $na < m < nb$.

Let $S = \{k \in \mathbb{Z} \mid k \leq na\}$, so completeness axiom tells you $\exists\, \sup S \in \mathbb{R}$

Since $S \subset \mathbb{Z}$, $\sup S$ is also the maximum element in S, and hence is an integer.

Define $m = \sup S + 1$

By def of $\sup S$, $m \notin S$ and hence $m > na$

Also, $m \leq na + 1 < nb$

Therefore, we just found $m, n \in \mathbb{N}$ s.t. $a < \frac{m}{n} < b$.

properties

a) Suppose S is bounded above.

$-S = \{x \in \mathbb{R} \mid x = -s \text{ for some } s \in S\}$

Then $\sup S = -(\inf(-S))$

b) Suppose A and B are bounded above. Then, if $A+B = \{x \in \mathbb{R} \mid x = a+b, \text{ for some } a \in A, b \in B\}$

$\sup(A+B) = \sup A + \sup B$.

pf) need to show $\sup A + \sup B$ is sup of $A+B$

- $\sup A + \sup B \geq x$, $\forall x \in A+B$
- $\forall \varepsilon > 0$, $\sup A + \sup B - \varepsilon < x$ for some $x \in A+B$

① $\sup A + \sup B \geq x$, $\forall x \in A+B$

write $x = a+b$ and $a \leq \sup A$ & $b \leq \sup B$,

we know $x \leq \sup A + \sup B$

② By def of sup, $\exists a \in A$ s.t. $a > \sup A - \frac{\varepsilon}{2}$

and $b \in B$ s.t. $b > \sup B - \frac{\varepsilon}{2}$

and $x = a + b > \sup A + \sup B - \varepsilon$

c) Suppose A and B are bounded below. Then if $A+B = \{ // \}$, then $\inf(A+B) = \inf A + \inf B$.

☆ Terminology ⇒ If S is not bounded above, $\sup S = +\infty$.

If S is not bounded below, $\inf S = -\infty$.

$\{S_n\}_{n \in a} = \{S_1, S_2, \cdots\}$ is a map from $N \to R$.

ex) $S_n = \frac{1}{n^2}$

ex) $S_n = (-1)^n$

formula for Fibonacci

$S_1 = C_1 \phi_1^1 + C_2 \phi_2^1$ → C_1 & C_2

$S_2 = C_1 \phi_1^2 + C_2 \phi_2^2$ → C_1 & C_2

$S_3 = C_1 \phi_1^3 + \phi_2^3$

ex)** S_n can be written in terms of $S_1, S_2, \cdots S_{n-1}$

(Recursion)

↳ ex) Fibonacci #: $S_1 = 1$, $S_2 = 2$, $S_n = S_{n-1} + S_{n-2}$ for $n \geq 3$

$S_n = C_1 \phi_1^n + C_2 \phi_2^n$ where ϕ_1 and ϕ_2 are roots of $x^2 - x - 1 = 0$.

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Def) $\{S_n\}_{n\in N}$ is converged to $S\in R$ if $\forall \varepsilon>0$, $\exists N$ s.t. $|S_n-S|<\varepsilon$ whenever $n>N$ (N depends on ε) we write $\lim_{n\to\infty} S_n = S$.

ex) $S_n = \frac{1}{n^2}$

$\lim_{n\to\infty} S_n = 0$

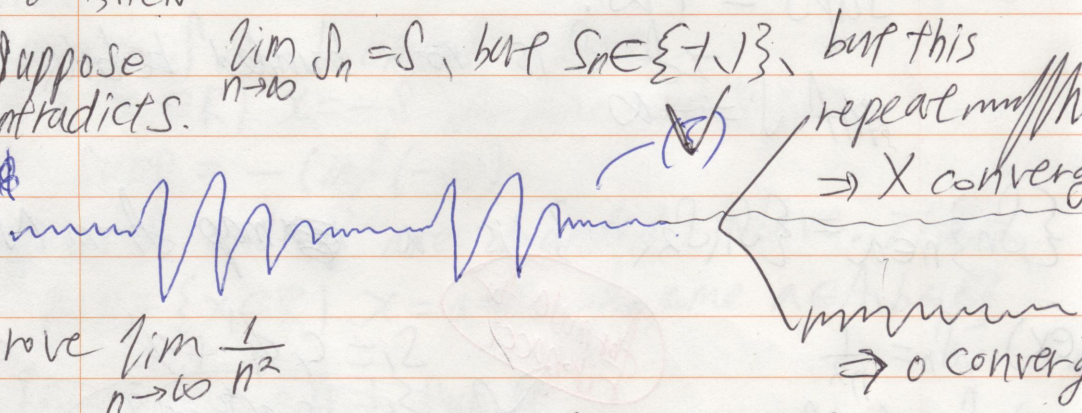
ex) $S_n = \frac{3n+1}{7n-4}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(3+\frac{1}{n})}{n(7-\frac{4}{n})} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{7-\frac{4}{n}} = \frac{3+0}{7-0} = \frac{3}{7}$

ex) $S_n = (-1)^n$

$\{S_n\}_{n\in N}$ X converge.

pf) Suppose $\lim_{n\to\infty} S_n = S$, but $S_n \in \{-1,1\}$, but this contradicts.

☆ex) 

ex) prove $\lim_{n\to\infty} \frac{1}{n^2}$

proof) given $\varepsilon>0$, we need to determine N s.t.

$n>N$ s.t. $\left|\frac{1}{n^2}-0\right|<\varepsilon$

so, $\frac{1}{n^2}<\varepsilon$, and $\frac{1}{\varepsilon}<n^2$, and $\sqrt{\frac{1}{\varepsilon}}<n$.

(so as $\varepsilon\downarrow$, $N\uparrow$)

choose $N\in N$ s.t. $N>\frac{1}{\sqrt{\varepsilon}}$, then $\forall n>N$ $\lim_{n\to\infty}\frac{1}{n^2}=0$

ex) $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$

pf) Given $\varepsilon > 0$, we need to determine N s.t. $\forall n > N$.

$$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \varepsilon$$

And, $\left|\frac{7(3n+1)-3(7n-4)}{7(7n-4)}\right| = \left|\frac{19}{7(7n-4)}\right| < \varepsilon$

So, $n > \frac{1}{7}\left(\frac{19}{7\varepsilon} + 4\right)$.

choose $N \in \mathbb{N}$ and then $\left|S_n - \frac{3}{7}\right| < \varepsilon$ for $n > N$.