Homework1

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1. Exponential Density and Survival-related Functions

(a) Let $\hat{\lambda}$ be the maximum likelihood estimator of the parameter λ.
For relapse time:
$\hat{\lambda} = \frac{6}{5+8+12+24+32+17+16+17+19+30} \approx 0.033$
This indicates that the rate of relapse is about 3.33% per month.
For relapse time:
$\hat{\lambda} = \frac{3}{10+12+15+33+45+28+16+17+19+30} \approx 0.013$
This indicates that the rate of death is about 1.33% per month.

(b)

i. Mean is $\int_0^\infty t\lambda e^{-\lambda t}dt = \frac{1}{\lambda}$ and I will use $\hat{\lambda}$ to derive the following values.
Mean time to relapse:
$\frac{1}{0.033} \approx 30.303$ months Mean survival time:
$\frac{1}{0.013} \approx 76.923$ months

ii. Median is $0.5 = e^{-\lambda\tau} \Rightarrow \tau = \frac{-log(0.5)}{\lambda}$. By $\hat{\lambda}$,
Median time to relapse:
$\frac{-log(0.5)}{0.033} \approx 21.004$ months Median survival time:
$\frac{-log(0.5)}{0.013} \approx 53.319$ months

iii. The survival function of exponential distribution is $S(t) = e^{-\lambda t}$. For relapse:
$S_R(12) = e^{-0.033*12} = 0.67$
$S_R(24) = e^{-0.033*24} = 0.449$
For death:
$S_D(12) = e^{-0.013*12} = 0.852$
$S_D(24) = e^{-0.013*24} = 0.726$

iv. The cumulative probabilities can be calculated as: $F(t) = \int_0^t \lambda(u)du = \lambda t$
For relapse:
$F_R(12) = 0.033 \times 12 = 0.4$
$F_R(24) = 0.033 \times 24$ 0.8
For death:
$F_D(12) = 0.013 \times 12 = 0.16$
$F_D(24) = 0.013 \times 24 = 0.32$

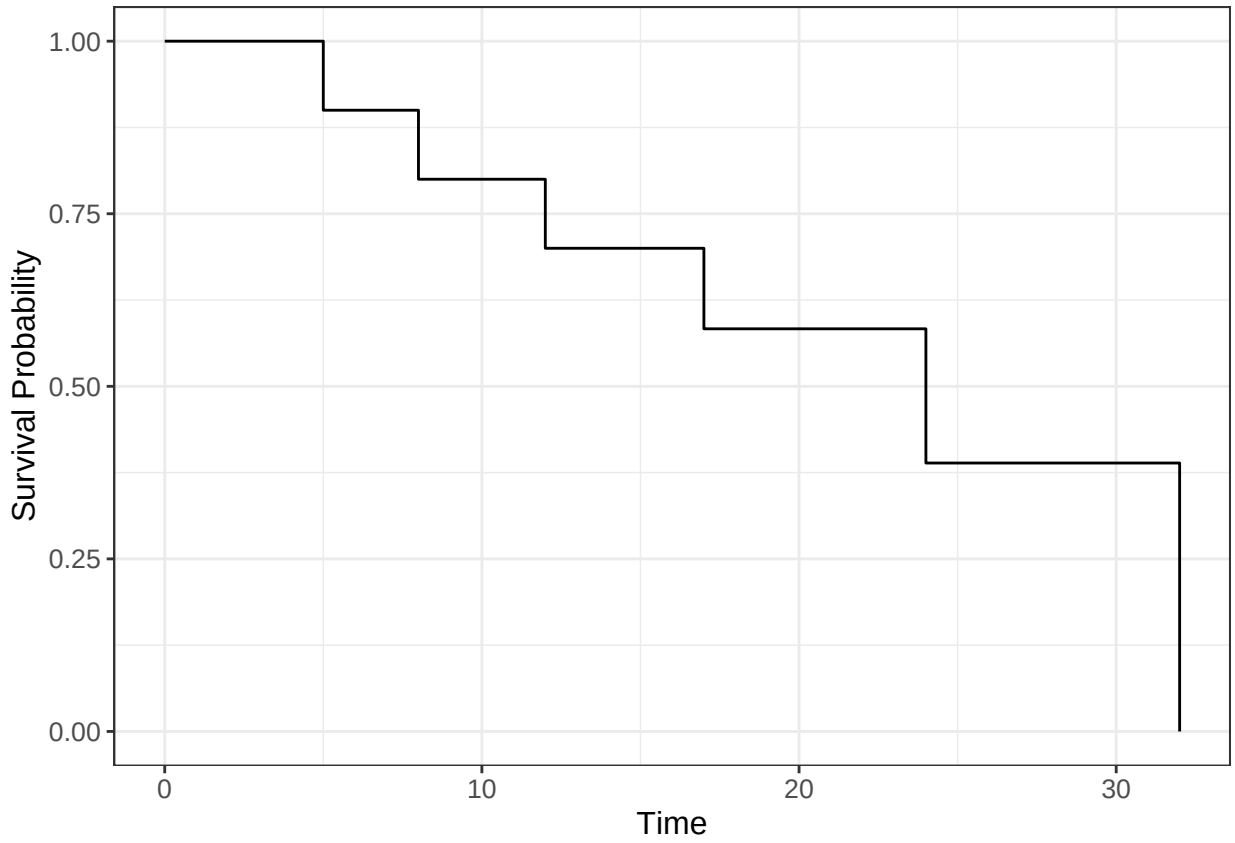
v. The conditional probability can be expressed as $P(T > 24|T > 12) = \frac{S_R(24)}{S_R(12)} = 0.67$
It is the same as what we observed in (iii) $S_R(12)$. This means that the conditional probability of being relapse-free after 2 years given that one has remained relapse-free for at least 1 year simplifies to the survival function for the remaining time period (memoryless property of the exponential distribution).

(c) We can use Kaplan-Meier estimator to estimate median time to relapse. I will calculate them using R.

```r
library(survival)
library(ggsurvfit)

# set up data
df1 <- data.frame(
  relapse_time = c(5, 8, 12, 24, 32, 17, 16, 17, 19, 30),
  relapse_censored = c(1, 1, 1, 1, 1, 1, 0, 0, 0, 0) # 1: event, 0: censored
)

# fit KM curve
relapse_surv <- Surv(df1$relapse_time, df1$relapse_censored)
relapse_km <- survfit(relapse_surv ~ 1)
relapse_km |>
  ggsurvfit()
```



This tells us that the median time τ s.t. $\hat{S}_R(\tau) \leq 0.50$ is 24 months.
Meanwhile, we cannot estimate median time to death because the survival probability does not go below 0.50.

2. Kaplan-Meier Survival Estimate

(a) I will make a table with a row for every death or censoring time.
t_j: distinct death or censoring times
d_j: the number of death at t_j

r_j: the number of individuals at risk right before the j-th death time
c_j: the number of censored observations between the j-th and $(j+1)$-st death time

t_j	d_j	c_j	r_j	$1-(d_j/r_j)$	$\hat{S}(t_j)$
2	1	0	17	0.941	0.941
3	1	0	16	0.938	0.882
4	1	0	15	0.933	0.824
12	1	0	14	0.929	0.765
22	1	0	13	0.923	0.706
48	1	0	12	0.917	0.647
51	0	1	11	1	0.647
56	0	1	10	1	0.647
80	1	0	9	0.889	0.575
85	1	0	8	0.875	0.503
90	1	0	7	0.857	0.431
94	0	1	6	1	0.431
160	1	0	5	0.8	0.345
171	1	0	4	0.75	0.259
180	1	1	3	0.667	0.173
238	1	0	1	0	0

(b) I will use R to calculate $\hat{S}(t)$ and their pointwise 95% confidence intervals using the "log-log" approach ad the linear approach.

```r
# set up data
df2 <- data.frame(
  t = c(2,3,4,12,22,48,51,56,80,85,90,94,160,171,180,180,238),
  c = c(1,1,1,1,1,1,0,0,1,1,1,0,1,1,1,0,1) # 1: event, 0: censored
)

# fit KM curve
surv <- Surv(df2$t, df2$c)
km <- survfit(surv ~ 1)

# log-log approach to obtain 95%CI
l_loglog <- km.ci(km, method = "loglog")$lower
u_loglog <- km.ci(km, method = "loglog")$upper

# linear approach to obtain 95%CI
l_linear <- km.ci(km, method = "linear")$lower
u_linear <- km.ci(km, method = "linear")$upper

# Create a table
tb <- data.frame(
  t = round(km$time,3), # time
  st = round(km$surv,3), # survival
  l_loglog = round(l_loglog, 3),
  u_loglog = round(u_loglog, 3),
  l_linear = round(l_linear, 3),
  u_linear = round(u_linear, 3)
)
kable(tb, col.names = c("t", "S(t)", "Lower 95%CI (log-log)", "Upper 95%CI (log-log)", "Lower 95%CI (li
```

t	S(t)	Lower 95%CI (log-log)	Upper 95%CI (log-log)	Lower 95%CI (linear)	Upper 95%CI (linear)
2	0.941	0.650	0.991	0.829	1.053
3	0.882	0.606	0.969	0.729	1.036
4	0.824	0.547	0.939	0.642	1.005
12	0.765	0.488	0.904	0.563	0.966
22	0.706	0.431	0.866	0.489	0.922
48	0.647	0.377	0.823	0.420	0.874
51	0.647	0.377	0.823	0.420	0.874
56	0.647	0.377	0.823	0.420	0.874
80	0.575	0.307	0.772	0.333	0.817
85	0.503	0.244	0.716	0.254	0.752
90	0.431	0.187	0.656	0.181	0.682
94	0.431	0.187	0.656	0.181	0.682
160	0.345	0.122	0.584	0.094	0.596
171	0.259	0.069	0.505	0.020	0.497
180	0.173	0.030	0.416	-0.038	0.383
238	0.000	NaN	NaN	NaN	NaN

Log-log approach returns CIs within $[0, 1]$, but linear approach returns some values outside $[0, 1]$.

(c)