Assignment3

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Problem 1

$X \sim N(\mu, \Sigma)$ where $X = (X_1, ...X_p)$
The conditional distribution $X_i, X_j | X_s = x_s$ is still a Gaussian distribution with mean and covariance given by: $X_i, X_j | X_s = x_s \sim N(\mu_{ij|S}, \Sigma_{ij|S})$

$\left(\Sigma_{\{ijS\},\{ijS\}}\right)^{-1} = \begin{pmatrix} K_{\{i,j\},\{i,j\}} & K_{\{i,j\},S} \\ K_{S,\{i,j\}} & K_S \end{pmatrix}$

When $X_i \perp\!\!\!\perp X_j | X_s$ is true, $K_{ij|S} = 0$, which makes $\left(\Sigma_{\{ijS\},\{ijS\}}\right)^{-1}$ diagonal. Therefore, $X_i \perp\!\!\!\perp X_j | X_s \Rightarrow (\Sigma_{\{ijS\},\{ijS\}})^{-1}_{ij} = 0$ holds.
When $(\Sigma_{\{ijS\},\{ijS\}})^{-1}_{ij} = 0$ is true, $K_{ij|S} = (\Sigma_{ij|S})^{-1} = 0$. $(\Sigma_{ij|S})^{-1}$ is the precision matrix for the conditional distribution $X_i, X_j | X_s = x_s$, so X_i and X_j are independent given X_s.
Therefore, $X_i \perp\!\!\!\perp X_j | X_s \Leftrightarrow (\Sigma_{\{ijS\},\{ijS\}})^{-1}_{ij} = 0$.

Problem 2

```
# simulate data from a given MRF independence model
set.seed(123)
K <- cbind(c(10,7,7,0),c(7,20,0,7),c(7,0,30,7),c(0,7,7,40))
data <- as.data.frame(mvrnorm(n=10000,mu=c(0,0,0,0),Sigma=solve(K)))
colnames(data) <- c("X1","X2","X3","X4")
K
```

```
##      [,1] [,2] [,3] [,4]
## [1,]   10    7    7    0
## [2,]    7   20    0    7
## [3,]    7    0   30    7
## [4,]    0    7    7   40
```

In the precision matrix, $K_{ij} = 0$ implies that variable X_i and X_j are conditionally independent given all other variables. Given the precision matrix K, $K_{14} = K_{41} = 0$ and $K_{23} = K_{32} = 0$. Therefore the following conditional independencies are represented by K:
$X_1 \perp\!\!\!\perp X_4 | X \backslash \{X_1, X_4\}$
$X_2 \perp\!\!\!\perp X_3 | X \backslash \{X_2, X_3\}$ The corresponding graph is an undirected graph that has no edges between X_1 and

X_4, and X_2 and X_3. All other pairs of variables are connected by edges.
Now, I will verify the conditional independence constraints by using linear regression.
$X_1 \perp\!\!\!\perp X_4 | X \backslash \{X_1, X_4\}$:

```
# conditional independence of X1 and X4 given X2, X3
m14 = lm(X1 ~ X2 + X3 + X4, data = data)
summary(m14)
```

```
##
## Call:
## lm(formula = X1 ~ X2 + X3 + X4, data = data)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.36729 -0.21127  0.00304  0.21389  1.20994
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.001934   0.003141   0.616    0.538
## X2          -0.682729   0.012203 -55.950   <2e-16 ***
## X3          -0.695282   0.015540 -44.741   <2e-16 ***
## X4           0.007927   0.020037   0.396    0.692
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.3141 on 9996 degrees of freedom
## Multiple R-squared:  0.4564, Adjusted R-squared:  0.4563
## F-statistic:  2798 on 3 and 9996 DF,  p-value: < 2.2e-16
```

In this linear model, the coefficient of $X4$ turned out to be non-significant with p-value <0.05.

$X_2 \perp\!\!\!\perp X_3 | X \backslash \{X_2, X_3\}$:

```
# conditional independence of X2 and X3 given X1, X4
m23 = lm(X2 ~ X1 + X3 + X4, data = data)
summary(m23)
```

```
##
## Call:
## lm(formula = X2 ~ X1 + X3 + X4, data = data)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.90282 -0.15318  0.00188  0.15342  0.85952
##
## Coefficients:
```

```
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.001141   0.002247   0.508    0.612
## X1          -0.349303   0.006243 -55.950   <2e-16 ***
## X3           0.012316   0.012177   1.011    0.312
## X4          -0.352810   0.013891 -25.398   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.2246 on 9996 degrees of freedom
## Multiple R-squared:  0.3841, Adjusted R-squared:  0.3839
## F-statistic:  2078 on 3 and 9996 DF,  p-value: < 2.2e-16
```

In this linear model, the coefficient of $X3$ turned out to be non-significant with p-value <0.05.

Therefore, the conditional independencies are verified.

The list of edges are $X_1 - X_2$, $X_1 - X_3$, $X_2 - X_4$, $X_3 - X_4$.

```r
# fit the model (estimate the precision matrix subject to the graph constraints)
glist <- list(
  c("X1", "X2"),
  c("X1", "X3"),
  c("X2", "X4"),
  c("X3", "X4")
)
ddd <- cov.wt(data, method="ML")
fit <- ggmfit(ddd$cov, ddd$n.obs, glist) # Estimate parameters using IPF
fit$K # estimated precision matrix
```

```
##           X1        X2        X3        X4
## X1 10.182411  6.988142  7.140856  0.000000
## X2  6.988142 19.832337  0.000000  7.076402
## X3  7.140856  0.000000 29.394792  6.852069
## X4  0.000000  7.076402  6.852069 40.745105
```

It appears that the model fitting worked because we can see that the estimated precision matrix has $K_{14} = K_{41} = 0$ and $K_{23} = K_{32} = 0$, and everything else non-zero, indicating that the above conditional independencies hold.

Problem 3

```r
# Gaussian Bayesian Network model
# covariance matrix
set.seed(123)
```

```r
Sig <- cbind(c(3,-1.4,0,0),c(-1.4,3,1.4,1.4),c(0,1.4,3,0),c(0,1.4,0,3))
data <- as.data.frame(mvrnorm(n=10000,mu=c(0,0,0,0),Sigma=Sig))
colnames(data) <- c("X1","X2","X3","X4")
```

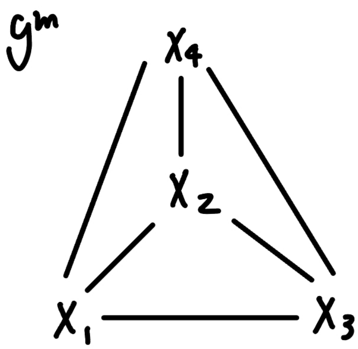
DAG $\mathcal{G}$: $X_1 \to X_2 \leftarrow X_3$ and $X_4 \to X_2$

(a) Given the model, correlation constraints are:
$X_1 \perp\!\!\!\perp X_3 | X_2$
$X_1 \perp\!\!\!\perp X_4 | X_2$
$X_3 \perp\!\!\!\perp X_4 | X_2$

```r
# estimate the correlation
cor(data)
```

```
##             X1         X2          X3           X4
## X1  1.00000000 -0.4661188 0.012187930 -0.011504563
## X2 -0.46611880  1.0000000 0.463034923  0.473314198
## X3  0.01218793  0.4630349 1.000000000  0.006392376
## X4 -0.01150456  0.4733142 0.006392376  1.000000000
```

We can see that the correlation between X_1 and X_3, X_1 and X_4, X_3 and X_4 are very close to zero.



(b)

The precision matrix K for $\mathcal{G}^m$ is 4 by 4 matrix with non-zero off-diagonal elements because all the pairs in the moralized graph are connected by edges. This indicates that all the pairs are conditionally dependent given the rest of other variables, meaning that they have non-zero partial correlations. Even though the covariance matrix Σ shows zero direct covariances between some pairs of variables, the precision matrix K reflects the conditional dependencies imposed by the moralized graph $\mathcal{G}^m$ where all pairs of variables are connected.

(c)

```r
# estimate precision matrix K
estK = ggmfit(cov(data), 10000, glist = glist)$K

# output estimated K
estK
```

```
##                X1          X2           X3           X4
## X1  0.427062054  0.2001158 -0.004798362  0.000000000
## X2  0.200115797  0.5291187  0.000000000 -0.202113521
## X3 -0.004798362  0.0000000  0.337140362 -0.003159533
## X4  0.000000000 -0.2021135 -0.003159533  0.418804400
```

```r
# take the inverse of K
solve(estK)
```

```
##               X1           X2           X3          X4
## X1  2.99202134 -1.387219505  0.036312655 -0.66919328
## X2 -1.38721950  2.960277610 -0.006355695  1.42857155
## X3  0.03631266 -0.006355695  2.966821484  0.01931498
## X4 -0.66919328  1.428571551  0.019314981  3.07731880
```

```r
# output the true covariance matrix
Sig
```

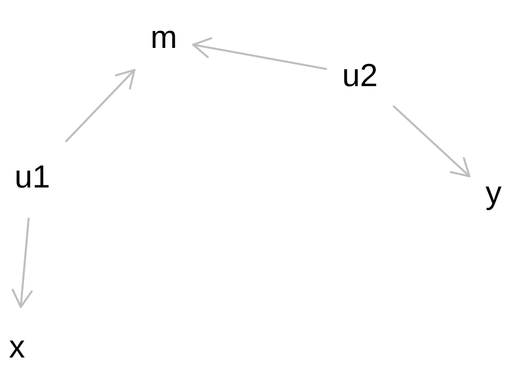
```
##      [,1] [,2] [,3] [,4]
## [1,]  3.0 -1.4  0.0  0.0
## [2,] -1.4  3.0  1.4  1.4
## [3,]  0.0  1.4  3.0  0.0
## [4,]  0.0  1.4  0.0  3.0
```

The estimated covariance matrix is mostly similar to true covariance matrix but differ slightly on X_2 and X_3, and X_1 and X_4.

Problem 4

```r
library(dagitty)

# simulate 10000 observations from the following graph
g <- dagitty( "dag{ x <- u1; u1 -> m <- u2 ; u2 -> y }" )
plot(g)
```



Problem 5