1 The construction of the densest double lattice packing

1.1 Double lattices

Definition 1.1. *A chord of a convex body K is a line segment whose endpoints lie on the boundary of K. A chord is an affine diameter if there is no longer chord parallel to it.*

Definition 1.2. *An inscribed parallelogram is a half-length parallelogram in the direction θ if one pair of edges is parallel with the line through the origin at an angle θ above the x-axis and their length is half the length of an affine diameter parallel to them.*

Note that any two half-length parallelograms in the direction θ have equal area, and we can define that area as a function $A(\theta)$ of the direction.

Definition 1.3. *A cocompact discrete subgroup of the Euclidean group consisting of translations and point reflections is a double lattice if it includes at least one point reflection.*

A double lattice is generated by a lattice and a point reflection, or alternatively by three point reflections.

Theorem 1.1 (Kuperberg and Kuperberg)**.** *For a convex K, an admissible double lattice of smallest mean area has mean area $4 \min_\theta A(\theta)$ and is generated by reflection about the vertices of a half-length parallelogram.*

Note that Kuperberg and Kuperberg make use of extensive parallelograms, inscribed parallelograms with edge length greater than half the affine diameter in their edge directions. Mount gives an explicit proof of the fact that it suffices to consider only the half-length parallelograms associated with affine diameters as a function of θ.

For our purposes, it is illustrative to consider the algorithm used find the half-length parallelogram of minimal area not only as a method of finding the best double lattice packing, but also as a way to explore the configuration space of double lattices. More specifically, the behavior of the affine diameter is fairly well behaved but non-trivial as θ increases while the behavior of the vertices of the half parallelogram are described by an *interspersing property*; they are non-decreasing functions $[0, 2\pi) \to [0, 2\pi)$.

We begin with a simple proposition

Proposition 1.1. *there is an affine diameter of a convex polygon K in every direction that meets a vertex.*

This is a consequence of the convexity K, for if a chord does not meet a vertex, it either lies within an edge or it meets two edges. If it lies within an edge, it is not an affine diameter. If it meets two edges, it is possible increase its length while parallel translating it in a non-decreasing direction of the cone defined by the two edges until it meets a vertex.

From this, it is possible to determine a