1 Introduction

references for packing [1] [2] references for anisotropic packing problems The packing problem for centrally symmetric bodies in the plane is mostly understood. [3] [6] [4] [7] references for mathematica [9] [5] references for linear programming

We prove the Kuperberg and Kuperberg double lattice construction for polygons is an isolated local maximum for density in the full space of packings. That is, the density of nearby packings are of strictly lower density.

The structure of the paper is as follows.

We review the required results from [] lattices, packing...

We review Kuperberg and Kuperberg

We give a local parametrization of the neighborhood of the double lattice in the space of packings and give a characterization of the a correction function and show that the optimality of the local configuration problem implies the global density result.

We prove a stability result for non-linear programming problems of the type described.

the density function can be replaced with a function of the correct type and which implies the density result.

2 Theoretical Considerations

2.1 Local Stability

Definition 2.1. *Let Ξ be a set of isometries. Its mean volume is the limit*

$$d(\Xi) = \lim_{r\to\infty} \frac{\mathrm{vol}B(0,r)}{|\{\xi \in \Xi : \xi(0) \in B(0,r)\}|}. \tag{1}$$

The upper and lower mean volumes are the corresponding limits superior and inferior. We say Ξ is a (r,R)-set if the point set $\{\xi(0) : \xi \in \Xi\}$ has a packing radius r at least r and a covering radius at most R.

Definition 2.2. *Let K be a compact set with interior. We say that Ξ is admissible for K if the interiors of $\xi(K)$ and $\xi'(K)$ are disjoint for any two distinct isometries $\xi, \xi' \in \Xi$. We say furthermore that Ξ is saturated if there is no ξ such that $\Xi \cup \{\xi\}$ is again admissible.*

There are $r(K)$ and $R(K)$ such that when Ξ is admissible and saturated, then Ξ is a $(r(K), R(K))$-set.

Definition 2.3. *Given two (r',R')-sets Ξ and Ξ' of isometries, we define the premetric*

$$\begin{aligned}\delta_R(\Xi,\Xi') = \inf_{enum.} \sup\{&||\xi_i^{-1}\xi_j - \xi_i'^{-1}\xi_j'|| : \\ &i,j \text{ such that } ||\xi_i(0) - \xi_j(0)|| < 2R \text{ or } ||\xi_i'(0) - \xi_j'(0)|| < 2R\}.\end{aligned} \tag{2}$$

The infimum is over all enumerations $\mathbb{N} \to \Xi$ and $\mathbb{N} \to \Xi$.

When $R > R'$, $\delta_R(\Xi, \Xi') = 0$ if and only if $\xi_i = \phi\xi'_i$ for some $\phi \in E(n)$ and some enumerations. Consider a body K. When $R > R(K)$, $\delta_R(\Xi, \Xi')$ is a metric on the space of admissible (r, R)-sets up to overall isometry, which includes the saturated sets as a subset.

Definition 2.4. *We say an admissible and saturated set Ξ is strongly extreme for K if it minimizes the mean volume among admissible elements in a neighborhood of Ξ.*

Note that the above definition is independent of R.

Our notion of strong extremality is meant to generalize weaker notions of local optimality for packing. Specifically, these notions apply to lattices and periodic packings.

Remark 2.1. *If a lattice Λ is strongly extreme for K, then Λ is extreme for K [**?**].*

Remark 2.2. *If a periodic set $\Xi = \{T_{\mathbf{l}}\xi_i : \mathbf{l} \in \Lambda, i = 1, \dots, N\}$ is strongly extreme, then it is periodic extreme for K [**?**].*

Definition 2.5. *Let Ξ be a countable set of isometries and fix an enumeration $\Xi = \{\xi_i : i \in \mathbb{N}\}$. Let $\mathcal{P}$ be a polyhedral complex whose underlying space is $\mathbb{R}^n$. For every face F of $\mathcal{P}$, let $I_F = \{i : \xi_i(0) \in F\}$. We say $\mathcal{P}$ is a honeycomb of Ξ if each n-face (cell) P is the convex hull of $\{\xi_i(0) : i \in I_P\}$.*

Theorem 2.1. *Let Ξ be admissible for K and let $\mathcal{P}$ be a honeycomb of Ξ. For every cell P, consider the optimization problem of minimizing $f_P(\Xi_P) = \operatorname{vol}\operatorname{conv}_{i \in I_P}\xi'_i(0)$ over the assignment of isometries ξ'_i, $i \in I_P$, such that this finite set is admissible. If $\xi'_i = \xi_i$, $i \in I_P$, is a local minimum for each cell P, then Ξ is strongly extreme.*

Theorem 2.2. *Let $g_F(\Xi_F)$ be a real-valued function over $\Xi_F = (\xi'_i)_{i \in I_F}$ for each oriented $(n-1)$-faces (ridge) of $\mathcal{P}$, such that $g_F(\Xi_F) = -g_{-F}(\Xi_F)$, where $-F$ is the orientation-reversed version of F. If we replace $f_P(\Xi_P)$ in the previous theorem with $f'_P(\Xi_P) = f_P(\Xi_P) + \sum_{F \in \partial P} g_F(\Xi_F)$, then again, if $\xi'_i = \xi_i$, $i \in I_P$, is a local minimum for each cell P, then Ξ is strongly extreme.*

2.1.1 Summary of [6, §2] on double lattices.

Definition 2.6. *A chord of a convex body K is a line segment whose endpoints lie on the boundary of K. A chord is an affine diameter if there is no longer chord parallel to it.*

Definition 2.7. *An inscribed parallelogram is a half-length parallelogram in the direction θ if one pair of edges is parallel with the line through the origin at an angle θ above the x-axis and their length is half the length of an affine diameter parallel to them.*

Note that any two half-length parallelograms in the direction θ have equal area, and we can define that area as a function $A(\theta)$ of the direction.

Definition 2.8. *A cocompact discrete subgroup of the Euclidean group consisting of translations and point reflections is a double lattice if it includes at least one point reflection.*

A double lattice is generated by a lattice and a point reflection, or alternatively by three point reflections.

Theorem 2.3 (Kuperberg and Kuperberg)**.** *For a convex K, an admissible double lattice of smallest mean area has mean area $4\min_\theta A(\theta)$ and is generated by reflection about the vertices of a half-length parallelogram.*

The densest double lattice packing of a convex polygon K can be constructed in time proportional to the number of vertices by an algorithm of Mount [8]. The goal of this paper is to show that this configuration is not only a local maximum of density among double lattices, but is in fact a local maximum in a broader sense, strong extremality.

To achieve this goal, we start by describing a honeycomb associated with the double lattice. Let K be a convex polygon and let $\mathbf{p}_2\mathbf{p}_3\mathbf{p}_5\mathbf{p}_6$ be a half-length parallelogram, such that $\mathbf{p}_3\mathbf{p}_2$ and $\mathbf{p}_5\mathbf{p}_6$ are half the length of and parallel to the affine diameter $\mathbf{p}_4\mathbf{p}_1$. The double lattice generated by reflections about the vertices of the parallelogram is Ξ and the subgroup of translations is the lattice Λ. Let $P = 0I_{\mathbf{p}_2}(0)I_{\mathbf{p}_6}(\mathbf{p}_1 - \mathbf{p}_4)$, then $\{\xi(P) : \xi \in \Xi\}$ are the cells of polyhedral complex which is a honeycomb for Ξ. Note that the optimization problem of minimizing $f_{\xi(P)}$ over ξ_i', $i \in I_{\xi(P)}$, is mathematically equivalent for every $\xi \in \Xi$. Therefore, to show that Theorem X applies, it suffices to show that $\xi_i' = \xi_i$, $i \in I_P$, is a local optimum over admissible assignments of ξ_i', $i \in I_P$.

For every convex body K and double lattice Ξ we now have a concrete optimization problem to solve: we wish to minimize the area of the quadrilateral $\xi_0'(0)\xi_6'(0)\xi_1'(0)\xi_2'(0)$ subject to the constraints that $\xi_i'(K)$ and $\xi_j'(K)$ do not overlap. Since we are only interested in certifying that the initial configuration is a local minimum, we can replace the constraints with ones that are equivalent in the neighborhood.

Lemma 2.1. *Let K and K' be two polygons that intersect at a segment. The endpoints of the segments are $\mathbf{x}$ a vertex of K and $\mathbf{y}$ a vertex of K'. Let $\mathbf{y}\mathbf{y}'$ and $\mathbf{x}\mathbf{x}'$ be the edges of K and K' containing the intersection. Let $\mathbf{x}'\mathbf{y}\mathbf{x}\mathbf{y}'$ be oriented counterclockwise from the point of view of the interior of K (otherwise switch K and K'). There is some $\epsilon > 0$ such that whenever $||\xi||, ||\xi'|| < \epsilon$, then $\xi(K)$ and $\xi'(K')$ have disjoint interiors if and only if $\alpha(\xi(\mathbf{x})\xi(\mathbf{x}')\xi'(\mathbf{y})) \geq 0$ and $\alpha(\xi'(\mathbf{y}')\xi'(\mathbf{y})\xi(\mathbf{x})) \geq 0$, where α is the signed area of the oriented triangle.*

Lemma 2.2. *Let K and K' be two polygons that intersect at a point and not at a segment. The intersection point $\mathbf{y}$ is a vertex of one polygon, which we let be K', and sits in the relative interior of an edge $\mathbf{x}'\mathbf{x}$ of K, oriented counter clockwise from the point of view of the interior of K. There is some $\epsilon > 0$ such that whenever $||\xi||, ||\xi'|| < \epsilon$, then $\xi(K)$ and $\xi'(K')$ have disjoint interiors if and only if $\alpha(\xi(\mathbf{x})\xi(\mathbf{x}')\xi'(\mathbf{y})) \geq 0$.*

Note that the case of an intersection at a point that is a vertex of both polygons is not treated. We exclude this case in the following lemma.

Lemma 2.3. *Let K be a convex polygon and let θ be such that $A(\theta)$ is minimal, then there is a half-length parallelogram in the direction θ that shares no vertex with K and an affine diameter with no more than one vertex shared with K.*

2.1.2 other theorems

Theorem 2.4. *Programs satisfying conditions (*) have a local maximum at 0.*

3 Calculation

3.1 Pentagons

rewrite in terms of section 1

In the case of packings by regular pentagons, one can consider the configuration space of four pentagons with respect to the density function taken with respect to the Delaunay triangulation and parametrized by .... This can be shown to satisfy the conditions of Section **??**....

3.1.1 exhibit the construction given by 2.3

3.1.2 parametrize neighborhood of four pentagons in correct configuration

the parametrization is important, should be consistent with the later sections. Can this be hidden in the code, and to just state that there exists a parametrization that satisfies the linear program theorem? Still need a sketch here.

3.1.3 constrained optimization problem

3.1.4 density function

3.1.5 verification

3.2 Heptagons

Outline of the method for heptagons.

For heptagons, the cost function is non-trivial. This is because there is a motion in the configuration space of four heptagons that increases the double Delauney density.

(include figure)

3.2.1 find the construction given by Theorem 2.3.

3.2.2 parametrize neighborhood of four heptagons in correct configuration

include figures

3.2.3 constrained optimization problem

3.2.4 the cost functions

Construct a cost function that satisfies the condition of correction theorems and makes the program satisfy the conditions of the LP theorems.

YOU GET THE COST FUNCTION FROM THE PROGRAM, AND IT SATISFIES THE PROPERTIES OF SECTION 1....

This area could be confusing as the objective functions become fairly complicated. Need to justify replacing objective functions with modifications. i.e.,

Sketch: Minimize the area of the Delaunay triangles. If there is no nearby configuration that decreases area done. (can we increase area and still increase density? possibly locally, so need to trade between nearby double triangles. But by symmetry, introduce a cost function between the "outer" heptagons, i.e. penalize rotation in opposite directions.) In general, average area of double triangles is minimized implies average density is maximized.

The function optimized in this case is the (modified) area of the Delaunay triangles.

3.3 general (2n+1)-gons

failure in the case of the enneagon, the construction of K+K is problematic. This requires additional analysis as

four polygons with corrected area function and an SL(2,R) motion hessian? The SL2 motion is required for the solution part, i.e. finding the initial configuration for regular polygons. Check this makes sense.

4 Formal methods

4.1 symbolic computation in an extension field

The Wolfram System supports symbolic computation over extension fields via it's pattern matching system.

5 Slicing nonlinear programs

A non-linear programing problem satisfying certain conditions can be certified as locally optimal by a linear programming problem. For the geometric problems considered, there are *a priori* configurations given by the maximal density configurations on a subspace of configuration space, namely subsets of the double lattice packings. To produce a certificate of local optimality for this type of problem it is possible to parametrize a neighborhood of the conjectured optimal configuration and analyze the associated non-linear programming problem

$$\max_{x\in\mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0, r \in I$$

in a neighborhood of 0. An appropriate choice of parametrization allows the full non-linear program to be sliced into a one-parameter family of non-linear programs that are subordinate to the linearization of the main program at 0. The following conditions are required.

Conditions 5.1. [1]

1. *Let I be a finite index set.*
2. *Let e_1 be the standard unit vector $\{1, 0, \ldots, 0\}$ in $\mathbb{R}^n$.*
3. *For r in I, let f and g_r be analytic functions on a neighborhood of 0.*
4. *Assume $f(0) = g_r(0) = 0$ for all r in I.*
5. *Let $F(t) = \nabla f(te_1)$.*
6. *Let $G_r(t) = \nabla g_r(te_1)$.*
7. *Assume the linear program*

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

   has a bounded solution and that the maximum is attained at 0.

8. *Assume that the set of solutions in $\mathbb{R}^n$ to*

$$F(0) \cdot x = 0 \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

   is

$$E := \{te_1 : t \in \mathbb{R}\}.$$

9. *Let H be the orthogonal complement of E so that $\mathbb{R}^n = E \oplus H$.*
10. *Assume there is an $\epsilon > 0$ so the functions $g_r(te_1) = 0$ for all $t \in (-\epsilon, \epsilon)$, for all r in I.*
11. *Assume $\frac{\partial}{\partial t} f(0) = 0$, $\frac{\partial^2}{\partial t^2} f(0) < 0$.*

Lemma 5.1. *Given Conditions 5.1, the linear program*

$$\max_{x \in H} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

has a unique maximum at $x = 0$

[1] These are the conditions that are required for the packing problems addressed. There are a number of ways they might be weakened, e.g. the condition that E be 1-dimensional is not essential.

Proof. By conditions 7 and 8, the linear program

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on E. The feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I \text{ and } x \in H\}$ is a subset of the feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I\}$. Thus, the program

$$\max_{x \in H} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on the non-empty intersection

$$E \cap \{x : G_r(0) \cdot x \geq 0, r \in I\} \cap H = 0.$$

□

Definition 5.1. *A* finitely generated cone *is a subset of* $\mathbb{R}^n$ *which is the non-negative span of a finite set of non-zero vectors* $\{v_1, \ldots, v_m\}$ *in* $\mathbb{R}^n$*, which are called the* generators *of the cone.*

Definition 5.2. *A* conical linear program *is a linear program with a constraint set that is a finitely generated cone.*

The linear programs described throughout this section are always constrained to be on the intersection of half-spaces with 0 on the boundary. These are conical programs.

Definition 5.3. *For a cone* C*, the set* $C^p := \{x \in \mathbb{R}^n : v \cdot x \leq 0 \text{ for all } v \in C\}$ *is the* polar cone *of* C*.*

Lemma 5.2. *A conical linear program with* $F \neq 0$ *given by*

$$\max_{x \in \mathbb{R}^n} F \cdot x \text{ subject to } G_r \cdot x \geq 0, r \in I$$

(a) has a unique[2] *maximum at* $x = 0$ *iff* F *is in the interior of the polar cone* C^p *of* $C = \{x : G_r \cdot x \geq 0, r \in I\}$ *(b) has a bounded solution iff* F *is in the polar cone* C^p *of* $C = \{x : G_r \cdot x \geq 0, r \in I\}$ *and attains its maximum exactly on the span of the generators* v_i *such that* $F \cdot v_i = 0$*.*

Proof. If F is in the interior of the polar cone C^p, then $F \cdot v_i < 0$ for all generators v_i. Therefore $F \cdot x$ is uniquely maximized in C at the vertex. If F is on the boundary of the polar cone, then $F \cdot x$ is maximized in C exactly on the span of the generators v_i for which $F \cdot v_i = 0$ as $F \cdot v_j < 0$ otherwise. If F is outside the polar cone, then $F \cdot v_i > 0$ for some generator v_i. Then $F \cdot x$ is unbounded in C. □

[2] The maximum satisfies a stronger uniqueness condition. It is stable under perturbations of F and G_k.

Lemma 5.3. *Given Conditions 5.1, there exists $\epsilon > 0$ such that for all t in $(-\epsilon, \epsilon)$, the linear program*

$$\max_{y_t \in H} F(t) \cdot y_t$$

subject to

$$G_r(t) \cdot y_t \geq 0, r \in I$$

has a unique maximum at $y_t = 0$.[3]

Proof. The program for $t \in (-\epsilon, \epsilon)$, for y_t in H, for each fixed t in $(-\epsilon, \epsilon)$, for some $\epsilon > 0$, can be written as a conical program on all of $\mathbb{R}^n$ with a cone C_t in $\mathbb{R}^n$ of co-dimension ≥ 1 by introducing further constraints $e_1 \cdot y_t \geq 0$ and $-e_1 \cdot y_t \geq 0$. By 5.1 and 5.2, $F(0)$ is in the polar cone of $C_0 = \{y_0 : G_r(0) \cdot y_0 \geq 0, e_1 \cdot y_0 \geq 0, -e_1 \cdot y_0 \geq 0\}$. As $f, g_r \in C^\omega$, the condition of $F(t)$ being in the interior of the polar cone C_t^p is open and the condition of the feasible set $C_t = \{y_t : G_r(t) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0\}$ being conical is open.[4] Therefore, by 5.2 the program has a unique maximum at $y_t = 0$ for each fixed t in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. □

Lemma 5.4. *Given Conditions 5.1 and ϵ as in Lemma 5.3, for all $t \in (-\epsilon, \epsilon)$ there exists $\delta(t) > 0$ and a cube $Q(t) \subset \mathbb{R}^n$ of side length $2\delta(t)$ such that*

$$\{(F(t) + Q(t)) \cap (\partial(C_t^p) + Q(t))\} = \emptyset.$$

Proof. This follows from 5.3, which shows $F(t)$ is in the interior of the polar cone C_t^p. Then $F(t)$ and the boundary of C_t^p can be separated and the existence of Q is trivial. □

Corollary 5.1. *Given 5.1 and ϵ as in Lemma 5.3, for all $t \in (-\epsilon, \epsilon)$,*

$$(F(t) + \Delta) \cdot y_t \leq 0$$

whenever y_t satisfies

$$(G_r(t) + \Delta_r) \cdot y_t \geq 0, r \in I \text{ and } e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0$$

where Δ and Δ_r are any points in the $2\delta(t)$-cube $Q(t)$ and y_t is in H.

Proof. By 5.4, $F(t) + \Delta$ is in the interior of the polar cone $C_{t,\Delta}^p$, where $C_{t,\Delta} = \{y_t : (G_r(t) + \Delta_r) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0, r \in I\}$. □

[3] Here y_t is a dummy variable and does not depend on t. It is labeled y_t to ease later exposition.

[4] The relationships between the constraint cone, the generators v_i and the constraint gradients G_k is subtle, but the openness of the condition follows from the continuity of the distance function.

Lemma 5.5. *Given Conditions 5.1 and ϵ as in Lemma 5.3, for all $t \in (-\epsilon, \epsilon)$, let $y_t = x - te_1 \in H$. Choose $\Delta = \Delta(y_t)$ and $\Delta_r = \Delta_r(y_t)$ in the $2\delta(t)$-cube $Q(t)$ to be the corner given by the sign of $x - te_1 = y_t$. Then there is an ϵ_t for which*

$$(F(t) + \Delta(y_t)) \cdot y_t \leq 0 \implies f(x) - f(te_1) \leq 0$$

and

$$(G_r(t) + \Delta_r(y_t)) \cdot y_t \leq 0 \implies g_r(x) - g_r(te_1) = g_r(x) \leq 0$$

for all $\|y_t\| \leq \epsilon_t$.

Proof. This follows from the local expansions of the nonlinear program. By this choice of $\Delta(y_t)$ and $\Delta_r(y_t)$,

$$\begin{aligned} f(x) - f(te_1) &= F(t) \cdot (x - te_1) + O(t^2) = F(t) \cdot y_t + O(t^2) \\ &\leq F(t) \cdot y_t + \delta(t)\|y_t\|_1 = (F(t) + \Delta(y_t)) \cdot y_t \end{aligned}$$

and using assumption 10,

$$\begin{aligned} g_r(x) = g_r(x) - g_r(te_1) &= G_r(t) \cdot (x - te_1) + O(t^2) = G_r(t) \cdot y_t + O(t^2) \\ &\leq G_r(t) \cdot y_t + \delta(t)\|y_t\|_1 = (G_r(t) + \Delta_r(y_t)) \cdot y_t. \end{aligned}$$

□

By Lemma 5.4 and Corollary 5.1, for t in $(-\epsilon, \epsilon)$, the program

$$\max_{y_t \in H} (F(t) + \Delta) \cdot y_t \text{ subject to } (G_r + \Delta_r) \cdot y_t$$

is uniquely maximized at $y_t = 0$ for any choice of Δ, Δ_r in the $2\delta(t)$ cube $Q(t)$. Combined with 5.5, there is an ϵ_t neighborhood of 0 where $f(y_t + te_1)$ is less than $f(te_1)$ on $\cup_{\Delta_r \in Q(t)} \{y_t : (G_r + \Delta_r) \cdot y_t \geq 0, r \in I, y_t \in H\}$, which contains the feasible set $\{y_t : g_r(y_t + te_1) \geq 0, r \in I, y_t \in H\}$. Therefore the nonlinear programs $f(y_t + te_1)$ subject to $g_r(y_t + te_1) \geq 0$, $y_t \in H$, which are parameterized by t in $(-\epsilon, \epsilon)$, have local maxima at $y_t = 0$. This gives the following:

Theorem 5.1. *Given Conditions 5.1, a fixed t in $(-\epsilon, \epsilon)$ and choosing Δ and Δ_r as in Lemma 5.5, for x satisfying $g_r(x) \geq 0$ for all r in I and $y_t = x - te_1$ in H, there exist linear programs*[5]

$$\max_{y_t \in H} (F(t) + \Delta(y_t)) \cdot y_t \textit{ subject to } (G_r(t) + \Delta_r(y_t)) \cdot y_t \geq 0$$

that give solutions to the nonlinear programs

$$\max_{x \in H + te_1} f(x) \textit{ subject to } g_r(x) \geq 0$$

in an ϵ_t neighborhood of te_1 in $H + te_1$. □

[5] These programs may depend on a choice of $y_t \in H$, but $f(x)$ is always less then $f(te_1)$ by Lemma 5.5.

By choice of a sufficiently small ϵ and a minimal[6] non-zero ϵ_t, Theorem 5.1 gives an open neighborhood of 0 in which the maximum value of the original nonlinear program occurs on E. The conditions for the first and second t-derivatives at 0 shows 0 to be a local maximum for the nonlinear program

$$\max_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0.$$

Theorem 5.2. *A nonlinear program satisfying Conditions 5.1 has an isolated local maximum at 0 with f(0) = 0.* □

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