1 Introduction

This paper began as an investigation of the optimality of the double lattice packing for pentagons and heptagons. In [6], G. and W. Kuperberg describe a double lattice packing of congruent planar convex bodies with maximal density that can be constructed algorithmically[1]. As an example, they construct the densest double lattice packing for regular pentagons and show that it has a density of $(5 - \sqrt{5})/3 = 0.92131\ldots$ which is the current record and possibly the best *general* packing of the plane.

In the early part of the 2000s, there has been a significant push both theoretically and computationally to answer some of the most naive yet perplexing questions in packing [2] Along with the proof and formal verification of the Kepler conjecture [], a number of other results have proved to be both illuminating and frustrating (Cohn-Elkies LP bounds, Lower bounds...). For packings by congruent anisotropic bodies, sharp results are limited mostly to the plane, where the best packings of all 0-symmetric bodies are achieved by lattices [3], and a series of space results in higher dimensions.

For non-centrally symmetric bodies, the problem of finding the best packing of regular pentagon serves as a toy model for problems like the best packing of regular tetrahedra. However, it is still not a tractable one. The first explicit upper bounds for the packing of regular tetrahedra [4] predate (?) the first explicit upper bounds for pentagons (as a test?) [7], albeit by only a few years.

Still, the upper bound on density of 0.98103 for pentagons and $1 - 2.6...10^-25$ for regular tetrahedra are quite far from their best known packings: 0.856437... for tetrahedra and 0.92131... as previously mentioned for pentagons.

Even in the plane, it is an open question to find the global pessimal convex body, that is the shape that has the lowest maximum packing density. In the class of 0-symmetric bodies, it is Reinhardt who conjectured that a smoothed octagon is the minimizer. In the class of general convex bodies, it is conjectured to be the regular heptagon. However, even thought conjectured maximum packing density of the regular heptagon is the maximal density double lattice, it has also resisted proofs its global optimality.

This motivates the following: we consider the densest double lattice packings and extend the optimality result for polygons to the a neighborhood in the space of packings. This demonstrates that while the double lattices are perhaps not globally optimal, they are at the very least locally optimal.

Theorem 1.1. *Given a convex planar polygon, there is a double lattice packing that is locally optimal. Modulo vertex condition...*

In that light, we must establish the correct notion of neighborhood. This neighborhood should be broader than simply the Hausdorff distance between two packings. To that end, we recall a topology on the space of sequences ambient isometries

[1]For the case of polygons, this turns out to be linear in the number of vertices [9].

[2]for a background on packing problems, see [1] [2][**?**].

We recast the relevant portions of [6].

We prove a stability result for non-linear programming problems.

We characterize the pentagon.

We give a local parametrization of the neighborhood of the double lattice in the space of packings and give a characterization of the a correction function and show that the optimality of the local configuration problem implies the global density result.

2 Theoretical Considerations

2.1 Local Stability

Definition 2.1. *Let Ξ be a set of isometries. Its mean volume is the limit*

$$d(\Xi) = \lim_{r\to\infty} \frac{\mathrm{vol} B(0,r)}{|\{\xi \in \Xi : \xi(0) \in B(0,r)\}|}. \tag{1}$$

The upper and lower mean volumes are the corresponding limits superior and inferior. We say Ξ is a (r, R)-set if the point set $\{\xi(0) : \xi \in \Xi\}$ has a packing radius r at least r and a covering radius at most R.

We will look at packings of congruent copies of a convex body K. That is, every element of the packing is given by $\xi(K)$, where ξ is an isometry of Euclidean space. It will be convenient to assume that the reference body K is situated so that its interior contains the origin.

Definition 2.2. *Let K be a compact set with interior. We say that Ξ is admissible for K if the interiors of $\xi(K)$ and $\xi'(K)$ are disjoint for any two distinct isometries $\xi, \xi' \in \Xi$. We say furthermore that Ξ is saturated if there is no ξ such that $\Xi \cup \{\xi\}$ is again admissible.*

There are $r(K)$ and $R(K)$ such that when Ξ is admissible and saturated, then Ξ is a $(r(K), R(K))$-set.

Definition 2.3. *Given two (r', R')-sets Ξ and Ξ' of isometries, we define the premetric*

$$\begin{aligned}\delta_R(\Xi, \Xi') = \inf_{enum.} \sup\{&||\xi_i^{-1}\xi_j - \xi_i'^{-1}\xi_j'|| : \\ &i, j \text{ such that } ||\xi_i(0) - \xi_j(0)|| < 2R \text{ or } ||\xi_i'(0) - \xi_j'(0)|| < 2R\}.\end{aligned} \tag{2}$$

The infimum is over all enumerations $\mathbb{N} \to \Xi$ and $\mathbb{N} \to \Xi$.

When $R > R'$, $\delta_R(\Xi, \Xi') = 0$ if and only if $\xi_i = \phi\xi_i'$ for some $\phi \in E(n)$ and some enumerations. Consider a body K. When $R > R(K)$, $\delta_R(\Xi, \Xi')$ is a metric on the space of admissible (r, R)-sets up to overall isometry, which includes the saturated sets as a subset.

Definition 2.4. *We say an admissible and saturated set Ξ is strongly extreme for K if it minimizes the mean volume among admissible elements in a neighborhood of Ξ.*

Note that the above definition is independent of R.

Our notion of strong extremality is meant to generalize weaker notions of local optimality for packing. Specifically, these notions apply to lattices and periodic packings.

Remark 2.1. *If a lattice Λ is strongly extreme for K, then Λ is extreme for K [8].*

Remark 2.2. *If a periodic set $\Xi = \{T_{\mathbf{l}}\xi_i : \mathbf{l} \in \Lambda, i = 1, \ldots, N\}$ is strongly extreme, then it is periodic extreme for K [10].*

Definition 2.5. *Let Ξ be a countable set of isometries and fix an enumeration $\Xi = \{\xi_i : i \in \mathbb{N}\}$. Let $\mathcal{P}$ be a polyhedral complex whose underlying space is $\mathbb{R}^n$. For every face F of $\mathcal{P}$, let $I_F = \{i : \xi_i(0) \in F\}$. We say $\mathcal{P}$ is a honeycomb of Ξ if each n-face (cell) P is the convex hull of $\{\xi_i(0) : i \in I_P\}$.*

Theorem 2.1. *Let Ξ be admissible for K and let $\mathcal{P}$ be a honeycomb of Ξ. For every cell P, consider the optimization problem of minimizing $f_P(\Xi_P) = \operatorname{vol}\operatorname{conv}_{i \in I_P}\xi'_i(0)$ over the assignment of isometries ξ'_i, $i \in I_P$, such that this finite set is admissible. If $\xi'_i = \xi_i$, $i \in I_P$, is a local minimum for each cell P, then Ξ is strongly extreme.*

Theorem 2.2. *Let $g_F(\Xi_F)$ be a real-valued function over $\Xi_F = (\xi'_i)_{i \in I_F}$ for each oriented $(n-1)$-faces (ridge) of $\mathcal{P}$, such that $g_F(\Xi_F) = -g_{-F}(\Xi_F)$, where $-F$ is the orientation-reversed version of F. If we replace $f_P(\Xi_P)$ in the previous theorem with $f'_P(\Xi_P) = f_P(\Xi_P) + \sum_{F \in \partial P} g_F(\Xi_F)$, then again, if $\xi'_i = \xi_i$, $i \in I_P$, is a local minimum for each cell P, then Ξ is strongly extreme.*

2.2 Double lattices

Definition 2.6. *A chord of a convex body K is a line segment whose endpoints lie on the boundary of K. A chord is an affine diameter if there is no longer chord parallel to it.*

Definition 2.7. *An inscribed parallelogram is a half-length parallelogram in the direction θ if one pair of edges is parallel with the line through the origin at an angle θ above the x-axis and their length is half the length of an affine diameter parallel to them.*

Note that any two half-length parallelograms in the direction θ have equal area, and we can define that area as a function $A(\theta)$ of the direction.

Definition 2.8. *A cocompact discrete subgroup of the Euclidean group consisting of translations and point reflections is a double lattice if it includes at least one point reflection.*

A double lattice is generated by a lattice and a point reflection, or alternatively by three point reflections.

Theorem 2.3 (Kuperberg and Kuperberg)**.** *For a convex K, an admissible double lattice of smallest mean area has mean area $4\min_\theta A(\theta)$ and is generated by reflection about the vertices of a half-length parallelogram.*

The densest double lattice packing of a convex polygon K can be constructed in time proportional to the number of vertices by an algorithm of Mount [9]. The goal of this paper is to show that this configuration is not only a local maximum of density among double lattices, but is in fact a local maximum in a broader sense, strong extremality.

2.3 General setup

To achieve this goal, we start by describing a honeycomb associated with the double lattice. Let K be a convex polygon and let $\mathbf{p}_2\mathbf{p}_3\mathbf{p}_5\mathbf{p}_6$ be a half-length parallelogram, such that $\mathbf{p}_3\mathbf{p}_2$ and $\mathbf{p}_5\mathbf{p}_6$ are half the length of and parallel to the affine diameter $\mathbf{p}_4\mathbf{p}_1$. The double lattice generated by reflections about the vertices of the parallelogram is Ξ and the subgroup of translations is the lattice Λ. Let $P = 0I_{\mathbf{p}_2}(0)I_{\mathbf{p}_6}(\mathbf{p}_1 - \mathbf{p}_4)$, then $\{\xi(P) : \xi \in \Xi\}$ are the cells of polyhedral complex which is a honeycomb for Ξ. Note that the optimization problem of minimizing $f_{\xi(P)}$ over ξ_i', $i \in I_{\xi(P)}$, is mathematically equivalent for every $\xi \in \Xi$. Therefore, to show that Theorem X applies, it suffices to show that $\xi_i' = \xi_i$, $i \in I_P$, is a local optimum over admissible assignments of ξ_i', $i \in I_P$.

For every convex body K and double lattice Ξ we now have a concrete optimization problem to solve: we wish to minimize the area of the quadrilateral $\xi_0'(0)\xi_6'(0)\xi_1'(0)\xi_2'(0)$ subject to the constraints that $\xi_i'(K)$ and $\xi_j'(K)$ do not overlap. Since the objective and the constraints are invariant under common isometry, we may fix $\xi_i' = \xi_i$ for one i. We parametrize $\xi_i' = T_{\mathbf{r}_i}\xi_i R_{\theta_i}$, where R_θ is a rotation by θ about the origin, and $T_{(x,y)}$ is a translation by $\mathbf{r}_i$. Since we are only interested in certifying that the initial configuration is a local minimum, we can replace the constraints with ones that are equivalent in the neighborhood.

Lemma 2.1. *Let K and K' be two polygons that intersect at a segment. The endpoints of the segments are $\mathbf{x}$ a vertex of K and $\mathbf{y}$ a vertex of K'. Let $\mathbf{y}\mathbf{y}'$ and $\mathbf{x}\mathbf{x}'$ be the edges of K and K' containing the intersection. Let $\mathbf{x}'\mathbf{y}\mathbf{x}\mathbf{y}'$ be oriented counterclockwise from the point of view of the interior of K (otherwise switch K and K'). There is some $\epsilon > 0$ such that whenever $||\xi||, ||\xi'|| < \epsilon$, then $\xi(K)$ and $\xi'(K')$ have disjoint interiors if and only if $\alpha(\xi(\mathbf{x})\xi(\mathbf{x}')\xi'(\mathbf{y})) \geq 0$ and $\alpha(\xi'(\mathbf{y}')\xi'(\mathbf{y})\xi(\mathbf{x})) \geq 0$, where α is the signed area of the oriented triangle.*

Lemma 2.2. *Let K and K' be two polygons that intersect at a point and not at a segment. The intersection point $\mathbf{y}$ is a vertex of one polygon, which we let be K', and sits in the relative interior of an edge $\mathbf{x}'\mathbf{x}$ of K, oriented counter clockwise from the point of view of the interior of K. There is some $\epsilon > 0$ such that whenever $||\xi||, ||\xi'|| < \epsilon$, then $\xi(K)$ and $\xi'(K')$ have disjoint interiors if and only if $\alpha(\xi(\mathbf{x})\xi(\mathbf{x}')\xi'(\mathbf{y})) \geq 0$.*

Note that the case of an intersection at a point that is a vertex of both polygons is not treated. In the first two cases we treat, the regular pentagon and the regular heptagon, there are no such intersections. When we generalize the calculation to all convex polygons, we will show that such an intersection only occurs in special cases that can be treated separately.

2.4 other theorems

We will show that optimization problems we obtain fall into a convenient form, where linear stability holds along all but one direction. Along the direction of vanishing linear stability, the construction of Kuperberg and Kuperberg will be shown to guaranty stability.

Theorem 2.4. *Programs satisfying conditions (*) have a local maximum at 0.*

3 Calculation

3.1 Pentagons

Let us fix a regular pentagon $K = \text{conv}\{\mathbf{k}_i : i = 0, \dots 4\}$, where $\mathbf{k}_i = R_{2\pi i/5}(1,0)$. In this subsection, we do all the calculations in the extension field $\mathbb{Q}(u,v)$, where $u = \cos\pi/5$ and $v = \sin\pi/5$.

One minimum-area half-length parallelogram corresponds to the affine diameter $\mathbf{p}_1\mathbf{p}_4$, where $\mathbf{p}_1 = \mathbf{k}_0$ and $\mathbf{p}_4 = \frac{1}{2}(\mathbf{k}_2 + \mathbf{k}_3)$. The vertices of the parallelogram are given by $\mathbf{p}_2 = \frac{1}{4}\mathbf{k}_0 + \frac{3}{4}\mathbf{k}_1$, $\mathbf{p}_3 = \frac{3-2u}{4}\mathbf{k}_1 + \frac{1+2u}{4}\mathbf{k}_2$, $\mathbf{p}_5 = \frac{1+2u}{4}\mathbf{k}_3 + \frac{3-2u}{4}\mathbf{k}_4$, and $\mathbf{p}_6 = \frac{3}{4}\mathbf{k}_4 + \frac{1}{4}\mathbf{k}_0$.

The four pentagons that surround our primitive honeycomb cell are $\xi_i(K)$, $i = 0,1,2,6$, where $\xi_0 = \text{Id}$, $\xi_1 = T_{\mathbf{p}_1-\mathbf{p}_4}$, $\xi_2 = I_{\mathbf{p}_2}$, and $\xi_6 = I_{\mathbf{p}_6}$. We are interested in showing that the assignment $\xi'_i = \xi_i$, $i = 0,1,2,6$ locally minimizes the area of the quadrilateral $\xi_0(0)\xi_6(0)\xi_1(0)\xi_2(0)$, subject to the nonoverlap constraints. As explained in the previous section, we may fix $\xi'_1 = \xi_1$ and replace the nonoverlap constraints by signed area constriants. We obtain the following optimization problem:

$$
\begin{aligned}
\text{maximize}\,&\alpha(\xi'_0(0),\xi'_1(0),\xi'_2(0)) - \alpha(\xi'_0(0),\xi'_1(0)+\xi'_6(0)) \\
\text{subj. to}\,&\alpha(\xi'_0(\mathbf{k}_1),\xi'_0(\mathbf{k}_0),\xi'_2(\mathbf{k}_1)) \geq 0 \\
&\alpha(\xi'_2(\mathbf{k}_1),\xi'_2(\mathbf{k}_0),\xi'_0(\mathbf{k}_1)) \geq 0 \\
&\alpha(\xi'_0(\mathbf{k}_0),\xi'_0(\mathbf{k}_4),\xi'_6(\mathbf{k}_4)) \geq 0 \\
&\alpha(\xi'_6(\mathbf{k}_0),\xi'_6(\mathbf{k}_4),\xi'_0(\mathbf{k}_4)) \geq 0 \\
&\alpha(\xi'_1(\mathbf{k}_2),\xi'_1(\mathbf{k}_1),\xi'_2(\mathbf{k}_2)) \geq 0 \\
&\alpha(\xi'_2(\mathbf{k}_2),\xi'_2(\mathbf{k}_1),\xi'_1(\mathbf{k}_2)) \geq 0 \\
&\alpha(\xi'_1(\mathbf{k}_4),\xi'_1(\mathbf{k}_3),\xi'_6(\mathbf{k}_3)) \geq 0 \\
&\alpha(\xi'_6(\mathbf{k}_4),\xi'_6(\mathbf{k}_3),\xi'_1(\mathbf{k}_3)) \geq 0 \\
&\alpha(\xi'_1(\mathbf{k}_3),\xi'_1(\mathbf{k}_2),\xi'_0(\mathbf{k}_0)) \geq 0,
\end{aligned}
\tag{3}
$$

where $\xi'_i = T_{(x_i,y_i)}\xi_i R_\theta$ for $i = 0,2,6$, and $\xi'_1 = \xi_1$. We adopt a condensed notation for the free variables $X = (x_0, y_0, \theta_0, x_2, y_2, \theta_2, x_6, y_6, \theta_6)$.

The linearized version of 3 is locally optimized .... Since 3 satisfies conditions (*), we have:

Theorem 3.1. *The optimal double-lattice packing of regular pentagons, illustrated in Figure X, is strongly extreme.*

3.2 Heptagons

Outline of the method for heptagons.

For heptagons, the cost function is non-trivial. This is because there is a motion in the configuration space of four heptagons that increases the double Delauney density.

(include figure)

3.2.1 find the construction given by Theorem 2.3.

3.2.2 parametrize neighborhood of four heptagons in correct configuration

include figures

3.2.3 constrained optimization problem

3.2.4 the cost functions

Construct a cost function that satisfies the condition of correction theorems and makes the program satisfy the conditions of the LP theorems.

YOU GET THE COST FUNCTION FROM THE PROGRAM, AND IT SATISFIES THE PROPERTIES OF SECTION 1....

This area could be confusing as the objective functions become fairly complicated. Need to justify replacing objective functions with modifications. i.e.,

Sketch: Minimize the area of the Delaunay triangles. If there is no nearby configuration that decreases area done. (can we increase area and still increase density? possibly locally, so need to trade between nearby double triangles. But by symmetry, introduce a cost function between the "outer" heptagons, i.e. penalize rotation in opposite directions.) In general, average area of double triangles is minimized implies average density is maximized.

The function optimized in this case is the (modified) area of the Delaunay triangles.

3.3 general (2n+1)-gons

failure in the case of the enneagon, the construction of K+K is problematic. This requires additional analysis as

four polygons with corrected area function and an SL(2,R) motion hessian? The SL2 motion is required for the solution part, i.e. finding the initial configuration for regular polygons. Check this makes sense.

4 Formal methods

4.1 symbolic computation in an extension field

The Wolfram System supports symbolic computation over extension fields via it's pattern matching system.

5 Slicing nonlinear programs

A non-linear programing problem satisfying certain conditions can be certified as locally optimal by a linear programming problem. For the geometric problems considered, there are *a priori* configurations given by the maximal density configurations on a subspace of configuration space, namely subsets of the double lattice packings. To produce a certificate of local optimality for this type of problem it is possible to parametrize a neighborhood of the conjectured optimal configuration and analyze the associated non-linear programming problem

$$\max_{x\in\mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0, r \in I$$

in a neighborhood of 0. An appropriate choice of parametrization allows the full non-linear program to be sliced into a one-parameter family of non-linear programs that are subordinate to the linearization of the main program at 0. The following conditions are required.

Conditions 5.1. [3]

1. *Let I be a finite index set.*
2. *Let e_1 be the standard unit vector $\{1, 0, \ldots, 0\}$ in $\mathbb{R}^n$.*
3. *For r in I, let f and g_r be analytic functions on a neighborhood of 0.*
4. *Assume $f(0) = g_r(0) = 0$ for all r in I.*
5. *Let $F(t) = \nabla f(te_1)$.*
6. *Let $G_r(t) = \nabla g_r(te_1)$.*
7. *Assume the linear program*

$$\max_{x\in\mathbb{R}^n} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

has a bounded solution and that the maximum is attained at 0.

[3]These are the conditions that are required for the packing problems addressed. There are a number of ways they might be weakened, e.g. the condition that E be 1-dimensional is not essential.

8. *Assume that the set of solutions in* $\mathbb{R}^n$ *to*

$$F(0) \cdot x = 0 \textit{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is

$$E := \{te_1 : t \in \mathbb{R}\}.$$

9. *Let* H *be the orthogonal complement of* E *so that* $\mathbb{R}^n = E \oplus H$.

10. *Assume there is an* $\epsilon > 0$ *so the functions* $g_r(te_1) = 0$ *for all* $t \in (-\epsilon, \epsilon)$, *for all* r *in* I.

11. *Assume* $\frac{\partial}{\partial t} f(0) = 0$, $\frac{\partial^2}{\partial t^2} f(0) < 0$.

Lemma 5.1. *Given Conditions 5.1, the linear program*

$$\max_{x \in H} F(0) \cdot x \textit{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

has a unique maximum at $x = 0$

Proof. By conditions 7 and 8, the linear program

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on E. The feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I \text{ and } x \in H\}$ is a subset of the feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I\}$. Thus, the program

$$\max_{x \in H} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on the non-empty intersection

$$E \cap \{x : G_r(0) \cdot x \geq 0, r \in I\} \cap H = 0.$$

□

Definition 5.1. *A* finitely generated cone *is a subset of* $\mathbb{R}^n$ *which is the non-negative span of a finite set of non-zero vectors* $\{v_1, \ldots, v_m\}$ *in* $\mathbb{R}^n$, *which are called the* generators *of the cone.*

Definition 5.2. *A* conical linear program *is a linear program with a constraint set that is a finitely generated cone.*

The linear programs described throughout this section are always constrained to be on the intersection of half-spaces with 0 on the boundary. These are conical programs.

Definition 5.3. *For a cone C, the set $C^p := \{x \in \mathbb{R}^n : v \cdot x \leq 0$ for all $v \in C\}$ is the* polar cone *of C.*

Lemma 5.2. *A conical linear program with $F \neq 0$ given by*

$$\max_{x \in \mathbb{R}^n} F \cdot x \text{ subject to } G_r \cdot x \geq 0, r \in I$$

(a) has a unique[4] *maximum at $x = 0$ iff F is in the interior of the polar cone C^p of $C = \{x : G_r \cdot x \geq 0, r \in I\}$ (b) has a bounded solution iff F is in the polar cone C^p of $C = \{x : G_r \cdot x \geq 0, r \in I\}$ and attains its maximum exactly on the span of the generators v_i such that $F \cdot v_i = 0$.*

Proof. If F is in the interior of the polar cone C^p, then $F \cdot v_i < 0$ for all generators v_i. Therefore $F \cdot x$ is uniquely maximized in C at the vertex. If F is on the boundary of the polar cone, then $F \cdot x$ is maximized in C exactly on the span of the generators v_i for which $F \cdot v_i = 0$ as $F \cdot v_j < 0$ otherwise. If F is outside the polar cone, then $F \cdot v_i > 0$ for some generator v_i. Then $F \cdot x$ is unbounded in C. □

Lemma 5.3. *Given Conditions 5.1, there exists $\epsilon > 0$ such that for all t in $(-\epsilon, \epsilon)$, the linear program*

$$\max_{y_t \in H} F(t) \cdot y_t$$

subject to

$$G_r(t) \cdot y_t \geq 0, r \in I$$

has a unique maximum at $y_t = 0$.[5]

Proof. The program for $t \in (-\epsilon, \epsilon)$, for y_t in H, for each fixed t in $(-\epsilon, \epsilon)$, for some $\epsilon > 0$, can be written as a conical program on all of $\mathbb{R}^n$ with a cone C_t in $\mathbb{R}^n$ of co-dimension ≥ 1 by introducing further constraints $e_1 \cdot y_t \geq 0$ and $-e_1 \cdot y_t \geq 0$. By 5.1 and 5.2, $F(0)$ is in the polar cone of $C_0 = \{y_0 : G_r(0) \cdot y_0 \geq 0, e_1 \cdot y_0 \geq 0, -e_1 \cdot y_0 \geq 0\}$. As $f, g_r \in C^\omega$, the condition of $F(t)$ being in the interior of the polar cone C_t^p is open and the condition of the feasible set $C_t = \{y_t : G_r(t) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0\}$ being conical is open.[6] Therefore, by 5.2 the program has a unique maximum at $y_t = 0$ for each fixed t in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. □

Lemma 5.4. *Given Conditions 5.1 and ϵ as in Lemma 5.3, for all $t \in (-\epsilon, \epsilon)$ there exists $\delta(t) > 0$ and a cube $Q(t) \subset \mathbb{R}^n$ of side length $2\delta(t)$ such that*

$$\{(F(t) + Q(t)) \cap (\partial(C_t^p) + Q(t))\} = \emptyset.$$

[4] The maximum satisfies a stronger uniqueness condition. It is stable under perturbations of F and G_k.

[5] Here y_t is a dummy variable and does not depend on t. It is labeled y_t to ease later exposition.

[6] The relationships between the constraint cone, the generators v_i and the constraint gradients G_k is subtle, but the openness of the condition follows from the continuity of the distance function.

Proof. This follows from 5.3, which shows $F(t)$ is in the interior of the polar cone C_t^p. Then $F(t)$ and the boundary of C_t^p can be separated and the existence of Q is trivial. $\square$

Corollary 5.1. *Given Conditions5.1 and ϵ as in Lemma 5.3, for all $t \in (-\epsilon, \epsilon)$,*

$$(F(t) + \Delta) \cdot y_t \leq 0$$

whenever y_t satisfies

$$(G_r(t) + \Delta_r) \cdot y_t \geq 0, r \in I \text{ and } e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0$$

where Δ and Δ_r are any points in the $2\delta(t)$-cube $Q(t)$ and y_t is in H.

Proof. By 5.4, $F(t) + \Delta$ is in the interior of the polar cone $C_{t,\Delta}^p$, where $C_{t,\Delta} = \{y_t : (G_r(t) + \Delta_r) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0, r \in I\}$. $\square$

Lemma 5.5. *Given Conditions 5.1 and ϵ as in Lemma 5.3, for all $t \in (-\epsilon, \epsilon)$, let $y_t = x - te_1 \in H$. Choose $\Delta = \Delta(y_t)$ and $\Delta_r = \Delta_r(y_t)$ in the $2\delta(t)$-cube $Q(t)$ to be the corner given by the sign of $x - te_1 = y_t$. Then there is an ϵ_t for which*

$$(F(t) + \Delta(y_t)) \cdot y_t \leq 0 \implies f(x) - f(te_1) \leq 0$$

and

$$(G_r(t) + \Delta_r(y_t)) \cdot y_t \leq 0 \implies g_r(x) - g_r(te_1) = g_r(x) \leq 0$$

for all $\|y_t\| \leq \epsilon_t$.

Proof. This follows from the local expansions of the nonlinear program. By this choice of $\Delta(y_t)$ and $\Delta_r(y_t)$,

$$\begin{aligned} f(x) - f(te_1) &= F(t) \cdot (x - te_1) + O(t^2) = F(t) \cdot y_t + O(t^2) \\ &\leq F(t) \cdot y_t + \delta(t)\|y_t\|_1 = (F(t) + \Delta(y_t)) \cdot y_t \end{aligned}$$

and using assumption 10,

$$g_r(x) = g_r(x) - g_r(te_1) = G_r(t) \cdot (x - te_1) + O(t^2) = G_r(t) \cdot y_t + O(t^2)$$

$$\leq G_r(t) \cdot y_t + \delta(t)\|y_t\|_1 = (G_r(t) + \Delta_r(y_t)) \cdot y_t.$$

$\square$

By Lemma 5.4 and Corollary 5.1, for t in $(-\epsilon, \epsilon)$, the program

$$\max_{y_t \in H} (F(t) + \Delta) \cdot y_t \text{ subject to } (G_r + \Delta_r) \cdot y_t$$

is uniquely maximized at $y_t = 0$ for any choice of Δ, Δ_r in the $2\delta(t)$ cube $Q(t)$. Combined with 5.5, there is an ϵ_t neighborhood of 0 where $f(y_t + te_1)$ is less than $f(te_1)$ on $\cup_{\Delta_r \in Q(t)}\{y_t : (G_r + \Delta_r) \cdot y_t \geq 0, r \in I, y_t \in H\}$, which contains the feasible set $\{y_t : g_r(y_t + te_1) \geq 0, r \in I, y_t \in H\}$. Therefore the nonlinear programs $f(y_t + te_1)$ subject to $g_r(y_t + te_1) \geq 0$, $y_t \in H$, which are parameterized by t in $(-\epsilon, \epsilon)$, have local maxima at $y_t = 0$. This gives the following:

Theorem 5.1. *Given Conditions 5.1, a fixed t in $(-\epsilon, \epsilon)$ and choosing Δ and Δ_r as in Lemma 5.5, for x satisfying $g_r(x) \geq 0$ for all r in I and $y_t = x - te_1$ in H, there exist linear programs*[7]

$$\max_{y_t \in H}(F(t) + \Delta(y_t)) \cdot y_t \textit{ subject to } (G_r(t) + \Delta_r(y_t)) \cdot y_t \geq 0$$

that give solutions to the nonlinear programs

$$\max_{x \in H + te_1} f(x) \textit{ subject to } g_r(x) \geq 0$$

in an ϵ_t neighborhood of te_1 in $H + te_1$. □

By choice of a sufficiently small ϵ and a minimal[8] non-zero ϵ_t, Theorem 5.1 gives an open neighborhood of 0 in which the maximum value of the original nonlinear program occurs on E. The conditions for the first and second t-derivatives at 0 shows 0 to be a local maximum for the nonlinear program

$$\max_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0.$$

Theorem 5.2. *A nonlinear program satisfying Conditions 5.1 has an isolated local maximum at 0 with f(0) = 0.* □

References

[1] P Brass, W Moser, and J Pach. *Research Problems in Discrete Geometry*, volume 1. Springer, 2005.

[2] J H Conway, C Goodman-Strauss, and N Sloane. Recent progress in sphere packing. *Current Developments in Mathematics*, pages 37–76, 1999.

[3] L Fejes Tóth. Some packing and covering theorems. *Acta Sci. Math. Szeged*, 12(Leopoldo Fejer et Frederico Riesz LXX annos natis dedicatus, Pars A):62–67, 1950.

[7] These programs may depend on a choice of $y_t \in H$, but $f(x)$ is always less then $f(te_1)$ by Lemma 5.5.
[8] This exists by a compactness argument.

[4] S Gravel, V Elser, and Y Kallus. Upper bound on the packing density of regular tetrahedra and octahedra. *Discrete & Computational Geometry*, 46(4):799–818, 2011.

[5] J B Keiper. Interval arithmetic in mathematica. *Mathematica Journal*, 5(2):66–71, 1995.

[6] G Kuperberg and W Kuperberg. Double-lattice packings of convex bodies in the plane. *Discrete & Computational Geometry*, 5(1):389–397, 1990.

[7] F Mário de Oliveira Filho and F Vallentin. Computing upper bounds for the packing density of congruent copies of a convex body I. *arXiv preprint arXiv:1308.4893*, 2013.

[8] Jacques Martinet. *Perfect lattices in Euclidean spaces*, volume 327. Springer Science & Business Media, 2003.

[9] David M Mount. The densest double-lattice packing of a convex polygon. *Discrete and Computational Geometry: Papers from the DIMACS Special Year*, 6:245–262, 1991.

[10] Achill Schürmann. Strict periodic extreme lattices. *Diophantine Methods, Lattices, and Arithmetic Theory of Quadratic Forms*, 587:185, 2013.

[11] Wolfram Research. *Mathematica 10.0.* Wolfram Research, Inc., Champaign, Illinois, 2014.