1 The Vertex-Vertex Lemma

Lemma 1.1. *Given an isolated minimum for the area $A(\theta)$ of the half-length parallelogram at θ_0, then $\mathbf{p_1}$ is the only vertex of the affine diameter to meet a vertex of K, and no vertices of the half-length parallelogram meet a vertex of K.*

Proof. To any vertex of K we may associate an entering angle and a leaving angle, the direction from which the sweeping affine diameter and the moving vertices of the half-length parallelogram enter and leave the vertex.

We work with a parameter t which is proportional to θ as in the analysis of the rate constants in section **??**, but may change across critical angles. Consider some configuration where some subset $\mathcal{P}$ of the vertices $\mathbf{p_2}$, $\mathbf{p_3}$, $\mathbf{p_4}$, $\mathbf{p_5}$, $\mathbf{p_6}$ meet vertices of K.

We may assume vertices of the half-length parallelogram do not coincide with the vertices of K for any sufficiently small neighborhood of in the domain of θ_0. Otherwise, we are in the situation described in Lemma **??** and there is a family of minima.

Since this configuration is an isolated local minimum of $A(t)$ or equivalently of $A(t)$, the left derivative of $A(t)$ is negative and the right derivative is positive. Then the stability condition **??** determines a family of motions which define a family of virtual polygons with the same affine diameter and the same half-length parallelogram but for which the area of the half length quadrilateral is constant. We will show that there is a virtual polygon that contains the vertices $\mathcal{P}$ in the interior of its edges.

We compute dA/dt to be

$$\begin{aligned}
&-\frac{1}{2}p_{21}\sin(\phi_3)\sin(\phi_2-\phi_4)\csc(\phi_2-\phi_3)+\frac{1}{2}p_{21}\sin(\phi_2)\sin(\phi_3-\phi_4)\csc(\phi_2-\phi_3)+\\
&\frac{1}{2}p_{22}\sin(\phi_2-\phi_4)\cos(\phi_3)\csc(\phi_2-\phi_3)-\frac{1}{2}p_{22}\sin(\phi_3-\phi_4)\cos(\phi_2)\csc(\phi_2-\phi_3)-\\
&\frac{1}{2}p_{61}\sin(\phi_5)\sin(\phi_4-\phi_6)\csc(\phi_5-\phi_6)+\frac{1}{2}p_{61}\sin(\phi_4-\phi_5)\sin(\phi_6)\csc(\phi_5-\phi_6)-\\
&\frac{1}{2}p_{62}\sin(\phi_4-\phi_5)\cos(\phi_6)\csc(\phi_5-\phi_6)+\frac{1}{2}p_{62}\sin(\phi_4-\phi_6)\cos(\phi_5)\csc(\phi_5-\phi_6)+\\
&\frac{1}{4}\sin(\phi_2)\sin(\phi_3-\phi_4)\csc(\phi_2-\phi_3)+\frac{1}{4}\sin(\phi_4-\phi_5)\sin(\phi_6)\csc(\phi_5-\phi_6)
\end{aligned} \tag{1}$$

We will refer to a collection of directions $\{\phi_i\}$ for which $dA/dt = 0$ as a set of support angles.

Provided the coincidences of the support angles $\phi_2 = \phi_3 \bmod \pi$ or $\phi_5 = \phi_6 \bmod \pi$ can be excluded, the derivative is a continuous function with of those angles. But these occur only when points $\mathbf{p}_i$ meet adjacent vertices of K, so the half-length parallelogram shares a full edge with K. By Lemma **??**....

If the left derivative is negative, so $dA/dt <= 0$ when evaluated at the entering, and similarly the right derivative is strictly positive, so the $dA/dt > 0$ then by the intermediate value principle, we can take the convex combination of the vector of entering angles and the vector of leaving angles and find a set of support angles that lie between them. By definition the area of the half-length parallelogram is preserved to first order as $dA/d\theta = 0$ if the vertices move along such an edge, so any increase in volume must be quadratic in t. However, the area lost by the discrepancy between the support angles and the leaving angles is linear in t.

In the case where the affine diameter does not meet a vertex, this is clear...

based on the signs of the we know there exists a set of slopes consistent with the property that increasing θ gives only a quadratic increase in area (the gain from the $d\theta = 0$ motion), but a linear decrease (the discrepancy in the the $d\theta$ slope and the slopes of the polygon.). See this by noticing that the

In the case where the affine diameter meets a vertex and $\mathbf{p_1}$ is fixed, a similar argument holds but care is needed to show that the en

In the case where the affine diameter meets a vertex and the moving end switches, and $\mathbf{p_1}$ switches, a similar argument holds, where there is a symmetry of the body....

The places where this argument breaks down are exactly when there are vertices of the half-length parallelogram or the moving vertex of the affine diameter that have parallel support. This corresponds to a ... $\square$