1 Introduction

This paper began as an investigation of the optimality of the double lattice packing for pentagons and heptagons. In [7], G. and W. Kuperberg describe a double lattice packing of congruent planar convex bodies with maximal density that can be constructed algorithmically[1]. As an example, they construct the densest double lattice packing for regular pentagons and show that it has a density of $(5-\sqrt{5})/3 = 0.92131\ldots$ which is the current record and possibly the best *general* packing of the plane by regular pentagons.

YK: what do you mean here by "algorithmically"? How is a nonpolygon represented computationally? WK: perhaps "procedurally" for "nice enough" bodies?

In the early part of the 2000s, there has been a significant push both theoretically and computationally to answer some of the most naive yet perplexing questions in packing [2] Along with the proof and formal verification of the Kepler conjecture [6], a number of other results have proved to be both illuminating and frustrating (Cohn-Elkies LP bounds, Lower bounds...). For packings by congruent anisotropic bodies, sharp results are limited mostly to the plane, where the best packings of all 0-symmetric bodies are achieved by lattices [3], and a series of sparce results in higher dimensions.

For non-centrally symmetric bodies, the problem of finding the best packing of regular pentagons serves as a toy model for problems like the best packing of regular tetrahedra. However, it is still not a tractable one. Explicit upper bounds for the packing of regular tetrahedra and octahedra are better than the trivial unity upper bound by minuscule margins [4]. A semidefinite programming approach has been suggested by Oliveira and Vallentin [8]. Though the SDP method has not yet yielded a nontrivial upper bound for packing of tetrahedra, it has been used to obtain an upper bound of 0.98103 on the density of regular pentagon packings. There remains a large gap between the highest density achieved for pentagon packings and this upper bound.

Even in the plane, it is an open question to find the global pessimal convex body, that is the shape that has the lowest maximum packing density. In the class of 0-symmetric bodies, it is Reinhardt [11] who conjectured that a smoothed octagon is the minimizer. In the class of general convex bodies, it is conjectured to be the regular heptagon. However, even though the conjectured maximum packing density of the regular heptagon is the maximal density double lattice, it has also resisted proofs its global optimality.

The regular pentagon and heptagons are cases of special interest, and we initially sought out to investigate whether their optimal double-lattice packing can be shown to be also optimal among a broader class of packings. We were able to show that these packings are optimal at least in some neighborhood in the space of all packings. Furthermore, we discovered that our method can be generalized to all convex polygons. We demonstrate that, while double lattices are in general not globally optimal, they are always at least locally optimal.

Theorem 1.1. *Given a convex planar polygon, there is a double lattice packing that is*

[1] For the case of polygons, this turns out to be linear in the number of vertices [10].

[2] for a background on packing problems, see [1] [2][5].

locally optimal. Modulo vertex condition...

In that light, we must establish the correct notion of neighborhood. This neighborhood should be broader than simply the Hausdorff distance between two packings. To that end, we recall a topology on the space of sequences ambient isometries

We recast the relevant portions of [7].

We prove a stability result for non-linear programming problems.

We characterize the pentagon.

We give a local parametrization of the neighborhood of the double lattice in the space of packings and give a characterization of the a correction function and show that the optimality of the local configuration problem implies the global density result.

2 Theoretical Considerations

2.1 Local Stability

Definition 2.1. *Let Ξ be a set of isometries. Its mean volume is the limit*

$$d(\Xi) = \lim_{r\to\infty} \frac{\mathrm{vol} B(0,r)}{|\{\xi \in \Xi : \xi(0) \in B(0,r)\}|}. \tag{1}$$

The upper and lower mean volumes are the corresponding limits superior and inferior. We say Ξ is a (r,R)-set if the point set $\{\xi(0) : \xi \in \Xi\}$ has a packing radius r at least r and a covering radius at most R.

We will look at packings of congruent copies of a convex body K. That is, every element of the packing is given by $\xi(K)$, where ξ is an isometry of Euclidean space. It will be convenient to assume that the reference body K is situated so that its interior contains the origin.

Definition 2.2. *Let K be a compact set with interior. We say that Ξ is admissible for K if the interiors of $\xi(K)$ and $\xi'(K)$ are disjoint for any two distinct isometries $\xi, \xi' \in \Xi$. We say furthermore that Ξ is saturated if there is no ξ such that $\Xi \cup \{\xi\}$ is again admissible.*

There are $r(K)$ and $R(K)$ such that when Ξ is admissible and saturated, then Ξ is a $(r(K), R(K))$-set.

Definition 2.3. *Given two (r', R')-sets Ξ and Ξ' of isometries, we define the premetric*

$$\begin{aligned}\delta_R(\Xi,\Xi') = \inf_{enum.} \sup\{&||\xi_i^{-1}\xi_j - \xi_i'^{-1}\xi_j'|| : \\ &i,j \text{ such that } ||\xi_i(0)-\xi_j(0)|| < 2R \text{ or } ||\xi_i'(0)-\xi_j'(0)|| < 2R\}.\end{aligned} \tag{2}$$

The infimum is over all enumerations $\mathbb{N} \to \Xi$ and $\mathbb{N} \to \Xi'$.

When $R > R'$, $\delta_R(\Xi, \Xi') = 0$ if and only if $\xi_i = \phi \xi'_i$ for some $\phi \in E(n)$ and some enumerations. Consider a body K. When $R > R(K)$, $\delta_R(\Xi, \Xi')$ is a metric on the space of admissible (r, R)-sets up to overall isometry, which includes the saturated sets as a subset.

Definition 2.4. *We say an admissible and saturated set Ξ is strongly extreme for K if it minimizes the mean volume among admissible elements in a neighborhood of Ξ.*

Note that the above definition is independent of R.

We note that there are packings that might be intuitively thought of as not locally optimal, but which are strongly extreme under our definition. One example is constructed by decorating a cylinder with a screw on its bottom base and a screw hole boring into its top base. Consider the packing where each cylinder is screwed into a cylinder below it, in such a way that the two cylinders are related to each other by a translation, and the screw is not completely screwed in. This creates a column of cylinders, and we arrange copies of this column in a triangular grid. Intuitively, since the screw is not completely screwed in, the density of the packing can be improved by screwing it in further. Since the interlayer spacing is related to the relative rotation between cylinders on the two layers, even an arbitrarily small consistent decrease in interlayer spacing will cause some layers to be rotated by at least some constant angle. Such a motion is not continuous in the topology we defined. Therefore, it is worth noting what local improvements to the density our results of strong extremality rule out and which are not ruled out.

However, our notion of strong extremality is stronger than previously introduced notions of local optimality local optimality of packings. The notions of an *extreme* lattice packing and a *periodic-extreme* periodic packing apply only to special classes of packings. We show that strong extremality, which applies more generally, implies extremality and periodic-extremality in these special classes.

Theorem 2.1. *If a lattice Λ is strongly extreme for K, then Λ is extreme for K [9].*

Theorem 2.2. *If a periodic set $\Xi = \{T_{\mathbf{l}}\xi_i : \mathbf{l} \in \Lambda, i = 1, \dots, N\}$ is strongly extreme, then it is periodic extreme for K [12].*

We now derive a general method for proving strong extremality which we will use in the following sections.

Definition 2.5. *Let Ξ be a countable set of isometries and fix an enumeration $\Xi = \{\xi_i : i \in \mathbb{N}\}$. Let $\mathcal{P}$ be a polyhedral complex whose underlying space is $\mathbb{R}^n$. For every face F of $\mathcal{P}$, let $I_F = \{i : \xi_i(0) \in F\}$. We say $\mathcal{P}$ is a honeycomb of Ξ if each n-face (cell) P is the convex hull of $\{\xi_i(0) : i \in I_P\}$.*

Theorem 2.3. *Let Ξ be admissible for K and let $\mathcal{P}$ be a honeycomb of Ξ. For every cell P, consider the optimization problem of minimizing $f_P(\Xi_P) = \operatorname{vol} \operatorname{conv}_{i \in I_P} \xi'_i(0)$ over the assignment of isometries ξ'_i, $i \in I_P$, such that this finite set is admissible. If $\xi'_i = \xi_i$, $i \in I_P$, is a local minimum for each cell P, then Ξ is strongly extreme.*

Theorem 2.4. *Let $g_F(\Xi_F)$ be a real-valued function over $\Xi_F = (\xi_i')_{i \in I_F}$ for each oriented $(n-1)$-faces (ridge) of $\mathcal{P}$, such that $g_F(\Xi_F) = -g_{-F}(\Xi_F)$, where $-F$ is the orientation-reversed version of F. If we replace $f_P(\Xi_P)$ in the previous theorem with $f_P'(\Xi_P) = f_P(\Xi_P) + \sum_{F \in \partial P} g_F(\Xi_F)$, then again, if $\xi_i' = \xi_i$, $i \in I_P$, is a local minimum for each cell P, then Ξ is strongly extreme.*

2.2 Double lattices

Definition 2.6. *A chord of a convex body K is a line segment whose endpoints lie on the boundary of K. A chord is an affine diameter if there is no longer chord parallel to it.*

Definition 2.7. *An inscribed parallelogram is a half-length parallelogram in the direction θ if one pair of edges is parallel with the line through the origin at an angle θ above the x-axis and their length is half the length of an affine diameter parallel to them.*

Note that, while the half-length parallelogram is not unique if K is not strictly convex, any non-unique half-length parallelograms associated to a particular direction have at least one half-length chord contained in an edge of K and are equivalent by sliding motions along of the chord along an edge of K. Thus, any two half-length parallelograms in the direction θ have equal area, and we can define that area as a function $A(\theta)$ of the direction.

Definition 2.8. *A cocompact discrete subgroup of the Euclidean group consisting of translations and point reflections is a double lattice if it includes at least one point reflection.*

A double lattice is generated by a lattice and a point reflection, or alternatively by three point reflections.

Theorem 2.5 (Kuperberg and Kuperberg)**.** *For a convex K, an admissible double lattice of smallest mean area has mean area $4\min_\theta A(\theta)$ and is generated by reflection about the vertices of a half-length parallelogram.*

Note that Kuperberg and Kuperberg make use of extensive parallelograms, inscribed parallelograms with edge length greater than half the affine diameter in their edge directions, but then restrict their analysis to the set of half-length parallelograms. Mount gives an explicit proof that it suffices to consider only the half-length parallelograms associated with affine diameters as a set valued function from θ.

For our purposes, it is illustrative to consider the algorithm used find the half-length parallelogram of minimal area for convex polygons, not only as a method of finding the best double lattice packing thereof, but also as a way to explore some of the configuration space of double lattice packings. More specifically, the affine diameter of a convex polygon is well behaved but with some non-trivial considerations as θ increases. The behavior of the vertices of the half-length parallelogram are described by an *interspersing property*; they are non-decreasing functions $[0, 2\pi) \to [0, 2\pi)$.

We begin with a simple proposition:

Proposition 2.1. *there is an affine diameter of a convex polygon K in every direction that meets a vertex.*

This is a consequence of the convexity K, for if a chord does not meet a vertex, it either lies within an edge or it meets two edges. If it lies within an edge, it is not an affine diameter. If it meets two edges, it is possible increase its length while parallel translating it in a non-decreasing direction of the cone defined by the two edges until it meets a vertex.

From this, it is possible to determine an initial affine diameter in a particular direction θ. This is done by extending a set of parallel rays in the direction θ from all the vertices into the interior of K and selecting the longest. However, it does not give the initial configuration of the vertices of the half-length parallelogram or the area thereof.

2.3 Tracking the affine diameter

In most situations, given an initial affine diameter in a particular direction, the affine diameter can be tracked by sweeping out the ray in a counter-clockwise direction from the initial vertex until it meets another vertex. From there, it is possible that the affine diameter continues to extend from the original vertex or that it begins to sweep out counter-clockwise from the opposite vertex.

The conditions for staying at the same vertex or switching to the opposite vertex are given exactly by the direction of the convex cone of the edges which the affine diameter will enter as it rotates. The vertex from which it sweeps is that which is closer to the vertex of the cone. In the special case where the affine diameter meets parallel edges, It is not uniquely determined, exactly because this cone is degenerate. However, in such cases, the non-unique behavior is restricted to sliding the affine diameter along the parallel edges, and as the affine diameter continues to rotate, it will meet a new cone, although the new cone may also be degenerate. Thus, aside from the degenerate sliding configurations, the affine diameter has a choice of a *moving end* and a *fixed end* with respect to an increase in the angle θ.

With regards to the double lattice packing, the affine diameter defines a column of convex bodies K, and as the affine diameter is swept out from a vertex, that vertex remains the point of contact between the bodies in the column.

2.4 Tracking the half length parallelogram

In order to generate the densest double lattice packing, one must also find the minimal area half-length parallelogram. This is done by moving between *critical angles* of the affine diameter: those angles where a vertex of the half-length parallelogram or both vertices of the affine diameter meet vertices of K. It is possible to tracking the motion via a parametrization of the slopes along which the vertices must travel between certain critical angles where the the area function becomes non-analytic. This is exactly where the moving end of the affine diameter or a vertex of the half length parallelogram meet a vertex of K.

Note that these also correspond to the degenerate situations not treated in Lemmas 2.1 and 2.2.

We parametrize the motion of the vertices $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_5$, $\mathbf{p}_6$ relative to a motion of the moving end of the affine diameter $\mathbf{p}_4$. The moving end of the affine diameter moves linearly, as do the vertices of the half-length parallelogram, as it they are constrained to be on the edges of K. The length of the chords is also a linear in t. Therefore, their motion is described by their initial position plus some constant velocity motion $\mathbf{v}_i$, provided that the vertices of the half-length parallelogram and the moving end of the affine diameter remain on their initial edges. If the affine diameter or one of the vertices of the half-length parallelogram encounter a vertex, the parametrization changes to account for the new direction of motion. In the case of the affine diameter, this may also require that the moving end become the fixed end and vice-versa. Away from those critical angles, we have the following:

$$\begin{aligned}
&\mathbf{p}_i(t) = \mathbf{p}_i^{inital} + k_i t \mathbf{v}_i \\
&\text{satisfying} \\
&\frac{1}{2}\mathbf{p}_4(t) - \mathbf{p}_1 = \mathbf{p}_3(t) - \mathbf{p}_2(t) \\
&\text{and} \\
&\frac{1}{2}\mathbf{p}_4 - \mathbf{p}_1 = \mathbf{p}_5(t) - \mathbf{p}_6(t)
\end{aligned} \tag{3}$$

This system can be solved for the rate constants k_i which give conditions on the motion of the parallelogram. We fix the affine diameter as a horizontal chord of length one, define variable motions of the point $\mathbf{p}_i$ at inclination ϕ_i and let the moving end of the affine diameter move at unit speed.

Solving this system yields rate constants

$$\begin{aligned}
&\alpha_1 = 0 \\
&{}_2 = \frac{1}{2}\sin(\phi_3 - \phi_4)\csc(\phi_2 - \phi_3) \\
&{}_3 = \frac{1}{2}\sin(\phi_2 - \phi_4)\csc(\phi_2 - \phi_3) \\
&{}_4 = 1 \\
&k_5 = \frac{1}{2}\sin(\phi_4 - \phi_6)\csc(\phi_5 - \phi_6) \\
&k_6 = \frac{1}{2}\sin(\phi_4 - \phi_5)\csc(\phi_5 - \phi_6)
\end{aligned} \tag{4}$$

Since the critical angles where the parametrization changes can be determined, the the minimum value of $A(\theta)$ between critical angles can be solved as an optimization problem determined completely by the data at the critical angles.

It is clear that there are points which have rate constant $k_i = 0$ only when there are vertices of the half-area parallelogram that are on edges parallel to the edge on which meets the moving end. Also, the functions become singular exactly when there are two vertices of the half-parallelogram sharing an edge. This is exactly when there is a free sliding motion that preserves area.

The densest double lattice packing of a convex polygon K can be constructed in time proportional to the number of vertices by an algorithm of Mount [10]. The goal of this paper is to show that this configuration is not only a local maximum of density among double lattices, but is in fact a local maximum in a broader sense, strong extremality.

2.5 General setup

To achieve this goal, we start by describing a honeycomb associated with the double lattice. Let K be a convex polygon and let $\mathbf{p}_2\mathbf{p}_3\mathbf{p}_5\mathbf{p}_6$ be a half-length parallelogram, such that $\mathbf{p}_3\mathbf{p}_2$ and $\mathbf{p}_5\mathbf{p}_6$ are half the length of and parallel to the affine diameter $\mathbf{p}_4\mathbf{p}_1$. The double lattice generated by reflections about the vertices of the parallelogram is Ξ and the subgroup of translations is the lattice Λ. Let $P = 0I_{\mathbf{p}_2}(0)I_{\mathbf{p}_6}(\mathbf{p}_1 - \mathbf{p}_4)$, then $\{\xi(P) : \xi \in \Xi\}$ are the cells of polyhedral complex which is a honeycomb for Ξ. Note that the optimization problem of minimizing $f_{\xi(P)}$ over ξ'_i, $i \in I_{\xi(P)}$, is mathematically equivalent for every $\xi \in \Xi$. Therefore, to show that Theorem X applies, it suffices to show that $\xi'_i = \xi_i$, $i \in I_P$, is a local optimum over admissible assignments of ξ'_i, $i \in I_P$.

For every convex body K and double lattice Ξ we now have a concrete optimization problem to solve: we wish to minimize the area of the quadrilateral $\xi'_0(0)\xi'_6(0)\xi'_1(0)\xi'_2(0)$ subject to the constraints that $\xi'_i(K)$ and $\xi'_j(K)$ do not overlap. Since the objective and the constraints are invariant under common isometry, we may fix $\xi'_i = \xi_i$ for one i. We parametrize $\xi'_i = T_{\mathbf{r}_i}\xi_i R_{\theta_i}$, where R_θ is a rotation by θ about the origin, and $T_{(x,y)}$ is a translation by $\mathbf{r}_i$. Since we are only interested in certifying that the initial configuration is a local minimum, we can replace the constraints with ones that are equivalent in the neighborhood.

Lemma 2.1. *Let K and K' be two polygons that intersect at a segment. The endpoints of the segments are $\mathbf{x}$ a vertex of K and $\mathbf{y}$ a vertex of K'. Let $\mathbf{y}\mathbf{y}'$ and $\mathbf{x}\mathbf{x}'$ be the edges of K and K' containing the intersection. Let $\mathbf{x}'\mathbf{y}\mathbf{x}\mathbf{y}'$ be oriented counterclockwise from the point of view of the interior of K (otherwise switch K and K'). There is some $\epsilon > 0$ such that whenever $||\xi||, ||\xi'|| < \epsilon$, then $\xi(K)$ and $\xi'(K')$ have disjoint interiors if and only if $\alpha(\xi(\mathbf{x})\xi(\mathbf{x}')\xi'(\mathbf{y})) \geq 0$ and $\alpha(\xi'(\mathbf{y}')\xi'(\mathbf{y})\xi(\mathbf{x})) \geq 0$, where α is the signed area of the oriented triangle.*

Lemma 2.2. *Let K and K' be two polygons that intersect at a point and not at a segment. The intersection point $\mathbf{y}$ is a vertex of one polygon, which we let be K', and sits in the relative interior of an edge $\mathbf{x}'\mathbf{x}$ of K, oriented counter clockwise from the point of view of* *the interior of K. There is some $\epsilon > 0$ such that whenever $||\xi||, ||\xi'|| < \epsilon$, then $\xi(K)$ and $\xi'(K')$ have disjoint interiors if and only if $\alpha(\xi(\mathbf{x})\xi(\mathbf{x}')\xi'(\mathbf{y})) \geq 0$.*

Note that the case of an intersection at a point that is a vertex of both polygons is not treated. In the first two cases we treat, the regular pentagon and the regular heptagon, there are no such intersections. When we generalize the calculation to all convex polygons, we will show that such an intersection only occurs in special cases that can be treated separately.

2.6 other theorems

We will show that optimization problems we obtain fall into a convenient form, where linear stability holds along all but one direction. Along the direction of vanishing linear stability, the construction of Kuperberg and Kuperberg will be shown to guaranty stability.

Theorem 2.6. *Programs satisfying conditions (*) have a local maximum at 0.*

3 Calculation

3.1 Pentagons

Let us fix a regular pentagon $K = \mathrm{conv}\{\mathbf{k}_i : i = 0, \dots 4\}$, where $\mathbf{k}_i = R_{2\pi i/5}(1, 0)$. In this subsection, we do all the calculations in the extension field $\mathbb{Q}(u, v)$, where $u = \cos \pi/5$ and $v = \sin \pi/5$.

One minimum-area half-length parallelogram corresponds to the affine diameter $\mathbf{p}_1\mathbf{p}_4$, where $\mathbf{p}_1 = \mathbf{k}_0$ and $\mathbf{p}_4 = \frac{1}{2}(\mathbf{k}_2 + \mathbf{k}_3)$. The vertices of the parallelogram are given by $\mathbf{p}_2 = \frac{1}{4}\mathbf{k}_0 + \frac{3}{4}\mathbf{k}_1$, $\mathbf{p}_3 = \frac{3-2u}{4}\mathbf{k}_1 + \frac{1+2u}{4}\mathbf{k}_2$, $\mathbf{p}_5 = \frac{1+2u}{4}\mathbf{k}_3 + \frac{3-2u}{4}\mathbf{k}_4$, and $\mathbf{p}_6 = \frac{3}{4}\mathbf{k}_4 + \frac{1}{4}\mathbf{k}_0$.

The four pentagons that surround our primitive honeycomb cell are $\xi_i(K)$, $i = 0, 1, 2, 6$, where $\xi_0 = \mathrm{Id}$, $\xi_1 = T_{\mathbf{p}_1 - \mathbf{p}_4}$, $\xi_2 = I_{\mathbf{p}_2}$, and $\xi_6 = I_{\mathbf{p}_6}$. We are interested in showing that the assignment $\xi'_i = \xi_i$, $i = 0, 1, 2, 6$ locally minimizes the area of the quadrilateral $\xi_0(0)\xi_6(0)\xi_1(0)\xi_2(0)$, subject to the nonoverlap constraints. As explained in the previous section, we may fix $\xi'_1 = \xi_1$ and replace the nonoverlap constraints by signed area constraints. We obtain the following optimization problem:

$$
\begin{aligned}
\text{minimize } & f(z) = \alpha(\xi_0'(0), \xi_1'(0), \xi_2'(0)) - \alpha(\xi_0'(0), \xi_1'(0) + \xi_6'(0)) \\
\text{subj. to } & g_1(z) = \alpha(\xi_0'(\mathbf{k}_1), \xi_0'(\mathbf{k}_0), \xi_2'(\mathbf{k}_1)) \geq 0 \\
& g_2(z) = \alpha(\xi_2'(\mathbf{k}_1), \xi_2'(\mathbf{k}_0), \xi_0'(\mathbf{k}_1)) \geq 0 \\
& g_3(z) = \alpha(\xi_0'(\mathbf{k}_0), \xi_0'(\mathbf{k}_4), \xi_6'(\mathbf{k}_4)) \geq 0 \\
& g_4(z) = \alpha(\xi_6'(\mathbf{k}_0), \xi_6'(\mathbf{k}_4), \xi_0'(\mathbf{k}_4)) \geq 0 \\
& g_5(z) = \alpha(\xi_1'(\mathbf{k}_2), \xi_1'(\mathbf{k}_1), \xi_2'(\mathbf{k}_2)) \geq 0 \\
& g_6(z) = \alpha(\xi_2'(\mathbf{k}_2), \xi_2'(\mathbf{k}_1), \xi_1'(\mathbf{k}_2)) \geq 0 \\
& g_7(z) = \alpha(\xi_1'(\mathbf{k}_4), \xi_1'(\mathbf{k}_3), \xi_6'(\mathbf{k}_3)) \geq 0 \\
& g_8(z) = \alpha(\xi_6'(\mathbf{k}_4), \xi_6'(\mathbf{k}_3), \xi_1'(\mathbf{k}_3)) \geq 0 \\
& g_9(z) = \alpha(\xi_1'(\mathbf{k}_3), \xi_1'(\mathbf{k}_2), \xi_0'(\mathbf{k}_0)) \geq 0,
\end{aligned}
\tag{5}
$$

where $\xi_i' = T_{(x_i,y_i)}\xi_i R_\theta$ for $i = 0, 2, 6$, and $\xi_1' = \xi_1$. We adopt a condensed notation for the free variables $X = (x_0, y_0, \theta_0, x_2, y_2, \theta_2, x_6, y_6, \theta_6)$.

We consider the linearization of 5 around the point $z = 0 \in \mathbb{R}^9$. This gives a problem of the form

$$
\text{minimize } c \cdot z \text{ subject to } Gz \geq 0, \tag{6}
$$

where $c \in \mathbb{R}^9$, $G \in \mathbb{R}^{9\times 9}$ and we use the linear programming notation ≥ 0 to denote a vector lying in the closed positive orthant. We can show by direct calculation a vector $\eta > 0$ lying in the open positive orthant exists such that $c = \eta^T G$. By the fundamental theorem of linear algebra, this observation implies that $Gz \geq 0$ and $c \cdot z \leq 0$ if and only if $Gz = 0$ and $c \cdot z = 0$. We can show that $\operatorname{rank} G = 8$, and so the program 6 is minimized exactly by the null space of G and is suboptimal elsewhere in the cone $Gz \geq 0$. The null space corresponds precisely to the rearrangement given by choosing a nearby half-length parallelogram in the double lattice construction to the minimum-area one. Let $z^{(0)}$ generate the null space, then $t_0^{(0)} = t_2^{(0)} = t_6^{(0)} = 0$. We can verify directly that $f(tz^{(0)})$ is a quadratic function of t minimized at $t = 0$, and that $g_r(tz^{(0)}) = 0$ identically for $r = 1, \ldots, 9$. Indeed, perturbing the half-length parallelogram away from the minimum-area one increases the area of the resulting cell and maintains all the contacts.

Therefore, 5 satisfies all the conditions of **??**, and we have:

Theorem 3.1. *The optimal double-lattice packing of regular pentagons, illustrated in Figure X, is strongly extreme.*

3.2 Heptagons

The calculation for the regular heptagon starts out the same as the calculation presented above for regular pentagons. However, it will turn out that the linear program equivalent to 6 is in this case not minimized at $z = 0$, and so we will need to add auxiliary cost functions to the area as allowed for in **??**.

We fix a regular heptagon $K = \mathrm{conv}\{\mathbf{k}_i : i = 0, \dots 6\}$, where $\mathbf{k}_i = R_{2\pi i/7}(1, 0)$. In this case, our calculations are performed in the extension field $\mathbb{Q}(u, v)$, where $u = \cos \pi/7$ and $v = \sin \pi/7$. A minimum-area half-length parallelogram corresponds to the affine diameter $\mathbf{p}_1\mathbf{p}_4$, where $\mathbf{p}_1 = \mathbf{k}_0$ and $\mathbf{p}_4 = \frac{1}{2}(\mathbf{k}_3 + \mathbf{k}_4)$. The vertices of the parallelogram are given by $\mathbf{p}_2 = (1-a)\mathbf{k}_1 + a\mathbf{k}_2$, $\mathbf{p}_3 = (1-b)\mathbf{k}_2 + b\mathbf{k}_3$, $\mathbf{p}_5 = b\mathbf{k}_4 + (1-b)\mathbf{k}_5$, and $\mathbf{p}_6 = a\mathbf{k}_5 + (1-a)\mathbf{k}_6$.

The four heptagons that surround our primitive honeycomb cell are again $\xi_i(K)$, $i = 0, 1, 2, 6$, where $\xi_0 = \mathrm{Id}$, $\xi_1 = T_{\mathbf{p}_1 - \mathbf{p}_4}$, $\xi_2 = I_{\mathbf{p}_2}$, and $\xi_6 = I_{\mathbf{p}_6}$. We will investigate whether the assignment $\xi'_i = \xi_i$, $i = 0, 1, 2, 6$ locally minimizes the area of the quadrilateral $\xi_0(0)\xi_6(0)\xi_1(0)\xi_2(0)$, subject to the nonoverlap constraints. Again, we fix $\xi'_1 = \xi_1$ and replace the nonoverlap constraints by signed area constraints. We obtain the following optimization problem:

$$
\begin{aligned}
\text{minimize } & f(z) = \alpha(\xi'_0(0), \xi'_1(0), \xi'_2(0)) - \alpha(\xi'_0(0), \xi'_1(0) + \xi'_6(0)) \\
\text{subj. to } & g_1(z) = \alpha(\xi'_0(\mathbf{k}_2), \xi'_0(\mathbf{k}_1), \xi'_2(\mathbf{k}_1)) \geq 0 \\
& g_2(z) = \alpha(\xi'_2(\mathbf{k}_2), \xi'_2(\mathbf{k}_1), \xi'_0(\mathbf{k}_1)) \geq 0 \\
& g_3(z) = \alpha(\xi'_0(\mathbf{k}_6), \xi'_0(\mathbf{k}_5), \xi'_6(\mathbf{k}_6)) \geq 0 \\
& g_4(z) = \alpha(\xi'_6(\mathbf{k}_6), \xi'_6(\mathbf{k}_5), \xi'_0(\mathbf{k}_6)) \geq 0 \\
& g_5(z) = \alpha(\xi'_1(\mathbf{k}_3), \xi'_1(\mathbf{k}_2), \xi'_2(\mathbf{k}_3)) \geq 0 \\
& g_6(z) = \alpha(\xi'_2(\mathbf{k}_3), \xi'_2(\mathbf{k}_2), \xi'_1(\mathbf{k}_3)) \geq 0 \\
& g_7(z) = \alpha(\xi'_1(\mathbf{k}_5), \xi'_1(\mathbf{k}_4), \xi'_6(\mathbf{k}_4)) \geq 0 \\
& g_8(z) = \alpha(\xi'_6(\mathbf{k}_5), \xi'_6(\mathbf{k}_4), \xi'_1(\mathbf{k}_4)) \geq 0 \\
& g_9(z) = \alpha(\xi'_1(\mathbf{k}_4), \xi'_1(\mathbf{k}_3), \xi'_0(\mathbf{k}_0)) \geq 0.
\end{aligned}
\tag{7}
$$

The linearization of 7 around the point $z = 0 \in \mathbb{R}^9$ gives

$$\text{minimize } c \cdot z \text{ subject to } Gz \geq 0. \tag{8}$$

Unfortunately, **??** is unbounded. This can be shown by producing some z_u such that $c \cdot z_u < 0$ and $Gz_u \geq 0$. In the dual setting, this implies that there is no η such that $c = \eta^T G$ and $\eta > 0$.

Due to **??**, we are allowed to modify the cost function $f(z)$ by adding auxiliary functions. In order for the new problem to be locally minimized, we need the new gradient c' to lie in the cone $\{\eta^T G : \eta > 0\}$. We will take the following simple form for our modified problem

$$
\begin{gathered}
\text{minimize } f'(z) = f(z) + \sum_{r=1}^{9} \mu_r g_r(z) \\
\text{subj. to same constraints as in 7.}
\end{gathered}
\tag{9}
$$

For a cell $\xi(P)$ other than the primitive cell P, the modified version is the same as 9, except we replace ξ'_i everywhere with $\xi'_i = \xi \circ \xi'_i$. Since the problem is invariant under common isometry, this is equivalent for all the cells and it is enough to show that 9 is locally minimized. Note that $\xi_2 \circ \xi_0 = \xi_2$ and $\xi_2 \circ \xi_2 = \xi_0$, so $g_1^P(z) = g_2^{\xi_2(P)}(z)$ and $g_2^P(z) = g_1^{\xi_2(P)}(z)$. Therefore, if $\mu_1 = -\mu_2$, the auxiliary addition $g_{F_2}(z) = \mu_1 g_1(z) + \mu_2 g_2(z)$ is equal in magnitude and opposite in sign for the two cells P and $\xi_2(P)$ sharing the face F_2, as required in **??** for an auxiliary function attached to a cell face. By similar observation, we note that we should have $\mu_3 = -\mu_4$, $\mu_5 = -\mu_6$, and $\mu_7 = \mu_8$. The term $\mu_9 g_9(z)$ does not cancel with any neighboring cells, so we set $\mu_9 = 0$. A choice for μ_r satisfying these condition and such that c' lies in the cone $\{\eta^T G : \eta > 0\}$ exists if and only if there is some η such that $c = \eta^T G$ and $\eta_1 + \eta_2 > 0$, $\eta_3 + \eta_4 > 0$, $\eta_5 + \eta_6 > 0$, $\eta_7 + \eta_8 > 0$, and $\eta_9 > 0$. We can show directly that such η exists.

We now have that $Gz \geq 0$ and $c' \cdot z \leq 0$ if and only if $Gz = 0$ and $c' \cdot z = 0$. The rank of the constraint matrix is again rank$G = 8$, and so the program 9 is minimized exactly at the one-dimensional null space of G. Let $z^{(0)}$, then we can verify directly that $f(tz^{(0)})$ is a quadratic function of t minimized at $t = 0$, and that $g_r(tz^{(0)}) = 0$ identically for $r = 1, \ldots, 9$.

Therefore, 9 satisfies all the conditions of **??**, and we have:

Theorem 3.2. *The optimal double-lattice packing of regular heptagons, illustrated in Figure X, is strongly extreme.*

3.3 General polygons

The structure of the solution in the cases of pentagons and heptagons suggests that it might be possible to extend the result to general convex polygons. Assuming that the minimum-area half-length parallelogram belongs to the generic case, the only data about the polygon that enters into the calculation are the following:

1. The coordinates of vertices of the minimum-area half-length parallelogram and the vertices of the corresponding affine diameter.

2. The direction of the polygon edges on which the vertices above lie, except for the end of the affine diameter that lies on a polygon vertex.

3. The distance and direction along the edge from those points to the nearest polygon vertex.

4 Formal methods

4.1 symbolic computation in an extension field

The Wolfram System supports symbolic computation over extension fields via its pattern matching system.

5 Slicing nonlinear programs

A non-linear programing problem satisfying certain conditions can be certified as locally optimal by a linear programming problem. For the geometric problems considered, there are *a priori* configurations given by the maximal density configurations on a subspace of configuration space, namely subsets of the double lattice packings. To produce a certificate of local optimality for this type of problem it is possible to parametrize a neighborhood of the conjectured optimal configuration and analyze the associated non-linear programming problem

$$\max_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0, r \in I$$

in a neighborhood of 0. An appropriate choice of parametrization allows the full non-linear program to be sliced into a one-parameter family of non-linear programs that are subordinate to the linearization of the main program at 0. The following conditions are required.

Conditions 5.1. [3]

1. *Let I be a finite index set.*
2. *Let e_1 be the standard unit vector $\{1, 0, \ldots, 0\}$ in $\mathbb{R}^n$.*
3. *For r in I, let f and g_r be analytic functions on a neighborhood of 0.*
4. *Assume $f(0) = g_r(0) = 0$ for all r in I.*
5. *Let $F(t) = \nabla f(te_1)$.*
6. *Let $G_r(t) = \nabla g_r(te_1)$.*
7. *Assume the linear program*

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \textit{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

   has a bounded solution and that the maximum is attained at 0.

8. *Assume that the set of solutions in $\mathbb{R}^n$ to*

$$F(0) \cdot x = 0 \textit{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

   is

$$E := \{te_1 : t \in \mathbb{R}\}.$$

[3] These are the conditions that are required for the packing problems addressed. There are a number of ways they might be weakened, e.g. the condition that E be 1-dimensional is not essential.

9. *Let H be the orthogonal complement of E so that $\mathbb{R}^n = E \oplus H$.*

10. *Assume there is an $\epsilon > 0$ so the functions $g_r(te_1) = 0$ for all $t \in (-\epsilon, \epsilon)$, for all r in I.*

11. *Assume $\frac{\partial f}{\partial t}(0) = 0$, $\frac{\partial^2 f}{\partial t^2}(0) < 0$.*

Lemma 5.1. *Given Conditions 5.1, the linear program*

$$\max_{x \in H} F(0) \cdot x \textit{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

has a unique maximum at $x = 0$

Proof. By conditions 7 and 8, the linear program

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on E. The feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I \text{ and } x \in H\}$ is a subset of the feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I\}$. Thus, the program

$$\max_{x \in H} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on the non-empty intersection

$$E \cap \{x : G_r(0) \cdot x \geq 0, r \in I\} \cap H = 0.$$

□

Definition 5.1. *A* finitely generated cone *is a subset of $\mathbb{R}^n$ which is the non-negative span of a finite set of non-zero vectors $\{v_1, \ldots, v_m\}$ in $\mathbb{R}^n$, which are called the* generators *of the cone.*

Definition 5.2. *A* conical linear program *is a linear program with a constraint set that is a finitely generated cone.*

The linear programs described throughout this section are always constrained to be on the intersection of half-spaces with 0 on the boundary. These are conical programs.

Definition 5.3. *For a cone C, the set $C^p := \{x \in \mathbb{R}^n : v \cdot x \leq 0$ for all $v \in C\}$ is the* polar cone *of C.*

Lemma 5.2. *A conical linear program with $F \neq 0$ given by*

$$\max_{x \in \mathbb{R}^n} F \cdot x \textit{ subject to } G_r \cdot x \geq 0, r \in I$$

(a) has a unique[4] *maximum at* $x = 0$ *iff* F *is in the interior of the polar cone* C^p *of* $C = \{x : G_r \cdot x \geq 0, r \in I\}$ *(b) has a bounded solution iff* F *is in the polar cone* C^p *of* $C = \{x : G_r \cdot x \geq 0, r \in I\}$ *and attains its maximum exactly on the span of the generators* v_i *such that* $F \cdot v_i = 0$.

Proof. If F is in the interior of the polar cone C^p, then $F \cdot v_i < 0$ for all generators v_i. Therefore $F \cdot x$ is uniquely maximized in C at the vertex. If F is on the boundary of the polar cone, then $F \cdot x$ is maximized in C exactly on the span of the generators v_i for which $F \cdot v_i = 0$ as $F \cdot v_j < 0$ otherwise. If F is outside the polar cone, then $F \cdot v_i > 0$ for some generator v_i. Then $F \cdot x$ is unbounded in C. □

Lemma 5.3. *Given Conditions 5.1, there exists* $\epsilon > 0$ *such that for all* t *in* $(-\epsilon, \epsilon)$*, the linear program*

$$\max_{y_t \in H} F(t) \cdot y_t$$

subject to

$$G_r(t) \cdot y_t \geq 0, r \in I$$

has a unique maximum at $y_t = 0$.[5]

Proof. The program for $t \in (-\epsilon, \epsilon)$, for y_t in H, for each fixed t in $(-\epsilon, \epsilon)$, for some $\epsilon > 0$, can be written as a conical program on all of $\mathbb{R}^n$ with a cone C_t in $\mathbb{R}^n$ of co-dimension ≥ 1 by introducing further constraints $e_1 \cdot y_t \geq 0$ and $-e_1 \cdot y_t \geq 0$. By Lemmas 5.1 and 5.2, $F(0)$ is in the polar cone of $C_0 = \{y_0 : G_r(0) \cdot y_0 \geq 0, e_1 \cdot y_0 \geq 0, -e_1 \cdot y_0 \geq 0\}$. As $f, g_r \in C^\omega$, the condition of $F(t)$ being in the interior of the polar cone C_t^p is open and the condition of the feasible set $C_t = \{y_t : G_r(t) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0\}$ being conical is open.[6] Therefore, by Lemma 5.2 the program has a unique maximum at $y_t = 0$ for each fixed t in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. □

Lemma 5.4. *Given Conditions 5.1 and* ϵ *as in Lemma 5.3, for all* $t \in (-\epsilon, \epsilon)$ *there exists* $\delta(t) > 0$ *and a cube* $Q(t) \subset \mathbb{R}^n$ *of side length* $2\delta(t)$ *such that*

$$\{(F(t) + Q(t)) \cap (\partial(C_t^p) + Q(t))\} = \emptyset.$$

Proof. This follows from Lemma 5.3, which shows $F(t)$ is in the interior of the polar cone C_t^p. Then $F(t)$ and the boundary of C_t^p can be separated and the existence of Q is trivial. □

[4]The maximum satisfies a stronger uniqueness condition. It is stable under perturbations of F and G_k.

[5]Here y_t is a dummy variable and does not depend on t. It is labeled y_t to ease later exposition.

[6]The relationships between the constraint cone, the generators v_i and the constraint gradients G_k are nontrivial as t varies, but the openness of this condition follows from the continuity of the distance function.

Corollary 5.1. *Given Conditions 5.1 and ϵ as in Lemma 5.3, for all $t \in (-\epsilon, \epsilon)$,*

$$(F(t) + \Delta) \cdot y_t \leq 0$$

whenever y_t satisfies

$$(G_r(t) + \Delta_r) \cdot y_t \geq 0, r \in I \text{ and } e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0$$

where Δ and Δ_r are any points in the $2\delta(t)$-cube $Q(t)$ and y_t is in H.

Proof. By Lemma 5.4, $F(t) + \Delta$ is in the interior of the polar cone $C^p_{t,\Delta}$, where $C_{t,\Delta} = \{y_t : (G_r(t) + \Delta_r) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0, r \in I\}$. □

Lemma 5.5. *Given Conditions 5.1 and ϵ as in Lemma 5.3, for all $t \in (-\epsilon, \epsilon)$, let $y_t = x - te_1 \in H$. Choose $\Delta = \Delta(y_t)$ and $\Delta_r = \Delta_r(y_t)$ in the $2\delta(t)$-cube $Q(t)$ to be the corner given by the sign of $x - te_1 = y_t$. Then there is an ϵ_t for which*

$$(F(t) + \Delta(y_t)) \cdot y_t \leq 0 \implies f(x) - f(te_1) \leq 0$$

and

$$(G_r(t) + \Delta_r(y_t)) \cdot y_t \leq 0 \implies g_r(x) - g_r(te_1) = g_r(x) \leq 0$$

for all $\|y_t\| \leq \epsilon_t$.

Proof. This follows from the local expansions of the nonlinear program. By this choice of $\Delta(y_t)$ and $\Delta_r(y_t)$,

$$\begin{aligned} f(x) - f(te_1) &= F(t) \cdot (x - te_1) + O(t^2) = F(t) \cdot y_t + O(t^2) \\ &\leq F(t) \cdot y_t + \delta(t)\|y_t\|_1 = (F(t) + \Delta(y_t)) \cdot y_t \end{aligned}$$

and using condition 10,

$$\begin{aligned} g_r(x) = g_r(x) - g_r(te_1) &= G_r(t) \cdot (x - te_1) + O(t^2) = G_r(t) \cdot y_t + O(t^2) \\ &\leq G_r(t) \cdot y_t + \delta(t)\|y_t\|_1 = (G_r(t) + \Delta_r(y_t)) \cdot y_t. \end{aligned}$$

□

By Lemma 5.4 and Corollary 5.1, for t in $(-\epsilon, \epsilon)$, the program

$$\max_{y_t \in H} (F(t) + \Delta) \cdot y_t \text{ subject to } (G_r + \Delta_r) \cdot y_t$$

is uniquely maximized at $y_t = 0$ for any choice of Δ, Δ_r in the $2\delta(t)$ cube $Q(t)$. Combined with Lemma 5.5, there is an ϵ_t neighborhood of 0 where $f(y_t + te_1)$ is less than $f(te_1)$ on $\cup_{\Delta_r \in Q(t)} \{y_t : (G_r + \Delta_r) \cdot y_t \geq 0, r \in I, y_t \in H\}$, which contains the feasible set $\{y_t : g_r(y_t + te_1) \geq 0, r \in I, y_t \in H\}$. Therefore the nonlinear programs $f(y_t + te_1)$ subject to $g_r(y_t + te_1) \geq 0$, $y_t \in H$, which are parameterized by t in $(-\epsilon, \epsilon)$, have local maxima at $y_t = 0$. This gives the following:

Theorem 5.1. *Given Conditions 5.1, a fixed t in $(-\epsilon, \epsilon)$ and choosing Δ and Δ_r as in Lemma 5.5, for x satisfying $g_r(x) \geq 0$ for all r in I and $y_t = x - te_1$ in H, there exist linear programs*[7]

$$\max_{y_t \in H} (F(t) + \Delta(y_t)) \cdot y_t \text{ subject to } (G_r(t) + \Delta_r(y_t)) \cdot y_t \geq 0$$

that give solutions to the nonlinear programs

$$\max_{x \in H + te_1} f(x) \text{ subject to } g_r(x) \geq 0$$

in an ϵ_t neighborhood of te_1 in $H + te_1$. □

By choice of a sufficiently small ϵ and a minimal[8] non-zero ϵ_t, Theorem 5.1 gives an open neighborhood of 0 in which the maximum value of the original nonlinear program occurs on E. The conditions for the first and second t-derivatives at 0 shows 0 to be a local maximum for the nonlinear program

$$\max_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0.$$

Theorem 5.2. *A nonlinear program satisfying Conditions 5.1 has an isolated local maximum at 0 with f(0) = 0.* □

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