1 The construction of the densest double lattice packing

1.1 Double lattices

Definition 1.1. *A chord of a convex body K is a line segment whose endpoints lie on the boundary of K. A chord is an affine diameter if there is no longer chord parallel to it.*

Definition 1.2. *An inscribed parallelogram is a half-length parallelogram in the direction θ if one pair of edges is parallel with the line through the origin at an angle θ above the x-axis and their length is half the length of an affine diameter parallel to them.*

Note that any two half-length parallelograms in the direction θ have equal area, and we can define that area as a function $A(\theta)$ of the direction.

Definition 1.3. *A cocompact discrete subgroup of the Euclidean group consisting of translations and point reflections is a double lattice if it includes at least one point reflection.*

A double lattice is generated by a lattice and a point reflection, or alternatively by three point reflections.

Theorem 1.1 (Kuperberg and Kuperberg)**.** *For a convex K, an admissible double lattice of smallest mean area has mean area $4 \min_\theta A(\theta)$ and is generated by reflection about the vertices of a half-length parallelogram.*

Kuperberg and Kuperberg make use of extensive parallelograms, inscribed parallelograms with edge length greater than half the affine diameter in their edge directions, but make use of half-length parallelograms. Mount gives an explicit proof that, when minimizing the volume of an extensive parallelogram, it suffices to consider only the half-length parallelograms associated with affine diameters as a function of θ.

From the definition of an extensive parallelogram, one observes that this is exactly the condition that produces an admissible, connected?tight? double lattice packing, and that the density of the packing increases as the area of the extensive parallelogram decreases. Thus, there is a direct correspondence between the admissible motions of the double lattice packing and the motions of inscribed, extensive parallelograms.

For example, sliding motions of a double lattice packing that preserves density and contact correspond to motions of an inscribed extensive parallelogram that preserve area.

For our purposes, it is illustrative to consider the algorithm used find the half-length parallelogram of minimal area not only as a method of finding the best double lattice packing, but also as a way to explore part of the configuration space of double lattices. More specifically, the behavior of the affine diameter is fairly well behaved but non-trivial as θ increases while the behavior of the vertices of the half parallelogram are described by an *interspersing property*; they are non-decreasing functions $[0, 2\pi) \to [0, 2\pi)$.

We begin with a simple proposition:

Proposition 1.1. *there is an affine diameter of a convex polygon K in every direction that meets a vertex.*

This is a consequence of the convexity K, for if a chord does not meet a vertex, it either lies within an edge or it meets two edges. If it lies within an edge, it is not an affine diameter. If it meets two edges, it is possible increase its length while parallel translating it in a non-decreasing direction of the cone defined by the two edges until it meets a vertex.

From this, it is possible to determine an initial affine diameter in a particular direction. Furthermore, this is done by geometrically satisfying procedure of extending a set of parallel rays in a from all vertices into the interior of K and selecting the longest. However, it does not give

1.2 Tracking the affine diameter

Walking around the vertices

Aside from the degenerate sliding configurations, the affine diameter has a choice of a *moving end* and a *fixed end* with respect to an increase in θ.

Parallel edges and non-uniqueness, a sliding motion.

1.3 Tracking the affine diameter

tracking the half length parallelogram

1.4 critical angles

In order to generate the densest double lattice packing, one must also find the minimal area half length parallelogram. This is done by moving between critical angles of the affine diameter where a vertex of the half-length parallelogram or both vertices of the affine diameter meet vertices of K.

Tracking the motion via a parametrization of the slopes.

Note that these also correspond to the degenerate situations for our stability condition.

1.5 stability condition

At any particular configuration, as *theta* varies, there is a relationship between the linear motions of the points given by the system of equations defining the half-length parallelogram. That is, the

Consider the vertex of the half length parallelogram. As theta varies,

We parametrize the motion of the vertices $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_5$, $\mathbf{p}_6$ relative to a motion of the moving end of the affine diameter $\mathbf{p}_4$.

The moving end of the affine diameter moves linearly, as do the vertices of the half-length parallelogram, as it they are constrained to be on the edges of K. The length of the chords is also a linear function in θ. Therefore, their motion is described by their initial position plus some constant velocity motion $\mathbf{v}_i$, provided that the vertices of the half-length parallelogram and the moving end of the affine diameter remain on their initial edges.

$$\begin{aligned}
&\mathbf{p}_i(t) = \mathbf{p}_i^{inital} + k_i t \mathbf{v}_i \\
&\text{satisfying} \\
&\frac{1}{2}\mathbf{p}_4(t) - \mathbf{p}_1 = \mathbf{p}_3(t) - \mathbf{p}_2(t) \\
&\text{and} \\
&\frac{1}{2}\mathbf{p}_4 - \mathbf{p}_1 = \mathbf{p}_5(t) - \mathbf{p}_6(t)
\end{aligned} \tag{1}$$

This system can be solved for the rate constants k_i which give conditions on the motion of the parallelogram. We fix the affine diameter as a horizontal chord of length one, define variable motions of the point $\mathbf{p}_i$ at inclination ϕ_i and let the moving end of the affine diameter move at unit speed.

Solving this system yields rate constants

$$\begin{aligned}
k_1 &= 0 \\
k_2 &= \frac{1}{2}\sin(\phi_3 - \phi_4)\csc(\phi_1 - \phi_3) \\
k_3 &= \frac{1}{2}\sin(\phi_1 - \phi_4)\csc(\phi_1 - \phi_3) \\
k_4 &= 1 \\
k_5 &= \frac{1}{2}\sin(\phi_4 - \phi_6)\csc(\phi_5 - \phi_6) \\
k_6 &= \frac{1}{2}\sin(\phi_4 - \phi_5)\csc(\phi_5 - \phi_6)
\end{aligned} \tag{2}$$

The densest double lattice packing of a convex polygon K can be constructed in time proportional to the number of vertices by an algorithm of Mount [**?**]. The goal of this paper is to show that this configuration is not only a local maximum of density among double lattices, but is in fact a local maximum in a broader sense, strong extremality.