0.1 Local Stability

Definition 0.1. *Let Ξ be a set of isometries. Its mean volume is the limit*

$$d(\Xi) = \lim_{s\to\infty} \frac{\mathrm{vol}B(0,s)}{|\{\xi \in \Xi : \xi(0) \in B(0,s)\}|}. \tag{1}$$

The upper and lower mean volumes are the corresponding limits superior and inferior. We say Ξ is a (r,R)-set if the point set $\{\xi(0) : \xi \in \Xi\}$ has a packing radius at least r and a covering radius at most R.

We will look at packings of congruent copies of a convex body K. That is, every element of the packing is given by $\xi(K)$, where ξ is an isometry of Euclidean space. It will be convenient to assume that the reference body K is situated so that its interior contains the origin.

Definition 0.2. *Let K be a compact set with interior. We say that Ξ is admissible for K if the interiors of $\xi(K)$ and $\xi'(K)$ are disjoint for any two distinct isometries $\xi, \xi' \in \Xi$. We say furthermore that Ξ is saturated if there is no $\xi \notin \Xi$ such that $\Xi \cup \{\xi\}$ is again admissible.*

There are radii $r(K)$ and $R(K)$ such that when Ξ is admissible and saturated, then Ξ is a $(r(K), R(K))$-set. As a consequence, such sets are countable.

Definition 0.3. *Given two (r', R')-sets Ξ and Ξ' of isometries, we define the premetric*

$$\begin{aligned}\delta_R(\Xi,\Xi') = \inf_{enum.} \sup\{||\xi_i^{-1}\xi_j - \xi_i'^{-1}\xi_j'|| :& \\ i,j \text{ such that } ||\xi_i(0) - \xi_j(0)|| < 2R \text{ or } ||\xi_i'(0) - \xi_j'(0)|| < 2R\}.&\end{aligned} \tag{2}$$

The infimum is over all enumerations $\mathbb{N} \to \Xi$ and $\mathbb{N} \to \Xi'$.

When $R > R'$, $\delta_R(\Xi,\Xi') = 0$ if and only if $\xi_i = \hat{\xi}\xi_i'$ for some $\hat{\xi} \in E(n)$ and some enumerations. Consider a body K. When $R > R(K)$, $\delta_R(\Xi,\Xi')$ is a metric on the space of admissible (r,R)-sets up to overall isometry, which includes the saturated sets as a subset.

Definition 0.4. *We say an admissible and saturated set Ξ is strongly extreme for K if it minimizes the mean volume among admissible elements in a neighborhood of Ξ in the metric space given by δ_R for some $R > R(K)$.*

Theorem 0.1. *If a lattice Λ is strongly extreme for K, then Λ is extreme for K [1].*

Theorem 0.2. *If a periodic set $\Xi = \{T_{\mathbf{l}}\xi_i : \mathbf{l} \in \Lambda, i = 1, \ldots, N\}$ is strongly extreme, then it is periodic extreme for K [2].*

We now derive a general method for proving strong extremality which we will use in the following sections.

Definition 0.5. *Let Ξ be a countable set of isometries and fix an enumeration $\Xi = \{\xi_i : i \in \mathbb{N}\}$. Let $\mathcal{P}$ be a polyhedral complex whose underlying space is $\mathbb{R}^n$. For every face F of $\mathcal{P}$, let $I_F = \{i : \xi_i(0) \in F\}$. We say $\mathcal{P}$ is a honeycomb of Ξ if each n-face (cell) P is the convex hull of $\{\xi_i(0) : i \in I_P\}$.*

Theorem 0.3. *Let Ξ be admissible for K and let $\mathcal{P}$ be a honeycomb of Ξ. For every cell P, consider the optimization problem of minimizing $f_P(\Xi_P) = \operatorname{vol}\operatorname{conv}_{i\in I_P}\xi'_i(0)$ over the assignment of isometries ξ'_i, $i \in I_P$, such that this finite set is admissible. If $\xi'_i = \xi_i$, $i \in I_P$, is a local minimum for each cell P on a uniform neighborhood for each isometry in Ξ, then Ξ is strongly extreme.*

By assumption, all cells P of the honeycomb $\mathcal{P}$ of Ξ are local minima for f in a neighborhood of P given by a uniform ϵ perturbation of any $\xi_i, i \in I_P$. From the definition of δ, it follows that $f(P_\Xi) \leq f(P_{\Xi'})$ for any Ξ' satisfying $\delta(\Xi, \Xi') \leq \epsilon$.

$$f^+(\Xi) = \limsup_{r\to\infty} \sum_{P\subset B(0,r)} \frac{\operatorname{vol}(P)}{|P|} \leq \limsup_{r\to\infty} \sum_{P'\subset B(0,r)} \frac{\operatorname{vol}(P')}{|P'|} = f^+(\Xi')$$

Then (use asympt. definition mean volume) $f(\Xi) \leq f(\Xi')$ and Ξ is strongly extreme.

Theorem 0.4. *Let $g_F(\Xi_F)$ be a real-valued function over $\Xi_F = (\xi'_i)_{i\in I_F}$ for each oriented $(n-1)$-faces (ridge) of $\mathcal{P}$, such that $g_F(\Xi_F) = -g_{-F}(\Xi_F)$, where $-F$ is the orientation-reversed version of F. Provided that g is uniformly bounded in a uniform neighborhood, we may replace $f_P(\Xi_P)$ in the previous theorem with $f'_P(\Xi_P) = f_P(\Xi_P) + \sum_{F\in\partial P} g_F(\Xi_F)$, then again, if $\xi'_i = \xi_i$, $i \in I_P$, is a local minimum for each cell P, then Ξ is strongly extreme.*

Following a similar argument to the previous theorem, all cells P of the honeycomb $\mathcal{P}$ of Ξ are minima for f' in neighborhood of P given by a uniform ϵ perturbation of any $\xi_i, i \in I_P$. From the definition of δ, it follows that $f'(P_\Xi) \leq f'(P_{\Xi'})$ for any Ξ' satisfying $\delta(\Xi, \Xi') \leq \epsilon$.

but with more boundary terms. there is an explicit separation between P and P' so the there is a large enough r so that

$$f^+(\Xi) = \limsup_{r\to\infty}(\sum_{P\subset B(0,r)} \frac{\operatorname{vol}(P)}{|P|} + g|\partial B) \leq \limsup_{r\to\infty}(\sum_{P'\subset B(0,r)} \frac{\operatorname{vol}(P')}{|P'|} + g|\partial B) = f^+(\Xi')$$

References

[1] Jacques Martinet. *Perfect lattices in Euclidean spaces*, volume 327. Springer Science & Business Media, 2003.

[2] Achill Schürmann. Strict periodic extreme lattices. *Diophantine Methods, Lattices, and Arithmetic Theory of Quadratic Forms*, 587:185, 2013.