Nonconvex optimization and jamming

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Sphere packing in 3D

Theorem (Hales, 1998–2015)

Every nonoverlapping arrangement of spheres in $\mathbb{R}^3$ fills at most $\pi/\sqrt{18} = 0.7405$ of space.

Sphere packing in 3D and higher dimensions

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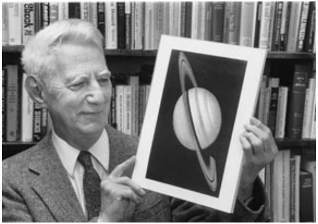
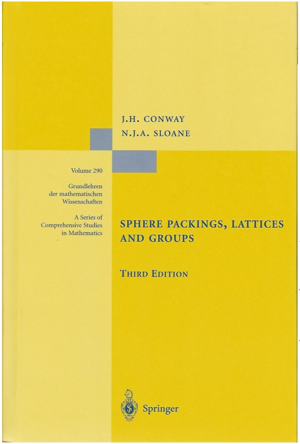
Applications in higher dimensions to transmitting, storing, and digitizing signals.

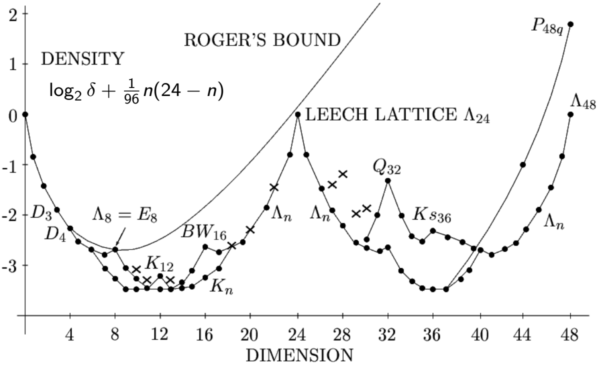


Curse of dimensionality

How to search for dense packings in high d? dof $\sim \exp(d)$.

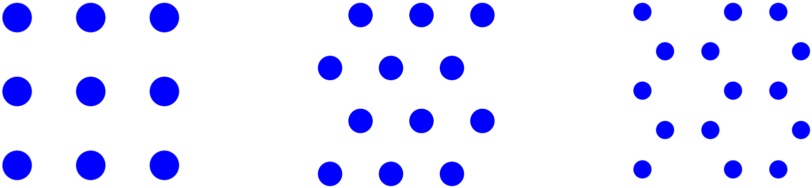
Solution: good packings are often lattices. dof $\sim d^2$





What do I mean by a lattice

$$L = A\mathbb{Z}^n = \{\sum_{i=1}^{n} m_i \mathbf{a}_i : m_i \in \mathbb{Z}\}$$



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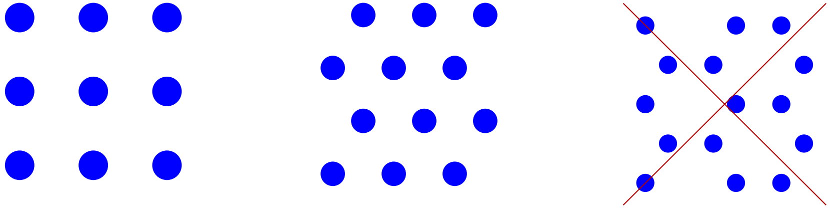

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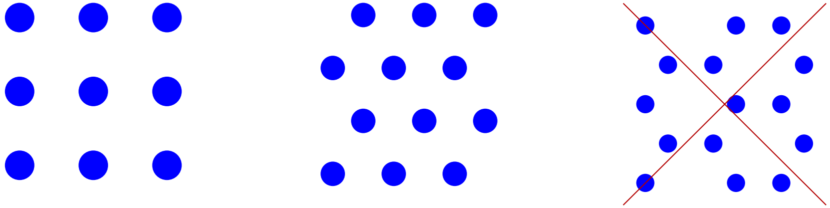
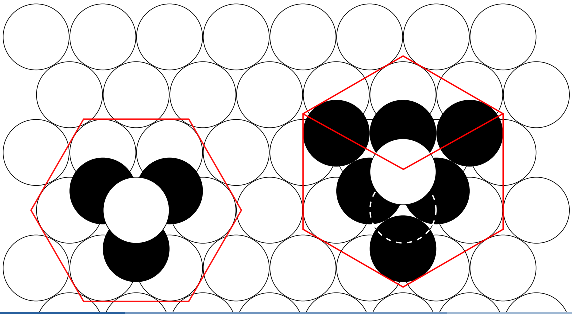
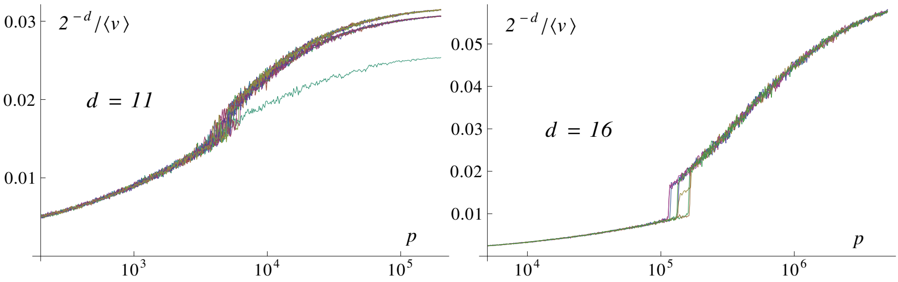




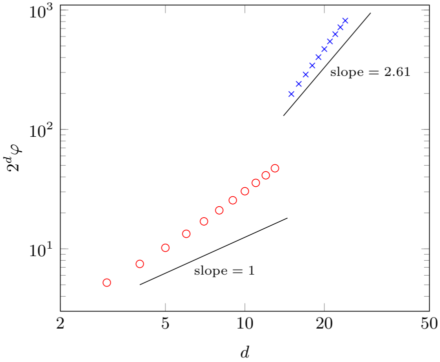
Thermodynamics of hard-sphere lattices

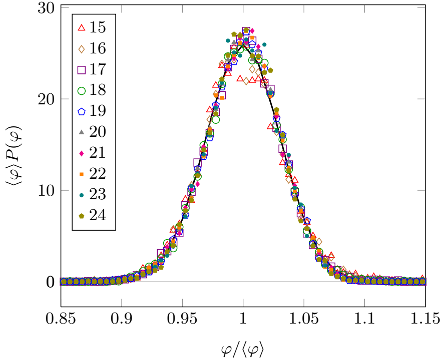
MC sampling from the space of lattices with probability $\sim \exp(-pv)$



Kallus, Phys. Rev. E ***87****, 063307 (2013)*

Lattice RCP

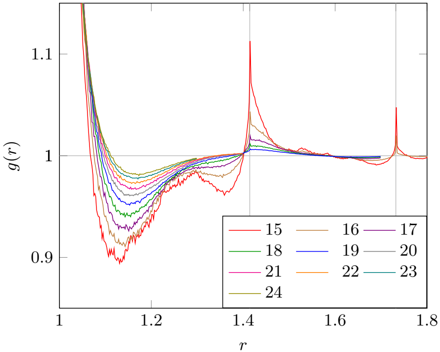


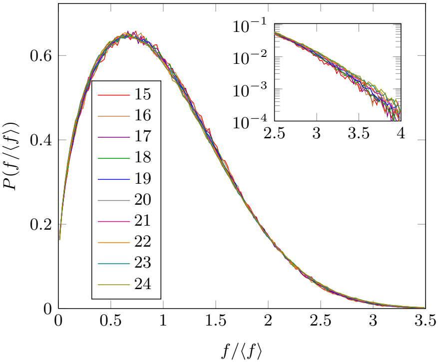


Best theoretic lower bound: $\varphi \gtrsim d2^{-d}$

Kallus, Marcotte, & Torquato, Phys. Rev. E **88**, *062151 (2013)*

Pair correlations and force distribution





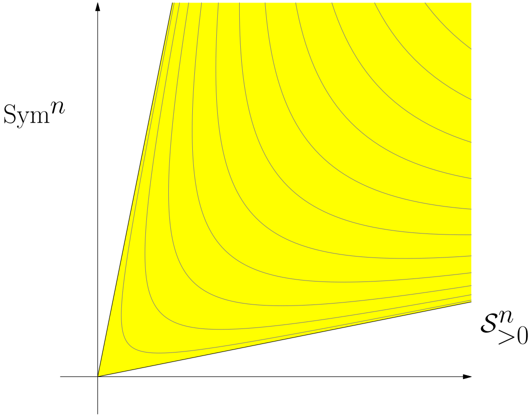
$$g(r) \sim (r-1)^{-\gamma}$$
$$\gamma = 0.314 \pm 0.004$$

$$P(f) \sim f^{\theta}$$
$$\theta = 0.371 \pm 0.010$$

Kallus, Marcotte, & Torquato, Phys. Rev. E **88**, *062151 (2013)*

Almost a linear program

Recall: $L = A\mathbb{Z}^n$, but we only care about the $G = A^T A$.
We need $\mathbf{n}^T G \mathbf{n} \geq 1$ for all $\mathbf{n} \in \mathbb{Z}^n$, and want to minimize $\det G$ (concave).



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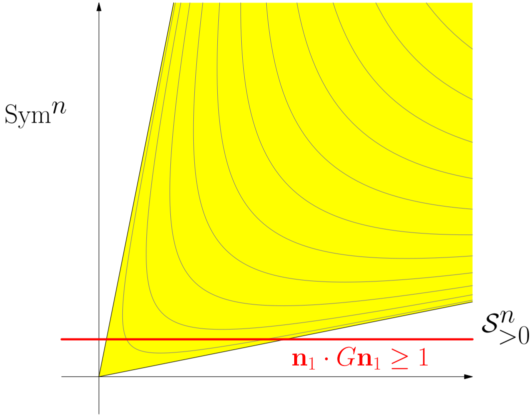

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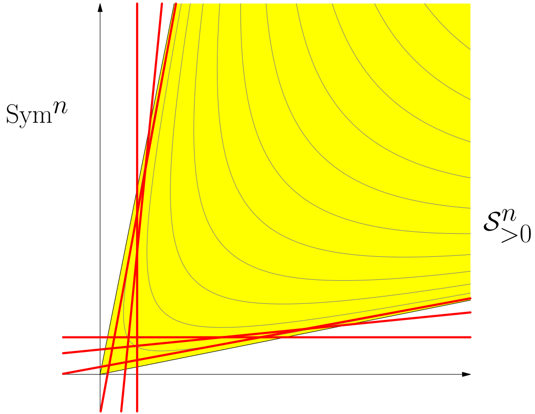

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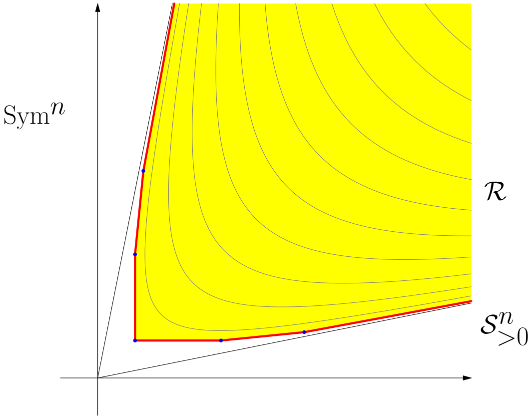

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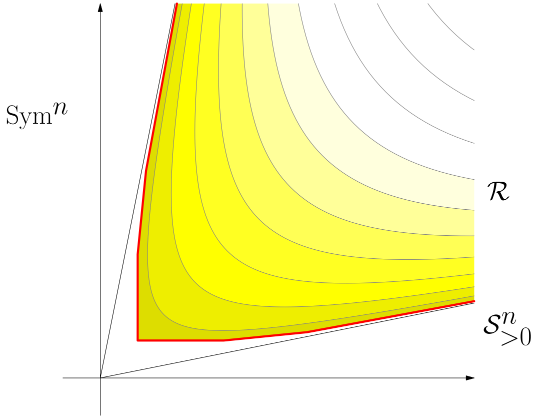
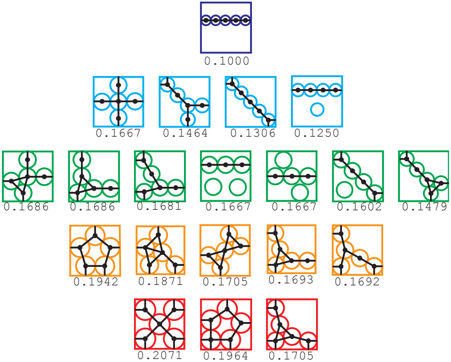
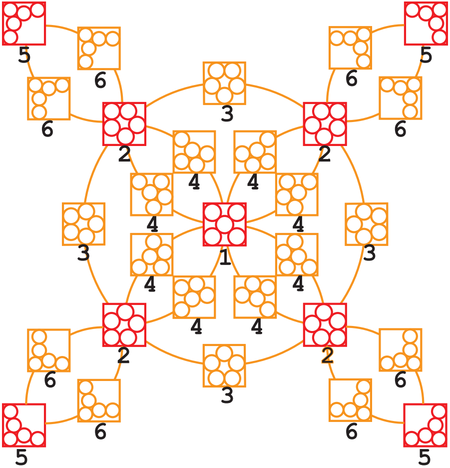


The topological theory of phase transitions





Carlsson, Gorham, Kahle, & Mason, Phys. Rev. E ***85****, 011303 (2012)*

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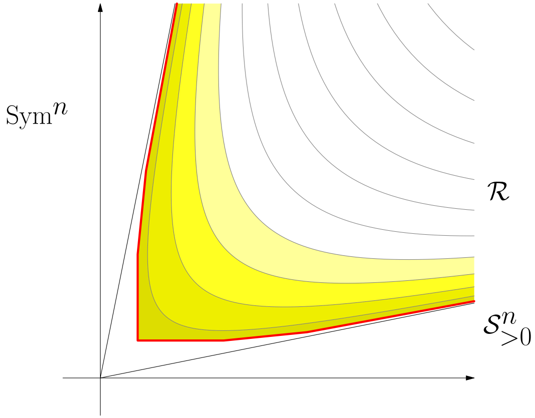
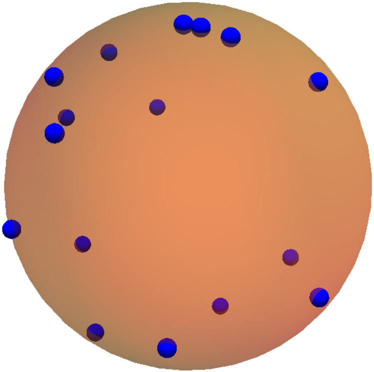


The "simplest model of jamming"

Place $m = \alpha n$ points randomly on the $(n-1)$-sphere, and try to find the point farthest from all of these.



Franz & Parisi, arXiv:1501.03397

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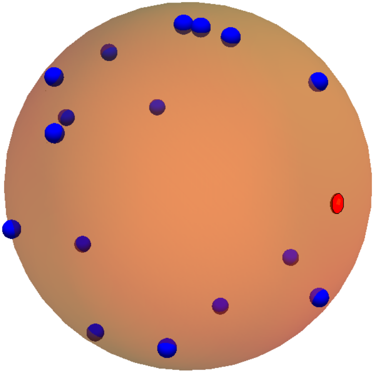

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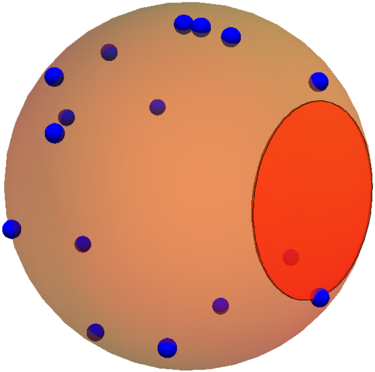

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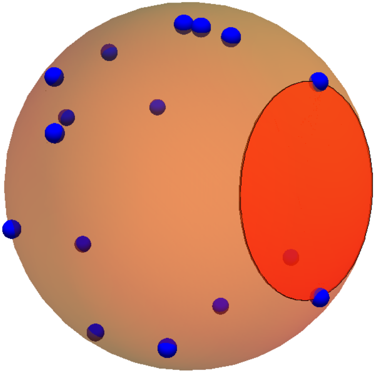

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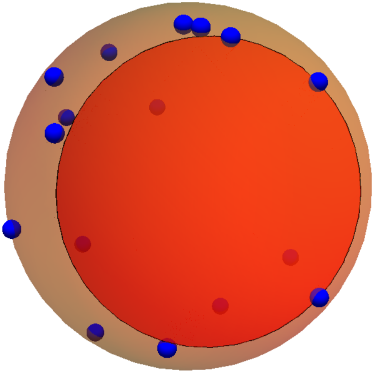

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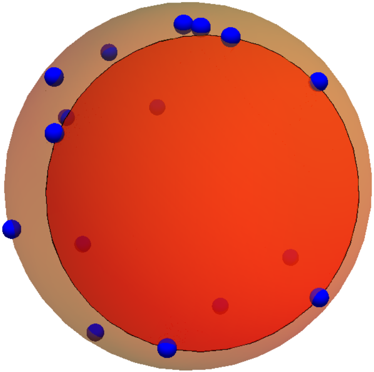
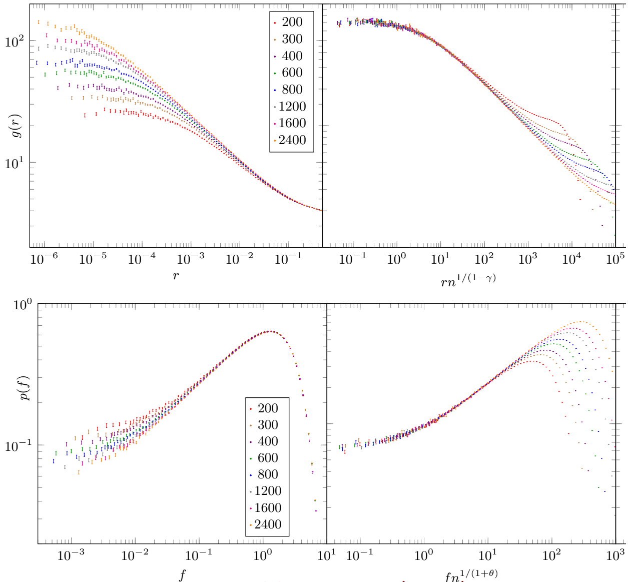


Same universality class as (off-lattice) sphere packing in $d \to \infty$

Franz & Parisi, arXiv:1501.03397

Finite-size scaling



Kallus, Phys. Rev. E ***93****, 012902 (2016)*

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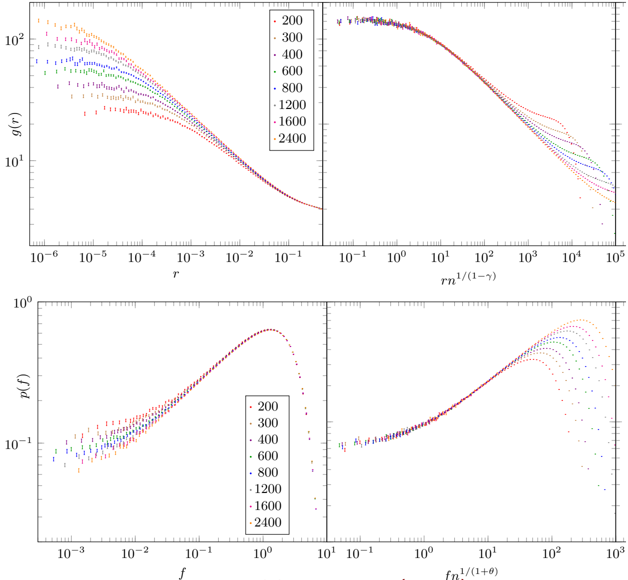

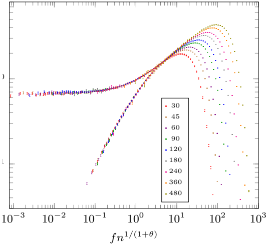

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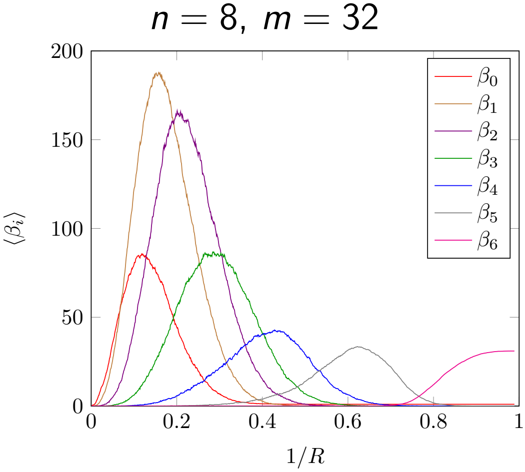
Computational topology



Conclusions

Lattice sphere packing

- random packings have high density
- no quenched disorder

The "simplest" model

- same universality class as off-lattice packing

- almost linear (minimizing a concave objective over a convex polytope)
- nice framework to study: topology, isostaticity, marginal stability, finite-size scaling