The Value of State Dependent Control in Ride-sharing Systems

Siddhartha Banerjee* Yash Kanoria† Pengyu Qian‡

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Abstract

We study the design of state-dependent control for a closed queueing network model of ride-sharing systems. We focus on the dispatch policy, where the platform can choose which vehicle to assign when a customer request comes in, and assume that this is the exclusive control lever available. The vehicle once again becomes available at the destination after dropping the customer. We consider the proportion of dropped demand in steady state as the performance measure.

We propose a family of simple and explicit state-dependent policies called Scaled MaxWeight (SMW) policies and prove that under the complete resource pooling (CRP) condition (analogous to the condition in Hall's marriage theorem), each SMW policy leads to exponential decay of demand-dropping probability as the number of vehicles scales to infinity. We further show that there is an SMW policy that achieves the optimal exponent among all dispatch policies, and analytically specify this policy in terms of the customer request arrival rates for all source-destination pairs. The optimal SMW policy protects structurally under-supplied locations.

Keywords: ride-sharing, maximum weight policy, closed queueing network, control, dispatch, Lyapunov function

1 Introduction

Recently there is an increasing interest in the control of ride-sharing platforms such as Uber and Lyft. These platforms are dynamic two-sided markets where customers (demands) arrive at different physical locations stochastically over time, and vehicles (supplies) circulate in the system as a result of driving demands to their destinations.[1] The platform's goal is to maximize throughput (proportion of demands fulfilled), revenue or other objectives by employing various types of controls.

The main inefficiency comes from the geographic mismatch of vehicles and customers: when a customer arrives, he has to be matched immediately with a nearby vehicle, otherwise the customer will abandon the request due to impatience. There are two sources of spatial supply-demand asymmetry: *structural imbalance* and *stochasticity*. The former is dominant during rush hours in the city when most of the demand pickups concentrate in a particular region of the city while dropoffs concentrate on others, otherwise the latter source often dominates, see [17]. *We propose a dispatch control policy that deals with both sources simultaneously (if this is possible): the choice of*

*Cornell University, Ithaca, NY, Email: sbanerjee@cornell.edu
†Columbia Business School, New York, NY, Email: ykanoria@columbia.edu
‡Columbia Business School, New York, NY, Email: PQian20@gsb.columbia.edu

[1] We will use demands and customers interchangeably, and similarly use vehicles and supplies interchangeably throughout the paper.

policy parameters accounts for the structural imbalance, while its state dependence nature manages stochasticity optimally.

Many control mechanisms have been proposed and analyzed in the literature. *Pricing*, for example, enjoys great popularity in both academia and industry, see, e.g., [10, 4, 3, 17]. By adjusting prices of rides, the system can indirectly re-balance supply and demand. *Empty-vehicle routing* [9] focus on sending available servers to under-supplied locations in order to meet more demand.

In this paper, we study another important form of control, *dispatch*. When customer requests a ride, the platform can decide from where to dispatch a vehicle, which will in turn influences the platform's future ability to fulfil demands. Previous work has studied dispatch decisions made in a state-independent manner – by optimizing the system's fluid limit, the platform can calculate the probability of dispatching from any compatible locations when a demand arises, and realize it by randomization [23, 3]. However, this approach requires exact knowledge of arrival rates (which is infeasible in practice), fails to react to the stochastic variation in the system and creates additional variance due to randomization. Although this control guarantees asymptotic optimality in the Law of Large Numbers sense, it converges only slowly to the fluid limit [3]. To counter these issues, *we study the state-dependent dispatch control of ride-sharing systems.*

We model the system as a closed queueing network with n servers representing physical locations, and K "jobs" that stands for vehicles. This is a common model for ride-sharing systems, see for example [3, 9, 31]. For each location i there are some *compatible* supply locations that are close enough, from where the platform can dispatch vehicles to serve demand at i. Demands arrives at the system stochastically, each has a destination in mind. Each time a customer arrives, the platform makes a dispatch decision from a compatible supply location based on the current spatial distribution of available supplies. After a vehicle picks up a customer, it drops her at the destination and becomes available again. (Supplies do not enter or leave the system.) The platform's goal is to maintain adequate supply in all neighborhoods and hence meet as much as demand as possible, therefore we adopt the (global) proportion of dropped demands as a measure of efficiency. For the formal description of our model, see section 2.

To study state-dependent spatial rebalancing of supply while keeping the size of the state space manageable, we make a key simplification – we assume that pickup and service of demand are both instantaneous. This allows us to get away from the complexity of tracking the positions of in transit vehicles, while retaining the essence of our focal challenge, that of ensuring that all neighborhoods have supplies at (almost) all times. To obtain tight characterizations, we further consider the asymptotic regime where the number of vehicles in the system K goes to infinity. It's worth noting that the large supply regime (demand arrival rates stay fixed while $K \to \infty$) and the large market regime (demand arrival rate scales with K as $K \to \infty$) are equivalent in our model. The reason is that we assume instantaneous completion of rides, hence the large market regime is simply a speed-up of the infinite supply regime.

A main assumption in our model is a complete resource pooling (CRP) condition. CRP is a standard assumption in the heavy traffic analysis of queueing systems (see e.g. [18, 12, 25]). It can be interpreted as requiring enough overlapping in the processing ability of servers so that they form a "pooled server". For the model considered in this paper, the CRP condition is closely related to the condition in Hall's marriage theorem in bipartite matching theory. We show that the CRP condition is necessary for any dispatch policy to have demand dropping probability that converges to zero.

One key difficulty in the analysis is the necessity to deal with a multi-dimensional system in the limit. In many existing works that seek to minimize the workload process or holding costs of a queueing system, asymptotic optimality of a certain policy relies on “collapse” of the system state to a lower dimensional space in the heavy traffic limit under the CRP condition. This is not the case for the objective we are considering, i.e., minimizing demand dropping probability, or maximizing throughput. Since the exponent of demand dropping probability depends on the most likely event that leads to demand dropping event, we need to “protect” all the subsets of locations simultaneously. An interesting observation drawn from our analysis is a “critical subset” property: given the current state, the most likely way a demand may be dropped is via the draining a certain critical subset of locations. The critical subset changes with the state of the system.

1.1 Main Contributions

As a function of system primitives, we derive a large deviation rate-optimal dispatch policy that minimizes demand dropping (maximizes throughput). Our optimal policy is strikingly simple and its parameters depend in a natural way on demand arrival rates. Our contribution is threefold:

1. Achievability: We propose a family of state-dependent dispatch policies called scaled MaxWeight (SMW) policies, and prove that all of them guarantee exponential decay of demand-dropping probability under CRP condition. The proof is based on a family of novel Lyapunov functions (a different one for each SMW policy) which are used to analyze a multi-dimensional variational problem. An SMW policy is parameterized by an n-dimensional vector consisting of a scaling factor for each location; each demand is served by dispatching a supply from the compatible location with the largest scaled number of cars. We obtain an explicit specification for the optimal scaling factors based on location compatibilities and demand arrival rates. Further, we obtain the surprising finding that the optimal SMW policy is, in fact, exponent-optimal among all state-dependent policies (Theorem 1). SMW policies are simple, explicit and appear promising for practical applications.

2. Converse bounds: We provide lower bound of demand dropping probability for any dispatch policy using random-walk related inequalities. We first show that no state-independent dispatch policy can achieve exponential decay rate (Proposition 3), which demonstrates the value of state-dependent control — even a naive state-dependent dispatch policy with no knowledge of demand arrival rates beats the best state-independent dispatch policy asymptotically. Then we justify the CRP assumption by showing that it is a necessary condition for exponential convergence (Proposition 1; in fact if any of the inequalities is reversed, a positive fraction of demands must be dropped for that instance even as $K \to \infty$). Finally, we obtain an upper bound on the demand dropping probability exponent for any state-dependent policy that matches the achievable exponent of the optimal SMW policy, thus proving that the best SMW policy is, in fact, exponent-optimal.

3. Qualitative insights: We characterize the system behavior under SMW policies as $K \to \infty$, which is technically challenging since the problem remains n dimensional even in the limit. We establish the *critical subset* property of the problem: given a system state (in the limit), there exists a (state-dependent) subset J of demand locations that are most likely to be depleted of supply in compatible locations, hence leading to demand dropping. The optimal SMW policy avoids using supply from locations compatible with J for demand arising outside of J.

1.2 Literature Review

Ride-sharing platforms. Optimization of ride-sharing system has drawn attention in recent years. Ozkan and Ward [23] studied revenue-maximizing state-independent dispatch control by solving a minimum cost flow problem in the fluid limit. Braverman et. al. [9] modeled the system by a closed queueing network and derived the optimal static routing policy that sends empty vehicles to under-supplied locations. Banerjee, Freund and Lykouris[3] adopted Gordon-Newell closed queueing network model and considered static pricing policy that maximize throughput, welfare or revenue. In contrast to our work, which studies state-dependent control, these works consider static control that completely relies on system parameters. In terms of convergence rate to the fluid-based solution, [23] did not show the order of convergence rate of their policy, [9] proved an $O(1/\sqrt{K})$ rate as number of servers in the closed system K goes to infinity, and [3] showed an $O(1/K)$ convergence rate as $K \to \infty$. We show that no state-independent policy can do better than $O(1/K^2)$, while our state-dependent policy achieves an $O(e^{-\gamma K})$ convergence rate with optimal $\gamma > 0$.

There are many other works that also studied ride-sharing platforms but focus on pricing aspects, see e.g. [1, 6, 31, 10, 17]. We remark briefly that our CRP condition has some similarity with the notion of "balancedness" in [6], although balancedness holds on a knife edge (requiring an exact balance between inflows and outflows for each location), whereas the CRP condition does not, due to the flexibility from having multiple compatible supply locations for a demand location.

MaxWeight scheduling. MaxWeight policy is a simple scheduling policy in constrained queueing networks that exhibits many good properties. It attaches a weight to each schedule activity, which is a function of current queue-lengths $\{f_u(\mathbf{X})\}$; in many cases $f_u(\mathbf{X})$ is simply the queue length at the scheduled server (source). At each time period it activates the admissible activities with the largest total weight.

MaxWeight scheduling has been studied intensively since the seminal work by Tassiulas and Ephremides [29], which showed MaxWeight (with weight defined as a constant multiple of source-destination queue-length difference) achieves the entire stability region for an open one-pass system. Dai and Lin showed that MaxWeight (with the same choice of weight as [29]) achieves throughput optimality in open stochastic processing networks in fluid limit under mild conditions [13]

Stolyar [27] showed that for one-hop open network, any MaxWeight policy with weight $f_u(\mathbf{X})$ chosen as a positive multiple of the β-th ($\beta > 0$) power of source queue length, minimizes workload (weighted sum of queue lengths) under Resource Pooling (RP) condition in diffusion limit. A similar result was obtained in [12], where they showed that for a more general family of open networks, MaxWeight policy with weight defined as in [13] minimizes workload in diffusion limit under CRP condition.

Eryilmaz and Srikant [15] showed that for one-hop network, MaxWeight policy with weight $f_u(\mathbf{X}) = c_{\text{source}(u)}\mathbf{X}_{\text{source}(u)}$ minimizes the expectation of $\sum_i c_i \mathbf{X}_i$ in heavy traffic limit with one-dimensional state-space collapse. Maguluri and Srikant [20] showed that for cross-bar switch MaxWeight policy with $f_u(\mathbf{X}) = \mathbf{X}_{\text{source}(u)}$ achieves the optimal scaling of total queue-length in heavy traffic with multi-dimensional state space collapse. Shi et. al. [25] considered a model similar to [15] with focus on design of network topology.

In contrast of the many of the above works where asymptotic optimality is guaranteed for any choice of constant $c_{\text{source}(u)} > 0$ in weight $f_u(\mathbf{X})$, we show that in our model the coefficients need to be chosen carefully in order to achieve optimal exponent, though any choice of positive constants will result in exponential decay under CRP condition. For example, vanilla MaxWeight ($f_u(\mathbf{X}) = \mathbf{X}_{\text{source}(u)}$) can result in non-trivial sub-optimality of system performance. This is because for open queueing networks under CRP condition, though the queue lengths can collapse to different subspaces under different MaxWeight policies, the workload process always converges to the same weak limit which is uniquely defined by the model primitives. In our case, however, different state space collapse usually leads to different system performance (exponent); and evaluation of each MaxWeight policy requires analysis of policy-specific most-likely sample paths.

There are other variants of MaxWeight policy that are less related to our work, e.g. [22].

Large deviations in queueing systems. There is a large literature on characterizing the decay rate of probability of building up large queue lengths in open queueing networks. To the best of our knowledge, we are the first to consider large deviation of controlled closed queueing networks.

Stolyar and Ramanan [28] showed the exponent optimality of Largest Weighted Delay First scheduling in minimizing the probability of waiting times exceeding large values for multi-class single server queueing system, and Stolyar [26] extended the result to multi-class multi-server queueing networks. Bodas et al. [8] considered the large deviation optimal scheduling of parallel servers, but the asymptotic regime is different in that they are scaling up the size of network while keeping buffer size fixed. Compared with these works, the difficulty of analyzing our model comes from its complex dynamics: servers circulate in the system endlessly (in contrast to one-hop system in [8]), and each server can be matched to demands at different nodes (in contrast to multi-class model in [28][26]). As a result, the techniques used in these works cannot be directly applied here. Our result is also qualitatively different: in [26] the analysis of queueing networks with arbitrary topology reduces to studying one node in isolation, but in our work the optimal exponent include terms for each subset of nodes.

The closest work to ours is that of Venkataramanan and Lin [30], who established the relationship between Lyapunov function and buffer overflow probability for open queueing networks. Our Lyapunov function approach is inspired by their work, and we devise a family of Lyapunov functions such that the decay rate of demand dropping probability is the same as that of Lyapunov function exceeding certain threshold. The key difficulty of extending the Lyapunov approach to closed queueing networks is the lack of natural reference state where the Lyapunov function equals to 0 (in open queueing network it's simply $\mathbf{0}$). It turns out when optimizing the MaxWeight parameters we are also solving for the best reference state.

There are also works that study the large deviation behavior of queueing networks without control aspect, see e.g. [21, 7].

1.3 Organization

The remainder of our paper is organized as follows. In section 2 we introduce the basic notation used throughout the paper and formally describe our model together with performance measure. Some background on sample path large deviation principle will also be provided. In section 3 we introduce the family of Scaled MaxWeight policies. In section 4 we present our main theoretical result, i.e., exponent optimality of SMW policy. In section 5 we prove the achievability bounds of of SMW policies. In section 6 we prove the converse bounds and show the exponent optimality of SMW policy. We also present there the converse bounds for state-independent policies and cases where CRP condition is violated.

2 Setting and Preliminaries

2.1 Notation

Wherever possible, we reserve capital letters for random quantities and small letters for their realizations; we also use boldface letters to indicate column vectors. We use $\mathbf{e}_i$ to denote the i-th unit vector, and $\mathbf{1}$ the all-1 vector. If vector $\mathbf{b}$ is strictly larger than $\mathbf{a}$ component-wise, we write $\mathbf{b} > \mathbf{a}$. For index set A, define $\mathbf{1}_A \triangleq \sum_{i\in A} \mathbf{e}_i$. For a set Ω in Euclidean space $\mathbb{R}^n$, denote its relative interior by $\mathrm{relint}(\Omega)$. We use $B(\mathbf{x}, \epsilon)$ to denote a ball centered at $\mathbf{x} \in \mathbb{R}^n$ with radius $\epsilon > 0$.

For event C, we define the indicator random variable $\mathbb{1}\{C\}$ to equal 1 when C is true, else 0.

2.2 Basic Setting

Underlying Model and Simplifications: We model the ridesharing system as a finite-state Markov process, comprising of a fixed number of identical *supplies* (i.e., vehicles) circulating among n *nodes* (i.e., a given partition of a city into neighborhoods). Customers (i.e., prospective passengers) arrive at each node i with desired destination j according to independent Poisson processes with rate $\hat{\phi}_{ij}$. To serve an arriving customer, the platform immediately dispatches a vehicle from a "neighboring" station of i (i.e., one among a set of nearby stations, defined formally below), and subsequently, after serving the customer, the vehicle becomes available at the destination node j. If however there are no supplies available in the neighboring nodes of i, then we experience a *demand drop*, wherein the customer leaves the system without being served. Customers do not wait. The aim of the platform is to dispatch supplies so as to minimize the fraction of a demands dropped. Intuitively, to achieve this objective, the platform should ensure that it maintains adequate supply in (or near) all neighborhoods, i.e., it needs to manage the spatial distribution of supply.

The study the design of dispatch rules in the above model, we make two simplifications. First, we assume that dispatch and service are instantaneous. This allows us to reformulate the above model as a *discrete-time* Markov chain (the so-called *jump chain* of the continuous-time process), where in each time-slot $t \in \mathbb{N}$, with probability proportional to $\hat{\phi}_{ij}$, exactly one customer arrives to the system at node i with desired destination j. The customer is then served by a dispatched vehicle from a neighboring station of i, which then becomes available at node j at the beginning of time-slot $t + 1$. This simplification removes the high-dimensionality required for tracking the positions of all in-transit vehicles, while still retaining the complex supply externalities between stations, which is the hallmark of ridesharing systems. Unfortunately, however, even after this simplification, the setting still does not admit any amenable way to characterize the performance of complex dispatch policies. To circumvent this we make a second simplification, we study the performance of dispatch policies as the number of supplies K grow to infinity, while fixing all other parameters.

Formal System Definition: We define $\phi \in \mathbb{R}^{n\times n}$ to be the arrival rate matrix with a row for each origin and a column for each destination, normalized[2] such that $\mathbf{1}^T\phi\mathbf{1} = 1$. We denote the i-th column (i.e., the arrival rates from different origins of customers to destination i) as $\phi_{(i)}$, and the transpose of the i'-th row of the arrival matrix (i.e., the arrival rates of customers to node i' with different destination nodes) as $\phi_{i'}$. Thus, the probability a customer arrives at node i' is $\mathbf{1}^T\phi_{i'}$, and, assuming all customers are matched, the rate of vehicles arriving at node i is $\mathbf{1}^T\phi_{(i)}$.

[2]This can always be achieved by appropriately re-scaling the arrival rates $\{\hat{\phi}_{ij}\}$, which produces an equivalent setting since pickups and dropoffs are instantaneous.

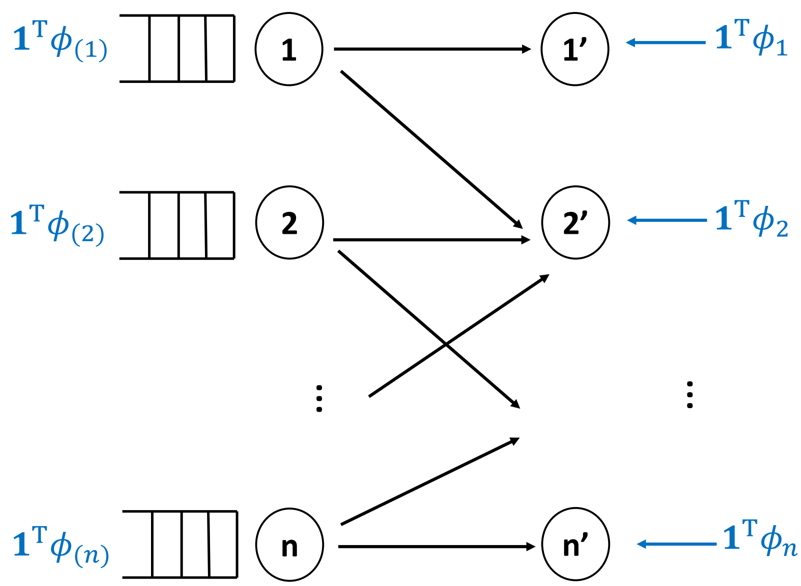

Figure 1: The bipartite compatibility graph for the dispatch problem: On the left are supply nodes i, and on the right are demand nodes i'. Customers arrive to i' with distributions chosen according to $\phi_{i'}$; the total probability of a customer arrival at i' is thus $\mathbf{1}^T\phi_{i'}$. Similarly, assuming no demand is dropped, the total probability a vehicle arrives at i is $\mathbf{1}^T\phi_{(i)}$. The edges entering a node i' encode compatible (i.e., nearby) nodes that can supply node i.

As mentioned above, we consider a sequence of systems parameterized by the number of supplies K. For the K-th system, its state at any time $t \in \mathbb{N}$ is given by $\mathbf{X}^K[t]$, a vector that tracks the number of supplies at each location in time-slot t. The state space of the K-th system is thus given by:

$$\Omega_K \triangleq \{\mathbf{x} \in \mathbb{R}^n | \mathbf{1}^T\mathbf{x} = K\} \cap \mathbb{N}^n$$

Note that the normalized state $\mathbf{X}^K/K$ lies in the n-simplex $\Omega = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \geq 0, \mathbf{1}^T\mathbf{x} = 1\}$. Henceforth, we drop explicit dependence on t when clear from context.

2.3 Dispatch Policies and System Dynamics

The main idea behind the dispatch problem in ridesharing is that most arriving customers have a maximum tolerance (say 7 minutes) for the pickup time or ETA (i.e., expected time of arrival) of a matched vehicle, but are essentially indifferent if the ETA is less than that. Thus when a customer arrives at a node i, then any vehicle located at a node which is within 7 minutes of i is a feasible match, while other vehicles which are further away are infeasible. This suggests a natural model for dispatch via a bipartite compatibility graph, as depicted in Figure 1.

Compatibility Graph: For pedagogical reasons, we move to a setup where we distinguish formally between demand locations V_D where customers arrive and supply locations V_S where vehicles wait (and where customers are dropped off).

We encode the compatibility graph as a bipartite graph $G(V_S \cup V_D, E)$, wherein each station $i \in V$ is replicated as a supply node $i \in V_S$ and a demand node $i' \in V_D$.

An edge $(i, j') \in E$ represents a compatible pair of supply and demand nodes, i.e., supplies stationed at i can serve demand arriving at j'. We denote the neighborhood of a supply node $i \in V_S$ (resp. demand node $j' \in V_D$) in G as $\partial(i)$ (resp. $\partial(j')$); thus, for a supply node i, its compatible demands are given by $\partial(i) = \{j' \in V_D | (i, j') \in E\}$, and similarly for each demand node. By default, we assume $(i, i') \in E \, \forall i$, and hence $i \in \partial(i')$, and $i' \in \partial(i)$. Moreover, for any set of supply nodes $A \subseteq V_S$, we also use $\partial(A)$ to denote its demand neighborhood (and vice versa).

We make some mild assumptions on arrival rates ϕ:

Assumption 1. *The following holds:*

1. *Matrix $[\phi_{ij}]$ is irreducible.*[3]

2. *There exists an origin-destination pair $i' \in V_D$ and $j \in V_S$ such that $j \notin \partial(i')$ and $\phi_{i'j} > 0$.*

Remark 1. The irreducibility assumption ensures that system states have one positive recurrent class under our policies. The second assumption is made to ensure that the dispatch control problem at hand is non-trivial. (If it is violated, and if the system starts with at least one car in each location, then we can "reserve" a car for serving each demand origin location $i' \in V_D$, and each such car will never leave the corresponding neighborhood $\partial(i')$, ensuring that no demand is ever dropped.)

Dispatch policies: Given the above setting, the problem we want to study is how to design dispatch policies which minimize the probability of dropped demand. For each $t \in \mathbb{N}$, $i \in V_D$, a dispatch policy U includes a series of mappings U^K for each system $K \in \mathbb{N}$, which maps the entire history of queue lengths, arrivals and dispatch decisions before the t-th period to $U^K[t](i) \in \partial(i) \cup \{\emptyset\}$. Here $U^K[t](i) = j$ denotes that given the entire history before t-th period, we dispatch a vehicle from $j \in \partial(i)$ to fulfill demand at i, and $U^K[t](i) = \emptyset$ means that the platform does not dispatch to i and hence any arriving demand at i is dropped. For simplicity of notation, we refer to the policies by U instead of U^K.

System Evolution: We can now formally define the evolution of the Markov chain we want to study. At the beginning of time-slot t, the state of the system is $\mathbf{X}^K[t-1]$; note that this incorporates the state-change due to serving the demand in time-slot $t-1$. Now suppose the platform uses a dispatch policy U, and in time-slot t, a customer arrives at origin node $o[t]$ with destination $d[t]$ (chosen from arrival matrix ϕ). If $U^K(t, d[t]) \neq \emptyset$, then a vehicle from $U^K[t](o[t])$ will pick up the demand and relocate to $d[t]$ instantly. Let $S[t] \triangleq U^K[t](o[t])$ be the chosen supply node (potentially $\emptyset$). Formally, we have

$$\mathbf{X}^K[t] = \begin{cases} \mathbf{X}^K[t-1] - \mathbf{e}_{S[t]} + \mathbf{e}_{d[t]}, & \text{if } S[t] \in V_S. \\ \mathbf{X}^K[t-1], & \text{if } S[t] = \emptyset. \end{cases}$$

2.4 Performance Measure

The platform's goal is to find a dispatch policy that drops as few demands as possible in steady-state. In this section we formally define the performance measure for all dispatch policies. We first introduce some necessary notations.

For each subset $A \subseteq V_S$ (resp. $A' \subseteq V_D$), define:

$$A^c \triangleq \{v \in V_S : v \notin A\} \quad (\text{resp. } A^c \triangleq \{v \in V_D : v \notin A\}). \tag{1}$$

[3] Replacing non-zero entries in the matrix by one, and viewing the matrix as the adjacency matrix of a directed graph, the matrix is irreducible if and only if this directed graph is strongly connected.

Recall that we define the scaled state as $\bar{\mathbf{X}}^K \triangleq \frac{1}{K}\mathbf{X}^K \in \Omega$. Now for a given state $\mathbf{x} \in \Omega$, we define the set of empty stations as

$$I_{\text{em}}(\mathbf{x}) \triangleq \{v \in V_S : \mathbf{x}_v = 0\}.$$

We now have the following simple observation:

Observation 1. *If the scaled state at time t is $\bar{\mathbf{X}}^K[t] = \mathbf{x}[t]$, then*

$$\mathbb{P}[\textit{demand dropped at } t+1] = \sum_{i':\partial(i')\subseteq I_{\text{em}}(\mathbf{x}), j \in V_S} \phi_{i'j} = \mathbf{1}^T_{\{i':\partial(i')\subseteq I_{\text{em}}(\mathbf{x})\}}\phi\mathbf{1}.$$

This follows from observing that demand arriving in period t to stations in $\{i : \partial(i) \subseteq I_{\text{em}}(\bar{\mathbf{X}}^K[t])\}$ must be dropped.

Given the above setting, a natural performance measure is the *long-run average demand-drop probability*. Formally, we define

$$\underline{P}^{K,U} \triangleq \min_{\mathbf{X}^{K,U}[0]\in\Omega_K} \mathbb{E}\left(\liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T}\left[\mathbf{1}^{\mathrm{T}}_{\{i:\partial(i)\subseteq I_{\text{em}}(\mathbf{x})\}}\phi\mathbf{1}\right]\right). \tag{2}$$

For policies that induce a unique stationary distribution over states, $\underline{P}^{K,U}$ is simply the demand dropping probability in the unique steady state (see "Performance guarantees" below). Our positive results will be for policies that satisfy this condition. On the other hand, even in the absence of a stationary distribution for any given policy, the above definition gives a lower bound for the demand-dropping probabilities, and hence is suitable for converse results. In particular, it means that our converse bounds are lower bounds on this lower bound on demand dropping probability, i.e., our converse bounds have a strong interpretation.

One issue however is that the exact expression of (2) may be complicated for a fixed K. To this end, the main performance measure of interest in this work is the decay rate of $\underline{P}^{K,U}$ as $K \to \infty$:

$$\gamma(U) = -\liminf_{K\to\infty} \frac{1}{K} \log \underline{P}^{K,U}. \tag{3}$$

For brevity, we henceforth refer to this as the *demand-drop exponent*. Again the definition is suited to strong converse results, whereas for our positive results the $\liminf$ will, in fact, be the limit, and hence the same as the $\limsup$.

For any given policy U, the demand-drop exponent can be simplified further in terms of the long-run average behavior of $\bar{\mathbf{X}}^{K,U}[t]$.

Now let $D_A \triangleq \{\mathbf{x} \in \Omega : \mathbf{x}_i = 0 \text{ for } i \in A, \mathbf{x}_j > 0 \text{ for } j \notin A\}$, and let $\mathcal{A} = \{A \subsetneq V_S : \exists i' \in V_D \text{ such that } \mathbf{1}^{\mathrm{T}}\phi_{i'} > 0 \text{ and } A \supseteq \partial(i')\}$ and let $D \triangleq \cup_{A\in\mathcal{A}} D_A$. If words, D is the set of (normalized) states satisfying the following requirement — that there is at least one location where demand arrives with positive rate which is currently starved of supply at compatible locations. The demand-drop exponent can now be simplified by the following observation:

Observation 2.

$$\gamma(U) = -\liminf_{K\to\infty} \frac{1}{K} \log\left(\min_{\mathbf{X}^{K,U}[0]\in\Omega_K} \mathbb{E}\left(\liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \mathbb{1}\left\{\bar{\mathbf{X}}^{K,U}[t] \in D\right\}\right)\right) \tag{4}$$

Observation 2 says that the probability of demand dropping has the same exponential decay rate as the probability that there exists a node without any vehicle at a compatible location. The key idea of the proof is to bound the ratio of the two probabilities by a constant that doesn't scale with K. We conclude with two relevant notions:

- *Performance guarantees with stationary policies*: Suppose scaled queue-lengths have a unique stationary distribution under policy U for any K, denoted by $\bar{\mathbf{X}}^{K,U}[\infty]$. Then the policy U has the associated performance/exponent guarantee

$$-\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}\left[\bar{\mathbf{X}}^{K,U}[\infty] \in D\right].$$

- *Benchmark over all policies*: No policy can achieve a demand-drop exponent larger than

$$\max_{U\in\mathcal{U}} \left(\gamma(U)\right). \tag{5}$$

2.5 Sample Path Large Deviation Principle

Our result relies on classical large deviation theory, which we briefly introduce in this subsection.

For each fixed $K \in \mathbb{N}_+$, define $\bar{\mathbf{A}}^K(\cdot) \in (L^\infty[0,T])^{n^2}$ where $\bar{\mathbf{A}}^K(0) = \mathbf{0}$. For $t = 1/K, 2/K, \ldots, \lceil KT \rceil/K$,

$$\bar{\mathbf{A}}^K_{ij}(t) \triangleq \frac{1}{K}\sum_{\tau=1}^{Kt} \mathbb{1}\{o[\tau]=i, d[\tau]=j\}.$$

$\bar{\mathbf{A}}^K(t)$ is defined by linear interpolation for other t's. Here $\bar{\mathbf{A}}^K(t)$ is the fluid-scale accumulated demand arrival process of the K-th system.

Let μ_K be the law of $\bar{\mathbf{A}}^K(\cdot)$ in $(L^\infty[0,T])^{n^2}$. Let $\Lambda(\lambda)$ be the cumulant generating function of $\bar{\mathbf{A}}^1(1)$:

$$\Lambda(\lambda) \triangleq \log \mathbb{E} e^{\langle \lambda, \bar{\mathbf{A}} \rangle} = \log\left(\sum_{i=1}^n \sum_{j=1}^n \phi_{ij} e^{\lambda_{ij}}\right) \qquad \lambda \in \mathbb{R}^{n\times n}.$$

Let $\Lambda^*(\mathbf{f})$ be the Legendre-Fenchel transform of $\Lambda(\cdot)$, then

$$\Lambda^*(\mathbf{f}) \triangleq \sup_{\lambda\in\mathbb{R}^n} \langle \lambda, \mathbf{f}\rangle - \Lambda(\lambda) = \begin{cases} D_{KL}(\mathbf{f}||\phi) & \text{if } \mathbf{f} \geq 0, \mathbf{1}^T\mathbf{f}\mathbf{1} = 1 \\ \infty & \text{otherwise.} \end{cases}$$

Here $D_{KL}(\mathbf{f}||\phi)$ is Kullback-Leibler divergence defined as:

$$D_{KL}(\mathbf{f}||\phi) = \sum_{i=1}^n \sum_{j=1}^n \mathbf{f}_{ij} \log \frac{\mathbf{f}_{ij}}{\phi_{ij}}.$$

For any set Γ, define $\bar{\Gamma}$ as its closure, Γ^o as its interior.

Below is the sample path large deviation principle (a.k.a. Mogulskii's theorem) (see [14]):

Fact 1. *For measures $\{\mu_K\}$ defined above, and any arbitrary measurable set $\Gamma \subseteq (L^\infty[0,T])^{n^2}$, we have*

$$-\inf_{\bar{\mathbf{A}}\in\Gamma^o} I_T(\bar{\mathbf{A}}) \leq \liminf_{K\to\infty} \frac{1}{K}\log \mu_K(\Gamma) \leq \limsup_{K\to\infty}\frac{1}{K}\log\mu_K(\Gamma) \leq -\inf_{\bar{\mathbf{A}}\in\bar{\Gamma}} I_T(\bar{\mathbf{A}}),$$

where the rate function is:

$$I_T(\bar{\mathbf{A}}) = \begin{cases} \int_0^T \Lambda^* \left(\frac{d}{dt} \bar{\mathbf{A}}(t) \right) dt, & \text{if } \bar{\mathbf{A}}(\cdot) \in AC[0,T], \bar{\mathbf{A}}(0) = \mathbf{0} \\ \infty, & \text{otherwise.} \end{cases}$$

Here $AC[0,T]$ is the space of absolutely continuous functions on $[0,T]$.

Remark 2. Since absolutely continuous functions are differentiable almost everywhere, the rate function is well-defined.

3 Scaled MaxWeight Policies

We now introduce the family of scaled MaxWeight (SMW) policies.

The traditional MaxWeight policy (hereafter referred to as vanilla MaxWeight) is a dynamic scheduling rule that allocates the service capacity to the queue(s) with largest "weight" (where weight can be any relevant parameter such as queue-length, sum of queue-lengths, head-of-the-line waiting time, etc.). In our setting, vanilla MaxWeight would correspond to dispatching from the compatible location with most supplies (with appropriate tie-breaking rules).

The popularity of MaxWeight scheduling stems from the fact that it is known to be optimal for different metrics in various problem setting (e.g., refer [26, 27, 25, 20]). However, in our setting, vanilla MaxWeight is suboptimal (and further, we will show that it does not achieve the optimal exponent). The suboptimality of vanilla MaxWeight can be seen from the following simple example.

Example 1. *Consider a network with two nodes $\{1,2\}$, compatibility graph $G = (V_S, V_D, E) = (\{1,2\}, \{1',2'\}, \{11',12',22'\})$ and demand arrival rates $\phi_{ij} = 1/4, i,j \in \{1,2\}$, as shown in Figure 2. Suppose at time t we have $\mathbf{X}_1[t] > \mathbf{X}_2[t]$ and a demand arrives at node 2.*

Under vanilla MaxWeight policy, we would dispatch from node 1 since there are more vehicles there. However, we claim that vanilla MaxWeight is dominated (in terms of minimizing demand dropping probability) by another policy where one always dispatch from node 2 to serve demand at 2 as long as $\mathbf{X}_2 > 0$. We call this policy the priority policy. To see this intuitively, note that under both policies demand at node 2 will never be dropped, hence demand dropping happens if and only if supply at node 1 is depleted. The priority policy tries to keep all the servers at node 1 while vanilla MaxWeight tries to equal the number of servers on both nodes, hence the priority policy drops less demand. In fact, as we will show formally later, the exponent of demand dropping under the priority policy is twice as large as the exponent under vanilla MaxWeight.

To deal with this issue, we slightly generalize vanilla MaxWeight by attaching a positive scale-down parameter w_i to each queue $i \in V$, and dispatch from the compatible queue with largest *weighted* queue length $\mathbf{X}_i/w_i$. Without loss of generality, we normalize w s.t.

$$\mathbf{1}^T \mathbf{w} = 1, \quad \text{or equivalently,} \quad \mathbf{w} \in \text{relint}(\Omega).$$

We call this family of policies *Scaled MaxWeight (SMW) policies*, and denote SMW with parameter $\mathbf{w}$ as SMW($\mathbf{w}$). Going back to Example 1, we can approximate the priority policy by attaching a much larger scale-down parameter at node 1 than at node 2.

The formal definition of SMW is as follows.

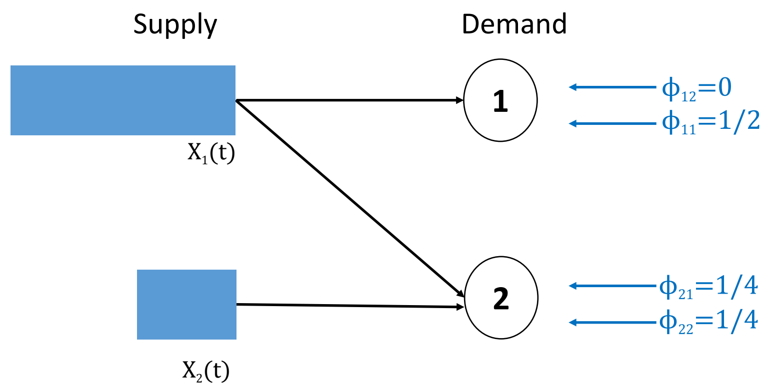

Figure 2: An example of the sub-optimality of vanilla MaxWeight policy.

Definition 1 (Scaled MaxWeight)**.** Given system state $\mathbf{X}[t-1]$ at the start of t-th period and for demand arriving at $o[t]$, SMW($\mathbf{w}$) dispatches from

$$\mathrm{argmax}_{i\in\partial(o[t])}\frac{\mathbf{X}_i[t-1]}{w_i}$$

if $\max_{i\in\partial(o[t])}\frac{\mathbf{X}_i[t-1]}{w_i} > 0$; otherwise the demand is dropped. If there are ties when determining the argmax, dispatch from the location with highest index.

The following fact (which we formalize in Section 5, on the way to proving our main result) gives some intuition about SMW policies.

Remark 3 (Resting point of state under $SMW(\mathbf{w})$). If Assumption 2 holds, the $SMW(\mathbf{w})$ policy causes the normalized system state $\mathbf{X}^K/K$ to drift towards $\mathbf{w}$, where all the scaled queue lengths are equal (to 1).

4 Optimality of Scaled MaxWeight

In this section we present our main result.

4.1 Complete Resource Pooling Condition

The following is the main assumption of this paper.

Assumption 2. *We assume that for any $J \subsetneq V_D$ where $J \neq \emptyset$,*

$$\sum_{i\in\partial(J)} \mathbf{1}^T\phi_{(i)} > \sum_{j'\in J} \mathbf{1}^T\phi_{j'}. \tag{6}$$

The intuition behind this assumption is clear: it assumes the system is "balanceable" in that for each subset $J \subsetneq V_D$ of demand locations we have enough supply at neighboring locations to meet the demand. Assumption 2 is equivalent to a strict version of the condition in Hall's marriage theorem. It is also closely related to Complete Resource Pooling (CRP) condition in queueing literature ([18, 27, 2, 16]). This assumption marks the limit of dispatch policies — no dispatch policy can achieve exponentially decaying demand dropping probability when assumption 4.1 is violated. (if the sign of the inequality is *reversed* in (6) for any subset J, then it is easy to see that, in fact, an $\Omega(1)$ fraction of demand must be dropped under any policy).

Proposition 1. *For any G and ϕ's such that Assumption 2 is violated, it holds that for any policy U, the demand dropping probability does not decay exponentially, i.e., $\gamma(U) = 0$ where $\gamma(U)$ was defined in* (3). *In fact, if the inequality* (6) *in Assumption 2 is strictly* reversed *for some $J \subsetneq V_D$, then we have* $\liminf_{K\to\infty} \underline{P}^{K,U} \geq \epsilon > 0$ *for* $\epsilon = \sum_{j'\in J} \mathbf{1}^T \phi_{j'} - \sum_{i\in\partial(J)} \mathbf{1}^T \phi_{(i)}$.

In other words, if Assumption 2 is violated, this means the system has significant spatial imbalance of demand and stronger forms of control like pricing or repositioning should be employed to restore spatial balance.

4.2 Rate Optimality of Scaled MaxWeight

In the fluid limit, it is easy to find a dispatch rule such that no demand is dropped (see, e.g., [3]). Our goal here is to approach this utopian world as fast as possible as the number of vehicles K grows. Define

$$\mathcal{J} \triangleq \left\{ J \subsetneq V_D : \sum_{i'\in J} \sum_{j\notin\partial(J)} \phi_{i'j} > 0 \right\}, \tag{7}$$

we have $\mathcal{J} \neq \emptyset$ by Assumption 1. The following result characterizes the rate at which the fraction of dropped demand falls as $K \to \infty$.

Theorem 1. *Under Assumptions 1 and 2, the following statements are true:*

1. ***Achievability**: For any $\mathbf{w} \in \text{relint}(\Omega)$, SMW($\mathbf{w}$) achieves exponential decay of the demand dropping probability with exponent*[4,5]

$$\gamma(\mathbf{w}) = \min_{J\in\mathcal{J}} (\mathbf{1}^T_{\partial(J)}\mathbf{w}) \log\left(\frac{\sum_{i'\notin J}\sum_{j\in\partial(J)} \phi_{i'j}}{\sum_{i'\in J}\sum_{j\notin\partial(J)} \phi_{i'j}} \right) > 0\,. \tag{8}$$

2. ***Converse**: Under any policy U, it must be that*

$$\gamma(U) \leq \gamma^*\,,$$

where

$$\gamma^* = \sup_{\mathbf{w}\in\text{relint}(\Omega)} \gamma(\mathbf{w})\,. \tag{9}$$

In words, there is an SMW policy that achieves an exponent arbitrarily close to the optimal one.

The first part of the theorem states that for any SMW policy with $\mathbf{w} \in \text{relint}(\Omega)$, the policy achieves an explicitly specified positive exponent $\gamma(\mathbf{w})$ such that the demand dropping probability decreases at the rate $\Theta(e^{-\gamma(\mathbf{w})K})$ as $K \to \infty$. The second part of the theorem provides a universal upper bound γ^* on the exponent that any policy can achieve, i.e., for any dispatch policy U, demand dropping probability must be $\Omega(e^{-\gamma^* K})$. Crucially, γ^* is in fact identical to the supremum over w

[4] We emphasize that for SMW policies, the $\liminf$ in (2) is a limit, which further does not depend on the starting state $X^{K,U}[0]$, because we show a unique stationary distribution with single positive recurrent class of states (Lemma 7). Further, we prove that the $\liminf$ in (3) is also actually a limit for SMW policies (see Section 6.2). Thus, we are not "cheating" on either count.

[5] Note that the argument of the logarithm has a larger numerator than denominator for every $J \subsetneq V_D$ since Assumption 2 holds, implying that $\gamma(\mathbf{w})$ is the minimum of several positive numbers, and hence is positive. (Also see Remark 4.)

of $\gamma(\mathbf{w})$. In other words, *there is an (almost) exponent optimal SMW policy*, and moreover, this policy can be obtained as the solution to an analytically specified optimization problem.

We note that this result is somewhat different from the numerous results showing optimality of maximum weight matching in various open queuing network settings. Here, our objective is a natural objective that is symmetric in all the queues. Our result says that there is an optimal maximum weight policy, but it is *not* symmetric; rather, it is asymmetric via specific scaling factors in a way that optimally accounts for the primitives of the setting at hand by protecting structurally undersupplied locations.

The following remark provides some intuition regarding the expression for $\gamma(\mathbf{w})$.

Remark 4 (Intuition for $\gamma(\mathbf{w})$). Consider the expression for $\gamma(\mathbf{w})$ in (8). It is a minimum of a "robustness" terms for each subset $J \in \mathcal{J}$ of demand locations. For subset J, the robustness of $SMW(\mathbf{w})$'s ability to serve demand arising in J is the product of two terms:

- Robustness arising from $\mathbf{w}$: At the resting point $\mathbf{w}$ (see Remark 3) of $SMW(\mathbf{w})$, the supply at neighboring locations is $(\mathbf{1}_{\partial(J)}^T\mathbf{w})$, and the larger that is, the more unlikely it is that the subset will be deprived of supply.

- Robustness arising from excess supply: The logarithmic term captures how vulnerable that subset is to being drained of supply. The numerator of the argument of the logarithm can be interpreted as maximum[6] average rate at which supply can come in to $\partial(J)$ from $V_S\backslash\partial(J)$, whereas the denominator can be interpreted as the minimum[6] average rate at which supply must go from $\delta(J)$ to $V_S\backslash\partial(J)$, unless demand is dropped (the larger the ratio, the more oversupplied and hence robust J is). To see why the term appears in this form, recall that the equilibrium distribution of the length of a stable M/M/1 queue is geometric, and hence has an exponentially decaying tail with the exponent corresponding to log(service rate/arrival rate). The "magic" here is that $SMW(\mathbf{w})$ achieves an exponent such that it suffers no loss from the need to protecting multiple J's simultaneously. (The converse result is somewhat more intuitive, pursuing this reasoning further.)

Similarly, we also try and give some intuition regarding the optimal choice of the resting state $\mathbf{w}$.

Remark 5 (Intuition for optimal $\mathbf{w}$). To develop some intuition regarding the optimal choice of $\mathbf{w}$, consider (informally) the special case of a "heavy traffic" type setting where there is just one subset J which has a vanishing logarithmic term (because it is only very slightly oversupplied, the numerator being only slightly more than the denominator), whereas each other subset of V_D has a logarithmic term that converges to some positive number. Then the optimal choice of $\mathbf{w}$ will satisfy $\mathbf{1}_{\partial(J)}^T\mathbf{w} \to 1$, i.e., all but a vanishing fraction of the weight will go to supply locations that can serve J. The intuition is that the random walk for the supply at $\partial(J)$ has only slightly positive drift even if the dispatch protects it, and hence it is optimal to keep the total supply at these locations at a high resting point, to minimize the likelihood that the supply at these locations will be depleted. We think an optimal policy for such a special case is itself interesting; what is more remarkable is that the optimal policy characterized in Theorem 1 solves the general n-dimensional problem considering *all* subsets simultaneously.

Before closing, we point out a few more features of our result:

[6]Which arises if the dispatch policy uses supply at $\partial(J)$ only to serve demand in J, i.e., the policy is maximally "protecting" J.

Novel Lyapunov analysis for a closed queueing network. A key technical challenge we face in our closed queueing network setting is that it is a priori unclear what the "best" state is for the system to be in. This is in contrast to open queueing network settings in which the best state is typically the one in which all queues are empty, and the Lyapunov functions considered typically achieve their minimum at this state. We get around this issue via an innovative approach where we define a tailored Lyapunov function that achieves its minimum at the resting point of the SMW policy we are analyzing, and use this Lyapunov function to characterize its exponent $\gamma(\mathbf{w})$. Moreover, given the optimal choice of $\mathbf{w}$, our tailored Lyapunov function corresponding to this choice of helps us establish our converse result. Our analysis is described in the next section.

Choosing $\mathbf{w}$ *where* ϕ *is imperfectly known.* Another key issue is that of knowledge of the true arrival rates ϕ. We remark that if these rates are entirely unknown, but Assumption 2 holds, the platform can pick an SMW policy such as vanilla max weight, and be sure to achieve exponential decay of the demand dropping probability (albeit with a suboptimal exponent). This is already an improvement over the state-independent control policies studied thus far [23, 3] – in particular, any state-independent policy will drop an $\Omega(1)$ fraction of demand if there is any model misspecification whatsoever. If ϕ is not precisely known but is known to lie within some set, that setting may lend itself to a natural robust optimization problem of maximizing the demand dropping exponent, worse case over possible ϕ. We leave this as an open question.

Future directions: Computational challenges. A limitation of Theorem 1 is that it does not prescribe how to compute $\mathbf{w}$. It simply specifies a concave maximization problem that must be solved, one where the objective is the minimum over an exponential number of linear functions. We suspect that structural properties of "realistic" G and ϕ can be exploited to make this problem tractable; for the moment, we leave it as an open question.

5 Analysis of Scaled MaxWeight Policies

In this section, we formally characterize the system behavior under SMW in fluid scale, and prove an achievability bound of SMW policies using a novel family of Lyapunov functions.

5.1 Fluid Sample Paths

Under any dispatch policy U defined in section 2, the system dynamics is as follows:

$$\begin{aligned}\mathbf{X}_i^K[t+1]-\mathbf{X}_i^K[t]=&\sum_{j'\in V_D}\mathbb{1}\{o[t+1]=j',d[t+1]=i\}\mathbb{1}\{U[t+1](j')\neq\emptyset\}\\&-\sum_{k'\in\partial(i),j\in V_S}\mathbb{1}\{o[t+1]=k',d[t+1]=j\}\mathbb{1}\{i=U[t+1](k')\}.\end{aligned}\tag{10}$$

For SMW($\mathbf{w}$), we write the dispatch mapping as $U^{\mathbf{w}}(\bar{\mathbf{X}},k)$ for scaled queue length $\bar{\mathbf{X}}$ and node k since it only depends on the current scaled state. Let $\mathbf{A}_{i'j}[t]$ be the total number of type (i',j) demands arriving in system during the first t periods ($t=1,2,\cdots$), and let

$$\mathbf{A}_{i'j}^K[t]\triangleq\frac{1}{K}\mathbf{A}_{i'j}[Kt]$$

for $t=0,1/K,2/K,\cdots$. We can rewrite equation (10) in terms of scaled queue-length process

$\bar{\mathbf{X}}^K[t]$:

$$\begin{aligned}\bar{\mathbf{X}}_i^K[t+1/K]-\bar{\mathbf{X}}_i^K[t] = &\sum_{j'\in V_D}\left(\bar{\mathbf{A}}_{j'i}^K[t+1/K]-\bar{\mathbf{A}}_{j'i}^K[t]\right)\mathbb{1}\{U^{\mathbf{w}}(\bar{\mathbf{X}}^K[t],j')\neq\emptyset\}\\ &-\sum_{k'\in\partial(i),j\in V_S}\left(\bar{\mathbf{A}}_{k'j}^K[t+1/K]-\bar{\mathbf{A}}_{k'jl}^K[t]\right)\mathbb{1}\{i=U^{\mathbf{w}}(\bar{\mathbf{X}}^K[t],k')\},\end{aligned} \tag{11}$$

where $t=0,1/K,2/K,\cdots$. For other $t\geq 0$, $\bar{\mathbf{X}}^K[t]$ is defined by linear interpolation.

Definition 2 (Set of demand arrival sample paths)**.** Define $\Gamma^{K,T}\subset C^{n^2}[0,T]$ as the set of all scaled demand arrival sample paths of the K-th system for $\lfloor TK\rfloor$ periods. Mathematically, any $\bar{\mathbf{A}}[\cdot]\in\Gamma^{K,T}$ satisfies:

1. $\bar{\mathbf{A}}(0)=\mathbf{0}$.

2. For any $t=0,1/K,2/K,\cdots,\lfloor TK\rfloor/K$, $\bar{\mathbf{A}}_{i'j}[t]=\bar{\mathbf{A}}_{i'j}[t+1/K]$ for but one pair $(i'_0,j_0)=(o[K(t+1)],d[K(t+1)])$. We have $\bar{\mathbf{A}}_{i'_0j_0}[t+1/K]=\bar{\mathbf{A}}_{i'_0j_0}[T]+1/K$.

3. $\bar{\mathbf{A}}[t]$ is defined by linear interpolation for other values of t.

Definition 3 (Queue-length correspondence of SMW($\mathbf{w}$))**.** For each given demand arrival sample path $\bar{\mathbf{A}}^K[\cdot]\in\Gamma^{K,T}$ and initial state $\bar{\mathbf{X}}^K[0]$, scaled queue length $\bar{\mathbf{X}}^K[\cdot]$ is uniquely (recursively) defined by equation (11). Denote this correspondence by the mapping:

$$\begin{aligned}G^{\mathbf{w}}:C^{n^2}[0,T]\times\Omega&\to C^n[0,T]\\ (\bar{\mathbf{A}}^K[\cdot],\bar{\mathbf{X}}^K[0])&\mapsto\bar{\mathbf{X}}^K[\cdot].\end{aligned}$$

To obtain a large deviation result, we need to look at the queue-length process at the fluid scaling. We take the standard approach of *fluid sample paths* (FSP) (see [26][30]).

Definition 4 (Fluid sample paths)**.** We call a pair $(\bar{\mathbf{A}}[\cdot],\bar{\mathbf{X}}[\cdot])$ a *fluid sample path (under SMW($\mathbf{w}$))* if there exists a sequence $(\bar{\mathbf{A}}^K[\cdot],\bar{\mathbf{X}}^K[0],G^{\mathbf{w}}(\bar{\mathbf{A}}^K[\cdot],\bar{\mathbf{X}}^K[0]))$ where $\bar{\mathbf{A}}^K[\cdot]\in\Gamma^{K,T},\bar{\mathbf{X}}^K[0]\in\Omega$, such that it has a subsequence which converges to $(\bar{\mathbf{A}}[\cdot],\bar{\mathbf{X}}[0],\bar{\mathbf{X}}[\cdot])$ uniformly on $[0,T]$.

Remark 6. Since for any such sequence all elements $(\bar{\mathbf{A}}^K[\cdot],\bar{\mathbf{X}}^K[0],G^{\mathbf{w}}(\bar{\mathbf{A}}^K[\cdot],\bar{\mathbf{X}}^K[0]))$ are Lipschitz continuous with Lipschitz constant 1, it must have a subsequence that converges uniformly to a limit by Arzelà-Ascoli theorem. Meanwhile, uniform convergence passes the Lipschitz continuity to the limit [24], hence all FSPs are Lipschitz continuous.

Remark 7. One must take extra care when stating results involving FSP. Although each $\bar{\mathbf{A}}^K[\cdot]$ uniquely defines a queue-length sample path $\bar{\mathbf{X}}^K[\cdot]$, it is not necessarily true that the $\bar{\mathbf{A}}[\cdot]$ component in FSP uniquely defines the $\bar{\mathbf{X}}[\cdot]$ component. In other words, it is possible that for different $\bar{\mathbf{X}}_1[\cdot]$ and $\bar{\mathbf{X}}_2[\cdot]$, $(\bar{\mathbf{A}}[\cdot],\bar{\mathbf{X}}_1[\cdot])$ and $(\bar{\mathbf{A}}[\cdot],\bar{\mathbf{X}}_2[\cdot])$ are both FSPs.

5.2 Fluid Limits

The fluid limits of the system characterizes its behavior on the Law of Large Numbers scale, i.e. for typical sample paths. Except for the fact that our queueing network is closed rather than open, the results in this section are similar to [13].

For the K-th system, denote $\mathbf{A}_{j'k}^K[t]$ as the total number of type (j',k) demands arriving in system during the first t periods, denote the number of times i is dispatched to serve type (j',k)

demand as $\mathbf{E}^K_{ij'k}[t]$; denote the number of times type (j',k) demand being dropped as $\mathbf{D}^K_{j'k}[t]$. The following equations hold for the system dynamics:

$$\begin{aligned}
&\mathbf{X}^K_i[t] = \mathbf{X}^K_i[0] - \sum_{j'\in\partial(i),k\in V_S} \mathbf{E}^K_{ij'k}[t] + \sum_{j\in V_S}\sum_{k'\in\partial(j)} \mathbf{E}^K_{jk'i}[t],\\
&\quad \forall t\in\{1,2,\cdots\}, i\in V_S\\
&\mathbf{X}^K_i[t]\geq 0, \quad \forall t\in\{1,2,\cdots\}, i\in V_S\\
&\sum_{i\in V_S}\mathbf{X}^K_i[t] = K, \quad \forall t\in\{1,2,\cdots\}\\
&\sum_{i\in\partial(j')} \mathbf{E}^K_{ij'k}[t] + \mathbf{D}^K_{j'k}[t] = \mathbf{A}_{j'k}[t], \quad \forall t\in\{1,2,\cdots\}, j'\in V_D, k\in V_S\\
&\mathbf{E},\mathbf{D} \text{ are non-decreasing and } \mathbf{E}_{ij'k}(0) = \mathbf{D}_{j'k}(0) = \mathbf{0}, \forall j'\in V_D, i,k\in V_S.
\end{aligned}$$

For the class of non-idling policies, i.e. $U[t](o[t])\neq\emptyset$ if $\exists i\in\partial(o[t])$ such that $\mathbf{X}_i[t-1]>0$, the following additional equations hold:

$$\left(\sum_{i\in\partial(j')}\mathbf{X}^K_i[t-1]\right)\left(\mathbf{D}^K_{j'k}[t]-\mathbf{D}^K_{j'k}[t-1]\right) = 0,$$
$$\forall t\in\{1,2,\cdots\}, j'\in V_D, k\in V_S.$$

It's obvious that $SMW(\mathbf{w})$ is non-idling for any $\mathbf{w}\in\mathrm{relint}(\Omega)$.

Proposition 2. *The following holds almost surely. Denote*

$$\begin{aligned}
\bar{\mathbf{X}}^K[t] &= \frac{1}{K}\mathbf{X}^K(\lfloor Kt\rfloor),\\
\bar{\mathbf{E}}^K[t] &= \frac{1}{K}\mathbf{E}^K(\lfloor Kt\rfloor),\\
\bar{\mathbf{D}}^K[t] &= \frac{1}{K}\mathbf{D}^K(\lfloor Kt\rfloor).
\end{aligned}$$

For every sequence of initial conditions $\{\bar{\mathbf{X}}^K[0]\}$, there exists a subsequence indexed by K_r such that as $r\to\infty$,

$$(\bar{\mathbf{X}}^K[\cdot],\bar{\mathbf{E}}^K[\cdot],\bar{\mathbf{D}}^K[\cdot]) \Rightarrow (\bar{\mathbf{X}}[\cdot],\bar{\mathbf{E}}[\cdot],\bar{\mathbf{D}}[\cdot]),$$

where $\Rightarrow$ is uniform convergence on compact sets on the space of RCLL functions on $[0,T]$. All the limits are called fluid limits and are Lipschitz continuous hence almost everywhere differentiable. The differentiable points are called regular points. The fluid limits satisfy the following relations at *the regular points:*

$$
\begin{aligned}
&\frac{d}{dt}\bar{\mathbf{X}}_i[t] = -\sum_{j'\in\partial(i),k\in V_S}\frac{d}{dt}\bar{\mathbf{E}}_{ij'k}[t] + \sum_{j\in V_S}\sum_{k'\in\partial(j)}\frac{d}{dt}\bar{\mathbf{E}}_{jk'i}[t],\\
&\qquad\qquad \forall t\in[0,T], i\in V_S\\
&\bar{\mathbf{X}}_i[t]\geq 0,\quad \forall t\in[0,T], i\in V_S\\
&\sum_i \bar{\mathbf{X}}_i[t] = 1,\quad \forall t\in[0,T]\\
&\sum_{i\in\partial(j')}\frac{d}{dt}\bar{\mathbf{E}}_{ij'k}[t] + \frac{d}{dt}\bar{\mathbf{D}}_{j'k}[t] = \phi_{j'k},\quad \forall t\in[0,T], j'\in V_D, k\in V_S\\
&\bar{\mathbf{E}},\ \bar{\mathbf{D}} \text{ are non-decreasing and } \bar{\mathbf{E}}_{ij'k}(0) = \bar{\mathbf{D}}_{j'k}(0) = 0. \forall j'\in V_D, i,k\in V_S.
\end{aligned}
$$

For non-idling policies, it satisfies additional equations:

$$\int_0^T \left(\sum_{i\in\partial(j')}\bar{\mathbf{X}}_i(t)\right)\frac{d}{dt}\bar{\mathbf{D}}_{j'k}(t) = 0,\quad \forall j'\in V_D, k\in V_S.$$

Proof. The lemma can be proved using an adaptation of proof of Theorem 3.3.1 in [19] and [13]. □

Lemma 1. *Let $\bar{\mathbf{X}}[\cdot]$ be a fluid limit of the system under SMW($\mathbf{w}$) policy. Suppose $t\in[0,T]$ is a regular point, then for any $i\in V_S$ such that $\bar{\mathbf{X}}_i[t]=0$, we have*

$$\frac{d}{dt}\bar{\mathbf{X}}_i[t] > 0.$$

Lemma 1 says that in the fluid limit, any node will be replenished with supplies 'immediately' once it becomes empty. However, one should be careful when interpreting this result because a fluid-scale time interval with length δ corresponds to $\lfloor \delta K \rfloor$ time periods in the original system. Lemma 1 is proved by exploiting the CRP condition and properties of SMW policies.

5.3 Lyapunov Functions for Scaled MaxWeight

Lyapunov functions are useful tool for analyzing complex stochastic systems. In the literature on the MaxWeight policy, the quadratic Lyapunov function is a popular choice to facilitate analysis ([29],[15],[20] etc.); others have also used piecewise linear Lyapunov functions ([5],[30],etc.). None of these suffice for our application, however, and so we need to define a novel family of piecewise linear Lyapunov functions, as follows.

Definition 5. For each $\mathbf{w}\in\mathrm{relint}(\Omega)$, define Lyapunov function $L_{\mathbf{w}}(\mathbf{x})$ as:

$$L_{\mathbf{w}}(\mathbf{x}) \triangleq 1 - \min_i \frac{x_i}{w_i}. \tag{12}$$

Note that for each $\mathbf{x}\in\Omega$, $L_{\mathbf{w}}(\mathbf{x})\in[0,1]$. Furthermore, the intersection of sub-level set $\{\mathbf{x} : L_{\mathbf{w}}(\mathbf{x})\leq 1\}$ and hyperplane $h_1 \triangleq \{\mathbf{x} : \mathbf{1}^{\mathrm{T}}\mathbf{x} = 1\}$ is exactly Ω, as is proved in the next lemma.

Lemma 2. *For any $\mathbf{w}\in\mathrm{relint}(\Omega)$, Lyapunov function $L_{\mathbf{w}}(\mathbf{x})$ satisfies:*

$$\{\mathbf{x} : L_{\mathbf{w}}(\mathbf{x})\leq 1\}\cap h_1 = \Omega.$$

To establish system stability using a Lyapunov function, a key step is to show that it has negative drift along fluid limits. We first prove in Lemma 3 that the chosen Lyapunov function $L_{\mathbf{w}}$ has negative drift under SMW($\mathbf{w}$) when all queue lengths are positive.

Lemma 3. *Let* $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$ *be any FSP on* $[0, T]$, $\mathbf{w} \in \mathrm{relint}(\Omega)$. *For a regular* $t \in [0, T]$ *such that* $L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) < 1$, *define:*

$$A_1(\bar{\mathbf{X}}[t]) \triangleq \left\{ i \in V_S : \bar{\mathbf{X}}_i[t] \in \mathrm{argmin} \frac{\bar{\mathbf{X}}_i[t]}{w_i} \right\},$$
$$A_2\left(\bar{\mathbf{X}}[t], \frac{d}{dt}\bar{\mathbf{X}}[t]\right) \triangleq \left\{ i \in A_1(\bar{\mathbf{X}}[t]) : \frac{d}{dt}\bar{\mathbf{X}}_i[t] \in \mathrm{argmin} \frac{1}{w_i}\frac{d}{dt}\bar{\mathbf{X}}_i[t] \right\}.$$

All the derivatives are well defined since t *is regular.*

Denote

$$c \triangleq \mathrm{argmin}_{i \in A_1(\bar{\mathbf{X}}[t])} \frac{1}{w_i}\frac{d}{dt}\bar{\mathbf{X}}_i[t].$$

We have:

1. $\frac{d}{dt} L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) = -c.$
2. $c = \frac{1}{\mathbf{1}_{A_2}^{\mathrm{T}}\mathbf{w}} \left(\sum_{i' \in V_D, j \in A_2} \frac{d}{dt}\bar{\mathbf{A}}_{i'j}[t] - \sum_{i' : \partial(i') \subseteq A_2, j \in V} \frac{d}{dt}\bar{\mathbf{A}}_{i'j}[t] \right).$

The above lemma is about the Lyapunov drift along fluid sample paths; we now state an analogous lemma bounding the drift for fluid limits. We first need some additional notation.

We denote

$$\lambda_{\min} \triangleq \min_{i \in V_S} \mathbf{1}^T \phi_{(i)} \tag{13}$$

It follows from Assumption 1 that $\lambda_{\min} > 0$. Similarly, we denote

$$\xi \triangleq \min_{J \subsetneq V_D, J \neq \emptyset} \left(\sum_{i \in \partial(J)} \mathbf{1}^T \phi_{(i)} - \sum_{j' \in J} \mathbf{1}^T \phi_{j'} \right), \tag{14}$$

and based on assuming Assumption 2 holds, we have $\xi > 0$.

Lemma 4. *Let* t *be a regular point, then for any fluid limit* $\bar{\mathbf{X}}[\cdot]$*:*

$$\frac{d}{dt} L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) \leq -\min\{\xi, \lambda_{\min}\}.$$

Note that the above result holds for any $\bar{\mathbf{X}} \in \Omega$, including when there are empty queues. The fact that boundary points also achieve negative Lyapunov drift follows from Lemma 1. Finally, the following lemma shows that we still have negative Lyapunov drift if the arrival process is a small perturbation of fluid limits.

Lemma 5. *There exists* $\epsilon > 0$ *such that for all fluid sample paths* $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$ *and any* $t \in [0, T]$, *if* $\frac{d}{dt}\bar{\mathbf{A}}[t] \in B(\phi, \epsilon)$, *we have*

$$\frac{d}{dt} L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) \leq -\frac{1}{2}\min\{\xi, \lambda_{\min}\}.$$

The Lyapunov function we defined admits many nice properties. The following lemma is a collection of basic technical facts regarding $L_{\mathbf{w}}(\mathbf{x})$ that are useful for the following proofs. The norm $||\cdot||$ is Euclidean norm if note stated otherwise.

Lemma 6. *Let $L_{\mathbf{w}}(\mathbf{x})$ be the Lyapunov function defined in (12) ($\mathbf{w} \in \text{relint}(\Omega)$). Note that $L_{\mathbf{w}}(\mathbf{x})$'s domain is $h_1 = \{\mathbf{x} : \mathbf{1}^T\mathbf{x} = 1\}$. We have:*

1. *$L_{\mathbf{w}}(\mathbf{x})$ is a continuous function of $\mathbf{x}$.*

2. *$L_{\mathbf{w}}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in h_1$, and $L_{\mathbf{w}}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{w}$.*

3. *There exists $c_{\mathbf{w}} > 0, l_{\mathbf{w}} > 0, u_{\mathbf{w}} > 0$ such that*

$$\min_{\mathbf{x}:||\mathbf{x}-\mathbf{w}||\geq c_{\mathbf{w}}, \mathbf{x}\in h_1} L_{\mathbf{w}}(\mathbf{x}) \geq l_{\mathbf{w}}, \quad \max_{\mathbf{x}:||\mathbf{x}-\mathbf{w}||\leq c_{\mathbf{w}}, \mathbf{x}\in h_1} L_{\mathbf{w}}(\mathbf{x}) \leq u_{\mathbf{w}}.$$

4. *$L_{\mathbf{w}}(\mathbf{x})$ is globally Lipschitz on h_1, i.e.*

$$|V_{\mathbf{w}}(\mathbf{x}_1) - V_{\mathbf{w}}(\mathbf{x}_2)| \leq \frac{1}{\min_{i\in V_S} w_i}||\mathbf{x}_1 - \mathbf{x}_2||.$$

5. *For all fluid sample paths $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$, there exists $M > 0$ such that:*

$$\frac{d}{dt}L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) \leq M.$$

6. *(Centered linear-in-scale property) For any $\mathbf{z}$ such that $\mathbf{1}^{\mathrm{T}}\mathbf{z} = 0$ and $c > 0$, we have:*

$$L_{\mathbf{w}}(c\mathbf{z} + \mathbf{w}) = cL_{\mathbf{w}}(\mathbf{z} + \mathbf{w}).$$

An important property of SMW($\mathbf{w}$) polices is that they induce unique stationary distribution for each $\mathbf{w}$.

Lemma 7. *Suppose Assumption 1 and 2 hold, then for each K and arbitrary $\mathbf{w} \in \text{relint}(\Omega)$, the queue length process $\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[t]$ has a unique stationary distribution $\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty]$ for large enough K.*

The following lemma is an adaptation of Theorem 5 and Proposition 7 in [30] to our setting. It gives the achievability bound of the exponent of the stationary probability that normalized queue-length $\bar{\mathbf{X}}$ under SMW($\mathbf{w}$) escapes from level set $\{\mathbf{x} : L_{\mathbf{w}}(\mathbf{x}) \leq 1 - \epsilon\}$. To prove this lemma, we first obtain a bound of exponent which is the solution to a calculus-of-variations problem (analogous to Theorem 5 in [30]), where the nature tries to lift $L_{\mathbf{w}}$ from 0 to $1 - \epsilon$ against the negative Lyapunov drift by choosing a demand arrival sample path that is different from the fluid limits. Choosing sample paths incur a cost that is characterized by the large deviation rate function, and the exponent bound is the minimum total cost for the lifting. Then we proceed to show that the above problem can be greatly simplified due to the piecewise linearity of $L_{\mathbf{w}}$ - the least costly sample paths that increases $L_{\mathbf{w}}$ from 0 to $1 - \epsilon$ have a linear structure (Proposition 7 in [30]). See [28][26] for similar results in a different setting where the achievable exponent is given by a variation problem with piecewise linear solution (called 'simple elements' in [28]).

Lemma 8. *For the system being considered, we have* $\forall \epsilon > 0$*,*

$$-\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}\left(L_{\mathbf{w}}(\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty]) \geq 1-\epsilon\right) \geq (1-\epsilon)\gamma(\mathbf{w}),$$

where for fixed $T > 0$*,*

$$\begin{aligned}
\gamma(\mathbf{w}) = &\inf_{v>0, f, \bar{\mathbf{A}}, \bar{\mathbf{X}}} \frac{1}{v}\Lambda^*(\mathbf{f}), \\
&\quad s.t.\ (\bar{\mathbf{A}}, \bar{\mathbf{X}}) \textit{ is a fluid sample path on } [0,T] \textit{ under } \mathrm{SMW}(\mathbf{w}) \\
&\qquad \textit{such that for some regular } t \in [0,T] \\
&\qquad \frac{d}{dt}\bar{\mathbf{A}}[t] = \mathbf{f}, \quad L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) = 1-\epsilon, \quad \frac{d}{dt}L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) = v.
\end{aligned}$$

Remark 8. As is discussed in Remark 6, all FSPs $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$ are Lipschitz continuous, hence $L_{\mathbf{w}}(\cdot)$ is also Lipschitz continuous due to Lemma 6. As a result, $L_{\mathbf{w}}(\bar{\mathbf{X}}[t])$ is almost everywhere differentiable w.r.t. t ([24]).

The next lemma gives an explicit bound of $\gamma(\mathbf{w})$.

Lemma 9.

$$\gamma(\mathbf{w}) \geq \min_{J\in\mathcal{J}} (\mathbf{1}^{\mathrm{T}}_{\partial(J)}\mathbf{w}) g(\phi, J),$$

$$\text{where} \quad g(\phi, J) \triangleq \log\left(\frac{\sum_{i'\notin J}\sum_{j\in\partial(J)} \phi_{i'j}}{\sum_{i'\in J}\sum_{j\notin\partial(J)} \phi_{i'j}}\right)$$

Now we are ready to provide an achievability bound of demand-drop rate, which is part of the proof of Theorem 1 claim (1).

An achievability bound of SMW(w): Under SMW($\mathbf{w}$), we have

$$\begin{aligned}
\mathbb{P}(\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty] \in D) \leq &\mathbb{P}(\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty] \in \{\mathbf{x} : \exists i \in V_S\ s.t. \mathbf{x}_i = 0\}) \\
\leq &\mathbb{P}(L_{\mathbf{w}}(\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty]) \geq 1-\epsilon) \qquad (\epsilon > 0, \text{Lemma 2}).
\end{aligned}$$

Using Lemma 8 and Lemma 9, for any $\epsilon > 0$ we have

$$-\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}(\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty] \in D) \geq (1-\epsilon)\gamma(\mathbf{w}).$$

Let $\epsilon \to 0$, we have the following bound:

$$-\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}(\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty] \in D) \geq \gamma(\mathbf{w}). \tag{15}$$

□

Combined with the converse bound that will be derived in the next section, we can show that the lim sup in (15) is in fact the limit.

5.4 Performance of Vanilla MaxWeight

The following theorem shows that SMW with a naive choice of parameter $\mathbf{w} = \frac{1}{n}\mathbf{1}$ achieves an exponent of demand dropping rate that is no smaller than $\frac{1}{n}$ of γ^*.

Theorem 2. *Suppose Assumption 2 holds, then we have*

$$\gamma\left(\frac{1}{n}\mathbf{1}\right) \geq \frac{1}{n}\gamma^*.$$

6 Proofs of Converse Bounds

In this section, we will complete the proof of Theorem 1 and provide a converse bound for state independent policies.

The key observation in converse proofs is that the number of vehicles in a subset of supply locations is upper bounded by a random walk. Under Assumption 2, the best state-dependent policy can provide this random walk with a positive drift, while the best state-independent policy only leads to zero drift. As a result of this difference, the lower bound of demand-drop probability is exponential in K for state-dependent policies, and polynomial in K for state-independent policies.

6.1 Proof of Theorem 1 claim (2): Converse Bound of State-Dependent Policies

We first prove the following lemma that characterizes the probability of a positive drift random walk hitting a negative value with large magnitude.

Lemma 10. *Let $\{Z_i\}_{i=1}^{\infty}$ be i.i.d. random variables such that $\mathbb{P}(Z_1 = 1) = p$, $\mathbb{P}(Z_1 = -1) = q$, $\mathbb{P}(Z_1 = 0) = 1 - p - q$, where $0 < q < p < 1$, $p + q \leq 1$. Define $S_n \triangleq \sum_{i=1}^{n} Z_i$. For any $\kappa \leq 0$ we have*

$$\mathbb{P}(S_{\kappa/(q-p)} \leq \kappa) \geq e^{\kappa\epsilon\frac{-\log p}{(p-q)(1-(p-q))}} e^{\kappa\frac{\Lambda_1^*((q-p)-\epsilon)}{p-q}} \left(1 - e^{\frac{\kappa\epsilon^2}{2(p-q)}}\right)$$

for any $\epsilon > 0, \kappa \leq 0$.

Lemma 11. *For random variable Z_1 defined in Lemma 10, we have:*

$$\frac{\Lambda_1^*(q-p)}{p-q} = \log\left(\frac{p}{q}\right).$$

Now we are ready to give an explicit converse bound using the technical Lemma 10 and 11 regarding large deviation behavior of random walks. Note that the proof does not involve any dispatch policy.

Proof of Theorem 1, claim (2): For any $J \in \mathcal{J}$, define:

$$B_J \triangleq \left\{\mathbf{x} \in \Omega : (\mathbf{1}_{\partial(J)}^T\mathbf{x})g(\phi, J) = \operatorname{argmin}_{A\in\mathcal{J}}(\mathbf{1}_{\partial(A)}^T\mathbf{x})g(\phi, A)\right\}.$$

To make $\{B_J\}_{J\in\mathcal{J}}$ a non-overlapping partition, index J by natural numbers and only keep $\mathbf{x}$'s membership in the set with smallest index J.

Recall the definition ξ in (14), define

$$\xi_J \triangleq \sum_{i'\notin J}\sum_{j\in\partial(J)} \phi_{i'j} - \sum_{i'\in J}\sum_{j\notin\partial(J)} \phi_{i'j}.$$

By Assumption 2, we have:

$$1 > \xi_{\max} \triangleq \max_{J\in\mathcal{J}} \xi_J \geq \min_{J\in\mathcal{J}} \xi_J \geq \xi > 0.$$

For the K-th system, we divide the discrete periods into cycles with length K/ξ (ignoring the technicality regarding the integrality of K/ξ). We are going to lower bound the probability of demand dropping in any cycle for any state-dependent dispatch policy U, even including time-varying policies. Without loss of generality, consider the first cycle $[0, K/\xi - 1]$. Suppose $\bar{\mathbf{X}}^{K,U}[0] = \mathbf{y} \in B_J$.

Consider the random walk $S_J[t]$ with the following dynamics:

- $S_J[0] = \mathbf{1}^{\mathrm{T}}_{\partial(J)}\mathbf{y}$.
- $S_J[t+1] = S_J[t] + 1$ if $o[s] \notin J, d[s] \in \partial(J)$.
- $S_J[t+1] = S_J[t] - 1$ if $o[s] \in J, d[s] \notin \partial(J)$.
- $S_J[t+1] = S_J[t]$ if otherwise.

Observe that:

$$\mathbb{P}\left(\text{some is was dropped in } [0, K/\xi - 1]\right) \geq \mathbb{P}\left(S_J[t] = 0 \text{ for some } t' \in [0, K/\xi]\right).$$

This is because either (1) some demand is dropped before t', or (2) no demand is dropped before t', then $S_J[t] \geq \mathbf{1}^T_{\partial(J)}\bar{\mathbf{X}}^{K,U}[t]$ holds for all $t \leq t'$; $S_J(t') = 0$ implies that there is no supply in $\partial(J)$ at t'. Either way, the event on RHS implies the event on LHS.

Plug in Lemma 10, we have:

$$\begin{aligned}
&\mathbb{P}\left(S_J[t] - S_J[0] = -K(\mathbf{1}^T_{\partial(J)}\mathbf{y}) \text{ for some } t' \in [0, K/\xi]\right) \\
\geq &\mathbb{P}\left(S_J[t] - S_J[0] = -K(\mathbf{1}^T_{\partial(J)}\mathbf{y}) \text{ for } t' = K(\mathbf{1}^T_{\partial(J)}\mathbf{y})/\xi_J\right) \\
\geq &e^{-K\epsilon\frac{-\log\xi}{(1-\xi_{\max})\xi}} e^{-(\mathbf{1}^T_J\mathbf{y})K\frac{\Lambda_1^*(\xi_J - \epsilon)}{\xi_J}} \left(1 - e^{-\frac{(\mathbf{1}^T_{\partial(J)}\mathbf{y})K\epsilon^2}{2\xi_{\max}}}\right).
\end{aligned}$$

Fix ϵ. Note that if $\mathbf{1}^T_{\partial(J)}\mathbf{y} = 0$ then $\bar{\mathbf{X}}^{K,U}$ already hit D in this cycle; otherwise the last term is at least $1 - \exp(-\frac{\epsilon^2}{2\xi_{\max}}) > 0$, i.e. bounded away from zero.

Hence

$$\begin{aligned}
\liminf_{K\to\infty} \frac{1}{K} \log \underline{P}^{K,U} \geq &\liminf_{K\to\infty} \frac{1}{K} \log \frac{\mathbb{P}(\bar{\mathbf{X}}^{K,U}[t] \in D \text{ for some } t \in [0, K/\xi - 1])}{K/\xi} \\
\geq &-\epsilon\frac{-\log\xi}{(1-\xi_{\max})\xi} - \max_{\mathbf{y}\in\Omega}\left((\mathbf{1}^T_{\partial(J)}\mathbf{y})\frac{\Lambda_1^*(\xi_J - \epsilon)}{\xi_J}\right).
\end{aligned}$$

Since we can choose arbitrary $\epsilon > 0$ and $\Lambda_1^*(\cdot)$ is continuous, hence by letting $\epsilon \to 0$ we have:

$$\begin{aligned}
\liminf_{K\to\infty} \frac{1}{K} \log \underline{P}^{K,U} &\geq -\max_{\mathbf{y}\in\Omega}\left((\mathbf{1}^T_{\partial(J)}\mathbf{y})\frac{\Lambda_1^*(\xi_J)}{\xi_J}\right) \\
&= -\max_{\mathbf{y}\in\Omega}(\mathbf{1}^T_{\partial(J)}\mathbf{y})g(\phi, J) = -\max_{\mathbf{y}\in\Omega}\min_{J\in\mathcal{J}}\left((\mathbf{1}^T_{\partial(J)}\mathbf{y})g(\phi, J)\right) = \gamma^*.
\end{aligned}$$

Here the first equality is due to Lemma 11, the second equality comes from J's dependence on $\mathbf{y}$.

6.2 Proof of Theorem 1 claim (1): Exponent of SMW Policies

Recall that we proved in (15)

$$-\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}(\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty] \in D) \geq \gamma(\mathbf{w}).$$

To complete the proof of Theorem 1 claim (1), it suffices to show that

$$-\liminf_{K\to\infty} \frac{1}{K} \log \mathbb{P}(\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}[\infty] \in D) \leq \gamma(\mathbf{w}). \tag{16}$$

Proof of Theorem 1 Claim (1). In the proof of Lemma 7 we showed that the system has one positive recurrent class that includes $K\mathbf{w}$ (ignore the technicality of its intergrality for now, see the proof of Lemma 7 for more details). Divide the discrete periods into regenerative cycles by hitting time of $K\mathbf{w}$. To show (16), it suffices to lower bound the probability of $\bar{\mathbf{X}}^{K,SMW(\mathbf{w})}$ hitting D in each cycle. Using the same techniques as in section 6.1, we can show that the this probability is of order $\Omega\left(\exp^{-\gamma(\mathbf{w})}\right)$, the details are omitted here. Since the number of states in Ω_K is polynomial in K, the expected length of each regenerative cycle is also polynomial. Hence by SLLN of renewal-reward process we have (16). □

6.3 No State-Independent Policy Achieves Exponential Decay

In the following, we show that under no circumstances can any state-independent (static) dispatch policy lead to exponential decay of demand dropping probability. We first formally define the set of state-independent policies.

Definition 6. We call a dispatch policy state-independent if for any time t, $j' \in V_D$, if a demand arrives at j' at t, the platform randomly selects a dispatch location $i \in \partial(j')$ with probability $u_{(i,j')}[t]$, or drop it with probability $1 - \sum_{i\in\partial(j')} u_{(i,j')}[t]$. If there is no supply at the dispatch location, the demand is also dropped.

Remark 9. The state-independent policy defined above includes time-varying policies. It is the same as the randomized policy defined in [23], definition 2.

Proposition 3. *Under any state-independent dispatch policy π, we have:*

$$P^{K,\pi} = \Omega\left(\frac{1}{K^2}\right).$$

References

[1] Daniel Adelman. Price-directed control of a closed logistics queueing network. *Operations Research*, 55(6):1022–1038, 2007.

[2] Baris Ata, Sunil Kumar, et al. Heavy traffic analysis of open processing networks with complete resource pooling: asymptotic optimality of discrete review policies. *The Annals of Applied Probability*, 15(1A):331–391, 2005.

[3] Siddhartha Banerjee, Daniel Freund, and Thodoris Lykouris. Pricing and optimization in shared vehicle systems: An approximation framework. *CoRR abs/1608.06819*, 2016.

[4] Siddhartha Banerjee, Carlos Riquelme, and Ramesh Johari. Pricing in ride-share platforms: A queueing-theoretic approach. 2015.

[5] Dimitris Bertsimas, David Gamarnik, and John N Tsitsiklis. Performance of multiclass markovian queueing networks via piecewise linear lyapunov functions. *Annals of Applied Probability*, pages 1384–1428, 2001.

[6] Kostas Bimpikis, Ozan Candogan, and Saban Daniela. Spatial pricing in ride-sharing networks. 2016.

[7] Jose Blanchet. Optimal sampling of overflow paths in jackson networks. *Mathematics of Operations Research*, 38(4):698–719, 2013.

[8] Shreeshankar Bodas, Sanjay Shakkottai, Lei Ying, and R Srikant. Scheduling in multi-channel wireless networks: Rate function optimality in the small-buffer regime. *IEEE Transactions on Information Theory*, 60(2):1101–1125, 2014.

[9] Anton Braverman, JG Dai, Xin Liu, and Lei Ying. Empty-car routing in ridesharing systems. *arXiv preprint arXiv:1609.07219*, 2016.

[10] Gerard P Cachon, Kaitlin M Daniels, and Ruben Lobel. The role of surge pricing on a service platform with self-scheduling capacity. *Manufacturing & Service Operations Management*, 2017.

[11] Kai Lai Chung. *A course in probability theory*. Academic press, 2001.

[12] JG Dai, Wuqin Lin, et al. Asymptotic optimality of maximum pressure policies in stochastic processing networks. *The Annals of Applied Probability*, 18(6):2239–2299, 2008.

[13] Jim G Dai and Wuqin Lin. Maximum pressure policies in stochastic processing networks. *Operations Research*, 53(2):197–218, 2005.

[14] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Application of Mathematics*. Springer, 2nd edition, 1998.

[15] Atilla Eryilmaz and R Srikant. Asymptotically tight steady-state queue length bounds implied by drift conditions. *Queueing Systems*, 72(3-4):311–359, 2012.

[16] Itay Gurvich and Ward Whitt. Scheduling flexible servers with convex delay costs in many-server service systems. *Manufacturing & Service Operations Management*, 11(2):237–253, 2009.

[17] Jonathan Hall, Cory Kendrick, and Chris Nosko. The effects of uber's surge pricing: A case study. *The University of Chicago Booth School of Business*, 2015.

[18] J Michael Harrison and Marcel J López. Heavy traffic resource pooling in parallel-server systems. *Queueing systems*, 33(4):339–368, 1999.

[19] Sunil Kumar and PR Kumar. Closed queueing networks in heavy traffic: Fluid limits and efficiency. In *Stochastic Networks*, pages 41–64. Springer, 1996.

[20] Siva Theja Maguluri, R Srikant, et al. Heavy traffic queue length behavior in a switch under the maxweight algorithm. *Stochastic Systems*, 6(1):211–250, 2016.

[21] Kurt Majewski and Kavita Ramanan. How large queue lengths build up in a jackson network. 2008.

[22] Sean Meyn. Stability and asymptotic optimality of generalized maxweight policies. *SIAM Journal on control and optimization*, 47(6):3259–3294, 2009.

[23] Erhun Ozkan and Amy R Ward. Dynamic matching for real-time ridesharing. 2016.

[24] Walter Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1964.

[25] Cong Shi, Yehua Wei, and Yuan Zhong. Process flexibility for multi-period production systems. 2015.

[26] Alexander L Stolyar. Control of end-to-end delay tails in a multiclass network: Lwdf discipline optimality. *Annals of Applied Probability*, pages 1151–1206, 2003.

[27] Alexander L Stolyar et al. Maxweight scheduling in a generalized switch: State space collapse and workload minimization in heavy traffic. *The Annals of Applied Probability*, 14(1):1–53, 2004.

[28] Alexander L Stolyar and Kavita Ramanan. Largest weighted delay first scheduling: Large deviations and optimality. *Annals of Applied Probability*, pages 1–48, 2001.

[29] Leandros Tassiulas and Anthony Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE transactions on automatic control*, 37(12):1936–1948, 1992.

[30] VJ Venkataramanan and Xiaojun Lin. On the queue-overflow probability of wireless systems: A new approach combining large deviations with lyapunov functions. *IEEE Transactions on Information Theory*, 59(10):6367–6392, 2013.

[31] Ariel Waserhole and Vincent Jost. Pricing in vehicle sharing systems: Optimization in queuing networks with product forms. *EURO Journal on Transportation and Logistics*, 5(3):293–320, 2016.

A Technical Proofs

A.1 Proof of Observation 2

Proof. First, since $\mathbf{1}^{\mathrm{T}}\phi\mathbf{1} = 1$ and $D_{I_{\mathrm{em}}(\bar{\mathbf{X}}^{K,U}[t])} \subseteq D$, we have

$$\mathbf{1}^{\mathrm{T}}_{\{i':\partial(i')\subseteq I_{\mathrm{em}}(\bar{\mathbf{X}}^{K,U}[t])\}}\phi\mathbf{1} \leq \mathbb{1}\left\{\bar{\mathbf{X}}^{K,U}[t] \in D\right\}.$$

On the other hand, whenever $\bar{\mathbf{X}}^{K,U}[t] \in D$, by definition of D there exists $A \subset V_D$ where $\mathbf{1}^{\mathrm{T}}_A\phi\mathbf{1} > 0$ such that

$$\bar{\mathbf{X}}^{K,U}[t] \in D_{\partial(A)},$$

hence

$$\left(\min_{A\subseteq V_D, \mathbf{1}^{\mathrm{T}}_A\phi\mathbf{1}>0} \mathbf{1}^T_A\phi\mathbf{1}\right) \mathbb{1}\left\{\bar{\mathbf{X}}^{K,U}[t] \in D\right\} \leq \mathbf{1}^{\mathrm{T}}_{\{i':\partial(i')\subseteq I_{\mathrm{em}}(\bar{\mathbf{X}}^{K,U}[t])\}}\phi\mathbf{1}.$$

As a result, the ratio between argument of logarithm on both sides of (4) is bounded away from zero and infinity, hence they have the same exponential decay rate. □

A.2 Proof of Proposition 1

Proof. There are two cases:

1. If some inequality in (6) is strictly reversed, i.e., there exists $A \subsetneq V_D$ s.t. $\sum_{i\in\partial(J)} \mathbf{1}^{\mathrm{T}}\phi_{(i)} < \sum_{j'\in J} \mathbf{1}^{\mathrm{T}}\phi_{j'}$, consider the following balance equation:

$$\begin{aligned}\#\{\text{supply in } \partial(J) \text{ at } t\} =& \#\{\text{supply in } \partial(J) \text{ at } 0\} + \#\{\text{supply arrive to } \partial(J) \text{ in } [0,t]\} \\ &- \#\{\text{demand should be served by } \partial(J) \text{ in } [0,t]\} \\ &+ \#\{\text{dropped demand that should be served by } \partial(J) \text{ in } [0,t]\} \\ \le& \#\{\text{supply in } \partial(J) \text{ at } 0\} + \sum_{s=1}^{t} \mathbb{1}\{d[s]\in\partial(J)\} - \sum_{s=1}^{t} \mathbb{1}\{o[s]\in J\} \\ &+ \#\{\text{dropped demand in } [0,t]\}\end{aligned}$$

   Divide both sides by K and let $K\to\infty$, by SLLN we have:

$$\lim_{t\to\infty} \{\text{proportion of dropped demand in } [0,t]\} \ge \sum_{j'\in J} \mathbf{1}^{\mathrm{T}}\phi_{j'} - \sum_{i\in\partial(J)} \mathbf{1}^{\mathrm{T}}\phi_{(i)} > 0,$$

   hence a positive portion of demand will be dropped, and the second half of the proposition is proved.

2. If all the inequalities in (6) are equalities, by Assumption 1 there exists $J \in \mathcal{J}$ and $\sum_{i\in\partial(J)} \mathbf{1}^{\mathrm{T}}\phi_{(i)} = \sum_{j'\in J} \mathbf{1}^{\mathrm{T}}\phi_{j'}$, divide the discrete time periods into cycles with length MK^2, where

$$M \triangleq \frac{6}{\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}}.$$

   Without loss of generality, consider the first cycle $[0, MK^2-1]$.

   Define random walk $S_J[t]$ with the following dynamics:

   - $S_J[0] = \mathbf{1}^{\mathrm{T}}_{\partial(J)} X[0]$.
   - $S_J[t+1] = S_J[t] + 1$ if $o[s]\notin J, d[s]\in\partial(J)$.
   - $S_J[t+1] = S_J[t] - 1$ if $o[s]\in J, d[s]\notin\partial(J)$.
   - $S_J[t+1] = S_J[t]$ if otherwise.

   We have:

$$\mathbb{P}\left(\text{demand is dropped in } [0, MK^2-1]\right) \ge \mathbb{P}\left(S_J[t'] = 0 \text{ for some } t' \in [0, MK^2-1]\right).$$

   This is because either (1) some demand is dropped before t', or (2) no demand is dropped before t', then $S_J[t] \ge \mathbf{1}^{T}_{\partial(J)} \mathbf{X}^{K,U}[t]$ holds for all $t \le t'$. Either way, the event on RHS implies the event on LHS.

For J of interest, note that S_J is an unbiased random walk, hence a martingale. It's easy to verify that

$$MG_J[t] = S_J^2[t] - 2t \sum_{j' \notin J, i \in \partial(J)} \phi_{j'i}$$

is also a martingale. Apply Optional Stopping Theorem, we have the following result for the first time S_J hit $\{\mathbf{1}_{\partial(J)}^{\mathrm{T}}\mathbf{X}[0] - 2K, \mathbf{1}_{\partial(J)}^{\mathrm{T}}\mathbf{X}[0] + 2K\}$, denoted by τ:

$$\mathbb{E}[\tau] = \frac{2K^2}{\sum_{j' \notin J, i \in \partial(J)} \phi_{j'i}}.$$

Note that S_J hitting $\{\mathbf{1}_{\partial(J)}^{\mathrm{T}}\mathbf{X}[0] - 2K\}$ implies that it hit 0 before, we have

$$\begin{aligned}
&\mathbb{P}\left(S_J[t] = 0 \text{ for some } t' \in [0, MK^2 - 1]\right) \\
=&1 - \mathbb{P}(\tau > MK^2) - \mathbb{P}(\tau \leq MK^2)\mathbb{P}(\mathbf{X}[\tau] = \mathbf{1}_{\partial(J)}^{\mathrm{T}}\mathbf{X}[0] + 2K | \tau \leq MK^2) \\
\geq&1 - \mathbb{P}(\tau > 3\mathbb{E}[\tau]) - \frac{1}{2} \\
\geq&\frac{1}{6}. \quad \text{(Markov inequality)}
\end{aligned}$$

As a result, for any U, we have:

$$\underline{P}^{K,U} = \Omega\left(\frac{1}{K^2}\right),$$

hence $\gamma(U) = 0$ for any U.

□

A.3 Proof of Lemma 1

Proof. We prove the result by contradiction. The set V_S can be partitioned into three disjoint sets:

$$\begin{aligned}
A_1[t] &= \{i \in V_S : \bar{\mathbf{X}}_i[t] > 0\}, \\
A_2[t] &= \{i \in V_S : \bar{\mathbf{X}}_i[t] = 0, \frac{d}{dt}\bar{\mathbf{X}}_i[t] > 0\}, \\
A_3[t] &= \{i \in V_S : \bar{\mathbf{X}}_i[t] = 0, \frac{d}{dt}\bar{\mathbf{X}}_i[t] = 0\}.
\end{aligned}$$

Suppose $A_3[t] \neq \emptyset$.

Denote:

$$\begin{aligned}
x_{min}^{+} &\triangleq \min_{i \in A_1[t]} \frac{\bar{\mathbf{X}}_i[t]}{w_i}, \\
p_{min}^{+} &\triangleq \min_{i \in A_1[t]} \frac{d}{dt}\frac{\bar{\mathbf{X}}_i[t]}{w_i}, \\
p_{max}^{0} &\triangleq \max_{i \in A_2[t]} \frac{d}{dt}\frac{\bar{\mathbf{X}}_i[t]}{w_i}, \\
p_{min}^{0} &\triangleq \min_{i \in A_2[t]} \frac{d}{dt}\frac{\bar{\mathbf{X}}_i[t]}{w_i}.
\end{aligned}$$

By definition of derivatives, we know that for arbitrary $\delta > 0$, $\exists \epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, we have:

$$\begin{aligned}
&\frac{\bar{\mathbf{X}}_i[t+\epsilon]}{w_i} \geq \frac{\bar{\mathbf{X}}_i[t]}{w_i} - (p^+_{min} + \delta)\epsilon, && \forall i \in A_1[t] \\
&\frac{\bar{\mathbf{X}}_i[t+\epsilon]}{w_i} \leq (p^0_{max} + \delta)\epsilon, && \forall i \in A_2[t] \\
&\frac{\bar{\mathbf{X}}_i[t+\epsilon]}{w_i} \geq (p^0_{min} - \delta)\epsilon, && \forall i \in A_2[t] \\
&\frac{\bar{\mathbf{X}}_i[t+\epsilon]}{w_i} \leq \delta\epsilon, && \forall i \in A_3[t].
\end{aligned}$$

Choose $\delta = \frac{p^0_{min}}{3}$, $\epsilon_0 \leq \frac{x^+_{min}}{2(p^0_{max}+|p^+_{min}|+2\delta)}$.

For $M > 1$ and any subsequence $\bar{\mathbf{X}}^{K_r}[\cdot]$ (as in Proposition 2) that uniformly converges to $\bar{\mathbf{X}}[\cdot]$, there exists $K_0 > 0$ such that for any $K_r > K_0$, we have:

$$\sup_{t\in[0,T]} \left| \frac{\bar{\mathbf{X}}_i[t]}{w_i} - \frac{\bar{\mathbf{X}}^{K_r}_i[t]}{w_i} \right| < \frac{\delta\epsilon_0}{6M}, \quad \forall i \in V_S.$$

As a result, for $K_r > N_0$ in the uniformly converging subsequence, the K_r-th system satisfies:

$$\frac{\bar{\mathbf{X}}_i[\tau]}{w_i} < \frac{\bar{\mathbf{X}}_j[\tau]}{w_j}, \forall i \in A_3[t], j \in A_1[t] \cup A_2[t]$$

for $\tau \in \{\lfloor K_r(t+\frac{\epsilon_0}{M})\rfloor, \cdots, \lfloor K_r[t+\epsilon_0]\rfloor\}$. Under SMW($\mathbf{w}$), all the demands arriving at $d(A_1[t] \cup A_2[t])$ will be met by supply within $A_1[t] \cup A_2[t]$ during $\{\lfloor K_r(t+\frac{\epsilon_0}{M})\rfloor, \cdots, \lfloor K_r[t+\epsilon_0]\rfloor\}$ for the K_r-th system. Moreoever, supplies in $A_1[t] \cup A_2[t]$ will not be depleted during this period. As a result, we have:

$$\begin{aligned}
&\sum_{i\in A_1[t]\cup A_2[t]} \bar{\mathbf{X}}_i(\lfloor K_r[t+\epsilon_0]\rfloor) - \sum_{i\in A_1[t]\cup A_2[t]} \bar{\mathbf{X}}_i(\lfloor K_r t\rfloor) \\
\leq &\lfloor K_r \frac{\epsilon_0}{M}\rfloor + \sum_{\tau=\lfloor K_r(t+\frac{\epsilon_0}{M})\rfloor}^{\lfloor K_r(t+\epsilon_0)\rfloor} \left(\sum_{j'\in V_D, i\in A_1[t]\cup A_2[t]} \mathbb{1}\{o[\tau]=j', d[\tau]=i\} \right. \\
&\left. - \sum_{j'\in\partial(A_1[t]\cup A_2[t]), i\in V_S} \mathbb{1}\{o[\tau]=j', d[\tau]=i\} \right).
\end{aligned}$$

LHS of the above inequality is the net increase of supplies in subset $A_1[t] \cup A_2[t]$, RHS is an upper-bound that let supplies accumulate in $A_1[t] \cup A_2[t]$ as the fastest speed possible (one supply per period) during the first $\frac{1}{M}$ fraction of the considered time interval, and use supplies in $A_1[t] \cup A_2[t]$ to serve all demands in $\partial(A_1[t] \cup A_2[t])$ during the rest of the period. Divide both sides by K_r and let $r \to \infty$. By SLLN we have almost surely,

$$\begin{aligned}
&\sum_{i\in A_1[t]\cup A_2[t]} \bar{\mathbf{X}}_i[t+\epsilon_0] - \sum_{i\in A_1[t]\cup A_2[t]} \bar{\mathbf{X}}_i[t] \\
\leq &\, t\frac{\epsilon_0}{M} + \frac{M-1}{M}\epsilon_0 \left(\sum_{j'\in V_D, i\in A_1[t]\cup A_2[t]} \phi_{j'i} - \sum_{j'\in\partial(A_1[t]\cup A_2[t]), i\in V_S} \phi_{j'i} \right).
\end{aligned}$$

Since we assume that $A_1[t] \cup A_2[t] \subsetneq V$, by Assumption 2 and (14) we have:

$$\sum_{i \in A_1[t] \cup A_2[t]} \bar{\mathbf{X}}_i[t+\epsilon_0] - \sum_{i \in A_1[t] \cup A_2[t]} \bar{\mathbf{X}}_i[t] \leq t\frac{\epsilon_0}{M} - \frac{M-1}{M}\epsilon_0 \xi.$$

Since M can be chosen arbitrarily, we choose increasingly large M and have:

$$\sum_{i \in A_1[t] \cup A_2[t]} \bar{\mathbf{X}}_i[t+\epsilon_0] - \sum_{i \in A_1[t] \cup A_2[t]} \bar{\mathbf{X}}_i[t] \leq -\epsilon_0 \xi.$$

Since the system is closed, this is equivalent to:

$$\sum_{i \in A_3[t]} \bar{\mathbf{X}}_i[t+\epsilon_0] - \sum_{i \in A_3[t]} \bar{\mathbf{X}}_i[t] \geq \epsilon_0 \xi. \tag{17}$$

But we derived above that

$$\frac{\bar{\mathbf{X}}_i[t+\epsilon_0]}{w_i} \leq \frac{\bar{\mathbf{X}}_i[t]}{w_i} + \delta\epsilon_0, \quad \forall i \in A_3[t].$$

Let $\delta < \frac{1}{2}\xi$, then the above implies

$$\sum_{i \in A_3[t]} \bar{\mathbf{X}}_i[t+\epsilon_0] - \sum_{i \in A_3[t]} \bar{\mathbf{X}}_i[t] \leq \frac{1}{2}\left(\sum_{i \in V_S} w_i\right)\epsilon_0 \xi = \frac{1}{2}\epsilon_0 \xi. \tag{18}$$

(17) and (18) contradicts each other, hence the lemma is proved. □

A.4 Proof of Lemma 2

Proof.

$$\{\mathbf{x} : L_{\mathbf{w}}(\mathbf{x}) \leq 1, \mathbf{x} \in h_1\} = \left\{\mathbf{x} : \min_{i \in V_S} \frac{x_i}{w_i} \geq 0, \mathbf{1}^T \mathbf{x} = 1\right\} = \left\{\mathbf{x} : \mathbf{x} \geq 0, \mathbf{1}^{\mathrm{T}} \mathbf{x} = 1\right\} = \Omega.$$

□

A.5 Proof of Lemma 3

Proof. For notation simplicity, we will write $A_1(\bar{\mathbf{X}}[t])$ as A_1, $A_2\left(\bar{\mathbf{X}}[t], \frac{d}{dt}\bar{\mathbf{X}}[t]\right)$ as A_2 in the following.

1. Denote $b \triangleq \operatorname{argmin}_{i \in V_S} \frac{\bar{\mathbf{X}}_i[t]}{w_i}$. By definition of derivatives, $\forall \delta > 0$, $\exists \epsilon_0 > 0$ such that $\forall \epsilon \in [0, \epsilon_0]$,

$$\frac{\bar{\mathbf{X}}_i(t+\epsilon)}{w_i} \geq b + (c-\delta)\epsilon \quad \forall i \in A_1.$$

Hence

$$\frac{d}{dt} L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) = -\lim_{\epsilon \to 0} \frac{\min_{i \in A_1} \frac{\bar{\mathbf{X}}_i(t+\epsilon)}{w_i} - b}{\epsilon} \leq -\lim_{\epsilon \to 0} \frac{b + (c-\delta)\epsilon - b}{\epsilon} = \delta - c.$$

Since $\delta > 0$ is chosen arbitrarily, we have $\frac{d}{dt} L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) \leq -c$. Similarly, we can show that $\frac{d}{dt} L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) \geq -c$, hence:

$$\frac{d}{dt} L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) = -c.$$

2. Define
$$c' \triangleq \text{argmin}_{i \in A_1 \backslash A_2} \frac{1}{w_i} \frac{d}{dt} \bar{\mathbf{X}}_i[t],$$
then $c' > c$. By definition of derivatives, there exists $\epsilon_0 > 0$ such that $\forall \epsilon \in [0, \epsilon_0]$,
$$\frac{\bar{\mathbf{X}}_k(t+\epsilon)}{w_k} \geq b + \left(c' - \frac{c'-c}{3}\right)\epsilon, \quad \forall k \in A_1 \backslash A_2 \tag{19}$$
$$\frac{\bar{\mathbf{X}}_l(t+\epsilon)}{w_l} \leq b + \left(c + \frac{c'-c}{3}\right)\epsilon, \quad \forall l \in A_2. \tag{20}$$
Let $(\bar{\mathbf{A}}^K[\cdot], \bar{\mathbf{X}}^K[0], G^{\mathbf{w}}(\bar{\mathbf{A}}^K[\cdot], \bar{\mathbf{X}}^K[0]))$ (where $\bar{\mathbf{A}}^K[\cdot] \in \Gamma^{K,T}$) converges uniformly to a FSP $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$. By definition of uniform convergence, for any $M > 1$ there exists K_0 such that for any $K > K_0$ and $t \in [0, T]$, we have $\forall i \in V_S$,
$$\left| \frac{\bar{\mathbf{X}}_i^K[t]}{w_i} - \frac{\bar{\mathbf{X}}_i[t]}{w_i} \right| < \frac{c'-c}{6} \frac{\epsilon_0}{M}. \tag{21}$$
As a result, for any $\epsilon \in \left[\frac{\epsilon_0}{M}, \epsilon_0\right]$ and $K \geq K_0$, let $k \in A_1 \backslash A_2$, $l \in A_2$, we have
$$\begin{aligned}
&\frac{\bar{\mathbf{X}}_k^K(t+\epsilon)}{w_k} \\
\geq& \frac{\bar{\mathbf{X}}_k(t+\epsilon)}{w_k} - \frac{c'-c}{6}\frac{\epsilon_0}{M} && \text{(plug in (21))} \\
>& b + \left(c' - \frac{c'-c}{3}\right)\epsilon - \frac{c'-c}{6}\epsilon && \text{(plug in (19))} \\
=& b + \left(c - \frac{c'-c}{2}\right)\epsilon \\
\geq& \frac{\bar{\mathbf{X}}_l(t+\epsilon)}{w_l} + \frac{c'-c}{6}\epsilon && \text{(plug in (20))} \\
\geq& \frac{\bar{\mathbf{X}}_l^K(t+\epsilon)}{w_l} && \text{(plug in (21)).}
\end{aligned}$$
By definition of SMW($\mathbf{w}$), the K-th system will not use dispatch within A_2 to serve demand outside $\{j' \in V_D : \partial(j') \in A_2\}$ during (scaled) time $[t + \frac{\epsilon_0}{M}, t + \epsilon_0]$. Also, note that since $L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) < 1$, we have $\bar{\mathbf{X}}[t] \in \text{relint}(\Omega)$ by Lemma 2; by uniform convergence, we know that for large enough K and s close enough to t, $U^{\mathbf{w}}(\bar{\mathbf{X}}^K[s], k)$ will never be $\emptyset$. Hence
$$\begin{aligned}
\sum_{i \in A_2} (\bar{\mathbf{X}}_i^K[t+\epsilon_0] - \bar{\mathbf{X}}_i^K[t + \frac{\epsilon_0}{M}]) = &- \sum_{j' \in V_D : \partial(j') \subseteq A_2, i \in V_S} (\bar{\mathbf{A}}_{j'i}^K[t+\epsilon_0] - \bar{\mathbf{A}}_{j'i}^K[t + \frac{\epsilon_0}{M}]) \\
&+ \sum_{j' \in V_D, i \in A_2} (\bar{\mathbf{A}}_{j'i}^K[t+\epsilon_0] - \bar{\mathbf{A}}_{j'i}^K[t + \frac{\epsilon_0}{M}]).
\end{aligned}$$
Using the Lipschitz property of $\bar{\mathbf{X}}^K[\cdot]$, we know
$$\left| \sum_{i \in A_2} (\bar{\mathbf{X}}_i^K[t + \frac{\epsilon_0}{M}] - \bar{\mathbf{X}}_i^K[t]) \right| \leq \frac{\epsilon_0}{M}.$$

Combined, we have:

$$\sum_{i\in A_2}(\bar{\mathbf{X}}_i^K[t+\epsilon_0]-\bar{\mathbf{X}}_i^K[t]) \leq -\sum_{j'\in V_D:\partial(j')\subseteq A_2, i\in V_S}(\bar{\mathbf{A}}_{j'i}^K[t+\epsilon_0]-\bar{\mathbf{A}}_{j'i}^K[t+\frac{\epsilon_0}{M}])$$
$$+\sum_{j'\in V_D, i\in A_2}(\bar{\mathbf{A}}_{j'i}^K[t+\epsilon_0]-\bar{\mathbf{A}}_{j'i}^K[t+\frac{\epsilon_0}{M}])+\frac{\epsilon_0}{M}.$$

First let $K\to\infty$, then let $M\to\infty$. Using the continuity of $\bar{\mathbf{A}}[\cdot]$, we have:

$$\sum_{k\in A_2}(\bar{\mathbf{X}}_i[t+\epsilon_0]-\bar{\mathbf{X}}_i[t]) \leq -\sum_{j'\in V_D:\partial(j')\subseteq A_2, i\in V_S}(\bar{\mathbf{A}}_{j'i}[t+\epsilon_0]-\bar{\mathbf{A}}_{j'i}[t])$$
$$+\sum_{j'\in V_D, i\in A_2}(\bar{\mathbf{A}}_{j'i}[t+\epsilon_0]-\bar{\mathbf{A}}_{j'i}[t])$$

Divided by ϵ_0 on both sides and let $\epsilon_0\to 0$. Since t is regular, we have:

$$\sum_{i\in A_2}\frac{d}{dt}\bar{\mathbf{X}}_i[t] \leq \left(\sum_{j'\in V_D, i\in A_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]-\sum_{j'\in V_D:\partial(j')\subseteq A_2, i\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]\right).$$

Similarly, we can show that

$$\sum_{i\in A_2}\frac{d}{dt}\bar{\mathbf{X}}_i[t] \geq \left(\sum_{j'\in V_D, i\in A_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]-\sum_{j'\in V_D:\partial(j')\subseteq A_2, i\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]\right).$$

Hence

$$\begin{aligned}\frac{d}{dt}L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) &= -\frac{1}{w_i}\frac{d}{dt}\bar{\mathbf{X}}_i[t], \quad \forall i\in A_2\\ &= -\frac{1}{\mathbf{1}_{A_2}\mathbf{w}}\sum_{i\in A_2}w_i\frac{1}{w_i}\frac{d}{dt}\bar{\mathbf{X}}_i[t]\\ &= \frac{1}{\mathbf{1}_{A_2}\mathbf{w}}\left(\sum_{j'\in V_D:\partial(j')\subseteq A_2, i\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]-\sum_{j'\in V_D, i\in A_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]\right).\end{aligned}$$

□

A.6 Proof of Lemma 4

Proof. Based on the definition of fluid sample paths and fluid limits, we can see that fluid limits are equivalent to FSP that satisfies:

$$\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]=\phi_{j'i}, \quad \forall j'\in V_D, i\in V_S, \text{for almost every } t\in[0,T].$$

There are two cases:

- $\bar{\mathbf{X}}[t] > 0$. In this case, plug in Lemma 3 and we have

$$\begin{aligned}\frac{d}{dt}L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) =& -\frac{1}{\mathbf{1}_{A_2}\mathbf{w}}\left(\sum_{j'\in V_D, i\in A_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t] - \sum_{j'\in V_D:\partial(j')\subseteq A_2, i\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]\right)\\ \leq& -\min_{A_2\subsetneq V_S}\frac{1}{\mathbf{1}_{A_2}\mathbf{w}}\left(\sum_{j'\in V_D, i\in A_2}\phi_{j'i} - \sum_{j'\in V_D:\partial(j')\subseteq A_2, i\in V_S}\phi_{j'i}\right)\\ \leq& -\xi.\end{aligned}$$

- There exists $i \in V$ such that $\bar{\mathbf{X}}_i[t] = 0$. Similar to the proof of Lemma 3, we focus on the behavior of the fluid limit during $[t, t+\epsilon_0]$ for small enough $\epsilon_0 > 0$, and partition it into two time intervals: $[t, t+\frac{\epsilon_0}{M}]$ and $[t+\frac{\epsilon_0}{M}, t+\epsilon_0]$ for $M > 1$. By Lemma 1, we know that $\frac{d}{dt}\bar{\mathbf{X}}_i[t] > 0$ for the nodes who have zero supply at time t, hence in the fluid limit the system will have positive supply at each node at $\bar{\mathbf{X}}_i[t+\frac{\epsilon_0}{M}] > 0$. We can derive the same Lyapunov drift as in the first case on $[t+\frac{\epsilon_0}{M}, t+\epsilon_0]$, and get the desired result by letting $M \to \infty$. The details are omitted here.

□

A.7 Proof of Lemma 5

Proof. From the proof of Lemma 4, we can see that the Lyapunov drift depends on the gap ξ defined in (14) and $\lambda_{\min}$ defined in (13). Mathematically, for fluid sample path $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$ and regular point t, we have:

$$\begin{aligned}\frac{d}{dt}L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) \leq& -\min_{A\subsetneq V_S}\left(\sum_{j'\in V_D, i\in A}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t] - \sum_{j':\partial(j')\subseteq A, i\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'i}[t]\right)\\ =& -\min\{\xi, \lambda_{\min}\}.\end{aligned} \tag{22}$$

Note that

$$G(\mathbf{f}) \triangleq \min_{A\subsetneq V_S}\left(\sum_{j'\in V_D, i\in A}\mathbf{f}_{j'i} - \sum_{j':\partial(j')\subseteq A, i\in V_S}\mathbf{f}_{j'i}\right)$$

is continuous for $\mathbf{f} \in \mathcal{M} \triangleq \{\mathbf{g} \in \mathbb{R}^{n\times n} : \mathbf{g} \geq 0, \mathbf{1}^T\mathbf{g}\mathbf{1} = 1\}$. Since $G(\phi) \leq -\min\{\xi, \lambda_{\min}\} < 0$, by continuity there must exist ϵ such that for any $\mathbf{f} \in M$ such that $\mathbf{f} \in B(\phi, \epsilon)$, we have

$$G(\mathbf{f}) \leq -\frac{1}{2}\min\{\xi, \lambda_{\min}\}.$$

□

A.8 Proof of Lemma 6

Proof. Property (1)(2)(4)(5)(6) are easy to verify, so we only prove (3) below. For fixed $\mathbf{w} \in$ relint(Ω):

(3) Let $c_{\mathbf{w}} \triangleq \min_i w_i$. For $\mathbf{x} \in \{\mathbf{x} : L_{\mathbf{w}}(\mathbf{w}) \leq \epsilon\} \cap h_1$, we have:

$$\begin{aligned}
||\mathbf{x} - \mathbf{w}|| \leq& \sqrt{n} \cdot \max_{i \in V} |x_i - w_i| \\
\leq& \sqrt{n} \cdot \max_{i \in V} \left(\max\{\epsilon w_i, \epsilon(1 - w_i)\}\right).
\end{aligned}$$

Hence for $\epsilon_0 \triangleq c_{\mathbf{w}} \left(2\sqrt{n} \cdot \max_{i \in V} \left(\max\{w_i, (1 - w_i)\}\right)\right)^{-1} > 0$, we have $\{\mathbf{x} : L_{\mathbf{w}}(\mathbf{x}) \leq \epsilon_0\} \cap h_1 \subset \{\mathbf{x} : ||\mathbf{x} - \mathbf{w}|| < c_{\mathbf{w}}\} \cap h_1$. Let $l_{\mathbf{w}} \triangleq \epsilon_0$, we have

$$\min_{\mathbf{x}:||\mathbf{x}-\mathbf{w}|| \geq c_{\mathbf{w}}, \mathbf{x} \in h_1} L_{\mathbf{w}}(\mathbf{x}) > l_{\mathbf{w}} > 0.$$

Also note that for any $\mathbf{x} \in \{\mathbf{x} : ||\mathbf{x} - \mathbf{w}|| \leq c_{\mathbf{w}}\} \cap h_1$, we have $x_i \geq w_i - c_{\mathbf{w}} \geq 0$, hence

$$\max_{\mathbf{x}:||\mathbf{x}-\mathbf{w}|| \leq c_{\mathbf{w}}, \mathbf{x} \in h_1} L_{\mathbf{w}}(\mathbf{x}) \leq u_{\mathbf{w}} \triangleq 1.$$

□

A.9 Proof of Lemma 7

Proof. We will prove the result by showing that the elements in finite state space Ω_K can be partitioned into two sets $\mathcal{R}$ and $\mathcal{T}$: any element in $\mathcal{T}$ is transient, while all elements in $\mathcal{R}$ belong to the same communication class. Let $K \geq (\min_{i \in V} w_i)^{-1} + 1$.

The proof consists of three steps:

1. *Define 'equilibrium' state* $\mathbf{x}^*$*.* To facilitate the exposition of the proof, we consider the case where $\mathbf{x}^* \triangleq K\mathbf{w}$ is a vector of integers. (When this is not the case, there is a quick fix where we can choose a state $\mathbf{x}^*$ in Ω_K that is close to $K\mathbf{w}$, and design a tie-breaking rule for $\bar{\mathbf{X}}^K$ close to $\mathbf{x}^*$ to ensure that $\mathbf{x}^*$ is reachable from any state. This is non-essential and we omit the details.)

2. *Show that starting from any state, the probability of hitting* $\mathbf{x}^*$ *is positive.* We will show that starting from any state $\mathbf{x}_0 \in \Omega_K$, we can construct a sample path $\mathbf{x}_0 \to \cdots \to \mathbf{x}^*$ that happens with positive probability. First define a potential function for any state $\mathbf{x} \in \Omega_K$:

   $$Q(\mathbf{x}) \triangleq \sum_{i \in V} [\mathbf{x}_i - \mathbf{x}_i^*]^+.$$

   Since given $\mathbf{x} \in \Omega_K$, $Q(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{x}^*$, we only need to show that starting from any state we can construct a sample path that reduces $Q(\mathbf{x})$ by at least 1 and happens with positive probability.

   Given $\mathbf{x}_0$ where $Q(\mathbf{x}_0) > 0$, arbitrarily select an $i \in V$ such that $\mathbf{x}_i \geq \mathbf{x}_i^* + 1$. By Assumption 1 we can construct a deterministic demand type sample path $(i, i_1), (i_1, i_2), , \cdots (i_\beta, i_{\beta+1}), , \cdots (i_{\nu-1}, i_\nu)$ where $\mathbf{x}_{i_\nu} < \mathbf{x}^*_{i_\nu}$, $\phi_{i_{\beta-1} i_\beta} > 0$ for any $\beta \leq \nu$. Note that by queue length correspondence in definition 3, the above demand sample path uniquely determines a queue length sample path $\mathbf{x}[\beta]$. Denote the smallest $\beta > 0$ such that $\mathbf{x}_{i_\beta}[\beta] \leq \mathbf{x}^*_{i_\beta}$ as τ, where $\tau \geq 1$. We have:

- For $\beta \leq \tau-1$, $Q(\mathbf{x}[\beta]) = Q(\mathbf{x}[\beta-1])$. To see this, note that in this step we dispatch from $\mathbf{x}_u$ and the supply is relocated to node i_β. Here we will not drop the demand because by definition of τ we have $\mathbf{x}_{i_{\beta-1}}[\beta-1] \geq \mathbf{x}^*_{i_{\beta-1}} \geq 1$. Note that the supply relocation will increase potential by 1, we only need to show that dispatch decrease potential by 1; in other words, it suffices to show that $\mathbf{x}_u[\beta-1] \geq \mathbf{x}^*_u + 1$. From the definition of $\mathbf{x}^*$ relationships

$$\mathbf{x}_{i_{\beta-1}}[\beta-1] \geq \mathbf{x}^*_{i_{\beta-1}} + 1,$$
$$\frac{\mathbf{x}_u[\beta-1]}{w_u} \geq \frac{\mathbf{x}_{i_{\beta-1}}[\beta-1]}{w_{i_{\beta-1}}}$$

we know that

$$\mathbf{x}_u[\beta-1] > w_u K = \mathbf{x}^*_u.$$

Hence the claim is proved.

- $Q(\mathbf{x}[\tau]) \leq Q(\mathbf{x}[\tau-1]) - 1$. We can perform the same analysis as above and show that dispatch reduce potential by 1. Since $\mathbf{x}_{i_\tau}[\tau] \leq \mathbf{x}^*_{i_\tau}$, relocation doesn't change the potential, hence the result.

3. *Conclude the result.* We already showed that any state in Ω_K has positive probability of hitting $\mathbf{x}^*$. Define $\mathcal{R}$ as the set of states reachable from $\mathbf{x}^*$, i.e. states that communicate with $\mathbf{x}^*$. Denote $\mathcal{T} = \Omega_K/\mathcal{R}$. Since states in $\mathcal{T}$ has positive probability of hitting $\mathcal{R}$, and it will not return to $\mathcal{T}$ once that happens, they must all be transient.

□

A.10 Proof of Lemma 8

Proof. The Lemma can be proved by an adaptation of the proof of Theorem 5 and Lemma 6 in [30]. Lemma 6 and 4 are parallel to Assumption 1 in [30]; Lemma 5 is parallel to Assumption 2, while Lemma 6(6) is parallel to Assumption 3, both in [30]. The Harris recurrence of open queueing networks used in the proof of Theorem [30] can be replaced by the Harris recurrence of our model, i.e., Lemma 7. □

A.11 Proof of Lemma 9

Proof. Let t be regular and $\mathbf{f} \triangleq \frac{d}{dt}\bar{\mathbf{A}}[t]$. In the following, denote

$$\mathrm{gap}_A(\mathbf{f}) \triangleq \sum_{j':\partial(j')\subseteq A, i\in V_S} \mathbf{f}_{j'i} - \sum_{j'\in V_D, i\in A} \mathbf{f}_{j'i}.$$

Using the result of Lemma 3, we have:

$$\frac{d}{dt} L_{\mathbf{w}}(\bar{\mathbf{X}}[t]) \leq \bar{v}(\mathbf{f}) \triangleq \max_{A\subseteq V_S} \left\{ \frac{1}{\mathbf{1}_A^T\mathbf{w}} \mathrm{gap}_A(\mathbf{f}) \right\}.$$

Note that in the definition of $\gamma(\mathbf{w})$ in Lemma 8, if we replace any feasible pair $(v, \mathbf{f})$ (where $v > 0$) with $(\bar{v}(\mathbf{f}), \mathbf{f})$, since $v \leq \bar{v}(\mathbf{f})$ and therefore $v > 0 \Rightarrow \bar{v}(\mathbf{f}) > 0$, we have

$$\frac{1}{v}\Lambda^*(\mathbf{f}) \geq \frac{1}{\bar{v}(\mathbf{f})}\Lambda^*(\mathbf{f}).$$

Combine the above observation with Lemma 8, we have

$$
\begin{aligned}
\gamma(\mathbf{w}) &\geq \min_{f:\bar{v}(\mathbf{f})>0} \frac{\Lambda^*(\mathbf{f})}{\bar{v}(\mathbf{f})} \\
&\geq \min_{A\subseteq V_S} \left\{ \min_{\mathbf{f}:\mathrm{gap}_A(\mathbf{f})>0} \frac{\Lambda^*(\mathbf{f})}{\frac{1}{\mathbf{1}_A^T\mathbf{w}}\mathrm{gap}_A(\mathbf{f})} \right\} \\
&= \min_{J\in\mathcal{J}} \left\{ (\mathbf{1}_{\partial(J)}^T\mathbf{w}) \left\{ \min_{\mathbf{f}:\mathrm{gap}_{\partial(J)}(\mathbf{f})>0} \frac{\Lambda^*(\mathbf{f})}{\mathrm{gap}_{\partial(J)}(\mathbf{f})} \right\} \right\}.
\end{aligned}
$$

The last equality holds because for any $A \subseteq V_S$ such that $\{j' \in V_D : \partial(j') \subseteq A, \exists i \notin \partial(j') \; s.t. \phi_{j'i} > 0\} = \emptyset$, we always have $\mathrm{gap}_A(\mathbf{f}) \leq 0$. Denote the optimal value of the inner minimization problem as $g(\phi, J) > 0$, then we have:

$$
\min_{\mathbf{f}:\mathrm{gap}_{\partial(J)}(\mathbf{f})>0} \Lambda^*(\mathbf{f}) - g(\phi,J)\left(\sum_{j'\in J, i\in V_S} f_{j'i} - \sum_{j'\in V_D, i\in\partial(J)} f_{j'i}\right) = 0. \tag{23}
$$

We can get rid of the constraint on $\mathbf{f}$ since any $\mathbf{f}$ that violates it will not achieve minimum in (23). Then using Legendre transform, we have:

$$
\begin{aligned}
&\min_{\mathbf{f}} \Lambda^*(\mathbf{f}) - g(\phi,J)\left(\sum_{j'\in J, i\in V_S} f_{j'i} - \sum_{j'\in V_D, i\in\partial(J)} f_{j'i}\right) \\
=&\min_{\mathbf{f}} \Lambda^*(\mathbf{f}) - \left\langle \mathbf{f}, g(\phi,J)\sum_{j'\in J, i\in V_S} \mathbf{e}_{j'i} - g(\phi,J)\sum_{j'\in V_D, i\in\partial(J)} \mathbf{e}_{j'i} \right\rangle \\
=&-\Lambda\left(g(\phi,J)\sum_{j'\in J, i\in V_S} \mathbf{e}_{j'i} - g(\phi,J)\sum_{j'\in V_D, i\in\partial(J)} \mathbf{e}_{j'i}\right) \\
=&-\log\left(\sum_{j'\in V_D, i\in V_S} \phi_{j'i} e^{g(\phi,J)\mathbb{1}\{j'\in J\} - g(\phi,J)\mathbb{1}\{i\in\partial(J)\}}\right).
\end{aligned}
$$

Hence equation (23) reduces to nonlinear equation:

$$
\left(\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}\right) e^{-g(\phi,J)} + \left(\sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}\right) e^{g(\phi,J)} = \sum_{j'\notin J, i\in\partial(J)} \phi_{j'i} + \sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}.
$$

Let $y \triangleq e^{g(\phi,J)}$, this becomes a quadratic equation:

$$
\left(\sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}\right) y^2 - \left(\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i} + \sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}\right) y + \left(\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}\right) = 0.
$$

Hence

$$
y = \frac{\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}}{\sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}} \text{ or } 1.
$$

Since $g(\phi, J) > 0$, we have

$$g(\phi, J) = \log\left(\frac{\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}}{\sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}}\right).$$

Plug in the original inequality, we have:

$$\gamma(\mathbf{w}) \geq \min_{J\in\mathcal{J}} (\mathbf{1}_{\partial(J)}^T \mathbf{w}) \log\left(\frac{\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}}{\sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}}\right).$$

□

A.12 Proof of Theorem 2

Proof. We have

$$\text{LHS} = \min_{J\in\mathcal{J}} \frac{|\partial(J)|}{n} \log\left(\frac{\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}}{\sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}}\right), \tag{24}$$

$$\gamma^* = \max_{\mathbf{w}\in\Omega} \min_{J\in\mathcal{J}} (\mathbf{1}_{\partial(J)}^{\mathrm{T}} \mathbf{w}) \log\left(\frac{\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}}{\sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}}\right). \tag{25}$$

Let $J^* \subsetneq V_D$ be one of the minimizer of (24), $\mathbf{w}^*$ is the maximizer of (25). Hence

$$\begin{aligned}
\gamma^* =& \min_{J\in\mathcal{J}} (\mathbf{1}_{\partial(J)}^{\mathrm{T}} \mathbf{w}^*) \log\left(\frac{\sum_{j'\notin J, i\in\partial(J)} \phi_{j'i}}{\sum_{j'\in J, i\notin\partial(J)} \phi_{j'i}}\right) \\
\leq& (\mathbf{1}_{\partial(J^*)}^{\mathrm{T}} \mathbf{w}^*) \log\left(\frac{\sum_{j'\notin J^*, i\in\partial(J^*)} \phi_{j'i}}{\sum_{j'\in J^*, i\notin\partial(J^*)} \phi_{j'i}}\right) \\
\leq& |\partial(J^*)| \log\left(\frac{\sum_{j'\notin J^*, i\in\partial(J^*)} \phi_{j'i}}{\sum_{j'\in J^*, i\notin\partial(J^*)} \phi_{j'i}}\right) \\
=& \gamma\left(\frac{1}{n}\mathbf{1}\right).
\end{aligned}$$

□

A.13 Proof of Lemma 10

Proof. Let $\Lambda_1^*(\cdot)$ as the rate function of Z_1. For $r \in (0, p-q)$, define $\{Z_i^r\}_{i=1}^\infty$ as i.i.d. random variables where Z_1^r is derived from Z_1 by change of measure such that it has mean r and support $\{-1, 0, 1\}$, as in Theorem 2.2.3 of[14]. Denote the partial sum of Z_i^r by S_i^r. Let $\theta_0 \triangleq \text{argmax}_{\theta\geq 0} \left(\theta \cdot r - \Lambda_1(\theta)\right)$, where $\Lambda_1(\cdot)$ is the log moment generating function of Z_1. A simple bound of θ_0 is:

$$\theta_0 \in \{\theta \geq 0 : \theta \cdot r - \Lambda_1(\theta) \geq 0\} \in \left[0, \frac{-\log p}{1-r}\right].$$

Follow the proof of Cramér's Theorem in [14] Theorem 2.2.3, for small $\epsilon > 0$ we have:

$$\begin{aligned}
\mathbb{P}(S_n \leq n(r+\epsilon)) \geq& e^{-n\epsilon\theta_0} e^{-n\Lambda_1^*(r)} \mathbb{P}\left(S_n^r \leq n(r+\epsilon)\right) \qquad (\epsilon \in (0, p-q-r)) \\
\geq& e^{-n\epsilon\frac{-\log p}{1-r}} e^{-n\Lambda_1^*(r)} \left(1 - \mathbb{P}\left(S_n^r > n(r+\epsilon)\right)\right) \\
\geq& e^{-n\epsilon\frac{-\log p}{1-(p-q)}} e^{-n\Lambda_1^*(r)} \left(1 - e^{-\frac{n\epsilon^2}{2}}\right) \qquad \text{(Azuma's inequality)}.
\end{aligned}$$

Plug in $n = \lfloor \kappa/(q-p) \rfloor$, $r = (q-p) - \epsilon$, we have: (ignore the technicality of integrality of $\kappa/(p-q)$ below)

$$\mathbb{P}(S_{\kappa/(q-p)} \leq \kappa) \geq e^{\kappa\epsilon \frac{-\log p}{(p-q)(1-(p-q))}} e^{\kappa \frac{\Lambda_1^*((q-p)-\epsilon)}{p-q}} \left(1 - e^{\frac{\kappa\epsilon^2}{2(p-q)}}\right)$$

for any $\epsilon > 0, \kappa \leq 0$. □

A.14 Proof of Lemma 11

Proof. Consider the optimization problem of minimizing $\Lambda_1^*(x)/x$ for $x < 0$. Using the same technique as in Lemma 9, the optimal value of the optimization problem $r < 0$ satisfies

$$\Lambda_1(r) = 0.$$

Again, apply the same technique in Lemma 9, we have $r = \log(q/p) < 0$. Note that the minimizing x, denoted as x^*, also solves:

$$\min_x \Lambda_1^*(x) - rx.$$

First order condition implies that:

$$r = \text{argmax}_\theta \theta x^* - \Lambda_1(\theta).$$

Expand the above expression, we have:

$$x^* = \frac{pe^r - qe^{-r}}{pe^r + qe^{-r} + (1-p-q)} = q - p.$$

Hence $\Lambda_1^*(q-p)/(p-q) = -r = \log(p/q)$. □

A.15 Proof of Proposition 3

Proof. We first define an 'augmented' policy $\tilde{\pi}$ for any state-independent policy π. Policy $\tilde{\pi}$ is also state independent, where:

$$\tilde{u}_{(i,j')}[t] = u_{(i,j')}[t] + \frac{1}{|\partial(j')|}\left(1 - \sum_{i \in \partial(j')} u_{(i,j')}[t]\right).$$

Note that $\tilde{\pi}$ is a non-idling policy, and $\tilde{\pi} = \pi$ if π is non-idling. In the following analysis, we couple π and $\tilde{\pi}$ in such a way that if π dispatch from i to serve the t-th demand, then $\tilde{\pi}$ will do the same.

We first divide the discrete periods into cycles with length K^2. We will lower bound the probability of demand-drop on any cycle. Without loss of generality, consider cycle $[0, K^2 - 1]$. Suppose $X^{K,\pi}[0] = \mathbf{X}_0$. By Assumption 1, there exists a subset $J \in \mathcal{J}$. Consider the 'virtual' process of change of number of supplies in $\partial(J)$, denoted by $S_J[t]$:

- $S_J[0] = 0$.
- $S_J[t+1] = S_J[t] + 1$ if $d[t] \in \partial(J)$ and policy $\tilde{\pi}$ dispatches a vehicle from $\partial(J)^c$ to serve the t-th demand (regardless of whether the demand is fulfilled).
- $S_J[t+1] = S_J[t] - 1$ if $d[t] \in \partial(J)^c$ and policy $\tilde{\pi}$ dispatches a vehicle from $\partial(J)$ to serve the t-th demand (regardless of whether the demand is fulfilled).

- $S_J[t+1] = S_J[t]$ if otherwise.

Observe that:

$$\mathbb{P}\left(\text{some demand is dropped in } \in [0, K^2-1]\right) \geq \mathbb{P}\left(S_J(K^2-1) + \mathbf{1}_{\partial(J)}^T \mathbf{X}_0 \geq K \text{ or } \leq 0\right). \quad (26)$$

This is because : (1) if $S_J[t] + \mathbf{1}_{\partial(J)}^T \bar{\mathbf{X}}_0 = \mathbf{1}_{\partial(J)}^T \bar{\mathbf{X}}^{K,\pi}[t]$ for all $t \leq T$, then $S_J(K^2) + \mathbf{1}_{\partial(J)}^T \mathbf{X}_0$ is exactly the number of supplies in $\partial(J)$ at K^2, and demand will be dropped if $\partial(J)$ is empty or $\partial(J)^c$ is empty; (2)if $S_J[t] + \mathbf{1}_{\partial(J)}^T \mathbf{X}_0 \neq \mathbf{1}_{\partial(J)}^T \bar{\mathbf{X}}^{K,\pi}[t]$ for some $t < K^2$, it means a unit of demand was already dropped at the smallest such t. Either way, RHS implies LHS.

Note that $S_J(K^2)$ is the sum of K^2 independent random variables Z_t, where $Z_t = S_J[t+1] - S_J[t]$. Here Z_t has support $\{-1, 0, 1\}$ and satisfies:

$$\mathbb{P}(Z_t = -1) \geq \mathbb{P}(o[t] \in J, d[t] \in \partial(J)^c)$$

There are two scenarios:

- If $\mathbb{E}[S_J(K^2)] \leq -\frac{K^2}{2}$, then for $K > 8$,

$$\begin{aligned}
\mathbb{P}\left(S_J(K^2) + \mathbf{1}_{\partial(J)}^T \mathbf{X}_0 \geq K \text{ or } \leq 0\right) \geq & 1 - \mathbb{P}\left(S_J(K^2) \in [-K, K]\right) \\
\geq & 1 - \mathbb{P}\left(S_J(K^2) - \mathbb{E}[S_J(K^2)] \geq -K + \frac{K^2}{2}\right) \\
\geq & 1 - 2\exp\left(-\frac{K^2}{32}\right) \quad \text{(Hoeffding's inequality)} \\
\geq & \frac{1}{2}.
\end{aligned}$$

- If $\mathbb{E}[S_J(K^2)] > -\frac{K^2}{2}$, then:

$$\text{the number of } t\text{'s where } \mathbb{E}[Z_t] \geq -\frac{3}{4} \text{ is at least } \frac{K^2}{7}. \quad (27)$$

  Denote the set of these t's as $\mathcal{T}$. Hence

$$\text{Var}(S_J(K^2)) = \sum_{t=1}^{K^2} \text{Var}(\xi_t) \geq \sum_{t \in \mathcal{T}} \text{Var}(\xi_t) \geq \frac{K^2}{7} \cdot \delta \left(1 - \frac{3}{4}\right)^2 = \frac{\delta K^2}{102}. \quad (28)$$

  Apply Theorem 7.4.1 in [11] (Berry-Esseen Theorem), we have:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(S_J(K^2) - \mathbb{E}(S_J(K^2)) \leq x\sqrt{\text{Var}(S_J(K^2))}\right) - \Phi(x) \right| \leq \frac{\sum_{t=1}^{K^2} \mathbb{E}|Z_t - \mathbb{E}Z_t|^3}{\text{Var}(S_J(K^2))^{3/2}}, \quad (29)$$

  where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution. Plug in (27) and (28), we can upper bound the RHS of (29) by $10000\delta^{-3/2}\frac{1}{K}$. For $K > \frac{40000 A_0 \delta^{-3/2}}{2\bar{\Phi}(50/\sqrt{\delta})}$:

$$\mathbb{P}(S_J(K^2) \in B\left(\mathbb{E}(S_J(K^2)), 4K\right) \in B\left(2\bar{\Phi}\left(\frac{4K}{\text{Var}(S_J(K^2))}\right), \frac{1}{2}\bar{\Phi}\left(\frac{4K}{\text{Var}(S_J(K^2))}\right)\right)$$
$$\mathbb{P}(S_J(K^2) \in B\left(\mathbb{E}(S_J(K^2)), 2K\right) \in B\left(2\bar{\Phi}\left(\frac{2K}{\text{Var}(S_J(K^2))}\right), \frac{1}{2}\bar{\Phi}\left(\frac{4K}{\text{Var}(S_J(K^2))}\right)\right).$$

Here $B(x,a) \triangleq [x-a, x+a]$. Now we are ready to bound demand-drop probability. Note that one of the three cases will happen: $[-K,K] \subset B\left(\mathbb{E}(S_J(K^2)), 4K\right)$, $[-K,K] \cap B\left(\mathbb{E}(S_J(K^2)), 4K\right) = \emptyset$, $[-K,K] \cap B\left(\mathbb{E}(S_J(K^2)), 2K\right) = \emptyset$. In any case, the probability of $S_J(K^2)$ fall into an interval with width $2K$ is uniformly bounded away from 1, denoted by $c < 1$. Hence

$$\mathbb{P}\left(\text{some demand is dropped in } \in [1, K^2]\right) = c > 0.$$

Combine the two cases, we have the desired result. $\square$