

# MANAGING CONGESTION IN MATCHING MARKETS

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**ABSTRACT.** Participants in matching markets face search and screening costs which prevent the market from clearing efficiently. In many settings, the rise of online matching platforms has dramatically reduced the cost of finding and contacting potential partners. While one might expect both sides of the market to benefit from reduced search costs, this is far from guaranteed. In particular, this change may force participants to screen more potential partners before finding one who is willing to accept their offer. Thus, for a fixed screening cost, reductions in search and application costs may actually *decrease* aggregate welfare.

We illustrate this fact using a model in which agents apply and screen asynchronously. In our model, it is possible that an agent on one side of the market (an “employer”) identifies a suitable match on the other side (an “applicant”), only to find that this applicant has already matched. We find that equilibrium is generically inefficient for both employers and applicants. Most notably, when application costs are sufficiently small, uncertainty about applicant availability may drive equilibrium employer welfare to *zero*.

We consider a simple intervention available to the platform: limiting the visibility of applicants. We find that this intervention can significantly improve the welfare of agents on both sides of the market; applicants pay lower application costs, while employers are less likely to find that the applicants they screen have already matched. Somewhat counterintuitively, the benefits of showing fewer applicants to each employer are greatest in markets in which there is a shortage of applicants.

*Keywords:* Matching, decentralized, market design, mean field

## 1. INTRODUCTION

Thanks to massive advances in information technology, online platforms are revolutionizing how trading partners find each other in a wide range of matching markets. These platforms use a variety of mechanisms to help agents find each other, learn about each other, and ultimately consummate matches. For example, sites such as LinkedIn, Upwork<sup>1</sup>, and TaskRabbit help match employers and employees; travelers looking for short-term housing options can turn to Airbnb and VRBO; and sites such

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<sup>1</sup>Previously Elance-oDesk

as Match.com and OkCupid allow those seeking a romantic partner to browse profiles of nearby users.

Although these platforms provide access to hundreds or thousands of potential matches, searching through these listings and contacting promising options remains a costly and time-consuming process. The challenge of finding a partner is exacerbated by the fact that each user’s demand for matches typically fluctuates over time. A worker on Upwork may find themselves temporarily overwhelmed with projects, or suddenly in need of extra work to pay the month’s expenses. A listing on Airbnb may be unavailable for a particular date because of a visit from the host’s in-laws. An OkCupid user might fail to respond to a message while on vacation, or due to a newly-formed relationship.

Typically, these changes in a user’s status are not immediately reflected on their profile. As a result, a great deal of effort may be wasted screening profiles of users who are currently unavailable. Recent empirical work suggests that lack of information about user availability is prevalent, and that this fact has significant welfare consequences.<sup>2</sup>

Motivated by this observation, this work formulates a model of an asynchronous dynamic matching market with two key features: (1) contacting and screening potential partners is costly, and (2) users cannot determine, before screening, which options are available to them. We refer to the two sides of our market as “applicants” and “employers,” and study the effect of changes in the application cost. One might expect lower application costs to benefit both applicants (who pay less to apply) and employers (who expect to receive more applications).

Our work shows that this need not be the case: indeed, our first main result shows that holding fixed screening costs, *increased search by applicants may lower employer welfare*. In fact, we find that unless there is a large surplus of applicants, employer welfare is identically *zero* for all sufficiently small application costs. The intuition for this result is that as applicants search more broadly, employers will increasingly find that applicants to whom they make offers have already matched elsewhere and are therefore unavailable. As a result, employers must screen more candidates in order to hire successfully, and this cost more than offsets the benefits from having

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<sup>2</sup>See, e.g., Horton (2015) for analysis of this phenomenon on the online labor market oDesk, and Fradkin (2015) for this issue in the context of Airbnb. For example, Fradkin (2015) shows that on Airbnb, 49% of inquiries are rejected or ignored by the host, and only 15% of inquiries eventually lead to a transaction. This work concludes that one of the most common reasons for rejection is a “stale” (or unavailable) listing, and that an initial rejection decreases the probability that the guest eventually books *any* listing by 50%.

a larger application pool. When employer welfare is driven to zero, although some employers continue to screen and match, others choose to exit the market rather than continuing their search. This striking conclusion is (to our knowledge) unparalleled in the literature, and perhaps explains why participants in many online matching markets leave the platform before finding a match.<sup>3</sup>

One might hope that even if employers must pay high costs to match, reduced application costs at should least benefit applicants. Our model concludes that this is true, but only weakly: if employers earn zero surplus in equilibrium, then reductions in application costs result in fiercer competition, without increasing the number of filled positions. As a result, the welfare of both sides remains unchanged.<sup>4</sup>

In such cases, our work suggests that it may be welfare-enhancing for the platform to add friction to the application process. Such friction could take several forms. The platform could, for example, charge a monetary fee for each application sent. In settings where such charges are unappealing or infeasible, the application process could be made more difficult, for example by requiring the applicant to complete an extensive application form for each job. One drawback of either of these approaches is that adding frictions through application fees and forms increases the burden on applicants. Within the context of our model, we find that unless payments are redistributed or forms provide information that reduces screening costs, these interventions necessarily *lower* applicant welfare. While this may be a worthwhile price to pay if employer welfare is raised substantially, it is natural to ask whether there is a simple intervention that simultaneously benefits both sides.

We find that simply restricting the number of applications that each person can send may do just that: our second main result shows that if application costs are small enough and screening costs large enough that employers earn no surplus in equilibrium, *an application limit can raise the welfare of both sides*. Indeed, in such cases,

<sup>3</sup>Fradkin (2015), for example, finds that on Airbnb, hosts rejecting proposals from guests can cause “searchers to leave the market although there are potentially good matches remaining.” Horton (2015) finds that on the online labor market oDesk, when a recruited potential worker turns down the employer, “that [employer] is substantially less likely to form a match, despite having access to many substitute workers.”

<sup>4</sup>This is descriptively reminiscent of recent trends in admissions processes at selective colleges: the introduction of the common application has triggered a cycle in which each student applies to more schools and admissions rates and yield plunge, offsetting many of the gains that a streamlined application process could potentially offer. A recent New York Times panel considered several possible solutions, including limiting the number of schools to which an individual applies (<http://www.nytimes.com/roomfordebate/2015/03/31/how-to-improve-the-college-admissions-process>).

an appropriately chosen application limit raises employer welfare to a constrained efficient benchmark, at little or no cost to applicants.

In our model, welfare losses arise because screening is costly, and employers are unable to determine applicant availability in advance. Thus it is natural to ask whether it is possible to obtain similar or superior results by providing tools to facilitate the screening and offer stages. The platform might be able to reduce screening costs by providing employers information about applicants’ skills and job experience. It could also try to infer which agents are available to match, or ask agents to self-report this information. Each of these approaches is worth pursuing; indeed, many platforms have instantiated versions of such features already.<sup>5</sup> We discuss some of these approaches further in Section 7, and note that they often leave a fair amount of residual uncertainty. Our work indicates that in such cases, the simple intervention of limiting visibility may be an effective tool in the platform operator’s toolbox.

In the remainder of the paper, we develop a stochastic matching model and analysis that addresses the issues raised above. We begin in Section 2 with a review of related work. Section 3 presents a stochastic game theoretic model of a matching market that captures the two salient features described above: namely that screening is costly, and employers cannot determine availability in advance.

The model we develop is complex both because of strategic and stochastic considerations. A key contribution of our work is to *simplify analysis of our stochastic matching model via a large market (“mean field”) scaling*. In Section 4, we present a mean field analysis of our game, and in Section 5, we prove that the mean field analysis holds asymptotically in large markets. The proof employs a novel stochastic contraction argument to establish convergence to the mean field limit uniformly over time.

Section 6 presents our main results concerning the welfare of employers and applicants in equilibrium, and the value of intervention by the platform. Section 7 concludes with a discussion of ways to operationalize the ideas presented in this work, and attendant practical considerations.

## 2. RELATED WORK

Our work relates to several strands of literature. We discuss each in turn below.

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<sup>5</sup>For example, OkCupid promotes recently active users in search results. Airbnb displays information about hosts’ response rates, and asks hosts to keep an updated calendar showing dates of availability. UpWork encourages freelancers to declare their “work status” (i.e. whether they are currently looking for new projects).

**2.1. Frictions in Labor Markets.** Our model closely resembles those used to study labor markets with costly search. This line of work was initiated by Diamond (1982a,b); Mortensen (1982a,b) and Pissarides (1984a,b), and has since received a great deal of attention. See Rogerson et al. (2005) for a helpful survey.

Although these papers make different assumptions about the matching and bargaining process, one common theme is that increased search by workers is either assumed or proven to generate positive externalities for firms. Considering one example, Mortensen (1982a) models the matching process in reduced form, and assumes that the rate of match formation is an (exogenously given) increasing function of the search costs paid by each side. One implication of this assumption is that increases in applicant search intensity reduce employers' average search cost per realized match. By contrast, our work reaches the *opposite* conclusion: as applicants search more intensively, the per-match cost paid by employers *increases*. Consequently, whereas much of the search literature finds that equilibrium match formation is inefficiently low (because agents do not consider the benefits that their search delivers to potential partners), we conclude that equilibrium search effort is inefficiently *high*, and that efficiency can be improved by having slightly *fewer* matches.

Why does our model conclude that increased search by applicants may harm employers, and why is this effect absent in previous work? When applicants search more intensively, they match at higher rates. In our model, this implies that each employer must screen more applicants in order to find one who will accept. Most models of search do not have this feature, as they assume that workers apply for only one vacancy per period, and thus firm offers never collide (see, for example, Moen (1997), Burdett et al. (2001), and Julien et al. (2000)<sup>6</sup>). Albrecht et al. (2006) permit workers to contact multiple firms simultaneously, but assume that firms can make only a single offer (rather than sequentially making offers until one is accepted). As we discuss in the conclusion, this may dramatically reduce negative externalities from workers sending many applications. Kircher (2009) also allows workers to apply to multiple firms simultaneously, and allows firms to continue making offers to applicants until one accepts. The main difference between his work and our own is that he assumes that screening and making offers is costless to firms, so that repeated rejection does not hurt firms.

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<sup>6</sup>To avoid confusion, we note that the latter papers reverse the terminology used in our work, so that firms contact a single worker, and workers then choose among the firms that contacted them.

**2.2. Signaling in Matching Markets.** Closer in message to our work is the body of literature on congestion and preference signaling in matching markets. This line of inquiry originated with Roth and Xing (1997). They note that thick markets may suffer from *congestion*, meaning that it is not possible to evaluate all potential transactions. If firms have limited ability to react to rejections, they benefit from targeting workers who are more likely to accept their offers. In some cases, firms may be able to identify these workers based on observable characteristics - for example, Che and Koh (2015) model an admissions game in which colleges manage yield by strategically targeting students who are less likely to receive other offers.

In other cases, the most reasonable way for firms to identify workers who are likely to accept is to have these workers signal their interest. Lee and Schwarz (2007) note that if signals are unrestricted and private, workers may choose to signal interest to all firms, and thus convey no useful information. If, however, the number of signals sent by each worker is limited, then signaling may improve equilibrium outcomes (Coles et al., 2013). The efficacy of signaling schemes has been tested empirically in the context of online dating markets (Lee et al., 2011) and the academic job market for economists (Coles et al., 2010).

Viewed within the context of these papers, we model applications as costly signals. When application costs are small, workers “signal” (apply to) many firms, and firms may find that most workers do not accept their offers. Like the work above, we find that limiting workers’ ability to signal firms may be beneficial. However, unlike in previous work, e.g. Coles et al. (2013), we explicitly model costs associated with applications and screening, thus endogenizing the level of application and screening activity by agents on both sides of the market, and allowing us to study the overall welfare of agents, instead of proxies like the number of matches. This leads us to different conclusions as well. For instance, Coles et al. (2013) assume that individuals costlessly know their own preferences, and find that signaling mechanisms increase both match quantity and quality (as assessed by workers). In contrast we conclude that limiting applications may dramatically reduce search costs, thereby benefiting both sides of the market (despite a small reduction in the number of matches, and no increase in their quality). Furthermore, we find that in many parameter regimes, firm welfare endogenously drops to the value of the outside option - a conclusion that does not hold when screening is costless. A key additional feature of our model that allows us to study the effect of a lack of availability information, is the dynamic arrival and departure of agents. Thus, our work complements the signaling literature

by identifying a novel reason (uncertainty regarding availability) that limiting the signals sent by workers may be beneficial, via modelling and analysis of the interplay between costly search and dynamics.

### 2.3. Online Platform Markets.

There are also papers that specifically focus on online matching platforms, and demonstrate (theoretically or empirically) that maximizing the number of potential matches shown to each user may not be desirable. For example, Halaburda et al. (2015) use a simple model to illustrate that showing users more potential matches decreases the likelihood that any of these matches is available. Although this cost is similar to the one that we identify, their matching process is somewhat artificial, and implies that a user who is shown  $n$  options matches with probability at most  $1/(n + 1)$ . Our model allows rejected users to continue to search.

Recent empirical work by Fradkin (2015) and Horton (2015) concludes that on both Airbnb and oDesk, it is common for users to contact unavailable partners, and that this fact has significant welfare consequences. On AirBnB, guests are uncertain about host availability because transactions take time to complete, and hosts may not reliably update the calendars. Fradkin (2015) notes that only 15% of inquiries eventually lead to a transaction, and that an initial rejection decreases the probability that the guest eventually books *any* listing by 50%. On oDesk, potential employers can browse profiles and invite workers to apply, but in practice, the best workers are often too busy to accept the invitation. Horton (2015) concludes that employers are often unable to discern which workers are busy, that this fact causes many invitations to be declined, and that spurned employers are much less likely to eventually find a match.

There is an emergent line of work on online two-sided markets with dynamic arrivals in the operations literature, that includes the current paper. The overall goal may be stated as that of optimally managing the inventory of a marketplace, where the market participants themselves constitute the inventory, and are also the ones whose needs the market must meet. A paper by Allon et al. (2012) is similar in spirit to our own, showing that in service marketplaces, operational efficiency may hurt market efficiency due to the involvement of strategic agents. There are also recent studies of optimal centralized dynamic matching with no strategic behavior. Hu and Zhou (2015) consider a two-sided market with horizontal and vertical differentiation, and a platform that knows the agent types and wants to execute an optimal centralized dynamic matching policy. There is also work motivated by kidney exchange, e.g., see

Ashlagi et al. (2013); Anderson et al. (2015), and work on matching queues, e.g. the paper by Gurvich et al. (2014).

#### 2.4. Mean Field Approximations.

In Section 4, we present a *mean field* analysis of our game. In particular, we assume that from the point of view of an employer, applicants are independently available with some fixed probability  $q$ ; and from the point of view of an applicant, each application yields an offer independently with probability  $p$ . Under these two assumptions, solving for the optimal strategies of employers and applicants becomes straightforward. On the other hand, both  $p$  and  $q$  must satisfy consistency checks that ensure they arise from the optimal decisions of employers and applicants. Taken together, this pair of conditions—optimality and consistency—define a notion of equilibrium for our mean field model that we call *mean field equilibrium* (MFE); several recent papers (Weintraub et al., 2008; Iyer et al., 2013; Balseiro et al., 2013; Bodoh-Creed, 2013) have used a similar modeling approach.

In Section 5, we prove that the mean field agent assumptions hold asymptotically in large markets. The proof is a significant technical contribution of our work, as much of the existing search literature relies on “large market” assumptions that are not formally stated or justified. Existing papers that provide such justification (see for example, Galenianos and Kircher (2012)) consider a static setting; establishing similar results in our dynamic model presents a number of novel technical challenges. In addition, in our model each applicant will contact multiple firms, which Galenianos and Kircher acknowledge presents many difficulties even in the static setting.

### 3. THE MODEL

In the market we consider, employers and applicants arrive over time, interact with each other, and eventually depart. Informally, we aim to capture the following behavior:

- (1) Employers arrive to the market, and post an *opening*.
- (2) When applicants arrive, they *apply* to a subset of the employers currently in the market.
- (3) Upon exit from the system, employers may screen candidates to learn whether they are *compatible* with the job. They may make offers to compatible applicants.
- (4) Whenever an applicant receives an offer, he chooses whether to accept it.



One could construct a static model in which the above stages happen sequentially, with every applicant applying simultaneously, and every employer making offers simultaneously. Instead, we model a dynamic and asynchronous market, the timing of which is described in more detail below. In Section 7 and Appendix A, we explain how a static model where each stage occurs once does not capture key features of such a market.

**3.1. Arrivals.** Our market starts empty at  $t = 0$ . Our dynamic markets are parameterized by  $n > 0$ , which describes the market size. Individual employers arrive at intervals of  $1/n$ , and applicants arrive at intervals of  $1/(rn)$ . Here  $r > 0$  is a parameter that controls the relative magnitude of the two sides of the market. Employers remain in the system for a unit lifetime. Applicants depart the system according to a process that we describe below. We endow employers and applicants with unique IDs, which convey no other information about the agent.

Upon arrival, employers post an opening. They do not make any decisions at this time. Upon arrival, each applicant selects a value  $m_a \in [0, n]$ , and applies to each employer currently in the system with probability  $m_a/n$ . Note that for all  $t \geq 1$ , there are exactly  $n$  employers in the system, and thus the expected number of applications sent by an applicant who arrives after time  $t \geq 1$  is  $m_a$ .<sup>7</sup>

**3.2. Applicant departure.** Applicants remain in the system for a maximum of one time unit. If they receive and accept an offer from any employer (as described below), they depart from the market at that time. Recall that (excluding application costs) an applicant earns a payoff of 1 as long as she is matched, and zero otherwise. Thus, *it is a dominant strategy for applicants to accept the first offer they receive*. We assume henceforth that applicants follow this strategy.

**3.3. Employer departure.** Each employer stays in the system for one time unit, and then departs. Upon departure, employers see the set of applicants to their opening. Initially, they do not know which of these applicants are compatible for their job, nor do they know which would accept the job if offered it (i.e. which applicants remain in the system). The employer takes a sequence of “screening” and “offer” actions, instantaneously learning the result of each, until they choose to exit the market.

At each stage of the employer’s sequential decision process, she may screen any unscreened applicant, thereby learning whether this applicant is compatible for the

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<sup>7</sup>An alternate model might allow applicants to directly choose the number of applications sent. We choose a probabilistic specification primarily for technical convenience - see Section 7 for further discussion on this point.

job. If the employer has yet to match, she may also make an offer to any applicant whom she has found to be compatible (the applicant responds immediately to the offer; as mentioned above, we assume that they accept it if and only if it is the first offer that they receive). Employers are also allowed to exit the market at any point.

Note that before making an offer to any applicant, employers must screen this applicant and find him compatible (this assumption is justified if, for example, the cost of hiring an unqualified worker is sufficiently high). We assume that each employer-applicant pair is compatible with probability  $\beta$  (independently across all such pairs), and that this is common knowledge.

**3.4. Utility.** If a compatible pair matches to each other, the employer earn  $v$  and the applicant earns  $w$ . Without loss of generality, we normalize  $v = w = 1$ .<sup>8</sup> Applicants pay a cost  $c_a$  for each application that they send, and employers pay a cost  $c_s$  for each applicant that they screen. The net utility to an agent will be the difference between value obtained from any match, and costs incurred, and agents act so as to maximize their expected utility.

**Assumption 1.**  $\max(c_a, c_s) < \beta$ .

This assumption rules out the uninteresting case where costs are so high that no activity occurs in the marketplace. If  $c_s \geq \beta$ , then regardless of applicant behavior, it would be optimal for employers to exit the market rather than screening. Similarly, if  $c_a \geq \beta$ , then because employers hire only compatible applicants, no applicant strategy can earn positive surplus.

For later reference, it will be useful to consider *normalized* versions of the screening and application costs, given by

$$(1) \quad c'_s = c_s/\beta, \quad c'_a = c_a/\beta.$$

#### 4. THE LARGE MARKET: A STATIONARY MEAN FIELD MODEL

In principle, the strategic choices facing an agent in the model described above may be quite complicated. Consider the case of an employer who knows that he has only one competitor. If he finds that his first applicant has already accepted another offer, he learns that every other applicant is still looking for a job. Similar logic suggests that in thin markets, information revealed during screening may induce significant shifts

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<sup>8</sup>The assumption that  $v = w$  is without loss of generality because we never compare absolute welfare of agents on opposite sides of the market.

in the employer’s beliefs. This could conceivably cause optimal employer behavior to be quite complex.

As the market thickens, however, one might expect that the correlations between agents on the same side of the market become weak. In particular, employers screening a pool of applicants might reasonably assume that learning that one applicant has already accepted an offer does not inform them about the availability of other applicants. Further, if the employer cannot distinguish individual applicants, each one should appear to be available with equal probability. Similarly, applicants who know nothing about individual employers may be justified in assuming that each of their applications convert to offers independently and with equal probability  $p$ .

In this section we develop a formal stationary *mean field* model for our dynamic matching market, and introduce a notion of game-theoretic equilibrium for this model; in particular, we study a model that arises from a limiting regime where the market thickens.

In our formal model, agents make the following assumptions.

**Mean Field Assumption 1** (Employer Mean Field Assumption). *Each applicant in an employer’s applicant set is available with probability  $q$ , and the availability of applicants in the applicant set is independent.*

**Mean Field Assumption 2** (Applicant Mean Field Assumption). *Each application yields an offer with probability  $p$ , independently across applications to different openings.*

**Mean Field Assumption 3** (Large Market Assumption). *The number of applications sent by an applicant who chooses  $m_a = m$  is Poisson distributed with mean  $m$ . If all applicants select  $m_a = m$ , the number of applications received by each employer is Poisson distributed with mean  $rm$ .*

Under these assumptions, optimal agent behavior simplifies greatly. We describe optimal employer and applicant responses in Section 4.1. For applicants, we show that there exists a unique optimal choice of  $m$ , given  $p$ ; and for employers, we show that given  $q$ , their optimal response is either to employ a simple sequential screening strategy or to exit immediately (they may also randomize when indifferent between these options).

Of course  $p$  and  $q$  are not given exogenously, but rather arise endogenously from the choices made by agents. In Section 4.2, we derive “consistency checks” that  $p$

and  $q$  should satisfy, if they indeed arise from the conjectured employer and applicant strategies.

The work in Sections 4.1 and 4.2 allows us to define a *mean field equilibrium* (MFE) in Section 4.3. Informally, a mean field equilibrium consists of strategies for employers and applicants that are best responses to the stationary market dynamics that they induce. We prove that there exists a *unique* MFE. We conclude with Section 4.4, which discusses a simple intervention available to the market operator: placing a limit  $\ell$  on the value  $m_a$  chosen by each applicant. We show that unique MFE continue to exist in this setting.

We emphasize that a key result of our paper is that our mean field model is in fact the correct limit of our dynamic market as the thickness  $n$  grows. In particular, Propositions 4 and 5 in Section 5 justify our study of MFE: they state that the mean field assumptions hold as  $n$  approaches infinity, and that as a consequence any MFE is an approximate equilibrium in the game with finite but sufficiently large  $n$ .

**4.1. Optimal decision rules.** We first study how agents respond when confronted with a world where the mean-field assumptions hold.

**4.1.1. Applicants.** As discussed in Section 3, it is a dominant strategy for applicants to accept the first offer (if any) that they receive, and we assume applicants follow this rule. Therefore the only decision an applicant  $a$  needs to make on arrival is her choice of  $m_a$ , the expected number of applications sent.

If an applicant  $a$  chooses  $m_a = m$ , they incur an expected cost of  $c_a \cdot m$ . If the applicant applies to  $\text{Poisson}(m)$  employers, and each application independently yields an offer with probability  $p$ , then at least one offer is received—i.e., the applicant matches to an employer—with probability  $1 - e^{-mp}$ . Thus, the expected payout of an applicant in the mean-field environment who selects  $m_a = m$  is  $1 - e^{-mp} - c_a m$ . Applicants choose  $m \geq 0$  to maximize this payoff. Because their objective is strictly concave and decays to  $-\infty$  as  $m \rightarrow \infty$ , this problem possesses a unique optimal solution identified by first-order conditions. If  $p \leq c_a$ , the optimal choice is  $m = 0$ . Otherwise, applicants select  $m = \frac{1}{p} \log \left( \frac{p}{c_a} \right)$ . We define  $\mathcal{M}$  to be the function that maps  $p$  to the unique optimal value of  $m$ :

$$(2) \quad \mathcal{M}(p) = \begin{cases} 0, & \text{if } p \leq c_a; \\ \frac{1}{p} \log \left( \frac{p}{c_a} \right), & \text{if } p > c_a. \end{cases}$$

**4.1.2. Employers.** Next, we consider the optimal strategy for employers, when Mean Field Assumption 1 holds. We consider a simple strategy, which we denote  $\phi^1$ . A

employer playing  $\phi^1$  sequentially screens candidates in her applicant list. When she finds a compatible applicant, she makes an offer to this candidate; otherwise, she considers the next candidate. This process repeats until one applicant accepts or no more applicants remain.

The optimal strategy for the employers is straightforward to characterize. First suppose an employer has exactly one applicant. The employer will prefer to screen the applicant if  $\beta q - c_s > 0$ , i.e., if  $q > c'_s$ ; exit if  $q < c'_s$ ; and is indifferent if  $q = c'_s$ . Now it is clear that if an employer has more than one applicant in her list, since all applicants are *ex ante* homogeneous from the perspective of the employer, the same reasoning holds: the employer will screen or exit immediately according to whether  $q$  is larger or smaller than  $c'_s$ , respectively. (Note the essential use of Mean Field Assumption 1: if there is correlation in the availability of successive applicants in the employer's list, the preceding reasoning no longer holds.) The following proposition summarizes the preceding discussion.

**Proposition 1.** *Let  $\phi^1$  be the strategy of sequentially screening applicants, offering them the job if and only if they are qualified, until either an applicant is hired or no more applicants remain. Then  $\phi^1$  is uniquely optimal if and only if  $q > c'_s$ , exiting immediately is uniquely optimal if and only if  $q < c'_s$ , and any mixture of these strategies is optimal if  $q = c'_s$ .*

Motivated by this proposition, we define  $\phi^\alpha$  to be the strategy that plays  $\phi^1$  with probability  $\alpha$  and exits immediately otherwise. Define the correspondence  $\mathcal{A}(q)$  by:

$$(3) \quad \mathcal{A}(q) = \begin{cases} \{0\} & \text{if } q < c'_s \\ [0, 1] & \text{if } q = c'_s \\ \{1\} & \text{if } q > c'_s. \end{cases}$$

This correspondence captures the optimal employer response, as described in Proposition 1, so that  $\mathcal{A}(q) = \{\alpha \in [0, 1] : \phi^\alpha \text{ is optimal for the employer, given } q\}$ .

**4.2. Consistency.** In the previous section, we discussed the best responses available to employers and applicants when the mean field assumptions hold; that is, given  $p$  and  $q$ , we found the strategies that agents would adopt. However,  $p$  and  $q$  are clearly *determined* by agent strategies. In this section we identify consistency conditions that  $p$  and  $q$  must satisfy, given specified agent strategies.

We focus on strategies that could conceivably be optimal, as identified in the preceding section. We assume that all applicants choose the same  $m \geq 0$ , and that

all employers play  $\phi^\alpha$ , i.e., they play  $\phi^1$  with probability  $\alpha$  and exit immediately otherwise. From any  $m$  and  $\alpha$ , we derive a unique prediction for the pair  $(p, q)$ .

We emphasize at the outset that our analysis aims only to derive the correct consistency conditions under the mean field assumptions. We provide rigorous justification for these assumptions via the propositions in Section 5. As a consequence, those propositions also justify the consistency conditions described below.

We start by deriving a consistency condition for  $q$ , given  $p$  and the strategy adopted by applicants. Intuitively,  $q$  should be equal to the long-run fraction of offers that are accepted. Fix the value of  $m$  chosen by applicants, and let  $X$  be the number of offers received by a single applicant. This applicant will accept an offer if and only if  $X > 0$ , so the expected fraction of offers that are accepted is  $P(X > 0)/E[X]$ . If Mean Field Assumptions 2 and 3 hold, then  $X$  is Poisson with mean  $mp$ , so we should have

$$(4) \quad q = \frac{1 - e^{-mp}}{mp}.$$

To derive a consistency condition for  $p$ , note that when employers follow  $\phi^\alpha$ , only compatible applicants receive offers. Thus,  $p$  should equal  $\beta$  times the long-run fraction of qualified applications that result in offers. Because an applicant's availability does not influence whether they receive an offer (as it is unobserved by prospective employers), this should be equal to  $\beta$  times the fraction of applications by qualified *available* applicants that result in offers. Fix an employer playing  $\phi^\alpha$ , and let  $Y$  be the number of qualified, available applicants received by this employer. This employer successfully hires if and only if  $Y > 0$  and she decides to screen. Thus, the fraction of qualified available applicants who receive offers should be  $\alpha P(Y > 0)/E[Y]$ . We conclude that

$$(5) \quad p = \alpha \beta \frac{P(Y > 0)}{E[Y]} = \alpha \beta \frac{1 - e^{-rm\beta q}}{rm\beta q},$$

where the final step comes from Mean Field Assumptions 1 and 3, which jointly imply that  $Y$  is distributed as a Poisson random variable with mean  $rm\beta q$ .

The equations (4) and (5) are a system for  $p$  and  $q$ , given the values of  $m$  and  $\alpha$  (as well as the parameters  $r$  and  $\beta$ ). The following proposition states that the pair of consistency equations (4) and (5) have a unique solution.

**Proposition 2.** *For fixed  $m, \alpha, r$ , and  $\beta$ , there exists a unique solution  $(p, q)$  to (4) and (5).*

We refer to the unique pair  $(p, q)$  that solve (4) and (5) as a *mean field steady state* (MFSS). This pair provides a prediction of how a large market should behave, given specific strategic choices of the agents. For later reference, given strategies  $m$  and  $\alpha$  (and parameters  $r$  and  $\beta$ ), let  $\mathcal{P}(m, \alpha; r, \beta)$  and  $\mathcal{Q}(m, \alpha; r, \beta)$  denote the unique values of  $p$  and  $q$  guaranteed by Proposition 2, respectively. Because our analysis is conducted with  $r$  and  $\beta$  fixed, we will omit the dependence on  $r$  and  $\beta$  in favor of the more concise  $\mathcal{P}(m, \alpha)$  and  $\mathcal{Q}(m, \alpha)$ .

**4.3. Mean field equilibrium.** In this section we define *mean field equilibrium* (MFE), a notion of game theoretic equilibrium for our stationary mean field model. Informally, a MFE should be a pair of strategies such that (1) agents play optimally given their beliefs about the marketplace, i.e., the values  $p$  and  $q$  in the mean field assumptions; and (2) agent beliefs are consistent with the strategies being played, i.e.,  $(p, q)$  is an MFSS corresponding to the agents' strategies. Section 4.1 addressed the first point; and Section 4.2 addressed the second. We define a mean field equilibrium by composing the maps defined in those sections.

**Definition 1.** A mean field equilibrium (MFE) is a pair  $(m^*, \alpha^*)$  such that  $m^* = \mathcal{M}(\mathcal{P}(m^*, \alpha^*))$  and  $\alpha^* \in \mathcal{A}(\mathcal{Q}(m^*, \alpha^*))$ .

In an MFE,  $m^*$  and  $\alpha^*$  are optimal responses (under the mean field assumptions) to the steady-state  $(p, q)$  that they induce. For future reference, we define  $p^* = \mathcal{P}(m^*, \alpha^*)$  and  $q^* = \mathcal{Q}(m^*, \alpha^*)$ . Our main theorem in this section establishes existence and uniqueness of MFE.

**Theorem 1.** Fix  $r, \beta, c_a, c_s$ , and suppose Assumption 1 holds. Then there exists a unique mean field equilibrium  $(m^*, \alpha^*)$ .

**4.4. A market intervention: application limits.** As noted in the Introduction, we are interested in comparing the outcome of the market described above to the outcome when the platform operator intervenes to try to improve the welfare of employers and/or applicants. We consider a particular type of intervention: a limit on the number of applications that can be sent by any individual.

In our model with application limits, agent payoffs are identical to before, as are the strategies available to employers. Applicants, however, are restricted to selecting  $m_a \leq \ell$ . In the corresponding mean field model, given  $p$ , applicants choose  $m_a$  to maximize  $1 - e^{-m_a p} - c_a m_a$  (their expected payoff), subject to  $m_a \in [0, \ell]$ . The

applicant objective is concave in  $m_a$ , so this problem has a unique solution given by

$$(6) \quad \mathcal{M}_\ell(p) = \min(\ell, \mathcal{M}(p)).$$

The consistency conditions are identical to those in Section 4.2. We define a *mean field equilibrium* of the market with application limit  $\ell$  as a pair  $(m_\ell^*, \alpha_\ell^*)$  solving the following pair of equations:

$$(7) \quad m_\ell^* = \mathcal{M}_\ell(\mathcal{P}(m_\ell^*, \alpha_\ell^*)), \quad \alpha_\ell^* \in \mathcal{A}(\mathcal{Q}(m_\ell^*, \alpha_\ell^*)).$$

The following proposition is an analog of Theorem 1 for the market with an application limit.

**Proposition 3.** *Fix  $r, \beta, c_a, c_s$  such that Assumption 1 holds, and let  $(m^*, \alpha^*)$  be the corresponding MFE in the market with no application limit. Then for any  $\ell \geq 0$  there exists a unique mean field equilibrium in the market with application limit  $\ell$ . If  $m^* \leq \ell$ , then  $(m_\ell^*, \alpha_\ell^*) = (m^*, \alpha^*)$ . Otherwise,  $m_\ell^* = \ell$  and  $\alpha_\ell^*$  is the unique solution to  $\alpha_\ell^* \in \mathcal{A}(\mathcal{Q}(\ell, \alpha_\ell^*))$ .*

For future reference, we define  $p_\ell^* = \mathcal{P}(m_\ell^*, \alpha_\ell^*)$ ,  $q_\ell^* = \mathcal{Q}(m_\ell^*, \alpha_\ell^*)$ .

## 5. MEAN FIELD APPROXIMATION

In this section, we show that our mean field model is (in an appropriate sense) a reasonable approximation to our finite system when the market grows large (i.e., when  $n \rightarrow \infty$ ). Formally, we show two results. First, we show that the mean field assumptions hold as  $n \rightarrow \infty$ , as long as all applicants  $a$  choose  $m_a = m$ , and all employers choose  $\alpha_e = \alpha$ . Note that once we fix  $m$  and  $\alpha$ , we have removed any strategic element from the evolution of the  $n$ -th system; these results are limit theorems about a certain sequence of stochastic processes. Second, we use the preceding results to show that any MFE is an approximate equilibrium in sufficiently large but finite markets.

We require the following notation. We let  $\text{Binomial}(n, p)$  denote the binomial distribution with  $n$  trials and probability of success  $p$ , and let  $\text{Binomial}(n, p)_k = \mathbb{P}(\text{Binomial}(n, p) = k)$ . We let  $\text{Poisson}(a)$  denote the Poisson distribution with mean  $a$ , and let  $\text{Poisson}(a)_k = \mathbb{P}(\text{Poisson}(a) = k)$ .

In the following proposition, we show that Mean Field Assumption 1 holds as  $n \rightarrow \infty$ . In the process, we also show half of Mean Field Assumption 3: that in the limit the number of applications received by an employer is Poisson distributed.



**Proposition 4.** *Fix  $r, \beta, m$ , and  $\alpha$ . Suppose that the  $n$ -th system is initialized in its steady state distribution. Consider any employer  $e$  that arrives at  $t_b \geq 0$ . Let  $R_b^{(n)}$  denote the number of applications received by employer  $e$  in the  $n$ -th system, and let  $A_b^{(n)}$  be the number of these applicants that are still available when the employer screens.*

*Then as  $n \rightarrow \infty$ , the pair  $(R_b^{(n)}, A_b^{(n)})$  converges in total variation distance to  $(R, A)$ , where  $R \sim \text{Poisson}(rm)$ , and conditional on  $R$ , we let  $A \sim \text{Binomial}(R, q)$ .*

Analogously, we have the following proposition, where we show that Mean Field Assumption 2 holds as  $n \rightarrow \infty$ . In the process, we also show the other half of Mean Field Assumption 3: that in the limit the number of applications sent by an applicant is Poisson distributed.

**Proposition 5.** *Fix  $r, \beta, m$ , and  $\alpha$  and any  $m_0 < \infty$ . Suppose that the  $n$ -th system is initialized in its steady state distribution. Consider any applicant  $a$  arriving at time  $t_s \geq 0$ , denote the value chosen by applicant  $a$  by  $m_a$  (all other applicants are assumed to choose  $m$ , and all employers are assumed to follow  $\phi^\alpha$ ). Let  $T_s^{(n)}$  denote the number of applications sent by applicant  $a$  in the  $n$ -th system, and let  $Q_s^{(n)}$  be the number of these applications that generate offers. Let  $T \sim \text{Poisson}(m_a)$ , and conditional on  $T$ , we let  $Q \sim \text{Binomial}(T, p)$ . Then*

$$(8) \quad \lim_{n \rightarrow \infty} \left\{ \max_{m_a \in [0, m_0]} d_{\text{TV}}((T_s^{(n)}, Q_s^{(n)}), (T, Q)) \right\} = 0,$$

where  $d_{\text{TV}}(X, Y)$  denotes the total variation distance between the distributions of random variables  $X$  and  $Y$  that take values in the same countable set.

In establishing these propositions, the fundamental result that we prove is that the evolution of the “state” of the  $n$ -th system satisfies a stochastic contraction condition, and therefore remains “close” to an appropriately defined fixed point for all time. This result allows us to establish that the mean field assumptions hold asymptotically. Informally, our technical argument proceeds as follows. We define the state of the system to track the residual lifetimes of all applicants in the system. To simplify this tracking, we use a “binned” version of the remaining lifetime of each applicant, and consider the number of available applicants in each bin. As a result we discretize time in our analysis, considering snapshots at intervals that are separated by the bin width. The proof of the stochastic contraction result has two parts: First, we consider the deterministic continuum limit of state evolution, showing that the state contracts towards a unique fixed point that corresponds to the unique mean field

equilibrium.<sup>9</sup> The second, more tedious part of our analysis consists of controlling stochastic deviations from the continuum limit that occur in a finite system. We expect that this contractive behavior as well as our technical approach apply more generally in matching contexts.

We conclude by using the preceding results to establish the following theorem, which establishes (in an appropriate sense) that any MFE is an approximate equilibrium in sufficiently large finite markets.

**Theorem 2.** *Fix  $r$ ,  $\beta$ ,  $c_s$  and  $c_a$ . Let the MFE be  $(m^*, \alpha^*)$ . For any  $\varepsilon > 0$ , and any nonnegative integer  $R_0$ , there exists  $n_0$  such that for all  $n \geq n_0$  the following hold:*

- (1) *For all  $a$ , if all employers and applicants other than  $a$  follow the prescribed mean field strategies, then applicant  $a$  can increase her expected payoff by no more than  $\varepsilon$  by deviating to any  $m \neq m^*$ .*
- (2) *For all  $e$ , if  $e$  receives no more than  $R_0$  applications, and if all employers and applicants other than  $e$  follow the prescribed mean field strategies, then in any state in the corresponding dynamic optimization problem solved by employer  $e$ , employer  $e$  can increase her expected payoff by no more than  $\varepsilon$  by deviating to any strategy other than  $\phi^{\alpha^*}$ .*

The preceding result shows that for sufficiently large  $n$ , the mean field equilibrium is (essentially) an  $\varepsilon$ -approximate equilibrium in the finite market of size  $n$ . The first statement in the result states that applicants cannot appreciably gain by changing the number of applications they send. The second statement in the result makes an analogous claim for employers. In particular, if employers follow  $\phi^{\alpha^*}$ , they will either exit immediately; or sequentially screen and make offers to compatible candidates until such an offer is accepted, or the applicant pool is exhausted. In doing so, they will obtain information about the compatibility and availability of the subset of applicants they have already screened. Our result states that at any stage in this dynamic optimization problem, the employer cannot appreciably increase their payoff by deviating.

An apparent limitation of the second statement is that it applies only to employers who receive no more than  $R$  applications. The fraction of employers who receive more than  $R$  applications scales as  $\exp(-\Omega(R))$ ; thus by choosing  $R$  large enough, this fraction can be made as small as desired. Since we show that the  $n$ -th system satisfies an appropriate stochastic contraction condition, we expect that in fact, for

<sup>9</sup>In particular, the fixed point converges to the unique MFE as the bin width tends to zero.

sufficiently large  $R$ , both statements in Theorem 2 hold even under arbitrary behavior by employers who receive more than  $R$  applications. In the interest of brevity we choose to omit this slightly stronger result.

## 6. WELFARE ANALYSIS

One of the chief advantages of MFE is that they are amenable to analysis. In particular, we can gain significant insight into employer and applicant welfare, and the potential value of intervention for each side of the market. In this section, we study employer and applicant welfare in MFE, and quantify the effects of limiting the number of applications sent by each applicant. We have two main goals. *First*, by studying the equilibrium market outcome, we characterize the welfare losses due to the two aspects of congestion described in the introduction: (1) many applications may be sent that are never even screened; and (2) many applicants may be screened that prove to be unavailable. The former leads to welfare losses for applicants; the latter leads to welfare losses for employers. *Second*, we study the effect of market intervention, and show that limiting the number of applications can lead to significant welfare gains for both sides of the market.

### 6.1. Welfare Benchmarks: Efficiency and Constrained Efficiency.

Fix parameter values  $r, \beta, c_a$ , and  $c_s$ . For given agent strategies  $m$  and  $\alpha$ , we let  $\Pi_a(m, \alpha)$  and  $\Pi_e(m, \alpha)$  denote the mean field applicant and employer welfare, respectively (Formal expressions for these payoffs can be found in Appendix D). We define  $\Pi_a^*, \Pi_e^*$  to be the expected applicant and employer surplus in mean-field equilibrium, respectively. In other words,

$$(9) \quad \Pi_a^* = \Pi_a(m^*, \alpha^*), \quad \Pi_e^* = \Pi_e(m^*, \alpha^*),$$

Analogously, let  $\Pi_a^\ell$  and  $\Pi_e^\ell$  be expected applicant and employer payoffs in the equilibrium of the market with application limit  $\ell$ . In other words,

$$(10) \quad \Pi_a^\ell = \Pi_a(m_\ell^*, \alpha_\ell^*), \quad \Pi_e^\ell = \Pi_e(m_\ell^*, \alpha_\ell^*).$$

When evaluating employer and applicant welfare in equilibrium, we need a point of comparison: what constitutes “good” welfare for each side of the market? In the search literature, there are two common comparison points: the “efficient” outcome and the “constrained efficient” outcome. In our setting, the former corresponds to the hypothetical situation in which the platform directly observes compatibility and costlessly matches individuals, so that applicants get  $\min(1, 1/r)$  and employers  $\min(1, r)$ .

While platform operators often have information that they use to infer compatibility and suggest matches, it is not reasonable to assume that they can costlessly and perfectly match agents. This motivates the concept of *constrained efficiency*, where the designer is subject to the informational constraints and costs of the decentralized market, but has the power to specify application and screening strategies for each side. Motivated by this approach, we define

$$\Pi_e^{\text{opt}} = \sup_{m, \alpha} \Pi_e(m, \alpha); \quad \Pi_a^{\text{opt}} = \sup_{m, \alpha} \Pi_a(m, \alpha).$$

It is clear that  $\Pi_e^{\text{opt}}$  (resp.,  $\Pi_a^{\text{opt}}$ ) is an upper-bound for employer (resp., applicant) welfare in equilibrium. These bounds are optimistic in at least two ways. First,  $\Pi_e^{\text{opt}}$  is the best that the social planner can obtain *when optimizing for employers* and  $\Pi_a^{\text{opt}}$  is the best that the social planner can obtain *when optimizing for applicants*; there is in general no reason to believe that the choices of  $m$  and  $\alpha$  that induce employer welfare of  $\Pi_e^{\text{opt}}$  are the same strategies that yield a payoff of  $\Pi_a^{\text{opt}}$  for applicants. Second, the value  $\Pi_a^{\text{opt}}$  does not incorporate the screening cost  $c'_s$ . If this cost is high, then the employer strategy required to produce applicant welfare of  $\Pi_a^{\text{opt}}$  may not be individually rational (and thus not attainable in practice). Similarly, the value  $\Pi_e^{\text{opt}}$  does not depend on the application cost  $c'_a$ , and it may be that no individually rational applicant strategy can yield employer surplus of  $\Pi_e^{\text{opt}}$ .

We employ  $\Pi_e^{\text{opt}}$  and  $\Pi_a^{\text{opt}}$  as our constrained efficient benchmarks.<sup>10</sup> For most of this section, we will discuss employer and applicant welfare in terms of the *fraction* of  $\Pi_e^{\text{opt}}$  and  $\Pi_a^{\text{opt}}$  earned by employers and applicants, respectively. In Section 6.2 we focus on employer welfare, and turn our attention to applicants in Section 6.3. In both cases, we conclude that equilibrium payouts are (often substantially) below the constrained efficient benchmarks, and provide conditions under which the simple intervention of employing an application limit can raise employer welfare to  $\Pi_e^{\text{opt}}$  or applicant welfare to  $\Pi_a^{\text{opt}}$ . Although the employer-optimal and applicant-optimal limits do not coincide, in Section 6.4 we provide conditions under which an application limit can deliver Pareto improvements. Furthermore, we prove that for a wide range of parameters, the tradeoff between optimizing for employers and for applicants is never too severe. Finally, in Section 6.5 we consider the effect of an alternative intervention: *raising*

<sup>10</sup>One possible measure for the performance of the market is the *aggregate* welfare given by  $r\Pi_a + \Pi_e$ . However, note that we have not imposed any constraints on the relative scale of employer and applicant payoffs; indeed, their units may not even be comparable. By focusing on  $\Pi_a^{\text{opt}}$  and  $\Pi_e^{\text{opt}}$  individually, we provide welfare analysis that holds for any relative weighting of employer and applicant welfare.

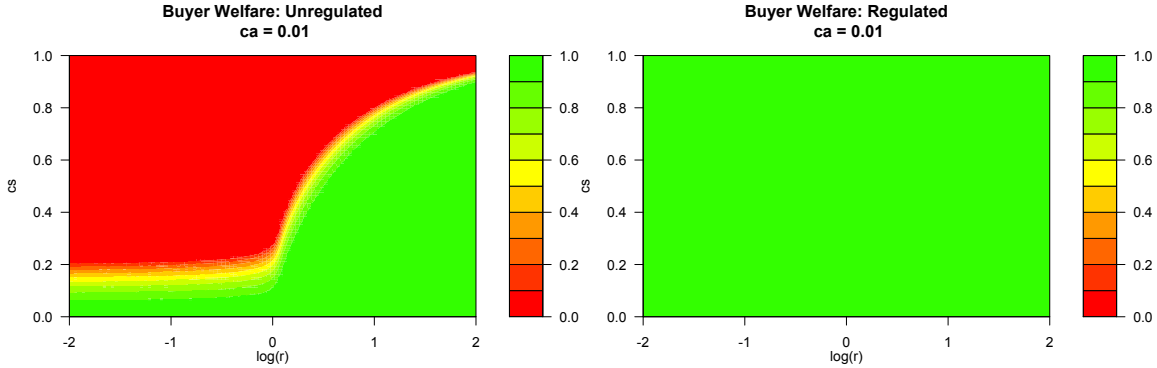


FIGURE 1. Normalized employer welfare in equilibrium ( $\Pi_e^*/\Pi_e^{\text{opt}}$ , left) and with an employer-optimal application limit ( $\sup_\ell \Pi_e^\ell/\Pi_e^{\text{opt}}$ , right). We fix the value  $c'_a = 0.01$ , and vary the values of  $\log r$  (x-axis) and  $c'_s$  (y-axis). Note that there are effectively two regions: one in which employers do very well in equilibrium, and another in which they do very poorly. In the latter case, intervention can substantially raise employer surplus. See Theorem 3.

the application cost  $c_a$ . We show that this intervention is equivalent to imposing an application limit from the perspective of employers, but always reduces applicant welfare.

Before moving on, we note that although our model has four parameters  $(r, c_a, c_s, \beta)$ , it turns out that there is some redundancy, in that the quantities  $\Pi_a^*$ ,  $\Pi_e^*$ ,  $\Pi_a^{\text{opt}}$ ,  $\Pi_e^{\text{opt}}$ ,  $\sup_\ell \Pi_a^\ell$ , and  $\sup_\ell \Pi_e^\ell$  depend only on  $r, c'_a = c_a/\beta$ , and  $c'_s = c_s/\beta$ . We characterize welfare in terms of this reduced parameter set.

## 6.2. Welfare Analysis: Employers.

In this section, we state a theorem regarding equilibrium employer welfare, and the welfare that is attainable through the intervention of enforcing an application limit. We preview our results using Figure 1, which displays  $\Pi_e^*/\Pi_e^{\text{opt}}$  as  $r$  and  $c'_s$  vary. The plot depicts a rather sharp transition for employers: for most parameter values, they either do very well or quite poorly. Furthermore, in the regions where equilibrium employer welfare is low, suitably limiting applications can substantially improve employer welfare. We formalize these observations in the following theorem.

### Theorem 3.

Employer welfare in equilibrium:

*There exists a function  $f(r, c'_a) \in (0, 1)$ , increasing in both arguments, such that*

$\Pi_e^* = 0$  if and only if  $c'_s \geq f(r, c'_a)$ .

Employer welfare with an application limit:

If  $\Pi_e^* = 0$ , there exists  $\ell$  such that  $\Pi_e^\ell = \Pi_e^{\text{opt}}$ .

The preceding result partitions the parameter space into two regions, via the function  $f(r, c'_a)$ . The region where  $c'_s > f(r, c'_a)$  corresponds to the red area in the left image in Figure 1. Here employers obtain zero welfare in equilibrium, but intervention (in the form of an appropriate application limit) can raise employer welfare all the way to the constrained efficient benchmark  $\Pi_e^{\text{opt}}$ . We note that equilibrium employer welfare may be zero even if the screening cost  $c'_s$  is small. Indeed, if  $r < 1$ , then  $f(r, c'_a) \rightarrow 0$  as  $c'_a \rightarrow 0$ , implying that regardless of  $c'_s$ , employer welfare may be zero if applicants send sufficiently many applications.

The intuition behind the theorem is as follows. When applicants send more applications, the effects on employers are two-fold. First, each employer expects to receive a higher number of qualified applicants, and is therefore more likely to match. However, each of these applicants is less likely to be available, as they have applied to many other employers; this is precisely the congestion effect (negative externality) on employers due to additional applicant applications. Thus, each employer pays a higher expected screening cost per successful hire. The theorem captures the phenomenon that when this expected screening cost is too high, employers obtain zero welfare in equilibrium.

The preceding discussion motivates the definition of the function  $f$ . In particular, as a thought experiment, consider *fixing* the employer screening strategy to be  $\phi^1$ : i.e., suppose employers always screen applicants before exiting. For fixed  $r$  and  $c'_a$ , we can study the resulting market via a “partial” equilibrium where applicants optimally choose  $m$ . Note that this partial equilibrium *does not* depend on  $c_s$ , since the employer strategy is already fixed. Let  $\tilde{q}$  denote the applicant availability in this partial equilibrium.<sup>11</sup>

The function  $f(r, c'_a)$  is exactly the availability  $\tilde{q}$  derived in the preceding partial equilibrium analysis. Following Proposition 1, we see that if availability is high enough (i.e., if  $\tilde{q} > c'_s$ ), then the strategy  $\phi^1$  is optimal for employers, and the partial equilibrium is in fact a general equilibrium; thus when  $f(r, c'_a) > c'_s$ , employers receive positive welfare in equilibrium. On the other hand, if availability is too low (i.e., if  $\tilde{q} < c'_s$ ), then employers would strictly prefer to exit immediately, and thus the partial

<sup>11</sup>Formally, such a partial equilibrium can be defined as a solution to  $m = \mathcal{M}(\mathcal{P}(m, 1))$ . We show in Appendix B that a unique solution  $\tilde{m}$  exists to this fixed point equation; we then let  $\tilde{q} = \mathcal{Q}(\tilde{m}, 1)$ .

equilibrium is not a general equilibrium. In this case, the market equilibrates just to the point that employers are indifferent between exiting and screening. In other words, when  $f(r, c'_a) < c'_s$ , the general equilibrium will involve employers receiving zero welfare, and mixing between exiting and screening. This is exactly the dichotomy captured by Theorem 3.

We now provide intuition for the monotonicity of  $f$ . The larger the value of  $r$ , the more likely it is that applicants are available. Thus, large values of  $r$  imply that employers become indifferent about whether or not to screen only if  $c'_s$  is also large. Meanwhile, when  $c'_a$  is larger, applicants send fewer applications and are therefore more likely to be available; hence,  $f$  is increasing in  $c'_a$ .

Due to the search and screening costs  $c'_a$  and  $c'_s$ , agents on both sides of the market remain unmatched in equilibrium.<sup>12</sup> One way to interpret the preceding definition of  $f$  is in terms of *which cost* (screening or application) constrains the equilibrium number of matches formed. If  $c'_s > f(r, c'_a)$ , then the market is *screening-limited*: a marginal decrease in  $c'_s$  will increase the number of matches formed in equilibrium, but a marginal change in  $c'_a$  will have no effect on the number of matches. This is because an increase in the number of applications sent (due to lower application costs) is exactly offset by a reduction in screening by employers, such that the availability  $q$  (and the total number of matches formed) remains the same. On the other hand, if  $c'_s < f(r, c'_a)$ , the market is *application-limited*: a marginal change in  $c'_s$  will have no effect on the number of matches formed, but a marginal decrease of  $c'_a$  will increase the equilibrium number of matches. This is because in this region employers receive positive welfare and never exit; reducing the screening cost thus cannot improve the likelihood an employer hires successfully, but increasing the number of applications can improve the likelihood of a match.

The fact that intervention can raise employer welfare from zero to  $\Pi_e^{\text{opt}}$  follows from Proposition 3, which states that whenever a limit  $\ell < m^*$  is enforced, applicants select  $m = \ell$  in the equilibrium of the resulting game. If  $\Pi_e^* = 0$ , it follows that the employer-optimal choice of  $m$  is less than  $m^*$ , and thus  $\Pi_e^{\text{opt}}$  can be attained via appropriate intervention.

### 6.3. Welfare Analysis: Applicants.

<sup>12</sup>Indeed, one of the leading motivations behind the modeling of search frictions in labor markets is to explain the simultaneous presence of vacancies and unemployed workers.

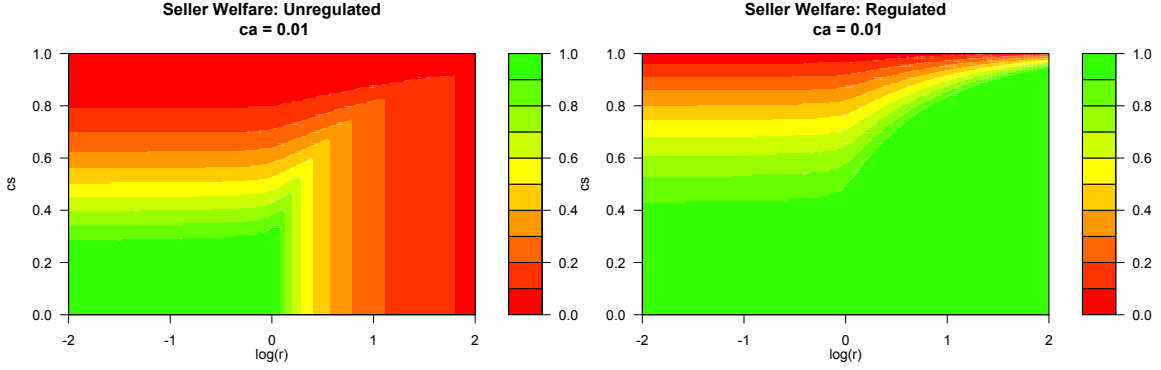


FIGURE 2. Normalized applicant welfare in equilibrium ( $\Pi_a^*/\Pi_a^{\text{opt}}$ , left) and with an applicant-optimal application limit ( $\sup_{\ell} \Pi_a^{\ell}/\Pi_a^{\text{opt}}$ , right). We fix the value  $c'_a = 0.01$ , and vary the values of  $\log r$  (x-axis) and  $c'_s$  (y-axis). Note that normalized equilibrium applicant welfare is low whenever  $c'_s$  is high or  $r$  is large. In the latter case, intervention can substantially raise applicant welfare. We formalize these statements in Theorem 4.

In this section, we state a theorem regarding equilibrium applicant welfare, and the welfare that is attainable through the intervention of enforcing an application limit. We preview our results using Figure 2, which displays  $\Pi_a^*/\Pi_a^{\text{opt}}$  as  $r$  and  $c'_s$  vary.

Figure 2 suggests that normalized applicant welfare in equilibrium is low whenever  $r$  is large. Of course, when  $r > 1$ , the fraction of applicants who eventually match is at most  $1/r$ . However, we normalize applicant welfare by  $\Pi_a^{\text{opt}}$ , which accounts for this fact. Applicants do badly *relative to*  $\Pi_a^{\text{opt}}$  when  $r$  is large because in this case, they compete fiercely for the limited number of open positions, and many of their applications are not even read. This is the more widely studied congestion effect in matching markets: applicants find themselves in a “tragedy of the commons” due to the negative externality their applications impose on each other.

Figure 2 also suggests that equilibrium normalized applicant welfare is low whenever  $c'_s$  is large. There are two reasons for this effect. First, high screening costs ensure that relatively few employers screen and hire applicants (a fact that is not accounted for in the computation of  $\Pi_a^{\text{opt}}$ ). Second, because many employers leave the marketplace without screening, the “effective” market imbalance (i.e.  $r/\alpha^*$ ) is large, so applicants again compete fiercely for limited spots. While enforcing an application limit cannot cause more applicants to match, it can eliminate the wasteful



competition among applicants. For this reason, an appropriate application limit can always boost applicant welfare.

We formalize these observations in Theorem 4, which roughly states that applicant welfare is low whenever  $c'_s$  is large or  $r$  is notably above one. Theorem 4 also states that whenever employers all screen in equilibrium, intervention can boost applicant welfare to  $\Pi_a^{\text{opt}}$ .

**Theorem 4.**

Applicant welfare in equilibrium:

- (1) Let  $\gamma$  be the unique solution to  $(1 - e^{-\gamma})/\gamma = c'_s$ . Then  $\Pi_a^* \leq 1 - (1 + \gamma)e^{-\gamma}$ .
- (2) If  $r > 1$ , then  $\Pi_a^* \leq \frac{1}{r} (1 - (r - 1) \ln(\frac{r}{r-1}))$  (Trivially,  $\Pi_a^* \leq 1$  for  $r \leq 1$ .)

Applicant welfare with an application limit:

- (1) There always exists  $\ell$  such that  $\Pi_a^\ell > \Pi_a^*$ .
- (2) If  $c'_s \leq f(r, c'_a)$ , there exists  $\ell$  such that  $\Pi_a^\ell = \Pi_a^{\text{opt}}$ .

To illustrate the magnitude of this effect, we consider an example where  $r = 1.4$  and application costs are near-zero. In this case, Theorem 4 implies that  $\Pi_a^* \leq \frac{1}{2r}$ , whereas an appropriate application limit  $\ell$  can ensure that applicants match at minimal costs, so that  $\Pi_a^\ell \approx \frac{1}{r}$ . Thus, in this case, intervention can roughly double applicant welfare (without notably changing the number of matches that form). If  $r = 1.9$ , the same reasoning implies that intervention can triple applicant welfare.

#### 6.4. Pareto Improvements.

The previous section separately addressed the employer and applicant welfare attainable through intervention. One natural concern is that while there may exist choices of  $\ell$  that result in high employer welfare and choices of  $\ell$  that result in high applicant welfare, these choices of  $\ell$  might not coincide. In this section, we address this concern.

Theorem 5 states that whenever the market is “screening-limited”, it is possible to choose a single application limit such that both employers and applicants are better off than they would be without any intervention. In such cases, we show that the tradeoff between optimizing for employers and for applicants cannot be too severe.

**Theorem 5.** If  $c'_s \geq f(r, c'_a)$ , then there exists  $\ell$  such that  $\Pi_e^\ell > \Pi_e^*$  and  $\Pi_a^\ell > \Pi_a^*$ .

One might wonder whether Pareto improvements are possible when the market is “application limited” (i.e.  $c'_s < f(r, c'_a)$ ). The answer turns out to be, “not always.” Indeed, if either  $r$  or  $c'_a$  is large enough, then availability never becomes a pressing

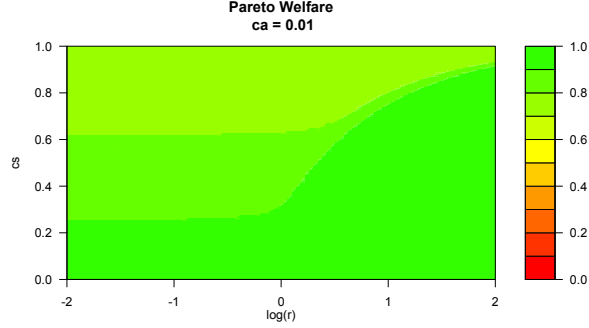


FIGURE 3. For  $c'_a = 0.01$  and varying values of  $c'_s$  and  $r$ , this figure shows the largest fraction  $\delta$  such that there exists a single limit  $\ell'$  for which employers earn an expected surplus of  $\delta \sup_{\ell} \Pi_e^{\ell}$  and applicants (simultaneously) earn an expected surplus of  $\delta \sup_{\ell} \Pi_a^{\ell}$ . We conjecture that for all parameter values,  $\delta \geq 3/4$ .

concern to employers, and *any* binding application limit lowers employer welfare.<sup>13</sup> While one might conclude that enforcing an application limit is undesirable in these cases, Figure 3 shows numerically that an application limit can often substantially improve applicant welfare at little cost to employers.

In this figure, we plot the largest fraction  $\delta$  such that there exists a single limit  $\ell'$  where employers earn an expected surplus of  $\delta \sup_{\ell} \Pi_e^{\ell}$  and applicants (simultaneously) earn an expected surplus of  $\delta \sup_{\ell} \Pi_a^{\ell}$ . Observe that  $\delta$  is never below  $3/4$ , and is often much higher. Although this plot shows only the case  $c'_a = 0.01$ , this fact holds generally, suggesting that the tradeoff between optimizing for employers and applicants is never too severe.<sup>14</sup>

**6.5. An Alternate Intervention.** For reasons not modeled in this paper, placing a limit on the number of applications sent by each applicant may be impractical or undesirable. For example, this limit may be unenforcable if applicants can easily create multiple anonymous accounts. Additionally, a “one size fits all” limit may be too coarse if applicants vary in quality or in the number of jobs that they wish to hold. These thoughts motivate a second type of intervention that reduces the number

<sup>13</sup>This holds, for example, if  $c'_s < \max \{1 - 1/r, c'_a\}$ .

<sup>14</sup>Intuitively, the wishes of applicants and employers are most at odds when  $c_a$  is small and  $c_s$  is large, or vice versa. Although we have no formal proof that this intuition is correct, numerical results show that the worst cases are when  $c'_a \rightarrow 0$  and  $c'_s \rightarrow 1$ , or when  $c'_a \rightarrow 1$  and  $c'_s \rightarrow 0$ . We have proven that in each of these cases,  $\delta \geq 3/4$ .

of applications sent: raising the application cost.<sup>15</sup> This intervention would dispense with any incentive to create multiple accounts, and would allow for applicants to tailor their strategy to their own characteristics.

From the perspective of the employer, limiting applications and raising application costs have similar effects. To applicants, however, they look different. Although both interventions reduce the amount of competition for each opening, raising the application cost also directly harms applicants. A priori, it is not obvious which effect dominates. Proposition 6 shows that within our model, raising application costs can only hurt applicants.<sup>16</sup> Additionally, we formalize the idea that the two interventions are equivalent from the perspective of employers.

**Proposition 6.** *Fix values of  $r, c_s, \beta$  satisfying  $c_s < \beta$ , and let  $\Pi_e^*(c)$  and  $\Pi_a^*(c)$  be employer and applicant welfare in the MFE when  $c'_a = c$ . Then*

- *For each  $\ell \in \mathbb{R}_+$ , there exists a unique  $\tilde{c}'_a \geq c'_a$  such that  $\Pi_e^\ell = \Pi_e^*(\tilde{c}'_a)$ .*
- *For each  $\tilde{c}'_a > c'_a$ , there exists a unique  $\ell \in \mathbb{R}_+$  such that  $\Pi_e^\ell = \Pi_e^*(\tilde{c}'_a)$ .*
- *$\Pi_a^*(c)$  is a decreasing function.*

Though we focus primarily on the intervention of imposing an application limit, Proposition 6 establishes that in cases where the intent of the intervention is to help employers, raising application costs is a viable alternative approach. In practice, the relative merits of these two forms of intervention depend on a number of considerations not modeled in this paper, and choice between them should be determined by the operator's objectives and capabilities.

## 7. DISCUSSION

In this section we interpret our analytical results, and also discuss a range of observations, extensions, and open directions related to our analysis. The following are our main insights.

- (1) *No intervention: Employer welfare can drop to zero.* One of our main conclusions is that when employers are unable to observe availability, congestion can

<sup>15</sup>This is effectively a “tax” on the externality that applicants impose on employers. This tax could take the form of an explicit monetary payment to the platform (presumably benefiting the platform, though we do not model the operator as a strategic agent), or could simply be an additional form or questionnaire that the applicant must complete for each employer contacted.

<sup>16</sup>We assume that the costs paid by applicants cannot be redistributed. This is reasonable if the increased cost is not a monetary transfer, but instead an additional barrier to application (such as additional questions that each applicant must answer). Even when the additional cost is monetary, redistribution may be logistically challenging, create incentives for individuals to create multiple accounts, and may be undesirable from the point of view of the platform operator.

drive employer welfare to zero. Theorem 3 reveals that unless there is a large surplus of applicants, this occurs whenever application costs are sufficiently small.

- (2) *No intervention: The applicants' see a "tragedy of the commons"*. Theorem 4 reveals that applicant welfare is bounded away from  $\min\{1, 1/r\}$ . This holds because applicants do not internalize the competition externality. This externality is particularly severe when either  $r$  or  $c'_s$  is large. Whenever equilibrium employer welfare is zero, any benefits from lower application costs are entirely offset by increasing competition.
- (3) *The value of intervention: Constrained efficiency*. Whenever employer welfare is zero in the market without intervention, an appropriate application limit can raise employer welfare to the constrained efficient benchmark (cf. Theorem 3). Theorem 4 reveals that an appropriately chosen application limit can always help applicants, and in some regimes can raise applicant welfare to the constrained efficient benchmark.
- (4) *Pareto improvements*. Theorem 5 shows that whenever employer welfare is zero in the market without intervention, there exists a choice of application limit that simultaneously improves welfare for both sides of the market. We show numerically that a limit can be chosen such that both sides get at least  $3/4$  of their constrained efficient benchmarks. In other words, tradeoffs between optimizing for employers and applicants are minimal.

We conclude the paper by discussing the implications of some of our modeling assumptions, as well as some open directions for future work.

**Observability of availability.** Our model assumes that availability is unobservable, and explores the consequences of this fact. Given the significant costs described above, one might wonder whether platforms could reveal the availability of their participants. For example, within the context of our model, information about when each applicant applied would provide employers with a signal of availability. Better yet, the platform could display matches as they form, so that employers screen only available applicants.

One challenge with this approach in practice is that properly detecting availability may be a difficult problem. For example, Horton (2015) shows that workers on oDesk vary widely in the amount of work that they accept, so those with many ongoing projects may still be available. Meanwhile, an Airbnb listing may be unavailable

even if is not already booked (for example, if the host is away on vacation, or simply too tired to clean for a guest).

Rather than trying to infer availability, the platform could solicit this information from its users. While some sites (including oDesk and Airbnb) do just that, this approach comes with its own challenges. To encourage truthful reporting, there must be some form of punishment for those who turn down offers after declaring themselves available. However, in markets with heterogeneous preferences, punishing users who decline invitations may prove unpalatable.

Finally, even if availability is inferred by the platform (or accurately reported by users), this information may arrive too late to influence the screening process. Although our model represents screening as happening instantaneously, this is largely a technical device. In practice, screening takes time, and there may be several employers screening the same applicant simultaneously. In such a situation, the platform can't inform other employers that an applicant is unavailable, because a match has not yet been consummated.

**Ex-Ante Heterogeneity.** In our model, we assume that each applicant is compatible with a given employer with probability  $\beta$ , i.i.d. across all employer-applicant pairs. In fact, we can relax this; all of our results hold if we only assume that *conditional on applicant  $a$  having applied to an employer  $e$ ,  $e$  finds  $a$  acceptable with probability  $\beta$ .*

In practice, we expect that employers do not look identical to applicants, and that applicants (or the platform) can direct applications to the most “promising” openings. In this case, an application limit encourages applicants to apply to the openings for which they are the best fit. Under suitable assumptions, this could be captured in our model by allowing  $\beta = \beta(m)$  to decrease as  $m$  increases. In such a setting, smaller  $m$  would lead to a larger  $\beta$ , leading to an additional improvement in welfare. It is somewhat surprising that limiting visibility helps even when the platform has no agent-specific or match-specific information about compatibility, and applications are sent randomly by uninformed agents: this only strengthens our belief in the potential benefits of limiting applications.

**Value of Improved Search.** As mentioned above, all of our results hold if we only assume that  $\beta$  is the probability that applicant  $a$  is acceptable to employer  $e$ , *conditioned on the event that  $a$  applied to  $e$ .* Thus, the parameter  $\beta$  is not necessarily exogenous to the platform: instead, it can be interpreted as the quality of the platform's recommendation and/or search algorithms. In other words, one way to model

the effect of improved recommendation algorithms is that they simply increase the value of  $\beta$ . In our paper we have not focused on the role of compatibility, but given the central role of search and recommendations in online matching platforms, it would be worthwhile to compare the potential welfare benefits of improved compatibility with the benefits of application limits.

**Dynamic vs. static models.** Ours is not the first work to conclude that restricting visibility may prove beneficial, but our dynamic model provides quantitative insights that differ substantially from those of the static models that preceded it. For concreteness, we consider a “one-shot” version of our dynamic model: applicants choose  $m$ , and apply to (in expectation)  $m$  employers; employers screen applicants, and make at most one offer; and applicants accept an offer at random among those received (this is similar, for example, to the model studied by Albrecht et al. (2006)).

A mean field analysis of this model reveals that if  $c'_s < e^{-1/r}$ , then employer welfare is increasing in  $m$  (see Appendix A). This suggests that minimizing search frictions (the costs  $c_a, c_s$ ) is a reasonable proxy for maximizing welfare. Our results contrast sharply with this conclusion: if  $r \leq 1$ , then for any screening cost  $c'_s$ , employer welfare is zero if the number of applications is sufficiently large.

This disagreement arises because static models, while admitting the possibility that employer offers will be rejected, severely understate the likelihood of this event. In the static model, employers send at most one offer. Because employers do not coordinate their offers, unless there are far more employers than applicants, a large number of applicants will go unmatched. These applicants would accept any offer that they receive, so from the perspective of each employer, the probability that their offer is accepted cannot be too low.

In practice, many labor markets (and more generally, matching markets) operate asynchronously, and it seems reasonable to expect that an employer whose initial offer is turned down will make an offer to a second candidate. While the ability to continue searching is good for this employer, it imposes a negative externality on future employers who may choose to screen the same applicants. Our model, because it is dynamic, captures this effect.<sup>17</sup>

**Alternative Application Model.** Applicant behavior in our model has two features that can be relaxed at the cost of significant additional technical complexity.

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<sup>17</sup>In principle, a similar effect could be captured by a static model such as that used by Kircher (2009). He assumes that after applicants send applications, a stable match is formed (thereby implicitly allowing employers to make multiple offers). One key difference between his model and ours is that he assumes that this match forms costlessly (viewed in our setting, he sets  $c_s = 0$ ).

First, in our model applicants cannot directly choose the number of applications that they send; rather, they apply to each employer independently with probability  $m/n$ . Allowing applicants to select the number of employers to whom they apply introduces an additional source of correlation that further complicates the proofs of Propositions 4 and 5. The mean-field analysis for this alternative model, however, remains quite tractable: in our model the number of applications sent by each applicant is  $\text{Poisson}(m)$ , whereas in the alternate model it is exactly  $m$ . As a result, the probability an applicant receives an offer becomes  $1 - (1 - p)^m$ , instead of  $1 - e^{-mp}$ . As we focus on settings in which application costs are small (and thus  $m$  is large and  $p$  is small), this does not substantively change our results.

Second, note that in our model applicants cannot condition on the number of other applicants that apply to a given job. Again, this simplifies our technical arguments. If applicants do see the number of other applicants on a given job, then in the mean field analysis each employer would receive exactly  $rm$  applications (while in our model, the number of applications received by an employer follows a Poisson distribution with mean  $rm$ ). The probability that an employer matches in the alternate model becomes  $1 - (1 - \beta q)^{rm}$  (as opposed to  $1 - e^{-rm\beta q}$  in our model). Again, this does not substantively change our results.

**Symmetric behavior.** In our model, the applicants make the applications and employers do the screening. An important direction for future work is to instead consider a symmetric model where any agent on *either* side of the market can initially submit applications as well as post her availability, and later return to screen applicants if she does not receive any offers. While we believe our current model is appropriate for a range of markets where asymmetric screening and application behavior is inherent to the platform, a symmetric model may be more appropriate for settings (such as dating websites) where no such constraint exists *a priori*.

**Transfers.** Our model does not capture the possibility of monetary transfers, e.g., wages. Numerous papers (for example Moen (1997); Acemoglu and Shimer (1999); Kircher (2009)) have observed that endogenously determined wages can guide the market towards efficiency. These results do not extend directly to our setting, but analyzing a model in which workers include a wage in their application could prove to be an illuminating direction for future work.

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## APPENDIX A. STATIC MODEL

Consider a “one-shot” version of our dynamic model: applicants choose  $m$ , and apply to each employer independently with probability  $m/n$ ; employers screen applicants, and make at most one offer; and applicants accept an offer at random among those received.

A mean field analysis of this model can be carried out as before. We make mean field assumptions very similar to those in Section 4: Assumptions 2 and 3 are unchanged, and Assumption 1 is slightly modified to “Each applicant in an employer’s applicant set will accept an offer from the employer with probability  $q$ , independently across applicants in the applicant set.” Again assume that each employer screens with probability  $\alpha$ . The consistency condition for  $p$  is slightly modified to

$$p = \alpha\beta g(rm\beta),$$

where  $g(x) = (1 - e^{-x})/x$  and the argument of  $g(\cdot)$  now does not contain the term  $q$  since employers do not screen for availability (the absence of  $q$  from this condition causes the static model to behave very differently from our dynamic model, as we see below). The consistency condition for  $q$  remains

$$q = g(mp),$$

leading to

$$q = g(\alpha(1 - e^{-rm\beta})/r).$$

Employer welfare, when  $\alpha = 1$  is

$$\begin{aligned} (1 - e^{-rm\beta})(q - c'_s) &= (1 - e^{-rm\beta})(g((1 - e^{-rm\beta})/r) - c'_s) = r(1 - e^{-(1 - e^{-rm\beta})/r}) - c'_s(1 - e^{-rm\beta}) \\ &= r(1 - e^{-y/r}) - c'_s y, \end{aligned}$$

where  $y = 1 - e^{-rm\beta} < 1$ . Note that  $y$  is monotone increasing in  $m$ . The derivative of the employer welfare with respect to  $y$  is  $e^{-y/r} - c'_s > e^{-1/r} - c'_s$ . Thus, under the relatively mild condition  $c'_s < e^{-1/r}$ , the employer welfare is monotone increasing with respect to  $y$ , and hence with respect to  $m$ , for  $\alpha = 1$ . (For values of  $m$  such that the employer welfare is negative with  $\alpha = 1$ , the equilibrium value of  $\alpha$  will be less than 1 such that the resulting employer welfare is 0.) This is in contrast with our dynamic

model, where whenever  $r \leq 1$ , employer welfare is unimodal in  $m$ , and becomes zero if  $m$  is sufficiently large. Thus, the static model does not capture that reductions in application costs can severely reduce the marketplace's efficiency. See Section 7 for a further discussion of this issue.

## APPENDIX B. PROOFS: SECTION 4

**B.1. Section 4.2: Mean Field Steady States.** In this appendix we study mean field steady states (MFSS), i.e., solutions  $(p, q)$  to the equations (4) and (5). We prove Proposition 2, which states that for any  $r, \beta, m, \alpha$ , there exists a unique MFSS. We also show in Lemma 3 that the MFSS values  $p$  and  $q$  are monotonic in  $m$  and  $\alpha$ . This will be useful in proving results about mean field equilibria in Section B.2.

For notational convenience, we define the function  $g$  as follows:

$$(11) \quad g(0) = 1, \quad g(x) = \frac{1 - e^{-x}}{x} \text{ for } x > 0.$$

**Lemma 1.** *The function  $g : [0, \infty) \rightarrow (0, 1]$  defined by (11) is continuous and strictly decreasing. Further:*

- (1)  $g'(x) \geq -g(x)/x$  and  $g''(x)/g'(x) > -1$  for  $x > 0$ ;
- (2)  $\lim_{x \rightarrow 0} g'(x) = -1/2$ , and  $\lim_{x \rightarrow 0} g''(x) = 1/3$ ;
- (3)  $e^{-g^{-1}(c)} \leq c^2$  for  $0 < c < 1$ .

*Proof of Lemma 1.* Note first that  $g$  is continuous at zero. Differentiating, for  $x > 0$  we have

$$(12) \quad g'(x) = \frac{(1+x)e^{-x} - 1}{x^2} = \frac{e^{-x} - g(x)}{x}.$$

Monotonicity of  $g$  follows from applying the inequality  $1+x < e^x$  for  $x > 0$ , which implies (on rearranging terms) that  $e^{-x} < g(x)$ . Having established that the expressions in (12) are negative, it follows that  $g'(x) > -g(x)/x$  for  $x > 0$ .

We now prove that  $g''(x)/g'(x) > -1$ , which is equivalent to  $g'(x) + g''(x) < 0$  (the inequality has reversed because  $g' < 0$ ). Basic algebra reveals that

$$(13) \quad -x(g'(x) + g''(x)) = (g(x) + 2g'(x))$$

$$(14) \quad = \frac{1}{x^2} ((2+x)e^{-x} + x - 2)$$

Thus,  $g'(x) + g''(x) < 0$  if and only if  $(2+x)e^{-x} + x - 2 > 0$ . Differentiating this expression, we see that

$$\frac{d}{dx} ((2+x)e^{-x} + x - 2) = 1 - e^{-x}(1+x) \geq 0,$$

so the expression is minimized at  $x = 0$ , when it takes the value zero.

If we apply L'Hospital's rule to (12), we obtain

$$\lim_{x \rightarrow 0} g'(x) = \lim_{x \rightarrow 0} \frac{e^{-x} - g(x)}{x} = \lim_{x \rightarrow 0} -e^{-x} - g'(x).$$

Rearranging, we see that  $2 \lim_{x \rightarrow 0} g'(x) = -1$ , implying that  $\lim_{x \rightarrow 0} g'(x) = -1/2$ .

Rearranging (13) reveals that

$$(15) \quad \lim_{x \rightarrow 0} g''(x) = \lim_{x \rightarrow 0} -g'(x) - \frac{g(x) + 2g'(x)}{x} = \frac{1}{2} - \lim_{x \rightarrow 0} (g'(x) + 2g''(x)),$$

where we have used the fact that  $\lim_{x \rightarrow 0} g'(x) = -1/2$  and applied L'Hospital's rule.

Rearranging (15), we see that  $3 \lim_{x \rightarrow 0} g''(x) = 1$ .

Finally, by rearranging we observe that  $e^{-g^{-1}(c)} \leq c^2$  if and only if  $1 - c^2 + 2c \log c \geq 0$ . It is straightforward to check that the expression  $1 - c^2 + 2c \log c$  is decreasing in  $c$ , and is equal to zero at  $c = 1$ .  $\square$

*Proof of Proposition 2.* Recall from (4)-(5) that given  $r, m, \beta, \alpha$ , we define a MFSS as any solution  $(p, q)$  to

$$(16) \quad p = \alpha\beta g(rm\beta q), \quad q = g(mp).$$

Note that the preceding system trivially has a single solution when  $m = 0$ : namely,  $p = \alpha\beta$  and  $q = 1$ . Therefore we focus on the case where  $m > 0$ . We proceed by showing that the preceding system has a unique solution.

For this purpose it is useful to rewrite (16) as providing two functions that yield  $p$  in terms of  $q$ . In particular, for  $m > 0$ , we note that  $(p, q)$  is an MFSS if and only if  $p = p_1(q) = p_2(q)$ , where  $p_1, p_2 : [g(m), 1] \rightarrow [0, 1]$  are defined by

$$(17) \quad p_1(x; m, \alpha) = \alpha\beta g(rm\beta x), \quad p_2(x; m) = g^{-1}(x)/m.$$

Here the notation  $f(x; y)$  indicates that the function  $f$  is parameterized by  $y$ . The chosen lower bound of  $g(m)$  for the domain arises from the fact that any mean-field steady-state  $(p, q)$  corresponding to  $(m, \alpha)$  must satisfy  $q = g(mp) \geq g(m)$  (since  $g$  is strictly decreasing).

We show in Lemma 2 that for  $m > 0$  and any  $r, \beta, \alpha$ , there is a unique point where  $p_1$  and  $p_2$  intersect; this point is the unique MFSS.  $\square$

The following lemma, used in the preceding proof, also establishes some useful properties of MFSS.

**Lemma 2.** *Fix  $r, \beta$ , and  $\alpha$ , and  $m > 0$ . Then there exists a unique pair  $(p, q) \in [0, \alpha\beta] \times [g(m), 1)$  such that  $p = p_1(q; m, \alpha) = p_2(q; m)$ .*

*Furthermore: (1) the functions  $p_1$  and  $p_2$  defined in (17) are monotonically decreasing in  $q$ ; and (2) for  $q' < q$ , we have  $p < p_1(q'; m, \alpha) < p_2(q'; m)$ ; and for  $q' > q$ , we have  $p > p_1(q'; m, \alpha) > p_2(q'; m)$ .*

*Proof.* For the duration of the proof we suppress the dependence of  $p_1$  and  $p_2$  on  $\alpha$  and  $m$ .

Note that on the domain  $[g(m), 1)$ ,  $p_1$  begins “below”  $p_2$  and finishes “above” it, i.e. (because  $g(\cdot) \leq 1$  by Lemma 1), we have  $p_1(g(m)) \leq \alpha\beta < 1 = p_2(g(m))$  and  $p_1(1) > 0 = p_2(1)$ . Since  $p_1$  and  $p_2$  are continuous, this implies that a mean-field steady-state exists. This is illustrated in Figure 4.

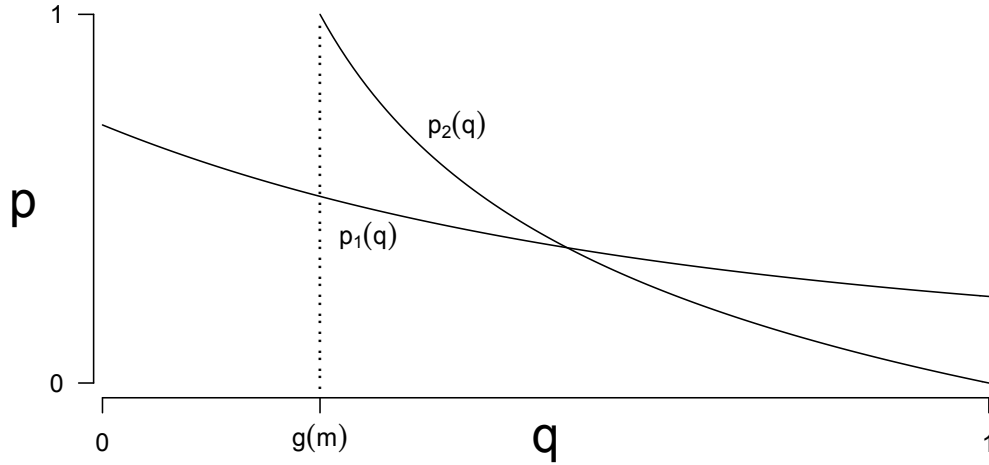


FIGURE 4. A depiction of  $p_1(q)$  and  $p_2(q)$ ; we prove in Lemma 2 that they have a unique intersection point.

Once we know that the intersection of  $p_1$  and  $p_2$  is unique, the final claim of the lemma follows immediately. Thus, all that remains to show is that  $p_1$  and  $p_2$  have a unique intersection point.

Note that because  $g$  is decreasing by Lemma 1, so are both  $p_1$  and  $p_2$ . To show uniqueness, we will show that whenever  $p_1$  and  $p_2$  intersect, the curve  $p_2$  is decreasing more “steeply,” i.e.  $\frac{\partial p_1}{\partial x} > \frac{\partial p_2}{\partial x}$ . Equivalently, we wish to show that whenever  $p_1(q) = p_2(q) = p$ , we have  $\frac{\partial p_1}{\partial x}(q) / \frac{\partial p_2}{\partial x}(q) < 1$  (the inequality reverses because both partials

are negative). But

$$(18) \quad \frac{\partial p_1}{\partial x}(q) \Big/ \frac{\partial p_2}{\partial x}(q) = (\alpha \beta r m \beta g'(r m \beta q)) (m g'(m p))$$

$$(19) \quad < \left( \frac{\alpha \beta g(r m \beta q)}{q} \right) \left( \frac{g(m p)}{p} \right).$$

$$(20) \quad = \left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = 1.$$

The first line follows from implicit differentiation of  $p_2$ ; the second from the inequality  $|g'(x)| < g(x)/x$  proven in Lemma 1; and the third from the fact that  $(p, q)$  is a mean-field steady-state, i.e.  $p_1(q) = p_2(q) = p$ .  $\square$

Having established the existence of a unique mean-field steady-state, it will be useful when proving later results to understand how  $\mathcal{P}(m, \alpha)$  and  $\mathcal{Q}(m, \alpha)$  vary with  $m$  and  $\alpha$ . We require the following identity, which can be easily derived by rearranging (4)-(5):

$$(21) \quad \alpha(1 - e^{-r m \beta q}) = r(1 - e^{-m p})$$

There is a straightforward interpretation of this identity. Each employer matches if and only if they screen (which they do with probability  $\alpha$ ), and they have a qualified available applicant (which occurs with probability  $1 - e^{-r m \beta q}$ , because each applicant is available with probability  $q$  and qualified with probability  $\beta$ ). On the other hand, each applicant matches if and only if one of her applications receives an offer (which occurs with probability  $1 - e^{-m p}$ ). Because the number of employers and applicants who match must be equal, we conclude that if  $p$  and  $q$  are the MFSS consistent with strategies  $m$  and  $\alpha$ , then (21) must hold.

The following lemma provides some monotonicity properties of  $\mathcal{P}(m, \alpha)$  and  $\mathcal{Q}(m, \alpha)$ .

**Lemma 3.** *For  $(m, \alpha) \in (0, \infty) \times (0, 1]$ , the function  $\mathcal{P}(m, \alpha)$  is strictly decreasing in  $m$  and strictly increasing in  $\alpha$ , and  $\mathcal{Q}(m, \alpha)$  is strictly decreasing in  $m$  and in  $\alpha$ . For  $\alpha > 0$ ,*

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathcal{P}(m, \alpha) &= \alpha \beta f_2(\alpha/r), \quad \mathcal{P}(0, \alpha) = \alpha \beta. \\ \lim_{m \rightarrow \infty} \mathcal{Q}(m, \alpha) &= f_2(r/\alpha), \quad \mathcal{Q}(0, \alpha) = 1, \end{aligned}$$

where  $f_2$  is defined by

$$(22) \quad f_2(x) = \begin{cases} 0 & : x \leq 1 \\ g(\log \frac{x}{x-1}) & : x > 1. \end{cases}$$

*Proof.* Define the functions  $p_1$  and  $p_2$  as in (17). Recall from Lemma 2 that  $\mathcal{P}(m, \alpha)$  and  $\mathcal{Q}(m, \alpha)$  are the unique solution  $(p, q)$  to  $p_1(q; m, \alpha) = p_2(q; m) = p$ . For  $x < \mathcal{Q}(m, \alpha)$ , we have  $p_1(x) < p_2(x)$  and for  $x > \mathcal{Q}(m, \alpha)$ , we have  $p_1(x) > p_2(x)$ .

We first prove the statements about monotonicity in  $\alpha$ . Suppose that  $\alpha < \alpha'$ . Then  $p_2(\mathcal{Q}(m, \alpha); m) = p_1(\mathcal{Q}(m, \alpha); m, \alpha) < p_1(\mathcal{Q}(m, \alpha); m, \alpha')$ . It follows immediately from Lemma 2 that  $\mathcal{Q}(m, \alpha) > \mathcal{Q}(m, \alpha')$ . Furthermore, since  $\mathcal{P}(m, \alpha) = p_2(\mathcal{Q}(m, \alpha); m)$  and  $\mathcal{P}(m, \alpha') = p_2(\mathcal{Q}(m, \alpha'); m)$  and  $p_2$  is monotonically decreasing, this implies that  $\mathcal{P}(m, \alpha) > \mathcal{P}(m, \alpha')$ .

We now prove monotonicity with respect to  $m$ . It follows from (21) that for fixed  $r$  and  $\alpha > 0$ ,

$$(23) \quad m' \mathcal{Q}(m', \alpha) < m \mathcal{Q}(m, \alpha) \Leftrightarrow m' \mathcal{P}(m', \alpha) < m \mathcal{P}(m, \alpha).$$

Let  $m' > m$ , and suppose that  $m' \mathcal{P}(m', \alpha) < m \mathcal{P}(m, \alpha)$ . It follows that

$$\mathcal{Q}(m', \alpha) = g(m' \mathcal{P}(m', \alpha)) > g(m \mathcal{P}(m, \alpha)) = \mathcal{Q}(m, \alpha),$$

which contradicts (23). Therefore, our supposition was incorrect; it must be that both  $m' \mathcal{P}(m', \alpha) > m \mathcal{P}(m, \alpha)$  and  $m' \mathcal{Q}(m', \alpha) > m \mathcal{Q}(m, \alpha)$ . By definition (see (4) and (5)), we have  $\mathcal{Q}(m, \alpha) = g(m \mathcal{P}(m, \alpha))$  and  $\mathcal{P}(m, \alpha) = \alpha \beta g(r \beta m \mathcal{Q}(m, \alpha))$ . Applying the fact that  $g$  is decreasing (see Lemma 1), we conclude that  $\mathcal{Q}(m', \alpha) < \mathcal{Q}(m, \alpha)$  and  $\mathcal{P}(m', \alpha) < \mathcal{P}(m, \alpha)$ , as claimed.

Having established monotonicity of  $\mathcal{P}$  and  $\mathcal{Q}$ , we move on to evaluating their limits as  $m \rightarrow \infty$ . When  $r < \alpha$ , (21) implies that  $1 - e^{r \beta m \mathcal{Q}(m, \alpha)} \leq r/\alpha$ , so  $m \mathcal{Q}(m, \alpha)$  is bounded above. This implies both that  $\mathcal{Q}(m, \alpha) \rightarrow 0$  as  $m \rightarrow \infty$  and that  $\mathcal{P}(m, \alpha) = \alpha \beta g(r \beta m \mathcal{Q}(m, \alpha))$  is bounded away from zero. It follows from (21) that

$$\lim_{m \rightarrow \infty} 1 - e^{-r \beta m \mathcal{Q}(m, \alpha)} = \lim_{m \rightarrow \infty} \frac{r}{\alpha} (1 - e^{-m \mathcal{P}(m, \alpha)}) = \frac{r}{\alpha},$$

so  $r \beta m \mathcal{Q}(m, \alpha) \rightarrow -\log(1 - \frac{r}{\alpha}) = \log\left(\frac{\alpha/r}{\alpha/r - 1}\right)$  and  $\mathcal{P}(m, \alpha) = \alpha \beta g(r \beta m \mathcal{Q}(m, \alpha)) \rightarrow \alpha \beta f_2(\alpha/r)$ .

Analogously, when  $r > \alpha$ , (21) implies that  $1 - e^{-m \mathcal{P}(m, \alpha)} \leq \alpha/r$ . This means that  $m \mathcal{P}(m, \alpha)$  is bounded above, so  $\mathcal{P}(m, \alpha) \rightarrow 0$  and  $\mathcal{Q}(m, \alpha) = g(m \mathcal{P}(m, \alpha))$  is bounded away from zero. Applying (21) we get that

$$\lim_{m \rightarrow \infty} 1 - e^{-m \mathcal{P}(m, \alpha)} = \lim_{m \rightarrow \infty} \frac{\alpha}{r} (1 - e^{-r m \beta \mathcal{Q}(m, \alpha)}) = \frac{\alpha}{r},$$

so  $m \mathcal{P}(m, \alpha) \rightarrow \log\left(\frac{r/\alpha}{r/\alpha - 1}\right)$  and  $\mathcal{Q}(m, \alpha) = g(m \mathcal{P}(m, \alpha)) \rightarrow f_2(r/\alpha)$ .

Finally, when  $r = \alpha$ , (21) implies that  $\alpha\beta\mathcal{Q}(m, \alpha) = \mathcal{P}(m, \alpha) = \alpha\beta g(r\beta m\mathcal{Q}(m, \alpha))$ . Since  $g$  is strictly decreasing,  $g(r\beta m\mathcal{Q}(m, \alpha)) \leq g(r\beta m)$ , and  $g(r\beta m) \rightarrow 0$  as  $m \rightarrow \infty$ . We conclude that  $g(r\beta m\mathcal{Q}(m, \alpha)) \rightarrow 0$  uniformly in  $q$  as  $m \rightarrow \infty$ ; since  $\mathcal{Q}(m, \alpha)$  is the solution to  $q = g(r\beta m\mathcal{Q}(m, \alpha))$ , it follows that  $\mathcal{Q}(m, \alpha) \rightarrow 0$  as well as  $m \rightarrow \infty$ . Since  $\mathcal{P}(m, \alpha) = \alpha\beta\mathcal{Q}(m, \alpha)$ , this implies  $\mathcal{P}(m, \alpha) \rightarrow 0 = f_2(1)$ , completing the proof.  $\square$

**B.2. Section 4.3: Mean Field Equilibrium.** We now prove Theorem 1, which states that mean field equilibria exist and are unique.

*Proof of Theorem 1.* We prove the theorem in two steps. First, we fix the employer strategy to be  $\phi^\alpha$ , and allow applicants to respond optimally. We show in Proposition 4 that for any  $\alpha \in [0, 1]$ , there exists a unique strategy  $m$  such that

$$(24) \quad m = \mathcal{M}(\mathcal{P}(m, \alpha)).$$

Let  $m_\alpha$  denote the unique choice of  $m$  satisfying (24).

Second, we endogenize the employer's choice of  $\alpha$ ; Lemma 7 shows that there is exactly one value of  $\alpha$  such that  $\alpha \in \mathcal{A}(\mathcal{Q}(m_\alpha, \alpha))$ , i.e. such that  $(m_\alpha, \alpha)$  is a mean field equilibrium.  $\square$

We start with the following lemma, regarding “partial” equilibrium among the applicants given a fixed value of  $\alpha$ .

**Lemma 4.** *Given  $\alpha \in [0, 1]$ , there exists a unique value  $m_\alpha$  satisfying:*

$$(25) \quad m_\alpha = \mathcal{M}(\mathcal{P}(m_\alpha, \alpha)).$$

*Furthermore: (1)  $m_\alpha = 0$  if and only if  $\alpha \leq c'_a$ ; and (2)  $\mathcal{P}(m_\alpha, \alpha)$  is strictly increasing in  $\alpha$ .*

*Proof.* Refer to Figure 5. Any solution to (24) is equivalent to finding a solution to the following system of equations:

$$p = \mathcal{P}(m, \alpha); \quad m = \mathcal{M}(p).$$

As discussed in Section 4.2, the value  $\mathcal{P}(m, \alpha)$  is the (unique) solution to

$$(26) \quad \mathcal{P}(m, \alpha) = \alpha\beta g(r\beta m\mathcal{P}(m, \alpha)) = \alpha\beta g(r\beta(1 - e^{-m\mathcal{P}(m, \alpha)})/\mathcal{P}(m, \alpha)).$$

We now incorporate the fact that  $m$  should be optimally chosen by applicants. Because  $g(x) \leq 1$  (see Lemma 1), we have  $\mathcal{P}(m, \alpha) \leq \alpha\beta$ . Thus, if  $\alpha \leq c'_a = c_a/\beta$ , then  $\mathcal{P}(m, \alpha) \leq c_a$  for all  $m$ , and thus  $\mathcal{M}(\mathcal{P}(m, \alpha)) = 0$  for all  $m$ , so  $m = 0$  is the unique solution to (24).



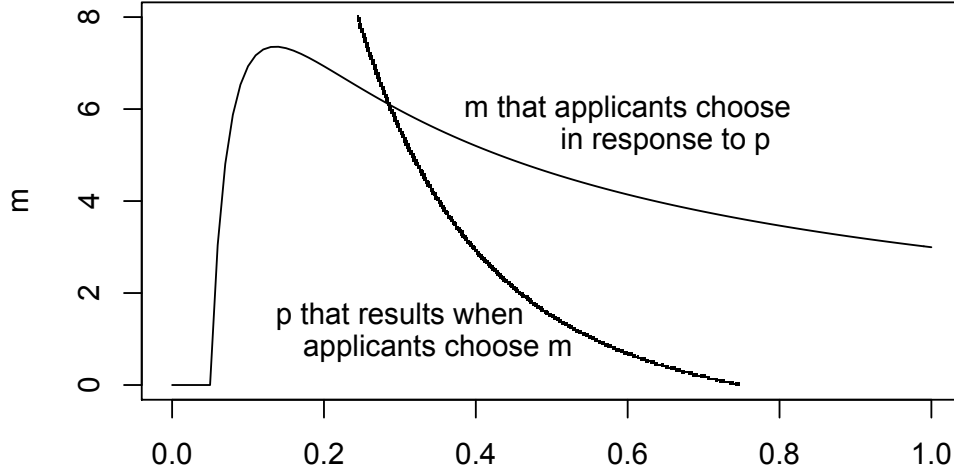


FIGURE 5. A visualization of Proposition 4. The value  $\mathcal{P}(m, \alpha)$  that results when applicants choose  $m$  is the solution to (26). Proposition 4 states that this line intersects with the applicant best response function  $\mathcal{M}(p)$  at a unique point.

Thus, we can substitute (24) into (2) to obtain:

$$(27) \quad p = \alpha\beta g(r\beta(1 - \beta c'_a/p)/p).$$

It suffices to show that (27) has a unique solution  $p \in (0, \alpha\beta)$ , as the desired (unique)  $m_\alpha$  is then  $\mathcal{M}(p)$ .

To see this, multiply each side by  $r/(\alpha p)$  and substitute  $x = \beta c'_a/p$  to get

$$(28) \quad \frac{r}{\alpha} = \frac{r}{c'_a} x g(rx(1-x)/c'_a) = \frac{1 - e^{-(r/c'_a)x(1-x)}}{(1-x)}.$$

By Lemma 5 (see below), the right side of (28) is strictly increasing in  $x$ , takes the value zero at  $x = 0$ , and approaches  $r/c'_a$  as  $x \rightarrow 1$ . Because  $\alpha > c'_a$ , this implies that there is a unique solution  $x$  to (28) and therefore a unique solution  $p$  to (27). Furthermore, evaluating the right side of (28) at  $x = c'_a/\alpha$  indicates that the solution  $x$  to (28) is greater than  $c'_a/\alpha$  and thus the solution  $p = \beta c'_a/x$  to (27) is less than  $\alpha\beta$ .

All that remains to show is that  $\mathcal{P}(m_\alpha, \alpha)$  is strictly increasing in  $\alpha$ . For  $\alpha \leq c'_a$ , we have  $m_\alpha = 0$ , so  $\mathcal{P}(m_\alpha, \alpha) = \mathcal{P}(0, \alpha) = \alpha\beta$ , which is strictly increasing. When  $\alpha > c'_a$ , we have  $\mathcal{P}(m_\alpha, \alpha) = \beta c'_a/x(\alpha)$ , where  $x(\alpha)$  is the solution to (28). Lemma 5 implies that the right side is strictly increasing in  $x$ , so  $x(\alpha)$  is strictly decreasing in  $\alpha$  and thus  $\mathcal{P}(m_\alpha, \alpha)$  is strictly increasing.  $\square$

Our proof of Lemma 4 used the following fact.

**Lemma 5.** *For any  $a > 0$  the function  $y(x) = \frac{1-e^{-ax(1-x)}}{1-x}$  for  $x \in [0, 1)$  is strictly increasing in  $x$ , taking the value 0 when  $x = 0$  and approaching  $a$  as  $x \rightarrow 1$ .*

*Proof.* Evaluating  $y(0)$  and  $\lim_{x \rightarrow 1} y(x)$  is straightforward, so we move on to proving that  $y(x)$  is strictly increasing. Differentiating with respect to  $x$  and rearranging terms, we get

$$(29) \quad \frac{dy}{dx} = \frac{ae^{-ax(1-x)}}{(1-x)^2} \left( 2x^2 + \frac{1}{a}(e^{ax(1-x)} - 1) - 3x + 1 \right).$$

We wish to show that this expression is positive for  $x \in (0, 1)$ . The first term is clearly positive, so let's consider the second term. Since  $e^{ax(1-x)} \geq 1 + ax(1-x)$ , we have for  $x \in (0, 1)$ :

$$(30) \quad \left( 2x^2 + \frac{1}{a}(e^{ax(1-x)} - 1) - 3x + 1 \right) \geq 2x^2 + x(1-x) - 3x + 1 = (1-x)^2 > 0.$$

□

Before proceeding, we require some properties of the solution  $m_\alpha$  to (25).

**Lemma 6.**  *$\mathcal{Q}(m_\alpha, \alpha)$  is nonincreasing in  $\alpha$ . If  $c'_a < 1$ ,  $\mathcal{Q}(m_\alpha, \alpha) = 1$  for  $\alpha \in [0, c'_a]$  and is strictly decreasing for  $\alpha \in (c'_a, 1]$ . If  $c'_a \geq 1$ ,  $\mathcal{Q}(m_\alpha, \alpha) = 1$  for all  $\alpha \in [0, 1]$ .*

*Proof.* From Lemma 4, we have that  $m_\alpha = 0$  if and only if  $\alpha \leq c'_a$ . When  $m_\alpha = 0$ , we have  $\mathcal{Q}(m_\alpha, \alpha) = g(0) = 1$ . If  $c'_a < 1$ , for  $\alpha > c'_a$ , Lemma 4 implies that  $m_\alpha > 0$  and thus by (2) we have:

$$(31) \quad m_\alpha \mathcal{P}(m_\alpha, \alpha) = \log \left( \frac{\mathcal{P}(m_\alpha, \alpha)}{\beta c'_a} \right).$$

Lemma 4 implies that the right hand side (and therefore the left hand side) is strictly increasing in  $\alpha$ . But

$$(32) \quad \mathcal{Q}(m_\alpha, \alpha) = g(m_\alpha \mathcal{P}(m_\alpha, \alpha)).$$

Because  $g$  is a strictly decreasing function (see Lemma 1), this completes the proof. □

The following lemma completes the proof of Theorem 1, by endogenizing the employers' choice of  $\alpha$ .

**Lemma 7.** *Define  $m_\alpha$  to be the unique solution to (24). If  $c'_s < 1$ , there is a unique value of  $\alpha \in [0, 1]$  such that*

$$(33) \quad \alpha \in \mathcal{A}(\mathcal{Q}(m_\alpha, \alpha)),$$

*i.e. such that  $(m_\alpha, \alpha)$  is a mean field equilibrium.*

*Proof.* Our proof leverages the following intuition. Consider fixing the employers' strategy to be  $\phi^1$ , i.e., employers always enter and screen. If in that case the resulting applicant availability is high enough (under the optimal seller response), then this will be an MFE. On the other hand, if the resulting applicant availability is too low, some employers will choose to exit the market. The key phenomenon that we exploit is that in this case, applicant availability is monotonically increasing as employers leave the market, increasing until exactly the point where employers are indifferent between entering and exiting. This indifference point is precisely where applicant availability  $q$  is equal to the (scaled) screening cost  $c'_s$ .

Formally, recall from Section 4.1.2 that

$$(34) \quad \mathcal{A}(q) = \begin{cases} \{0\}, & \text{if } q < c'_s; \\ [0, 1], & \text{if } q = c'_s; \\ \{1\}, & \text{if } q > c'_s. \end{cases}$$

If  $\mathcal{Q}(m_1, 1) \geq c'_s$ , then  $\alpha = 1$  solves (33). For any  $\alpha < 1$ , since  $c'_s < 1$ , it follows by Lemma 6 that regardless of the value of  $c_a$ , we have  $\mathcal{Q}(m_\alpha, \alpha) > c'_s$ . Thus, for any  $\alpha < 1$ ,  $\alpha \notin \mathcal{A}(\mathcal{Q}(m_\alpha, \alpha)) = \{1\}$ , so  $\alpha = 1$  is the unique solution to (33).

Now suppose that  $\mathcal{Q}(m_1, 1) < c'_s$ . By Lemma 6, for any  $\alpha \leq c'_a$ , we have  $\mathcal{Q}(m_\alpha, \alpha) = 1 > c'_s$ . By continuity and monotonicity of  $\mathcal{Q}$  (see Lemma 6), there exists exactly one value  $\alpha' \in (c'_a, 1)$  such that  $\mathcal{Q}(m_{\alpha'}, \alpha') = c'_s$ . Clearly,  $\alpha' \in \mathcal{A}(\mathcal{Q}(m_{\alpha'}, \alpha')) = \mathcal{A}(c'_s) = [0, 1]$ , so  $\alpha'$  solves (33). Furthermore, for any  $\alpha < \alpha'$ , we have  $\mathcal{Q}(m_\alpha, \alpha) > c'_s$  and thus  $\mathcal{A}(\mathcal{Q}(m_\alpha, \alpha)) = \{1\}$ . For any  $\alpha > \alpha'$ , we have  $\mathcal{Q}(m_\alpha, \alpha) < c'_s$  and thus  $\mathcal{A}(\mathcal{Q}(m_\alpha, \alpha)) = \{0\}$ . Therefore,  $\alpha'$  is the unique solution to (33). □

**B.3. Section 4.4: The Regulated Market.** In this section we prove Proposition 3, which states that there is a unique equilibrium in the regulated market. It is the same as the equilibrium of the unregulated market if the limit  $\ell$  does not bind, and otherwise involves applicants selecting  $m_a = \ell$ .

Our analysis requires a partial equilibrium characterization of the employers' behavior given a choice of  $m$  by applicants. In particular, for a fixed  $m$ , we show there

exists a unique value  $\alpha_m$  satisfying:

$$(35) \quad \alpha_m \in \mathcal{A}(\mathcal{Q}(m, \alpha_m)),$$

where we recall from Section 4.1.2 that:

$$(36) \quad \mathcal{A}(q) = \begin{cases} \{0\}, & \text{if } q < c'_s; \\ [0, 1], & \text{if } q = c'_s; \\ \{1\}, & \text{if } q > c'_s. \end{cases}$$

We have the following lemma.

**Lemma 8.** *For any fixed  $m$ , there exists a unique solution  $\alpha_m$  to (35).*

*Proof.* If  $\mathcal{Q}(m, 1) \geq c'_s$ , then  $\alpha_m = 1$  solves (35). Furthermore, for any  $\alpha < 1$ ,  $\mathcal{Q}(m, \alpha) > \mathcal{Q}(m, 1)$  by Lemma 3, and thus  $\alpha \notin \mathcal{A}(\mathcal{Q}(m, \alpha)) = \{1\}$ .

If  $\mathcal{Q}(m, 1) < c'_s$ , then  $1 \notin \mathcal{A}(\mathcal{Q}(m, 1)) = \{0\}$ , and  $0 \notin \mathcal{A}(\mathcal{Q}(m, 0)) = \mathcal{A}(1) = \{1\}$ . It follows that any solution  $\alpha_m$  to (35) must satisfy  $0 < \alpha_m < 1$ , and thus must satisfy  $\mathcal{Q}(m, \alpha_m) = c'_s$ . Lemma 3 states that  $\mathcal{Q}(m, \alpha)$  strictly increases continuously to 1 as  $\alpha$  decreases, implying that there is a unique  $\alpha_m$  such that  $\mathcal{Q}(m, \alpha_m) = c'_s$ .  $\square$

In addition, note that the pair  $(m, \alpha_m)$  is an MFE if and only if  $m \in \mathcal{M}(\mathcal{P}(m, \alpha_m))$ . For  $m > 0$ , from (2), this holds if and only if  $h(m) = -\log c'_a$ , where

$$(37) \quad h(m) = m\mathcal{P}(m, \alpha_m) - \log(\mathcal{P}(m, \alpha_m)/\beta).$$

The following lemma gives some basic properties of the function  $h$ .

**Lemma 9.** *The function  $h(m)$  given in (37) is strictly increasing. It takes the value zero at  $m = 0$ , and increases without bound as  $m \rightarrow \infty$ .*

*Proof.* Note that a mean-field equilibrium is exactly a pair  $(m, \alpha_m)$  such that  $m = \mathcal{M}(\mathcal{P}(m, \alpha_m))$ . From the definition of  $\mathcal{M}$  it follows that the pair  $(m, \alpha_m)$  with  $m > 0$  is an equilibrium if and only if  $m = \frac{1}{\mathcal{P}(m, \alpha_m)} \log\left(\frac{\mathcal{P}(m, \alpha_m)}{c_a}\right)$ , or equivalently

$$(38) \quad m\mathcal{P}(m, \alpha_m) - \log(\mathcal{P}(m, \alpha_m)/\beta) = -\log c'_a.$$

We already know from Theorem 1 that for any  $c'_a < 1$  and any  $\beta$ , there is a unique equilibrium with  $m > 0$ , i.e. a unique solution to  $h(m) = -\log c'_a$ . This implies that  $h$  is invertible (if  $h(m) = h(m')$  for  $m \neq m'$ , then for some  $c'_a$ , there would be multiple mean-field equilibria). Because  $h$  is continuous, this also implies that  $h$  is monotonic. Furthermore, the existence of a solution to  $h(m) = -\log c'_a$  for any  $c'_a < 1$  (by Theorem 1) implies that  $h(0) = 0$  and that  $h$  is unbounded.  $\square$

We now prove the desired result.

*Proof of Proposition 3.* Let  $(m^*, \alpha^*)$  be the mean-field equilibrium in the original game, and let  $(m_\ell^*, \alpha_\ell^*)$  be any equilibrium of the game with application limit  $\ell$ , meaning that  $m_\ell^* = \mathcal{M}_\ell(\mathcal{P}(m_\ell^*, \alpha_\ell^*))$ , and  $\alpha_\ell^* \in \mathcal{A}(\mathcal{Q}(m_\ell^*, \alpha_\ell^*))$ .

Immediately from (6) (which states that  $\mathcal{M}_\ell(p) = \min(\ell, \mathcal{M}(p))$ ), we get that if there is an equilibrium of the regulated market with  $m_\ell^* < \ell$ , it must be that  $(m_\ell^*, \alpha_\ell^*)$  is an equilibrium in the unregulated market and thus  $(m_\ell^*, \alpha_\ell^*) = (m^*, \alpha^*)$ . Therefore, the only candidate equilibria in the game with application limit  $\ell$  are  $(m^*, \alpha^*)$  and  $(\ell, \alpha_\ell)$ .

It is clear that  $(m^*, \alpha^*)$  is an equilibrium of the game with application limit  $\ell$  if and only if  $m^* \leq \ell$ . To complete the proof of Proposition 3, we claim that the pair  $(\ell, \alpha_\ell)$  is an equilibrium of the regulated market if and only if  $m^* \geq \ell$ .

To prove this claim, first suppose that  $m^* < \ell$ . Then  $-\log c'_a = h(m^*) < h(\ell)$  by Lemma 9. Conversely, if  $m^* \geq \ell$ , then  $-\log c'_a = h(m^*) \geq h(\ell)$ . In other words, for  $c'_a \leq 1$ ,

$$(39) \quad m^* \geq \ell \Leftrightarrow h(\ell) \leq -\log c'_a.$$

Straightforward manipulation of the equation (37) defining  $h$  and an application of (2) reveals that for  $\ell > 0$  and  $c'_a \leq 1$ ,

$$(40) \quad \ell \leq \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell)) \Leftrightarrow h(\ell) \leq -\log c'_a.$$

Combining (39) and (40), we get that  $m^* \geq \ell$  if and only if  $\ell \leq \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell))$ . Furthermore,  $\ell \leq \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell))$  implies  $\mathcal{M}_\ell(\mathcal{P}(\ell, \alpha_\ell)) = \ell$ , and conversely  $\ell > \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell))$  implies  $\mathcal{M}_\ell(\mathcal{P}(\ell, \alpha_\ell)) = \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell))$  (using  $\mathcal{M}_\ell(p) = \min(\ell, \mathcal{M}(p))$  from (6)). Putting it all together, we get that

$$(41) \quad m^* \geq \ell \Leftrightarrow \ell \leq \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell)) \Leftrightarrow \ell = \mathcal{M}_\ell(\mathcal{P}(\ell, \alpha_\ell)).$$

In other words,  $m^* \geq \ell$  if and only if  $(\ell, \alpha_\ell)$  is an equilibrium of the game with application limit  $\ell$ .  $\square$

## APPENDIX C. PROOFS: SECTION 5

In this section we develop the technology required to prove the approximation results (Propositions 4 and 5 and Theorem 2).

We begin by formalizing the stochastic process of interest, when  $m$  and  $\alpha$  are fixed. Note that in our original model, applicants decide where to apply when they arrive

to the system; however, for purposes of stochastic analysis, we obtain an equivalent system if we *realize applicant applications only when employers depart*. In particular, we consider the following stochastic system parameterized by  $n$ . Individual applicants arrive at intervals of length  $1/rn$ , as before. Let  $S(t)$  denote the number of applicants in the system at time  $t$ . In addition, we define  $\Sigma(t)$  as the *normalized* number of applicants in the system:

$$(42) \quad \Sigma(t) = S(t)/(rn);$$

note that  $\Sigma(t) \leq 1$  for any  $N(t)$  that can arise. At intervals of length  $1/n$  (corresponding to employer departures), there are opportunities for at most a single applicant in the system to depart early. At each such employer departure time  $t$ , the probability of a departure of an applicant is:

$$(43) \quad \alpha \left( 1 - \left( 1 - \frac{\beta m}{n} \right)^{S(t)} \right) = \alpha(1 - \rho^{\Sigma(t)}),$$

where we define  $\rho := (1 - \beta m/n)^{rn}$ . Note that  $\rho \rightarrow \exp(-\eta)$  as  $n \rightarrow \infty$ , where we define  $\eta := rm\beta$ .

The preceding equation (43) is derived as follows. As before, with probability  $\alpha$  an employer screens using strategy  $\phi$ , and exits immediately otherwise (in which case no applicant departs early). Every applicant that arrived in the last one time unit applied to the departing employer with probability  $m/n$ . Any such applicant that has already departed cannot match to the given employer. On the other hand, among the remaining employers, if even one of them is compatible with the employer, then employer following  $\phi$  is sure to find a match. Thus at least one departure occurs as long as there is at least one available, compatible applicant that applied to the departing employer. Note that under  $\phi$ , each of the applicants in the system at time  $t$  is equally likely to depart, so for each applicant the probability of departure is  $\alpha(1 - \rho^{\Sigma(t)})/(rn\Sigma(t))$ .

Note that to capture the state at time  $t$ , we must track the residual lifetimes of all applicants in the system. To simplify this tracking, a key instrument in our analysis is a “binned” version of the stochastic process  $S(t)$ , defined as follows. Fix an integer  $k$ , and let  $S_j(t)$  be the number of applicants that have been in the system for a time between  $j/k$  and  $(j+1)/k$  units, for  $j = 0, 1, \dots, k-1$ . Let  $X_j(t) = S_j(t)/(rn)$ ; note that  $\Sigma(t) = \sum_{j=0}^{k-1} X_j(t)$ . Our fundamental result proves a concentration result for the vector-valued stochastic process  $X(t)$ .

What does  $X(t)$  concentrate around? To develop intuition, let's think of the matching process from the perspective of the applicants that arrive in a fixed interval of length  $1/k$ . In the large market limit, each of these applicants should match in successive intervals of length  $1/k$  with a constant probability; or equivalently, their survival probability is a constant  $\gamma$  in each such interval. Looking back in time, then, in steady state we should expect that the vector  $X(t)$  satisfies  $X_j(t) = \gamma X_{j-1}(t)$  for  $j = 1, \dots, k-1$ , with  $X_0(t) = 1/k$ . With this inspiration (and in an abuse of notation), we define  $\Sigma(\gamma)$  as:

$$(44) \quad \Sigma(\gamma) = \frac{1}{k} \sum_{j=0}^{k-1} \gamma^j = \frac{1 - \gamma^k}{k(1 - \gamma)}.$$

(Note that  $\Sigma(1) = 1$ .)

On the other hand, as in our mean field analysis, we can develop a “consistency check” that  $\gamma$  must satisfy using (43). Assume that  $k \geq \eta/r$  and define  $\gamma(\Sigma)$  as follows:

$$(45) \quad \gamma(\Sigma) = 1 - \alpha \left( \frac{1 - e^{-\eta\Sigma}}{rk\Sigma} \right),$$

where we take  $\gamma(0) = 1 - \alpha\eta/(rk)$  (this is the limit of the preceding quantity as  $\Sigma \rightarrow 0$ ). This equation is an approximate version of (43):  $1 - \gamma(\Sigma)$  represents an estimate of the probability that an individual applicant in the system is matched in the next  $1/k$  time units, if the current (normalized) number of available applicants is  $\Sigma$ , and  $k$  and  $n$  are “large” (specifically  $k = \omega(1)$  and  $n = \omega(k)$ ).

The following two results are critical to our analysis: they establish the uniqueness of a solution to the preceding two equations, and show that in the limit  $k \rightarrow \infty$  this solution is the unique MFSS  $(p, q)$  guaranteed by Proposition 2. The proofs are in Section C.1.

**Lemma 10.** *Suppose that  $k \geq \eta/r$ . There is a unique pair of real numbers  $(\gamma^*, \Sigma^*)$  that simultaneously solve (44) and (45).*

**Lemma 11.** *Suppose that  $k \geq \eta/r$ . Let  $(\gamma_k^*, \Sigma_k^*)$  denote the unique solution to (44) and (45) guaranteed by Lemma 10. Then as  $k \rightarrow \infty$ ,  $k(1 - \gamma_k^*) \rightarrow mp$ , and  $\Sigma_k^* \rightarrow q$ , where  $(p, q)$  is the unique MFSS guaranteed by Proposition 2.*

The interpretation is as follows. Note that under the mean field assumptions, each applicant sends a Poisson distributed number of applications, with mean  $m$ ; and each application independently succeeds with probability  $p$ . Since the applicant's

applications are independent to each employer, it follows that in the mean field model the applicant's lifetime in the system is an exponential random variable of mean  $1/mp$  truncated to be less than or equal to 1 (since applicants only live for at most a unit lifetime). In other words, the rate of applicant departure is  $mp$ . On the other hand, for fixed  $k$  the rate of departure is approximated by  $k(1 - \gamma_k^*)$ , so we should expect the latter quantity to approach  $mp$ . Similarly, observe that  $\Sigma^*$  is meant to be an estimate of the steady state (normalized) number of applicants in the system; we should expect that this approaches the applicant availability  $q$  in the mean field model.

Let  $X^*$  be the vector given by  $X_j^* = (\gamma^*)^j/k$  for  $j = 0, \dots, k-1$ , so that  $\Sigma^* = \sum_{j=0}^{k-1} X_j^*$  from Eq. (44). Our main result is the following theorem, that shows that  $X(t)$  concentrates around  $X^*$ . Note that this is a very strong result, because it precludes drift of the  $X(t)$  process away from  $X^*$  as time grows. We achieve this result by using a stochastic concentration argument on the process  $X(t)$ .<sup>18</sup>

**Theorem 6.** *Fix  $m_0 \in [1, \infty)$  and  $\alpha \in [0, 1]$ . There exists  $C = C(r, m_0, \beta, \alpha) < \infty$  such that for any  $m \in [1/m_0, m_0]$ , for any  $n > C$ ,  $k = \lfloor n^{1/3} \rfloor$  the following is true: For any  $t > C \log n$ , and any starting state at time 0, we have  $\mathbb{E}[\|X(t) - X^*\|_1] \leq Cn^{-1/6}$ . If the starting state at time 0 is drawn from the steady state distribution, we have  $\mathbb{E}[\|X(t) - X^*\|_1] \leq Cn^{-1/6}$  for all  $t \geq 0$ .*

We use this theorem to establish that the mean field assumptions hold asymptotically (Propositions 4 and 5).

The remainder of this section is organized as follows. In Section C.1, we prove Lemmas 10 and 11.

The proof of Theorem 6 is presented in Section C.2. The theorem follows from a stochastic contraction result that shows that for any starting state that is significantly far from  $X^*$ , the distance between the state and  $X^*$  decays exponentially (a *contraction*) until the state becomes close to  $X^*$ . We prove this contraction with two key steps. First, Lemma 12 (proved in Section C.2.1) says that given a starting state, in the large system limit the state contracts towards  $X^*$  in time. Second, Lemma 13 (proved in Section C.2.2) says that the expected distance of the true state in a (stochastic) finite market from the state in the large system limit is small for time increments that are not large.

The proof of Lemma 12 is in Section C.2.1. In the limit, a fraction  $1 - \gamma$  of sellers match and leave in an interval of length  $1/k$ , and the remaining ones move into the

<sup>18</sup>Note that technically, the result only holds at rational time points  $t$ , given the definition of our process  $X(t)$ . However, we suppress this in the presentation for clarity.



next bin of length  $1/k$  (except for the last bin, from which all sellers leave). Crucially, we show that the measure of sellers that leave by virtue of matching,  $(1 - \gamma)\Sigma$ , is increasing in  $\Sigma$ , at a rate that is bounded below, cf. Lemma 14 part (4). Thus if the number of available sellers exceeds the steady state value, then sellers match faster than usual, whereas if the number of available sellers is less than the steady state value, then sellers match slower than usual; thus making the number trend towards the steady state value. Formally we use the  $L_1$  distance between states and careful analysis to show a contraction in the large market limit. We believe this contraction phenomenon and our analytical approach extends to matching setting beyond the one considered here.

The proof of Lemma 13 in Section C.2.2 involves careful control of the stochastic behavior of the finite system, and showing that the finite system is closely approximated by the large system limit.

Having completed the proof of Theorem 6, we use it to prove the main approximation results presented in the text. In Section C.3, we prove the mean field limit Propositions 4 and 5 using Theorem 6. In Section C.4 we prove Theorem 2, using Propositions 4 and 5.

### C.1. Proofs of Lemmas 10 and 11.

*Proof of Lemma 10.* For  $\alpha = 0$  we immediately find that  $\gamma^* = 1$  and  $\Sigma^* = 1$  is the unique solution.

Assume  $\alpha > 0$ . Rearranging (45), we get

$$(46) \quad r = \frac{\alpha(1 - e^{-\eta\Sigma})}{k(1 - \gamma)\Sigma} = \frac{\alpha(1 - e^{-\eta\Sigma})}{1 - \gamma^k},$$

where the final expression follows from substituting (44). Differentiate the expression on the right with respect to  $\gamma$ , thinking of  $\Sigma$  as the function of  $\gamma$  specified by Eq. (44): the result is

$$(47) \quad \frac{\alpha}{(1 - \gamma^k)^2} \left( k\gamma^{k-1}(1 - e^{-\eta\Sigma}) + \eta e^{-\eta\Sigma}(1 - \gamma^k) \frac{d\Sigma}{d\gamma} \right).$$

Since  $\frac{d\Sigma}{d\gamma} > 0$  for  $\Sigma$  given by Eq. (44), the expression above is positive, so the right side of (46) is increasing in  $\gamma$ . Further, since

$$(48) \quad \lim_{\gamma \rightarrow 0} \frac{\alpha(1 - e^{-\eta\Sigma})}{1 - \gamma^k} = \alpha(1 - e^{-\eta/k}) < \alpha\eta/k \leq r < \infty = \lim_{\gamma \rightarrow 1} \frac{\alpha(1 - e^{-\eta\Sigma})}{1 - \gamma^k},$$

there is a unique solution  $\gamma^*$  to (46), leading to a unique value  $\Sigma^*$  from Eq. (44).  $\square$

*Proof of Lemma 11.* Note that  $0 \leq \Sigma_k^* \leq 1$ . Further:

$$(49) \quad k(1 - \gamma_k^*) = \alpha \left( \frac{1 - e^{-\eta \Sigma_k^*}}{r \Sigma_k^*} \right).$$

This remains bounded for all  $\Sigma_k^* \in [0, 1]$ . Taking subsequences if necessary, therefore, we assume without loss of generality that  $k(1 - \gamma_k^*) \rightarrow m\hat{p}$  and  $\Sigma_k^* \rightarrow \hat{q}$  as  $k \rightarrow \infty$ .

To establish the result, we show that  $(\hat{p}, \hat{q})$  must satisfy (4) and (5). Rewriting and taking limits in (44), we have:

$$\Sigma_k^* = \frac{1 - \left(1 - \frac{k(1 - \gamma_k^*)}{k}\right)^k}{k(1 - \gamma_k^*)} \rightarrow \frac{1 - e^{-m\hat{p}}}{m\hat{p}}.$$

And similarly, taking limits in (49) and substituting for  $\eta$  we have:

$$m\hat{p} = \alpha \left( \frac{1 - e^{-rm\beta\hat{q}}}{r\hat{q}} \right).$$

Thus  $(\hat{p}, \hat{q})$  satisfy (4) and (5), as required.  $\square$

**C.2. Proof of Theorem 6.** Throughout our proof of Theorem 6, we think of the state of the system at  $t$  as being the subset of applicants who are still available at  $t$  (with their arrival times) from among the applicants who arrived during the last 1 time unit. The applications to a specific employer are ‘revealed’ just before the employer exits. If at time 0, the set of applications by applicants in the system has been revealed, we simply need to move forward 1 time unit before we can use our above description of the state. Thus, our analysis holds for times  $t \geq 1$ .

*Proof of Theorem 6.* In order to prove Theorem 6, we first show that for fixed  $m_0 \in [1, \infty)$  and  $\alpha \in [0, 1]$ , there exists  $\kappa' = \kappa'(r, m_0, \beta, \alpha) > 0$  and  $C' = C'(r, m_0, \beta) < \infty$  such that for any  $n > C'$ , for any  $k$ , any  $m \in [1/m_0, m_0]$ , and any starting state, we have

$$(50) \quad \mathbb{E}\|X(t + 1/k) - X^*\|_1 \leq (1 - \kappa'/k)\|X(t) - X^*\|_1 + C' \left( \frac{1}{\sqrt{n}} + \frac{1}{k^{3/2}} \right).$$

so that by iterating we have:

$$(51) \quad \mathbb{E}\|X(t + i/k) - X^*\|_1 \leq (1 - \kappa'/k)^i \|X(t) - X^*\|_1 + C' \left( \frac{1}{\sqrt{n}} + \frac{1}{k^{3/2}} \right) k/\kappa'.$$

To establish the existence of  $\kappa'$  that yields (50), define  $\hat{X}(t+1/k)$  by  $\hat{X}_0(t+1/k) = 1/k$ , and for  $0 \leq j \leq k-1$ ,  $\hat{X}_{j+1}(t+1/k) = \gamma X_j(t)$ , where  $\gamma$  is given by (45), i.e.,

$$(52) \quad \gamma = 1 - \frac{\alpha(1 - e^{-\eta\|X(t)\|_1})}{rk\|X(t)\|_1}.$$

We use the following two lemmas (proven below) to establish (50).

**Lemma 12.** *Fix  $m_0 \in [1, \infty)$ . There exists  $\kappa = \kappa(r, m_0, \beta) > 0$  such that for any  $k$ , any  $\alpha \in [0, 1]$ , any  $m \in [1/m_0, m_0]$ , and any  $X(t)$  we have*

$$(53) \quad \|\hat{X}(t+1/k) - X^*\|_1 \leq (1 - \alpha\kappa/k)\|X(t) - X^*\|.$$

**Lemma 13.** *Fix  $m_0 < \infty$ . There exists  $C' = C'(m_0, r, \beta) < \infty$  such that for  $n > C'$ , any  $m \leq m_0$ , any  $k$ , any  $\alpha \in [0, 1]$ , and any starting state, we have*

$$(54) \quad \mathbb{E}\|X(t+1/k) - \hat{X}(t+1/k)\|_1 \leq C'(n^{-1/2} + k^{-3/2}).$$

Now using the triangle inequality, we have

$$(55) \quad \mathbb{E}\|X(t+1/k) - X^*\|_1 \leq \|\hat{X}(t+1/k) - X^*\|_1 + \mathbb{E}\|X(t+1/k) - \hat{X}(t+1/k)\|_1.$$

The relation (50) follows immediately from Lemmas 12 and 13 with  $\kappa' = \alpha\kappa$ .

To complete the proof of Theorem 6, note that  $\|X(t_0) - X^*\|_1 \leq \|X(t_0)\|_1 + \|X^*\|_1 = \Sigma(X(t_0)) + \Sigma(X^*) \leq 2$  for any  $t_0$ . Use (51) to track the evolution from  $t_0 = t - i/k$  to  $t$  where  $i = \lfloor kC \log n \rfloor$ . For  $C = (1 + 2C')/\kappa'$  and using  $(1 - \kappa'/k)^i \|X(t_0) - X^*\|_1 \leq \exp(-C\kappa' \log n) \cdot 2 \leq 1/n$ , we have

$$(56) \quad \mathbb{E}[\|X(t) - X^*\|_1] \leq Cn^{-1/6},$$

yielding the first part of the theorem.

Now consider the second part of the theorem. For any starting state, the system reaches steady state as  $t \rightarrow \infty$ . Let  $X_{ss}$  be the steady state distribution of  $X$ . Then we have  $\lim_{t \rightarrow \infty} X(t) = X_{ss}$ . It follows from the dominated convergence theorem that  $\mathbb{E}[\|X_{ss} - X^*\|_1] = \lim_{t \rightarrow \infty} \mathbb{E}[\|X(t) - X^*\|_1] \leq Cn^{-1/6}$  using (56). The second part of the theorem follows immediately since if the starting state is distributed as per the steady state distribution, the state at time  $t$  is also distributed as per the steady state distribution, so the same bound holds for all  $t \geq 0$ .  $\square$

It remains to show Lemmas 12 and 13; we establish these below. To prove them, we use the following facts about Eq. (52).

**Lemma 14.** *Let  $h(\Sigma) = (1 - e^{-\eta\Sigma})/\Sigma$ , so that  $\gamma = 1 - \alpha h(\Sigma)/(rk)$ . Then*

- (1) We have  $-\eta^2 \leq \frac{dh}{dx} \leq 0$ .
- (2) We have  $0 \leq \frac{d\gamma}{d\Sigma} \leq \frac{\alpha\eta^2}{rk}$ .
- (3) We have  $\frac{\alpha(1-e^{-\eta})}{rk} \leq 1 - \gamma \leq \frac{\alpha\eta}{rk}$ .
- (4) We have  $\frac{d}{d\Sigma}(1 - \gamma)\Sigma \geq \frac{\alpha\eta e^{-\eta}}{rk}$ .

*Proof.* Note that  $\frac{dh}{dx} = \frac{(1+\eta x)e^{-\eta x}-1}{x^2}$ . The first inequality in item 1 comes from substituting  $e^{-\eta x} \geq 1 - \eta x$ , the second from substituting  $1 + \eta x \leq e^{\eta x}$ . Item 2 follows from the fact that  $\gamma = 1 - \frac{\alpha}{rk}h(\Sigma)$ . The third item comes from the fact that  $(1 - \gamma)$  is monotone in  $\Sigma$  (from item 2), and  $\Sigma \in [0, 1]$ . The final item comes from the fact that  $\frac{d}{d\Sigma}(1 - \gamma)\Sigma = \frac{d}{d\Sigma} \frac{\alpha(1-e^{-\eta\Sigma})}{rk} = \frac{\alpha\eta e^{-\eta\Sigma}}{rk}$ , which is decreasing in  $\Sigma$  (and  $\Sigma \leq 1$ ).  $\square$

**Lemma 15.** Recall  $\rho = (1 - m\beta/n)^{rn}$ . Let  $\tilde{h}(\Sigma) = (1 - \rho^\Sigma)/\Sigma$ . Then we have  $(\log \rho)^2 \leq \frac{d\tilde{h}}{d\Sigma} \leq 0$ .

*Proof.* Analogous to the proof of Lemma 14 item 1.  $\square$

C.2.1. *Proof of Lemma 12.* In this subsection we prove Lemma 12; we prove Lemma 13 in the next subsection.

*Proof of Lemma 12.* For simplicity, we use  $X$  to denote  $X(t)$ ,  $\hat{X}$  for  $X(t + 1/k)$ ,  $\Sigma$  for  $\Sigma(X)$ , and  $\gamma$  for  $\gamma(\Sigma)$ .

We first prove Lemma 12 for the case  $\Sigma \geq \Sigma^*$ . By Lemma 14.2, this implies  $\gamma \geq \gamma^*$ . Note that

$$(57) \quad |a - b| = (b - a) + 2[a - b]_+,$$

which we use to get that

$$(58) \quad \|X - X^*\|_1 = \sum_{j=0}^{k-1} (X_j - X_j^*) + 2[X_j^* - X_j]_+ = \Sigma - \Sigma^* + 2 \sum_{j=0}^{k-1} [X_j^* - X_j]_+.$$

Recall that  $\hat{X}_0 = X_0^* = 1/k$ . It follows that

$$\begin{aligned}
 \|\hat{X} - X^*\|_1 &= \sum_{j=1}^{k-1} |X_j^* - \hat{X}_j| = \sum_{j=0}^{k-2} |\gamma^* X_j^* - \gamma X_j| \\
 &= \sum_{j=0}^{k-2} \gamma X_j - \gamma^* X_j^* + 2[\gamma^* X_j^* - \gamma X_j]_+ \\
 &= \gamma \Sigma - \gamma X_{k-1} - \gamma^* \Sigma^* + \gamma^* X_{k-1}^* + 2 \sum_{j=0}^{k-2} [\gamma^* X_j^* - \gamma X_j]_+ \\
 (59) \quad &= \gamma \Sigma - \gamma^* \Sigma^* - |\gamma^* X_{k-1}^* - \gamma X_{k-1}| + 2 \sum_{j=0}^{k-1} [\gamma^* X_j^* - \gamma X_j]_+,
 \end{aligned}$$

where both the first and last lines use (57). Now, by Lemma 14, item 4,

$$\begin{aligned}
 \gamma \Sigma - \gamma^* \Sigma^* &= \Sigma - \Sigma_* + [(1 - \gamma^*) \Sigma^* - (1 - \gamma) \Sigma] \\
 (60) \quad &\leq (\Sigma - \Sigma^*) \left(1 - \frac{\alpha \eta e^{-\eta}}{rk}\right).
 \end{aligned}$$

Also, using  $\gamma \geq \gamma^*$  and the upper bound on  $\gamma^*$  from Lemma 14, item 3, we have

$$\begin{aligned}
 [X_{j+1}^* - \hat{X}_{j+1}]_+ &= [\gamma^* X_j^* - \gamma X_j]_+ \\
 &\leq [\gamma^* X_j^* - \gamma^* X_j]_+ = \gamma^* [X_j^* - X_j]_+ \\
 (61) \quad &\leq \left(1 - \frac{\alpha(1 - e^{-\eta})}{rk}\right) [X_j^* - X_j]_+.
 \end{aligned}$$

If we define  $\kappa = \min_{m \in [1/m_0, m_0]} \frac{1}{r} \min(\eta e^{-\eta}, 1 - e^{-\eta}) > 0$ , then substituting (60) and (61) into (59) yields

$$\begin{aligned}
 \|\hat{X} - X^*\|_1 &\leq (1 - \alpha \kappa/k)(\Sigma - \Sigma^*) - |\gamma^* X_{k-1}^* - \gamma X_{k-1}| + 2(1 - \alpha \kappa/k) \sum_{j=0}^{k-1} [X_j^* - X_j]_+ \\
 (62) \quad &\leq (1 - \alpha \kappa/k) \|X - X^*\|_1,
 \end{aligned}$$

where the second line follows from dropping the absolute value term and applying (58).

The proof for the complementary case  $\Sigma \leq \Sigma^*$  is analogous.  $\square$

C.2.2. *Proof of Lemma 13.* Note that

$$(63) \quad \mathbb{E} |X_j - \hat{X}_j| \leq |\mathbb{E}[X_j] - \hat{X}_j| + \mathbb{E} |X_j - \mathbb{E}[X_j]| \leq |\mathbb{E}[X_j] - \hat{X}_j| + \sqrt{\text{Var}(X_j)},$$

where we have applied the triangle inequality and Jensen's inequality in turn. Sum over  $j$  to get

$$(64) \quad \mathbb{E}\|X(t+1/k) - \hat{X}(t+1/k)\|_1 \leq \|\mathbb{E}[X(t+1/k)] - \hat{X}(t+1/k)\|_1 + \sum_{j=0}^{k-1} \sqrt{\text{Var}(X_j(t+1/k))}.$$

Fix an applicant  $a$  who is in the system at time  $t$  and has arrived within the last  $1 - 1/k$  time units. Let  $I_a$  be the indicator that  $a$  is matched in the next  $1/k$  time units, and let  $\tilde{\gamma} = \mathbb{E}[1 - I_a]$  be the probability that this applicant is still in the system at time  $t + 1/k$ . To prove Lemma 13 we use the following two lemmas; Lemma 16 is used to prove Lemma 17. The proofs of both lemmas are deferred below.

**Lemma 16.** *Fix  $m_0 < \infty$ . There exists  $C = C(m_0, r, \beta) < \infty$  such that for  $n > C$ , any  $m \leq m_0$ , any  $k$ , any  $\alpha \in [0, 1]$ , and any starting state, the following hold:*

$$(65) \quad |\tilde{\gamma} - \gamma| \leq C/k^2$$

*For applicants  $a$  and  $a'$  who arrived between  $t - 1 + 1/k$  and  $t + 1/k$  we have*

$$(66) \quad \text{Var}(I_a) \leq C/k$$

$$(67) \quad |\text{Cov}(I_a, I_{a'})| \leq 1/k^3 \text{ for } a \neq a'$$

**Lemma 17.** *Fix  $m_0 < \infty$ . There exists  $C = C(m_0, r, \beta) < \infty$  such that for  $n > C$ , any  $m \leq m_0$ , any  $k$ , any  $\alpha \in [0, 1]$ , and any starting state, the following hold*

$$(68) \quad \|\mathbb{E}[X(t+1/k)] - \hat{X}(t+1/k)\|_1 \leq (C + \eta/r)/k^2 \leq (C + \eta/r)k^{-3/2},$$

and

$$(69) \quad \sum_{j=0}^{k-1} \sqrt{\text{Var}(X_j(t+1/k))} \leq \sqrt{C/r} n^{-1/2} + k^{-3/2}.$$

Here  $C$  is the same as in Lemma 16.

Using these two lemmas we can complete the proof of Lemma 13.

*Proof of Lemma 13.* Note that Lemma 17 implies that

$$(70) \quad \|\mathbb{E}[X(t+1/k)] - \hat{X}(t+1/k)\|_1 + \sum_{j=0}^{k-1} \sqrt{\text{Var}(X_j(t+1/k))} \leq \sqrt{C/r} n^{-1/2} + (\eta/r + C + 1)k^{-3/2},$$

using Lemma 16. If we take  $C' = \max(\sqrt{C/r}, \eta/r + C + 1)$ , then (64) and (70) imply Lemma 13.  $\square$

*Proof of Lemma 16.* Note that for all  $t' \in (t, t + 1/k)$ , we have

$$(71) \quad \Sigma(t) - 1/(rk) \leq \Sigma(t') \leq \Sigma(t) + 1/k,$$

since the number of new arrivals in the system is  $rn/k$ , the number of “match” departures is at most  $n/k$ , and  $\Sigma(t') = N(t')/(rn)$ .

Fix  $a$ , and suppose that there are  $rn\Sigma$  available applicants when an employer possibly screens and exits. The probability that  $a$  exits at this opportunity is  $\tilde{h}(\Sigma)/(rn) = \frac{\alpha(1-\rho^\Sigma)}{rn\Sigma}$  and this is a monotone decreasing function of  $\Sigma$  by Lemma 15 item 1. For convenience we define  $\tilde{h}(x) = \alpha \cdot (-\log \rho)/rn$  for  $x \leq 0$ . Since there are  $n/k$  opportunities to depart, Eq. (71) implies

$$(72) \quad \left(1 - \frac{\alpha}{rn} \tilde{h}(\Sigma(t) + 1/k)\right)^{n/k} \leq \mathbb{E}[1 - I_a] = \tilde{\gamma} \leq \left(1 - \frac{\alpha}{rn} \tilde{h}(\Sigma(t) - 1/(rk))\right)^{n/k}$$

Recall that  $\lim_{n \rightarrow \infty} \rho = \exp(-\eta)$ . By Lemma 15 item 1 we have that

$$(73) \quad \tilde{h}(\Sigma(t) - 1/(rk)) \leq \tilde{h}(\Sigma(t)) + \frac{2\eta^2}{rk}, \quad \text{and}$$

$$(74) \quad \tilde{h}(\Sigma(t) + 1/k) \geq \tilde{h}(\Sigma(t)) - \frac{2\eta^2}{k}.$$

for large enough  $n$ . It follows by substitution into (72) that

$$(75) \quad \left(1 - \frac{1}{rn} \tilde{h}(\Sigma(t)) - \frac{2\eta^2}{rnk}\right)^{n/k} \leq \tilde{\gamma} \leq \left(1 - \frac{1}{rn} \tilde{h}(\Sigma(t)) + \frac{2\eta^2}{r^2nk}\right)^{n/k}.$$

Using the inequality

$$(76) \quad 1 - m\varepsilon \leq (1 - \varepsilon)^m \leq 1 - m\varepsilon + \frac{1}{2}m^2\varepsilon^2,$$

we get

$$(77) \quad 1 - \frac{\alpha}{rk} \tilde{h}(\Sigma(t)) - \frac{2\alpha\eta^2}{rk^2} \leq \tilde{\gamma} \leq 1 - \frac{\alpha}{rk} \tilde{h}(\Sigma(t)) + \frac{2\alpha\eta^2}{r^2k^2} + \frac{1}{2} \left(\frac{\alpha}{rk}\right)^2 \tilde{h}(\Sigma(t))^2.$$

for large enough  $k$ . Now  $\rho = \exp(-\eta) + O(1/n)$ . Also,  $|\partial \tilde{h}/\partial \rho| = \rho^{\Sigma-1} \leq 1/\rho \leq 2\exp(\eta)$  for all  $\Sigma \in [0, 1]$ , for large enough  $n$ . It follows that

$$(78) \quad \tilde{h}(\Sigma) = h(\Sigma) + O(1/n).$$

Since we have  $\gamma = 1 - \frac{1-e^{-\eta\Sigma}}{rk\Sigma} = 1 - \frac{1}{rk}h(\Sigma)$ , combining Eqs. (77) and (78), we obtain  $|\tilde{\gamma} - \gamma|$  is  $O(\alpha/k^2)$ , establishing (65) in Lemma 16.

Note that

$$(79) \quad \text{Var}(I_a) = \tilde{\gamma}(1 - \tilde{\gamma}) \leq 1 - \tilde{\gamma}.$$

Using Eq. (77) and  $\tilde{h}(\Sigma(t)) \leq \tilde{h}(0) = -\log(\rho) = \eta + O(1/n)$  from Lemma 15, we have

$$1 - \tilde{\gamma} \leq \frac{\alpha\eta}{rk} + O(\alpha/k^2),$$

which proves (66) from Lemma 16 for applicants that arrived between  $t - 1 + 1/k$  and  $t$ , and were available at  $t$ . For any applicant  $a$  that arrived after  $t$ , the probability of still being available at  $t + 1/k$  is even larger than  $\tilde{\gamma}$ , i.e., this probability is in  $[\tilde{\gamma}, 1]$ , leading to  $\text{Var}(I_a) \leq \tilde{\gamma}(1 - \tilde{\gamma})$  since  $\tilde{\gamma} \geq 1/2$  leading to (66) for  $a$ .

Finally, we bound  $\text{Cov}(I_a, I_{a'})$ . We define a bipartite ‘interaction’ graph  $G = (V_S, V_B, E)$  whose vertex sets are  $V_S = \mathcal{S}(t) \cup \mathcal{S}(t + 1/k)$ , i.e., the applicants who arrive between  $t - 1$  and  $t + 1/k$ , and  $V_B = \mathcal{B}(t + 1/k) \setminus \mathcal{B}(t)$ , i.e., the set of employers who arrive between  $t$  and  $t + 1/k$ . For  $s \in V_S$  and  $b \in V_B$ , we decide on the presence of edge  $(s, b)$  as follows: Let  $\tau$  denote the time of arrival of  $a$ . Then if  $b \in \mathcal{B}(\tau)$ , i.e.,  $e$  is in the system when  $a$  arrives, we set  $\mathbb{I}((s, b) \in E) = \mathbb{I}(s \in M_b)$ , i.e., we include edge  $(s, b)$  if applicant  $a$  applies to employer  $e$ . If  $b \notin \mathcal{B}(\tau)$ , i.e.,  $e$  is not in the system when  $a$  arrives, we draw  $\mathbb{I}((s, b) \in E) \sim \text{Bernoulli}(m/n)$ , independent of everything else.

Note that the interaction graph  $G$  as defined above is a bipartite Erdos-Renyi graph with  $rn(1 + 1/k)$  vertices on one side,  $n/k$  vertices on the other, and edge probability  $m/n$  independently between vertices on the two sides. The following fact is immediate (see, e.g., Janson et al. (2011)) :

**Fact 1.** *Fix  $k$  such that  $r(1 + 1/k)(1/k)m < 1$ . There exists  $C < \infty$  such that the following occurs. For any  $\varepsilon > 0$  there exists  $n_0 < \infty$  such that for all  $n > n_0$ , with probability at least  $1 - \varepsilon$ , no connected component in  $G$  has more than  $C \log n$  vertices.*

Here the threshold for existence of a giant component is  $r(1 + 1/k)(1/k)m = 1$ . Thus it is sufficient to ensure that  $r(1 + 1/k)(1/k)m \leq 2rm/k < 1$ . In fact, fixing  $k$ , Fact 1 holds uniformly for all  $m < m_0 = k/(2r)$ , since the size of the largest connected component is monotone in the edge probability.

Clearly,  $I_a$  depends only on the connected component  $\mathcal{C}_a$  containing  $a$  and similarly for  $I_{a'}$ . Choose arbitrary  $\varepsilon > 0$ . Let  $\mathcal{E}_1$  be the event that no connected component has size more than  $C \log n$ . Fact 1 tells us that  $\mathbb{P}(\mathcal{E}_1^c) \leq \varepsilon$ . Let  $\mathcal{E}_2$  be the event that  $a' \notin \mathcal{C}_a$ . Clearly,  $\mathbb{P}(\mathcal{E}_2^c | \mathcal{E}_1) \leq C \log n / (rn) \leq \varepsilon$  for large enough  $n$ . We deduce that

$$(80) \quad \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c) = \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c \cap \mathcal{E}_1) \leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c | \mathcal{E}_1) \leq 2\varepsilon$$



Reveal  $\mathcal{C}_a$ . If  $|\mathcal{C}_a| \geq C \log n$  or  $a' \in \mathcal{C}_a$ , declare ‘failure’. Here  $|\mathcal{C}_a|$  denotes the number of vertices in  $\mathcal{C}_a$ . Suppose failure does not occur.

Let  $\mathcal{C}_s = \mathcal{C}$  be the revealed connected component. Since failure has not occurred we know that  $\mathcal{C}$  contains no more than  $C \log n$  nodes and does not contain  $a'$ . Now consider the conditional distribution of  $\mathcal{C}_{a'}$  given  $\mathcal{C}_s = \mathcal{C}$ .

**Claim 1.** *Consider any candidate connected component  $\mathcal{C}'$ , containing  $a'$  and not overlapping with  $\mathcal{C}$ . We have*

$$\mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') = \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}' | \mathcal{C}_s = \mathcal{C}) (1 - m/n)^{|\mathcal{C}'|_a |\mathcal{C}|_b + |\mathcal{C}'|_b |\mathcal{C}|_a},$$

where  $|\mathcal{C}|_a$  denotes the number of applicants in component  $\mathcal{C}$ , and  $|\mathcal{C}|_b$  denotes the number of employers in component  $\mathcal{C}$ .

*Proof.* The distribution of the rest of  $G$  conditioned on  $\mathcal{C}_s = \mathcal{C}$  has edge  $(s, b)$  present iid with probability  $m/n$  if both  $a$  and  $e$  are not in  $\mathcal{C}$ , and not present if one of  $a$  or  $e$  is present in  $\mathcal{C}$ . The result follows from a standard revelation argument on  $\mathcal{C}_{a'}$ .  $\square$

We have

$$(81) \quad \mathbb{E}(I_{a'}) = \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}(I_{a'} | \mathcal{C}_{a'} = \mathcal{C}'),$$

and

$$(82) \quad \mathbb{E}[I_{a'} | I_a = 1] = \sum_{\mathcal{C} \ni s, \mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_a = \mathcal{C} | I_a = 1) \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}' | \mathcal{C}_a = \mathcal{C}) \mathbb{E}[I_{a'} | \mathcal{C}_{a'} = \mathcal{C}']$$

using the fact that  $I_a - \mathcal{C}_a - \mathcal{C}_{a'} - I_{a'}$  form a Markov chain. Below we argue that the sum in Eq. (82) is very close to the sum in Eq. (81).

Let  $\pi \equiv \mathbb{E}[I_a] = \mathbb{E}[I_{a'}]$ . If  $\pi = 0$ , which might occur for instance if  $a$  arrives just before  $t + 1/k$ , we immediately have  $\text{Cov}(I_a, I_{a'}) = 0$ . As such, we assume  $\pi > 0$  in what follows. We deduce from Eq. (80) that

$$(83) \quad \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c | I_s = 1) \leq 2\varepsilon/\pi$$

It follows from Eqs. (82) and (83) that

$$(84) \quad \begin{aligned} \mathbb{E}[I_{a'} | I_a = 1] = & \sum_{\substack{\mathcal{C} \ni s, \mathcal{C}' \ni a' \text{ s.t. } \mathcal{C} \cap \mathcal{C}' = \emptyset, \\ |\mathcal{C}| \leq C \log n, |\mathcal{C}'| \leq C \log n}} \mathbb{P}(\mathcal{C}_a = \mathcal{C} | I_a = 1) \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}' | \mathcal{C}_a = \mathcal{C}) \mathbb{E}[I_{a'} | \mathcal{C}_{a'} = \mathcal{C}'] \\ & + \delta_1, \end{aligned}$$

where  $0 \leq \delta_1 \leq \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c | I_a) \leq 2\varepsilon/\pi$ . Now using Claim 1, we know that for such  $(\mathcal{C}, \mathcal{C}')$  we have

$$\mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}' | \mathcal{C}_a = \mathcal{C}) = \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}')(1 + \delta_{\mathcal{C}, \mathcal{C}'}),$$

where  $0 \leq \delta_{\mathcal{C}, \mathcal{C}'} \leq \varepsilon$  for large enough  $n$ . It follows that

$$\begin{aligned} & \sum_{\substack{\mathcal{C}' \ni a' \text{ s.t. } \mathcal{C} \cap \mathcal{C}' = \Phi, \\ |\mathcal{C}'| \leq C \log n}} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}' | \mathcal{C}_a = \mathcal{C}) \mathbb{E}[I_{a'} | \mathcal{C}_{a'} = \mathcal{C}'] \\ (85) \quad &= (1 + \delta_2) \sum_{\substack{\mathcal{C}' \ni a' \text{ s.t. } \mathcal{C} \cap \mathcal{C}' = \Phi, \\ |\mathcal{C}'| \leq C \log n}} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'} | \mathcal{C}_{a'} = \mathcal{C}'] \end{aligned}$$

for some  $0 \leq \delta_2 \leq \varepsilon$ . Now,

$$(86) \quad \mathbb{P}(|\mathcal{C}_{a'}| \geq C \log n) \leq \varepsilon$$

and

$$\mathbb{P}(v \in \mathcal{C}_{a'} | |\mathcal{C}_{a'}| \leq C \log n) \leq C \log n / (kn)$$

for any agent  $v$ , whether employer or applicant, hence for any  $\mathcal{C}$  s.t.  $a' \notin \mathcal{C}$  and  $|\mathcal{C}| \leq C \log n$  we have

$$(87) \quad \mathbb{P}(\mathcal{C} \cap \mathcal{C}_{a'} \neq \Phi | |\mathcal{C}_{a'}| \leq C \log n) \leq (C \log n)^2 / (kn) \leq \varepsilon$$

for large enough  $n$ . Combining Eqs. (86) and (87), we obtain

$$(88) \quad \mathbb{P}((|\mathcal{C}_{a'}| \geq C \log n) \cup (\mathcal{C} \cap \mathcal{C}_{a'} \neq \Phi)) \leq 2\varepsilon.$$

Plugging in to Eq. (85) we obtain

$$\begin{aligned} & \sum_{\substack{\mathcal{C}' \ni a' \text{ s.t. } \mathcal{C} \cap \mathcal{C}' = \Phi, \\ |\mathcal{C}'| \leq C \log n}} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}' | \mathcal{C}_a = \mathcal{C}) \mathbb{E}[I_{a'} | \mathcal{C}_{a'} = \mathcal{C}'] \\ &= (1 + \delta_2) \left( -\delta_3 + \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'} | \mathcal{C}_{a'} = \mathcal{C}'] \right) \\ (89) \quad &= \delta_4 + \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'} | \mathcal{C}_{a'} = \mathcal{C}'] \end{aligned}$$

for some  $0 \leq \delta_3 \leq 2\varepsilon$ , leading to  $3\varepsilon \leq -2\varepsilon(1 + \varepsilon) \leq \delta_4 \leq \varepsilon$ . Plugging Eq. (89) back into Eq. (82), we obtain

$$\begin{aligned}
 \mathbb{E}[I_{a'}|I_a = 1] &= \sum_{\substack{\mathcal{C} \ni s \text{ s.t.} \\ |\mathcal{C}| \leq C \log n}} \mathbb{P}(\mathcal{C}_a = \mathcal{C}|I_a = 1) \left( \delta_4 + \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'}|\mathcal{C}_{a'} = \mathcal{C}'] \right) \\
 (90) \quad &= \delta_5 + \left( \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'}|\mathcal{C}_{a'} = \mathcal{C}'] \right) \left( \sum_{\substack{\mathcal{C} \ni s \text{ s.t.} \\ |\mathcal{C}| \leq C \log n}} \mathbb{P}(\mathcal{C}_a = \mathcal{C}|I_a = 1) \right)
 \end{aligned}$$

where  $|\delta_5| \leq |\delta_4| \leq 3\varepsilon$ . The first term in the product is simply  $\pi' \equiv \mathbb{P}(I_{a'} = 1) = \mathbb{E}[I_{a'}]$ , as noted in Eq. (81). The second term in the product is  $\mathbb{P}(|\mathcal{C}_a| \leq C \log n | I_a = 1) \geq 1 - \mathbb{P}(\mathcal{E}_1 | I_a = 1) \geq 1 - \varepsilon/\pi$ , where again  $\pi = \mathbb{P}(I_a = 1) \geq 1/(C_1 k)$ . We deduce that

$$(91) \quad \mathbb{E}[I_{a'}|I_a = 1] = \delta_6 + \pi'$$

where  $|\delta_6| \leq \varepsilon(3 + 1/\pi)$  for large enough  $n$ . We have

$$\begin{aligned}
 \text{Cov}(I_a, I_{a'}) &= \mathbb{E}[I_a I_{a'}] - \mathbb{E}[I_a] \mathbb{E}[I_{a'}] \\
 &= \mathbb{E}[I_a] \mathbb{E}[I_{a'}|I_a = 1] - \pi \pi' \\
 &= \pi(\pi' + \delta_6) - \pi \pi' = \pi \delta_6
 \end{aligned}$$

Hence

$$|\text{Cov}(I_a, I_{a'})| \leq \varepsilon(3\pi + 1) \leq 4\varepsilon$$

Choosing  $\varepsilon = 1/(4k^3)$  yields the desired result.  $\square$

*Proof of Lemma 17.* To establish (68), note that for  $j = 0, 1, \dots, k-1$

$$(92) \quad \left| \mathbb{E}[X_j(t + 1/k)] - \hat{X}_j(t + 1/k) \right| = |\tilde{\gamma} - \gamma| X_{j-1}(t) \leq |\tilde{\gamma} - \gamma|/k \leq C/k^3,$$

with the final inequality coming from Lemma 16 Eq. (65). Meanwhile, using the fact that newly arrived applicants are matched with probability no greater than  $\alpha\eta/(rk)$  in time  $(1/k)$ , we have that

$$(93) \quad \left| \mathbb{E}[X_0(t + 1/k)] - \hat{X}_0(t + 1/k) \right| \leq (1/k) \max(\alpha\eta/(rk), 1 - \gamma) = \alpha\eta/(rk^2),$$

where we used Lemma 14 item 2. Summing (92) over  $j$  and combining it with (93) yields (68).

We now turn our attention to the variance term, i.e. (69). For  $j \geq 1$ ,

$$\begin{aligned}
 \text{Var}(X_j(t + 1/k)) &= \frac{1}{(rn)^2} \text{Var}\left(\sum_{s \in N_{j-1}(t)} I_a\right) \\
 &= \frac{1}{(rn)^2} \left( |N_{j-1}(t)| \text{Var}(I_a) + 2 \binom{|N_{j-1}(t)|}{2} \text{Cov}(I_a, I_{a'}) \right) \\
 &\leq X_{j-1}(t) \text{Var}(I_a)/rn + X_{j-1}(t)^2 \text{Cov}(I_a, I_{a'}). \\
 (94) \quad &\leq \frac{C}{rnk^2} + \frac{1}{k^5},
 \end{aligned}$$

where the final line comes from the fact that  $X_{j-1}(t) \leq 1/k$  and Lemma 16 Eqs. (66) and (67). By the concavity of the square root function, this implies

$$(95) \quad \sqrt{\text{Var}(X_j(t + 1/k))} \leq \sqrt{C}/(k\sqrt{rn}) + 1/k^{5/2}$$

Summing (95) over  $j$  yields (69).  $\square$

**C.3. Proofs of Propositions 4 and 5.** We require the following lemma.

**Lemma 18** (Le Cam's inequality). *Let  $X_1, \dots, X_n$  be Bernoulli random variables with  $\mathbb{P}(X_i = 1) = p_i$ . Let  $S_n = \sum_j X_j$ , and  $\lambda_n = \sum_i p_i$ . Then:*

$$(96) \quad \sum_{j \geq 0} \left| \mathbb{P}(S_n = j) - \frac{e^{-\lambda_n} \lambda_n^j}{j!} \right| \leq 2 \sum_{i=1}^n p_i^2.$$

The preceding inequality is a concentration result for the Poisson approximation to the binomial distribution; in particular it implies the following result.

**Corollary 1.** *Fix  $\zeta_0 < \infty$ . For any  $\zeta < \infty$ , as  $n \rightarrow \infty$ , we have that  $\text{Binomial}(n, \zeta/n)$  converges in total variation distance to  $\text{Poisson}(\zeta)$ . Further, this convergence is uniform over  $\zeta \in [0, \zeta_0]$ .*

*Proof.* Let  $p_i = \zeta/n$  for all  $i$  in Lemma 18; then (96) bounds the total variation distance between  $\text{Binomial}(n, \zeta/n)$  and  $\text{Poisson}(\zeta)$  by  $\zeta^2/n \leq \zeta_0^2/n \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

*Proof of Proposition 4.* We proceed as follows:

$$\begin{aligned}
& \sum_{\ell, a} \left| \mathbb{P}(R_b^{(n)} = \ell, A_b^{(n)} = a) - \mathbb{P}(R = \ell, A = a) \right| \\
&= \sum_{\ell, a} \left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) \mathbb{P}(R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell) \mathbb{P}(R = \ell) \right| \\
&\leq \sum_{\ell, a} \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) \left| \mathbb{P}(R_b^{(n)} = \ell) - \mathbb{P}(R = \ell) \right| \\
(97) \quad &+ \sum_{\ell, a} \mathbb{P}(R = \ell) \left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell) \right|.
\end{aligned}$$

Since the system begins in steady state, any employer arriving to the system is visible to  $rn$  applicants that each apply to the given employer with probability  $m/n$ ; i.e., the number of applications  $R_b^{(n)}$  received by such an employer  $e$  follows a Binomial( $rn, m/n$ ) distribution. Thus  $R_b^{(n)}$  converges in total variation distance to  $R$  by Corollary 1, so the first summation in (97) approaches zero as  $n \rightarrow \infty$  (since for every  $\ell$ ,  $\sum_a \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) = 1$ ).

We thus focus on the second summation in (97). Note that for all  $a$  such that  $0 \leq a \leq \ell$ ,  $\mathbb{P}(A = a | R = \ell) = \binom{\ell}{a} q^a (1-q)^{\ell-a}$ . To simplify notation, let  $t = t_b + 1$  denote the exit time of employer  $e$ .

Given  $\varepsilon > 0$ , let  $C_n(\varepsilon)$  be the event that  $\|X(t) - X^*\|_1 \leq \varepsilon$  in the  $n$ -th system. Recall that we realize the randomness as follows: at the time of exit, we independently determine whether each applicant that arrived in the last time unit applied to this employer. Thus in particular the number of applications that employer  $e$  receives is independent of the state  $X(t)$  at her exit time  $t$ , and so we conclude that  $R_b^{(n)}$  is independent of  $C_n(\varepsilon)$ . Letting  $k = \lfloor n^{1/3} \rfloor$ , from Theorem 6, for  $n$  sufficiently large it follows that:

$$\mathbb{P}(C_n(\varepsilon) | R_b^{(n)} = \ell) \geq 1 - \frac{C}{\varepsilon n^{1/6}}$$

for an appropriate constant  $C$ , by Markov's inequality. Note that on  $C_n(\varepsilon)$  it follows that  $|\Sigma(t) - \Sigma^*| \leq \varepsilon$  as well, since  $\Sigma(t) = \|X(t)\|_1$  and  $\Sigma^* = \|X^*\|_1$ .

Now again, since we realize applications at the time of exit of the employer, note that conditional on the value of  $S(t)$  as well as  $R_b^{(n)} = s$ , the employer receives exactly  $a$  applications from available applicants with the following probability:

$$(98) \quad \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell, S(t)) = \frac{\binom{S(t)}{a} \binom{rn-S(t)}{\ell-a}}{\binom{rn}{\ell}}.$$

For the moment, assume that  $0 < q < 1$ , and choose  $n$  large enough and  $\varepsilon$  small enough so that  $0 < \Sigma^* - \varepsilon < \Sigma^* + \varepsilon < 1$  (cf. Lemma 11). On the event  $C_n(\varepsilon)$ , we know that  $\Sigma^* - \varepsilon < \Sigma(t) < \Sigma^* + \varepsilon$ ; since  $S(t) = rn\Sigma(t)$ , we conclude that on  $C_n(\varepsilon)$  both  $S(t)$  and  $rn - S(t)$  are  $\Theta(n)$ . Thus for fixed  $\ell$  and  $a$ , we can approximate the preceding probability with the equivalent calculation as if the available and unavailable applicants were sampled with replacement. It follows that:

$$\left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell, C_n(\varepsilon)) - \binom{\ell}{a} (\Sigma^*)^a (1 - \Sigma^*)^{\ell-a} \right| \leq f(\varepsilon),$$

where  $f(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Finally, taking  $n$  large enough, we can assume that  $|\Sigma^* - q| \leq \varepsilon$ , from which it follows that:

$$\left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell, C_n(\varepsilon)) - \binom{\ell}{a} q^a (1 - q)^{\ell-a} \right| \leq \hat{f}(\varepsilon),$$

where  $\hat{f}(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Putting the steps together, we find that:

$$\begin{aligned} & |\mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell)| \\ & \leq \left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell, C_n(\varepsilon)) - \binom{\ell}{a} q^a (1 - q)^{\ell-a} \right| + |\mathbb{P}(C_n(\varepsilon)^c)| \\ & \leq \hat{f}(\varepsilon) + \frac{C}{\varepsilon n^{1/6}}. \end{aligned}$$

Take  $n \rightarrow \infty$ , then  $\varepsilon \rightarrow 0$  to conclude that:

$$\lim_{n \rightarrow \infty} \left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell) \right| = 0.$$

In the case where  $q = 0$  or  $q = 1$ , a direct analysis of (98) can be used to establish the preceding result; we omit the details.

Finally, to conclude the proof, note that for each  $\ell$ :

$$\sum_{a=0}^{\ell} |\mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell)|$$

converges to zero as  $n \rightarrow \infty$ . Further, the preceding quantity is bounded above by  $\ell + 1$ , which is integrable against the  $\text{Poisson}(rm)$  distribution; so by the dominated convergence theorem we conclude that the second term in (97) converges to zero as  $n \rightarrow \infty$ , as required.  $\square$

*Proof of Proposition 5.* The result is trivial for  $m_a = 0$  so we assume  $m_a > 0$  henceforth.

We start with a similar approach to the proof of Proposition 4, as follows:

$$\begin{aligned}
 & \sum_{\ell, a} |\mathbb{P}(T_a^{(n)} = \ell, Q_a^{(n)} = a) - \mathbb{P}(T = \ell, Q = a)| \\
 & \leq \sum_{\ell, a} \mathbb{P}(Q_a^{(n)} = a | T_a^{(n)} = \ell) |\mathbb{P}(T_a^{(n)} = \ell) - \mathbb{P}(T = \ell)| \\
 (99) \quad & + \sum_{\ell, a} \mathbb{P}(T = \ell) |\mathbb{P}(Q_a^{(n)} = a | T_a^{(n)} = \ell) - \mathbb{P}(Q = a | T = \ell)|.
 \end{aligned}$$

Since the system begins in steady state, any arriving applicant will find  $n$  employers in the system upon arrival, and applies to each of these employers independently with probability  $m_a/n$ ; i.e., the number of applications  $T_a^{(n)}$  sent by such an applicant  $a$  follows a Binomial( $n, m_a/n$ ) distribution. Thus  $T_a^{(n)}$  converges in total variation distance to  $T$  by Corollary 1, so the first summation in (99) approaches zero as  $n \rightarrow \infty$ , uniformly for all  $m_a \in (0, m_0]$ .

As before, therefore, we focus our attention on the second summation in (99). Note that for all  $a$  such that  $0 \leq a \leq \ell$ , we have  $\mathbb{P}(Q = a | T = \ell) = \binom{\ell}{a} p^a (1-p)^{\ell-a}$ .

First, we fix an upper bound  $L$  on the number of applications that we consider by applicant  $a$ . In particular, note that we have:

$$\begin{aligned}
 (100) \quad & \sum_{\ell > L, a} \mathbb{P}(T = \ell) |\mathbb{P}(Q_a^{(n)} = a | T_a^{(n)} = \ell) - \mathbb{P}(Q = a | T = \ell)| \leq \sum_{\ell \geq L} (\ell + 1) \mathbb{P}(T = \ell),
 \end{aligned}$$

and the last expression does not depend on  $n$  and goes to zero as  $L \rightarrow \infty$ , uniformly for all  $m_a \leq m_0$ . Thus it suffices to show that for fixed  $L$ , the sum:

$$(101) \quad \sum_{\ell \leq L, a} \mathbb{P}(T = \ell) |\mathbb{P}(Q_a^{(n)} = a | T_a^{(n)} = \ell) - \mathbb{P}(Q = a | T = \ell)|$$

goes to zero as  $n \rightarrow \infty$ , uniformly for all  $m_a \leq m_0$ .

Let  $\mathcal{I}(\ell)$  denote the set of all  $2^\ell$  possible outcome vectors  $\mathbf{I}$  that are possible when the applicant sends  $\ell$  applications. Note that (101) is less than or equal to:

$$(102) \quad \sum_{\ell \leq L, \mathbf{I} \in \mathcal{I}(\ell)} \mathbb{P}(T = \ell) |\mathbb{P}(\mathbf{I} | T_a^{(n)} = \ell) - p^{\sum_i I_i} (1-p)^{\ell - \sum_i I_i}|,$$

where we take advantage of the fact that  $Q$  is distributed as Binomial( $\ell, p$ ) when  $T = \ell$ . Since  $\ell \leq L$ , it suffices to show that for any  $\ell$  and each  $\mathbf{I} \in \mathcal{I}(\ell)$ , the quantity

$$(103) \quad |\mathbb{P}(\mathbf{I} | T_a^{(n)} = \ell) - p^{\sum_i I_i} (1-p)^{\ell - \sum_i I_i}|$$

goes to zero as  $n \rightarrow \infty$ . Now note that the preceding quantity is less than or equal to:

$$(104) \quad \int |\mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell, \mathbf{t}) - p^{\sum_i I_i} (1-p)^{\ell - \sum_i I_i}| d\mathbb{P}_n(\mathbf{t}|T_a^{(n)} = \ell).$$

Here the integral is over all possible vectors of departure times for employers to whom the applicant submitted an application. Note that  $\mathbf{t}$  has an atomic distribution that varies with  $n$ , so the integral reduces to a sum over feasible  $\mathbf{t}$ . Note also that this quantity does not depend on  $m_a$  at all (since we are conditioning on applicant  $a$  making  $\ell$  applications). In fact, it turns out that we do not need to consider  $m_a$  for the rest of the proof.

We argue as follows. Fix  $k = \lfloor n^{1/3} \rfloor$ . When  $T_a^{(n)} = \ell$ , we use  $b_1, \dots, b_\ell$  to denote the employers that  $a$  applied to, and without loss of generality we let  $t_1 \leq t_2 \leq \dots \leq t_\ell$  denote the departure times of these employers, and let  $\mathbf{t}$  denote the vector of departure times. We define “rounded” departure times by rounding the true departure times up to the nearest multiple of  $1/k$ ; denote these as  $t'_i = \lceil kt_i \rceil / k$  for  $i = 1, 2, \dots, \ell$ , and let  $\mathbf{t}'$  denote the vector of rounded departure times. We use the rounded departure times so that we can apply the analysis that yields (51); that result applies to the “binned” process of available applicants, binned on time increments of length  $1/k$ .

In our analysis, we make use of a coupled process of available applicants, but with a particular employer  $b_i$  and applicant  $a$  removed during  $(t'_{i-1}, t'_i]$  (with  $t'_0 = 0$ ). Let  $S^{-i}(t)$  denote the state of this process at time  $t$  for  $t \leq t'_i$ . We couple this process to the original process so that each employer (besides  $b_i$ ) receives the same set of applications, and screens them in the same order (or does not screen at all in both systems). So  $S^{-i}(t)$  is identical to  $S(t)$  for  $t \leq t'_{i-1}$ , and further for  $t \in (t'_{i-1}, t'_i)$  except for the removal of  $a$ . The states are further nearly identical until  $t'_i$  if none of the applicants who applied to  $b_i$  applied to another employer who departed in  $(t_i, t'_i)$  (the event  $E_n(i+1)$  defined below): in this case the systems can differ only on applicant  $a$  and any applicant who received an offer from  $b_i$ .

We also use  $I_i$  to denote the outcome on the  $i$ 'th application; i.e.,  $I_i = 1$  if applicant  $a$  receives an offer from employer  $b_i$ , and  $I_i = 0$  otherwise. Let  $\mathbf{I}$  denote the vector of application outcomes. Fix  $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_\ell > 0$ . Let  $C_n(0)$  be the event that the  $\mathbf{t}'$  are all distinct. Let  $D_n(J)$  be the event that no more than  $2mrn/k$  distinct applicants apply to employers who depart during  $(t_J, t'_J)$ ; this will hold with high probability for large  $n$ . Define  $C_n(1)$  be the event that  $C_n(0)$  occurs; and in the  $S^{-1}(\cdot)$  process,  $D_n(1)$  occurs and  $\|X(t'_1) - X^*\|_1 \leq \varepsilon_1$ . Let  $C_n(J)$  for  $J > 1$  be iteratively defined as the event



that  $C_n(J-1)$  occurs; and in the  $S^{-J}(\cdot)$  process,  $D_n(J)$  occurs,  $\|X(t'_J) - X^*\|_1 \leq \varepsilon_J$ , and no applicant applies to both  $b_{J-1}$  and some other employer who departs in the interval  $(t_{J-1}, t'_{J-1}]$ . We remark here that we consider the modified process  $S^{-i}(\cdot)$ , and employ event  $D_n(J)$  in the definition of  $C_n(J)$  for convenience; these can be viewed as minor technical details.

In what follows, to economize on notation for a vector  $\mathbf{x}$ , we let  $\mathbf{x}_{i:j} = (x_i, x_{i+1}, \dots, x_j)$ .

We proceed as follows. First, we condition on  $\mathbf{I}_{1:i-1}$  and  $\Sigma(t_i)$ , and consider the probability employer  $b_i$  makes applicant  $a$  an offer. Recall that the employer only learns compatibility but not availability by screening;  $a$  receives an offer from  $b_i$  if and only if she is screened by employer  $b_i$  and is also compatible.

Given that the employer follows  $\phi^\alpha$ ,  $\Sigma(t_i)$  and  $\mathbf{I}_{1:i-1}$  form a sufficient statistic to determine whether employer  $i$  screens  $a$ , as follows. Let  $d$  denote the number of competing applicants that employer  $i$  receives. Applicant  $a$  receives an offer from employer  $b_i$  if she is compatible with  $b_i$ , and no compatible, available applicant is screened by employer  $b_i$  before  $a$ . The scaled number of available applicants besides  $a$  at time  $t_i$  is  $\Sigma(t_i)$  (possibly with a  $O(1/n)$  adjustment if applicant  $a$  is also available; we skip this minor detail). It follows that the probability  $a$  receives an offer is:

$$\beta \sum_{d=0}^{rn\Sigma(t_i)} \binom{rn\Sigma(t_i)}{d} \left(\frac{m}{n}\right)^d \left(1 - \frac{m}{n}\right)^{rn\Sigma(t_i)-d} \left\{ \frac{\alpha}{d+1} \sum_{j=0}^d (1-\beta)^j \right\}.$$

The expression in brackets simplifies to:

$$\frac{\alpha}{d+1} \cdot \frac{1 - (1-\beta)^{d+1}}{\beta},$$

so the probability that  $a$  receives an offer becomes:

$$(105) \quad \sum_{d=0}^{rn\Sigma(t_i)} \text{Binomial}\left(rn\Sigma(t_i), \frac{m}{n}\right)_d \left\{ \frac{\alpha(1 - (1-\beta)^{d+1})}{d+1} \right\}.$$

At this point we recall the derivation of (5); from that calculation it follows that:

$$(106) \quad \sum_{d=0}^{\infty} \text{Poisson}(rm\Sigma)_d \cdot \frac{\alpha(1 - (1-\beta)^{d+1})}{d+1} = \frac{\alpha(1 - e^{-rm\beta\Sigma})}{rm\Sigma} = \alpha\beta g(rm\beta\Sigma),$$

where  $g(x) = (1 - e^{-x})/x$ . Further, observe that if  $\Sigma = q$  in the preceding expression, then (5) implies the right hand side of the preceding expression is equal to  $p$ .

We proceed by showing that on  $C_n(i)$ , (105) is well approximated by (106) with  $\Sigma = q$ . This requires three steps: first, showing that the binomial distribution in

(105) is well approximated by a Poisson distribution; second, exploiting the fact that  $\Sigma(t_i)$  is close to  $\Sigma^*$ ; and third, taking  $n$  large enough so that  $\Sigma^*$  is close to  $q$ .

We proceed as follows. Let  $h(d) = \alpha(1 - (1 - \beta)^{d+1})/(d+1)$ . Note that  $0 \leq h(d) \leq 1$  for all  $d$ . Further, note that we have:

$$\begin{aligned} & \left| \left( \sum_{d=0}^{rn\Sigma(t_i)} h(d) \text{Binomial} \left( rn\Sigma(t_i), \frac{m}{n} \right)_d \right) - p \right| \\ & \leq \sum_{d=0}^{\infty} h(d) \left| \text{Binomial} \left( rn\Sigma(t_i), \frac{m}{n} \right)_d - \text{Poisson}(rm\Sigma(t_i))_d \right| \\ & \quad + \left| \sum_{d=0}^{\infty} h(d) (\text{Poisson}(rm\Sigma(t_i))_d - \text{Poisson}(rm\Sigma^*)_d) \right| \\ & \quad + \left| \sum_{d=0}^{\infty} h(d) (\text{Poisson}(rm\Sigma^*)_d - \text{Poisson}(rmq)_d) \right|. \end{aligned}$$

By Lemma 18, the first summation on the right is bounded above by a  $K/n$  for some constant  $K$ . We now use (106) to simplify the second and third summations. The second summation reduces to  $|\alpha\beta(g(rm\beta\Sigma(t_i)) - g(rm\beta\Sigma^*))|$ , which for large enough  $n$  on  $C_n(i)$ , using that  $|\Sigma(t_i) - \Sigma(t'_i)| = O(t'_i - t_i) = O(1/k)$ , is bounded above by  $f(\varepsilon_i + 1/k)$  for some  $f$  such that  $f(x) \rightarrow 0$  as  $x \rightarrow 0$ . Finally, the third summation above reduces to  $|\alpha\beta(g(rm\beta\Sigma^*) - g(rm\beta q))|$ , which can be made less than or equal to  $\varepsilon_i$  for  $n$  large enough. Summarizing then, by taking  $n$  large enough, we can ensure that on  $C_n(i)$  we have:

$$\left| \left( \sum_{d=0}^{rn\Sigma(t_i)} h(d) \text{Binomial} \left( rn\Sigma(t_i), \frac{m}{n} \right)_d \right) - p \right| \leq \hat{f}(\varepsilon_i + 1/k),$$

for some  $\hat{f}$  such that  $\hat{f}(\varepsilon_i) \rightarrow 0$  as  $\varepsilon_i \rightarrow 0$ .

Analogously, in the case where employer  $i$  does not make an offer to applicant  $a$ , it immediately follows that  $\mathbb{P}(I_i = 0 | T_a^{(n)} = \ell, \mathbf{t}, \mathbf{I}_{1:i-1}, C_n(i))$  is well approximated by  $1 - p$ .

To simplify notation, let  $A_{\ell, \mathbf{t}}^{(n)}$  be the event that  $T_a^{(n)} = \ell$  and the vector of employer departure times this applicant applied to is  $\mathbf{t}$ . For now, consider only those  $A_{\ell, \mathbf{t}}^{(n)}$  such that  $C_n(0)$  holds (this occurs with high probability, cf. Eq. (114) below). To summarize then, we conclude that by taking  $n$  large enough, we have:

$$(107) \quad \left| \mathbb{P}(I_i | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i)) - p^{I_i} (1 - p)^{1-I_i} \right| \leq \hat{f}(\varepsilon_i + 1/k).$$

Next, we consider the conditional probability  $\mathbb{P}(C_n(i)|A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1))$ . As a preliminary step, note that  $\mathbb{P}(D_n(i)|A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1)) \leq 1 - \exp(-\Theta(n/k))$  by the Chernoff bound, since each of  $(1 + 1/k)rn$  applicants has, independently, applied to some employer who departs in  $(t_i, t'_i)$  with probability no more than  $(m/n)(n/k) = m/k$ , so the likelihood that over  $2mrn/k$  distinct applicants have applied to this set of employers is  $\exp(-\Theta(n/k))$ . Also, let  $E_n(i)$  be the event that none of the applicants who applied to an employer who departs in  $(t_{i-1}, t'_{i-1})$  also applied to employer  $b_i$ . By the union bound, we have  $\mathbb{P}(E_n(i)|A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1), D_n(i)) \leq (m/n)(2mrn/k) = 2m^2/k$  under  $D_n(i-1)$ . It is straightforward to check that this bound remains  $O(1/k)$  even if we further condition on  $I_{i-1}$ , because the conditional probability  $\mathbb{P}(I_{i-1}|A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1), D_n(i))$  is bounded away from zero and one for sufficiently large  $n$ .

We now rely on (51) to control  $\mathbb{P}(C_n(i)|A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1))$ . Now observe that on  $C_n(i-1)$ , we know that  $\|X(t'_{i-1}) - X^*\|_1 \leq \varepsilon_{i-1}$  for the  $S^{-(i-1)}(\cdot)$  process. To iterate our argument to the departure of the  $i$ 'th employer, we need to control the state  $X(t'_i)$  in the process  $S^{-i}(\cdot)$ , i.e., for the system where employer  $b_i$  and applicant  $a$  are removed during  $(t'_{i-1}, t'_i)$  (but employer  $b_{i-1}$  is included). For this, we only need to make an adjustment to  $X(t'_{i-1})$  if employer  $b_{i-1}$  matched to an applicant  $a'$  other than  $a$  in the original system. If this indeed happened, then, on  $E_n(i)$ , we only need to adjust one coordinate of  $X(t'_{i-1})$  downward by  $1/n$  corresponding to  $a'$  being unavailable in  $S^{-i}(t'_{i-1})$ , and this small adjustment will not affect our analysis (we omit this detail below). For  $n$  large, it follows that an analogous result to (51) holds for the evolution of  $X(t)$  between  $t'_i$  and  $t'_{i-1}$ , from which we can conclude that for large enough  $n$  there holds:

$$\begin{aligned} \mathbb{E}[\|X(t'_i) - X^*\|_1 | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1), E_n(i)] \\ \leq \varepsilon_{i-1} + \frac{C'}{\kappa'} \frac{1}{n^{1/6}} + O(1/n). \end{aligned}$$

The first term follows from the contraction term in (51); the second term follows because  $k = \lfloor n^{1/3} \rfloor$ . By Markov's inequality we obtain:

$$(108) \quad \mathbb{P}(\|X(t'_i) - X^*\|_1 < \varepsilon_i | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1), E_n(i)) \geq 1 - \frac{\varepsilon_{i-1}}{\varepsilon_i} - \frac{2C'}{\kappa' \varepsilon_i} \frac{1}{n^{1/6}}.$$

Combining, we obtain

$$\begin{aligned}
& \mathbb{P}(C_n(i) | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1)) \\
& \geq \mathbb{P}(C_n(i) | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1), E_n(i)) (1 - \mathbb{P}(E_n(i)^c)) \\
& \geq \mathbb{P}(C_n(i) | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1), E_n(i)) - \mathbb{P}(E_n(i)^c) \\
(109) \quad & \geq 1 - \frac{\varepsilon_{i-1}}{\varepsilon_i} - \frac{3C'}{\kappa' \varepsilon_i} \frac{1}{n^{1/6}}
\end{aligned}$$

for large enough  $n$ . This bound applies for any  $i > 1$ . For  $i = 1$ —the first employer that applicant  $a$  applies to—we can directly apply Theorem 6 together with Markov's inequality to conclude that for sufficiently large  $n$ , we have:

$$(110) \quad \mathbb{P}(C_n(1) | A_{\ell, \mathbf{t}}^{(n)}, C_n(0)) \geq 1 - \frac{2}{\varepsilon_1 n^{1/6}}.$$

Observe that by the definition of conditional probability, together with the fact that  $C_n(i) \subset C_n(i-1)$ , we have:

$$\begin{aligned}
& \mathbb{P}(\mathbf{I}_{i:\ell} | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1)) \\
& = \mathbb{P}(\mathbf{I}_{i:\ell} | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i)) \cdot \mathbb{P}(C_n(i) | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1)) \\
(111) \quad & + \mathbb{P}(\mathbf{I}_{i:\ell} | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i)^c, C_n(i-1)) \cdot \mathbb{P}(C_n(i)^c | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1)).
\end{aligned}$$

If we consider the first term in the expansion above, we have again by the definition of conditional probability that:

$$\begin{aligned}
& \mathbb{P}(\mathbf{I}_{i:\ell} | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i)) \\
(112) \quad & = \mathbb{P}(\mathbf{I}_{i+1:\ell} | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i}, C_n(i)) \cdot \mathbb{P}(I_i | A_{\ell, \mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i)).
\end{aligned}$$

The last piece we need is the following simple bound on the difference of two products, where  $0 \leq a_i, b_i \leq 1$ :

$$(113) \quad |a_1 b_1 - a_2 b_2| \leq |a_1 - a_2| + |b_1 - b_2|.$$

We now return to our overall goal: namely, we wish to show that the absolute value

$$|\mathbb{P}(\mathbf{I} | A_{\ell, \mathbf{t}}^{(n)}) - p^{\sum_i I_i} (1-p)^{\ell - \sum_i I_i}|$$

becomes small for  $A_{\ell, \mathbf{t}}^{(n)}$  such that  $C_n(0)$  holds. To show this, we iterate and combine (107), (109), (110), (111), (112), and (113). For example, after the first step of this

iteration, we obtain that:

$$\begin{aligned}
|\mathbb{P}(\mathbf{I}|A_{\ell,\mathbf{t}}^{(n)}) - p^{\sum_i I_i} (1-p)^{\ell-\sum_i I_i}| &= |\mathbb{P}(\mathbf{I}|A_{\ell,\mathbf{t}}^{(n)}, C_n(1))\mathbb{P}(C_n(1)|A_{\ell,\mathbf{t}}^{(n)}) \\
&\quad + \mathbb{P}(\mathbf{I}|A_{\ell,\mathbf{t}}^{(n)}, C_n(1))\mathbb{P}(C_n(1)^c|A_{\ell,\mathbf{t}}^{(n)}) \\
&\quad - p^{\sum_i I_i} (1-p)^{\ell-\sum_i I_i} (\mathbb{P}(C_n(1)|A_{\ell,\mathbf{t}}^{(n)}) + \mathbb{P}(C_n(1)^c|A_{\ell,\mathbf{t}}^{(n)}))| \\
&\leq |\mathbb{P}(\mathbf{I}_{2:\ell}|A_{\ell,\mathbf{t}}^{(n)}, I_1, C_n(1))\mathbb{P}(I_1|A_{\ell,\mathbf{t}}^{(n)}, C_n(1)) \\
&\quad - p^{\sum_i I_i} (1-p)^{\ell-\sum_i I_i}| + \mathbb{P}(C_n(1)^c|A_{\ell,\mathbf{t}}^{(n)}) \\
&\leq |\mathbb{P}(\mathbf{I}_{2:\ell}|A_{\ell,\mathbf{t}}^{(n)}, I_1, C_n(1)) - p^{\sum_{i>1} I_i} (1-p)^{\ell-\sum_{i>1} I_i}| \\
&\quad + \hat{f}(\varepsilon_1 + 1/n^{1/3}) + O(n^{-1/6}).
\end{aligned}$$

The first step follows by conditioning, i.e., (111); the second step follows by (112), together with the fact that the absolute value of the difference of two probabilities is bounded above by one; and the third step follows by applying (113) to the absolute value, and then using (107) and (110). Continuing in this manner, we obtain:

$$\begin{aligned}
|\mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell, \mathbf{t} \text{ s.t. } C_n(0)) - p^{\sum_i I_i} (1-p)^{\ell-\sum_i I_i}| \\
\leq \sum_i \hat{f}(\varepsilon_i + 1/n^{1/3}) + \sum_{i=2}^{\ell} \frac{\varepsilon_{i-1}}{\varepsilon_i} + O(n^{-1/6}).
\end{aligned}$$

We know that

$$(114) \quad \mathbb{P}(C_n(0)^c|T_a^{(n)} = \ell) \leq L^2/k = O(n^{-1/3})$$

using an elementary union bound over events that two rounded departure times are identical. To complete the proof, returning to (104), and recalling the integral is in fact a sum over feasible  $\mathbf{t}$ , we deduce that:

$$\begin{aligned}
&\int |\mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell, \mathbf{t}) - p^{\sum_i I_i} (1-p)^{\ell-\sum_i I_i}| d\mathbb{P}_n(\mathbf{t}|T_a^{(n)} = \ell) \\
&\leq \sum_i \hat{f}(\varepsilon_i + 1/n^{1/3}) + \sum_{i=2}^{\ell} \frac{\varepsilon_{i-1}}{\varepsilon_i} + O(n^{-1/6}) + \mathbb{P}(C_n(0)^c|T_a^{(n)} = \ell) \\
&= \sum_i \hat{f}(\varepsilon_i + 1/n^{1/3}) + \sum_{i=2}^{\ell} \frac{\varepsilon_{i-1}}{\varepsilon_i} + O(n^{-1/6}).
\end{aligned}$$

Note that  $\varepsilon_1, \dots, \varepsilon_\ell$  were arbitrary; thus if we first take  $n \rightarrow \infty$ , and then take  $\varepsilon_i \rightarrow 0$  in such a way that every ratio  $\varepsilon_{i-1}/\varepsilon_i \rightarrow 0$  as well, then we conclude that the right hand side of the preceding expression approaches zero as  $n \rightarrow \infty$ . Returning to

(103), we conclude that for each  $\ell$  and  $\mathbf{I} \in \mathcal{I}(\ell)$  we have:

$$|\mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell) - p^{\sum_i I_i} (1-p)^{\ell - \sum_i I_i}| \rightarrow 0$$

as  $n \rightarrow \infty$ , as required.  $\square$

**C.4. Proof of Theorem 2.** We give a proof of Theorem 2, using Propositions 4 and 5.

*Proof of Theorem 2.* We first prove the result for applicants. Consider an applicant  $a$  who can choose the value of  $m_a$ , while other agents play their prescribed mean field strategies. Clearly, choosing  $m_a = 0$  leads to a utility of 0, whereas choosing  $m_a > 1/c_a$  leads to a negative expected utility since the expected application cost exceeds 1. (In particular, note that  $m^* < 1/c_a$ .) Hence, it suffices to show that:

**Claim 2.** *For large enough  $n$ , playing  $m_a = m^*$  is additively  $\varepsilon$ -optimal for applicant  $a$  among  $m_a \in [0, 1/c_a]$ .*

(Since  $m_a = 0$  has greater utility than any  $m_a > 1/c_a$ , this will imply that  $m_a = m^*$  is additively  $\varepsilon$ -optimal among  $m_a \in [0, \infty)$ .)

We prove the claim using Proposition 5 with  $m_0 = 1/c_a$ . Now, the expected utility of applicant  $a$  who chooses  $m_a$  is

$$\Pr(\text{Applicant } a \text{ gets at least one offer under } m_a) - c_a m_a.$$

Since  $m^*$  is a best response under mean field assumptions, it follows that

$$(115) \quad 1 \cdot \Pr(Q(m_a) > 0) - c_a m_a \leq \Pr(Q(m^*) > 0) - c_a m^*$$

for any  $m_a$ . Proposition 5 implies that there exists  $n_0$  such that for any  $n > n_0$ , we have

$$\max_{m_a \in [0, m_0]} d_{\text{TV}}((T_a^{(n)}(m_a), Q_a^{(n)}(m_a)), (T(m_a), Q(m_a))) \leq \varepsilon/2.$$

where we have made the dependence on  $m_a$  explicit for convenience. It follows that

$$|\Pr(Q(m_a) > 0) - \Pr(Q_a^{(n)}(m_a) > 0)| \leq \varepsilon/2 \quad \forall m_a \in [0, m_0].$$

Combining with Eq. (115), we obtain that

$$\Pr(Q_a^{(n)}(m_a) > 0) - c_a m_a \leq \Pr(Q_a^{(n)}(m^*) > 0) - c_a m^* + \varepsilon \quad \forall m_a \in [0, m_0],$$

implying the claim.

We now prove the result for employers who receive no more than  $R_0$  applications. Think of employer  $e$  as first selecting a uniformly random order among her applicants,

and then screening them (or not) in that order. With this order, denote the applicants by  $s_1, s_2, \dots, s_\ell$  (here  $\ell$  denotes the number of applications) and let  $\mathcal{S}$  denote the set of all applicants. Let  $\mathbf{I}_b^{(n)}$  denote the applicant availability vector at the time that  $e$  departs, with this order. (So  $\mathbf{I}_b^{(n)} \in \{0, 1\}^\ell$ , where a 1 represents that the corresponding applicant is available.) Let  $\mathbf{I}$  denote an analogously defined vector for a hypothetical employer who receives  $R \sim \text{Poisson}(rm)$  applications of which  $A \sim \text{Binomial}(R, q)$  are from applicants who are still available. It follows from Proposition 4, and the fact that applicants are considered in a uniformly random order, that:

**Claim 3.**  $(R_b^{(n)}, A_b^{(n)}, \mathbf{I}_b^{(n)})$  converges in total variation distance to  $(R, A, \mathbf{I})$ .

Let  $\mathcal{S}_i = \{s_j : j \leq i\}$ . Let  $\mathcal{S}'_i$  denote the subset of  $\mathcal{S}_i$  that  $e$  makes offers to. (So, for  $i < \ell$ , employer  $e$  does not screen  $s_{i+1}$  if one of  $\mathcal{S}'_{i'}$  accepts an offer.) The next claim follows from Claim 3 above.

**Claim 4.** Fix  $\varepsilon' > 0$  and  $R_0 < \infty$ . There exists  $n_1 < \infty$  such that the following holds. For any non-negative integer  $\ell \leq R_0$ , any  $i < \ell$  and any subset  $\mathcal{S}'_i \subseteq \mathcal{S}_i$  with fixed indices in  $\{1, 2, \dots, i\}$  we have

$$(116) \quad \left| \Pr(s_{i+1} \text{ is available} \mid |\mathcal{S}| = \ell, \mathcal{S}'_i = \mathcal{S}'_i, \mathcal{S}'_i \text{ are not available}) - q \right| \leq \varepsilon'.$$

The prescribed mean field strategy for employer  $e$  is  $\phi^{\alpha^*}$ . For  $\alpha^* < 1$ , we argue that not screening at all is additively  $\varepsilon$  optimal (for large enough  $n$ ): First, note that  $c'_s \geq \beta q$ , since not screening is a best response under the employer mean field assumption. Then we know, from Claim 4, that each time the employer chooses to screen an applicant, the net expected benefit is no more than  $\beta\varepsilon' \leq \varepsilon'$ . Using an elementary martingale stopping argument, it follows that the overall net utility of any other strategy (relative to the utility of 0 obtained by not screening) is no more than  $\mathbb{E}[\bar{R}\varepsilon'] \leq R_0\varepsilon'$ , where  $\bar{R}$  is the number of applicants screened under  $\phi^1$ , using  $\bar{R} \leq \ell \leq R_0$ . Choosing  $\varepsilon' \leq \varepsilon/R_0$  we obtain the desired result.

Similarly for  $\alpha^* > 0$ , we argue that  $\phi^1$  (keep screening until you hire or run out of applicants) is additively  $\varepsilon$ -optimal (for large enough  $n$ ): First, note that  $c'_s \leq \beta q$ , since screening is a best response under the employer mean field assumption. Then we know, from Claim 4, that each time the employer chooses to screen the next applicant, the net expected benefit is at least  $-\beta\varepsilon' \geq -\varepsilon'$ , irrespective of the realization of  $\mathcal{S}'_i$  (all these applicants rejected their offers). Any other strategy can be better than  $\phi^1$  by at most  $\varepsilon'$  in expectation for each applicant screened under  $\phi^1$  but not under the

alternative strategy Using an elementary martingale stopping argument, it follows that  $\phi^1$  is additively  $\mathbb{E}[\underline{R}]\varepsilon' \leq R_0\varepsilon'$  optimal, where  $\underline{R}$  is the number of applicants screened under  $\phi^1$ , but not under the alternative strategy. Choosing  $\varepsilon' \leq \varepsilon/R_0$  we obtain the desired result.

Finally, since  $\phi^0$  is  $\varepsilon$ -optimal for  $\alpha^* < 1$  and  $\phi^1$  is  $\varepsilon$ -optimal for  $\alpha^* > 0$ , we conclude that  $\phi^{\alpha^*}$  is  $\varepsilon$ -optimal, irrespective of the value of  $\alpha^*$ , for large enough  $n$ .  $\square$

#### APPENDIX D. PROOFS: SECTION 6

Consider an applicant in the mean field environment, where each application yields an offer with probability  $p$ . If the applicant chooses to send an average of  $m$  applications, they match with probability  $1 - e^{-mp}$  and incur expected application costs of  $c_a m$ . Thus, we define

$$(117) \quad \Pi_a(m, \alpha) = 1 - e^{-m\mathcal{P}(m, \alpha)} - c_a m.$$

Similarly, we can consider an employer in the mean field environment, where each applicant is available with probability  $q$ . If this employer chooses to screen, they match if and only if they have a qualified available applicant (which occurs with probability  $1 - e^{-rm\beta q}$ ). We claim that the expected number of applicants screened is equal to the probability that the employer matches, divided by  $q\beta$ .<sup>19</sup> Thus, we define

$$(118) \quad \Pi_e(m, \alpha) = \alpha(1 - e^{-rm\beta\mathcal{Q}(m, \alpha)})(1 - c'_s/\mathcal{Q}(m, \alpha)).$$

Note that in the expression for  $\Pi_a$ , both  $m$  and  $m\mathcal{P}(m, \alpha)$  appear. In the expression for  $\Pi_e$ , both  $\mathcal{Q}(m, \alpha)$  and  $m\mathcal{Q}(m, \alpha)$  appear. Because  $\mathcal{P}(m, \alpha)$  and  $\mathcal{Q}(m, \alpha)$  have no closed form, directly analyzing the expressions in (117) and (118) is difficult. For this reason, in Proposition 7, we re-express  $\Pi_a$  and  $\Pi_e$  as functions of only the model parameters  $(r, c'_a, c'_s)$ , the value  $\alpha$  selected by employers, and the quantity  $m\mathcal{P}(m, \alpha)$ .

**Proposition 7.** *For any  $m$  and  $\alpha$ , we have*

$$(119) \quad \Pi_a(m, \alpha) = 1 - e^{-m\mathcal{P}(m, \alpha)} - c'_a m\mathcal{P}(m, \alpha)/(\alpha g(-\log(1 - \frac{r}{\alpha}(1 - e^{-m\mathcal{P}(m, \alpha)}))))$$

$$(120) \quad \Pi_e(m, \alpha) = r(1 - e^{-m\mathcal{P}(m, \alpha)} - c'_s m\mathcal{P}(m, \alpha)).$$

<sup>19</sup>To see this, let  $X$  be geometric with parameter  $\beta q$  (this represents the number of applicants that would need to be screened before matching), and let  $Y$  be Poisson with parameter  $rm$  (this represents the number of applications received). Then employers who choose to screen expect to screen  $\mathbb{E}[\min\{X, Y\}] = \mathbb{E}[X] - \mathbb{E}[(X - Y)1_{X > Y}] = \mathbb{E}[X](1 - \mathbb{P}(X > Y))$  applicants, where we use the memoryless property of the geometric distribution in the second equality. But  $1 - \mathbb{P}(X > Y)$  is the probability that a match is found (conditioned on deciding to screen).



Furthermore,

$$(121) \quad \Pi_a^* = 1 - (1 + m^* p^*) e^{-m^* p^*}.$$

$$(122) \quad \Pi_e^* = r(1 - e^{-m^* p^*} - c'_s m^* p^*).$$

*Proof.* Recall that the mean field equations (4) and (5) imply that  $\mathcal{P}(m, \alpha)$  and  $\mathcal{Q}(m, \alpha)$  solve the following equations

$$(123) \quad rm\mathcal{P}(m, \alpha)\mathcal{Q}(m, \alpha) = \alpha(1 - e^{-rm\beta\mathcal{Q}(m, \alpha)}) = r(1 - e^{-m\mathcal{P}(m, \alpha)}).$$

Applying this to (118) yields (120), from which (122) follows immediately.

To get to (119), we note that  $c_a m = (\alpha\beta)c'_a m/\alpha$ . The mean-field equation (5) for  $\mathcal{P}(m, \alpha)$  implies that

$$\alpha\beta = \mathcal{P}(m, \alpha)/g(rm\beta\mathcal{Q}(m, \alpha)) = \mathcal{P}(m, \alpha)/g(-\log(1 - \frac{r}{\alpha}(1 - e^{-m\mathcal{P}(m, \alpha)}))),$$

where the final equality follows from solving (123) for  $rm\beta\mathcal{Q}(m, \alpha)$ . Combining these facts and substituting into (117) yields (119). We obtain (121) by applying (2) (which gives the applicant's best response as a function of  $p$ ) to (119).  $\square$

From Proposition 7, we have the following corollary.

**Proposition 8.** *For any  $\alpha < 1$  and any  $m > 0$ , there exists  $m' < m$  such that  $\Pi_a(m', 1) > \Pi_a(m, \alpha)$  and  $\Pi_e(m', 1) = \Pi_e(m, \alpha)$ .*

*Proof.* Note that by Lemma 3,  $\mathcal{Q}(m, 1) < \mathcal{Q}(m, \alpha) \leq \mathcal{Q}(0, \alpha) = 1$ , and furthermore  $\mathcal{Q}(\cdot, 1)$  is continuously decreasing. It follows that for some  $m' < m$ ,  $\mathcal{Q}(m', 1) = \mathcal{Q}(m, \alpha)$ . Because the mean-field consistency equation (4) states that  $\mathcal{Q} = g(m\mathcal{P})$ , it follows that  $m'\mathcal{P}(m', 1) = m\mathcal{P}(m, \alpha)$ . This fact, combined with (120) from Proposition 7, implies that  $\Pi_e(m', 1) = \Pi_e(m, \alpha)$ . Furthermore, applying (117) from Proposition 7, we see that

$$\Pi_a(m', 1) = 1 - e^{-m'\mathcal{P}(m', 1)} - c_a m' = 1 - e^{-m\mathcal{P}(m, \alpha)} - c'_a m' > 1 - e^{-m\mathcal{P}(m, \alpha)} - c_a m = \Pi_a(m, \alpha).$$

$\square$

Proposition 8 states that the Pareto frontier of  $(\Pi_e, \Pi_a)$  consists only of points where  $\alpha = 1$ . When  $\alpha = 1$ , Proposition 7 gives  $\Pi_a$  and  $\Pi_e$  as functions of only  $r, c'_a, c'_s$  and the quantity  $m\mathcal{P}(m, 1)$ . Motivated by this, for fixed  $r, c'_a, c'_s$  we define

$$(124) \quad SW(x) = 1 - e^{-x} - c'_a x / g(-\log(1 - r(1 - e^{-x})))$$

$$(125) \quad BW(x) = r(1 - e^{-x} - c'_s x)$$

Note that  $\Pi_a(m, 1) = SW(m\mathcal{P}(m, 1))$ , and  $\Pi_e(m, 1) = BW(m\mathcal{P}(m, 1))$ . Thus, we have reduced the problem of optimizing  $\Pi_a$  and  $\Pi_e$  to that of optimizing  $SW$  and  $BW$  over the set of values that the quantity  $m\mathcal{P}(m, 1)$  may attain.<sup>20</sup>

**Lemma 19.** *For any  $r, c'_a, c'_s$ , the functions  $\Pi_a(\cdot, 1)$ ,  $\Pi_e(\cdot, 1)$ ,  $SW(\cdot)$ , and  $BW(\cdot)$  are unimodal.  $\Pi_a$  and  $SW$  have a unique local maximum, and  $\Pi_e$  and  $BW$  either have a unique local maximum or are strictly increasing.*

*Proof.* Proposition 7 establishes that  $\Pi_e(m, 1) = BW(m\mathcal{P}(m, 1))$ , and  $\Pi_a(m, 1) = SW(m\mathcal{P}(m, 1))$ . Lemma 3 states that  $\mathcal{Q}(m, 1) = g(m\mathcal{P}(m, 1))$  is decreasing in  $m$ , implying that  $m\mathcal{P}(m, 1)$  is increasing in  $m$  (since  $g(\cdot)$  is decreasing). Thus, the unimodality of  $\Pi_a$  and  $\Pi_e$  follows from the unimodality of  $SW$  and  $BW$ .

It is straightforward to show that  $BW$  is concave. Thus, all that remains is to prove that for all  $r > 0$  and  $c'_a \in (0, 1)$ ,  $SW$  is unimodal. Because  $SW$  has a continuous first derivative,  $SW'(0) = 1 - c'_a > 0$ , and  $SW$  is negative for sufficiently large  $x$ , it suffices to show that there is a unique solution to  $SW'(x) = 0$ .

For the purposes of this proof, for fixed  $r$  we define  $h(x) = -\log(1 - r(1 - e^{-x}))$ ,  $b(x) = g(h(x))$ , and  $u(x) = x/b(x)$ , so that  $SW(x) = 1 - e^{-x} - c'_a u(x)$ . Then

$$(126) \quad SW'(x) = e^{-x} - c'_a u'(x) = 0 \Leftrightarrow u'(x)e^x = 1/c'_a.$$

Note that

$$(127) \quad u'(x) = \frac{1}{b(x)} - x \frac{b'(x)}{b(x)^2},$$

so  $u'(0)e^0 = 1$ . It follows from (126) that  $SW'(x) = 0$  has a unique solution for all  $c'_a \in (0, 1)$  if and only if  $u'(x)e^x$  is (strictly) increasing and unbounded. We see that

$$\frac{d}{dx} u'(x)e^x = e^x(u'(x) + u''(x)).$$

To show that  $u'(x)e^x$  is increasing and unbounded, we will show that  $u'(x) + u''(x) > 1$ .

By differentiating (127), we see that

$$u'(x) + u''(x) = \frac{1}{b(x)} \left( 1 - \frac{xb'}{b} - \frac{xb''}{b} + 2x \left( \frac{b'}{b} \right)^2 - 2 \frac{b'}{b} \right).$$

Note that  $b'(x) = g'(h(x))h'(x) < 0$ , so every term in the above sum except  $-x \frac{b''}{b}$  is clearly positive. Since  $b(x) \leq 1$ , to show that  $u'(x) + u''(x) > 1$ , it suffices to show

<sup>20</sup>Lemma 3 implies that if  $r \leq 1$ ,  $m\mathcal{P}(m, 1)$  is onto  $[0, \infty)$ ; if  $r > 1$ ,  $m\mathcal{P}(m, 1)$  is onto  $[0, \log \frac{r}{r-1})$ .

that  $-\frac{xb'}{b} - \frac{xb''}{b} > 0$ , or equivalently,  $b'(x) + b''(x) < 0$ , or equivalently

$$(128) \quad \frac{b''(x)}{b'(x)} > -1.$$

We note (omitting the algebra) that

$$(129) \quad b'(x) = g'(h(x))h'(x)$$

$$(130) \quad b''(x) = g''(h(x))h'(x)^2 + g'(h(x))h''(x).$$

$$(131) \quad h'(x) = re^{h(x)-x}$$

$$(132) \quad h''(x) = \frac{r-1}{r}e^x h'(x)^2 = (r-1)e^{h(x)}h'(x)$$

We apply (129), (130), followed by (132), to conclude that

$$(133) \quad \frac{b''(x)}{b'(x)} = h'(x) \frac{g''(h(x))}{g'(h(x))} + \frac{h''(x)}{h'(x)}$$

$$(134) \quad = h'(x) \frac{g''(h(x))}{g'(h(x))} + (r-1)e^{h(x)}.$$

Note that  $h'(x) > 0$ , and by Lemma 1,  $g''(h)/g'(h) > -1$ . Hence,

$$\frac{b''(x)}{b'(x)} > -h'(x) + (r-1)e^{h(x)}.$$

Now apply (131) and rearrange to get that

$$-h'(x) + (r-1)e^{h(x)} = -e^{h(x)}(1 - r(1 - e^{-x})) = -1,$$

completing the proof of (128). □

Motivated by Lemma 19, we define

$$(135) \quad m_a \in \arg \max_m \Pi_a(m, 1), \quad m_e \in \arg \max_m \Pi_e(m, 1).$$

Note that Lemma 19 implies that there is a unique value of  $m$  that maximizes  $\Pi_a(m, 1)$ . However, in some cases,  $\arg \max_m \Pi_e(m, 1)$  may be empty.<sup>21</sup> In this case, we define  $m_e = \infty$  and  $\Pi_e(\infty, 1) = \lim_{m \rightarrow \infty} \Pi_e(m, 1)$ .

We are now prepared to prove the Propositions from Section 6.

*Proof of Theorem 3.* For fixed  $r, c'_a$ , define  $f(r, c'_a) = \mathcal{Q}(m_1, 1)$  (recall that  $m_1$  is the value of  $m$  chosen by applicants when they respond optimally to employers playing

<sup>21</sup>This is precisely when the maximizer of  $BW$ , given by  $x = -\log c'_s$ , is outside of the domain of  $BW$ , as established by Lemma 3.

$\alpha = 1$ ). If  $\mathcal{Q}(m_1, 1) \geq c'_s$ , then  $1 \in \mathcal{A}(\mathcal{Q}(m_1, 1))$  and thus  $(m^*, \alpha^*) = (m_1, 1)$ . By Proposition 7,

$$(136) \quad \Pi_e(m_1, 1) = rm_1 \mathcal{P}(m_1, 1)(\mathcal{Q}(m_1, 1) - c'_s),$$

where we have applied the mean-field consistency equation (4) :  $\mathcal{Q}(m_1, 1) = g(m_1 \mathcal{P}(m_1, 1))$ . It follows that  $\Pi_e^* > 0$  if and only if  $\mathcal{Q}(m_1, 1) > c'_s$ . If  $\mathcal{Q}(m_1, 1) < c'_s$ , then  $(m_1, 1)$  is not an equilibrium, and thus  $\alpha^* < 1$ . This implies that employers are indifferent between screening and exiting, so we must have  $\Pi_e^* = 0$ .

We now turn to proving the monotonicity of  $f(r, c'_a) = \mathcal{Q}(m_1, 1)$ . So long as  $c'_a < 1$ , then  $m_1 > 0$ , so the applicant best response function (2) implies that  $m_1 \mathcal{P}(m_1, 1) = -\log(c_a / \mathcal{P}(m_1, 1))$ . Combining this with the consistency equation (4), we see that

$$(137) \quad f(r, c'_a) = \mathcal{Q}(m_1, 1) = g(-\log(c'_a \beta / \mathcal{P}(m_1, 1))).$$

Making use of (28) from Proposition 4,  $c'_a \beta / \mathcal{P}(m_1, 1)$  is the solution  $y$  to

$$(138) \quad c'_a = yg(ry(1-y)/c'_a),$$

or equivalently (after rearrangement),

$$(139) \quad r = \frac{1 - e^{-\frac{r}{c'_a} y(1-y)}}{1 - y}.$$

Because  $g$  is decreasing (Lemma 1), (137) implies that in order to prove that  $f(r, c'_a)$  is increasing in  $r$ , it suffices to show that the solution  $y$  to (138) is increasing in  $r$ . This follows because (by Lemma 5) the right side of (138) is increasing in  $y$  (for any  $r$ ), and decreasing in  $r$  (for fixed  $y$ ). Similarly, to prove that  $f(r, c'_a)$  is increasing in  $c'_a$ , it suffices to show that the solution  $y$  to (139) is increasing in  $c'_a$ . This follows because the right side of (139) is increasing in  $y$  (Lemma 5) and decreasing in  $c'_a$  (for fixed  $y$ ).

All that remains is to show that whenever  $\Pi_e^* = 0$ , a suitably chosen limit  $\ell$  can attain the employer-optimal outcome. By Proposition 3, we know that whenever  $m_e < m^*$ , this can be accomplished by setting  $\ell = m_e$ . Furthermore, by (136),  $\Pi_e^* = 0$  implies  $\mathcal{Q}(m^*, 1) \leq c'_s$  (whereas  $\mathcal{Q}(m_e, 1) > c'_s$  by the definition of  $m_e$ ). Using the fact that  $\mathcal{Q}(m, 1)$  is decreasing in  $m$  (Lemma 3), we deduce that  $m_e < m^*$ .  $\square$

*Proof of Theorem 4.* We begin by noting that the expression for  $\Pi_a^*$  given by (121) is increasing in  $m^* p^*$ . Thus, to derive upper-bounds on  $\Pi_a^*$ , it suffices to provide upper-bounds on  $m^* p^*$ .

Our first bound comes from the fact that in any equilibrium,  $g(m^*p^*) = q^* \geq c'_s$  (otherwise, employers would select  $\alpha = 0$ , implying  $q = 1$ ). It follows that  $m^*p^* \leq g^{-1}(c'_s) = \gamma$ , and therefore (121) implies that  $\Pi_a^* \leq 1 - (1 + \gamma)e^{-\gamma}$ .

Our second bound comes from Lemma 3, which states that when  $r > 1$ ,  $m\mathcal{P}(m, \alpha) \leq \log \frac{r}{r-1}$  for all  $m$  and  $\alpha$ .<sup>22</sup>

The statements about applicant welfare in the regulated market are proven in Lemma 20.  $\square$

*Proof of Theorem 5.* This theorem is restated and proven in Lemma 20.  $\square$

**Lemma 20.**

- (1) If  $c'_s > f(r, c'_a)$ , then there exists  $\ell$  such that  $\Pi_a^\ell > \Pi_a^*$  and  $\Pi_e^\ell > \Pi_e^*$ .
- (2) If  $c'_s \leq f(r, c'_a)$ , then there exists  $\ell$  such that  $\Pi_a^\ell = \Pi_a^{\text{opt}} > \Pi_a^*$ .

*Proof.* As established in Theorem 3, if  $c'_s \leq f(r, c'_a)$ , then  $\alpha^* = 1$  and  $q^* = g(m^*\mathcal{P}(m^*, 1)) \geq c'_s$ . Note that for any application limit  $\ell$ , we have  $m_\ell^* \leq m^*$ , and thus  $g(\ell\mathcal{P}(\ell, 1)) \geq g(m^*\mathcal{P}(m^*, 1)) \geq c'_s$ , so  $\alpha_\ell^* = 1$  for all  $\ell$ . Furthermore,

$$\begin{aligned} \frac{d}{dm}\Pi_a(m, 1)\big|_{m=m^*} &= \frac{d}{dm}(1 - e^{-m\mathcal{P}(m, 1)} - c'_a m\beta)\big|_{m=m^*} \\ &< \frac{d}{dm}(1 - e^{-mp^*} - c'_a m\beta)\big|_{m=m^*} = 0. \end{aligned}$$

The inequality follows because  $\frac{d}{dm}m\mathcal{P}(m, 1) = \mathcal{P}(m, 1) + m\frac{d}{dm}\mathcal{P}(m, 1) < \mathcal{P}(m, 1)$  (by Lemma 3), and the final equality follows because  $m^*$  is optimally chosen by applicants, who take  $p^*$  as given. Because  $\Pi_a(\cdot, 1)$  is unimodal (Lemma 19), it follows that  $m_a < m^*$ . By Theorem 3, if we set limit  $\ell = m_a$ , then we obtain  $\Pi_a^{\text{opt}}$  for applicants.

By Theorem 3, if  $c'_s > f(r, c'_a)$ , then  $\alpha^* < 1$  and  $\Pi_e^* = 0$ . It follows (from (120) in Proposition 7) that  $g(m^*p^*) = q^* = c'_s$ , and thus  $g^{-1}(c'_s) = m^*\mathcal{P}(m^*, \alpha^*) < m^*\mathcal{P}(m^*, 1)$  (the inequality follows from Lemma 3). Since  $\ell\mathcal{P}(\ell, 1)$  is increasing in  $\ell$  (Lemma 3), there exists  $\ell < m^*$  such that  $\ell\mathcal{P}(\ell, 1) = g^{-1}(c'_s)$ . Then  $1 \in \mathcal{A}(\mathcal{P}(\ell, 1))$ , and thus  $\alpha_\ell^* = 1$ . It follows from the definition of  $\Pi_a$  in (117) that

$$\Pi_a^\ell = \Pi_a(\ell, 1) = 1 - e^{-\ell\mathcal{P}(\ell, 1)} - c_a\ell = 1 - e^{-m^*p^*} - c_a\ell > 1 - e^{-m^*p^*} - c_am^* = \Pi_a^*.$$

By continuity, for all sufficiently small  $\varepsilon > 0$ ,  $\Pi_a^{\ell-\varepsilon} > \Pi_a^*$ . For any such  $\varepsilon$ ,  $q_{\ell-\varepsilon}^* = g((\ell - \varepsilon)\mathcal{P}(\ell - \varepsilon, 1)) > c'_s$ , so by (120),  $\Pi_e^{\ell-\varepsilon} > 0 = \Pi_e^*$ .  $\square$

<sup>22</sup>This has a simple interpretation. Of course, when  $r > 1$ , applicants match with probability at most  $1/r$ . But the proportion of applicants who match is  $1 - e^{-m\mathcal{P}(m, \alpha)}$ . From this, we conclude that  $r(1 - e^{-m\mathcal{P}(m, \alpha)}) \leq 1$ , which can be rearranged to give  $m\mathcal{P}(m, \alpha) \leq \log \frac{r}{r-1}$ .

**Lemma 21.** *Fix  $r, c'_s$ , and consider  $m^*$  and  $p^*$  as functions of  $c'_a$ . The quantity  $m^*$  is strictly decreasing in  $c'_a$ , with  $m^*(1) = 0$  and  $\lim_{c'_a \rightarrow 0} m^*(c'_a) = \infty$ . The quantity  $m^*p^*$  is weakly decreasing in  $c'_a$ .*

*Proof.* We know that  $m^*$  is the unique solution to  $h(m^*) = -\log c'_a$ . Recall that by Lemma 9, the function  $h : [0, \infty) \rightarrow [0, \infty)$  defined in (37) (which does not depend on  $c'_a$ ) satisfies  $h(0) = 0$  and is strictly increasing. From this, the statements about  $m^*$  follow.

We know from Theorem 3 and Proposition 7 that if  $c'_s \geq f(r, c'_a)$ , then  $g(m^*p^*) = c'_s$ , so  $m^*p^*$  is constant on the set  $\{c'_a : c'_s \geq f(r, c'_a)\}$ . For  $c'_a$  such that  $c'_s < f(r, c'_a)$ , we know that  $\alpha^* = 1$ , and (making use of (28) from Proposition 4) that  $c'_a\beta/p^*$  is the solution  $y$  to

$$(140) \quad r = \frac{1 - e^{-\frac{r}{c'_a}y(1-y)}}{1 - y}.$$

Furthermore, the applicant best-response function  $\mathcal{M}$  given by (2) implies that  $e^{-m^*p^*} = c'_a\beta/p^*$ . Thus, to show that  $m^*p^*$  is decreasing in  $c'_a$ , it is enough to show that  $c'_a\beta/p^*$  is increasing in  $c'_a$ . This holds because the right side of (140) is decreasing in  $c'_a$  (for fixed  $y > 0$ ), and increasing in  $y$  for fixed  $c'_a$  (Lemma 5).  $\square$

*Proof of Proposition 6.* The proof is straightforward. For a fixed applicant strategy, the value of  $c'_a$  is irrelevant for employers. Thus, it suffices to show that for each  $\ell \leq m^*(c'_a)$ , there exists  $\tilde{c}'_a \in [c'_a, 1]$  such that  $m^*(\tilde{c}'_a) = m^*_\ell$ ; and for each  $\tilde{c}'_a > c'_a$ , there exists a unique  $\ell \leq m^*(c'_a)$  such that  $m^*(\tilde{c}'_a) = m^*_\ell$ .

This follows because Proposition 3 implies that for  $\ell \leq m^*$ ,  $m^*_\ell = \ell$ , and Lemma 21 implies that for any desired application level  $\ell \in [0, m^*]$ , there exists a unique  $\tilde{c}'_a \geq c'_a$ , such that  $m^*(\tilde{c}'_a) = \ell$ .

As for applicant welfare, by Proposition 7,  $\Pi_a^* = 1 - (1 + m^*p^*)e^{-m^*p^*}$ , which is increasing in  $m^*p^*$ . Lemma 21 states that  $m^*p^*$  is decreasing in  $c'_a$ , implying that  $\Pi_a^*$  is, as well.  $\square$