Facilitating the Search for Partners on Matching Platforms
Supplementary appendix

G Supplementary material to Appendix C (Appendix to Section 2: Model)

Proof of Proposition 10. **Roadmap for the proof:** The bulk of this proof is devoted to showing that $(\kappa_w^t, \kappa_e^t, \hat{\kappa}_w^t, \hat{\kappa}_e^t)$ are uniquely determined. We establish this in each of the two relevant cases — when there is a positive mass of agents in each tier, and when one of the tiers has a mass zero of agents. At the end, we deduce that νs and ρs are also uniquely determined.

Fix a tier $t \in \{\mathrm{H}, \mathrm{L}\}$ of workers.

The "easy" case $N_e > 0$, $N_w^{\mathbf{H}} > 0$, $N_w^{\mathbf{L}} > 0$. In the "easy" case that $N_e > 0$, $N_w^{\mathrm{H}} > 0$, $N_w^{\mathrm{L}} > 0$, a candidate (belonging to the preferred tier) is available to any agent whenever their opportunity clock rings (which occurs at rate 1), and the distribution of the candidate's strategy is identical to the distribution of strategies in candidate's tier as per $\bar{N}$, which immediately leads to a unique specification of $(\kappa_w^t, \kappa_e^t)_{t=\mathrm{H},\mathrm{L}}$ as we make explicit below. As a result, $(\hat{\kappa}_w^t, \hat{\kappa}_e^t)_{t=\mathrm{H},\mathrm{L}}$ are also uniquely determined by the mass-balance requirements (17) and (18).

For workers in tier t seeing employer candidates, for all $\mathcal{S}' \subseteq \mathcal{S}_e$ we have

$$\kappa_w^t(\mathcal{S}'; s_w, \bar{N}, \bar{f}) = \mathbb{I}(s_w \in \tilde{\mathcal{S}}_w)\bar{N}_e(\mathcal{S}')/N_e\,,$$

where $\tilde{\mathcal{S}}_w \subseteq \mathcal{S}_w$ represents the set of strategies for which $a^{\mathrm{o}} \neq \mathrm{DN}$, that is, the set of strategies for which a tier-t worker considers opportunities and proposes with or without screening. As a result, the mass-balance requirement (18) pins down $\hat{\kappa}_e^t(\mathcal{S}; s_e, \bar{N}, \bar{f})$ as, for all $\mathcal{S} \subseteq \mathcal{S}_w^t$ we have

$$\hat{\kappa}_e^t(\mathcal{S}; s_e, \bar{N}, \bar{f}) = \bar{N}_w^t(\tilde{\mathcal{S}}_w \cap \mathcal{S})/N_e\,.$$

In words, the rate at which an employer is shown as a candidate to a worker in tier t with a strategy in $\mathcal{S}$ is proportional to the mass of such workers who want candidates, divided by the total mass of employers.

Next, we compute $\kappa_e^t(s_e, s_w; \bar{N})$ for $t \in \{\mathrm{H}, \mathrm{L}\}$ in the case that two tiers of workers are present. To that end, define the strategy set $\mathcal{S}_e(t, i)$ as

$$\mathcal{S}_e(t, i) \triangleq \{s \in \mathcal{S}_e:\ t \text{ is listed as the } i^{\text{th}}\text{-preferred tier in } a^{\mathrm{o}}(s)\} \quad \text{for } t \in \{\mathrm{H}, \mathrm{L}\}, i \in \{1, 2\}\,. \tag{49}$$

Let $\bar{t}$ denote the other tier of workers, i.e., if $t = \mathrm{H}$ then $\bar{t} = \mathrm{L}$ and vice versa. For employers seeing worker candidates, for each $t \in \{\mathrm{H}, \mathrm{L}\}$ and all $\mathcal{S} \in \mathcal{S}_w^t$, we have

$$\kappa_e^t(\mathcal{S}; s_e, \bar{N}, \bar{f}) = \begin{cases} \bar{N}_w^t(\mathcal{S})/N_w^t & \text{if } s_e \in \mathcal{S}_e(t, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Consequently, $\hat{\kappa}_w^t$ is uniquely determined by the mass-balance requirement (17), as

$$\hat{\kappa}_w^t(\mathcal{S}'; s_w, \bar{N}, \bar{f}) = \bar{N}_e(\mathcal{S}_e(t, 1) \cap \mathcal{S}')/N_w^t$$

for all $\mathcal{S}' \in \mathcal{S}_e$.

The case where one tier of agents has mass zero. We consider the case $N_w^{\mathrm{H}} = 0$, $N_w^{\mathrm{L}} > 0$, $N_e > 0$. The case $N_w^{\mathrm{H}} > 0$, $N_w^{\mathrm{L}} = 0$, $N_e > 0$ and the case $N_w^{\mathrm{H}} > 0$, $N_w^{\mathrm{L}} > 0$, $N_e = 0$ are analogous.

This case is slightly more subtle. There are now two possibilities:

(i) H-workers match slower than they arrive. Here, as in the easy case, a worker in tier H is available whenever an employer gets an opportunity. As a result κ_e^{H} is uniquely determined by f_w^{H} and $\bar{N}_e$, and $\hat{\kappa}_w^{\mathrm{H}}$ is then uniquely determined by the matching constraint (17). Note that for any strategy s_e that asks for H-worker candidates, the measure κ_e^{H} is a probability measure 21)(since the hazard rate of opportunities is 1) but is, in general, different from f_w^{H}. Instead, it is proportional to the rate of buildup of H-workers following strategy s_w in the system. Since this case $N_w^{\mathrm{H}} = 0$, $dN_w^{\mathrm{H}}/d\tau > 0$ cannot arise in steady state, and we study stationary equilibria, we omit the explicit calculations.

(ii) H-workers match as fast as they arrive. In this case $\hat{\kappa}_w^{\mathrm{H}}$ is uniquely determined by the arrival rate λ_w^{H}, the mix of strategies picked by arrivals f_w^{H}, and $\bar{N}$. We need to be careful, since $\hat{\kappa}_w^{\mathrm{H}}(\mathcal{S}_e; s_w, \bar{N}, \bar{f}) = \infty$ for all strategies s_w in the support of $\bar{N}_w^{\mathrm{H}}$, since workers match as soon as they arrive. Also, $N_w^{\mathrm{H}} = 0$. However, integrals capturing flows of such opportunities, and consequent proposals and matches are well defined. In particular, the differential $\hat{\kappa}_w^{\mathrm{H}}(\mathcal{S}'; s_w, \bar{N}, \bar{f}) d\bar{N}_w^{\mathrm{H}}(s_w)$ is valid and captures the differential *flow* of worker candidates of strategy s_w seen by employers following strategies in $\mathcal{S}' \subseteq \mathcal{S}_e$. We compute this differential as follows. We have

$$\lambda_w^{\mathrm{H}} df_w^{\mathrm{H}}(s_w) = \left(\int_{\mathcal{S}_e} \eta_e^t(s_e, s_w) d\hat{\kappa}_w^{\mathrm{H}}(s_e; s_w, \bar{N}, \bar{f}) \right) d\bar{N}_w^{\mathrm{H}}(s_w)\,, \tag{50}$$

since H-workers match and leave as soon as they enter, so the (differential) inflow of high workers equals the (differential) match flow involving high workers. (The term in brackets on the right-hand side is the notional match rate.) Furthermore, given Assumption 3, the (infinite) measure $\hat{\kappa}_w^{\mathrm{H}}(\,\cdot\,; s_w, \bar{N}, \bar{f})$ notionally satisfies

$$d\hat{\kappa}_w^{\mathrm{H}}(s_e; s_w, \bar{N}, \bar{f}) \propto \mathbb{I}(s_e \in \mathcal{S}_e(t,1)) d\bar{N}_e(s_e) \quad \Leftrightarrow \quad \hat{\kappa}_w^{\mathrm{H}}(\mathcal{S}'; s_w, \bar{N}, \bar{f}) \propto \bar{N}_e(\mathcal{S}' \cap \mathcal{S}_e(t,1))\,.$$

Using (50) the notional constant of proportionality is $\lambda_w^{\mathrm{H}} df_w^{\mathrm{H}}(s_w) / \big(d\bar{N}_w^{\mathrm{H}}(s_w) \int_{\mathcal{S}_e(t,1)} \eta_e^t(s_e, s_w) d\bar{N}_e(s_e)\big)$. We deduce the desired differential flow of H-worker candidates

$$\hat{\kappa}_w^{\mathrm{H}}(\mathcal{S}'; s_w, \bar{N}, \bar{f}) d\bar{N}_w^{\mathrm{H}}(s_w) = \frac{\bar{N}_e(\mathcal{S}' \cap \mathcal{S}_e(t,1)) \lambda_w^{\mathrm{H}} df_w^{\mathrm{H}}(s_w)}{\int_{\mathcal{S}_e(t,1)} \eta_e^t(s_e, s_w) d\bar{N}_e(s_e)} \qquad \forall \mathcal{S}' \subseteq \mathcal{S}_e\,, \tag{51}$$

$$\Rightarrow \int_{\mathcal{S}} \hat{\kappa}_w^{\mathrm{H}}(\mathcal{S}'; s_w, \bar{N}, \bar{f}) d\bar{N}_w^{\mathrm{H}}(s_w) = \bar{N}_e(\mathcal{S}' \cap \mathcal{S}_e(t,1)) \lambda_w^{\mathrm{H}} \int_{\mathcal{S}} \frac{df_w^{\mathrm{H}}(s_w)}{\int_{\mathcal{S}_e(t,1)} \eta_e^t(s_e, s_w) d\bar{N}_e(s_e)}$$
$$\forall \mathcal{S}' \subseteq \mathcal{S}_e,\ \mathcal{S} \subseteq \mathcal{S}_w\,. \tag{52}$$

And κ_e^{H} then gets determined immediately by the matching constraint (17) as

$$\kappa_e^{\mathrm{H}}(\mathcal{S}; s_e, \bar{N}, \bar{f}) = \mathbb{I}(s_e \in \mathcal{S}_e(t,1)) \beta^{\mathrm{H}}(\mathcal{S}) \quad \text{for } \beta^{\mathrm{H}}(\mathcal{S}) = \lambda_w^{\mathrm{H}} \int_{\mathcal{S}} \frac{df_w^{\mathrm{H}}(s_w)}{\int_{\mathcal{S}_e(t,1)} \eta_e^t(s_e, s_w) d\bar{N}_e(s_e)}\,. \tag{53}$$

The case which arises between the cases (i) and (ii) mentioned above is determined by doing the calculation for case (ii) and checking the value of $\beta^{\mathrm{H}}(\mathcal{S}_w)$ given by (53). If $\beta^{\mathrm{H}}(\mathcal{S}_w) \leq 1$, we deduce that H-worker candidates are indeed scarce and available only a fraction $\beta^{\mathrm{H}}(\mathcal{S}_w)$ of the time so case (ii) arises. Otherwise, case (i) arises. This completes the case $N_w^{\mathrm{H}} = 0$, $N_w^{\mathrm{L}} > 0$, $N_e > 0$.

Thus we have established that $(\kappa_w^t, \kappa_e^t, \hat{\kappa}_w^t, \hat{\kappa}_e^t)$ are uniquely determined for $t \in \{\mathrm{H}, \mathrm{L}\}$. We complete the proof by noting that the above explicit expressions for $(\kappa_w^t, \kappa_e^t, \hat{\kappa}_w^t, \hat{\kappa}_e^t)$ further yield:

1. For each tier t, an explicit expression for ν_w^t via (13), and for ν_e^t via (14).

2. An explicit expression for ρ^t_w via (15) and for ρ_e via (16).

$\square$

G.1 Proof of Proposition 1

In this subsection we prove Proposition 1. Thereafter, we deduce that the set of best responses is a product set, and formalize this observation in Lemma G.1.

Proof of Proposition 1. Consider a worker i in tier t, who sees an agent mix $\bar{N}$ and strategies distributions for new arrivals $\bar{f}$ when he enters the system.

Roadmap for the proof. Recall that Appendix C.3 cast the agent's problem for a fixed vector of actions $a_w \in \mathcal{A}_w$ as a discounted, discrete-time, infinite-horizon, bounded-cost, time-invariant MDP in a standard form. Our proof of the proposition uses this MDP and consists of two parts.

In the first part, we establish that, whenever a_w involves screening, the optimal strategy uses acceptability thresholds equal to the best-response expected utility given a_w; we do so by characterizing the optimal strategies using the Bellman equation.

In the second part, we simply compare the best-response expected utility across the finite set of possible $a_w \in \mathcal{A}_w$, and use the fact that the true best-response a_w corresponds to the highest expected utility among these. This results in a proof of the proposition.

Recall from Appendix C.1 and Observation 1 that $\kappa^t_w(\mathcal{S}_e; a_w, \bar{N}, \bar{f})$ denotes the rate at which the worker sees candidates and $\nu^t_w(\bar{N}, \bar{f})$ denotes the rate at which the worker receives proposals. These quantities will appear in the proof.

Part 1: Solution to the agent's MDP problem for fixed a_w. Note that the discrete-time MDP defined in Appendix C.3 is time invariant. We next characterize the solution to this MDP under two different regimes. We start by characterizing the solutions in the original system, which correspond to the *weak* best responses. We then refine these solutions to obtain the best responses in Definition 6.

Suppose that the problem has a discount factor (defined in (19)) less than 1. Recall that we have $\mu > 0$, and, from Appendix C.2 we have that $\kappa^t_w(\mathcal{S}_e; a_w, \bar{N}, \bar{f}) \leq 1$, but $\nu^t_w(\bar{N}, \bar{f})$ can be infinite; thus, the above condition is equivalent to requiring $\nu^t_w(\bar{N}, \bar{f}) < \infty$. Then, by the classic result (Bertsekas 2007, Proposition 1.2.5), there exists a time-invariant deterministic optimal policy. We use the same result to characterize the optimal policy.

Clearly, the continuation value of being in the Matched state is $J(\text{Matched}) = 0$. We now write down the Bellman equation for the system in the states where the worker has just screened an incoming proposal (IP), and so $v_{ij} \in \mathbb{R}$ is known:

$$J^*((\text{IP}, v_{ij})) = \max(-c + J^*(\text{Waiting}), v_{ij} - c)\,, \tag{54}$$

where the first term corresponds to the action Reject, and the second term corresponds to the action Accept (the continuation value here is $J^*(\text{Matched}) = 0$). Here $J^*(\text{Waiting})$ is identical to the value V of the MDP, since, in the original problem, the worker starts in the waiting state $x_{0+} = \text{Waiting}$. From (Bertsekas 2007, Proposition 1.2.5), it follows that any optimal strategy corresponds to a maximizer on the right-hand side and hence must Accept if v_{ij} exceeds $V = J^*(\text{Waiting})$ and Reject otherwise, as we wanted to show.

(Note that $V = J^*(\text{Waiting})$ can be easily expressed in terms of the continuation value in the states $\mathcal{X}$ as

$$V = J^*(\text{Waiting}) = r \cdot \frac{\nu^t_w(\bar{N}, \bar{f})\mathrm{E}_{v_{ij}}[J^*((\text{IP}, v_{ij}))] + \kappa^t_w\left(\mathcal{S}_e; a_w, \bar{N}, \bar{f}\right)\mathrm{E}_{v_{ij}}[J^*((\text{O}, v_{ij}))]}{\kappa^t_w\left(\mathcal{S}_e; a_w, \bar{N}, \bar{f}\right) + \nu^t_w(\bar{N}, \bar{f})} \tag{55}$$

where r is the discount factor defined in (19), which we can substitute back into (54) to get the Bellman equation in standard form with no reference to a post-epoch state.)

Next, we write down the Bellman equation for the system in the states where the worker has just screened a candidate, and so $v_{ij} \in \mathbb{R}$:

$$J^*((\mathrm{O}, v_{ij})) = \max\big(-c + J^*(\text{Waiting}),\, p^w_{\mathrm{acc},t}(v_{ij} - c) + (1 - p^w_{\mathrm{acc},t})(-c + J^*(\text{Waiting}))\big), \quad (56)$$

where the first term corresponds to the action DNP (do not propose), and the second term corresponds to the action Propose (the continuation value here is $J^*(\text{Matched}) = 0$ if the proposal is accepted and $J^*(\text{Waiting})$ otherwise). Thus, the optimal strategy is to Propose if and only if $p^w_{\mathrm{acc},t}(v_{ij} - c) + (1 - p^w_{\mathrm{acc},t})(-c + J^*(\text{Waiting}))$ exceeds $-c + J^*(\text{Waiting})$ or, equivalently, to Propose if and only if v_{ij} exceeds $V = J^*(\text{Waiting})$, as we wanted to show.

Finally, to satisfy the notion of best response in Definition 6 we require that, if the worker screens incoming proposals, then he uses an optimal threshold even if $\nu^t_w(\bar{N}, \bar{f}) = 0$. That is, condition 1 requires that the worker's best response strategy satisfies (54) whenever that strategy involves screening incoming proposals. Analogously, by condition 2, a best response strategy must satisfy (56) whenever the strategy specifies to screen+propose.

Next, suppose that the discount factor in (19) is equal to 1, that is, $\nu^t_w(\bar{N}, \bar{f}) = \infty$. In this case, (Bertsekas 2007, Proposition 1.2.5) does not apply. However, we are able to complete the proof of Step 1 by considering the following cases that may arise:

1. For $a^{\mathrm{i}}_w = \mathrm{I}$, incoming proposals have no actions or payoffs associated with them, so we modify our definition of the MDP so that epochs correspond exclusively to opportunities and the discount factor is defined as $r = 1 - \mu/(\kappa^t_w\left(\mathcal{S}_e; a_w, \bar{N}, \bar{f}\right) + \mu) < 1$. The proof of the remaining steps goes through with no other changes.
2. For $a^{\mathrm{i}}_w = \mathrm{A}$, an incoming proposal is immediately accepted, so in fact the screening threshold for opportunities is immaterial and has no effect on the utility as the worker never gets to see an opportunity. Thus, in this case, the expected value of the MDP for the worker is $V = 1/2$. Then, any best response strategy must be an optimal control for which the strategy to deal with opportunities also satisfies (56) if he screens, that is, he screens with a threshold of $1/2$.
3. If $a^{\mathrm{i}}_w = \mathrm{S{+}A/R}$, then in the above MDP we have that an IP w.p. 1 in each epoch (that is, all epochs correspond to incoming proposals), and a discount factor $r = 1$. We can nevertheless establish the result in this undiscounted setting as follows. In this case, opportunities never arise, and the "IP-only" MDP is an optimal stopping problem, i.e., in each epoch the choice is between stopping (by accepting the proposal) and continuing (by rejecting the proposal). In state $x_n = (\mathrm{IP}, v_{ij})$, the payoff from stopping (via $\pi_n = \text{Accept}$) is $v_{ij} - c$ (note that this payoff is bounded above by 1), and the payoff from continuing (via $\pi_n = \text{Reject}$) is $-c$. As noted in (Bertsekas 2007, Section 4.4, middle of page 236), since the payoff from stopping is bounded above and the payoff from continuing is strictly negative, (Bertsekas 2007, Proposition 4.1.5) applies, i.e., if a stationary policy is optimal it must solve Bellman's equation. This allows us to complete our proof of Step 1 as before using (54), including the fact that any best response strategy must be an optimal control for which the strategy to deal with opportunities also satisfies (56).

Therefore, we have shown that for fixed $a_w \in \mathcal{A}_w$, in any best response, subsequent to screening an incoming proposal/opportunity the worker uses a threshold strategy with acceptability threshold equal to his maximum achievable expected utility. In particular, if the worker screens both opportunities and incoming proposals, he uses the same acceptability threshold for both.

Part 2: Comparing across $a_w \in \mathcal{A}_w$. Above we characterized the best response for a worker when the action vector a_w fixed. In fact, the worker can choose a_w as part of his strategy as well. This is easy to account for. In any best response in the original system (1), the worker will adopt an a_w such that the corresponding expected utility V (the value of the MDP above) is the maximum achievable among all possible $a_w \in \mathcal{A}_w$ (note that the action space is discrete and finite, thus a maximum exists). Moreover, it can be the case that several a_w achieve V; in such a case, we must choose one that satisfies conditions (1) and (2) in Definition 6. As we argued above, for that choice of a_w, the worker will use an acceptability threshold of V for that a_w each time he screens a proposal/candidate in any best response.

Finally, in any stationary equilibrium, all agents see the same $\bar{N}$ when they enter the system, which corresponds to the steady state of the system. Hence, all workers in the same tier are able to achieve exactly the same best-response utility. As a result, all workers in this tier use an identical threshold. This completes the proof of the proposition for workers. The proof for employers is analogous and therefore omitted. $\square$

It follows from the proof of Proposition 1 above that a strategy is a best response if and only if it consists of a best response action for handling incoming proposals as well as a best response action for handling opportunities, which also means that the set of best responses is a product set. We now formalize this observation.

Lemma G.1. *Consider one tier on each side of the market and fix $\bar{N}, \bar{f}$. Then the corresponding best response anticipated utility for a worker (similarly for an employer) is $\theta = \max_{s_w \in \mathcal{S}_w} U_w^t(s_w, \bar{N}, \bar{f})$ and workers use acceptability threshold(s) θ in any best response (Proposition 1). Let $\mathcal{A}_{\mathrm{br}} \triangleq \{a \in \mathcal{A}_w : (a, \theta) \in \arg\max_{s_w \in \mathcal{S}_w} U_w^t(s_w, \bar{N}, \bar{f})\}$ be the best response actions. Then there exist sets $\mathcal{A}^{\mathrm{i}}_{\mathrm{br}} \subseteq \mathcal{A}^{\mathrm{i}} \triangleq \{\mathrm{I}, \mathrm{A}, \mathrm{S{+}A/R}\}$ and $\mathcal{A}^{\mathrm{o}}_{\mathrm{br}} \subseteq \mathcal{A}^{\mathrm{o}} \triangleq \{\mathrm{DN}, \mathrm{P\ w/o\ S}, \mathrm{S{+}P}\}$ such that $\mathcal{A}_{\mathrm{br}}$ is the product set $\mathcal{A}^{\mathrm{i}}_{\mathrm{br}} \times \mathcal{A}^{\mathrm{o}}_{\mathrm{br}}$.*

Proof of Lemma G.1. Because agents are solving a regenerative dynamic programming problem with a return to the same "Waiting" state between consecutive incoming proposals/opportunities until they match (see proof of Proposition 1), it follows that there is a set of best response actions to incoming proposals $\mathcal{A}^{\mathrm{i}}_{\mathrm{br}}$, and a set of best response actions to opportunities $\mathcal{A}^{\mathrm{o}}_{\mathrm{br}}$. The set of best response action vectors is simply $\mathcal{A}_{\mathrm{br}} = \mathcal{A}^{\mathrm{i}}_{\mathrm{br}} \times \mathcal{A}^{\mathrm{o}}_{\mathrm{br}}$, thus establishing the fact. $\square$

H Supplementary material to Appendix D (Some useful lemmas)

Proof of Lemma D.1. Consider the first system. Let $q' \triangleq 1 - F(\theta)$ be the likelihood that a value exceeds θ. The probability that a potential option is both requested and approved is qq'. Hence:

(i) The expected screening cost paid per obtained item is $c/(qq')$.

(ii) The likelihood of obtaining an item before death is the likelihood that a Poisson clock of rate $\eta qq'$ rings before the Poisson death clock of rate μ.

It is easy to check that the two parts of this description each apply also to the second system, since the probability of a value exceeding θ is again q'. Finally, the expected value of an obtained item is just $\mathbb{E}_{X\sim F}[X|X > \theta]$ in each system. Combining, we obtain the claim. □

Proof of Proposition 11. As all on the same side select the same strategy upon arrival, to ease exposition we will slightly abuse notation and suppress the dependence on the strategy by defining $\eta_w \triangleq \eta_w(s_w, s_e)$ and $\eta_e \triangleq \eta_e(s_e, s_w)$. Also note that, since all employers use the same strategy $L_e = \bar{L}_e(s_e)$ and similarly $L_w = \bar{L}_w(s_w)$. We will show convergence to a limiting mass of workers and employers, L_w and L_e, respectively. We calculate the limits assuming conditions (23) and (24). If $\eta_e = 0$, then condition (23) holds automatically. Suppose $\eta_e > 0$. We will find that the limiting values of L_w and L_e resulting from $\lambda_w \to \left(\frac{\eta_e\lambda_e}{\mu+\eta_e}\right)_+$ and $\lambda_w \to \left(\frac{\eta_e\lambda_e}{\mu+\eta_e}\right)_-$, holding everything else fixed, are identical. Though we omit the details, a coupling argument can be used to establish that this pair of values matches the L_w and L_e that arise from $\lambda_w = \left(\frac{\eta_e\lambda_e}{\mu+\eta_e}\right)$.

Note that the mass of workers in the system in steady state, L_w, is bounded above by λ_w/μ, since agents die at rate $\mu > 0$ (even if they don't leave by matching), and, similarly, $L_e \le \lambda_e/\mu$ for employers. Also, note that the only way agents can have a vanishing expected lifetime in the system is if they receive incoming proposals at a rate tending to ∞. All other agents have a positive expected lifetime in the system. We will argue that:

(i) All agents have a positive expected lifetime in the system if $\lambda_w > \frac{\eta_e\lambda_e}{\mu+\eta_e}$ and $\lambda_e > \frac{\eta_w\lambda_w}{\mu+\eta_w}$, i.e., the left-hand side is greater than the right-hand side in both conditions (23) and (24), and

(ii) If $\lambda_w < \frac{\eta_e\lambda_e}{\mu+\eta_e}$, then workers will have a vanishing lifetime in the system. Similarly, $\lambda_e < \frac{\eta_w\lambda_w}{\mu+\eta_w}$,, then employers will have a vanishing lifetime in the system.

To that end, suppose that $\lambda_w > \frac{\eta_e\lambda_e}{\mu+\eta_e}$ and $\lambda_e > \frac{\eta_w\lambda_w}{\mu+\eta_w}$ and suppose, for a contradiction, that workers' expected lifetime in the system is zero. In that case, workers must be leaving the system as soon as they arrive by matching after accepting an employer's proposal (workers do not get a chance to propose). Note that, even if employers see potential opportunities at every possible clock ring, their likelihood of matching before dying is only $\eta_e/(\mu + \eta_e)$. Hence, the maximum flow rate at which workers match due to proposals by employers is $\lambda_e\eta_e/(\mu + \eta_e)$. If the $\lambda_w > \frac{\eta_e\lambda_e}{\mu+\eta_e}$, then a positive fraction of workers do not match as a result of proposals by employers, which is a contradiction; thus, the mass of workers in the system must be positive. Analogously, if $\lambda_e > \frac{\eta_w\lambda_w}{\mu+\eta_w}$, then the mass of employers in the system is positive. Therefore, agents have a positive expected lifetime in the system whenever $\lambda_w > \frac{\eta_e\lambda_e}{\mu+\eta_e}$ and $\lambda_e > \frac{\eta_w\lambda_w}{\mu+\eta_w}$.

Next, suppose that $\lambda_w < \frac{\eta_e\lambda_e}{\mu+\eta_e}$. Then, we must have that $\eta_e > 0$ (employers do propose to workers) $\lambda_e > \lambda_w$, and so a simple argument can be used to show that at any time, the mass of employers in the system is positive (since employers die at a flow rate of at least $(\lambda_e - \lambda_w)$ for all $\tau \ge \tau_0$, for some τ_0). Suppose employers are provided a potential opportunity each time they ask for one. Then the likelihood an employer will match before she dies is at least $\eta_e/(\mu + \eta_e)$. But then the match flow is $\lambda_e\eta_e/(\mu + \eta_e) > \lambda_w$, a contradiction. Hence, employers are *not* provided a potential opportunity each time they ask for one, which means that the limiting mass of workers in the system is 0. Moreover, the number of workers will stay at that level: if it starts to build up, then employers will be able to issue proposals at a faster rate, and form matches at rate at least $\lambda_e\eta_e/(\mu+\eta_e) > \lambda_w$, reducing the mass of workers. Hence, in steady state, the mass of workers remains 0. Therefore, if $\lambda_w < \frac{\eta_e\lambda_e}{\mu+\eta_e}$, workers will never build up in the system. Note that the analogous argument can be applied in the case that $\lambda_e < \frac{\eta_w\lambda_w}{\mu+\eta_w}$.

Limiting steady state when $\lambda_w < \frac{\eta_e\lambda_e}{\mu+\eta_e}$. In fact, we can precisely characterize the steady state as follows. Since the mass of workers is 0, the flow of workers dying is 0, meaning that workers form matches at flow rate λ_w (moreover, these matches occur at a steady pace). Since the match flow is λ_w, the flow of employers dying is $\lambda_e - \lambda_w$, hence the mass of employers in the system is $(\lambda_e - \lambda_w)/\mu$, and in fact the mass of employers remains steady near this value since the match flow does not change over time. It follows that the mass of employers concentrates around the limiting value of $L_e = \frac{\lambda_e-\lambda_w}{\mu}$, whereas the mass of workers in the system is 0. Note that $\lambda_w < L_e\eta_e$, consistent with the mass of workers remaining 0.

Note that as $\lambda_w \to \left(\frac{\eta_e\lambda_e}{\mu+\eta_e}\right)_-$ we have $L_w = 0$ (in fact, this holds everywhere in this case) and $L_e \to \frac{\lambda_e}{\mu+\eta_e}$.

Limiting steady state when $\lambda_w > \frac{\eta_e\lambda_e}{\mu+\eta_e}$ and $\lambda_e > \frac{\eta_w\lambda_w}{\mu+\eta_w}$. Now $N_w = \bar{N}_w(s_w)$ is the mass of workers in the system at time τ since all workers are using the same strategy s_w, and similarly $N_e = \bar{N}_e(s_e)$. The limiting dynamical system when $\lambda_w > \frac{\eta_e\lambda_e}{\mu+\eta_e}$ and $\lambda_e > \frac{\eta_w\lambda_w}{\mu+\eta_w}$ is given by:

$$\frac{\mathrm{d}N_w}{\mathrm{d}\tau} = A\bar{N} + b\,, \text{ for}$$
$$\bar{N} = \begin{bmatrix} N_e \\ N_w \end{bmatrix}, \; b = \begin{bmatrix} \lambda_e \\ \lambda_w \end{bmatrix}, \; A = \begin{bmatrix} -\mu-\eta_e & -\eta_w \\ -\eta_e & -\mu-\eta_w \end{bmatrix}.$$

This is a pair of coupled linear differential equations in N_w and N_e; note that this system is a special case of (20). The match flow resulting from options shown to employers is $N_e\eta_e$ and the match flow resulting from options shown to workers is $N_w\eta_w$. In addition, individual agents die at rate μ, leading to the form of the equations.

The eigenvalues of A are $-\mu$ and $-\mu-\eta_e-\eta_w$. Since the eigenvalues are negative (Kreyszig 2010), we deduce that

$$\bar{L} = \begin{bmatrix} \frac{\lambda_e(\mu+\eta_w)-\lambda_w\eta_w}{\mu(\mu+\eta_e+\eta_w)} \\ \frac{\lambda_w(\mu+\eta_e)-\lambda_e\eta_e}{\mu(\mu+\eta_e+\eta_w)} \end{bmatrix},$$

which solves $A\bar{N} + b = 0$, is an attractive fixed point of the dynamical system with a global basin of attraction. Hence, the dynamical system converges globally to $\bar{L}$. □

I Planner's solution

In this appendix we characterize the the planner's solution with one tier on each side (Proposition 2). In fact, we prove a slightly more general version of Proposition 2 to facilitate a characterization of the planner's solution with two worker tiers as a corollary. Armed with this foundation, it is easy to characterize the planner's solution with two worker tiers (Proposition 7).

Before launching into the proofs, we provide a general definition of total welfare that does not require the system to be in steady state. (We will prove stronger versions of Propositions 2 and 7 that allow the planner to choose time-varying strategies.)

General definition of total welfare. Fix a time interval $[0, \tau_0]$. Let $\bar{N}_\tau = (\bar{N}^{\mathrm{H}}_{w,\tau}, \bar{N}^{\mathrm{L}}_{w,\tau}, \bar{N}_{e,\tau})$ be the agent mix at time τ, and let $\mathbf{N} = \mathbf{N}(\tau_0) \triangleq (\bar{N}_\tau)_{\tau\in[0,\tau_0]}$. Consider an arbitrary time varying mix of strategies for incoming arrivals $\mathbf{f} = \mathbf{f}(\tau_0) \triangleq (\bar{f}_\tau)_{\tau\in[0,\tau_0]}$ for $\bar{f}_\tau = (f^{\mathrm{H}}_{w,\tau}, f^{\mathrm{L}}_{w,\tau}, f_{e,\tau})$. Let $\tilde{U}_{e,\tau}(s_e; \mathbf{N}, \mathbf{f})$ denote the expected utility for an employer who enters at time τ, up to time τ_0. (We use $\tilde{U}_{e,0}(s_e; \mathbf{N}, \mathbf{f})$ to denote the expected utility for an employer who is present at time 0, up to time τ_0.) We define $\tilde{U}^t_{w,\tau}(s_w; \mathbf{N}, \mathbf{f})$ similarly for $t = \{\mathrm{H}, \mathrm{L}\}$. While expected utilities can be computed via a non-steady-state version of the calculation in Section 2 "agent utilities", we will take a different approach. In light of the exact LLN, the total welfare during the time interval $[0, \tau_0]$ is defined simply as the expected utility integrated over all the agents

$$\begin{aligned}\text{Tot-welf}(\mathbf{N}, \mathbf{f}) \triangleq& \int_{\mathcal{S}_e} \tilde{U}_{e,0}(s_e; \mathbf{N}, \mathbf{f}) d\bar{N}_{e,0}(s_e) + \lambda_e \int_0^{\tau_0} \int_{\mathcal{S}_e} \tilde{U}_{e,\tau}(s_e; \mathbf{N}, \mathbf{f}) df_{e,\tau}(s_e) d\tau \\ &+ \sum_{t\in\mathrm{H,L}} \left[\int_{\mathcal{S}_w} \tilde{U}^t_{w,0}(s^t_w; \mathbf{N}, \mathbf{f}) d\bar{N}^t_{w,0}(s^t_w) + \lambda^t_w \int_0^{\tau_0} \int_{\mathcal{S}_w} \tilde{U}^t_{w,\tau}(s^t_w; \mathbf{N}, \mathbf{f}) df^t_{w,\tau}(s^t_w) d\tau \right]. \end{aligned} \tag{57}$$

The average welfare over the interval is then defined as the the total welfare divided by the mass of agents who have arrived during that interval.

$$\text{Avg-welf}(\mathbf{N}, \mathbf{f}) \triangleq \frac{\text{Tot-welf}(\mathbf{N}, \mathbf{f})}{\tau_0(\lambda_e + \lambda^{\mathrm{H}}_w + \lambda^{\mathrm{L}}_w)}. \tag{58}$$

Note that (58) generalizes (4). (To see this, observe that in steady state, the numerator in the latter is the steady-state rate of generating utility systemwide times τ_0, i.e., equal to the numerator in the former times τ_0. The denominator in the latter is also τ_0 times the denominator in the former.)

I.1 Planner's solution in a symmetric market with homogeneous agents

In this subsection we consider a more general version of the planner's problem introduced in Section 3, where we allow workers to have arbitrary qualities q_w. We state and prove a corresponding more general version of Proposition 2, and reuse this result later when establishing the optimal solution to the planner's problem in markets with two tiers of workers (Proposition 7 in Section 4).

Generalized Planner's problem. In defining this more general version of the planner's problem, we find it useful to consider the screening thresholds on the idiosyncratic utilities u_{ij} instead of thresholds on the full match utilities v_{ij} (which are the sum of the quality term and the idiosyncratic utility). To avoid confusion, we denote such thresholds as $\tilde{\theta}^{\mathrm{i}}$ and $\tilde{\theta}^{\mathrm{o}}$, respectively. That is, if a worker screens an opportunity with threshold $\tilde{\theta}^{\mathrm{o}}$, he will propose only if $u_{ij} \geq \tilde{\theta}^{\mathrm{o}}$. (In other words, we have $\tilde{\theta}^{\mathrm{i}} = \theta^{\mathrm{i}} - q_w$ and $\tilde{\theta}^{\mathrm{o}} = \theta^{\mathrm{o}} - q_w$ for employers, whereas since $q_e = 0$ we have $\tilde{\theta}^{\mathrm{i}} = \theta^{\mathrm{i}}$ and $\tilde{\theta}^{\mathrm{o}} = \theta^{\mathrm{o}}$ for workers.) The definition of the planner's problem with general qualities is then analogous to the one in Definition 5. That is, we consider a single tier of employers with quality $q_e = 0$ and a single tier of workers with quality $q_w \geq 0$ and the set of strategies $\mathcal{P}$ available to the planner is defined as

$$\mathcal{P} = \left\{ \ (a, \tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}}) : \quad a = (a^{\mathrm{o}}, a^{\mathrm{i}}) \in \mathcal{A}, \ \tilde{\theta}^{\mathrm{o}} \in [0,1], \ \tilde{\theta}^{\mathrm{i}} \in [0,1] \ \right\}$$

where $\mathcal{A}$ is the possible set actions for $(a^{\mathrm{i}}, a^{\mathrm{o}})$ and $\tilde{\theta}^{\mathrm{o}}$ ($\tilde{\theta}^{\mathrm{i}}$) denotes the *threshold on idiosyncratic values* used to screen opportunities (incoming proposals) if a^{o} (a^{i}) includes screening.

We next present a more general version of Proposition 2, which we then proceed to prove.

Proposition 18 (Planner's solution with general qualities)**.** *Consider a market with only one tier of agents on each side with qualities $q_w \geq 0$ and $q_e = 0$, respectively. Let h_w, h_e denote probability measures over $\mathcal{P}$, representing the distribution over strategies adopted by workers and employers upon entering, and let $\bar{L}(h_w, h_e)$ be the induced steady state agent mix. Then, for every fixed $c > 0$ and $\mu > 0$, we have the upper bound*

$$\sup_{\substack{h_w \in \Delta(\mathcal{P}) \\ h_e \in \Delta(\mathcal{P})}} \text{Avg-welf}(\bar{L}(h_w, h_e), (h_w, h_e)) \leq \frac{\min(\lambda_w, \lambda_e)\mathcal{W}(c)}{\lambda_w + \lambda_e},$$

where

$$\mathcal{W}(c) \triangleq \begin{cases} q_w + g(c) & \text{if } c \in (0, \tilde{c}) \,, \\ q_w + 3/2 - \sqrt{2c} & \text{if } c \in \left[\tilde{c}, \frac{1}{8}\right) \,, \\ q_w + 1 & \text{if } c \geq \frac{1}{8} \,, \end{cases} \tag{59}$$

$$g(c) \triangleq \sup_{(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}}) \in (0,1)^2} \frac{1 + \tilde{\theta}^{\mathrm{o}}}{2} - \frac{c}{(1 - \tilde{\theta}^{\mathrm{o}})(1 - \tilde{\theta}^{\mathrm{i}})} + \frac{1 + \tilde{\theta}^{\mathrm{i}}}{2} - \frac{c}{(1 - \tilde{\theta}^{\mathrm{i}})},$$

and $\tilde{c}$ is defined as the unique solution to $3/2 - \sqrt{2\tilde{c}} = g(\tilde{c})$.

Moreover, the following policy (as a function of c) achieves a limiting welfare equal to the upper bound $\frac{\min(\lambda_w, \lambda_e)}{\lambda_w + \lambda_e}\mathcal{W}(c)$ as $\mu \to 0$:

(i) $\mathbf{c} \in (\mathbf{0}, \tilde{\mathbf{c}})$: All agents screen and propose $a^{\mathrm{o}} = \mathrm{S{+}P}$ with threshold $\tilde{\theta}^{\mathrm{o}}(c)$ on idiosyncratic utilities, and screen incoming proposals $a^{\mathrm{i}} = \mathrm{S{+}A/R}$ with threshold $\tilde{\theta}^{\mathrm{i}}(c)$ on idiosyncratic utilities, where $(\tilde{\theta}^{\mathrm{o}}(c), \tilde{\theta}^{\mathrm{i}}(c)) \in (0,1)^2$ is the unique solution in $(0,1)^2$ to the following equations:

$$\left(1 - \tilde{\theta}^{\mathrm{o}}(c)\right)\left(1 + \left(1 - \tilde{\theta}^{\mathrm{o}}(c)\right)\right) = \left(1 - \tilde{\theta}^{\mathrm{i}}(c)\right) \quad \text{and} \quad \left(1 - \tilde{\theta}^{\mathrm{i}}(c)\right)^3\left(1 + \left(1 - \tilde{\theta}^{\mathrm{i}}(c)\right)\right) = 2c\,. \tag{60}$$

(ii) $\mathbf{c} \in [\tilde{\mathbf{c}}, \mathbf{1/8})$: All agents propose without screening $a^{\mathrm{o}} = \mathrm{P\ w/o\ S}$, and screen incoming proposals $a^{\mathrm{i}} = \mathrm{S{+}A/R}$ with threshold on idiosyncratic utilities $\tilde{\theta}^{\mathrm{i}}(c) = 1 - \sqrt{2c}$.

(iii) $\mathbf{c} \geq \mathbf{1/8}$: All agents propose and accept without screening $a = (a^{\mathrm{i}} = \mathrm{A}, a^{\mathrm{o}} = \mathrm{P\ w/o\ S})$.

Moreover, for any $\lambda_e \neq \lambda_w$, for each $c > 0$ there exists $\mu_0 = \mu_0(c) > 0$ such that for all $\mu < \mu_0$ the above policy produces average welfare exactly equal to the upper bound.

Roadmap for the proof of Proposition 18. We first prove the upper bound on the planner's average welfare, and then prove that the specified policy achieves the upper bound.

In proving our upper bound, we prove a slightly stronger result that allows a more powerful planner who can choose arbitrary time varying strategy distributions for incoming agents. We find that, despite allowing such flexibility, the planner simply chooses the single time-invariant pair of strategies characterized in Proposition 18.

Proof of Proposition 18. Fix $c > 0$. We first prove the upper bound, and then establish achievability as $\mu \to 0$.

Upper bound to the average welfare that can be achieved by the planner. We construct an upper bound to the average welfare that can be achieved by the planner. In fact, we allow the planner to adopt arbitrary time varying distributions over strategies for arriving agents, and upper bound the resulting long run average welfare.

Fix an arbitrary initial agent mix $\bar{N}_0$. For all $\tau \geq 0$, denote by $\bar{h}_\tau = (h_{w,\tau}, h_{e,\tau}) \in \Delta(\mathcal{P}) \times \Delta(\mathcal{P})$ the planner's choice of distributions of over strategies adopted by arriving agents at time τ. Consider the resulting average welfare during the interval $[0, \tau_0]$ as defined in (58). Our goal is to upper bound

$$\limsup_{\tau_0 \to \infty} \text{Avg-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)), \quad \text{where } \mathbf{N}(\tau_0) \triangleq (\bar{N}_\tau)_{\tau \in [0,\tau_0]}, \ \mathbf{h}(\tau_0) \triangleq (\bar{h}_\tau)_{\tau \in [0,\tau_0]}. \tag{61}$$

We will now find a convenient way to express Avg-welf$(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0))$ that will facilitate our upper bound.

Let

$$\mathcal{P}_{\text{pr}} \triangleq \{s = (a, \tilde{\theta}^{\text{o}}, \tilde{\theta}^{\text{i}}) \in \mathcal{P} : a^{\text{o}} \in \{\text{S+P}, \text{P w/o S}\}\} \tag{62}$$

be the set of planner strategies which propose, and let

$$\mathcal{P}_{\text{acc}} \triangleq \{s = (a, \tilde{\theta}^{\text{o}}, \tilde{\theta}^{\text{i}}) \in \mathcal{P} : a^{\text{i}} \in \{\text{S+A/R}, \text{A}\}\} \tag{63}$$

be the set of planner strategies which consider incoming proposals.

Let $s_w = (a_w, \tilde{\theta}^{\text{o}}_w, \tilde{\theta}^{\text{i}}_w) \in \mathcal{P}_{\text{pr}}$, $s_e = (a_e, \tilde{\theta}^{\text{o}}_e, \tilde{\theta}^{\text{i}}_{e,r}) \in \mathcal{P}_{\text{acc}}$ denote a pair of strategies such that s_w proposes and s_e considers incoming proposals. (Such strategy pairs are the only ones that can lead to match formation via a proposal by a worker.) Let $V_w(s_w, s_e)$ denote the *expected per-match utility* of matches resulting from workers following strategy s_w proposing to employers following strategy s_e, including both the sum of expected match values as well as the expected screening cost incurred enroute to the formation of one match. Then,

$$V_w(s_w, s_e) \triangleq \begin{cases} \gamma_w(s_w) + \hat{\gamma}_e(s_e) - \frac{c}{1-\tilde{\theta}^{\text{i}}_e} & \text{if } a^{\text{i}}_e = \text{S+A/R}, \\ \quad - \frac{c \cdot \mathbb{I}(s_w \text{involves S+P})}{(1-\tilde{\theta}^{\text{o}}_w \mathbb{I}(s_w \text{involves S+P}))(1-\tilde{\theta}^{\text{i}}_e)} & \\ \gamma_w(s_w) + \hat{\gamma}_e(s_e) - \frac{c \cdot \mathbb{I}(s_w \text{involves S+P})}{(1-\tilde{\theta}^{\text{o}}_w \mathbb{I}(s_w \text{involves S+P}))} & \text{if } a^{\text{i}}_e = \text{A w/o S}. \end{cases} \tag{64}$$

where

$$\gamma_w(s_w) = \mathbb{I}(s_w \text{ involves S+P}) \mathrm{E}\big[v_{ij} \big| v_{ij} \geq \tilde{\theta}^{\text{o}}_w\big] + \mathbb{I}(s_w \text{ involves P w/o S}) \mathrm{E}\big[v_{ij}\big] = \frac{1 + \mathbb{I}(s_w \text{ involves S+P}) \tilde{\theta}^{\text{o}}_w}{2},$$

$$\begin{aligned} \hat{\gamma}_e(s_e) &= \mathbb{I}(s_e \text{ involves S+A/R}) \mathrm{E}\big[v_{ji} \big| v_{ji} \geq \tilde{\theta}^{\text{i}}_e + q_w\big] + \mathbb{I}(s_e \text{ involves A proposals}) \mathrm{E}\big[v_{ji}\big] \\ &= \frac{q_w + 1 + \mathbb{I}(s_e \text{ involves S+A/R}) \tilde{\theta}^{\text{i}}_e}{2}, \end{aligned} \tag{65}$$

using $v_{ij} \sim \text{Uniform}(0, 1)$ in calculating $\gamma_w(s_w)$ and $v_{ji} \sim \text{Uniform}(q_w, q_w + 1)$ in calculating $\hat{\gamma}_e(s_e)$. Define $V_e(s_e, s_w)$ analogously.

The negative terms in (64) represent the expected total screening cost incurred by workers and employers per successful match: If $a^{\text{i}}_e = \text{S+A/R}$, then the employers screen $\frac{1}{1-\tilde{\theta}^{\text{i}}_e}$ proposals in expectation per successful match, each at a cost c. If $a^{\text{o}}_w = \text{S+P}$ then workers screen $1/(1 - \tilde{\theta}^{\text{o}}_w \mathbb{I}(s_w \text{involves S+P}))$ candidates per proposal (and further need to make $\frac{1}{1-\tilde{\theta}^{\text{i}}_e}$ proposals in expectation for a match to form if the employers also screen).

Fix τ_0. Let matches be labeled with (proposer's strategy, recipient's strategy) and let $\rho_w(\cdot, \cdot)$ be the measure over match labels in $\mathcal{P} \times \mathcal{P}$ realized during $[0, \tau_0]$ where the proposer is a worker.

(For any subsets $\mathcal{S} \subseteq \mathcal{P}$ and $\mathcal{S}' \subseteq \mathcal{P}$, $\rho_w(\mathcal{S}, \mathcal{S}')$ is the mass of matches formed from a worker with strategy in $\mathcal{S}$ proposing to an employer with strategy in $\mathcal{S}'$.) Similarly, let $\rho_e(\cdot, \cdot)$ be the measure over match labels in $\mathcal{P} \times \mathcal{P}$ realized during $[0, \tau_0]$ where the proposer is an employer. Then the total welfare realized during the time interval $[0, \tau_0]$ is bounded as

$$
\begin{aligned}
&\text{Tot-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)) \\
&\leq \int_{(s_w, s_e) \in \mathcal{P}_{\text{pr}} \times \mathcal{P}_{\text{acc}}} V_w(s_w, s_e) d\rho_w(s_w, s_e) + \int_{(s_e, s_w) \in \mathcal{P}_{\text{pr}} \times \mathcal{P}_{\text{acc}}} V_e(s_e, s_w) d\rho_e(s_e, s_w)\,, \qquad (66) \\
&\leq \min(\lambda_e \tau_0 + N_{e,0}, \lambda_w \tau_0 + N_{w,0}) \cdot \max \left(\sup_{s_w \in \mathcal{P}_{\text{pr}}, s_e \in \mathcal{P}_{\text{acc}}} V_w(s_w, s_e), \sup_{s_e \in \mathcal{P}_{\text{pr}}, s_w \in \mathcal{P}_{\text{acc}}} V_e(s_e, s_w) \right). \qquad (67)
\end{aligned}
$$

The inequality (66) holds for the following reason. Using the exact LLN, the right-hand side in the first step is the total utility accrued from interactions where an agent who proposes (i.e., who uses a strategy in $\mathcal{P}_{\text{pr}}$) considers as candidate/proposes to an agent who considers incoming proposals (i.e., who uses a strategy in $\mathcal{P}_{\text{acc}}$). These are the only interactions that can produce matches. However, there may be additional, futile interactions where an agent wastes screening effort on a candidate who does not consider incoming proposals. The latter kind of interactions only waste screening effort and hence lower the total welfare, leading to inequality (66). The inequality (67) holds because the welfare per unit mass of matches cannot exceed

$$
\max \left(\sup_{s_w \in \mathcal{P}_{\text{pr}}, s_e \in \mathcal{P}_{\text{acc}}} V_w(s_w, s_e), \sup_{s_e \in \mathcal{P}_{\text{pr}}, s_w \in \mathcal{P}_{\text{acc}}} V_e(s_e, s_w) \right),
$$

and the total mass of matches cannot exceed the smaller of (the mass of workers) and (the mass of employers), where these masses include all agents who are present at any time during $[0, \tau_0]$. Note that for inequality (67) to be tight, no agent on the short side should leave unmatched.

It immediately follows from inequality (67) and the definition (58) of average welfare that

$$
\begin{aligned}
&\limsup_{\tau_0 \to \infty} \text{Avg-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)) \\
&\leq \frac{\min(\lambda_w, \lambda_e)}{\lambda_w + \lambda_e} \cdot \max \left(\sup_{s_w \in \mathcal{P}_{\text{pr}}, s_e \in \mathcal{P}_{\text{acc}}} V_w(s_w, s_e), \sup_{s_e \in \mathcal{P}_{\text{pr}}, s_w \in \mathcal{P}_{\text{acc}}} V_e(s_e, s_w) \right),
\end{aligned}
$$

since the initial masses $N_{e,0}$ and $N_{w,0}$ are fixed. This upper bound holds for arbitrary $(\bar{h}_\tau)_{\tau \geq 0}$ chosen by the planner.

We now prove that $\sup_{s_w \in \mathcal{P}_{\text{pr}}, s_e \in \mathcal{P}_{\text{acc}}} V_w(s_w, s_e) = \mathcal{W}(c)$. The same argument will also establish that $\sup_{s_e \in \mathcal{P}_{\text{pr}}, s_w \in \mathcal{P}_{\text{acc}}} V_e(s_e, s_w) = \mathcal{W}(c)$. To find the pair of strategies for which the expected per-match utility is maximized, we start by noting that each match must result from one of the following pair of actions: (1) None of the agents screen, (2) Only one of the agents screen, or (3) Both agents screen. For each of the three pair of actions just described, the maximum achievable expected per-match utility (when one or both agents are screening, this is the per-match utility that results from optimizing over the thresholds) for a given c can be calculated as follows:

1. *None of the agents screen*; one agent proposes and the other accepts the proposal. In this case, agents are matched to a random partner and they derive an expected utility of $1/2$ (workers) or $q_w + 1/2$ (employers). Hence, the total expected utility generated by such a match is $q_w + 1$, as agents do not incur any screening costs.

2. *Only one of the agents screen*, one agent proposes and the other accepts the proposal. In

this case, as non-screening agents neither internalize nor generate cross-side externalities, it is also irrelevant which side proposes and which side is receiving the proposal. Suppose employers screen using a threshold θ_e. Note that any successful match guarantees a utility of q_w to employers. Thus, instead of using threshold θ_e such that an employer agrees to match only if $q_w + u_{ij} \geq \theta_e$, it is equivalent to use threshold on idiosyncratic values $\tilde{\theta}_e \in [0,1]$ such that he matches only if $u_{ij} \geq \tilde{\theta}_e$. The expected total utility generated per match is $\frac{1}{2} + q_w + \frac{1+\tilde{\theta}_e}{2} - \frac{c}{(1-\tilde{\theta}_e)}$, where the first two terms correspond to the expected utility of the (non-screening) worker, and the last three terms represent the expected utility from the match minus the expected screening cost per match incurred by the screening employer. It is easy to see that the per-match utility is maximized when the screening employers use a threshold $\tilde{\theta}_e = 1 - \sqrt{2c}$ on the on idiosyncratic value, leading to a total per-match utility of $q_w + 3/2 - \sqrt{2c}$. Moreover, noting that the expected total per-match utility remains the same if workers are screening instead, we conclude that, if screening, they should use the same threshold on idiosyncratic values as employers.

3. *Agents on both sides screen*; assume that agents one side S+P (proposers) and agents on the other side S+A/R (recipients) with thresholds on the idiosyncratic values given by $\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}} \in [0,1)$ respectively. (It might be possible for agents on both sides to simultaneously propose and wait for proposals. It turns out that there will be a unique pair of idiosyncratic-value thresholds maximizing the per-match utility. Therefore, both sides must use the same strategy —threshold $\tilde{\theta}^{\mathrm{o}}$ to issue proposals, and $\tilde{\theta}^{\mathrm{i}}$ when receiving proposals— as described above.) The expected total per-match utility at c is given by

$$q_w + \ell_c(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}})$$

where $\ell_c(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}})$ is defined as $\ell_c(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}}) \triangleq \frac{1+\tilde{\theta}^{\mathrm{o}}}{2} - \frac{c}{(1-\tilde{\theta}^{\mathrm{o}})(1-\tilde{\theta}^{\mathrm{i}})} + \frac{1+\tilde{\theta}^{\mathrm{i}}}{2} - \frac{c}{(1-\tilde{\theta}^{\mathrm{i}})}$. The term q_w represents the guaranteed utility resulting from any match. The positive terms in ℓ_c represent the expected value of the idiosyncratic match utilities. Finally, the negative terms represent the expected cost collectively incurred for such a match to occur. Note that for every $c > 0$, the function ℓ_c is strictly concave[41] in $(0,1)^2$. One can easily verify that if $c \geq 1/8$, either agent screening leads to a suboptimal per match utility, since screening produces a per-match net utility of (weakly) less than $q_w + 1/2 + 1/2 = q_w + 1$, less than the per match utility in the first case. Hence, consider a fixed $c < 1/8$. We show that the supremum of ℓ_c over the values of $(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}}) \in (0,1)^2$ occurs at an interior point and characterize this point. The first order condition $\partial \ell_c / \partial \tilde{\theta}^{\mathrm{o}} = 0$ gives

$$(1-\tilde{\theta}^{\mathrm{i}})(1-\tilde{\theta}^{\mathrm{o}})^2 = 2c\,, \tag{68}$$

and the first order condition $\partial \ell_c / \partial \tilde{\theta}^{\mathrm{i}} = 0$ gives

$$(1-\tilde{\theta}^{\mathrm{i}})^2(1-\tilde{\theta}^{\mathrm{o}}) = 2c(1 + (1-\tilde{\theta}^{\mathrm{o}}))\,. \tag{69}$$

Substituting the expression for $2c$ from (68) into (69) gives the first part of (60), and substituting the expression for $(1-\tilde{\theta}^{\mathrm{i}})$ from (68) into (69) gives the second part of (60). Multiple solutions to (60) are ruled out since ℓ_c is strictly concave. We now show the existence of a solution in $(0,1)^2$ to the FOCs for all $c \in (0, 1/8)$, and since ℓ_c is strictly concave, this solution will be the maximizer of ℓ_c in $(0,1)^2$. Consider the second part of (60). It is easy to see that left-hand side $(1-\tilde{\theta}^{\mathrm{o}})^3(2-\tilde{\theta}^{\mathrm{o}})$ is monotone decreasing in $\tilde{\theta}^{\mathrm{o}} \in (0,1)$ from 2 to 0. Since the right-hand side is simply $2c$, the solution $\tilde{\theta}^{\mathrm{o}}$ to this condition is monotone decreasing in c. For $c \to 0^+$, we have $\tilde{\theta}^{\mathrm{o}} \to 1^-$ and for $c = 1/8$, we have $\tilde{\theta}^{\mathrm{o}} \approx 0.456$. Thus, for all $c \in (0, 1/8)$, we have $\tilde{\theta}^{\mathrm{o}} \in (0.45, 1) \subseteq (0,1)$ as needed. Now consider

[41] One can check that $\nabla^2 \ell_c(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}}) = -\begin{bmatrix} \frac{2c}{(1-\tilde{\theta}^{\mathrm{o}})^3(1-\tilde{\theta}^{\mathrm{i}})} & \frac{c}{(1-\tilde{\theta}^{\mathrm{o}})^2(1-\tilde{\theta}^{\mathrm{i}})^2} \\ \frac{c}{(1-\tilde{\theta}^{\mathrm{o}})^2(1-\tilde{\theta}^{\mathrm{i}})^2} & \frac{2c}{(1-\tilde{\theta}^{\mathrm{i}})^3} + \frac{2c}{(1-\tilde{\theta}^{\mathrm{o}})(1-\tilde{\theta}^{\mathrm{i}})^3} \end{bmatrix} \prec 0$ for $(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}}) \in (0,1)^2$.

the first part of (60). It expresses $\tilde{\theta}^{\mathrm{i}}$ as $1-(1-\tilde{\theta}^{\mathrm{o}})(2-\tilde{\theta}^{\mathrm{o}})$, a monotone increasing function of $\tilde{\theta}^{\mathrm{o}}$. Since $\tilde{\theta}^{\mathrm{o}} \in (0.45, 1)$, we immediately deduce that $1-0.55\cdot 1.55 = 0.1475 < \tilde{\theta}^{\mathrm{i}} < 1$, and in particular, $\tilde{\theta}^{\mathrm{i}} \in (0,1)$ for all $c \in (0, 1/8)$, as required. The function $g(c) = \ell_c(\tilde{\theta}^{\mathrm{o}}(c), \tilde{\theta}^{\mathrm{i}}(c))$ at this solution, for all $c \in (0, 1/8)$.

To show that the maximum per-match utility achievable is $\mathcal{W}(c)$, we will find, for every fixed c, the pair of actions with the corresponding optimal thresholds (from the three cases described above) that maximizes the per-match utility, and show that the resulting per-match utility is equal to $\mathcal{W}(c)$. We next show that $g(\tilde{c}) - 3/2 + \sqrt{2\tilde{c}} = 0$ has a unique solution, and that the third case maximizes expected utility for $c < \tilde{c}$, whereas the second case maximizes expected utility for $c \in [\tilde{c}, 1/8)$. The key step will be to show that the difference between the utilities under the third case and the second case $\hat{g}(c) \triangleq g(c) - 3/2 + \sqrt{2c}$ is a strictly decreasing function of c, which is positive for $c \to 0^+$ and negative for $c \to 1/8$.

Observe that when we use the FOCs (60) in the expression for $\ell_c(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}})$, we find that we can write $g(c) = 2 - 3(1-\tilde{\theta}^{\mathrm{o}})/2 - (1-\tilde{\theta}^{\mathrm{o}})^2$. Differentiating the second part of (60) with respect to c leads to

$$\frac{d\tilde{\theta}^{\mathrm{o}}}{dc} = -\frac{2}{(1-\tilde{\theta}^{\mathrm{o}})^2(3+4(1-\tilde{\theta}^{\mathrm{o}}))}.$$

Combining, we obtain that

$$g'(c) = \big(3/2 + 2(1-\tilde{\theta}^{\mathrm{o}})\big)\frac{d\tilde{\theta}^{\mathrm{o}}}{dc} = -\frac{1}{(1-\tilde{\theta}^{\mathrm{o}})^2}.$$

It follows that

$$\hat{g}'(c) = g'(c) + \frac{1}{\sqrt{2c}} = \frac{-\sqrt{2c} + (1-\tilde{\theta}^{\mathrm{o}})^2}{(1-\tilde{\theta}^{\mathrm{o}})^2\sqrt{2c}} < 0, \tag{70}$$

since the second part of (60) implies

$$(1-\tilde{\theta}^{\mathrm{o}})^4 < (1-\tilde{\theta}^{\mathrm{o}})^3 + (1-\tilde{\theta}^{\mathrm{o}})^4 = 2c \quad \Rightarrow \quad (1-\tilde{\theta}^{\mathrm{o}})^2 < \sqrt{2c}.$$

Clearly, $\hat{g}(0^+) = g(0^+) - 3/2 = 2 - 3/2 = 1/2 > 0$. As the final step we show $\hat{g}(1/8) < 0$. Note that for all $c \in (0, 1/8]$ we have

$$\ell_c(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}}) < \frac{1+\tilde{\theta}^{\mathrm{o}}}{2} - \frac{c}{(1-\tilde{\theta}^{\mathrm{o}})} + \frac{1+\tilde{\theta}^{\mathrm{i}}}{2} - \frac{c}{(1-\tilde{\theta}^{\mathrm{i}})} \leq 2\left(1-\sqrt{2c}\right),$$

using that $1/(1-\tilde{\theta}^{\mathrm{i}}) > 1.17$ for all candidate $\tilde{\theta}^{\mathrm{i}} \in (0.1475, 1)$, and $\max_{\theta\in(0,1)} \frac{1+\theta}{2} - \frac{c}{(1-\theta)} = 1 - \sqrt{2c}$. It follows that $g(c) = \sup_{(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}})\in(0,1)^2} \ell_c(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}}) < 2\left(1-\sqrt{2c}\right)$ and in particular, $g(1/8) < 1$. We deduce $\hat{g}(1/8) < 0$ as required.

It remains to consider the first case, that is, no agent screens. We know that the expected utility in if no agent screens is $q_w + 1$, and clearly this (weakly) exceeds the expected utility $q_w + 3/2 - \sqrt{2c}$ in the second case (only one agent screens) if and only if $c \geq 1/8$. We already checked that the third case (both agents screen) produces per match utility $q_w + g(1/8) < q_w + 1$ for $c = 1/8$, and the same upper bound holds for all $c > 1/8$ since the per match utility is monotone decreasing in c. Thus, for all $c \leq 1/8$ the no-agent-screens case beats the third case as well. Combining, it follows that the no-agent-screens case dominates for $c \geq 1/8$, whereas the one-agent-screens case wins for $c \in [\tilde{c}, 1/8)$ and the both-agents-screen case wins for $c < \tilde{c}$. Recalling the definition (59) of $\mathcal{W}(c)$, we immediately obtain that the maximum achievable expected per-match utility is $\mathcal{W}(c)$. Thus we have established the identity we set out to

$$\sup_{s_w \in \mathcal{P}_{\mathrm{pr}}, s_e \in \mathcal{P}_{\mathrm{acc}}} V_w(s_w, s_e) = \sup_{s_e \in \mathcal{P}_{\mathrm{pr}}, s_w \in \mathcal{P}_{\mathrm{acc}}} V_e(s_e, s_w) = \mathcal{W}(c)\,. \tag{71}$$

Planner's policy achieving the upper bound. The argument above tells us that:

1. For $c \in (0, \tilde{c})$, it is optimal to induce matches in which both agents screen (case 3). One possible implementation (as specified in the proposition statement) is that both sides screen and propose with threshold on idiosyncratic values $\tilde{\theta}^{\mathrm{o}}(c)$ and both sides screen and accept/reject with threshold on idiosyncratic values $\tilde{\theta}^{\mathrm{i}}(c)$ where $(\tilde{\theta}^{\mathrm{o}}(c), \tilde{\theta}^{\mathrm{i}}(c)) \in (0,1)^2$ is the unique maximizer of $\ell_c(\tilde{\theta}^{\mathrm{o}}, \tilde{\theta}^{\mathrm{i}})$.

2. For $c \in [\tilde{c}, 1/8)$, agents should screen only when issuing or when receiving proposals, but not both (case 2). In particular, this can be implemented (as specified in the proposition statement) by making all agents propose without screening and screen + accept/reject incoming proposals with threshold on idiosyncratic utilities $\tilde{\theta}^{\mathrm{i}} = 1 - \sqrt{2c}$.

3. Finally, for $c \geq 1/8$, no one should screen (case 1). This can be implemented (as specified in the proposition statement) by making both sides propose and accept without screening.

Suppose that the central planner instructs all agents to use the strategy specified in the proposition for the corresponding c. We already showed above that the net per-match utility as a function of c is then given by function $\mathcal{W}(c)$. It remains to show that the required flow of matches is achieved. As agents on each side follow a single strategy, Proposition 11 shows that a steady-state of the system exists and thus the average welfare is well-defined. First consider the case $\lambda_w \neq \lambda_e$. Proposition 11 tells us that in steady state, there exists $\mu_0 = \mu_0(c) > 0$ such that there is a mass 0 of agents on the short side in steady state for all $\mu < \mu_0$ and so no short side agents leave without matching, i.e., the match flow rate is *exactly* $\min(\lambda_w, \lambda_e)$ for all such μ. This establishes the last sentence of the proposition which covers the case $\lambda_w \neq \lambda_e$. If $\lambda_w = \lambda_e$, the steady state mass on each side is $\frac{1}{\mu+2\eta} = \Theta(1)$ for $\eta = (1 - \tilde{\theta}^{\mathrm{o}}\mathbb{I}(\text{proposer screens}))(1 - \tilde{\theta}^{\mathrm{o}}\mathbb{I}(\text{recipient screens}))$, so the flow of agents leaving without matching is $\frac{\mu}{\mu+2\eta} = \Theta(\mu)$, i.e., vanishing as $\mu \to 0$. It follows that average welfare equal to the upper bound is achieved in the limit $\mu \to 0$ if all agents use the prescribed strategy, as required. □

The planner's solution in vertically differentiated markets

We now characterize the planner's solution in the markets discussed in Section 4 as stated in Proposition 7. To establish this proposition, we will make heavy use of Appendix I.1 where we established the analogous result (Proposition 2) for markets with ex-ante homogeneous agents.

Proof of Proposition 7. The proof proceeds along the same lines as that of Proposition 2. Fix $c > 0$. We first show the upper bound and then establish achievability as $\mu \to 0$.

Upper bound to the average welfare that can be achieved by the planner. In fact, we allow the planner to adopt arbitrary time varying distributions over strategies for arriving agents, and upper bound the resulting long run average welfare. Fix an arbitrary initial agent mix $\bar{N}_0$. For all $\tau \geq 0$, denote by $\bar{h}_\tau = (h^{\mathrm{H}}_{w,\tau}, h^{\mathrm{L}}_{w,\tau}, h_{e,\tau}) \in \Delta(\mathcal{P}_w) \times \Delta(\mathcal{P}_w) \times \Delta(\mathcal{P}_e)$ the planner's choice of distributions of over strategies adopted by arriving agents at time τ, where $\mathcal{P}_w = \mathcal{P}$ and $\mathcal{P}_e$ is the analogous set of planner strategies for employers

$$\mathcal{P}_e \triangleq \left\{(a, \tilde{\Theta}) :\ a = (a^{\mathrm{i}}, a^{\mathrm{o}}) \in \mathcal{A}_e,\ \tilde{\Theta} = (\tilde{\theta}^{\mathrm{H,o}}, \tilde{\theta}^{\mathrm{L,o}}, \tilde{\theta}^{\mathrm{H,i}}, \tilde{\theta}^{\mathrm{L,i}}) \in [0,1]^4\right\} . \tag{72}$$

Consider the resulting average welfare during the interval $[0, \tau_0]$ as defined in (58). Our goal is to upper bound

$$\limsup_{\tau_0 \to \infty} \text{Avg-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)) , \quad \text{where } \mathbf{N}(\tau_0) \triangleq (\bar{N}_\tau)_{\tau \in [0,\tau_0]} ,\ \mathbf{h}(\tau_0) \triangleq (\bar{h}_\tau)_{\tau \in [0,\tau_0]} . \tag{73}$$

Let $\mathcal{P}_{\text{pr}}$ and $\mathcal{P}_{\text{acc}}$ be as defined in (62) and (63), respectively. Define the analogous strategy sets for employers that propose/consider incoming proposals from a specific worker tier as follows

$$\begin{aligned} \mathcal{P}^t_{\text{pr}} &\triangleq \{s = (a, \tilde{\Theta}) \in \mathcal{P}_e : a^{\mathrm{o}}_t \in \{\text{S+P}, \text{P w/o S}\}\} \quad \forall t \in \{\text{H}, \text{L}\} \\ \mathcal{P}^t_{\text{acc}} &\triangleq \{s = (a, \tilde{\Theta}) \in \mathcal{P}_e : a^{\mathrm{i}}_t \in \{\text{S+A/R}, \text{A}\}\} \qquad \forall t \in \{\text{H}, \text{L}\} . \end{aligned}$$

Using verbatim the argument leading to (66) in the proof of Proposition 18, we have the following analogous bound on the total welfare

$$\begin{aligned} &\text{Tot-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)) \\ &\leq \sum_{t \in \{\text{H},\text{L}\}} \left[\int_{(s_w,s_e) \in \mathcal{P}_{\text{pr}} \times \mathcal{P}^t_{\text{acc}}} V^t_w(s_w, s_e) d\rho^t_w(s_w, s_e) + \int_{(s_e,s_w) \in \mathcal{P}^t_{\text{pr}} \times \mathcal{P}_{\text{acc}}} V^t_e(s_e, s_w) d\rho^t_e(s_e, s_w)\right] , \end{aligned} \tag{74}$$

where $V^t_w(s_w, s_e)$ is given by (64) and (65) with

$$q_w = \begin{cases} q & \text{for } t = \text{H and} \\ 0 & \text{for } t = \text{L,} \end{cases} \tag{75}$$

and $\rho^t_w(\cdot, \cdot)$ is the measure over (proposer strategy, recipient strategy) for matches formed in $[0, \tau_0]$ where the proposer was a tier-t worker.

We now deduce a bound to total welfare analogous to (67), via a similar argument bounding the total mass of matches formed. Before considering the mass of matches, observe from the definition (64), (65) and (75) of $V^t_w(s_w, s_e)$ that $V^{\mathrm{H}}_w(s_w, s_e) = V^{\mathrm{L}}_w(s_w, s'_e) + q$ for any s_e, s'_e such that s_e treats incoming proposals from tier-H exactly the same as s'_e treats incoming proposals from tier-L (formally, we are assuming here that $(a^{\mathrm{i}}_{\mathrm{H}}, \tilde{\theta}^{\mathrm{i,H}})$ under s_e is identical to $(a^{\mathrm{i}}_{\mathrm{L}}, \tilde{\theta}^{\mathrm{i,L}})$ under s'_e). It follows immediately that

$$\sup_{(s_w,s_e) \in \mathcal{P}_{\text{pr}} \times \mathcal{P}^{\mathrm{H}}_{\text{acc}}} V^{\mathrm{H}}_w(s_w, s_e) = q + \sup_{(s_w,s_e) \in \mathcal{P}_{\text{pr}} \times \mathcal{P}^{\mathrm{L}}_{\text{acc}}} V^{\mathrm{L}}_w(s_w, s_e) \tag{76}$$

and similarly that

$$\sup_{(s_e,s_w) \in \mathcal{P}^{\mathrm{H}}_{\text{pr}} \times \mathcal{P}_{\text{acc}}} V^{\mathrm{H}}_e(s_e, s_w) = q + \sup_{(s_e,s_w) \in \mathcal{P}^{\mathrm{L}}_{\text{pr}} \times \mathcal{P}_{\text{acc}}} V^{\mathrm{L}}_e(s_e, s_w) . \tag{77}$$

From (74), it follows that

$$
\begin{aligned}
&\text{Tot-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)) \\
&\leq (\text{Mass of H-worker matches}) \max\Big(\sup_{(s_w,s_e)\in\mathcal{P}_{\text{pr}}\times\mathcal{P}^{\text{H}}_{\text{acc}}} V^{\text{H}}_w(s_w,s_e), \sup_{(s_e,s_w)\in\mathcal{P}^{\text{H}}_{\text{pr}}\times\mathcal{P}_{\text{acc}}} V^{\text{H}}_e(s_e,s_w)\Big) + \\
&\quad (\text{Mass of L-worker matches}) \max\Big(\sup_{(s_w,s_e)\in\mathcal{P}_{\text{pr}}\times\mathcal{P}^{\text{L}}_{\text{acc}}} V^{\text{L}}_w(s_w,s_e), \sup_{(s_e,s_w)\in\mathcal{P}^{\text{L}}_{\text{pr}}\times\mathcal{P}_{\text{acc}}} V^{\text{L}}_e(s_e,s_w)\Big) \\
&= q \times (\text{Mass of H-worker matches}) + \\
&\quad (\text{Total mass of matches}) \max\Big(\sup_{(s_w,s_e)\in\mathcal{P}_{\text{pr}}\times\mathcal{P}^{\text{L}}_{\text{acc}}} V^{\text{L}}_w(s_w,s_e), \sup_{(s_e,s_w)\in\mathcal{P}^{\text{L}}_{\text{pr}}\times\mathcal{P}_{\text{acc}}} V^{\text{L}}_e(s_e,s_w)\Big)\,. \qquad (78)
\end{aligned}
$$

where we used (76) and (77) in the second step. Clearly, the mass of matches formed during $[0, \tau_0]$ involving high-workers cannot exceed the mass of high-workers present at any time during $[0, \tau_0]$ which is $N^{\text{H}}_{w,0} + \tau_0\lambda^{\text{H}}_w$. Similarly, the total mass of matches formed during $[0, \tau_0]$ cannot exceed the mass of employers present at any time during $[0, \tau_0]$ which is $N_{e,0} + \tau_0\lambda_e$. We deduce from (78) that

$$
\begin{aligned}
&\text{Tot-welf}(\mathbf{N}(\tau_0),\mathbf{h}(\tau_0)) \leq q(N^{\text{H}}_{w,0} + \tau_0\lambda^{\text{H}}_w) + \\
&\qquad (N_{e,0} + \tau_0\lambda_e) \max\Big(\sup_{(s_w,s_e)\in\mathcal{P}_{\text{pr}}\times\mathcal{P}^{\text{L}}_{\text{acc}}} V^{\text{L}}_w(s_w,s_e), \sup_{(s_e,s_w)\in\mathcal{P}^{\text{L}}_{\text{pr}}\times\mathcal{P}_{\text{acc}}} V^{\text{L}}_e(s_e,s_w)\Big)\,. \qquad (79)
\end{aligned}
$$

But recalling the bound (71) established in the proof of Proposition 18, and plugging in $q_w = 0$, we have that

$$
\sup_{(s_w,s_e)\in\mathcal{P}_{\text{pr}}\times\mathcal{P}^{\text{L}}_{\text{acc}}} V^{\text{L}}_w(s_w,s_e) = \sup_{(s_e,s_w)\in\mathcal{P}^{\text{L}}_{\text{pr}}\times\mathcal{P}_{\text{acc}}} V^{\text{L}}_e(s_e,s_w) = \mathcal{W}(c) = g(c)\,, \qquad (80)
$$

for $c < \tilde{c}$. Note here that $\tilde{c} \approx 0.060$ whereas from the definition (40) of $c_0(q)$ we can infer $c_0(q) \leq 1/32 \approx 0.031$, so $c_0(q) < \tilde{c}$ and we deduce that (80) holds in particular for all $c < c_0(q)$. Combining (79) and (80), we have

$$
\text{Tot-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)) \leq q(N^{\text{H}}_{w,0} + \tau_0\lambda^{\text{H}}_w) \;+ (N_{e,0} + \tau_0\lambda_e)g(c) \quad \text{for all } c \leq c_0(q)\,. \qquad (81)
$$

We immediately deduce that

$$
\begin{aligned}
&\text{Avg-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)) = \frac{\text{Tot-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0))}{\tau_0(\lambda^{\text{H}}_w + \lambda^{\text{L}}_w + \lambda_e)} \leq \frac{q(N^{\text{H}}_{w,0} + \tau_0\lambda^{\text{H}}_w) \;+ (N_{e,0} + \tau_0\lambda_e)g(c)}{\tau_0(\lambda^{\text{H}}_w + \lambda^{\text{L}}_w + \lambda_e)} \\
\Rightarrow\quad &\limsup_{\tau_0\to\infty} \text{Avg-welf}(\mathbf{N}(\tau_0), \mathbf{h}(\tau_0)) \leq \frac{\lambda^{\text{H}}_w q + \lambda_e g(c)}{\lambda^{\text{H}}_w + \lambda^{\text{L}}_w + \lambda_e}
\end{aligned}
$$

for all $c \leq c_0(q)$, which is the upper bound we sought.

Planner's policy achieving the upper bound. Fix $c < c_0(q)$. Suppose all agents follow the policy prescribed in the statement of Proposition 7, i.e., low workers do not propose, employers and high workers screen and propose with threshold $\tilde{\theta}^{\text{o}}$ on idiosyncratic utilities with employers preferring to propose to high workers, and all agents screen incoming proposals with threshold $\tilde{\theta}^{\text{i}}$ on idiosyncratic utilites. Suppose, for simplicity,[42] that the system starts at some fixed initial state $N^{\text{H}}_w = 0, N_e > \frac{\lambda^{\text{H}}_w}{(1-\tilde{\theta}^{\text{i}})(1-\tilde{\theta}^{\text{o}})}, N^{\text{L}}_w > N_e$. We show that an average welfare equal to the upper bound is achieved as $\mu \to 0$ by establishing a series of properties. Throughout we assume the exact LLN.

[42]It is a straightforward but tedious exercise to extend the analysis to an arbitrary initial state

Property 1. *The mass of workers always exceeds the mass of employers in the system, i.e.,* $N_w = N_w^{\mathrm{H}} + N_w^{\mathrm{L}} > N_e$ *for all* $\tau \geq 0$.
Proof: The property clearly holds at time 0. Furthermore, we will show that there exists $\epsilon > 0$ such that any time it holds that $N_w < N_e + \epsilon$, we have $dN_w/d\tau > dN_e/d\tau$. Clearly, the property will follow for all $\tau > 0$ if we are able to establish the existence of such an ϵ, in fact, we will immediately deduce that $N_w \geq N_e + \min(\epsilon, N_{w,0} - N_{e,0})$ at all times $\tau \geq 0$.

The reason that $N_w < N_e + \epsilon \Rightarrow dN_w/d\tau > dN_e/d\tau$ is as follows. The flow of agents departing due to matching is equal on the two sides of the market by mass balance, whereas the flow of agents leaving without matching is $\mu(N_w - N_e) < \mu\epsilon$ more for workers than for employers. On the other hand, the arrival flow of workers is $\lambda_w^{\mathrm{H}} - \lambda_w^{\mathrm{L}} - \lambda_e > 0$ more than that for employers. Choosing $\epsilon = (\lambda_w^{\mathrm{H}} - \lambda_w^{\mathrm{L}} - \lambda_e)/\mu > 0$ ensures that the difference in arrival flows exceeds the difference in departure flows, and the desired inequality $dN_w/d\tau > dN_e/d\tau$ follows.

Property 2. *For small enough* μ *the following holds. For all* $\tau > 0$, *we have* $N_e > \frac{\lambda_w^{\mathrm{H}}}{(1-\tilde{\theta}^{\mathrm{i}})(1-\tilde{\theta}^{\mathrm{o}})}$ *while the mass of high workers in the system remains zero* $N_w^{\mathrm{H}} = 0$ *because all high workers who arrive to the system immediately match and leave. Furthermore,* $\lim_{\tau\to\infty} N_e = \frac{\lambda_e}{\mu+(1-\tilde{\theta}^{\mathrm{i}})(1-\tilde{\theta}^{\mathrm{o}})} < \frac{\lambda_e}{(1-\tilde{\theta}^{\mathrm{i}})(1-\tilde{\theta}^{\mathrm{o}})}$.
Proof: The flow of opportunities to employers is N_e at any time. Since $N_w > N_e \geq 0$, a worker candidate is always available, so each opportunity converts into a match with probability $(1 - \tilde{\theta}^{\mathrm{i}})(1 - \tilde{\theta}^{\mathrm{o}})$. Thus, the total flow of matches due to employers proposing is $N_e(1 - \tilde{\theta}^{\mathrm{i}})(1 - \tilde{\theta}^{\mathrm{o}})$. At $\tau = 0$ by assumption we have $N_{e,0}(1 - \tilde{\theta}^{\mathrm{i}})(1 - \tilde{\theta}^{\mathrm{o}}) > \lambda_w^{\mathrm{H}}$, and so for all times $\tau < \tau_1$ for

$$\tau_1 \triangleq \inf\left\{\tau : N_{e,\tau} < \frac{\lambda_w^{\mathrm{H}}}{(1-\tilde{\theta}^{\mathrm{i}})(1-\tilde{\theta}^{\mathrm{o}})}\right\} > 0 \tag{82}$$

we have that high-workers match as soon as they arrive, since employers prefer high worker candidates. We will show $\tau_1 = \infty$. Observe that for $\tau < \tau_1$, matches are formed only due to proposals by employers, so the flow of employers matching and leaving is $N_e(1 - \tilde{\theta}^{\mathrm{i}})(1 - \tilde{\theta}^{\mathrm{o}})$ while the flow of employers leaving without matching is μN_e. Overall, for $\tau < \tau_1$ we have

$$\begin{aligned} & \frac{dN_e}{d\tau} = \lambda_e - N_e(\mu + (1 - \tilde{\theta}^{\mathrm{i}})(1 - \tilde{\theta}^{\mathrm{o}})) \\ \Rightarrow \quad & N_e(\tau) = \frac{\lambda_e}{\zeta} + \left(N_{e,0} - \frac{\lambda_e}{\zeta}\right)e^{-\zeta\tau} \quad \text{for } \zeta \triangleq \mu + (1 - \tilde{\theta}^{\mathrm{i}})(1 - \tilde{\theta}^{\mathrm{o}})\,. \end{aligned}$$

We immediately obtain that for small enough μ, we have $\lambda_e/\zeta > \frac{\lambda_w^{\mathrm{H}}}{(1-\tilde{\theta}^{\mathrm{i}})(1-\tilde{\theta}^{\mathrm{o}})}$ since $\lambda_e > \lambda_w^{\mathrm{H}}$ which leads to $N_e \geq \min(N_{e,0}, \lambda_e/\zeta) > \frac{\lambda_w^{\mathrm{H}}}{(1-\tilde{\theta}^{\mathrm{i}})(1-\tilde{\theta}^{\mathrm{o}})}$ for all τ, i.e., $\tau_1 = \infty$ and all high workers match as soon as they arrive, as we wanted to show. It also follows that $\lim_{\tau\to\infty} N_e = \lambda_e/\zeta < \frac{\lambda_e}{(1-\tilde{\theta}^{\mathrm{i}})(1-\tilde{\theta}^{\mathrm{o}})}$, which is the last line of the property. In particular, the system converges to a steady state.

Property 3. *The flow of matches involving high workers is* λ_w^{H} *whereas the total flow of matches is* $\lambda_e - O(\mu)$ *in steady state.*
Proof: This property follows from Property 2. Since high workers match immediately upon arrival, the flow of matches involving high workers is λ_w^{H}. Furthermore, the mass of employers converges to a steady state limit which is $O(1)$ so the steady state flow of employers leaving without matching is $O(\mu)$, and so the steady state total match flow is $\lambda_e - O(\mu)$.

Property 4. *For any* $c < c_0(q)$ *matches involving high workers earn per-match net utility* $q + g(c)$ *on average, while matches involving low workers earn per match net utility* $g(c)$ *on average. No screening effort is wasted in screening candidates who will not consider incoming proposals.*

Proof: The first sentence of the property is immediate from the definition of the planner's policy and from the proof of Proposition 18 where we showed that per match net utility is maximized by screening and proposing with threshold $\tilde{\theta}^{\mathrm{o}}$ and screening incoming proposals with threshold $\tilde{\theta}^{\mathrm{i}}$, producing per match net utility $g(c) + q_w$ for $c < \tilde{c}$ (recall that $c_0(q) < \tilde{c}$ as we showed above). The second sentence follows since all agents consider incoming proposals under the planner's policy.

It follows from Properties 3 and 4 that the steady state average welfare under the planner's policy as $\mu \to 0$ is $\frac{\lambda_w^{\mathrm{H}} q + \lambda_e g(c)}{\lambda_w^{\mathrm{H}} + \lambda_w^{\mathrm{L}} + \lambda_e}$, i.e., equal to the upper bound. $\square$

J Supplementary material to Appendix E (Appendix to Section 3: ex-ante homogeneous agents)

J.1 Auxiliary lemmas for characterizing equilibria in strictly unbalanced markets

The next few lemmas will help us to characterize equilibria in strictly unbalanced markets. Without loss of generality we assume $\lambda_w > \lambda_e$.

Lemma J.1. *Fix $R = \lambda_w/\lambda_e > 1$, $c \leq 1/8$. Consider a setting where employers use strategy s_e, in which they do not propose and accept incoming proposals with probability at least $\sqrt{2c}$. That is, let $\hat{p}(s_e)$ be the probability of accepting an incoming proposal*

$$\hat{p}(s_e) \triangleq \mathbb{I}(s_e \text{ does not involve I}) - \theta_e \mathbb{I}(s_e \text{ involves S+A/R}),$$

and require $\hat{p}(s_e) \geq \sqrt{2c}$, i.e., if employers screen, they must use a threshold $\theta_e \leq 1 - \sqrt{2c}$.

Let s_w be any worker strategy in which workers propose with probability at least $\sqrt{2c}$, i.e, let $p(s_w)$ be the probability of proposing to a candidate

$$p(s_w) \triangleq \mathbb{I}(s_w \text{ does not involve DN}) - \theta_w \mathbb{I}(s_w \text{ involves S+P}),$$

and require $p(s_w) \geq \sqrt{2c}$.

Suppose that, upon arrival to the system, all workers select strategy s_w and all employers select strategy s_e. Then, there exists a $\mu_0(c) > 0$ such that for every μ with $0 < \mu < \mu_0(c)$, we have that:

1. *A steady state $\bar{L} = [\bar{L}_e(s_e), \bar{L}_w(s_w)]'$ exists and, moreover, $\bar{L}_e(s_e) = 0$ and $\bar{L}_w(s_w) = (R-1)/\mu$.*
2. *Let p be the likelihood that a worker using strategy s with $a^o \neq$ DN (i.e., strategy s that considers candidates) is presented with a candidate before leaving unmatched. Then,*

$$\mathrm{p} = \frac{1}{p(s_w)\hat{p}(s_e)(R-1) + 1}.$$

Correspondingly, the hazard rate of a worker being presented with a candidate is $\frac{\mu}{p(s_w)\hat{p}(s_e)(R-1)}$.

Remark 7. *When characterizing equilibrium, it is without loss of generality to assume that, if $\hat{p}(s_e), p(s_w) > 0$ then $\hat{p}(s_e), p(s_w) \geq \sqrt{2c}$.*

Note that $\hat{p}(s_e) \geq \sqrt{2c}$ is immediate in equilibrium from Lemma E.1. To see why $p(s_w) \geq \sqrt{2c}$, note that if $p(s_w) > 0$, then $p(s_w) < 1$ only if s_e screens incoming opportunities. We know that, in any best response, if opportunities are screened with threshold θ then this threshold must be equal to the expected utility. By Lemma D.2, we know that the expected utility is upper bounded by $\max(1/2, 1 - \sqrt{2c})$ and therefore, $p(s_w) = 1 - \theta \geq \sqrt{2c}$.

Proof of Lemma J.1. Fix $c \leq 1/8$. Let $\bar{f}$ be such that f_e has a point mass of 1 at s_e, and f_w has a point mass of 1 at s_w. Let $\eta_w \triangleq \eta_w(s_w, s_e)$ and let $\eta_e \triangleq \eta_e(s_e, s_w)$.

Proof of part 1. The existence of a steady state $\bar{L}$ follows from Proposition 11. In particular, by assumption, we have that the match success rate from proposals issued by workers is $\eta_w = p(s_w)\hat{p}(s_e) \geq 2c$ and that $\mu < (R-1)2c$. Therefore, using that $\lambda_e = 1$ and $\lambda_w = R$, we have that condition $\lambda_e = 1 < \frac{\eta_w \lambda_w}{\mu + \eta_w}$ is satisfied. Therefore, the system converges to the steady state in (25), i.e., $\bar{L} = \begin{bmatrix} 0 \\ \frac{(R-1)}{\mu} \end{bmatrix}$, as desired.

Proof of part 2. To prove the second part of the claim, we will rely on the notation introduced in Appendix C. For a quick reference, the reader can find a summary of such notation in Appendix B. Note that a worker following s where he requests a candidate if the opportunity arise will be shown candidates at rate

$$\kappa_w \triangleq \kappa_w(s_e, s, \bar{L}, \bar{f}) = \frac{1}{\eta_w L_w} = \frac{1}{\eta_w (R-1)/\mu} = \frac{\mu}{\eta_w (R-1)},$$

where we used the calculations in Appendix C.2 together with the characterization of $\bar{L}$ to derive an expression for κ_w. (For the reader's convenience: The main idea is simply that the flow of matches in the system is equal to $\lambda_e = 1$ and so the flow of chances to see candidates must be $\lambda_e/\eta_w = 1/\eta_w$. This flow is divided equally among the mass L_w of candidates.) Therefore, we have that

$$\mathrm{p} = \frac{\kappa_w}{\mu + \kappa_w} = \frac{1}{\eta_w (R-1) + 1}, \tag{83}$$

together with the fact that $\eta_w = p(s_w)\hat{p}(s_e)$ by definition. □

We now provide a proof of Lemma E.6, which was introduced in the main appendix.

Proof of Lemma E.6. Suppose all employers do not propose and accept incoming proposals without screening. Suppose all workers (except one particular worker) use strategy s_w and propose with probability $p(s_w)$. Then we characterize the worker best response (focusing on proposals alone, since employers do not propose) among strategies that use a^{o} = S+P, i.e., among strategies that screen and propose.

Consider the given worker who must decide which threshold to use. By Proposition 1 we know that the worker will solve an MDP; let V be the value of the MDP. Then, the screening threshold θ is a best response if and only if it satisfies the following Bellman equation:

$$V = \underbrace{\mathrm{p}}_{\substack{\text{likelihood of getting}\\ \text{a candidate}}} \left(\underbrace{-c}_{\substack{\text{screening}\\ \text{cost}}} + \underbrace{(1-\theta)}_{\substack{\text{prob. that proposal}\\ \text{is issued and accepted}}} \underbrace{\frac{(1+\theta)}{2}}_{\substack{\text{expected}\\ \text{match value}}} + \underbrace{\theta}_{\substack{\text{prob. no}\\ \text{proposal}\\ \text{is issued}}} \underbrace{V}_{\substack{\text{continuation}\\ \text{value}}} \right)$$

where $\mathrm{p} = \frac{1}{p(s_w)(R-1)+1}$ is the likelihood of seeing another candidate for small enough μ, using Lemma J.1. Moreover, by Proposition 1, we have that the threshold θ must be equal to the continuation value V. Therefore, we have that θ is a best response if and only if

$$0 = -2c + 1 - 2\left(p(s_w)(R-1) + 1\right)\theta + \theta^2 . \tag{84}$$

Thus we have characterized best responses among strategies that screen and propose.

It remains to compare with the strategy of proposing with screening. If the worker uses a^{o} = P w/o S then his expected utility is

$$\theta = \frac{1}{2(p(s_w)(R-1)+1)} . \tag{85}$$

We are now ready to characterize symmetric (i.e., pure) equilibria. In order to have a symmetric equilibrium where workers screen and propose, it must be that screening with threshold θ is a best response when $p(s_w) = 1 - \theta$. Substituting $p(s_w) = 1 - \theta$ in Eq. (84) we get

$$0 = 1 - 2c - 2R\theta + (2R - 1)\theta^2, \tag{86}$$

and solving it gives

$$\theta = \frac{R - \sqrt{R^2 - (2R-1)(1-2c)}}{2R - 1} = \xi(R, c)\,.$$

This constitutes a symmetric equilibrium if $\xi(R, c)$ exceeds the utility from proposing without screening (given by substituting $p(s_w) = 1 - \xi(R, c)$ in (85) and noticing that the expected utility is 1/2 if the worker is presented with a candidate), i.e.,

$$\xi(R, c) > \frac{1}{2((1 - \xi(R, c))(R - 1) + 1)}\,. \tag{87}$$

This condition simplifies to

$$2\xi(R, c)^2(R - 1) - 2R\theta + 1 > 0 \qquad \Leftrightarrow \qquad \xi(R, c) > \frac{R - \sqrt{1 + (R-1)^2}}{2(R - 1)}\,, \tag{88}$$

where we used $\xi(R, c) \leq 1$. From (37) we know that $\xi(R, c)$ is monotone decreasing in c so (88) tells us that the equilibrium exists for all c less than a certain maximum value. Substituting (37) in (88) followed by routine but tedious calculations tells us that this maximum value of c is exactly $\bar{c}$ given by (38). Thus we have established that the Equilibrium 1 exists if $c \in (0, \bar{c})$, and that the workers are playing a *strict* best response in this equilibrium for any $c < \bar{c}$. Since the best response is strict, this also implies evolutionary stability. On the other hand, since $\xi(R, c)$ is strictly decreasing in c for $c > \bar{c}$ so we have $\xi(R, c) < \frac{R - \sqrt{1+(R-1)^2}}{2(R-1)}$, and so workers are *not* playing a best response and Equilibrium 1 does *not* exist for $c > \bar{c}$.

Now consider the (possible) symmetric equilibrium where all workers propose without screening. In order for this to be an equilibrium, it must be that proposing without screening produces a higher utility that screening and proposing. The utility from proposing without screening is $\theta = \frac{1}{2R}$ using $p(s_w) = 1$ in (85). By (Bertsekas 2007, Proposition 1.2.5) this is the best response strategy if deviating to "the first time you are presented with a candidate, screen and propose with threshold θ, thereafter propose without screening" produces a smaller utility. The probability of getting to see a candidate is $1/R$. Hence the equilibrium condition is

$$\frac{1}{R}\Big(- c + (1 - \theta)\frac{1 + \theta}{2} + \theta\frac{1}{2R}\Big) < \frac{1}{2R} \tag{89}$$

for $\theta = \frac{1}{2R}$. This simplifies to $c > \underline{c} = \frac{1}{8R^2}$. Thus we have established that the Equilibrium 2 exists if $c > \underline{c}$. Moreover, since workers are playing a strict best response, the equilibrium is also evolutionarily stable. On the other hand we also have that workers can strictly improve their utility by deviating if $c > \underline{c}$ since inequality (89) gets reversed, and so Equilibrium 2 does *not* exist for $c > \underline{c}$.

Finally, we rule out mixed equilibria by showing that they are not evolutionarily stable. Mixing must be between proposing with or without screening, since we already know from Proposition 1 that all workers use the same screening threshold in equilibrium. Suppose there is a mixed equilibrium where a fraction $\in (0, 1)$ of workers screen with threshold θ (call this strategy s_w) and the rest of the workers do not screen. Then, analogous to the proof of Lemma J.1 part 2, we have that the hazard rate for a worker of being presented with a candidate is

$$\kappa_w = \frac{\mu}{N_w(\text{P w/o S}) + (1-\theta)N_w(s_w)} \tag{90}$$

Here we have written the hazard rate in terms of the *current* masses (and not the steady state masses) since we are going to check evolutionary stability. Intuitively, one would expect that an increase in κ_w above the steady state value will cause the expected utility under screening to increase more than the expected utility under not screening (and hence s_w will become more attractive that P w/o S), since more candidates are available to a worker. We now formalize this intuition. As before $\text{p} = \frac{\kappa_w}{\mu+\kappa_w}$. The expected utility of proposing without screening is simply p/2. The expected utility of screening and proposing is

$$\Pr(\text{Matching before leaving under } s_w)\Big(-\frac{c}{1-\theta}+\frac{1+\theta}{2}\Big) = \frac{\kappa_w(1-\theta)}{\kappa_w(1-\theta)+\mu}\Big(-\frac{c}{1-\theta}+\frac{1+\theta}{2}\Big),$$

where we used that s_w screens $1/(1-\theta)$ candidates on average for every match formed and that the expected match utility is $(1+\theta)/2$. Now in the steady state corresponding to the equilibrium we know that the expected utility from s_w is equal to the expected utility from P w/o S, so it suffices to show that $\frac{d(\log(\text{Expected utility}))}{d\kappa_w}$ is larger for s_w for all $\kappa_w > 0$. But this is true, since we have

$$\begin{aligned}\frac{d(\log(\text{Expected utility}(\text{P w/o S})))}{d\kappa_w} &= \frac{d(\log(\text{p}))}{d\kappa_w} = \frac{1}{\kappa_w} - \frac{1}{\mu+\kappa_w}\\ \frac{d(\log(\text{Expected utility}(s_w)))}{d\kappa_w} &= \frac{1}{\kappa_w} - \frac{1}{\frac{\mu}{1-\theta}+\kappa_w} > \frac{1}{\kappa_w} - \frac{1}{\mu+\kappa_w}.\end{aligned}$$

Hence any increase in κ_w above its equilibrium value will cause incoming workers to prefer s_w.

We now demonstrate that the mixed equilibrium cannot be evolutionarily stable. Suppose we perturb the system by slightly increasing the proportion of s_w workers present. The total mass of workers in the system remains exactly equal to its steady state value of $(R-1)/\mu$ throughout since the arrival flow is always R, the flow of matching and leaving is always equal to the rate of arrival of employers 1, and the flow of leaving without matching is $\mu N_w = R-1$, hence the total mass of workers in the system does not change. But since the *proportion* $N_w(s_w)/N_w = N_w(s_w)/(R-1)$ of workers increases slightly, this increases κ_w as per (90), which causes incoming workers to prefer s_w over P w/o S, which further increases $N_w(s_w)/N_w$ and hence κ_w. Hence, the mixed equilibrium is not an attractive fixed point of the evolutionary dynamics (20), i.e., it is not stable. Thus we have shown that there are no stable mixed equilibria. □

In the next lemma, we state some consequences of Lemma E.6 that will help us in proving our main results.

Lemma J.2. *Fix $R = \lambda_w/\lambda_e > 1$. Suppose employers mechanically propose with probability $\text{p}_e \geq \sqrt{2c}$ at each opportunity (e.g., if they screen with threshold θ then $\text{p}_e = 1-\theta$) and that workers are not allowed to propose. We allow p_e to be drawn i.i.d. across employers from an arbitrary distribution on $[\sqrt{2c}, 1]$. Then the following are the equilibria between workers:*

1. *If $c < \bar{c}$ there exists $\mu_0(c) > 0$ such that for $\mu < \mu_0(c)$ there is an equilibrium where all workers screen incoming proposals $a^{\text{i}} = \text{S+A/R}$ with threshold (and expected utility) $\theta_w \xrightarrow{\mu\to 0} \xi(R,c)$.*
2. *If $c > \underline{c}$, there is an equilibrium where all workers accept incoming proposals without screening and earn expected utility $\theta_w \xrightarrow{\mu\to 0} \frac{1}{2R}$.*

Moreover there are no other equilibria. (We omit to characterize whether Equilibrium 1 exists at the boundary $c = \bar{c}$ and whether Equilibrium 2 exists at the boundary $c = \underline{c}$.)

Proof of Lemma J.2. The proof follows the same steps as that of Lemma E.6 with only minor modifications.

Suppose all workers (except one particular worker) use strategy s_w under which they accept incoming proposals with probability $p(s_w) \geq \sqrt{2c}$. (Worker behavior in any equilibrium satisfies this restriction by Lemma E.1.)

Let p be the probability that the worker receives a proposal before leaving without matching. We now show that

$$\mathrm{p} \xrightarrow{\mu \to 0} \frac{1}{p(s_w)(R-1)+1} \tag{91}$$

using the proof of Lemma J.1 part 2 with minor modifications — Every time an employer's opportunity clock rings, there is a candidate available (since $R > 1$) and so a match is formed with probability $p_e p(s_w) \geq \sqrt{2c}\sqrt{2c} = 2c$. It follows that the likelihood that an individual employer leaves without matching is at most $\frac{\mu}{\mu+2c}$, so the flow of employers leaving without matching is at most $\frac{\mu}{\mu+2c}$ (the arrival flow is 1), and the steady state mass of employers is just $1/\mu$ times this flow, i.e., $L_e \leq \frac{1}{\mu+2c} = O(1)$. The match flow is $\lambda_e - O(\mu) = 1 - O(\mu)$. The flow of workers leaving without matching is then $(R-1)+O(\mu)$ and the steady state mass L_w is therefore $L_w = \frac{R-1}{\mu}(1+O(\mu))$. So incoming proposals arrive to workers at a rate

$$\nu_w = \frac{\text{Match flow}}{p(s_w)L_w} \xrightarrow{\mu \to 0} \frac{\mu}{p(s_w)(R-1)}\,.$$

Since $\mathrm{p} = \frac{\nu_w}{\mu+\nu_w}$ (this is analogous to (83) in Lemma J.1 part 2) we obtain (91).

From here, the rest of the proof of Lemma E.6 goes through unchanged with each equality and inequality we have stated now holding in the limit $\mu \to 0$. For Equilibrium 1, we have a *strict* inequality (in the limit) for any $c < \bar{c}$ showing that workers are playing a best response, and so it follows that Equilibrium 1 exists for small enough μ. And for $c > \bar{c}$ we have a strict inequality showing that workers are *not* playing a best response so it follows that Equilibrium 1 breaks for $c > \bar{c}$ for small enough μ. The same thing happens for Equilibrium 2 with a boundary value of $\underline{c}$. The argument ruling out mixed equilibria also goes through unchanged. □

We now prove Lemmas E.7 and E.8 which were introduced in the main appendix.

Proof of Lemma E.7. Here we make use of Lemma J.1 part 2, Lemma D.1 and Lemma E.6. Let $p_e = \sqrt{2c}$ denote the probability that an employer accepts a proposal. If workers all propose with probability $p(s_w)$, then by Lemma J.1 part 2, the hazard rate of a worker being presented with a candidate is

$$\kappa_w = \frac{\mu}{p_e p(s_w)(R-1)}\,. \tag{92}$$

(Since all employers are using the same strategy and we assume that the worker asks for a candidate, we suppress the arguments of κ_w.) When a worker proposes, the proposal is accepted with probability $\sqrt{2c}$. By Lemma D.1, for any choice of threshold θ_w the expected utility of the worker in this setting is exactly equal to his expected utility in a setting where the worker is presented with candidates at rate $p_e\kappa_w = \frac{\mu}{p(s_w)(R-1)}$ and the screening cost is $c/p_e = \sqrt{c/2}$ and proposals are accepted with probability 1. From the perspective of workers, the latter setting is exactly that of

Lemma E.6 but with a screening cost of $c_{\text{eff}} = \sqrt{c/2}$; in the setting of Lemma E.6 since employers are accepting all incoming proposals, when workers propose with probability $p(s_w)$, each worker is presented with candidates at hazard rate $\frac{\mu}{p(s_w)(R-1)}$ using Lemma J.1 part 2. Thus, we have shown that Lemma E.6 applies but with c_{eff} playing the role of c. The lemma follows: Equilibrium 1 exists for $c_{\text{eff}} < \bar{c} \Leftrightarrow c < 2\bar{c}^2$ and does not exist for $c > 2\bar{c}^2$, Equilibrium 2 exists for $c_{\text{eff}} > \underline{c} \Leftrightarrow c > 2\underline{c}^2$ and does not exist for $c < 2\underline{c}^2$, and there are no other equilibria. $\square$

Proof of Lemma E.8. We first observe that under the assumed behavior of workers, the best response of an employer is independent of what other employers do. After all, the mass of workers in the system always exceeds the mass of employers in the system so a candidate is always available to an employer whenever an opportunity arises. As such we only need to characterize the best response of employers. Employers can earn positive utility by proposing without screening which exceeds the utility of zero which they would earn by not proposing. Hence all employers propose as claimed in the lemma. It remains to characterize the condition under which they screen before proposing.

If employers screen: Given that workers use threshold θ_w and suppose that employers screen, then because a candidate is always available the situation is identical to the one leading to the best response (28) (see the proof of Lemma E.3) with the roles of employers and workers reversed, i.e., we have that the employers will choose a threshold (and earn expected utility)

$$\theta_e = l_e(\theta_w) \quad \text{where } l_e(\theta) \triangleq 1 + \frac{\mu}{1-\theta} - \sqrt{\frac{2c}{1-\theta} + \frac{2\mu}{1-\theta} + \left(\frac{\mu}{1-\theta}\right)^2}. \tag{93}$$

Clearly $\lim_{\mu\to 0}\theta_w = \xi(R,c)$ leads to $\lim_{\mu\to 0}\theta_e = 1 - \sqrt{2c/(1-\xi(R,c))}$.

If employers do not screen: In this case, the hazard rate of matching is $1-\theta_w$ whereas the hazard rate of leaving without matching is μ so the likelihood of matching before leaving is $\frac{1-\theta_w}{\mu+1-\theta_w}$. The expected utility in case a match is formed is $1/2$ hence $\theta_e = \frac{1-\theta_w}{2(\mu+1-\theta_w)} \xrightarrow{\mu\to 0} 1/2$.

Next we compare the limiting utilities under screening versus not screening as $\mu \to 0$. The limiting utility under screening is higher if

$$\begin{aligned} & 1 - \sqrt{2c/(1-\xi(R,c))} > 1/2 \\ \Leftrightarrow \quad & \xi(R,c) - 1 + 8c < 0 \\ \Leftrightarrow \quad & -(R-1) + 8c(2R-1) < \sqrt{(R-1)^2 + 2c(2R-1)} \end{aligned}$$

where we plugged in the definition (37) of $\xi(R,c)$. We can square both sides of the last inequality to obtain an equivalent inequality: the reason for the equivalence is that the right-hand side above is positive, whereas the left-hand side may be negative but in that case the above inequality holds and the squared inequality also holds because the above LHS satisfies $|\text{LHS}| < R-1 < |\text{RHS}|$. Squaring both sides and simplifying we obtain the equivalent condition $c < \frac{8R-7}{32(2R-1)} = \hat{c}$. Thus the limiting utility under screening is strictly higher if and only if $c < \hat{c}$, completing the proof of part 1 (we already showed $\lim_{\mu\to 0}\theta_e = 1 - \sqrt{2c/(1-\xi(R,c))}$).

The limiting utility under not screening is higher if the reverse inequality

$$1 - \sqrt{2c/(1-\xi(R,c))} < 1/2 \quad \Leftrightarrow \quad -(R-1) + 8c(2R-1) > \sqrt{(R-1)^2 + 2c(2R-1)}$$

holds. As above we find that this is true if and only if $c > \hat{c}$, completing the proof of part 2 of the lemma (we already characterized θ_e as claimed in the lemma). $\square$

J.2 Equilibria under intervention: Proof of Theorem 14 (and hence Theorem 5)

We find it convenient to first establish Theorem 14 in this subsection, and then proceed to prove Theorem 13 in the next subsection.

After the proof of Theorem 14, we briefly discuss in Appendix J.4 the additional equilibria that might arise under the intervention of preventing workers from proposing.

Proof of Theorem 14. We divide the proof into three parts. In the first part if the proof, we fix $c < \min(\bar{c}, \hat{c})$ and look for equilibria where all workers use $a_w = (\mathrm{DN}, \mathrm{S{+}A/R})$ and all employers use $a_e = (\mathrm{S{+}P}, \mathrm{S{+}A/R})$. We show that an equilibrium where agents use those strategies exists for μ small enough, and characterize the (unique) equilibrium thresholds as function of μ. Finally, we argue that the equilibrium is stable.

In the second part, we show that, if $c > \min(\bar{c}, \hat{c})$, then there is no equilibrium in which all workers use $a_w = (\mathrm{DN}, \mathrm{S{+}A/R})$ and all employers use $a_e = (\mathrm{S{+}P}, \mathrm{S{+}A/R})$.

Finally, in the third part, we argue that the uniqueness of the equilibrium established in the first part for $c \in (0, \min(\bar{c}, \underline{c}, \hat{c}))$.

Existence, uniqueness, limiting thresholds, and stability for $c < \min(\bar{c}, \hat{c})$ and given a_w and a_e. Our proof of this part is very similar to the proof of Lemma E.3, but the roles of workers and employers interchanged.

Fix $c \in (0, \min(\bar{c}, \hat{c}))$. Our high level approach will be as follows. We first constrain all workers to use $a_w = (\mathrm{DN}, \mathrm{S{+}A/R})$ and all employers to use $a_e = (\mathrm{S{+}P}, \mathrm{S{+}A/R})$ and characterize (i) the equilibrium threshold $\theta_e := \theta_e(\mu)$ between employers for given θ_w, (ii) the equilibrium threshold $\theta_w := \theta_w(\mu)$ between workers for given θ_e, and then to show that the composition of the two single-side equilibrium operators is contractive for small enough μ, which will imply equilibrium existence, uniqueness as well as the limiting equilibrium description as $\mu \to 0$. At the end we will check that a_w and a_e constitute best responses and show stability.

For now we constrain workers to use $a_w = (\mathrm{DN}, \mathrm{S{+}A/R})$ and employers to use $a_e = (\mathrm{S{+}A/R}, \mathrm{S{+}A/R})$. We will characterize the resulting equilibrium thresholds on both sides of the market. From Proposition 1, we know that in any such equilibrium, all workers use the same threshold θ_w equal to their expected utility and similarly for employers with threshold θ_e. Using Lemma D.2 we further know that $\theta_w \le 1 - \sqrt{2c}$ and $\theta_e \le 1 - \sqrt{2c}$. Thus, we are searching for equilibria with $(\theta_w, \theta_e) \in [0, 1 - \sqrt{2c}]^2$.

We start by characterizing the equilibrium θ_e between employers for a given employer threshold θ_w. This part is verbatim from the proof of Lemma E.4 (with the roles of employers and workers interchanged); we provide the main steps again here for the convenience of the reader: Let V_e be the expected utility for an employer. The Bellman equation for an employer is

$$V_e = \frac{1}{1+\mu}\left(-c + (1-\theta_e)(1-\theta_w)\left(\frac{1+\theta_e}{2}\right) + \big(1 - (1-\theta_e)(1-\theta_w)\big)V_e\right).$$

Substituting $V_e = \theta_e$ and solving for θ_e we obtain

$$\theta_e = l_e(\theta_w) \quad \text{where } l_e(\theta) \triangleq 1 + \frac{\mu}{1-\theta} - \sqrt{\frac{2c}{1-\theta} + \frac{2\mu}{1-\theta} + \left(\frac{\mu}{1-\theta}\right)^2}. \tag{94}$$

We observe that for all $\mu > 0$ we have $l_e : [0, 1-\sqrt{2c}] \to [0, 1-\sqrt{2c}]$, i.e., l_e maps valid θ_w to valid θ_e, and that l_e is $(c^{-1/4})$-Lipschitz on its domain for small enough μ.

Now let us similarly characterize the equilibrium θ_w between workers for a given employer threshold θ_e. First note that given θ_w and θ_e, let $\eta_e \triangleq (1-\theta_e)(1-\theta_w)$. Then, by Proposition 11 part 2, the steady state is given by $\bar{L}' = \left[\frac{1}{\mu+\eta_e}, \ \frac{R-1}{\mu} + \frac{1}{\mu+\eta_e}\right]$.

It then follows that the hazard rate of receiving a proposal for a worker, denoted by $\hat{\kappa}_w$, is equal to the flow of proposals from employers, i.e., $L_e(1-\theta_e)$ divided by the mass of workers L_w, that is,

$$\hat{\kappa}_w = \frac{\mu(1-\theta_e)}{R\mu + (R-1)\eta_e} = \frac{\mu}{R\mu/(1-\theta_e) + (R-1)(1-\theta_w)}.$$

Let V_w be the expected utility for a worker. Now a worker receives a proposal before leaving without matching with probability $\hat{\kappa}_w/(\hat{\kappa}_w+\mu)$, in which case he screens it at a cost c, and accepts with probability $1-\theta_w$. If he rejects the proposal, he returns to the initial state and has future expected utility V_w. Hence,

$$V_w = \frac{\hat{\kappa}_w}{\hat{\kappa}_w+\mu}\left(-c + (1-\theta_w)\left(\frac{1+\theta_w}{2}\right) + \theta_w V_w\right). \tag{95}$$

Substituting $V_w = \theta_w$, solving the quadratic equation for θ_w and excluding the root which exceeds 1, we obtain

$$\theta_w = l_w(\theta_e) \quad \text{where } l_w(\theta) \triangleq \frac{R\left(\mu/(1-\theta)+1\right) - \sqrt{R^2\left(\mu/(1-\theta)+1\right)^2 - (2R-1)(1-2c)}}{2R-1}. \tag{96}$$

As above we find that for all $\mu > 0$, we have $l_w : [0, 1-\sqrt{2c}] \to [0, 1-\sqrt{2c}]$, i.e., l_w maps valid θ_e to valid θ_w. A routine calculation of $l'_w(\theta)$ suffices to establish that $|l'_w(\theta)| < \mu/c^{3/2}$ for small enough μ and all $R \geq 1$ and $\theta \in [0, 1-\sqrt{2c}]$, implying that l_w is $(\mu/c^{3/2})$-Lipschitz on its domain for small enough μ. (The exact Lipschitz constant will not matter for our argument except that it is decaying as $\mu \to 0$.)

Now consider the composition of the two operators $l \triangleq l_w \circ l_e$, $l : [0, 1-\sqrt{2c}] \to [0, 1-\sqrt{2c}]$. The θ_w in any equilibrium must be fixed point of l. Now, for small enough μ, since each of the individual operators is Lipschitz with the given constants, the composition l is Lipschitz with constant equal to the product $(\mu/c^{7/4})$ of the constants. It follows that for μ small enough, the operator l is $(1/2)$-Lipschitz, i.e., it is contractive since its Lipschitz constant is less than 1. Since the range of l is identical to its domain, we deduce that it has a unique fixed point.

We now characterize this fixed point. Note that $\lim_{\mu\to 0} l_e(\xi(R,c)) = 1 - \sqrt{2c/(1-\xi(R,c))}$, and

$$\lim_{\mu\to 0} l(\xi(R,c)) = \xi(R,c). \tag{97}$$

For clarity we use $\theta_w^* = \theta_w^*(\mu)$ to denote the unique fixed point of l. We will now show that $\lim_{\mu\to 0}\theta_w^*(\mu) = \xi(R,c)$. Fix any $\epsilon > 0$. Using (97), for small enough μ we have that $|l(\xi(R,c)) - \xi(R,c)| < \epsilon/4$. Further, since l is $(1/2)$-Lipschitz it then follows that the fixed point satisfies

$$|\theta_w^*(\mu) - \xi(R,c)| \leq \epsilon/4 \times 1/(1-1/2) = \epsilon/2 < \epsilon$$

for small enough μ, as needed.

Finally, note that $\lim_{\mu\to 0}\theta_w^*(\mu) = \xi(R,c)$ implies also that the equilibrium threshold $\theta_e^* = \theta_e^*(\mu)$ of employers satisfies $\lim_{\mu\to 0}\theta_e^*(\mu) = \lim_{\mu\to 0} l_e(\theta_w^*(\mu)) = 1 - \sqrt{2c/(1-\xi(R,c))}$. Thus we have obtained the required limiting characterizations of equilibrium thresholds.

Now consider the employers. From Lemma E.8 part 1 we know that employers' proposal strategy above (screen and propose with threshold θ_e) constitutes a best response for small enough μ since $c < \hat{c}$. It remains to characterize how an employer would handle a "bonus" incoming proposal (this issue is purely technical, since workers are not permitted to propose): The expected utility from screening and accepting/rejecting is $-c + (1-\theta_e)(1+\theta_e)/2 + \theta_e^2 = -c + (1+\theta_e^2)/2 > \theta_e > 1/2$ for small enough μ using $\lim_{\mu\to 0}\theta_e > 1/2$ (the inequality was established in Lemma E.8 for $c < \hat{c}$) and $c < \hat{c} < 1/8$. Therefore, we conclude that employers should prefer to screen the incoming proposal over accepting/rejecting without screening.

Moving on to the workers, it is immediate from Lemma J.2 part 1 that workers are playing a strict best response for small enough μ.

The steady state agent mix $\bar{L}$ is immediate from Proposition 11. In showing that a_e and a_w are best responses, we in fact showed that they are strict best responses. As a result, they are also best responses for any agent mix $\bar{N}$ that is a small perturbation of the steady state $\bar{L}$, which implies evolutionary stability (Definition 7).

This completes our proof that for small enough μ there is a unique equilibrium with the given description, and that the equilibrium is stable.

No equilibrium with a_w and a_e for $c > \min(\bar{c}, \hat{c})$. Fix any $c > \min(\bar{c}, \hat{c})$. If $\bar{c} = \min(\bar{c}, \hat{c})$ we know from Lemma J.2 part 1 that workers are *not* playing a best response, breaking the equilibrium.

On the other hand, if $\hat{c} = \min(\bar{c}, \hat{c})$ then again we can rule out any such equilibrium for $c > \hat{c}$. Above we showed that in such an equilibrium we must have $\lim_{\mu\to 0}\theta_w^* = \xi(R, c)$ but then we know from Lemma E.8 part 2 that employers prefer to propose without screening for $c > \hat{c}$ and small enough μ.

Uniqueness for $c \in (0, \min(\bar{c}, \underline{c}, \hat{c}))$. Fix $c < \min(\bar{c}, \underline{c}, \hat{c})$. We start by showing that all employers screen and propose in any equilibrium. First note that workers will not use a threshold that exceeds $\xi(R, c)$; after all by Lemma E.6 they screen with this threshold for $c < \min(\bar{c}, \underline{c})$ in the hypothetical setting where employers accept all incoming proposals and, in the hypothetical setting, the probability of a worker leaving matched is larger as no employer leaves without matching. From Lemma E.8 part 1 we can deduce that it is strict a best response (for μ small enough) for employers to screen before proposing if workers use a threshold of at most $\xi(R, c)$ for all $c < \hat{c}$; after all if workers are less selective, this only makes screening more attractive (and a worker candidate is always available so the strategies of other employers have no impact on an employer). Therefore, employers must screen and propose.

Now consider the workers. Since $c < \underline{c}$ and $c < \bar{c}$ and knowing that the employers will screen and propose, we know from Lemma J.2 that all workers screen and accept/reject in any equilibrium. (Lemma J.2 eliminates Equilibrium 2 for $c < \underline{c}$ as well as any other possible equilibria including mixed equilibria between workers.) □

J.3 No-intervention Equilibria in Unbalanced Markets: Proof of Theorem 13 (and hence Theorem 4)

Roadmap for the proof of Theorem 13. We proof is established through a sequence of claims.

We first establish the existence and correctness of the steady states claimed in the statement of the theorem (Lemma J.3).

Lemmas J.4, J.5, and J.6 then show that the claimed equilibria exist for the specified intervals of c and, moreover, do not exist outside those intervals (we omit to characterize whether the equilibria exist at the boundary values of c). Moreover, these lemmas establish some additional facts that are helpful later in ruling out other equilibria.

We then prove an additional result Lemma J.9, via intermediate results Lemmas J.7 and J.8, to help us rule out other equilibria.

Finally, in the proof of Theorem 13, we show that there are no other equilibria besides the ones captured by Lemmas J.4, J.5, and J.6.

Lemma J.3. *For every claimed equilibrium in the statement of Theorem 13, if the agents play according to the specified strategies, there is a unique steady state $\bar{L}$ which is the one stated in*

Theorem 13.

Proof of Lemma J.3. The claim immediately follows from Proposition 11 part 1. For $R > 1$, the steady state of $L = [0, (R-1)/\mu]'$ in each of the claimed equilibria follows from the symmetric counterpart of (25). (Note that the value of $\eta_e \triangleq \eta_e(s_e, s_w)$ (and therefore how the employers handle opportunities a_e^{o}) plays no role in the steady state characterization for $R > 1$. As per Remark 3 the characterization extends even to the case where different employers use different a_e^{o}, as may be the case in Equilibria 2 and 3.) $\square$

The next three lemmas characterize the equilibria under different set of actions. These lemmas are analogous to those of Lemmas E.3, E.4, E.5, respectively, that characterize the equilibria under the same sets of actions for balanced markets.

Lemma J.4 (Pure equilibrium where workers S+P and employers S+A/R)**.** *The following is a characterization of equilibria where all workers use* $a^{\mathrm{o}} = \mathrm{S{+}P}$ *and all employers use* $a^{\mathrm{i}} = \mathrm{S{+}A/R}$.

(i) For any $c \in (0, 2\bar{c}^2)$ *there exists* $\mu_0 = \mu_0(c) > 0$ *such that for all* $\mu < \mu_0$, *there is a unique such equilibrium. In this equilibrium, all workers use* $s_w = (a^{\mathrm{o}} = \mathrm{S{+}P}, a^{\mathrm{i}} = \mathrm{S{+}A/R}, \theta_w = \xi(R, \sqrt{c/2}))$, *and all employers use* $s_e = (a^{\mathrm{o}} = \mathrm{DN}, a^{\mathrm{i}} = \mathrm{S{+}A/R}, \theta_e = 1 - \sqrt{2c})$. *The agent expected utilities are given by* θ_w *and* θ_e *and the equilibrium is stable.*

(ii) For any $c > 2\bar{c}^2$ *there exists* $\mu_0 = \mu_0(c) > 0$ *such that for all* $\mu < \mu_0$ *there is no equilibrium where all workers use* $a^{\mathrm{o}} = \mathrm{S{+}P}$ *and all employers use* $a^{\mathrm{i}} = \mathrm{S{+}A/R}$.

Additionally, for $c < 1/2$, *in any equilibrium where all workers propose, all employers use threshold* $\theta_e = 1 - \sqrt{2c}$ *and use* $a^{\mathrm{i}} = \mathrm{S{+}A/R}$, *i.e., they screen incoming proposals.*

Proof of Lemma J.4. Let $\bar{f}$ be such that f_e has a point mass of 1 at s_e, and f_w has a point mass of 1 at s_w and let $\bar{L}$ be as defined in the statement of the theorem. We will first show that s_e is a strict best response for the employers and that s_w is a strict best response for workers for $c < 2\bar{c}^2$, and then that the equilibrium is stable. We then observe that fixing a_w and a_e as in the lemma, the thresholds θ_e and θ_w in the lemma statement are the only ones possible in equilibrium. Also, we will note that the best responses identified do not depend on how employers handle opportunities, allowing us to show uniqueness. Finally, for $c > 2\bar{c}^2$ we find that s_w is no longer a worker best response, breaking the equilibrium.

$\mathbf{s_e}$ is a strict best response for the employers. By Lemma D.2, we have that the utility of the employers is upperbounded by $\max(1/2, 1-\sqrt{2c})$; as $c < 1/8$, this upper bound is equal to $1 - \sqrt{2c}$. We show that the employers' strategy achieves the desired upper bound.

By Lemma J.3, we have that $L_e = 0$ for every c and that the rate at which employers receive proposals is $\nu_e(\bar{L}, \bar{f}) = \infty$. This means that an employer following s_e is guaranteed to match, and the utility can be expressed as:

$$U_e(s_e, \bar{L}, \bar{f}) = \frac{1+\theta_e}{2} - \frac{c}{1-\theta_e} = 1 - \sqrt{2c},$$

where the first term captures the expected utility of matching conditional on the partner being above the threshold θ, and the second term is c times the expected number of proposals that must be screened so as to find such a partner. Therefore, we conclude that s_e is a utility-maximizing strategy. Moreover, by Lemma D.2, we know that if such upper bound is achieved, it must be through screening incoming proposals (if incoming proposals are accepted without screening, the expected utility is $1/2 < 1 - \sqrt{2c}$ and, if incoming proposals are ignored, the employer leaves without matching with positive probability and thus the upper bound is strict). Therefore, the choice of a^{i} is strictly utility-maximizing.

We pause briefly here to notice that we have actually established the last sentence of the lemma: Since $\nu_e(\bar{L}, \bar{f}) = \infty$ holds in any equilibrium where all workers propose, and this is the only aspect of worker strategies we have used so far, we have actually shown that employers use $a^{\mathrm{i}} = \mathrm{S{+}A/R}$ and $\theta_e = 1 - \sqrt{2c}$ in any equilibrium where all workers propose (with or without screening).

We now return to our task of establishing that all employers using s_e and all workers using s_w is an equilibrium: To conclude that s_e is indeed a best response, we must argue that the choice of a^{o} is also optimal as per Definition 6. (Note that as employers do not get the chance to propose, this does not impact the utility calculation.) In particular, if the employer gets a bonus candidate to propose to, what should he do? One can easily see that, as workers are screening incoming proposals, the employer is better off by not proposing. If he where to issue a proposal he must screen the candidate; otherwise, if such proposal gets accepted he gets 1/2 in expectation, which we already argued is less that θ_e, his current expected value. However, if he screens the candidate, then he incurs the screening cost and risks rejection, so overall utility does not increase. This pins down a^{o}=DN, as desired.

$\mathbf{s_w}$ **is a strict best response for the workers.** We have assumed that all employers use s_e and all workers use s_w. This allows us to make use Lemma E.7 since we are in exactly that setting. Equilibrium 1 there shows that $a^{\mathrm{o}} = \mathrm{S{+}P}$ with threshold $\theta_w = \xi(R, \sqrt{c/2})$ is a strict best response for workers if $c < 2\bar{c}^2$ (for small enough μ) and that it is not a best response for workers for $c > 2\bar{c}^2$ (for small enough μ).

We now need to argue that $s_w = (a^{\mathrm{o}} = \mathrm{S{+}P}, a^{\mathrm{i}} = \mathrm{S{+}A/R}, \theta_w = \xi(R, \sqrt{\frac{c}{2}}))$ is indeed the unique best response strategy as per Definition 6. Note that, as employers do not propose, all worker strategies that screen and propose with threshold θ_w achieve the same utility regardless of the choice of a^{i}. However, a^{i} is uniquely pinned down by Definition 6. In particular, suppose the worker gets a "bonus" proposal. If he does not screen it, he gets 1/2. If he does screen it, by Proposition 1, we have that he must use threshold $\theta_w = \xi(R, \sqrt{\frac{c}{2}})$ and thus his expected value is

$$-c + (1-\theta_w)\frac{1+\theta_w}{2} + \theta_w \cdot \theta_w = -c + \frac{1}{2} + \frac{\theta_w^2}{2} > \frac{1}{2}, \tag{98}$$

whenever c is in the desired range. The reason for the inequality is that for a given R, the expression $-c + \xi(R, \sqrt{c/2})^2/2 = -2c_{\text{eff}}^2 + \xi(R, c_{\text{eff}})^2/2$ is monotone decreasing in c_{eff} and since $c_{\text{eff}} < \bar{c}$, we have $-c + \xi(R, \sqrt{c/2})^2/2 > -2\bar{c}^2 + \xi(R, \bar{c})^2/2$ for $c < 2\bar{c}^2$. One can show that $-2\bar{c}^2 + \xi(R, \bar{c})^2/2$ is a monotone decreasing function of R so we only need to ensure that it remains positive as $R \to \infty$. As $R \to \infty$ we have $\bar{c} \to 0$ like $1/(8R^2)$, formally $\lim_{R\to\infty} 8R^2\bar{c} = 0$. On the other hand $\xi(R, \bar{c}) \xrightarrow{R\to\infty} 0$ like $\frac{1}{2R}$, formally $\lim_{R\to\infty} 2R\xi(R, \bar{c}) = 1$. As a result the $\xi(R, \bar{c})^2/2$ term dominates and the overall function remains positive, formally

$$\lim_{R\to\infty} 8R^2\big(-2\bar{c}^2 + \xi(R, \bar{c})^2/2\big) = 1\,,$$

which implies inequality (98).

Evolutionary stability. In showing that s_e and s_w are strict best responses, we in fact also established that the associated actions a_e and a_w are strict best responses. As a result, these actions are also best responses for any agent mix $\bar{N}$ that is a small perturbation of the steady state $\bar{L}$, which implies evolutionary stability (Definition 7).

Thresholds θ_w and θ_e are uniquely determined. Fix a_w and a_e as in the lemma. We now observe that the thresholds in the above established equilibrium are the only ones possible: Using Lemma D.2, we know that $\theta_w \le 1 - \sqrt{2c}$, so workers propose to each candidate with probability at least $\sqrt{2c}$, and so employers receive proposals at a hazard rate ∞. Since $\nu_e(\bar{L}, \bar{f}) = \infty$, as above we have $U_e(s_e, \bar{L}, \bar{f}) = \frac{1+\theta_e}{2} - \frac{c}{(1-\theta_e)}$, which is maximized for $\theta_e = 1 - \sqrt{2c}$. But then we already established above that given that employers use s_e and workers screen and propose (the latter holds under a_w), it must be that $\theta_w = \xi(R, \sqrt{c/2})$.

Uniqueness. Can there be any other equilibria where workers screen and propose and that employers screen incoming proposals? The argument above uses the infinite rate of incoming proposals to uniquely pin down that employers screen incoming proposals with threshold $\theta_e = 1 - \sqrt{2c}$. Given this, the worker best response is uniquely determined as above (it does not depend on how employers handle opportunities, after all there is zero mass of employers in the system) to be s_w. But this then implies as above that employers ignore opportunities, thus showing that s_w and s_e is the only candidate equilibrium where workers screen and propose and employers screen incoming proposals.

No equilibrium with a_w and a_e for $c > 2\bar{c}^2$. As observed above using Lemma E.7, workers are not playing a best response if $c > 2\bar{c}^2$ (in particular, they would rather propose without screening). □

Lemma J.5 (Equilibrium where all workers P w/o S and all employers S+A/R)**.** *For any $c \in \left(2\underline{c}^2, \underline{c}\right) \cup \left(\underline{c}, \frac{1}{8}\right)$ there exists $\mu_0 = \mu_0(c) > 0$ such that for all $\mu < \mu_0$, there is an equilibrium where all workers use $s_w = (a^{\mathrm{o}} = \mathrm{P\ w/o\ S}, a^{\mathrm{i}} = a^{\mathrm{i}}_w, \theta_w = 1/2R)$, with $a^{\mathrm{i}}_w = \mathrm{S{+}A/R}$ if $c \in (2\underline{c}^2, \underline{c})$ and $a^{\mathrm{i}}_w = \mathrm{A}$ if $c \in (\underline{c}, 1/8)$, and all employers use $s_e = (a^{\mathrm{o}} = a^{\mathrm{o}}_e, a^{\mathrm{i}} = \mathrm{S{+}A/R}, \theta_e = 1 - \sqrt{2c})$, with $a^{\mathrm{o}}_e = \mathrm{DN}$ if $c \in (2\underline{c}^2, \underline{c})$ and $a^{\mathrm{o}}_e \in \{\mathrm{DN}, \mathrm{S{+}P}\}$ if $c \in (\underline{c}, 1/8)$. The equilibrium is stable. Moreover, for $c \in \left(2\underline{c}^2, \underline{c}\right) \cup \left(\underline{c}, \frac{1}{8}\right)$ these are the only equilibria where all workers use $a^{\mathrm{o}} = \mathrm{P\ w/o\ S}$ and all employers use $a^{\mathrm{i}} = \mathrm{S{+}A/R}$.*

For any $c \notin \left[2\underline{c}^2, \frac{1}{8}\right]$ there exists $\mu_0 = \mu_0(c) > 0$ such that for all $\mu < \mu_0$ there is no equilibrium where all workers use $a^{\mathrm{o}} = \mathrm{P\ w/o\ S}$ and all employers use $a^{\mathrm{i}} = \mathrm{S{+}A/R}$.

Proof of Lemma J.5. For $c \in \left(2\underline{c}^2, \frac{1}{8}\right)$, we will first show that s_e is a strict best response for the employers and that s_w is a strict best response for workers, and that this is the only equilibrium where all workers use $a^{\mathrm{o}} = \mathrm{P\ w/o\ S}$ and all employers use $a^{\mathrm{i}} = \mathrm{S{+}A/R}$. We then argue that the equilibrium is evolutionarily stable. Finally, we establish that for $c \notin \left[2\underline{c}^2, \frac{1}{8}\right]$ there is no such equilibrium.

Fix $c \in \left(2\underline{c}^2, \frac{1}{8}\right)$. Suppose all employers use s_e and all workers use s_w as specified in the lemma.

$\mathbf{s_e}$ is a strict best response for the employers. The proof that s_e is a utility-maximizing response for the employers follows exactly the same arguments as in the proof of Lemma J.4. In particular, by Lemma D.2, we have that the utility of the employers is upper-bounded by $\max(1/2, 1 - \sqrt{2c})$; as $c < 1/8$, this upper bound is equal to $1 - \sqrt{2c}$. One can easily verify that the employers' strategy achieves the desired upper bound and, moreover, this can only be achieved by strategies that screen incoming proposals. Note also that $\theta_e = 1 - \sqrt{2c}$ is uniquely determined given that all workers propose without screening.

To conclude that s_e is indeed a best response, we must argue that the choice of a^{o} is also optimal as per Definition 6. (Note that as employers do not get the chance to propose, this does not impact the utility calculation.) In particular, if the employer gets a bonus candidate to propose to, what should he do? As we established in the proof of Lemma J.4, when workers are screening incoming proposals, the employer is better off by not proposing. Therefore, for $c < \underline{c}$, we have a^{o}=DN. Whenever $c \in (\underline{c}, 1/8)$, workers accept without screening. If he where to issue a proposal he must screen the candidate; otherwise, if such proposal gets accepted he gets 1/2 in expectation, which we already argued is less that θ_w. Note that, since workers are not screening incoming proposals, the employers does not face any risk of rejection; therefore, this bonus opportunity is the same as having just received a proposal, which we know he screens. Alternatively, he could "do nothing" with the bonus opportunity and still achieve the same utility; this is because, as $L_e = 0$, employers are guaranteed to leaving matched and thus, by doing nothing, he does not risk dying. Hence, the employer is indifferent between a^{o}=DN and a^{o}=S+P in this regime. Note that we have not only established that s_e is a best response, but characterized all equilibria between employers given that workers use s_w.

$\mathbf{s_w}$ is a strict best response for the workers. We assumed that all employers follow s_e and all workers follow s_w. Since the mass of employers in the system is zero, employer proposals play no role. It follows that we find ourselves in the setting of Lemma E.7 Equilibrium 2 from the perspective of workers. It follows that for $c > 2\underline{c}^2$, the workers strictly maximize their utility by using $a^{\mathrm{o}} = \mathrm{P\ w/o\ S}$, whereas for $c < 2\underline{c}^2$, this is not a best response for workers. This completes the required characterization of a^{o} for workers in a best response.

Note that workers achieve the same utility regardless of the choice of a^{i}. However, our additional requirements on best response (see Definition 6) will pin down a^{i} as follows. Suppose the worker receives a "bonus" proposal before joining the system. By Proposition 1, if he screens it, he must use a threshold equal to the continuation utility, which in this case is $1/(2R)$, in which case he obtains a limiting expected utility of

$$-c + \frac{1+\frac{1}{2R}}{2}\left(1 - \frac{1}{2R}\right) + \frac{1}{2R}\frac{1}{2R},$$

since the likelihood of accepting is $1 - 1/(2R)$, the expected match value conditional on accepting is $\frac{1+\frac{1}{2R}}{2}$, and the continuation value if he rejects (which happens with probability $\frac{1}{2R}$) is $\frac{1}{2R}$. Alternatively, if he does not screen it, then he obtains an expected utility of exactly $1/2$. Therefore, a worker screens incoming proposals if and only if

$$-c + \frac{1+\frac{1}{2R}}{2}\left(1 - \frac{1}{2R}\right) + \frac{1}{2R}\frac{1}{2R} = -c + \frac{1-\frac{1}{4R^2}}{2} + \frac{1}{4R^2} > \frac{1}{2} \quad \iff \quad \frac{1}{8R^2} > c$$

as desired. Thus, we have not only shown that workers are playing a best response, but characterized worker behavior in any equilibrium where workers propose without screening and employers screen incoming proposals.

Evolutionary stability. In showing that s_e and s_w are strict best responses, we in fact also established that the associated actions a_e^{i} and a_w are strict best responses. As a result, these actions are also best responses for any agent mix $\bar{N}$ that is a small perturbation of the steady state $\bar{L}$, which implies evolutionary stability (Definition 7). (Multiplicity of best responses in a_e^{o} for $c \in (\underline{c}, 1/8)$ does not affect this conclusion since the (a_e^{o})s have no effect on match flows and agent utilities, and the specified equilibrium allows employers to choose any $a_e^{\mathrm{o}} \in \{\mathrm{DN}, \mathrm{S{+}P}\}$.)

For $c \notin \left[2\underline{c}^2, \frac{1}{8}\right]$, no equilibrium where all workers use $a^{\mathrm{o}} = \mathrm{P\ w/o\ S}$ and all employers use $a^{\mathrm{i}} = \mathrm{S{+}A/R}$. Fix that all workers use $a^{\mathrm{o}} = \mathrm{P\ w/o\ S}$ and all employers use $a^{\mathrm{i}} = \mathrm{S{+}A/R}$. For $c < 2\underline{c}^2$ we found above that it is no longer a best response for workers to propose without screening, breaking the equilibrium. On the other hand, for $c > 1/2$, employers can earn $1/2 > 1 - \sqrt{2c}$ by accepting without screening, breaking the equilibrium.

This completes our proof of the lemma. □

Lemma J.6 (Equilibrium where no agent screens)**.** *For any $c > 1/8$ there exists $\mu_0 = \mu_0(c) > 0$ such that for all $\mu < \mu_0$, there is an equilibrium where all workers choose strategy $s_w = (a^{\mathrm{o}} = \mathrm{P\ w/o\ S}, a^{\mathrm{i}} = \mathrm{A})$ and earn expected utility $\theta_w = 1/(2R)$, and all employers choose strategy $s_e = (a^{\mathrm{o}} = a_e^{\mathrm{o}}, a^{\mathrm{i}} = \mathrm{A})$ with $a_e^{\mathrm{o}} \in \{\mathrm{P\ w/o\ S}, \mathrm{DN}\}$ and earn expected utility $\theta_e = 1/2$.*

Moreover, there is no other equilibrium for such c.

Proof of Lemma J.6. Let $\bar{f}$ be such that f_e has a point mass of 1 at s_e, and f_w has a point mass of 1 at s_w and let $\bar{L}$ be as defined in the statement of the theorem. We will first show that s_e is a strict best response for the employers and that s_w is a strict best response for workers. Finally, we argue that the equilibrium is evolutionary stable and the uniqueness claim.

$\mathbf{s_e}$ **is a strict best response for the employers.** By Lemma D.2, we have that the utility of the employers is upper-bounded by $\max(1/2, 1-\sqrt{2c}) = 1/2$, as $c > 1/8$. By the statement of the theorem, we have that $L_e = 0$ for every c and that the rate at which employers receive proposals is $\nu_e(\bar{L}, \bar{f}) = \infty$. This means that an employer that accepts proposals without screening is guaranteed to match and, moreover, his expected utility equals the upper bound.

Moreover, in a setting where $\nu_e(\bar{L}, \bar{f}) = \infty$, he never wants to screen; let $s_e(\theta_e)$ denote a strategy that screens and accept/rejects with threshold θ_e, we have that

$$\sup_{\theta_e} U_e(s_e(\theta_e), \bar{L}, \bar{f}) = \sup_{\theta_e} \frac{1+\theta_e}{2} - \frac{c}{(1-\theta_e)} = 1 - \sqrt{2c} < 1/8$$

for $c > 1/8$. Therefore, we conclude that a^{i} is a strict utility-maximizing choice of actions for employers whenever $c > 1/8$.

To conclude that s_e is indeed a best response, we must argue that the choice of a^{o} is also optimal as per Definition 6. The argument is identical to the one in Lemma J.5 for $c \geq 1/(8R^2)$ and it is therefore omitted.

$\mathbf{s_w}$ **is a strict best response for the workers.** Since the mass of employers in the system is zero, employer proposals play no role. So this setting matches the hypothetical setting in Lemma E.6, where we showed that all workers proposing without screening is an equilibrium between workers (where workers are moreover playing a strict best response) for $c > \underline{c} = 1/(8R^2)$ and in particular for $c > 1/2$. Moreover, since a "bonus" incoming proposal here is the same as an opportunity in the hypothetical setting, this implies that if a workers receives a "bonus" proposal he should accept it without screening. Therefore, s_w is a strict best response in our setting.

Evolutionary stability. In showing that s_e and s_w are strict best responses, we in fact also established that the associated actions a_e and a_w are strict best responses. As a result, these actions are also best responses for any agent mix $\bar{N}$ that is a small perturbation of the steady state $\bar{L}$, which implies evolutionary stability (Definition 7).

No other equilibrium for $c > \frac{1}{8}$**.** Fix $c > 1/8$. We know by Lemma E.1 that all employers consider incoming proposals. An employer will not want to screen incoming proposals by an argument similar to the above: after all they can earn the maximum possible utility of 1/2 by accepting without screening, whereas if they screen incoming proposals and proposals arrive at rate ∞ then their utility is strictly below 1/2 as above. Also if incoming proposals arrive at a finite rate then they leave without matching with a positive probability and so Lemma D.2 tells that their utility is strictly below 1/2. We conclude employers strictly prefer to accept incoming proposals without screening, $a^{\mathrm{i}} = \mathrm{A}$. This implies that, if they were to propose, they will not screen either. As $R > 1$ if workers are all proposing employers have $\nu_e(\bar{L}, \bar{f})$ and a zero lifetime, so they also have a choice of not proposing without affecting their utility, and these are the only two possible choices for a^{o}.

Now consider workers. Opportunities and incoming proposals are equivalent since employers accept without screening. By the same argument as for employers, no workers will want to screen, regardless of what other workers are doing. Being on the long side, the workers have a positive lifetime in the system. Thus, workers have a unique choice for both a^{i} and a^{o}, i.e., all workers use s_w specified in the lemma, and this is independent of the strategy chosen by other workers. This completes our proof that there are no other equilibria. □

We now prove some additional lemmas which will help us rule out other equilibria.

Lemma J.7 (Some workers propose in any equilibrium)**.** *For any $R > 1.25$ and any c, there exists $\mu_0 = \mu_0(R, c) > 0$ such that for any $\mu < \mu_0$ there is no equilibrium where all workers use $a^{\mathrm{o}} = \mathrm{DN}$, i.e., where all workers do not propose.*

Proof of Lemma J.7. Suppose there is an equilibrium where all workers use $a^{\mathrm{o}} = \mathrm{DN}$. We establish a series a properties that any such equilibrium must have, which eventually allows us to show that such an equilibrium cannot exist for $R > 1.25$.

Property 1: *Workers earn expected utility $\theta_w \geq 1/2$.*
Proof: If not, when an opportunity arises, a candidate will be available (as other workers are not proposing, there must be a positive mass of employers in steady state). By proposing without screening, and reverting to the equilibrium strategy thereafter, the worker earns expected utility

$$p_e(1/2) + (1 - p_e)\theta_w > \theta_w\,,$$

where p_e is the probability that the employer will accept the proposal. Here we used our initial assumption $\theta_w < 1/2$ as well as Lemma E.1 which guarantees that $p_e > 0$. Thus we have a contradiction, and have hence established that $\theta_w \geq 1/2$ holds.

Property 2: *All workers screen incoming proposals, i.e., they use $a^{\mathrm{i}} = \mathrm{S{+}A/R}$ (along with $a^{\mathrm{o}} = \mathrm{DN}$).*
Proof: Since the expected utility under $a^{\mathrm{o}} = \mathrm{DN}, a^{\mathrm{i}} = \mathrm{A}$ would be strictly below $1/2$ (if the worker matches, the expected match utility is $1/2$, but since $R > 1$ there is always a positive mass of workers in the system, so there is a positive probability of leaving without matching, causing the overall expected utility to be below $1/2$).

Property 3: *All employers propose, i.e., they use $a^{\mathrm{o}} \in \{\mathrm{S{+}P}, \mathrm{P\ w/o\ S}\}$.*
Proof: Since workers do not propose, we know that all employers propose by Lemma E.2.

Property 4: *Workers earn expected utility $\theta_w \xrightarrow{\mu\to\infty} \xi(R, c)$.*
Proof: By Properties 2 and 3, and using Lemma J.2, we know that the equilibrium between workers is Lemma J.2 Equilibrium 1 and so workers earn limiting expected utility $\lim_{\mu\to 0} \theta_w = \xi(R, c)$.

Property 5: *Employers face an effective screening cost of $c_{\mathrm{eff}} = \frac{c}{1-\theta_w} \xrightarrow{\mu\to 0} \frac{c}{1-\xi(R,c)}$ and receive options (i.e., candidates who will accept their proposal) at a rate $\Theta(1)$.*
Proof: As a result of Properties 2 and 4 and using Lemma D.1, we know that employers face an effective screening cost of $c_{\mathrm{eff}} = \frac{c}{1-\theta_w} \xrightarrow{\mu\to 0} \frac{c}{1-\xi(R,c)}$ and receive options at a rate $1 - \theta_w \in [\sqrt{2c}, 1]$. Here we used Lemma D.2 and that a candidate is available at each opportunity given that $R > 1$, and that opportunities arrive at rate 1.

We consider the possible choices of a^{i} by employers, knowing that $a^{\mathrm{i}} \neq \mathrm{I}$ by Lemma E.1. (Initially we assume all employers make the same choice and show that workers prefer to screen and propose in each case. As a result we conclude that workers prefer to screen and propose even if employers mix between $a^{\mathrm{i}} = \mathrm{S{+}A/R}$ and $a^{\mathrm{i}} = \mathrm{A}$.)

Case 1: All employers use $a^{\mathrm{i}} = \mathbf{S{+}A/R}$

In this case the situation for employers is identical to that in the equilibrium in Theorem 14 and we have $\theta_e = \ell_e(\theta_w)$ for $l_e(\cdot)$ given by (94). Since $\theta_w \leq 1 - \sqrt{2c}$ we obtain $\theta_e \xrightarrow{\mu\to 0} 1 - \sqrt{\frac{2c}{1-\xi(R,c)}}$. (Note that this is unsurprising given Property 5.) Now consider a worker who gets an opportunity to propose. Suppose he follows the following policy: screen and propose to the candidate using threshold $\theta_w(\mu)$ and, if his proposal is rejected, go back to using the strategy s_w. Then, his expected utility is given by

$$-c + (1-\theta_w)(1-\theta_e)\frac{(1+\theta_w)}{2} + \big(1 - (1-\theta_w)(1-\theta_e)\big)\theta_w = -2c + (1-\theta_w)^2(1-\theta_e) + \theta_w\,,$$

as such a proposal will result in a match with probability $(1-\theta_w)(1-\theta_e)$. Therefore, the worker would rather use this strategy if

$$-2c + (1-\theta_w)^2(1-\theta_e) + \theta_w > \theta_w \qquad \Leftrightarrow \qquad -2c + (1-\theta_w)^2(1-\theta_e) > 0\,. \tag{99}$$

Using the limiting values of θ_w and θ_e, the limiting value of the left-hand side as $\mu \to 0$ is

$$-2c + (1-\xi(R,c))^2\sqrt{\frac{2c}{1-\xi(R,c)}} = \sqrt{2c}\left(-\sqrt{2c} + (1-\xi(R,c))^{3/2}\right)\,. \tag{100}$$

We will show that this limit is strictly positive for $\xi(R,c) > \max(3-2R, 0)$. But then, for $R > 1.25$, the limit is strictly positive for $\xi(R,c) \geq 1/2$, meaning that for small enough μ there is no equilibrium where all workers ignore opportunities using Property 1, because workers prefer to screen and propose.

In the following calculation we suppress the arguments of ξ to reduce notational burden. Let us consider when $(1-\xi)^3 - 2c > 0$ in order to know when expression (100) is strictly positive. Since ξ is a solution of (86), we have $-2c = -1 + 2R\xi - (2R-1)\xi^2 = ((2R-1)\xi - 1)(1-\xi)$. So we have

$$\begin{aligned}(1-\xi)^3 - 2c &= (1-\xi)\big((1-\xi)^2 + (2R-1)\xi - 1\big)\\ &= (1-\xi)\xi(\xi + 2R - 3)\end{aligned}$$

We know that $1-\xi > 0$ so the above quantity is greater than 0 if $\xi > \max(3-2R, 0)$ as we wanted to show.

Case 2: All employers use $a^{\mathrm{i}} = \mathrm{A}$

In this case, for a worker an opportunity when a candidate is available is fully equivalent to an incoming proposal, so workers will want to propose, breaking the equilibrium. Moreover, since workers strictly prefer to screen incoming proposals (Property 2), we know that they will want to screen and propose.

Since workers prefer to screen and propose in each case where all employers use the same a^{i}, we conclude that workers prefer to screen and propose even if employers mix between $a^{\mathrm{i}} = \mathrm{S{+}A/R}$ and $a^{\mathrm{i}} = \mathrm{A}$. This completes our proof of the lemma. □

Lemma J.8. *Fix $R = \lambda_w/\lambda_e > 1$ and $c > 0$. Then there exists $\mu_0 = \mu_0(R,c) > 0$ such that there is no equilibrium where workers mix between proposing and not proposing.*

Proof. Suppose $R > 1$ and there is an equilibrium where workers (long side) are mixing between proposing and not proposing. (We know from Lemma E.2 that, in such a case, all employers must be proposing.) Now, a slight increase in the fraction of workers proposing reduces the mass of employers present, and hence makes things worse for the other workers and in particular reduces more the utility from not proposing than that from proposing (we use here that f_w and f_e are product distributions). As a result, incoming workers prefer to propose under the evolutionary dynamics (20), i.e., the equilibrium is not stable. □

Together, the previous two lemmas imply that *all* workers propose in all equilibria for $R > 1.25$ and small enough μ.

Lemma J.9. *Fix $R = \lambda_w/\lambda_e > 1.25$ and $c > 0$. Then there exists $\mu_0 = \mu_0(R, c) > 0$ such that all workers propose in any equilibrium.*

Proof. The result is immediate from Lemmas J.7 and J.8. □

We can now proceed to the proof of Theorem 13.

Proof of Theorem 13. The existence of the claimed steady states is established in Lemma J.3. Lemmas J.4, J.5, and J.6 show that the claimed equilibria exist for the specified intervals of c and do not exist outside those intervals (we omit to characterize whether the equilibria exist at the boundary).

We now show that there are no other equilibria. We already know from Lemma J.6 that for $c > 1/8$ and small enough μ, there is no other equilibrium besides Theorem 13 Equilibrium 3. It remains to consider $c < 1/2$.

No other equilibrium for $c < 1/2$. Fix any $c < 1/2$. Lemma J.9 shows that all workers propose in all equilibria for $R > 1.25$. By the last sentence of Lemma J.4, it follows that employers use threshold $\theta_e = 1 - \sqrt{2c}$ and screen incoming proposals. It then follows from Lemma E.7 that there are only two possibilities for equilibria:

1. Equilibria where workers screen and propose corresponding to Lemma E.7 Equilibrium 1. This case is fully covered by Lemma J.4 (except at the boundary $c = 2\bar{c}^2$, which is also excluded from the theorem statement) which rules out any such equilibria other than Theorem 13 Equilibrium 1.

2. Equilibria where workers propose without screening corresponding to Lemma E.7 Equilibrium 2. This case is fully covered by Lemma J.5 (except at the boundaries $c \in \{2\underline{c}^2, \bar{c}, 1/8\}$) which rules out any such equilibria other than Theorem 13 Equilibrium 2.

□

J.4 Additional equilibria in unbalanced markets

Additional equilibria under no intervention

Consider the setting of Theorem 13. For $R \in [1, 1.25]$, in addition to the equilibria described in the theorem some additional equilibria may arise. Employers propose in all these additional equilibria (these equilibria are related to those which arise when workers are not allowed to propose). In the interest of brevity, we omit the lengthy formal analysis and restrict ourselves to an informal description of these equilibria in the following remark.

Remark 8 (Additional no-intervention equilibria)**.** *Consider the market defined in the statement of Theorem 13. Then, in addition to those specified in Theorem 13, the following are all the equilibria as a function of c, for small enough μ:*

1. $\mathbf{c} \in (\bar{\mathbf{c}}_2, \hat{\mathbf{c}})$: *(Employers screen + propose, Workers screen + accept/reject) with thresholds $\theta_w \xrightarrow{\mu\to 0} \xi(R, c)$ and $\theta_e \xrightarrow{\mu\to 0} 1 - \sqrt{2c/(1 - \xi(R, c))}$. Workers do not propose.*[43] *The steady state $\bar{L} = [L_e, L_w]'$ is given by $L = \left[\frac{1}{\mu+\eta}, \frac{R-1}{\mu} + \frac{1}{\mu+\eta}\right]'$ where $\eta = (1-\theta_w)(1-\theta_e)$. Furthermore, the interval of c for which this equilibrium exists is non-empty if and only if $R < 1.25$.*

[43] Employers screen + accept/reject incoming proposals with threshold θ_e.

2. $\mathbf{c} \in [\hat{\mathbf{c}}, \bar{\mathbf{c}}_{\mathbf{3}})$*: (Employers propose w/o screening, Workers screen + accept/reject) with threshold* $\theta_w \xrightarrow{\mu \to 0} \xi(R, c)$ *is an equilibrium. Workers do not propose.*[44] *Employers earn expected utility* $\theta_e \xrightarrow{\mu \to 0} 1/2$. *The corresponding steady state* $\bar{L} = [L_e, L_w]'$ *is given by* $\bar{L} = \left[\frac{1}{\mu+1-\theta_w}, \frac{R-1}{\mu} + \frac{1}{\mu+1-\theta_w}\right]'$. *Furthermore, the interval of c for which this equilibrium exists is non-empty if and only if* $R < 1.25$.

Here $\xi(\cdot, \cdot)$, $\bar{c}$ *and* $\hat{c}$ *are defined by* (37), (38) *and* (39) *respectively, and*

$$\bar{c}_2 \triangleq 4(R-1)^3 , \tag{101}$$

$$\bar{c}_3 \triangleq \frac{(3-2R)}{8} . \tag{102}$$

Furthermore, (Employers screen + propose, Workers accept without screening) is not an equilibrium for any c under no intervention.

Additional equilibria under the "block workers from proposing" intervention in unbalanced markets

Suppose the workers are prevented from proposing. The characterization of equilibria for small c was formally established in Theorem 14. Since the characterization for larger values of c does not advance the main message of the paper (and hence does not appear in the main paper), in the interest of brevity we provide it only informally without proof, in the following remark.

Remark 9 (Other intervention equilibria). *Consider the market described in the statement of Theorem 14: i.e., arrival rates* $\lambda_e = 1$ *and* $\lambda_w = R > 1$ *with workers not allowed to propose.*

The equilibrium described in Theorem 14 exists for $c \in (0, \min(\bar{c}, \hat{c}))$ *and does not exist for* $c > \min(\bar{c}, \hat{c})$. *Call it Equilibrium 1. Besides Equilibrium 1, all the stable stationary equilibria as a function of c are as follows for small enough* μ:

2. $\mathbf{c} \in [\underline{\mathbf{c}}, \frac{\mathbf{1}}{\mathbf{8}})$ *: (Employers screen + propose, Workers accept w/o screening) with threshold* $\theta_e \xrightarrow{\mu \to 0} 1 - \sqrt{2c}$. *Each worker earns an expected utility of* $\theta_w \xrightarrow{\mu \to 0} 1/(2R)$. *The corresponding steady state* $L = [L_e, L_w]'$ *is given by* $L = \left[\frac{1}{\mu+1-\theta_e}, \frac{R-1}{\mu} + \frac{1}{\mu+1-\theta_e}\right]'$.
3. $\mathbf{c} \in [\hat{\mathbf{c}}, \bar{\mathbf{c}})$ *: (Employers propose without screening, Workers screen + accept/reject) with threshold* $\theta_w \xrightarrow{\mu \to 0} \xi(R, c)$. *Each employer earns an expected utility of* $\theta_e \xrightarrow{\mu \to 0} 1/2$. *This equilibrium exists only if* $\hat{c} < \bar{c}$, *which occurs for* $R < 1.461$ *(rounded to three decimals). The corresponding steady state* $L = [L_e, L_w]'$ *is given by* $L = \left[\frac{1}{\mu+1-\theta_w}, \frac{R-1}{\mu} + \frac{1}{\mu+1-\theta_w}\right]'$.
4. $\mathbf{c} \geq \mathbf{1/8}$ *: (Employers propose without screening, Workers accept incoming proposals without screening). Employers' expected utility is* $\theta_e \xrightarrow{\mu \to 0} 1/2$ *and workers' expected utility is* $\theta_w \xrightarrow{\mu \to 0} 1/(2R)$. *The corresponding steady state* $L = [L_e, L_w]'$ *is given by* $L = \left[\frac{1}{\mu+1}, \frac{R-1}{\mu} + \frac{1}{\mu+1}\right]'$.

Here $\bar{c}$ *and* $\underline{c}$ *are as per* (38), $\xi(R, c)$ *is defined in* (37), *and* $\hat{c}$ *is defined in* (39). *Equilibrium 4 is the unique equilibrium for* $c \geq 1/8$. *Co-existence of equilibria for* $c < 1/8$ *is as follows, depending on the value of R:*

(i) For $R < 1.342$ *we have* $\hat{c} \leq \underline{c} < \bar{c} < 1/8$. *Equilibria 2 and 3 co-exist for* $c \in [\underline{c}, \bar{c})$.

(ii) For $R \in (1.342, 1.461)$ *we have* $\underline{c} < \hat{c} < \bar{c} < 1/8$. *Equilibria 1 and 2 co-exist for* $c \in (\underline{c}, \hat{c})$, *whereas Equilibria 2 and 3 co-exist for* $c \in [\hat{c}, \bar{c})$.

[44] Employers screen + accept/reject incoming proposals with threshold θ_e.

(iii) For $R > 1.461$ we have $\underline{c} < \bar{c} \le \hat{c} < 1/8$. Equilibria 1 and 2 co-exist for $c \in (\underline{c}, \bar{c})$. Equilibrium 3 does not exist for any c.

J.4.1 When does the intervention help?

Figure 4 uses equilibrium characterizations to depict for which (R, c) pairs the intervention (preventing workers from proposing) increases social welfare, for which pairs it decreases social welfare and for which pairs it leaves welfare unchanged.

For any $R > 1.25$ and small c, we show that preventing workers from proposing increases the average welfare, because the increase in the welfare of workers exceeds the decrease in the welfare of employers. We now state a detailed version of our result, Corollary 6, in terms of the detailed versions of Theorems 4 and 5, and then prove it.

Corollary 19 (Intervention boosts welfare for small c)**.** *Fix any $R > 1.25$. There exists $c_{\max} > 0$ such that for all $c \in (0, c_{\max})$ there exists $\mu_0 = \mu_0(c) > 0$ such that for all $\mu < \mu_0$ the intervention of preventing the long side from proposing strictly increases average welfare. More specifically, for $c < c_{\max}$ and $\mu < \mu_0(c)$, when workers are blocked from proposing the unique equilibrium is that described in Theorem 14, whereas under no intervention Theorem 13 Equilibrium 1 is the unique equilibrium, and the former has strictly larger average welfare.*

Proof. By choosing $c_{\max} \le \min(\hat{c}, 2\underline{c}^2)$, we can ensure that for all $c < c_{\max}$ and μ small enough, the only equilibrium under intervention is that described in Theorem 14, and that the only equilibrium under no intervention is Theorem 13 Equilibrium 1. It remains to show that average welfare is higher under the former equilibrium for small enough c. For small c, a Taylor expansion for $\xi(R, c)$ (defined in (10)) yields

$$\xi(R, c) = \frac{1}{2R-1} - \frac{c}{R-1} + O(c^2)\,. \tag{103}$$

It follows that

$$\begin{aligned}\lim_{\mu \to 0} (R+1) \times \text{Avg-welfare(Theorem 13 Equilibrium 1)} &= R\xi(R, \sqrt{c/2}) + 1 - \sqrt{2c} \\ &= \frac{R}{2R-1} - \frac{R\sqrt{c/2}}{(R-1)} + O(c) + 1 - \sqrt{2c}\,, \end{aligned} \tag{104}$$

and

$$\begin{aligned}\lim_{\mu \to 0} (R+1) \times \text{Avg-welfare(Theorem 14 equilibrium)} &= R\xi(R, c) + 1 - \sqrt{2c/(1-\xi(R, c))} \\ &= \frac{R}{2R-1} + 1 - \sqrt{\frac{c(2R-1)}{R-1}} - O(c)\,. \end{aligned} \tag{105}$$

Subtracting (104) from (105), we see that the constant order terms cancel out and the coefficient of $\sqrt{c}$ is positive, because

$$\sqrt{\frac{2R-1}{R-1}} < \left(2 + \frac{R}{R-1}\right)\frac{1}{\sqrt{2}}\,,$$

which can be seen by squaring both sides. It follows that there exists $c_{\max}$ such that for all $c < c_{\max}$

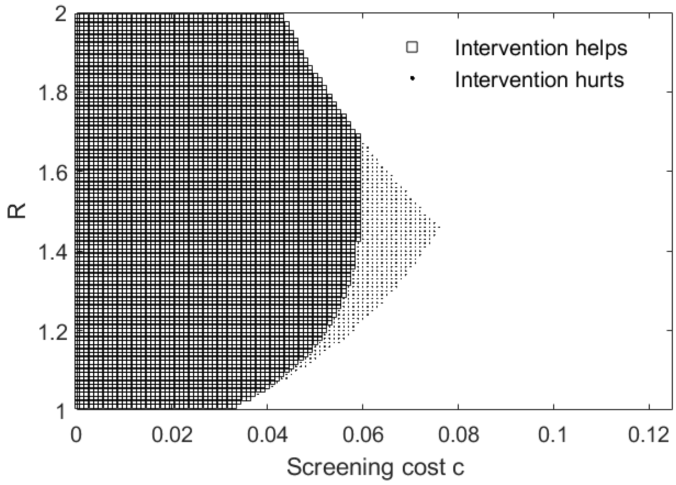

Figure 4: This figure shows the values of (R, c) for which the average welfare increases/decreases by blocking workers from proposing, based on the equilibria without and with the intervention (Theorems 13 and 14 along with Remarks 8 and 9) as $\mu \to 0$. The intervention will help if the parameters fall in the dark gray region (indicated by the squares), will hurt in the light gray region (indicated by the dots), and will not have an effect in the white area.

we have

$$\begin{aligned} &\lim_{\mu\to 0} \text{Avg-welfare}(\text{Theorem 14 equilibrium}) > \lim_{\mu\to 0} \text{Avg-welfare}(\text{Theorem 13 Equilibrium 1})\,, \\ \Rightarrow \quad &\text{Avg-welfare}(\text{Theorem 14 equilibrium}) > \text{Avg-welfare}(\text{Theorem 13 Equilibrium 1}) \end{aligned}$$

for small enough μ, completing the proof. $\square$

K Supplementary material to Appendix F (Appendix to Section 4: vertical differentiation)

K.1 Additional proofs for Appendix F.1 (No-intervention equilibrium)

Remark 10. *The eigendecomposition of A_1, provided in* (48)*, admits an elegant interpretation in the context of the dynamical system.*

1. *The first eigenvalue corresponds to the mass of reacher employers in the system,*[45] *and if this number deviates, it returns to zero via the resulting imbalance between the arrival and departure flows. Since arrival and matching flows are fixed, this imbalance per unit mass of deviation is equal to μ (the rate at which agents leave unmatched), and hence the value of λ_1.*
2. *The third eigenvalue corresponds to a short-lived imbalance in the arrival and matching flow rates of settler employers (a vanishing flow of settler employers leave without matching). If such an imbalance occurs, it quickly disappears, since the matching flow rate increases by nearly $L_w^{\mathrm{L}}/\bar{L}_e(s^{\mathrm{se}}) = \eta_{\mathrm{L}}(\lambda_w^{\mathrm{L}} - \lambda_e^{\mathrm{se}})/(\lambda_e^{\mathrm{re}} - \lambda_w^{\mathrm{H}}) = \gamma$ per unit mass increase in z_3, due to more proposals by low workers going to reacher employers. There is a small effect on the scaled mass of low workers present, and hence the μ-valued first coordinate of the eigenvector.*
3. *The second eigenvalue corresponds to a deviation in the mass of low workers, and how it disappears. As per the above bullet, the mass of settler employers adjusts and settles at a point such that the flow of matches is nearly identical to their arrival flow λ_e^{re}. Once the flow of matches has equilibrated, there is a slow adjustment of the mass of low workers in the system, similar to bullet 1, modulated by the rate of leaving unmatched, μ, since the arrival and matching flows are nearly fixed in time, and hence we have $\lambda_2 \approx \mu$.*

Proof of Claim 2. Suppose we start at $z(\tau) = z = z^* + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, where z^* is a fixed point of (47). Allowing the system to evolve for a (short) time Δ, we have that

$$z(\tau+\Delta) = z^* + \sum_{i=1}^{3} \alpha_i(\tau+\Delta) v_i = z^* + \sum_{i=1}^{3} \alpha_i(\tau)(1-\lambda_i\Delta) v_i + \begin{bmatrix} \epsilon_1(z(\tau)) - \epsilon_1(z^*) \\ \epsilon_2(z(\tau)) - \epsilon_2(z^*) \\ \epsilon_3(z(\tau)) - \epsilon_3(z^*) \end{bmatrix} \Delta + O(\Delta^2)\,.$$

Consider the expansion of the error term in terms of eigenvectors. The $\epsilon_3(z(\tau)) - \epsilon_3(z^*)$ term is of order $|z(\tau) - z^*|\Delta\mu = |\alpha(\tau)|\Delta\mu$, and when we express $[0\ 0\ \mu]'$ in terms of eigenvectors, we obtain a coefficient of order μ for v_3 and a coefficient of order μ^2 for v_2. (Other error terms can be handled similarly.) For each of the eigenvectors, we see that the ratio of the magnitude of the coefficient (due to the error term) to the magnitude of the eigenvalue is order $\mu|\alpha(\tau)|\Delta$. We deduce that:

1. The coefficient of v_2 in the error term is dominated by $\alpha_2(\tau)\lambda_2\Delta$ provided $\alpha_2(\tau)$ is large compared to $\mu|\alpha(\tau)|$.
2. The coefficient of v_3 is dominated by $\alpha_3(\tau)\lambda_3\Delta$ provided $\alpha_3(\tau)$ is large compared to $\mu|\alpha(\tau)|$.
3. The coefficient of v_1 is dominated by $\alpha_1(\tau)\lambda_1\Delta$ provided $\alpha_1(\tau)$ is large compared to $\mu|\alpha(\tau)|$.

In other words, we have:

$$\max(|\alpha_i(\tau+\Delta)|, \epsilon) \le \max(|\alpha_i(\tau)|, \epsilon) - \mathbb{I}(|\alpha_i(\tau)| > \epsilon)|\alpha_i(\tau)||\lambda_i|\Delta/2 \quad \text{for } i = 1,2,3$$
$$\text{where } \epsilon = O(\mu|\alpha(\tau)|)\,.$$

for Δ small enough. It follows that this condition remains valid for a fixed $\epsilon = O(\mu|\alpha(\tau)|)$ for all subsequent times. Hence, it must be that eventually all $|\alpha_i|$'s are no more than ϵ. At this point, we can reset the value of ϵ to a value that is $O(\mu)$ times the original value and repeat. It follows

[45] The scaling in the definition of z_2 does not affect the eigenvalue, since it affects both sides of the dynamical equation by the same factor.

that the system returns to the fixed point z^*. This argument goes through for all μ smaller than some value μ_0. $\square$

Proof of Claim 4 (evolutionary stability)

Evolutionary stability was defined in Appendix C.5 via Definition 7, which demands that the equilibrium steady state $\bar{L}$ be an *attractive* fixed point under the evolutionary dynamics (20).

We will start by rewriting the evolutionary dynamics in terms of $z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$ defined in (44). It is easy to see that, when the system deviates *slightly* from the fixed point z^*, the best response of the high workers and the low workers remains unchanged. (In particular, for a small enough perturbation, high workers will still match instantaneously, whereas low workers still have to propose and cannot afford to screen.) However, for entering employers, being a reacher may be more or less attractive than being a settler, the main determinant of the relative attractiveness being the mass of Reachers presently in the system. When there are more Reachers present, this increases the likelihood of leaving without matching for Reachers, and hence makes being a reacher less attractive. This reasoning implies that the evolutionary dynamics (20) are equivalent to the following dynamic evolution of z defined in (44):

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = A_1 z + \begin{bmatrix} 0 \\ \mu\big(\mathbb{I}(z_2 < \epsilon_6)\lambda_e - \lambda_e^{\mathrm{re}}\big) \\ -\mathbb{I}(z_2 < \epsilon_6)\lambda_e + \lambda_e^{\mathrm{re}} \end{bmatrix} + \begin{bmatrix} \epsilon_1(z) \\ \epsilon_4(z) \\ \epsilon_5(z) \end{bmatrix} ,$$
$$\text{where } \epsilon_6 = \epsilon_6(z_1, z_3) = O(\mu), \epsilon_5 = O(\mu), \epsilon_4 = O(\mu^2), \epsilon_1 = O(\mu^2) \,. \tag{106}$$

Again, each of the ϵ's is, in fact, Lipschitz continuous (for $|z| = O(1)$) with the Lipschitz constant bounded as $O(\mu)$ for ϵ_6 and ϵ_5 and $O(\mu^2)$ for ϵ_4 and ϵ_1. Recall that $\lambda_e^{\mathrm{re}} = \tilde{\lambda}^{\mathrm{H}} + O(\mu)$. We call the three terms above, in order, the linear term (unchanged from before), the best response term, and the error term.

To begin with, we show that, after an initial transient, the system will hit the boundary $z_2 = \epsilon_6(z_1, z_3)$ and stay there.

Claim 5. *Fix $\epsilon_9 > 0$. There exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$ the following holds for the evolutionary dynamics* (106)*. There exists $\epsilon_7 > 0$ and $C > 0$ (that can depend on all model primitives), such that for any starting point $z(0)$ satisfying $|z(0) - z^*| < \epsilon_7$, after an initial transient of duration $C|z(0) - z^*|$, the system will hit the best-response boundary $z_2 = \epsilon_6(z_1, z_3)$ and stay on the boundary thereafter. Moreover, when it hits the boundary, $|z(0) - z^*| < \epsilon_9$.*

Proof of Claim 5. Consider the following dynamical system that captures (47) in the region $z_2 < \epsilon_6$.

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = A_1 y + \begin{bmatrix} 0 \\ \mu\lambda_e^{\mathrm{se}} \\ -\lambda_e^{\mathrm{se}} \end{bmatrix} + \begin{bmatrix} \epsilon_1(y) \\ \epsilon_4(y) \\ \epsilon_5(y) \end{bmatrix} ,$$
$$\text{where } \epsilon_6 = \epsilon_6(y_1, y_3) = O(\mu), \epsilon_5 = O(\mu), \epsilon_4 = O(\mu^2), \epsilon_1 = O(\mu^2) \,. \tag{107}$$

Suppose the system starts at $y(0) = z(0)$ such that $y_2 < \epsilon_6$. (The complementary case can be handled via a very similar argument.) Consider the evolution of y_2. It increases at a rate of

$$\mu\lambda_e^{\mathrm{se}} + \frac{\partial\epsilon_4}{\partial y_1}\frac{\mathrm{d}y_1}{\mathrm{d}\tau} + \frac{\partial\epsilon_4}{\partial y_3}\frac{\mathrm{d}y_3}{\mathrm{d}\tau} \,.$$

To hit the boundary, we need $y_2 = \epsilon_6$. The initial distance from the boundary along the y_2

coordinate is bounded by $2|y(0)-y^*|$ for small enough μ, since ϵ_6 is $O(\mu)$-Lipschitz continuous in y_1 and y_3, and $z^* = y^*$ is on the boundary (in fact, it is a fixed point of (106)) bounded above by $\hat{\tau} = 2(2|y(0)-y^*|)/(\mu\lambda_e^{\text{se}}) = 4|y(0)-y^*|/(\mu\lambda_e^{\text{se}})$. To establish this, it will suffice to show that

$$\frac{\mathrm{d}\epsilon_6}{\mathrm{d}\tau} - \frac{\mathrm{d}\epsilon_4}{\mathrm{d}\tau} = \sum_{j=1,3} \left(\frac{\partial \epsilon_6}{\partial y_j} - \frac{\partial \epsilon_4}{\partial y_j} \right) \frac{\mathrm{d}y_j}{\mathrm{d}\tau} \leq \mu\lambda_e^{\text{se}}/2 \tag{108}$$

holds for all $\tau \leq \hat{\tau}$. Now, if ϵ_7 is sufficiently small, then so is $\hat{\tau}$. There exists $\epsilon_8 > 0$ such that while $|y(\tau)-y^*| \leq \epsilon_8$ we have $\left|\frac{\mathrm{d}y}{\mathrm{d}\tau}\right| \leq \lambda_e$. It follows that $|y(\tau)-y^*| \leq \epsilon_7 + \lambda_e t$ for such times, and hence $|y(\tau)-y^*| \leq \epsilon_9 \leq \epsilon_8$ holds up to $\hat{\tau}' = (\epsilon_9-\epsilon_7)/\lambda_e$ for any $\epsilon_9 \in (0,\epsilon_8)$. By choosing ϵ_7 and hence $\hat{\tau}$ sufficiently small, we can ensure that $\hat{\tau}' \geq \hat{\tau}$. It follows that $\left|\frac{\mathrm{d}y}{\mathrm{d}\tau}\right| \leq \lambda_e$ and $|y(\tau)-y^*| \leq \epsilon_9$ holds up to $\hat{\tau}$. We deduce that

$$\left|\frac{\partial \epsilon_i}{\partial y_j}\frac{\mathrm{d}y_j}{\mathrm{d}\tau}\right| = O(\mu^2) \leq \mu\lambda_e^{\text{re}}/6 \quad \text{for } \mu < \mu_0 \text{ and } (i,j) \in \{(4,1),(4,3),(6,1)\}\,.$$

For $i=4$ and $j=1,3$, we used that ϵ_4 is $O(\mu^2)$-Lipschitz continuous. For $i=6, j=1$, we used $\left|\frac{\mathrm{d}y_1}{\mathrm{d}\tau}\right| = O(\mu)$ and $|y_1-y_1^*| = O(\mu)$ for $\tau \leq \hat{\tau}$ and small enough ϵ_7. The remaining term is

$$\frac{\partial \epsilon_6}{\partial y_3}\frac{\mathrm{d}y_3}{\mathrm{d}\tau}\,.$$

Note that $\frac{\partial \epsilon_6}{\partial y_3} > 0$ since more Settlers in the system makes it less attractive for an incoming employer to be a settler, and is justified only at a higher value of y_2. Also, observe from (107) that $\frac{\mathrm{d}y_3}{\mathrm{d}\tau} < 0$ for $\tau \leq \hat{\tau}$. We deduce that this term is negative. It follows that Eq. (108) holds for all $\tau \leq \hat{\tau}$. We deduce that the system hits the boundary for the first time at $\tau = \hat{\tau}'' \leq \hat{\tau}$, and until it hits the boundary, the distance between the y_2 coordinate and ϵ_6 is always decreasing. After $\hat{\tau}''$ the system does not leave the boundary, since if it "tries," the result we just derived implies that the dynamics immediately pushes the system back to the boundary. Also, we observe that $|y(\hat{\tau}'')-y^*| < \epsilon_9$. □

We now write the equations for the two-dimensional evolutionary dynamics when the system is on the best-response boundary, and employ an argument similar to the one we used to prove Claim 2 (stability of the fixed point when there is a fixed mix of agent strategies), in order to complete the proof of evolutionary stability.

Proof of Claim 4. Claim 3 establishes that all agent categories are, in fact, playing a best response in the fixed point in Claim 1. It follows that the fixed point is also a fixed point of the evolutionary dynamics (106). We will now show evolutionary stability.

Suppose the evolutionary dynamics (106) begin from $z(0)$ such that $|z(0)-z^*| < \epsilon_7$. By Claim 5, we know that the dynamics hit the best-response boundary at a point that is at most ϵ_9 from z^*, at some time $\hat{\tau}''$. Thereafter, we can follow the proof of Claim 2 to control system dynamics on the boundary. The dynamical system is now two-dimensional, in terms of z_1 and z_3, since $z_2 = \epsilon_6$ remains true. The evolutionary dynamics are now

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \begin{bmatrix} -\mu(1+\xi) & -\mu\gamma \\ -\frac{\xi}{1+\rho} & -\frac{\gamma}{1+\rho} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} \epsilon_{10}(x) \\ \epsilon_{11}(x) \end{bmatrix} \qquad \text{where } \rho = \frac{1}{\mu}\left.\frac{\partial \epsilon_6}{\partial x_3}\right|_{x^*} > 0 \tag{109}$$

Here, $x(\tau) = [x_1(t)\ x_3(t)]' = [z_1(t+\hat{\tau}'')\ z_3(t+\hat{\tau}'')]'$. We know $|x(0)| \le \epsilon_9$ and also have that $\epsilon_{10}(x)$ is $O(\mu^2)$-Lipschitz continuous and $\epsilon_{11}(x)$ is $O(\mu)$ at x^* and $O(\mu + \epsilon_9)$-Lipschitz continuous. The reason for this is as follows. Note that $\rho = \Theta(1)$. We expect $\left|\frac{\mathrm{d}x_3}{\mathrm{d}\tau}\right| = \Theta(|x(t) - x^*|)$. If this holds, using Taylor expansion to control $\frac{\partial \epsilon_6}{\partial x_3}$, we obtain that

$$\frac{\mathrm{d}z_2}{\mathrm{d}\tau} = \frac{\partial \epsilon_6}{\partial x_3}\frac{\mathrm{d}x_3}{\mathrm{d}\tau} + O(\mu^2) = \mu\rho\frac{\mathrm{d}x_3}{\mathrm{d}\tau} + O(\mu(\mu + \epsilon_9^2))\,.$$

Since the best-response term in (106) produces the leading-order component of $\frac{\mathrm{d}z_2}{\mathrm{d}\tau}$, it must contribute an opposite push to $z_3 = x_3$ that is $1/\mu$ times larger in magnitude, to leading order, by definition. This push is then $-\rho\frac{\mathrm{d}x_3}{\mathrm{d}\tau} + O(\mu)$. It follows that

$$\frac{\mathrm{d}x_3}{\mathrm{d}\tau} = \begin{bmatrix}-\xi & -\gamma\end{bmatrix} x \,-\, \rho\frac{\mathrm{d}x_3}{\mathrm{d}\tau} \,+\, O(\mu + \epsilon_9^2).$$

Rearranging leads to the $\frac{\mathrm{d}x_3}{\mathrm{d}\tau}$ expression in (109). The best response of low workers remains unchanged, and hence there is no similar leading-order "correction" to $\frac{\mathrm{d}x_1}{\mathrm{d}\tau}$. The eigenvalues of the 2x2 matrix capturing the linear part are, defining $\tilde{\gamma} = \gamma/(1+\rho)$, the same as the eigenvalues λ_1 and λ_3 of A_1, see (48), when γ is replaced by $\tilde{\gamma}$. In particular, the eigenvalues are negative.

By using the arguments in the proof of Claim 2, we can now show that for ϵ_9 and μ_0 small enough, this implies convergence to the fixed point x^* and hence to z^* for the overall system. □

K.2 Proof of Theorem 16 (Block workers from proposing)

Roadmap for the proof. The theorem is established by proving a series of properties that hold in any equilibrium, for small enough c:

1. All employers screen and propose to high workers whenever possible, high workers screen and accept/reject with threshold $\theta_w^{\mathrm{H}} = 1 - \sqrt{2c}$, and $L_w^{\mathrm{H}} = 0$.
2. A positive fraction of employers (settlers) propose to low workers (if no high worker is available).
3. All settlers match in the limit $\mu \to 0$.
4. Settlers screen and propose to low workers.
5. As $\mu \to 0$, the arrival flow of reachers is $\lambda_w^{\mathrm{H}} + \tilde{\delta}^{\mathrm{H}} = \lambda_w^{\mathrm{H}}(1 + q/2) + o_c(1)$. Also, $\bar{L}_e(s^{\mathrm{re}})$ and L_w^{L} are as specified in the theorem.
6. Low workers screen and accept/reject with a threshold θ_w^{L} which satisfies $\lim_{\mu\to 0}\theta_w^{\mathrm{L}} = 1/(2R_\delta^{\mathrm{L}} - 1) - o_c(1)$. Also, θ_e and $\bar{L}_e(s^{\mathrm{se}})$ are as specified in the theorem.

Having established all these properties, we finally show existence and uniqueness of equilibrium.

Proof of Theorem 16. To prove the theorem, we establish a succession of properties that hold in any equilibrium, for small enough c. At the end we establish existence and uniqueness of equilibrium. We quantify the expected utilities and other relevant quantities taking in the limit $\mu \to 0$ and then $c \to 0$ as per the theorem statement.

Property 1: *All employers screen and propose to high workers whenever possible, high workers screen and accept/reject with threshold $\theta_w^{\mathrm{H}} = 1 - \sqrt{2c}$, and $L_w^{\mathrm{H}} = 0$.*

Proof: Note that high workers will accept/reject proposals (with or without screening) with some threshold $\theta_w^{\mathrm{H}} \le 1 - \sqrt{2c}$ in any (robust) best response, since ignoring incoming proposals would lead to a utility of 0 (if they do not screen, we think of the threshold as $\theta_w^{\mathrm{H}} = 0$). As a result, the expected net utility per match with a high worker that results from an employer screening and proposing with threshold $1 + q - \sqrt{2c/(1-\theta_w^{\mathrm{H}})}$ is $1 + q - \sqrt{2c/(1-\theta_w^{\mathrm{H}})} \geq 1 + q - (2c)^{1/4} > 1$ for small enough c, which exceeds the largest possible net utility per match with low workers. (Here we used that employers now face an effective screening cost of $c/(1-\theta_w^{\mathrm{H}})$ with respect to high workers; see Lemma D.1.) It follows that employers screen and propose to high workers whenever they get the chance (we cannot yet deduce what threshold employers use, just that they screen and propose). Since $\lambda_e > \lambda_w^{\mathrm{H}}$, high workers are flooded with proposals as soon as they arrive, and so they employ an optimal threshold of $\theta_w^{\mathrm{H}} = 1 - \sqrt{2c}$. The match and leave immediately, so the mass of high workers in the system is $L_w^{\mathrm{H}} = 0$.

Property 2: *A positive fraction of employers (settlers) propose to low workers (if no high worker is available).*
Proof: We prove this fact by contradiction. Suppose no employers propose to low workers, i.e., all employers are reachers: they only screen and propose to high workers. Let the arrival flow of employers who do not match with low workers be $\lambda_w^{\mathrm{H}} + \tilde{\delta}^{\mathrm{H}}$; our assumption that no employers propose to low workers implies $\tilde{\delta}^{\mathrm{H}} = \lambda_e - \lambda_w^{\mathrm{H}} > \lambda_w^{\mathrm{H}} q/2$. We can write the average net utility $\theta_e^{\mathrm{re}} = \theta_e$ for a reacher from matching with high workers as follows:

$$\begin{aligned}
\theta_e^{\mathrm{re}} &= \text{(Likelihood of matching)(Expected net utility of a match)} \\
&= \frac{\lambda_w^{\mathrm{H}}}{\lambda_w^{\mathrm{H}} + \tilde{\delta}^{\mathrm{H}}} \left(\frac{q + 1 + \theta_e^{\mathrm{re}}}{2} - \frac{c}{\sqrt{2c}(q + 1 - \theta_e^{\mathrm{re}})} \right),
\end{aligned} \tag{110}$$

using that the expected match utility is $\frac{q+1+\theta_e^{\mathrm{re}}}{2}$ and the expected screening cost per match is $\frac{c_{\mathrm{eff}}}{\text{Likelihood that high worker exceeds } \theta_e^{\mathrm{re}}}$ where $c_{\mathrm{eff}} = c/(1-\theta_w^{\mathrm{H}}) = c/\sqrt{2c}$ and the likelihood that a high worker clears θ_e^{re} is $q+1-\theta_e^{\mathrm{re}}$. Considering only the first term on the right and using $\epsilon = \tilde{\delta}^{\mathrm{H}} - \lambda_w^{\mathrm{H}} q/2 > 0$ we deduce that $\theta_e^{\mathrm{re}} < \frac{1}{1+q/2+\epsilon} \left(\frac{q+1+\theta_e^{\mathrm{re}}}{2} \right) \Rightarrow \theta_e^{\mathrm{re}} < 1 - \epsilon'$ for some $\epsilon' > 0$. Since $\epsilon' > 0$ does not depend on c, we deduce that for small enough c, we have $\theta_e^{\mathrm{re}} < 1 - \sqrt{2c}$. Given that no employers propose to low workers, the (robust) best response of low workers is to accept an incoming proposal without screening. As a result, an employer can improve her utility by screening and proposing to an low worker (e.g., with threshold θ_e^{re}) if no high worker is available. Thus, we have obtained a contradiction. It follows that a positive fraction of employers are settlers who propose to low workers if no high worker is available.

Property 3: *All settlers match in the limit $\mu \to 0$.*
Proof: Again, we prove this property by contradiction. Suppose a positive flow of settlers leave without matching even as $\mu \to 0$. In any equilibrium, all agents are playing a best response so the threshold employed by settlers while proposing is no more than $1-\sqrt{2c}$, and the threshold employed by low workers for incoming proposals is no more than $1 - \sqrt{2c}$. As a result, a fraction at least $\sqrt{2c}\sqrt{2c} = 2c$ of opportunities that arise for settlers when a worker is available produce matches (if a high worker is available, the likelihood of a match is even higher). Then since a positive flow of settlers leave without matching even as $\mu \to 0$, it must be that workers are typically *not* available, and hence it must be that all workers, including all low workers, match and leave. Thus, a flow λ_w^{L} of settlers match with low workers. It follows that the arrival flow of employers who do not match with low workers $\lambda_w^{\mathrm{H}} + \tilde{\delta}^{\mathrm{H}}$ is $\lambda_e - \lambda_w^{\mathrm{L}} < \lambda_w^{\mathrm{H}}(1 + q/2)$, meaning that $\tilde{\delta}^{\mathrm{H}} < \lambda_w^{\mathrm{H}} q/2$. Plugging into (110) and recalling $\theta_e \leq 1 - \sqrt{2c}$ we observe that for c small enough, we have $\theta_e \geq 1$, which contradicts the fact that settlers are playing a best response when they propose to low workers (producing net utility per match less than 1). We deduce that all settlers match and leave in the limit $\mu \to 0$.

Property 4: *Settlers screen and propose to low workers.*

Proof: Given that settlers do not run the risk of leaving without matching as $\mu \to 0$(and so they must get many opportunities since a flow 0 leave without matching), and since we know that $\theta_w^{\mathrm{L}} \le 1 - \sqrt{2c}$, for small enough c settlers have a best response of screening and proposing to low workers with a threshold $\theta_e^{\mathrm{se}} = \theta_e$ which satisfies

$$\lim_{\mu \to 0} \theta_e^{\mathrm{se}} = 1 - \sqrt{\frac{2c}{1 - \theta_w^{\mathrm{L}}}} = 1 - o_c(1)\,. \tag{111}$$

Here we used that settlers face an effective screening cost of $c/(1-\theta_w^{\mathrm{L}})$ (see Lemma D.1).

Property 5: *As $\mu \to 0$, the arrival flow of reachers is $\lambda_w^{\mathrm{H}} + \tilde{\delta}^{\mathrm{H}} = \lambda_w^{\mathrm{H}}(1+q/2) + o_c(1)$. Also, $\bar{L}_e(s^{\mathrm{re}})$ and L_w^{L} are as specified in the theorem.*
Proof: Since all settlers match with low workers, by definition of $\tilde{\delta}^{\mathrm{H}}$, the arrival flow of reachers is $\lambda_w^{\mathrm{H}} + \tilde{\delta}^{\mathrm{H}}$. Recall that all high workers match and leave in equilibrium, and since settlers match with them at a flow of 0, it follows that reachers match with them at a flow of λ_w^{H}. (Informally, there is effective "High submarket" of reachers and high workers.) We deduce that $\tilde{\delta}^{\mathrm{H}} \ge 0$. Solving (110) gives

$$\begin{aligned}
\theta_e^{\mathrm{re}} &= \frac{(q+1)\frac{\lambda_w^{\mathrm{H}}+\tilde{\delta}^{\mathrm{H}}}{\lambda_w^{\mathrm{H}}} - \sqrt{(q+1)^2\left(\frac{\lambda_w^{\mathrm{H}}+\tilde{\delta}^{\mathrm{H}}}{\lambda_w^{\mathrm{H}}} - 1\right)^2 + \sqrt{2c}\left(2\cdot\frac{\lambda_w^{\mathrm{H}}+\tilde{\delta}^{\mathrm{H}}}{\lambda_w^{\mathrm{H}}} - 1\right)}}{2\cdot\frac{\lambda_w^{\mathrm{H}}+\tilde{\delta}^{\mathrm{H}}}{\lambda_w^{\mathrm{H}}} - 1}\,, \\
&= \frac{(q+1)^2 - \sqrt{2c}}{(q+1)\frac{\lambda_w^{\mathrm{H}}+\tilde{\delta}^{\mathrm{H}}}{\lambda_w^{\mathrm{H}}} + \sqrt{(q+1)^2\left(\frac{\lambda_w^{\mathrm{H}}+\tilde{\delta}^{\mathrm{H}}}{\lambda_w^{\mathrm{H}}} - 1\right)^2 + \sqrt{2c}\left(2\cdot\frac{\lambda_w^{\mathrm{H}}+\tilde{\delta}^{\mathrm{H}}}{\lambda_w^{\mathrm{H}}} - 1\right)}}
\end{aligned} \tag{112}$$

since the other root of the quadratic equation strictly exceeds $1+q$ for any $\tilde{\delta}^{\mathrm{H}} \ge 0$. Note that the right-hand side is a strictly decreasing function of $\tilde{\delta}^{\mathrm{H}}$. Note that the right hand side of (112) has a limiting value of 1 as $c \to 0$ if and only if $\tilde{\delta}^{\mathrm{H}} = \lambda_w^{\mathrm{H}} q/2 + o_c(1)$. But from (111) we know that in equilibrium, we must have $\lim_{c\to 0}\lim_{\mu\to 0}\theta_e = \lim_{c\to 0}\lim_{\mu\to 0}\theta_e^{\mathrm{se}} = 1$, so we infer that $\lim_{\mu\to 0}\tilde{\delta}^{\mathrm{H}} = \lambda_w^{\mathrm{H}} q/2 + o_c(1)$ holds in equilibrium. We immediately deduce that the arrival flow of reachers is $\lambda_w^{\mathrm{H}} + \tilde{\delta}^{\mathrm{H}} = \lambda_w^{\mathrm{H}}(1+q/2) + o_c(1)$ and of settlers is $\lambda_e - \lambda_w^{\mathrm{H}}(1+q/2) - o_c(1)$ as $\mu \to 0$. We also infer that a flow $\tilde{\delta}^{\mathrm{H}} = \lambda_w^{\mathrm{H}} q/2 + o_c(1)$ of reachers leave without matching as $\mu \to 0$, and hence that the steady state mass of reachers is given by $\lim_{\mu\to 0}\mu\bar{L}_e(s^{\mathrm{re}}) = \frac{\lambda_w^{\mathrm{H}} q}{2}(1+o_c(1))$. Finally, we infer that the match flow in the "low submarket" constituted by settlers and low workers is equal to the arrival rate of settlers $\lambda_e - \lambda_w^{\mathrm{H}}(1+q/2) - o_c(1)$, and hence a flow $\lambda_w^{\mathrm{L}} - \lambda_e + \lambda_w^{\mathrm{H}}(1+q/2) + o_c(1)$ of low workers leave without matching as $\mu \to 0$ and their steady state mass in the system is given by $\lim_{\mu\to 0}\mu L_w^{\mathrm{L}} = (\lambda_w^{\mathrm{L}} + \lambda_w^{\mathrm{H}}(1+q/2) - \lambda_e)(1+o_c(1))$.

Property 6: *Low workers screen and accept/reject with a threshold θ_w^{L} which satisfies $\lim_{\mu\to 0}\theta_w^{\mathrm{L}} = 1/(2R_\delta^{\mathrm{L}} - 1) - o_c(1)$. Also, θ_e and $\bar{L}_e(s^{\mathrm{se}})$ are as specified in the theorem.*
Proof: As $\mu \to 0$, the low submarket (settlers and low workers) has an imbalance of

$$\tilde{R}_\delta^{\mathrm{L}} = \frac{\lambda_w^{\mathrm{L}}}{\lambda_e - \lambda_w^{\mathrm{H}} - \tilde{\delta}^{\mathrm{H}}} = R_\delta^{\mathrm{L}} + o_c(1)\,. \tag{113}$$

Given this imbalance, and provided that c is small enough, we can use Theorem 5 to (uniquely) determine that low workers will now screen with threshold θ_w^{L} satisfying

$$\lim_{\mu\to 0}\theta_w^{\mathrm{L}} = \xi(\tilde{R}_\delta^{\mathrm{L}}, c) \tag{114}$$

$$= \xi(R_\delta^{\mathrm{L}}, 0) - o_c(1) = \tfrac{1}{2R_\delta^{\mathrm{L}}-1}\left(1 - o_c(1)\right), \tag{115}$$

using that $\xi(\cdot,\cdot)$ is Lipschitz in both arguments. Plugging (115) into (111), we obtain

$$\theta_e = 1 - \sqrt{\tfrac{c(2R_\delta^{\mathrm{L}}-1)}{R_\delta^{\mathrm{L}}-1}}\left(1 + o_c(1)\right).$$

Further, the steady state mass of settlers is (since the flow of opportunities equal to this mass)

$$\begin{aligned}\lim_{\mu\to 0}\bar{L}_e(s^{\mathrm{se}}) &= \frac{\text{Match flow in the low submarket}}{\text{Likelihood that an low worker opportunity results in a match}}\\ &= \frac{\lambda_e - \lambda_w^{\mathrm{H}}(1+q/2)}{(1-\theta_e)(1-\theta_w^{\mathrm{L}})} - o_c(1)\\ &= (\lambda_e - \lambda_w^{\mathrm{H}}(1+q/2))\sqrt{\tfrac{2R_\delta^{\mathrm{L}}-1}{4c(R_\delta^{\mathrm{L}}-1)}}\,(1+o_c(1)).\end{aligned} \tag{116}$$

Existence and uniqueness of equilibrium. We have already established that any equilibrium must have the leading order description provided in the theorem statement. We now establish that there exists a unique equilibrium with this description. Treating the right hand sides of (113), (114), (111), and (112) as functions of $\tilde{\delta}^{\mathrm{H}} \in (\lambda_w^{\mathrm{H}}q/2 - o_c(1), \lambda_w^{\mathrm{H}}q/2 + o_c(1))$, we see that as $\mu \to 0$, we have that $\tilde{R}_\delta^{\mathrm{L}}$ is strictly increasing in $\tilde{\delta}^{\mathrm{H}}$, θ_w^{L} is strictly decreasing in $\tilde{\delta}^{\mathrm{H}}$, and hence $\lim_{\mu\to 0}\theta_e^{\mathrm{se}}$ as per (111) (based on the settlers) is strictly increasing in $\tilde{\delta}^{\mathrm{H}}$. On the other hand we saw that θ_e^{re} as per (112) (based on the reachers) is strictly decreasing in $\tilde{\delta}^{\mathrm{H}}$. Thus, and using that $d\theta_e^{\mathrm{se}}/d\tilde{\delta}^{\mathrm{H}}$ is continuous in μ for settlers and θ_e^{re} does not depend on μ for reachers, we deduce that for small enough μ there is at most a unique value of $\tilde{\delta}^{\mathrm{H}}$ such that (111) and (112) are consistent with each other. To obtain existence of an equilibrium, it remains to show that such a value exists. (Note that we have already fixed behavior of both low workers —as per (114)— and high workers to be best responses to employer behavior. Furthermore, employers are playing best responses within each submarket, consistency of θ_e^{se} and θ_e^{re} is the only pending condition to ensure that all employers are playing a global best response.)

Note that the right-hand side of (111) is always in the interval $[1-(2c)^{1/4}, 1-\sqrt{2c}]$, and so $\lim_{\mu\to 0}\theta_e^{\mathrm{se}} = 1$ for all $\tilde{\delta}^{\mathrm{H}} \in [0, \lambda_e - \lambda_w^{\mathrm{H}}]$ as $\mu \to 0$ and $c \to 0$. On the other hand, θ_e^{re} given by (112) does not depend on μ and as $c \to 0$, has limiting value of (I) $\lim_{c\to 0}\theta_e^{\mathrm{re}} = 1$ when $\tilde{\delta}^{\mathrm{H}} = q\lambda_w^{\mathrm{H}}/2$, (II) $\lim_{c\to 0}\theta_e^{\mathrm{re}} > 1$ for fixed $\tilde{\delta}^{\mathrm{H}} < q\lambda_w^{\mathrm{H}}/2$, and (II) $\lim_{c\to 0}\theta_e^{\mathrm{re}} < 1$ for fixed $\tilde{\delta}^{\mathrm{H}} > q\lambda_w^{\mathrm{H}}/2$. It follows, using the intermediate value theorem that for small enough c, a (unique, by the above) feasible solution $\tilde{\delta}^{\mathrm{H}}$ exists such that (111) and (112) are consistent, and this solution is $\tilde{\delta}^{\mathrm{H}} = \frac{q\lambda_w^{\mathrm{H}}}{2}(1+o_c(1))$ to leading order in c. In fact the argument establishes the existence of $\tilde{\delta}^{\mathrm{H}}$ such that $\theta_e^{\mathrm{se}} = \theta_e^{\mathrm{re}}$ for all small enough μ for each small enough c, leveraging continuity in[46] μ. This completes our proof of existence.

Evolutionary stability can be established similar to the proof of Theorem 8 and, for brevity, we omit the calculations. The basic intuition is that, if the proportion of reachers increases slightly, this makes it less attractive to be a reacher due to competition for scarce high workers, and so the

[46] In particular, for any fixed c and $\mu > 0$, $\theta_e^{\mathrm{se}}(\tilde{\delta}^{\mathrm{H}}) : [0, \lambda_e - \lambda_w^{\mathrm{H}}] \to (0,1)$ for settlers is monotone increasing (the corresponding intuition is that larger $\tilde{\delta}^{\mathrm{H}}$ means fewer settlers and so low workers are less selective, helping settlers) and hence bounded below by $\theta_e^{\mathrm{se}}(0)$, and this value is continuous in μ.

proportion returns to its equilibrium value. □

K.3 Proof of Theorem 17 (Block workers from proposing and hiding quality information from employers)

Roadmap for the proof. The theorem is established by proving a series of (increasingly refined) properties that hold in any equilibrium, for small enough c:

1. Employers screen and propose.
2. Upper bound $\theta_e = 1 - \Omega_c(c)$.
3. All employers match as $\mu \to 0$.
4. Low workers accept an incoming proposal with likelihood $\Omega_c(1)$ for small enough μ.
5. Tighter bound $\theta_e = 1 - \Theta_c(c)$ as $\mu \to 0$. Accordingly, let $\lim_{\mu\to 0} \theta_e = 1 - \tilde{K}_e c$, where $\tilde{K}_e = \tilde{K}_e(c) = \Theta_c(1)$.
6. High workers are selective $\theta_w^{\mathrm{H}} = 1 - \Theta_c(\sqrt{c})$ as $\mu \to 0$. Accordingly, let $\lim_{\mu\to 0} \theta_w^{\mathrm{H}} = 1 - \tilde{K}_{\mathrm{H}}\sqrt{c}$, where $\tilde{K}_{\mathrm{H}} = \tilde{K}_{\mathrm{H}}(c) = \Theta_c(1)$.
7. The fraction ζ_{H} of workers present who are high workers is $o_c(1)$ as $\mu \to 0$.
8. Precise estimates of match flow rates, and low worker best response, θ_w^{L}, and L_w^{L} as $\mu \to 0$.
9. Uniqueness of equilibrium, precise estimate of $\tilde{K}_e$, employer best response and L_e as $\mu \to 0$.
10. High worker best response and precise estimate of $\tilde{K}_{\mathrm{H}}$ as $\mu \to 0$. Existence of equilibrium.
11. Precise estimate of ζ_{H}, fraction of high workers who leave without matching and L_w^{H}.

Here we use the Big-O and allied notations for small c with their standard meanings, with the subscript c indicating small c. "Precise estimate" in each case refers to an estimate matching the quantification in the theorem statement. Along the way, properties 9 and 10 further establish uniqueness and existence of equilibrium for c small enough, and μ small enough for the given c.

Proof of Theorem 17. To prove the theorem, we establish a succession of properties that hold in any equilibrium, for small enough c. Along the way we establish uniqueness and existence of equilibrium. We emphasize that we quantify the equilibrium for $\mu \to 0$ for fixed c, and furthermore consider small c. In addition to $o_c(\cdot)$, we employ the allied Big-O notations $O_c(\cdot)$, $\Theta_c(\cdot)$, $\omega_c(\cdot)$ and $\Omega_c(\cdot)$, for small c, with their standard meanings.

Property 1: *Employers screen and propose.*
Proof: In any equilibrium, all employers propose (otherwise they will never match) and all workers (high and low) accept/reject with or without screening. Next, notice that there are always workers in the market, as they arrive faster than employers overall, i.e., $\lambda_e < \lambda_w^{\mathrm{H}} + \lambda_w^{\mathrm{L}}$. In any steady state, the mix of workers (high versus low) is fixed. Also recall that the platform hides the quality tier of workers from employers. As such, from an employer's perspective, a potential candidate worker has an ex-ante match value drawn i.i.d. from a fixed distribution (a mixture between Uniform$(0, 1)$ and Uniform$(q, q + 1)$). Given our assumptions on the parameter space, it follows that, for small enough c and μ, the employer will obtain a higher expected utility by screening before proposing.

Property 2: *Upper bound $\theta_e = 1 - \Omega_c(c)$.*
Proof: Fix the threshold θ_e used by employers. We first prove an upper bound of $\theta_e = 1 - \Omega_c(c)$, employing a proof by contradiction. If θ_e exceeds 1, then low workers are never proposed to. Thus, as employers only match with high workers, a flow at least $\lambda_e - \lambda_w^{\mathrm{H}}$ of employers leave without matching. This implies there is a mass over $(\lambda_e - \lambda_w^{\mathrm{H}})/\mu$ of employers present in the market in steady state. As there are always workers present in the system ($\lambda_e < \lambda_w^{\mathrm{H}} + \lambda_w^{\mathrm{L}}$), a flow over $(\lambda_e - \lambda_w^{\mathrm{H}})/\mu$ of candidates being presented to employers and candidates are screened, so screening cost by employers accrues at a flow of at least $(\lambda_e - \lambda_w^{\mathrm{H}})c/\mu$. It follows that the expected screening cost per employer is at least $(\lambda_e - \lambda_w^{\mathrm{H}})c/(\lambda_e\mu) \xrightarrow{\mu\to 0} \infty$ for every fixed c, implying that the expected

utility of employers converges to $-\infty$ as $\mu \to 0$, a contradiction. A similar argument applies even if $\theta_e = 1 - o_c(c)$: if a flow at least $(\lambda_e - \lambda_w^{\mathrm{H}})/2$ of employers leaves without matching then the argument above yields a contradiction. If not, then then a flow at least $(\lambda_e - \lambda_w^{\mathrm{H}})/2$ of employers leaves by matching with low workers. But for every proposal to an low worker, an employer screens $1/(1-\theta_e) = \omega_c(1/c)$ candidates, and hence accrues $\omega_c(1)$ screening cost. Since a fraction $(\lambda_e - \lambda_w^{\mathrm{H}})/(2\lambda_e)$ of employers match with low workers, we deduce that the average screening cost per employer is $\omega_c(1)$ meaning that the average utility per employer is negative for small enough c, a contradiction.

Property 3: *All employers match as $\mu \to 0$.*
Proof: Note that both kinds of workers will accept incoming proposals with probability at least $\sqrt{2c}$ by Lemma D.2 and Proposition 1, and, for any candidate shown, an employer will propose with likelihood at least $1 - \theta_e = \Omega_c(c)$, meaning that each potential candidate produces a match with likelihood at least $\Omega_c(c^{3/2})$. Hence, for an employer the rate of leaving without matching μ vanishes relative to the rate $\Omega_c(c^{3/2})$ of matching and leaving, as $\mu \to 0$. It follows that all employers match and leave in the limit $\mu \to 0$; the flow of employers leaving without matching is 0.

Property 4: *Low workers accept an incoming proposal with likelihood $\Omega_c(1)$ for small enough μ.*
Proof: Fix any equilibrium. Consider a worker (high or low). Denote by p the probability that a worker receives an incoming proposal before she leaves without matching in the equilibrium steady state. Importantly, p will be larger for high workers than low workers, since both kinds of workers are shown as candidates to employers equally often, but a high worker is more likely to clear the threshold θ_e than an low worker. Also note that $p^{\mathrm{L}} = \Omega_c(1)$ even for low workers as $\mu \to 0$, since a fraction at least $(\lambda_e - \lambda_w^{\mathrm{H}} - o_c(1))/\lambda_e$ of them match and leave (this is because all employers match and leave, and only a flow λ_w^{H} can match with high workers). Then, for small enough c, the worker will want to screen and then accept/reject incoming proposals. Let θ denote the expected utility and screening threshold of the worker. We can write the worker's Bellman equation as $\mu \to 0$ as

$$\begin{aligned}
\theta &= p\big\{\Pr(\text{Proposer's } v > \theta)\mathrm{E}[v|v > \theta] - c + \Pr(\text{Proposer's } v \leq \theta)\theta\big\} \\
&= p\big\{(1-\theta)\tfrac{1+\theta}{2} - c + \theta^2\big\} \\
\Rightarrow \theta &= \tfrac{1-\sqrt{1-p^2(1-2c)}}{p}
\end{aligned} \tag{117}$$

Then the likelihood that the worker will match divided by the likelihood the worker will leave without matching (eventually, one of these two will occur) is

$$\tfrac{p\Pr(\text{Proposer's } v > \theta)}{1-p} = \tfrac{p(1-\theta)}{1-p} = -1 + \sqrt{1 - 2c + \tfrac{2}{1-p}(1-2c) + \tfrac{2c}{(1-p)^2}}\,,$$

which is increasing in p. It follows that the fraction of high workers who match and leave exceeds the fraction of low workers who match and leave. Hence, the fraction of low workers who match and leave is less than $\lambda_e/(\lambda_w^{\mathrm{H}} + \lambda_w^{\mathrm{L}}) < 1$. We deduce that low workers accept an incoming proposal with likelihood at least $1 - \xi((\lambda_w^{\mathrm{H}} + \lambda_w^{\mathrm{L}})/\lambda_e, c) \geq 1 - \xi((\lambda_w^{\mathrm{H}} + \lambda_w^{\mathrm{L}})/\lambda_e, 0) > 0$, i.e., the likelihood of acceptance is $\Omega_c(1)$ as $\mu \to 0$, and hence also for small enough μ (for each c) by continuity in μ.

Property 5: *Tighter bound $\theta_e = 1 - \Theta_c(c)$ as $\mu \to 0$.*
Proof: Next we prove that, in fact, $\theta_e = 1 - \Theta_c(c)$ as $\mu \to 0$ in any equilibrium. The expected

match utility that employers receive as $\mu \to 0$ is

$$\begin{aligned}&(\text{Fraction who match with high workers})\tfrac{1+q+\theta_e}{2}\\&\quad+(\text{Fraction who match with low workers})\tfrac{1+\theta_e}{2}\\&\geq \tfrac{\lambda_w^{\mathrm{H}}}{\lambda_w^{\mathrm{H}}+\lambda_w^{\mathrm{L}}}\tfrac{1+q+\theta_e}{2}+\tfrac{\lambda_w^{\mathrm{L}}}{\lambda_w^{\mathrm{H}}+\lambda_w^{\mathrm{L}}}\tfrac{1+\theta_e}{2}=\theta_e+\Theta_c(1)\,.\end{aligned}$$

Then, for the overall expected utility to be θ_e, the expected screening cost must also be $\Theta_c(1)$. Now, the expected screening cost per match with a high worker is bounded above by

$$\frac{c}{\Pr(\text{worker exceeds }\theta_e)\Pr(\text{employer exceeds }\theta_w^{\mathrm{H}})}\leq\frac{c}{q\sqrt{2c}}=O_c(\sqrt{c})\,.$$

We deduce that the expected screening cost per match with an low worker must be $\Theta_c(1)$, meaning that an employer must screen $\Theta_c(1/c)$ low workers per successful match. Recalling that low workers accept with likelihood $\Omega_c(1)$, we deduce that employers propose to low workers with likelihood only $\Theta_c(c)$, hence it must be that $\lim_{\mu\to 0}\theta_e = 1-\Theta_c(c)$. Let $\lim_{\mu\to 0}\theta_e = 1-\tilde{K}_e c$, where $\tilde{K}_e=\tilde{K}_e(c)=\Theta_c(1)$.

Property 6: *High workers are selective $\theta_w^{\mathrm{H}} = 1-\Theta_c(\sqrt{c})$ as $\mu\to 0$.*
Proof: Using that $\theta_e = 1-\Theta_c(c)$, we deduce that a high worker receive proposals at a rate that is $\Theta_c(1/c)$ times larger than that for an low worker. Recall that the likelihood of receiving a proposal before leaving for a low worker $p^{\mathrm{L}} = \Theta_c(1)$, and also $1-p^{\mathrm{L}}=\Theta_c(1) \Rightarrow \frac{p^{\mathrm{L}}}{1-p^{\mathrm{L}}}=\Theta_c(1)$ since low workers are not very selective and a fraction bounded away from 0 and 1 form matches. It follows that $\frac{p^{\mathrm{H}}}{1-p^{\mathrm{H}}}=\Theta_c\Big(\frac{p^{\mathrm{L}}}{(1-p^{\mathrm{L}})c}\Big)=\Theta_c(1/c)$, i.e., $p^{\mathrm{H}}=1-\Theta_c(c)$. Then (117) tells us that $\lim_{\mu\to 0}\theta_w^{\mathrm{H}}=1-\Theta_c(\sqrt{c})$. Let $\lim_{\mu\to 0}\theta_w^{\mathrm{H}}=1-\tilde{K}_{\mathrm{H}}\sqrt{c}$, where $\tilde{K}_{\mathrm{H}}=\tilde{K}_{\mathrm{H}}(c)=\Theta_c(1)$.

Now that we have a relatively narrow range of possible θ_e and θ_w^{H} in equilibrium, we proceed to sharpen our characterization further to prove each part of the theorem.

Property 7: *The fraction ζ_{H} of workers present who are high workers is $o_c(1)$ as $\mu\to 0$.*
Proof: Let ζ_{H} be the fraction of workers in the system at any time who are high workers. Then, when an employer is presented with an option (which she then proceeds to screen, not knowing whether it is a high or a low worker), it is a high worker with probability ζ_{H}. If it is a high worker, she proposes w.p. $q+o_c(1)$ and gets accepted w.p. $\tilde{K}_{\mathrm{H}}\sqrt{c}$, and so the overall likelihood of the option resulting in a match with a high worker is

$$\zeta_{\mathrm{H}} q\tilde{K}_{\mathrm{H}}\sqrt{c}\,(1+o_c(1))\,.$$

If the option is a low worker, she proposes w.p. $c\tilde{K}_e$ and gets accepted w.p. $\Theta_c(1)$, and so the overall likelihood of the option resulting in a match with a low worker is $\Theta_c(c)$. Suppose ζ_{H} is $\Omega(1)$. Then, with likelihood $1-O_c(\sqrt{c})$, an employer ends up matching with a high worker (since $\lambda_e<\lambda_w^{\mathrm{H}}+\lambda_w^{\mathrm{L}}$ there are always workers present in the market, so employers are presented with options at the highest possible rate), meaning that there is a flow $\lambda_e - o_c(1) > \lambda_w^{\mathrm{H}}$ of matches between employers and high workers, a contradiction. Hence, $\lim_{\mu\to 0}\zeta_{\mathrm{H}} = o_c(1)$.

Property 8: *Precise estimates of match flow rates, and low worker best response, θ_w^{L}, and L_w^{L} as $\mu\to 0$.*
Proof: Now, almost all employers form matches as proved above. It follows that the flow of workers leaving without matching is $\lambda_{\mathrm{L}}^w+\lambda_w^{\mathrm{H}}-\lambda_e$, and since $1-\zeta_{\mathrm{H}}=1-o_c(1)$ fraction of these are low workers. We deduce that the mass of low workers in the system in steady state is given by $\lim_{\mu\to 0}\mu L_w^{\mathrm{L}} = \lambda_w^{\mathrm{L}}+\lambda_w^{\mathrm{H}}-\lambda_e-o_c(1)$. Also, employers match with low workers at a flow $\lim_{\mu\to 0}\rho_w^{\mathrm{L}}=\lambda_e-\lambda_w^{\mathrm{H}}+o_c(1)$, and with high workers at rate $\lim_{\mu\to 0}\rho_w^{\mathrm{H}}=\lambda_w^{\mathrm{H}}-o_c(1)$. Recall the

definition $R^{\mathrm{L}} = \frac{\lambda_w^{\mathrm{L}}}{\lambda_e - \lambda_w^{\mathrm{H}}}$. We deduce that low workers screen and accept reject with a threshold

$$\lim_{\mu \to 0} \theta_w^{\mathrm{L}} = \xi(\lambda_w^{\mathrm{L}}/\rho_w^{\mathrm{L}}, c) = \xi(R^{\mathrm{L}} + o_c(1), c) = \xi(R^{\mathrm{L}}, c) + o_c(1) = \xi(R^{\mathrm{L}}, 0) + o_c(1) = \tfrac{1}{2R^{\mathrm{L}}-1} + o_c(1)\,,$$

where we have appealed to (10).

Property 9: *Uniqueness of equilibrium, precise estimate of $\tilde{K}_e$, employer best response and L_e as $\mu \to 0$.*
Proof: The expected match utility received by an employer is then

$$\frac{\lambda_w^{\mathrm{H}}}{\lambda_e} \frac{1 + q + \theta_e}{2} + \frac{\lambda_e - \lambda_w^{\mathrm{H}}}{\lambda_e} \frac{1 + \theta_e}{2} + o_c(1) = 1 + \frac{q\lambda_w^{\mathrm{H}}}{2\lambda_e} + o_c(1)\,.$$

The expected screening cost per employer is dominated by the cost of screening low workers as proved above. Per match with a low worker, the number of low workers screened is $\frac{1}{c\tilde{K}_e} \frac{2R^{\mathrm{L}}-1}{2(R^{\mathrm{L}}-1)}(1 + o_c(1))$, hence the expected screening cost per employer is

$$\frac{\lambda_e - \lambda_w^{\mathrm{H}}}{\lambda_e} \frac{2R^{\mathrm{L}} - 1}{2\tilde{K}_e(R^{\mathrm{L}} - 1)} + o_c(1)\,.$$

Hence, the total expected utility of employers is

$$1 + \frac{q\lambda_w^{\mathrm{H}}}{2\lambda_e} - \frac{\lambda_e - \lambda_w^{\mathrm{H}}}{\lambda_e} \frac{2R^{\mathrm{L}} - 1}{2\tilde{K}_e(R^{\mathrm{L}} - 1)} + o_c(1)\,.$$

Note that this is a strictly increasing function of $\tilde{K}_e$ (the derivative of the $o_c(1)$ term with respect to $\tilde{K}_e$ is also $o_c(1)$). In equilibrium, this expected utility must be equal to the threshold $1 - \tilde{K}_e c$, which is a decreasing function of $\tilde{K}_e$. We conclude that there is at most a unique equilibrium, and that $\tilde{K}_e = \frac{(\lambda_e - \lambda_w^{\mathrm{H}})}{q\lambda_w^{\mathrm{H}}} \cdot \frac{2R^{\mathrm{L}}-1}{R^{\mathrm{L}}-1} + o_c(1) = K_e + o_c(1)$. The argument extends to all μ small enough by continuity of all relevant quantities in μ. To prove existence of an equilibrium, it remains to check the high worker best response (we have already gone over the employer and low worker best responses).

Note, the likelihood that an option presented to an employer will result in a match with an low worker is

$$(1 - \zeta_{\mathrm{H}})\tilde{K}_e c\Big(1 - \frac{1}{2R^{\mathrm{L}} - 1} + o_c(1)\Big) = c\frac{2(\lambda_e - \lambda_w^{\mathrm{H}})}{q\lambda_w^{\mathrm{H}}}(1 + o_c(1))\,.$$

Given that the match flow between employers and low workers is $\lambda_e - \lambda_w^{\mathrm{H}} - o_c(1)$, it follows that the steady state mass of employers is

$$\begin{aligned}
\lim_{\mu \to 0} L_e &= \text{Flow of opportunities to employers} \\
&= \frac{\text{Match flow rate with low workers}}{\text{Likelihood of opportunity producing a match with an low worker}} \\
&= \frac{q\lambda_w^{\mathrm{H}}}{2c}(1 + o_c(1))\,.
\end{aligned}$$

Property 10: *High worker best response and precise estimate of $\tilde{K}_{\mathrm{H}}$ as $\mu \to 0$. Existence of equilibrium.*
Proof: The mass of low workers in steady state is given by $\lim_{\mu \to 0} \mu L_w^{\mathrm{L}} = \lambda_w^{\mathrm{L}} + \lambda_w^{\mathrm{H}} - \lambda_e + o_c(1)$

as obtained above, and this is also the total mass of workers to leading order since $\zeta_{\rm H} = o_c(1)$. The rate of arrival of proposals to a high worker, using that the total flow of opportunities (across all workers) to employers is L_e and a fraction $1 + q - \theta_e$ of high worker opportunities result in proposals, is

$$\tfrac{L_e}{L_w^{\rm H}+L_w^{\rm L}}(1+q-\theta_e) = \tfrac{q^2\lambda_w^{\rm H}}{2(\lambda_w^{\rm L}+\lambda_w^{\rm H}-\lambda_e)}\tfrac{\mu}{c}(1+o_c(1))\,.$$

It follows that the likelihood of getting an incoming proposal before leaving without matching is

$$p^{\rm H} = 1 - \tfrac{2(\lambda_w^{\rm L}+\lambda_w^{\rm H}-\lambda_e)}{q^2\lambda_w^{\rm H}}c(1+o_c(1))\,.$$

Substituting in (117) we obtain

$$\theta_w^{\rm H} = 1 - \sqrt{2c\left(\tfrac{2(\lambda_w^{\rm L}+\lambda_w^{\rm H}-\lambda_e)}{q^2\lambda_w^{\rm H}}+1\right)}(1+o_c(1)) \quad \Leftrightarrow \quad \tilde{K}_{\rm H} = K_{\rm H}(1+o_c(1))\,.$$

By employing threshold $\theta_w^{\rm H}$, high workers are playing a best response. This completes our proof of the existence of this equilibrium (we already proved uniqueness above). Again, the argument extends to all μ small enough by continuity of all relevant quantities in μ.

Property 11: *Precise estimate of $\zeta_{\rm H}$, fraction of high workers who leave without matching and $L_w^{\rm H}$.*
Proof: Taking the ratio of the likelihoods above that a high worker versus a low worker option will result in a match, and equating with the ratio of the match flows, we have

$$\begin{aligned}\zeta_{\rm H} q\tilde{K}_{\rm H}\sqrt{c}\cdot\tfrac{q\lambda_w^{\rm H}}{2(\lambda_e-\lambda_w^{\rm H})c}(1+o_c(1)) = \tfrac{\rho_w^{\rm H}}{\rho_w^{\rm L}} &= \tfrac{\lambda_w^{\rm H}}{\lambda_e-\lambda_w^{\rm H}} + o_c(1)\,,\\ \Rightarrow \zeta_{\rm H} &= \tfrac{2\sqrt{c}}{q^2\tilde{K}_{\rm H}}(1+o_c(1)) = \tfrac{\lambda_w^{\rm H}(K_{\rm H}/2-1/K_{\rm H})}{\lambda_w^{\rm H}+\lambda_w^{\rm L}-\lambda_e}\sqrt{c}\,(1+o_c(1))\,,\end{aligned}$$

Since the mass of all workers in steady state is $(\lambda_w^{\rm L}+\lambda_w^{\rm H}-\lambda_e+o_c(1))/\mu$, we multiply by $\zeta_{\rm H}$ to deduce that the mass of high workers in steady state is given by $\lim_{\mu\to 0}\mu L_w^{\rm H} = \lambda_w^{\rm H}(K_{\rm H}/2-1/K_{\rm H})\sqrt{c}(1+o_c(1))$, hence they leave without matching at a flow of μ times $L_w^{\rm H}$, i.e., $\lambda_w^{\rm H}(K_{\rm H}/2-1/K_{\rm H})\sqrt{c}(1+o_c(1))$.

Finally, evolutionary stability can be established as in the proof of Theorem 8. The intuition is as follows: suppose the fraction $\zeta_{\rm H}$ of workers in the system who are high workers falls below its steady state value. Then employers' expected utility will fall, they will lower their threshold for accepting a match, will form more matches with low workers, and, as a result, $\zeta_{\rm H}$ will rise toward its steady state value. □