Fast Convergence of Natural Bargaining Dynamics in Exchange Networks

Yashodhan Kanoria

Stanford University

joint work with

Mohsen Bayati, Christian Borgs, Jennifer Chayes, and Andrea Montanari

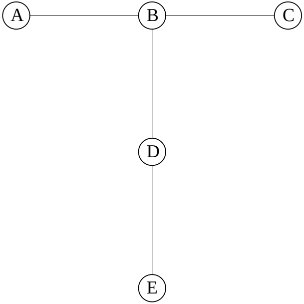
Outline

1. Exchange networks and Nash bargaining (NB) solns
2. Natural dynamics
3. Main results

Outline

1. Exchange networks and Nash bargaining (NB) solns
2. Natural dynamics
3. Main results

Setting

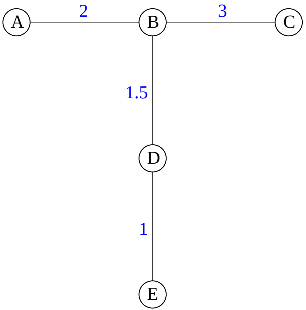


- $G = (V, E)$
- Edge weights $w_{ij} \in (0, W], (ij) \in E$
- $w_{ij} \equiv$ profit from $i - j$ trade
- 1-exchange rule
- Profit from trade split b/w partners

WHAT REALLY HAPPENS?

Setting

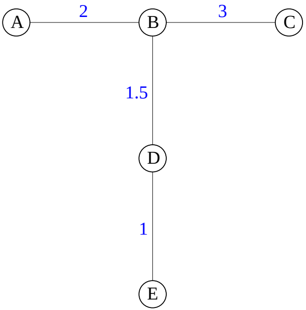
- $G = (V, E)$
- Edge weights $w_{ij} \in (0, W], (ij) \in E$
- $w_{ij} \equiv$ profit from $i - j$ trade
- 1-exchange rule
- Profit from trade split b/w partners



WHAT REALLY HAPPENS?

Setting

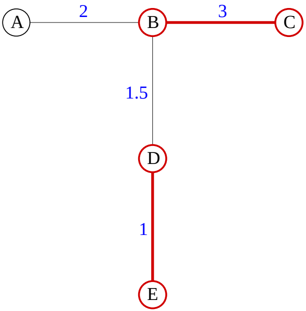
- $G = (V, E)$
- Edge weights $w_{ij} \in (0, W], (ij) \in E$
- $w_{ij} \equiv$ profit from $i - j$ trade
- 1-exchange rule
- Profit from trade split b/w partners



WHAT REALLY HAPPENS?

Setting

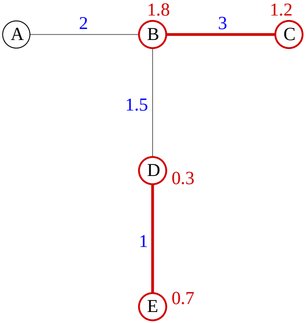
- $G = (V, E)$
- Edge weights $w_{ij} \in (0, W], (ij) \in E$
- $w_{ij} \equiv$ profit from $i - j$ trade
- 1-exchange rule
- Profit from trade split b/w partners



WHAT REALLY HAPPENS?

Setting

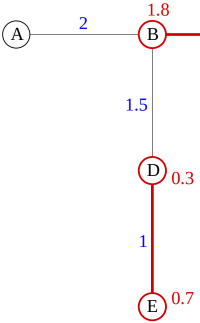
- $G = (V, E)$
- Edge weights $w_{ij} \in (0, W], (ij) \in E$
- $w_{ij} \equiv$ profit from $i - j$ trade
- 1-exchange rule
- Profit from trade split b/w partners



WHAT REALLY HAPPENS?

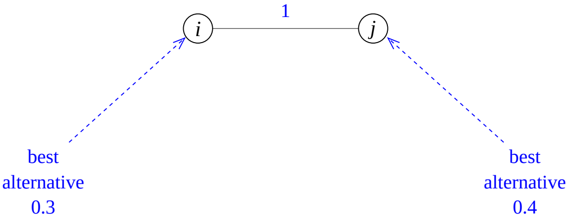
Setting

- $G = (V, E)$
- Edge weights $w_{ij} \in (0, W], (ij) \in E$
- $w_{ij} \equiv$ profit from $i - j$ trade
- 1-exchange rule
- Profit from trade split b/w partners

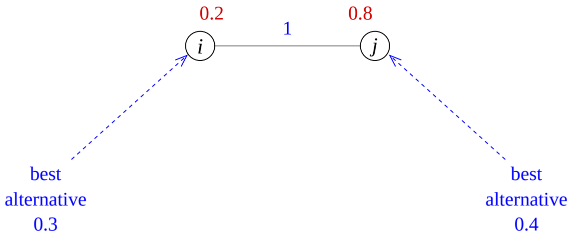


WHAT REALLY HAPPENS?

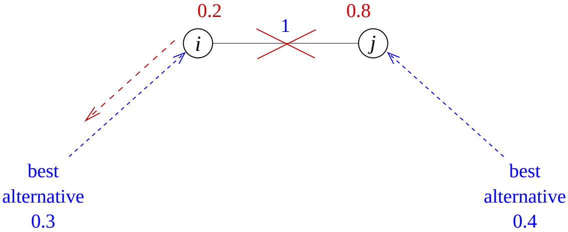
Stability and Balance



Stability and Balance



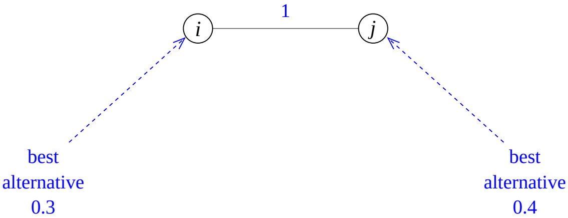
Stability and Balance



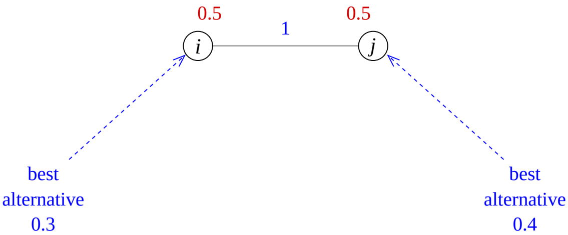
i prefers outside alternative

unstable!

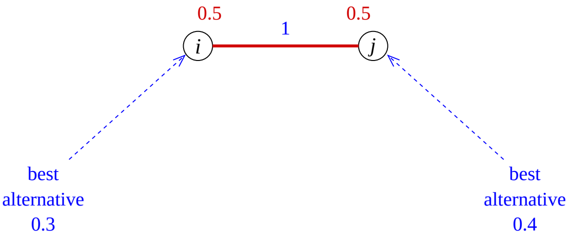
Stability and Balance



Stability and Balance



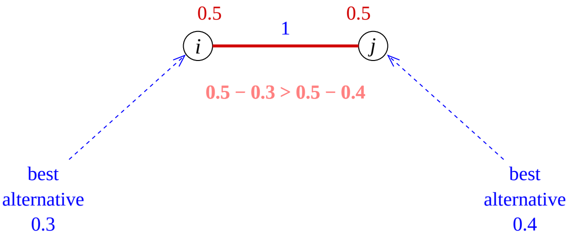
Stability and Balance



Both i and j are happy

i.e. **stable**

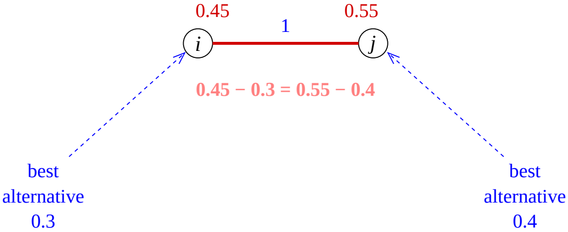
Stability and Balance



But the allocation lacks **balance**:

$$\text{surplus}(j) = 0.2$$
$$\text{surplus}(i) = 0.1$$

Stability and Balance



both stable and balanced

Nash bargaining (NB) solution for (i, j) pair

Nash bargaining solution on a graph

- $M \equiv$ set of trades, a matching
- $\gamma_i \equiv$ earning of node i

Definition

$(M, \underline{\gamma})$ is a *trade outcome* if:

- For all $(i, j) \in M$, $\gamma_i + \gamma_j = w_{ij}$
- For all unmatched i, $\gamma_i = 0$

NB soln on graph:
Pairwise NB soln for each pair in M, with outside γ's fixed

Definition

$(M, \underline{\gamma})$ is a **NB soln** if it is a **stable** and **balanced** trade outcome.

Nash bargaining solution on a graph

- $M \equiv$ set of trades, a matching
- $\gamma_i \equiv$ earning of node i

Definition

$(M, \underline{\gamma})$ is a *trade outcome* if:

- For all $(i,j) \in M$, $\gamma_i + \gamma_j = w_{ij}$
- For all unmatched i, $\gamma_i = 0$

NB soln on graph:

Pairwise NB soln for each pair in M, with outside γ's fixed

Definition

$(M, \underline{\gamma})$ is a **NB soln** if it is a **stable** and **balanced** trade outcome.

Nash bargaining solution on a graph

- $M \equiv$ set of trades, a matching
- $\gamma_i \equiv$ earning of node i

Definition

$(M, \underline{\gamma})$ is a *trade outcome* if:

- For all $(i, j) \in M$, $\gamma_i + \gamma_j = w_{ij}$
- For all unmatched i, $\gamma_i = 0$

NB soln on graph:

Pairwise NB soln for each pair in M, with outside γ's fixed

Definition

$(M, \underline{\gamma})$ is a **NB soln** if it is a **stable** and **balanced** trade outcome.

Nash bargaining solution on a graph

- For $(i, j) \notin M$, implicit $i \to j$ 'offer': $(w_{ij} - \gamma_i)_+$
- **Stability:** For all $(i, j) \in E$, $\gamma_i + \gamma_j \geq w_{ij}$.
  - Equality on matched edges
  - No pair has incentive to deviate
- **Balance:** For all $(i, j) \in M$,

$$\gamma_i - \max_{k \in \partial i \setminus j} (w_{ik} - \gamma_k)_+ = \gamma_j - \max_{l \in \partial j \setminus i} (w_{jl} - \gamma_l)_+$$

Nash bargaining solution on a graph

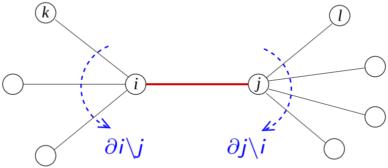
- For $(i,j) \notin M$, implicit $i \to j$ 'offer': $(w_{ij} - \gamma_i)_+$
- **Stability:** For all $(i,j) \in E$, $\gamma_i + \gamma_j \geq w_{ij}$.
  - Equality on matched edges
  - No pair has incentive to deviate
- **Balance:** For all $(i,j) \in M$,

$$\gamma_i - \max_{k \in \partial i \setminus j} (w_{ik} - \gamma_k)_+ = \gamma_j - \max_{l \in \partial j \setminus i} (w_{jl} - \gamma_l)_+$$

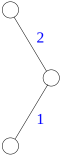
Nash bargaining solution on a graph

- For $(i,j) \notin M$, implicit $i \to j$ 'offer': $(w_{ij} - \gamma_i)_+$
- **Stability:** For all $(i,j) \in E$, $\gamma_i + \gamma_j \geq w_{ij}$.
  - Equality on matched edges
  - No pair has incentive to deviate
- **Balance:** For all $(i,j) \in M$,

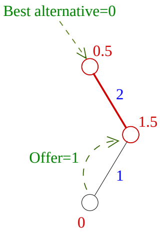
$$\gamma_i - \max_{k \in \partial i \backslash j} (w_{ik} - \gamma_k)_+ = \gamma_j - \max_{l \in \partial j \backslash i} (w_{jl} - \gamma_l)_+$$



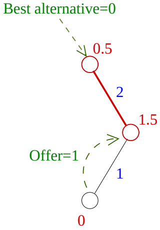
Examples



Examples

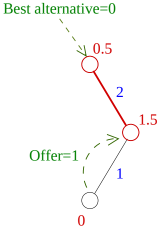


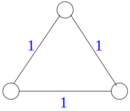
Examples



Unique NB soln

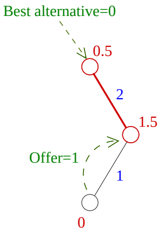
Examples



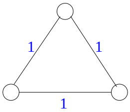


Unique NB soln

Examples

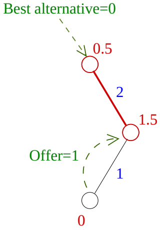


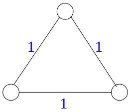
Unique NB soln

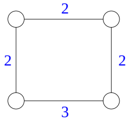


No stable soln

Examples



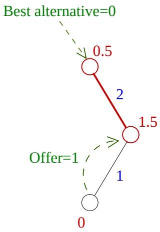




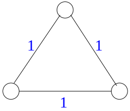
Unique NB soln

No stable soln

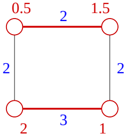
Examples



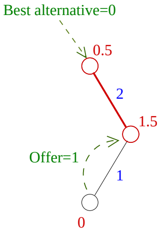
Unique NB soln



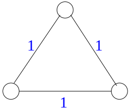
No stable soln



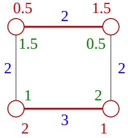
Examples



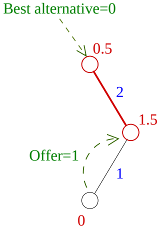
Unique NB soln



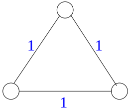
No stable soln



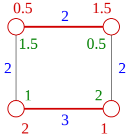
Examples



Unique NB soln



No stable soln



Multiple NB solns

Existence of NB solns

Lemma (Sotomayor '05, Kleinberg & Tardos '08)

Stable solutions correspond to maximum weight matching (MWM).

Theorem (Kleinberg & Tardos STOC '08)

$$\textit{NB soln exists} \quad \Leftrightarrow \quad \textit{Stable soln exists}$$

Existence of NB solns

Lemma (Sotomayor '05, Kleinberg & Tardos '08)

Stable solutions correspond to maximum weight matching (MWM).

Theorem (Kleinberg & Tardos STOC '08)

$$\textit{NB soln exists} \quad \Leftrightarrow \quad \textit{Stable soln exists}$$

Existence of NB solns

Lemma (Sotomayor '05, Kleinberg & Tardos '08)

Stable solutions correspond to maximum weight matching (MWM).

Theorem (Kleinberg & Tardos STOC '08)

NB soln exists $\Leftrightarrow$ *Stable soln exists* $\Leftrightarrow$ *LP relaxation to MWM problem is tight.*

Linear programming (LP) relaxation to MWM problem:

$$\text{maximize} \sum_{(ij)\in E} w_{ij} x_{ij},$$

$$\text{subject to} \sum_{j\in\partial i} x_{ij} \le 1 \quad \forall i \in V, \qquad x_{ij} \ge 0 \quad \forall (ij) \in E$$

Construction of NB solutions

Theorem (Kleinberg & Tardos '08)

NB solns can be constructed in polynomial time.

- **Global** algorithm

Construction of NB solutions

Theorem (Kleinberg & Tardos '08)

NB solns can be constructed in polynomial time.

- **Global** algorithm

Local bargaining processes?

Local bargaining processes?

Theorem (Azar, Birnbaum, Celis, Devanur, Peres FOCS '09)

There is a local algorithm that finds an ϵ-Nash bargaining solution in $O(|V|/g) + 2^{O(|V|+|E|)}/\epsilon$ iterations.

- Improved bounds in special cases [Celis, Devanur, Peres WINE '10]

Local bargaining processes?

Theorem (Azar, Birnbaum, Celis, Devanur, Peres FOCS '09)

There is a local algorithm that finds an ϵ-Nash bargaining solution in $O(|V|/g) + 2^{O(|V|+|E|)}/\epsilon$ iterations.

- Improved bounds in special cases [Celis, Devanur, Peres WINE '10]

Theorem (this paper)

There is a local bargaining process that converges to an ϵ-Nash bargaining solution in $O(|V|^4/g^2 + 1/\epsilon^2)$ iterations.

- Process is a natural model for agent behavior.

Outline

1. Exchange networks and Nash bargaining (NB) solns
2. Natural dynamics
3. Main results

A natural dynamics

- Nodes exchange information/messages *on graph*
- $\alpha^t_{i\backslash j}$: Current estimated **"best alternative"** of i to j
- $\underline{\alpha}^t$ constitutes network 'state'.
- $m^t_{i\to j}$: **"Offer"** from i to j determined as

$$m^t_{i\to j} = (w_{ij} - \alpha^t_{i\backslash j})_+ \ ?$$

A natural dynamics

- Nodes exchange information/messages *on graph*
- $\alpha^t_{i\backslash j}$: Current estimated **"best alternative"** of i to j
- $\underline{\alpha}^t$ constitutes network 'state'.
- $m^t_{i\to j}$: **"Offer"** from i to j determined as

$$m^t_{i\to j} = (w_{ij} - \alpha^t_{i\backslash j})_+ \ ?$$

A natural dynamics

- Nodes exchange information/messages *on graph*
- $\alpha^t_{i\backslash j}$: Current estimated **"best alternative"** of i to j
- $\underline{\alpha}^t$ constitutes network 'state'.
- $m^t_{i\to j}$: **"Offer"** from i to j determined as

$$m^t_{i\to j} = (w_{ij} - \alpha^t_{i\backslash j})_+ \ ?$$

A natural dynamics

- Nodes exchange information/messages *on graph*
- $\alpha^t_{i\backslash j}$: Current estimated **"best alternative"** of i to j
- $\underline{\alpha}^t$ constitutes network 'state'.
- $m^t_{i\to j}$: **"Offer"** from i to j determined as

$$m^t_{i\to j} = (w_{ij} - \alpha^t_{i\backslash j})_+ \ ?$$

A natural dynamics

- Nodes exchange information/messages *on graph*
- $\alpha^t_{i\backslash j}$: Current estimated **"best alternative"** of i to j
- $\underline{\alpha}^t$ constitutes network 'state'.
- $m^t_{i\to j}$: **"Offer"** from i to j determined as

$$m^t_{i\to j} = (w_{ij} - \alpha^t_{i\backslash j})_+ - \frac{1}{2}(w_{ij} - \alpha^t_{i\backslash j} - \alpha^t_{j\backslash i})_+$$

A natural dynamics

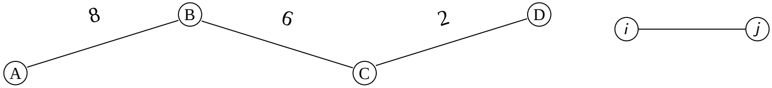
- Nodes exchange information/messages *on graph*
- $\alpha^t_{i\backslash j}$: Current estimated **"best alternative"** of i to j
- $\underline{\alpha}^t$ constitutes network 'state'.
- $m^t_{i\to j}$: **"Offer"** from i to j determined as

$$m^t_{i\to j} = (w_{ij} - \alpha^t_{i\backslash j})_+ - \frac{1}{2}(w_{ij} - \alpha^t_{i\backslash j} - \alpha^t_{j\backslash i})_+$$

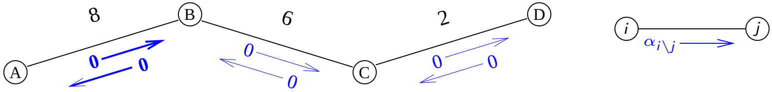
Update rule (synchronous):

$$\alpha^{t+1}_{i\backslash j} = \max_{k\in\partial i\backslash j} m^t_{k\to i}$$

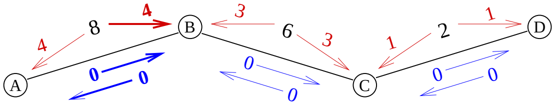
Example of update under natural dynamics

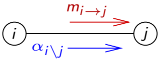


Example of update under natural dynamics

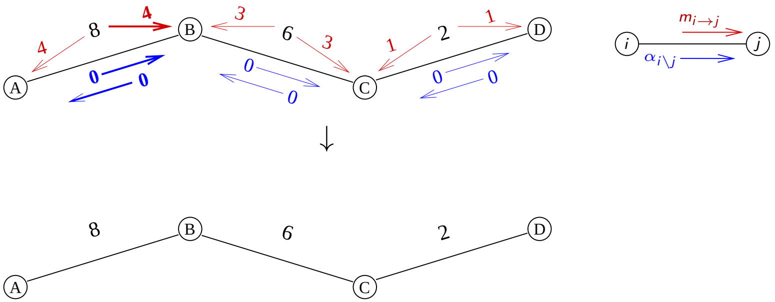


Example of update under natural dynamics

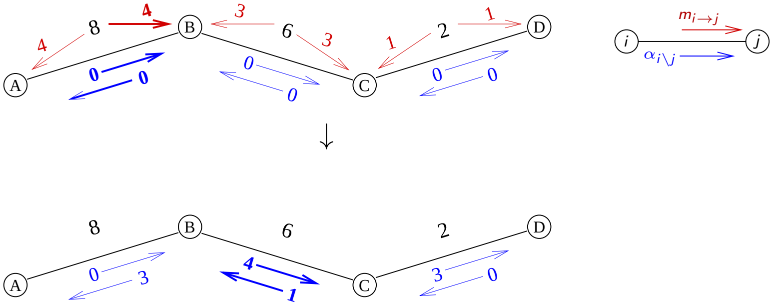




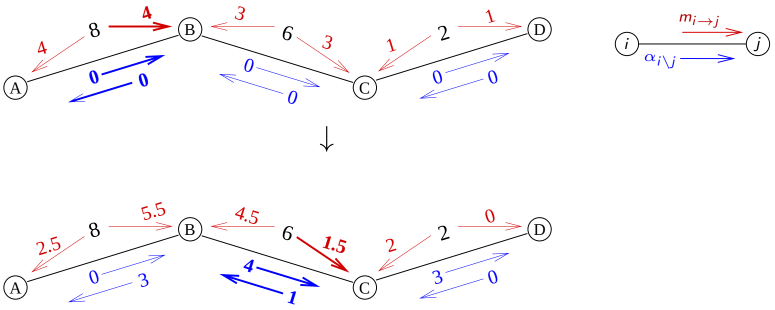
Example of update under natural dynamics



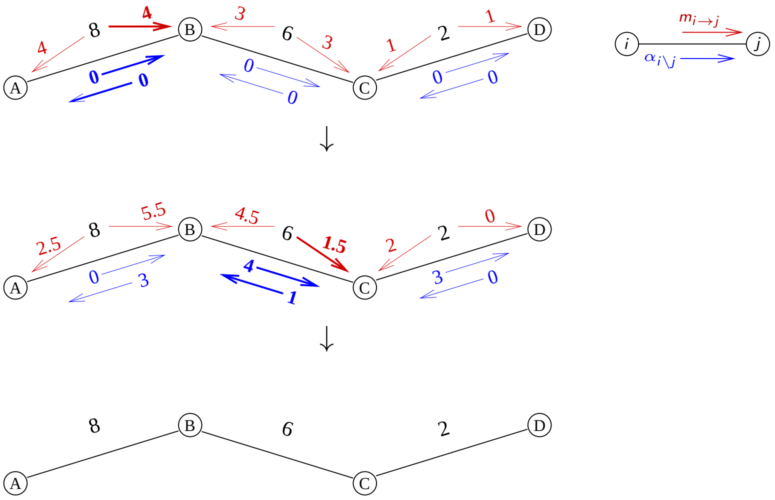
Example of update under natural dynamics



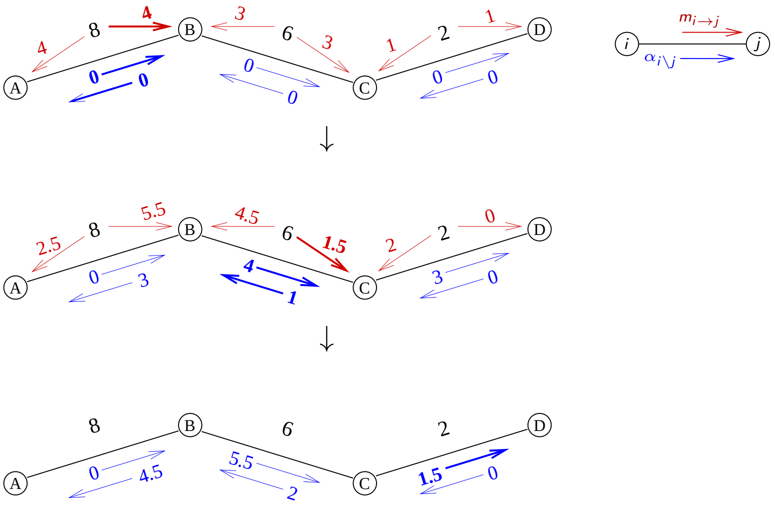
Example of update under natural dynamics



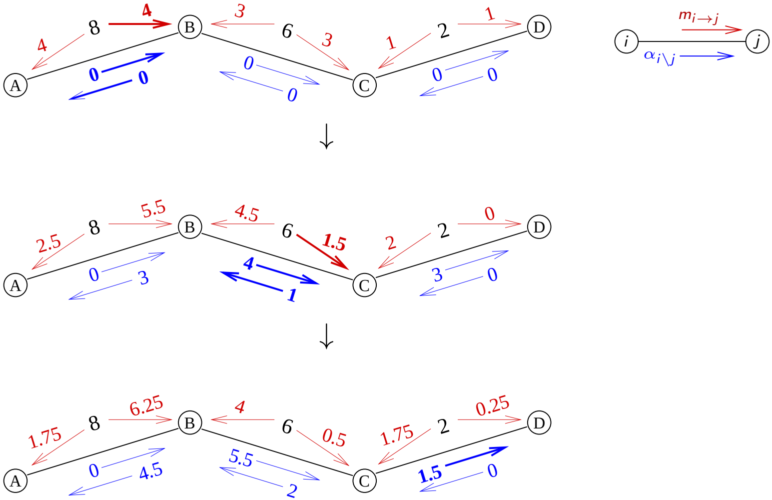
Example of update under natural dynamics



Example of update under natural dynamics



Example of update under natural dynamics



Natural dyamics (contd.)

- Oscillations on even cycles
- Hence, damping introduced in updates

$$\underline{\alpha}^{t+1} = \kappa\, \mathsf{T}\underline{\alpha}^{t} + (1-\kappa)\, \underline{\alpha}^{t}$$

$$(\mathsf{T}\underline{\alpha}^{t})_{i\backslash j} \equiv \max_{k \in \partial i \backslash j} m^{t}_{k\to i}\,,$$

$$\kappa \in (0,1)$$

- Interpretation as reluctance/inertia towards new info
- γ_i^t: Current earning estimate of i

$$\gamma_i^t \equiv \max_{k \in \partial i} m^{t}_{k\to i}$$

Natural dyamics (contd.)

- Oscillations on even cycles
- Hence, damping introduced in updates

$$\underline{\alpha}^{t+1} = \kappa \, \mathsf{T}\underline{\alpha}^{t} + (1-\kappa)\, \underline{\alpha}^{t}$$

$$(\mathsf{T}\underline{\alpha}^{t})_{i \backslash j} \equiv \max_{k \in \partial i \backslash j} m^{t}_{k \to i} \,,$$

$$\kappa \in (0,1)$$

- Interpretation as reluctance/inertia towards new info
- γ_i^t: Current earning estimate of i

$$\gamma_i^t \equiv \max_{k \in \partial i} m^t_{k \to i}$$

Natural dyamics (contd.)

- Oscillations on even cycles
- Hence, damping introduced in updates

$$\underline{\alpha}^{t+1} = \kappa \; \mathsf{T}\underline{\alpha}^{t} + (1-\kappa)\; \underline{\alpha}^{t}$$

$$(\mathsf{T}\underline{\alpha}^{t})_{i\setminus j} \equiv \max_{k\in\partial i\setminus j} m^{t}_{k\to i}\,,$$

$$\kappa \in (0,1)$$

- Interpretation as reluctance/inertia towards new info
- γ_i^t: Current earning estimate of i

$$\gamma_i^t \equiv \max_{k\in\partial i} m^{t}_{k\to i}$$

What can we hope for?

We want:

- Fast convergence
- Fixed points give NB solutions

Outline

Short summary

- FP $\leftrightarrow$ NB solns
- Dynamics converges rapidly to ϵ-FP
- ϵ-FP corresponds to (6ϵ)-NB soln for small ϵ

Implies **fast convergence to ϵ-NB soln**.

Short summary

- FP $\leftrightarrow$ NB solns
- Dynamics converges rapidly to ϵ-FP
- ϵ-FP corresponds to (6ϵ)-NB soln for small ϵ

Implies **fast convergence to ϵ-NB soln**.

Short summary

- FP $\leftrightarrow$ NB solns
- Dynamics converges rapidly to ϵ-FP
- ϵ-FP corresponds to (6ϵ)-NB soln for small ϵ

Implies **fast convergence to ϵ-NB soln**.

Short summary

- FP $\leftrightarrow$ NB solns
- Dynamics converges rapidly to ϵ-FP
- ϵ-FP corresponds to (6ϵ)-NB soln for small ϵ

Implies **fast convergence to ϵ-NB soln**.

Definitions

Definition (Fixed point)

$$\textit{FP if } \underline{\alpha} = \mathrm{T}\underline{\alpha}$$

where $(\mathrm{T}\underline{\alpha})_{i\backslash j} \equiv \max_{k\in\partial i\backslash j} m_{k\to i}$.

Definition (ϵ-Fixed point)

$$\epsilon\textit{-FP if } ||\underline{\alpha} - \mathrm{T}\underline{\alpha}||_\infty \le \epsilon$$

- FP, ϵ-FP independent of damping κ.

Definitions

Definition (Fixed point)

$$FP \text{ if } \underline{\alpha} = \mathsf{T}\underline{\alpha}$$

where $(\mathsf{T}\underline{\alpha})_{i\backslash j} \equiv \max_{k \in \partial i \backslash j} m_{k \to i}$.

Definition (ϵ-Fixed point)

$$\epsilon\text{-}FP \text{ if } ||\underline{\alpha} - \mathsf{T}\underline{\alpha}||_\infty \le \epsilon$$

- FP, ϵ-FP independent of damping κ.

Definitions

Definition (Fixed point)

$$FP \text{ if } \underline{\alpha} = \mathrm{T}\underline{\alpha}$$

where $(\mathrm{T}\underline{\alpha})_{i\backslash j} \equiv \max_{k \in \partial i \backslash j} m_{k \to i}$.

Definition (ϵ-Fixed point)

$$\epsilon\text{-}FP \text{ if } ||\underline{\alpha} - \mathrm{T}\underline{\alpha}||_\infty \leq \epsilon$$

- FP, ϵ-FP independent of damping κ.

Definitions

Definition (Induced matching)

$(\underline{\alpha}, \underline{m}, \underline{\gamma})$ *(or just* $\underline{\alpha}$*) induces a matching M if:*

- *Pairs in M receive unique (non-zero) best offers from each other.*
- *Unmatched nodes receive no non-zero offers.*

- General $\underline{\alpha}$ **may not induce any matching**.

Definitions

Definition (Induced matching)

$(\underline{\alpha}, \underline{m}, \underline{\gamma})$ *(or just* $\underline{\alpha}$*) induces a matching M if:*

- *Pairs in M receive unique (non-zero) best offers from each other.*
- *Unmatched nodes receive no non-zero offers.*

- General $\underline{\alpha}$ **may not induce any matching**.

Main results: Fixed points

FPs $\leftrightarrow$ NB solns:

Theorem

Suppose LP-MWM is tight with unique optimum M^*. Then

- $\underline{\alpha}$ *is FP* $\Rightarrow$ $\underline{\alpha}$ *induces* M^* *and* $(M^*, \underline{\gamma})$ *is NB soln.*
- $(M, \underline{\gamma}_{\text{NB}})$ *is NB soln* $\Rightarrow$ *(i)* $M = M^*$, *(ii) unique FP with earnings* $\underline{\gamma}_{\text{NB}}$

Note:

- Unique LP optimum is **generic** in instances with tight LP.

Main results: Fixed points

FPs $\leftrightarrow$ NB solns:

Theorem

Suppose LP-MWM is tight with unique optimum M^*. Then

- $\underline{\alpha}$ *is FP* $\Rightarrow$ $\underline{\alpha}$ *induces* M^* *and* $(M^*, \underline{\gamma})$ *is NB soln.*
- $(M, \underline{\gamma}_{\mathrm{NB}})$ *is NB soln* $\Rightarrow$ *(i)* $M = M^*$*, (ii) unique FP with earnings* $\underline{\gamma}_{\mathrm{NB}}$

Note:

- Unique LP optimum is **generic** in instances with tight LP.

Main results: Fixed points

FPs $\leftrightarrow$ NB solns:

Theorem

Suppose LP-MWM is tight with unique optimum M^*. Then

- $\underline{\alpha}$ *is FP* $\Rightarrow$ $\underline{\alpha}$ *induces* M^* *and* $(M^*, \underline{\gamma})$ *is NB soln.*
- $(M, \underline{\gamma}_{\mathrm{NB}})$ *is NB soln* $\Rightarrow$ *(i)* $M = M^*$, *(ii) unique FP with earnings* $\underline{\gamma}_{\mathrm{NB}}$

Note:

- Unique LP optimum is **generic** in instances with tight LP.

Convergence to FP

Theorem

For any initialization, $\underline{\alpha}^t$ converges to a FP.

- True for *all* problem instances.

Convergence to FP

Theorem

For any initialization, $\underline{\alpha}^t$ converges to a FP.

- True for *all* problem instances.

Convergence to FP

Theorem

For any initialization, $\underline{\alpha}^t$ converges to a FP.

- True for *all* problem instances.

Fast convergence to ϵ-FP

Theorem

Assume $w_{ij} \leq 1$ for all $(ij) \in E$. Take any initialization, any $\epsilon > 0$. Define

$$\tau^*(\epsilon) = \frac{1}{\pi\kappa(1-\kappa)\epsilon^2}$$

For all $t \geq \tau^*(\epsilon)$, $\underline{\alpha}^t$ is an ϵ-FP.

- True for all problem instances

Fast convergence to ϵ-FP

Theorem

Assume $w_{ij} \leq 1$ for all $(ij) \in E$. Take any initialization, any $\epsilon > 0$. Define

$$\tau^*(\epsilon) = \frac{1}{\pi\kappa(1-\kappa)\epsilon^2}$$

For all $t \geq \tau^*(\epsilon)$, $\underline{\alpha}^t$ is an ϵ-FP.

- True for all problem instances

Fast convergence to ϵ-FP

Theorem

Assume $w_{ij} \leq 1$ for all $(ij) \in E$. Take any initialization, any $\epsilon > 0$. Define

$$\tau^*(\epsilon) = \frac{1}{\pi\kappa(1-\kappa)\epsilon^2}$$

For all $t \geq \tau^*(\epsilon)$, $\underline{\alpha}^t$ is an ϵ-FP.

- True for all problem instances

Definition of Approximate NB soln

Definition (ϵ-NB soln)

$(M, \underline{\gamma})$ is ϵ-NB soln if it is:

- *A valid trade outcome*
- *Stable*
- *ϵ-balanced, i.e., for every $(i, j) \in M$*

$$\left|[\gamma_i - \max_{k \in \partial i \backslash j}(w_{ik} - \gamma_k)_+] - [\gamma_j - \max_{l \in \partial j \backslash i}(w_{jl} - \gamma_l)_+]\right| \le \epsilon$$

- Analogous to defn of ϵ-Nash equilibrium

Definition of Approximate NB soln

Definition (ϵ-NB soln)

$(M, \underline{\gamma})$ is ϵ-NB soln if it is:

- *A valid trade outcome*
- *Stable*
- *ϵ-balanced, i.e., for every $(i, j) \in M$*

$$\left|[\gamma_i - \max_{k \in \partial i \backslash j} (w_{ik} - \gamma_k)_+] - [\gamma_j - \max_{l \in \partial j \backslash i} (w_{jl} - \gamma_l)_+]\right| \le \epsilon$$

- Analogous to defn of ϵ-Nash equilibrium

Approximate fixed points

ϵ-FP $\to$ (6ϵ)-NB soln:

Theorem

Suppose LP-MWM is tight with unique optimum M^*, and gap $g > 0$. Let $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ be an ϵ-FP for $\epsilon < g/(6|V|^2)$. Then

- *$\underline{\alpha}$ induces matching M^**
- *$(M^*, \underline{\gamma})$ is a (6ϵ)-NB soln*

- $g \equiv$ Diff. betw. wts of M^* and next heaviest (half-integral) matching

Approximate fixed points

ϵ-FP $\rightarrow$ (6ϵ)-NB soln:

Theorem

Suppose LP-MWM is tight with unique optimum M^*, and gap $g > 0$. Let $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ be an ϵ-FP for $\epsilon < g/(6|V|^2)$. Then

- *$\underline{\alpha}$ induces matching M^**
- *$(M^*, \underline{\gamma})$ is a (6ϵ)-NB soln*

- $g \equiv$ Diff. betw. wts of M^* and next heaviest (half-integral) matching

Approximate fixed points

ϵ-FP $\to$ (6ϵ)-NB soln:

Theorem

Suppose LP-MWM is tight with unique optimum M^*, and gap $g > 0$. Let $(\underline{\alpha}, \underline{m}, \underline{\gamma})$ be an ϵ-FP for $\epsilon < g/(6|V|^2)$. Then

- *$\underline{\alpha}$ induces matching M^**
- *$(M^*, \underline{\gamma})$ is a (6ϵ)-NB soln*

- $g \equiv$ Diff. betw. wts of M^* and next heaviest (half-integral) matching

Fast convergence to approx NB soln

Fast conv to ϵ-FP

$+$

ϵ-FP $\rightarrow$ (6ϵ)-NB soln for small ϵ

$\downarrow$

Fast conv to approx NB soln!

Fast convergence to approx NB soln

Theorem (Fast convergence to ϵ-NB)

Assume $w_{ij} \leq 1$ for all $(ij) \in E$. Suppose LP-MWM is tight with unique optimum M^*, and gap $g > 0$. For appropriately chosen $C = C(\kappa)$, define

$$\tau^*(\epsilon) = C \max\left(|V|^4/g^2,\, 1/\epsilon^2\right)$$

For any $t \geq \tau^*(\epsilon)$ we have

- *$\underline{\alpha}^t$ induces matching M^**
- *$(M^*, \underline{\gamma}^t)$ is an ϵ-NB soln*

Can any local algorithm be faster?

ϵ-NB solution $= M^* +$ balanced allocation

- Extensive work on local algorithms for M^*
- Standard algorithms — max-product BP, auction alg.[Bertsekas '88]
  $O(|V|/g)$ iterations for convergence, tight in worst case
- Appears hard to escape polynomial dependence on $1/g$ and $|V|$

Can any local algorithm be faster?

$$\epsilon\text{-NB solution} = M^* + \text{balanced allocation}$$

- Extensive work on local algorithms for M^*
- Standard algorithms — max-product BP, auction alg.[Bertsekas '88]
  $O(|V|/g)$ iterations for convergence, tight in worst case
- Appears hard to escape polynomial dependence on $1/g$ and $|V|$

Example: Bipartite graphs

- NB soln always exists
- We show gap g is large whp under small **random perturbations**
- 'True' polynomial convergence

Proof sketch: Convergence

Natural dynamics is $\underline{\alpha}^{t+1} = \kappa\, \mathsf{T}\underline{\alpha}^t + (1-\kappa)\, \underline{\alpha}^t$

Lemma

T *is* **non-expansive** *in sup-norm, i.e.,*

$$||\mathsf{T}\underline{\alpha} - \mathsf{T}\underline{\beta}||_\infty \le ||\underline{\alpha} - \underline{\beta}||_\infty$$

- [Ishikawa '76] implies convergence to FP.
- [Baillon, Bruck '96] implies rapid convergence to ϵ-FP.

Lots more work to characterize FPs!

Proof sketch: Convergence

Natural dynamics is $\underline{\alpha}^{t+1} = \kappa\, \mathsf{T}\underline{\alpha}^t + (1-\kappa)\, \underline{\alpha}^t$

Lemma

T *is* **non-expansive** *in sup-norm, i.e.,*

$$||\mathsf{T}\underline{\alpha} - \mathsf{T}\underline{\beta}||_\infty \leq ||\underline{\alpha} - \underline{\beta}||_\infty$$

- [Ishikawa '76] implies convergence to FP.
- [Baillon, Bruck '96] implies rapid convergence to ϵ-FP.

Lots more work to characterize FPs!

Proof sketch: Convergence

Natural dynamics is $\underline{\alpha}^{t+1} = \kappa \, \mathsf{T}\underline{\alpha}^t + (1-\kappa)\, \underline{\alpha}^t$

Lemma

T *is* **non-expansive** *in sup-norm, i.e.,*

$$||\mathsf{T}\underline{\alpha} - \mathsf{T}\underline{\beta}||_\infty \leq ||\underline{\alpha} - \underline{\beta}||_\infty$$

- [Ishikawa '76] implies convergence to FP.
- [Baillon, Bruck '96] implies rapid convergence to ϵ-FP.

Lots more work to characterize FPs!

Proof sketch: Convergence

Natural dynamics is $\underline{\alpha}^{t+1} = \kappa\, \mathsf{T}\underline{\alpha}^t + (1-\kappa)\, \underline{\alpha}^t$

Lemma

T *is* **non-expansive** *in sup-norm, i.e.,*

$$||\mathsf{T}\underline{\alpha} - \mathsf{T}\underline{\beta}||_\infty \leq ||\underline{\alpha} - \underline{\beta}||_\infty$$

- [Ishikawa '76] implies convergence to FP.
- [Baillon, Bruck '96] implies rapid convergence to ϵ-FP.

Lots more work to characterize FPs!

Proof sketch: Convergence

Natural dynamics is $\underline{\alpha}^{t+1} = \kappa \, \mathsf{T}\underline{\alpha}^t + (1-\kappa)\, \underline{\alpha}^t$

Lemma

T *is* **non-expansive** *in sup-norm, i.e.,*

$$||\mathsf{T}\underline{\alpha} - \mathsf{T}\underline{\beta}||_\infty \le ||\underline{\alpha} - \underline{\beta}||_\infty$$

- [Ishikawa '76] implies convergence to FP.
- [Baillon, Bruck '96] implies rapid convergence to ϵ-FP.

Lots more work to characterize FPs!

Unequal bargaining powers

- Nodes may have different bargaining power
- No obvious generalization of KT algorithm.
- Natural dynamics can be modified
- **Theorem:** Unsymmetrical 'solutions' exist iff LP-MWM is tight.
- Exponential convergence time, fixed in [Kanoria WINE 2010].

Unequal bargaining powers

- Nodes may have different bargaining power
- No obvious generalization of KT algorithm.
- Natural dynamics can be modified
- **Theorem:** Unsymmetrical 'solutions' exist iff LP-MWM is tight.
- Exponential convergence time, fixed in [Kanoria WINE 2010].

Unequal bargaining powers

- Nodes may have different bargaining power
- No obvious generalization of KT algorithm.
- Natural dynamics can be modified
- **Theorem:** Unsymmetrical 'solutions' exist iff LP-MWM is tight.
- Exponential convergence time, fixed in [Kanoria WINE 2010].

Conclusion

- Convincing theoretical validation of NB solution.
- Introduced powerful technical tool: non-expansive iterations.

Thank you!

Conclusion

- Convincing theoretical validation of NB solution.
- Introduced powerful technical tool: non-expansive iterations.

Thank you!