Birkhoff's variety theorem for relative algebraic theories

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Single-sorted algebras

Definition

A (single-sorted) algebra consists of:

- a base set A;
- operators $\sigma\colon A^n \to A$ $(n \geq 0)$;
- equations.

Example

A group consists of:

- a base set G;
- operators $e\colon 1 \to G$, $\quad i\colon G \to G$, $\quad m\colon G^2 \to G$;
- equations $m(e,x) = x = m(x,e)$, $\quad m(x,i(x)) = e = m(i(x),x)$, $m(m(x,y),z) = m(x,m(y,z))$.

Multi-sorted algebras

Definition

S: a set. (the set of sorts)

An S-sorted algebra consists of:

- base sets $(A_s)_{s \in S}$ indexed by S;
- operators $\sigma\colon A_{s_1} \times \cdots \times A_{s_n} \to A_s$;
- equations.

Example

A chain complex consists of:

- base sets $(A_n)_{n \in \mathbb{Z}}$;
- operators $0_n\colon 1 \to A_n, \quad -_n\colon A_n \to A_n, \quad +_n\colon A_n \times A_n \to A_n,$
  $d_n\colon A_n \to A_{n+1}$;
- appropriate equations.

This is an $\mathbb{Z}$-sorted algebra.

The free-forgetful adjunctions

$$\begin{tikzcd} \mathbf{Grp} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Set} \arrow[u, bend left, "F"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Ch} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Set}^{\mathbb{Z}} \arrow[u, bend left, "F"] \end{tikzcd}$$

$$\begin{tikzcd} Alg(\Omega, E) \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Set}^{S} \arrow[u, bend left, "F"] \end{tikzcd}$$

$(\underbrace{\Omega}_{\text{operators}}, \underbrace{E}_{\text{equations}})$: an S-sorted algebraic theory.

Finitary monads

Definition

A monad on a category $\mathscr{C}$ consists of:

- an endofunctor $T : \mathscr{C} \to \mathscr{C}$,
- a natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow T$,
- a natural transformation $\mu : T^2 (:= T \circ T) \Rightarrow T$

such that the following commute.

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

Definition

A monad (T, η, μ) is finitary when the functor T preserves filtered colimits.

Linton's theorem

Theorem ([Lin69])

There is an equivalence

$$\mathbf{Th}^S \simeq \mathbf{Mnd_f}(\mathbf{Set}^S).$$

Here,

$\mathbf{Th}^S$: the category of S-sorted algebraic theories,

$\mathbf{Mnd_f}(\mathbf{Set}^S)$: the category of finitary monads on $\mathbf{Set}^S$.

S-sorted algebraic theory $=$ finitary monad on $\mathbf{Set}^S$

Algebras over a quiver

Example

A small category consists of:

- a base quiver $\mathrm{mor}\,\mathscr{C} \overset{\mathrm{d}}{\underset{\mathrm{c}}{\rightrightarrows}} \mathrm{ob}\,\mathscr{C}$;
- a total operator $\mathrm{id} \colon \mathrm{ob}\,\mathscr{C} \to \mathrm{mor}\,\mathscr{C}$;
- a partial operator $\circ \colon \mathrm{mor}\,\mathscr{C} \times \mathrm{mor}\,\mathscr{C} \to \mathrm{mor}\,\mathscr{C}$ such that

$$g \circ f \text{ is defined iff } \mathrm{d}(g) = \mathrm{c}(f)$$

which satisfy the following:

- $\mathrm{d}(\mathrm{id}(x)) = x$ and $\mathrm{c}(\mathrm{id}(x)) = x$;
- $\mathrm{d}(g \circ f) = \mathrm{d}(f)$ and $\mathrm{c}(g \circ f) = \mathrm{c}(g)$ whenever $\mathrm{d}(g) = \mathrm{c}(f)$;
- $f \circ \mathrm{id}(\mathrm{d}(f)) = f$ and $\mathrm{id}(\mathrm{c}(f)) \circ f = f$;
- $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $\mathrm{d}(h) = \mathrm{c}(g)$ and $\mathrm{d}(g) = \mathrm{c}(f)$.

Small categories are algebras over quivers.

Ordered algebras

Example ([Gol03])

A uniquely difference-ordered semiring consists of:

- a base poset $(R, \leq)$;
- total operators $+, \cdot \colon R \times R \to R$;
- constants $0, 1 \in R$;
- a partial operator $\ominus \colon R \times R \rightharpoonup R$ such that

$$b \ominus a \text{ is defined iff } a \leq b$$

which satisfy the following:

- $(R, +, \cdot, 0, 1)$ is a semiring;
- $a \leq a + b$;
- $(a + b) \ominus a = b$;
- $a + (b \ominus a) = b$ whenever $a \leq b$.

UDO semirings are algebras over posets.

Partial algebras

Example ([BH12])

A partial abelian group consists of:

- a base set A with a reflexive symmetric relation $\odot \subseteq A \times A$; (a set with commeasurability)
- a constant $0 \in A$;
- a total operator $-\colon A \to A$;
- a partial operator $+\colon A \times A \rightharpoonup A$ such that

$$a + b \text{ is defined iff } a \odot b$$

which satisfy the following:

- $a \odot 0$;
- $a \odot (-b)$ whenever $a \odot b$;
- $a \odot (b + c)$ whenever $a \odot b$, $b \odot c$, $c \odot a$;
- $(a + b) + c = a + (b + c)$ whenever $a \odot b$, $b \odot c$, $c \odot a$;
- $a + b = b + a$ whenever $a \odot b$;
- $a + 0 = a$ and $a \odot (-a) = 0$.

Definition

A monoid-graded set is a map $\mathrm{d}\colon X \to M$ from a set X to a monoid $(M, \cdot, e)$.

Example

A monoid-graded ring consists of:

- a base monoid-graded set $(X, \mathrm{d}, M, \cdot, e)$;
- a constant $1 \in X$;
- total operators $\otimes\colon X \times X \to X, \quad 0\colon M \to X, \quad -\colon X \to X$;
- a partial operator $+\colon X \times X \rightharpoonup X$ s.t. $x + y$ is defined iff $\mathrm{d}(x) = \mathrm{d}(y)$

which satisfy the following:

- $\mathrm{d}(1) = e, \quad \mathrm{d}(x \otimes y) = \mathrm{d}(x)\mathrm{d}(y), \quad \mathrm{d}(0(a)) = a, \quad \mathrm{d}(-x) = \mathrm{d}(x)$;
- $\mathrm{d}(x + y) = \mathrm{d}(x)$ whenever $\mathrm{d}(x) = \mathrm{d}(y)$;
- $(x \otimes y) \otimes z = x \otimes (y \otimes z), \quad 1 \otimes x = x = x \otimes 1$;
- $x + 0(\mathrm{d}(x)) = x, \quad x + (-x) = 0(\mathrm{d}(x))$;
- $(x + y) + z = x + (y + z)$ whenever $\mathrm{d}(x) = \mathrm{d}(y) = \mathrm{d}(z)$;
- $x + y = y + x$ whenever $\mathrm{d}(x) = \mathrm{d}(y)$;
- $(x + y) \otimes z = x \otimes z + y \otimes z$ and $z \otimes (x + y) = z \otimes x + z \otimes y$ whenever $\mathrm{d}(x) = \mathrm{d}(y)$.

Relative algebraic theories

Informal definition [Kaw23a]

$\mathscr{A}$: a (locally presentable) category
An $\mathscr{A}$-relative algebraic theory consists of:

- a set Ω of *partial operators*;
- a set E of *implications* $\cdots(\underbrace{\underline{\text{YYY}}}_{\text{postcondition}}$ whenever $\underbrace{\underline{\text{XXX}}}_{\text{precondition}})$

such that

- For each operator $\omega \in \Omega$, its domain must be defined by "$\mathscr{A}$'s language."
- For each implication in E, its precondition must be written in "$\mathscr{A}$'s language."

Example

- small categories $\rightsquigarrow$ a $\mathbf{Quiv}$-relative algebraic theory
- UDO semirings $\rightsquigarrow$ a $\mathbf{Pos}$-relative algebraic theory
- partial abelian groups $\rightsquigarrow$ an $\mathbf{RSRel}$-relative algebraic theory
- monoid-graded rings $\rightsquigarrow$ an $\mathbf{MGSet}$-relative algebraic theory

A generalized Linton's theorem

Theorem ([Kaw23a])

There is an equivalence

$$\mathbf{Th}^{\mathscr{A}} \simeq \mathbf{Mnd_f}(\mathscr{A}).$$

Here,

$$\begin{aligned} \mathbf{Th}^{\mathscr{A}} &: \text{ the category of } \mathscr{A}\text{-relative algebraic theories,} \\ \mathbf{Mnd_f}(\mathscr{A}) &: \text{ the category of finitary monads on } \mathscr{A}. \end{aligned}$$

$\uparrow$ generalize

Recall (Linton's theorem)

$$\mathbf{Th}^{S} \simeq \mathbf{Mnd_f}(\mathbf{Set}^{S}).$$

S-sorted algebraic theory $=$ $\mathbf{Set}^S$-relative algebraic theory

The free-forgetful adjunction

$$\begin{tikzcd} Alg(\Omega, E) \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Set}^S \arrow[u, bend left, "F"] \end{tikzcd}$$

(Ω, E): an S-sorted algebraic theory

$$\begin{tikzcd} Alg(\Omega, E) \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathscr{A} \arrow[u, bend left, "F"] \end{tikzcd}$$

(Ω, E): an $\mathscr{A}$-relative algebraic theory

Equational classes

Definition

(Ω, E): a single-sorted algebraic theory. A full subcategory $\mathscr{E} \subseteq Alg(\Omega, E)$ is definable (by equations) if $\mathscr{E} = Alg(\Omega, E + {}^{\exists}E')$, i.e., $\mathscr{E}$ can be defined by adding equations.

Example

$\{$commutative monoids$\} \subseteq \mathbf{Mon}$ is definable by the equation $xy = yx$.

Example

$\{$invertible monoids$\} \subseteq \mathbf{Mon}$ is not definable by equations. $\underbrace{\text{not definable by equations.}}_{\textit{How can we prove this?}}$

Birkhoff's variety theorem

Birkhoff's variety theorem [Bir35]

(Ω, E): a single-sorted algebraic theory. $\mathscr{E} \subseteq Alg(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq Alg(\Omega, E)$ is definable by equations.
2. $\mathscr{E} \subseteq Alg(\Omega, E)$ is closed under *products*, *subobjects*, and *quotients*.

closed under products: $A_i \in \mathscr{E} \implies \prod_i A_i \in \mathscr{E}$.
closed under subobjects: $B \subseteq A$: sub, $A \in \mathscr{E} \implies B \in \mathscr{E}$.
closed under quotients: $A \twoheadrightarrow B$: surj, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

Corollary

$\{$invertible monoids$\} \subseteq \mathbf{Mon}$ is not definable by equations.

Proof.

There is a submonoid $\mathbb{N} \subset \mathbb{Z}$. The additive monoid $\mathbb{N}$ is not invertible even though $\mathbb{Z}$ is invertible. Thus, $\{$invertible monoids$\} \subseteq \mathbf{Mon}$ is not closed under subobjects. □

A generalized Birkhoff's theorem

Theorem ([Kaw23a])

(Ω, E): an $\mathscr{A}$-relative algebraic theory. $\mathscr{E} \subseteq Alg(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq Alg(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq Alg(\Omega, E)$ is closed under *products*, *closed subobjects*, U*-local retracts*, and *filtered colimits*.

single-sorted alg. ($\mathbf{Set}$-relative alg.)		$\mathscr{A}$-relative alg.
products	$\rightsquigarrow$	products
subobjects	$\rightsquigarrow$	*closed subobjects*
quotients	$\rightsquigarrow$	U*-local retracts*
	$\rightsquigarrow$	*filtered colimits* (new)

The filtered colimit elimination problem

Question

Why can the closure property under filtered colimits be eliminated in the case of $\mathbf{Set}$-relative algebras?

Answer

The category $\mathbf{Set}$ satisfies a "noetherian" condition.

A noetherian condition for categories

Definition ([Kaw23b])

A category $\mathscr{A}$ satisfies the ascending chain condition (ACC) if it has no chain $A_0 \to A_1 \to A_2 \to \cdots$ of objects such that there is no morphism $A_n \leftarrow A_{n+1}$ for all n.

Definition

- Objects X and Y are strongly connected if there are morphisms $X \to Y$, $Y \to X$.
- An equivalence class under strong connectedness is called a strongly connected component.
- $\sigma(\mathscr{A})$: the large poset of all strongly connected components in a category $\mathscr{A}$. (the *posetification* of $\mathscr{A}$)

Proposition

A category $\mathscr{A}$ satisfies ACC $\Leftrightarrow$ the large poset $\sigma(\mathscr{A})$ satisfies ACC.

Filtered colimit elimination

Theorem ([Kaw23b])

(Ω, E): an $\mathscr{A}$-relative algebraic theory. $\mathscr{E} \subseteq Alg(\Omega, E)$: fullsub.
Assume that $\mathscr{A}$ satisfies ACC.
TFAE:

1. $\mathscr{E} \subseteq Alg(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq Alg(\Omega, E)$ is closed under *products*, *closed subobjects*, and U*-local retracts*.

Example

$\sigma(\mathbf{Set}) = \{$ the empty set, the non-empty sets $\}$.
$\rightsquigarrow$ $\mathbf{Set}$ satisfies ACC. $\rightsquigarrow$ fil.colim.elim. holds for single-sorted alg.

Further examples

Example

S: a set.
$\mathbf{Set}^S$ satisfies ACC $\quad \Leftrightarrow \quad S$ is finite.
$\rightsquigarrow$ fil.colim.elim. holds for *finite-sorted algebras.*
$\rightsquigarrow$ This generalizes a result in [ARV12].

Example

$\mathbf{Pos}$ satisfies ACC.
$\rightsquigarrow$ fil.colim.elim. holds for *ordered algebras.*
$\rightsquigarrow$ This generalizes a result in [Blo76].

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l.f.p. category	Objects	Number of srg.conn. components	satisfaction of ACC
$\mathbf{Set}$	sets	2	true
$\mathbf{Pos}$	posets	2	true
$\mathbf{Mon}$	monoids	1	true
$\mathbf{Grp}$	groups	1	true
$\mathbf{Ab}$	abelian groups	1	true
$\mathbf{SGrp}$	semigroups	infinity	false
$\mathbf{Ring}$	rings	infinity	false
$\mathbf{SLat}$	(join-)semilattices	2	true
$\mathbf{SLat_0}$	bounded (join-)semilattices	1	true
$\mathbf{Lat}$	lattices	2	true
$\mathbf{Lat_{0,1}}$	bounded lattices	infinity	false
$\mathbf{Set}^n$	n-sorted sets	2^n	true
$\mathbf{Set}^S$	S-sorted sets (S: an infinite set)	infinity	false
$\mathbf{Set}_*$	pointed sets	1	true
$n/\mathbf{Set}$	sets with n constants	the n-th Bell number	true
$S/\mathbf{Set}$	sets with S-indexed constants (S: an infinite set)	infinity	false
$\mathbf{End}$	sets with an endomorphism	infinity	false
$\mathbf{Idem}$	sets with an idempotent endomorphism	2	true
$\mathbf{Aut}$	sets with an automorphisms	infinity	false

l.f.p. category	Objects	Number of srg.conn. components	satisfaction of ACC
$\mathbf{Quiv}$	quivers (or directed graphs)	infinity	false
$\mathbf{RQuiv}$	reflexive quivers	2	true
$\mathbf{Cat}$	small categories	2	true
$\mathbf{Set}^{\to}$	maps	3	true
$\mathbf{Cospan}$	cospans of sets	6	true
$\mathbf{Set}^{\omega^{\mathrm{op}}}$	ω^{op}-chains of sets	infinity	false
$\mathbf{Set}^{\omega}$	ω-chains of sets	infinity	**true**
$\mathbf{Set}^{\Delta^{\mathrm{op}}}$	simplicial sets	2	true
$\mathbf{Set}^{\mathbb{G}^{\mathrm{op}}}$	globular sets	infinity	false
$\mathbf{Nom}$	nominal sets	infinity	false
$\mathbf{URel}$	sets with an unary relation	3	true
$\mathbf{BRel}$	sets with a binary relation	infinity	false
$\mathbf{SRel}$	sets with a symmetric relation	infinity	false
$\mathbf{RSRel}$	sets with a reflexive symmetric relation	2	true
$\mathbf{PER}$	sets with a symmetric transitive relation	3	true
$\mathbf{PreOrd}$	preordered sets	2	true
$\mathbf{ERel}$	sets with an equivalence relation	2	true