

# Formalizing accessibility and duality in a virtual equipment

Yuto Kawase (joint with Keisuke Hoshino)

RIMS, Kyoto University

June 28, 2024. CT2024



← Today's slides

This work is supported by JSPS KAKENHI Grant Numbers JP24KJ1462,JP23KJ1365.

- 1 The ordinary accessibility
- 2 Virtual equipments
- 3 Formal category theory in a virtual equipment
- 4 Ind-completions in a virtual equipment

## Definition

The *free cocompletion* of  $\mathbf{A}$  under filtered colimits  
... fullsub  $\mathbf{Ind}(\mathbf{A}) := \{\text{fil.colim of repr}\} \subseteq \mathbf{Set}^{\mathbf{A}^{\text{op}}}$   
(= **ind-completion** of  $\mathbf{A}$ .)

## Definition

$X \in \mathbf{X}$  is **finitely presentable (f.p.)**  
 $\stackrel{\text{def}}{\iff} \mathbf{X}(X, -)$  preserves filtered colimits.

## Fact

TFAE for a category  $\mathbf{X}$ :

- 1  $\mathbf{X}$  has filtered colimits, and every  $X \in \mathbf{X}$  is a filtered colimit of f.p.objects.
- 2  $\mathbf{X} \simeq \mathbf{Ind}(\mathbf{A})$  ( $\exists \mathbf{A}$ ).

$\Updownarrow$ def (if we ignore "size.")

$\mathbf{X}$  is **finitely accessible**.

## Definition

The *free cocompletion* of  $\mathbf{A}$  under  $\Phi$ -colimits  
... fullsub  $\mathbf{Ind}_{\Phi}(\mathbf{A}) := \{\Phi\text{-colim of repr}\} \subseteq \mathbf{Set}^{\mathbf{A}^{\text{op}}}$   
(= " **$\Phi$ -ind-completion**" of  $\mathbf{A}$ .)

## Definition

$X \in \mathbf{X}$  is  **$\Phi$ -atomic**  
 $\stackrel{\text{def}}{\iff} \mathbf{X}(X, -)$  preserves  $\Phi$ -colimits.

## Fact

TFAE for a category  $\mathbf{X}$ :

- 1  $\mathbf{X}$  has  $\Phi$ -colimits, and every  $X \in \mathbf{X}$  is a  $\Phi$ -colimit of  $\Phi$ -atomic obj.
- 2  $\mathbf{X} \simeq \mathbf{Ind}_{\Phi}(\mathbf{A})$  ( $\exists \mathbf{A}$ ).

$\Updownarrow$ def (if we ignore "size.")

$\mathbf{X}$  is  **$\Phi$ -accessible**. ( $\Phi$ : a class of shapes of colim)

# Duality

$\Phi$ : a class of shapes of colim.

## Definition (only for today)

A functor  $\mathbf{X} \xrightarrow{F} \mathbf{Y}$  is  $\Phi$ -weighty

$\stackrel{\text{def}}{\Leftrightarrow}$  (Pointwise) left Kan extensions along  $F$  are given by  $\Phi$ -colimits.

## Theorem (Duality in the $\Phi$ -accessible context)

There is a biequivalence of 2-categories:

$$\mathcal{Cau}_{\Phi}^{\text{co}} \simeq_{\text{bi}} \mathcal{Acc}_{\Phi}^{\text{op}}$$

The 2-category  $\mathcal{Cau}_{\Phi}$ :

- 0-cell  $\dots$  Cauchy complete small category
- 1-cell  $\dots$   $\Phi$ -weighty functor
- 2-cell  $\dots$  natural transformation

The 2-category  $\mathcal{Acc}_{\Phi}$ :

- 0-cell  $\dots$   $\Phi$ -accessible category
- 1-cell  $\dots$   $\Phi$ -cocontinuous right adjoint functor
- 2-cell  $\dots$  natural transformation

This is a “ $\Phi$ -modified” version of *Makkai–Paré duality* (Makkai and Paré 1989).

This duality has recently been generalized to the enriched context (Tendas 2023).

# Goal

## $(\mathcal{V}$ -enriched) accessibility

- duality
- ind-completion
- Cauchy completeness

= Accessibility in  $\mathcal{V}\text{-}\mathbf{Prof}$

The *virtual equipment*  $\mathcal{V}\text{-}\mathbf{Prof}$ :

- $\mathcal{V}$ -enriched categories
- $\mathcal{V}$ -functors
- $\mathcal{V}$ -profunctors

## Formal accessibility in a virtual equipment

$\mathcal{V}\text{-}\mathbf{Prof} \xrightarrow{\text{generalize}} \mathbb{E}$  (an arbitrary virtual equipment)

This extends the notion of accessibility to other category-theoretic contexts:

- bicategory-enriched categories
- fibered (or indexed) categories
- internal categories
- something that is no longer categories

# Why virtual equipments?

2-categories are suitable for capturing:

- ✓ ordinary limit and colimits,
- ✓ adjunctions,
- ✓ monads,
- ✓ Kan extensions and lifts.

2-categories are **not** suitable for capturing interactions of functors and profunctors:

- × weighted limits and colimits,
- × presheaves,
- × cocompletions,
- × pointwise Kan extensions,
- × Cauchy completeness,
- × commutation of weights.



*virtual equipments*

# Main features

In our formalization,

- We do **not** use *opposite categories*.  
     $\rightsquigarrow$  categories enriched over a non-symmetric monoidal category or a bicategory
- We do **not** require either the *smallness* of categories or the *compositions* of arbitrary profunctors.  
     $\rightsquigarrow$  overcoming the *size matters*
- We do **not** demand “(co)completeness” for the universe.  
     $\rightsquigarrow$  enrichment by a monoidal category that is neither (co)complete nor closed.

- 1 The ordinary accessibility
- 2 Virtual equipments
- 3 Formal category theory in a virtual equipment
- 4 Ind-completions in a virtual equipment



# The augmented virtual double category $\mathcal{V}\text{-}\mathbb{P}\text{rof}$

- $\mathcal{V}$ -categories  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ ;

- $\mathcal{V}$ -functors  $\begin{array}{c} \mathbf{A} \\ F \downarrow \\ \mathbf{B} \end{array}, \dots$  and their compositions and identities;

- $\mathcal{V}$ -profunctors  $\mathbf{A} \xrightarrow{P} \mathbf{B}, \dots$ ;

- $\binom{n}{1}$ - $\mathcal{V}$ -forms  $\begin{array}{ccccc} \mathbf{A}_0 & \xrightarrow{P_1} & \mathbf{A}_1 & \xrightarrow{P_2} & \dots & \xrightarrow{P_n} & \mathbf{A}_n \\ F \downarrow & & & \alpha & & & \downarrow G \\ \mathbf{B} & \xrightarrow{\quad\quad\quad} & \mathbf{C} \end{array} = \{P_1(A_0, A_1) \otimes \dots \otimes P_n(A_{n-1}, A_n) \rightarrow Q(FA_0, GA_n)\},$

- $\binom{n}{0}$ - $\mathcal{V}$ -forms  $\begin{array}{ccc} \mathbf{A}_0 & \xrightarrow{P_1} \dots \xrightarrow{P_n} & \mathbf{A}_n \\ F \searrow & \alpha & \swarrow G \\ & \mathbf{B} & \end{array} = \{P_1(A_0, A_1) \otimes \dots \otimes P_n(A_{n-1}, A_n) \rightarrow \mathbf{B}(FA_0, GA_n)\},$

- and their compositions  $\begin{array}{ccccccc} \mathbf{A}_0 & \xrightarrow{\vec{P}_1} & \mathbf{A}_1 & \xrightarrow{\vec{P}_2} & \dots & \xrightarrow{\vec{P}_n} & \mathbf{A}_n \\ F_0 \downarrow & \alpha_1 F_1 \downarrow & \alpha_2 & \dots & \alpha_n & \downarrow F_n & \\ \mathbf{B}_0 & \xrightarrow{\dots} & \mathbf{B}_1 & \xrightarrow{\dots} & \dots & \xrightarrow{\dots} & \mathbf{B}_n \\ G \downarrow & & Q_1 & \xrightarrow{\dots} & Q_2 & \xrightarrow{\dots} & Q_n \\ & & \beta & & & & \downarrow H \\ \mathbf{C} & \xrightarrow{\dots} & \mathbf{D} \end{array} \rightsquigarrow \begin{array}{ccc} \mathbf{A}_0 & \xrightarrow{\vec{P}_1} & \mathbf{A}_1 & \xrightarrow{\vec{P}_2} & \dots & \xrightarrow{\vec{P}_n} & \mathbf{A}_n \\ F_0 \circ G \downarrow & \vec{\alpha} \circ \beta & & & & \downarrow F_n \circ H & \\ \mathbf{C} & \xrightarrow{\quad\quad\quad} & \mathbf{D} \end{array}$

# An augmented virtual double category $\mathbb{X}$

- **objects**  $A, B, C, \dots$ ;

- **vertical arrows**  $\begin{array}{c} A \\ f \downarrow \\ B \end{array}, \dots$  and their compositions and identities;

- **horizontal arrows**  $A \xrightarrow{p} B, \dots$ ;

- **cells**:  $\binom{n}{1}$ -cells  $\begin{array}{ccccc} A_0 & \xrightarrow{p_1} & A_1 & \xrightarrow{p_2} & \dots & \xrightarrow{p_n} & A_n \\ f \downarrow & & & & \alpha & & \downarrow g \\ B & \xrightarrow{q} & & & & & C \end{array}, \dots$   $\binom{n}{0}$ -cells  $\begin{array}{ccc} A_0 & \xrightarrow{p_1} \dots \xrightarrow{p_n} & A_n \\ f \searrow & \alpha & \swarrow g \\ & B & \end{array}, \dots$

- and their compositions:  $\begin{array}{ccccccc} A_0 & \xrightarrow{\vec{p}_1} & A_1 & \xrightarrow{\vec{p}_2} & \dots & \xrightarrow{\vec{p}_n} & A_n \\ f_0 \downarrow & \alpha_1 f_1 \downarrow & \alpha_2 & \dots & \alpha_n & & \downarrow f_n \\ B_0 & \xrightarrow{q_1} & B_1 & \xrightarrow{q_2} & \dots & \xrightarrow{q_n} & B_n \\ g \downarrow & & \beta & & & & \downarrow h \\ C & \xrightarrow{r} & & & & & D \end{array} \rightsquigarrow \begin{array}{ccccccc} A_0 & \xrightarrow{\vec{p}_1} & A_1 & \xrightarrow{\vec{p}_2} & \dots & \xrightarrow{\vec{p}_n} & A_n \\ f_0 \circ g \downarrow & & \alpha \circ \beta & & & & \downarrow f_n \circ h \\ C & \xrightarrow{r} & & & & & D \end{array}$

and identity-cells:  $\begin{array}{ccc} X & & X \xrightarrow{p} Y \\ f \downarrow (=) \downarrow f & \parallel & \parallel \\ Y & & X \xrightarrow{p} Y \end{array}$

# Virtual equipments

Definition (Cruttwell and Shulman 2010; Koudenburg 2020)

A **virtual equipment** = an augmented virtual double category s.t.

$$\begin{array}{ccc} A & \xrightarrow{\exists p} & B \\ f \downarrow & \exists \text{cart} & \downarrow g \\ X & \xrightarrow[u]{\dots\dots\dots} & Y \end{array}$$

## Example

virtual equipment	object	vert.arrow	hor.arrow
$\mathcal{V}\text{-Prof}$ ( $\mathcal{V}$ : a monoidal cat)	$\mathcal{V}$ -enriched cat	$\mathcal{V}$ -functor	$\mathcal{V}$ -profunctor
$\mathcal{W}\text{-Prof}$ ( $\mathcal{W}$ : a bicategory)	$\mathcal{W}$ -enriched cat	$\mathcal{W}$ -functor	$\mathcal{W}$ -profunctor
$\text{Prof}(\mathbf{C})$ ( $\mathbf{C}$ : cat with p.b.)	$\mathbf{C}$ -internal cat	$\mathbf{C}$ -internal functor	$\mathbf{C}$ -internal profunctor
and so on.			

From now on, we fix a virtual equipment  $\mathbb{E}$ . (e.g.  $\mathbb{E} := \mathcal{V}\text{-Prof}$ )

- 1 The ordinary accessibility
- 2 Virtual equipments
- 3 Formal category theory in a virtual equipment
- 4 Ind-completions in a virtual equipment

# Zoo of cells

$$\begin{array}{ccc} A_0 & \xrightarrow{\vec{u}} & A_n \\ \parallel & \text{comp} & \parallel \\ A_0 & \xrightarrow{\quad} & A_n \end{array} : \text{composing}$$

$$\begin{array}{ccc} C & \xrightarrow{\quad} & B \xrightarrow{\vec{u}} A \\ \parallel & \text{lift} & \downarrow f \\ C & \xrightarrow{\quad} & X \end{array} : \text{lifting}$$

$$\begin{array}{ccc} B & \xrightarrow{u} & A \\ \searrow \text{ran} & \downarrow f & \\ & X & \end{array} : \text{right Kan ext.}$$

$$\left( \begin{array}{c} A \\ \swarrow \text{comp} \searrow \\ A \xrightarrow{\text{Id}_A} A \end{array} \right)$$

$$\begin{array}{ccccc} A & \xrightarrow{\vec{u}} & B & \xrightarrow{\quad} & C \\ f \downarrow & & \text{ext} & & \parallel \\ X & \xrightarrow{\quad} & & \xrightarrow{\quad} & C \end{array} : \text{extending}$$

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ f \downarrow & \swarrow \text{lan} & \\ X & & \end{array} : \text{left Kan ext.}$$

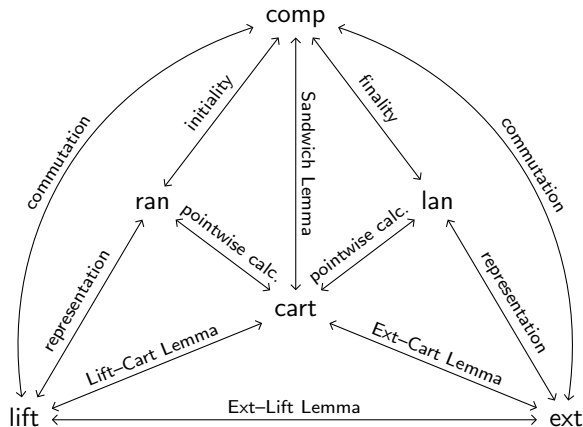
In  $\mathcal{V}\text{-Prof}$ ,

- $\rightsquigarrow$
- composition of profunctors,
  - colimits in  $\mathcal{V}$ .

- $\rightsquigarrow$
- lift/extension of profunctors,
  - limits in  $\mathcal{V}$ .

- $\rightsquigarrow$
- weighted limits/colimits,
  - pointwise Kan extensions.

# Techniques in a virtual equipment



Formal category theory in a virtual equipment

= A *puzzle* to be solved using some lemmas and relationships as above.

- 1 The ordinary accessibility
- 2 Virtual equipments
- 3 Formal category theory in a virtual equipment
- 4 Ind-completions in a virtual equipment

# (Co)completeness and (co)continuity

A **left weight**  $\cdots$  A horizontal arrow  $A \xrightarrow{\varphi} B$  satisfying a “nice” property.  
We regard  $A$  as a “shape of diagrams,” and  $\varphi$  as “weights parametrized by  $B$ .”

## Definition

$\Phi$ : a class of left weights

$$\textcircled{1} \quad X \text{ is } \Phi\text{-cocomplete} \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ f \downarrow & \text{lan} \swarrow & \\ X & & \exists \end{array} \quad (\forall \varphi \in \Phi, \forall f)$$

$$\textcircled{2} \quad \begin{array}{c} X \\ g \downarrow \\ Y \end{array} \text{ is } \Phi\text{-cocontinuous} \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ f \downarrow & \text{lan} \swarrow & \\ X & & \\ g \downarrow & & \\ Y & & \end{array} \text{ is a lan-cell } (\forall \varphi \in \Phi, \forall f).$$

A class  $\Phi$  of left weights plays a role as a class of “shapes” of colimits.



In  $\mathcal{V}\text{-Prof}$ ,

$\mathbf{A}$   
 $\downarrow$   
 $\mathbf{Ind}_\Phi(\mathbf{A}) = \{\varphi: \mathbf{A}^{\text{op}} \rightarrow \mathcal{V} \mid \varphi \text{ in } \Phi\}$  : the free cocompletion of  $\mathbf{A}$  under  $\Phi$ .

Then,  $\mathbf{B}$   
 $\downarrow$   
 $\mathbf{Ind}_\Phi(\mathbf{A})$   $\parallel$   $\mathbf{A}^{\text{op}} \otimes \mathbf{B} \xrightarrow{F} \mathcal{V} \text{ s.t. } F(-, \forall b) \in \Phi$   $\parallel$   $\mathbf{A} \xrightarrow{F} \mathbf{B} \text{ in } \Phi$

## Definition

$\Phi$ : a “good” class of left weights in  $\mathbb{E}$ .

$\mathbf{A}$   
 $\downarrow$   
 $\mathbf{X}$  is a  **$\Phi$ -ind-morphism**  $\stackrel{\text{def}}{\Leftrightarrow}$   $k$  yields the adj equiv:  $\mathbf{Hom}_{\mathbb{E}}\left(\frac{B}{X}\right) \xrightleftharpoons[\text{Lan}_- k]{X(k, -)} \mathbf{Hom}_\Phi(A, B). \quad (\forall B \in \mathbb{E})$

## Remark

$\Phi$ -ind-morphisms are a  $\Phi$ -modified version of *Yoneda morphisms* in the sense of (Koudenburg 2022).

## Notation

$A \rightarrow \Phi^\nabla A$ : a  $\Phi$ -ind-morphism. (up to vertical equivalences)

The “functor”  $A \mapsto \Phi^\nabla A$

### Question

- Does the assignment  $A \mapsto \Phi^\nabla A$  yield a “functor”?
- What are the domain and codomain of  $\Phi^\nabla$ ?
- What is the universality of  $\Phi^\nabla$ ?

# $\Phi^\nabla$ behaves like a left adjoint

## Observation 1

$$\frac{A \xrightarrow{\varphi} UB \text{ in } \Phi}{\Phi^\nabla A \xleftarrow{\hat{\varphi}} B} \text{ (by def. of } \Phi\text{-ind-mor.)}$$

$$\rightsquigarrow \Phi^\nabla \dashv U ? \quad (U: B \mapsto B)$$

## Observation 2

$$\begin{array}{c} A \\ f \downarrow \\ UB \end{array} \parallel \begin{array}{c} \Phi^\nabla A \\ \downarrow \hat{f}: \Phi\text{-cocts} \\ B \end{array} \quad (\Phi^\nabla \text{ is a “}\Phi\text{-cocompletion.”})$$

## Definition

$\Phi$ : a class of left weights in  $\mathbb{E}$ .

The pseudo-double category  $\mathbb{E}_\Phi$ :

- object  $\dots$  the same as  $\mathbb{E}$
- vert.arrow  $\dots$  the same as  $\mathbb{E}$
- hor.arrow  $\dots$  hor.arrow in  $\Phi$
- cell  $\dots$  the same as  $\mathbb{E}$

The (strict) dbl cat  $\mathbb{Q}\Phi^\nabla$  (*quintet const.*):

- object  $\dots$   $\Phi$ -cocomplete object in  $\mathbb{E}$
- vert.arrow  $\dots$   $\Phi$ -cocts vert.arrow in  $\mathbb{E}$
- hor.arrow  $X \rightarrowtail Y \dots$  vert.arrow  $X \leftarrow Y$

$$\bullet \text{ cell } \begin{array}{ccc} X & \xrightarrow{u} & Y \\ f \downarrow & \alpha & \downarrow g \\ Z & \xrightarrow{v} & W \end{array} \cdots \begin{array}{ccccc} & & Y & & \\ & u \swarrow & & \searrow g & \\ X & & \alpha & & W \\ & f \searrow & & \swarrow v & \\ & & Z & & \end{array} \text{ in } \mathbb{E}$$

# Relative companied biadjoints and duality

$$\begin{array}{ccc}
 \mathbb{E}_{\Phi}' & \subseteq & \mathbb{E}_{\Phi} \\
 \searrow \Phi^{\nabla} & \Downarrow \eta & \uparrow U \\
 & \mathbb{Q}\Phi^{\nabla} &
 \end{array}$$

$\Phi^{\nabla}$  behaves like a (relative) left adjoint of  $U$  w.r.t. the vertical and horizontal directions.

$\rightsquigarrow$  forms a **relative companied biadjunction**. (a new concept)

## Theorem (Duality)

There is an “equivalence” of double categories:

$$\begin{array}{ccc}
 \mathbb{Cau}\Phi^{\nabla} & \simeq & \mathbb{Acc}\Phi^{\nabla} \\
 \downarrow \cap & & \downarrow \cap \\
 ( \mathbb{E}_{\Phi}' & \xrightarrow{\Phi^{\nabla}} & \mathbb{Q}\Phi^{\nabla} )
 \end{array}$$

The (pseudo) double category  $\mathbb{Cau}\Phi^{\nabla}$ :

- obj ... A “Cauchy cpl” obj  $A \in \mathbb{E}$  s.t.  $\exists \Phi^{\nabla}A$
- vert.arr ... A vert.arrow  $f$  in  $\mathbb{E}$  s.t.  $f_* \in \Phi$
- hor.arr ... that in  $\Phi$

The (strict) double category  $\mathbb{Acc}\Phi^{\nabla}$ :

- obj ... objects in the image of  $\Phi^{\nabla}$   
( **$\Phi$ -accessible obj**)
- vert.arr ...  $\Phi$ -cocts right adjoint vert.arrow
- hor.arr ...  $\Phi$ -cocts vert.arrow  
(in the opposite direction)

## Ongoing works

- Developing a formal theory of “locally presentable objects”
- Exploring virtual equipments  $\mathbb{E}$  that provide interesting duality.
- Comparing with related work: formal accessibility in a 2-category with a “KZ context” (Di Liberti and Loregian 2023)

Thank you!



Today's slides

# References I



Adámek, J., F. W. Lawvere, and J. Rosický (2003). “On the duality between varieties and algebraic theories”. In: *Algebra Universalis* 49.1, pp. 35–49.



Adámek, J. and J. Rosický (2001). “On sifted colimits and generalized varieties”. In: *Theory Appl. Categ.* 8, pp. 33–53.



Adámek, J., F. Borceux, et al. (2002). “A classification of accessible categories”. In: vol. 175. 1-3. Special volume celebrating the 70th birthday of Professor Max Kelly, pp. 7–30.



Centazzo, C. (2004). *Generalised algebraic models*. Presses univ. de Louvain.



Cruttwell, G. S. H. and M. A. Shulman (2010). “A unified framework for generalized multicategories”. In: *Theory Appl. Categ.* 24, No. 21, 580–655.



Di Liberti, I. and F. Loregian (2023). “Accessibility and presentability in 2-categories”. In: *J. Pure Appl. Algebra* 227.1, Paper No. 107155, 25.



Fujii, S. and S. Lack (2022). *The oplax limit of an enriched category*. arXiv: 2211.12122 [math.CT].



Kelly, G. M. and V. Schmitt (2005). “Notes on enriched categories with colimits of some class”. In: *Theory Appl. Categ.* 14, no. 17, 399–423.



Koudenburg, S. R. (2020). “Augmented virtual double categories”. In: *Theory Appl. Categ.* 35, Paper No. 10, 261–325.



— (2022). *Formal category theory in augmented virtual double categories*. arXiv: 2205.04890 [math.CT].



Makkai, M. and R. Paré (1989). *Accessible categories: the foundations of categorical model theory*. Vol. 104. Contemporary Mathematics. American Mathematical Society, Providence, RI, pp. viii+176.

## References II



Street, R. (1983). “Absolute colimits in enriched categories”. In: *Cahiers Topologie Géom. Différentielle* 24.4, pp. 377–379.



Tendas, G. (2023). *Dualities in the theory of accessible categories*. [arXiv: 2302.06273 \[math.CT\]](#).



# Commutation of limits and colimits

## Commutation in Set

$\Phi$ : a class of “shapes” of colim,		$\Psi$ : a class of “shapes” of lim.
$\Phi$ -colimits		$=$ colim commuting with
		$\Psi$ -limits
In Set,	filtered colimits	finite limits
	$\kappa$ -filtered colimits	$\kappa$ -limits
	sifted colimits	finite products
	connected colimits	terminal
	coproducts of filtered colimits	finite connected limits
	absolute colimits	small limits
	small colimits	“nothing”

$\Psi_{//}$ : the class of “shapes” of colim commuting with  $\Psi$ -lim in Set.

- finitely accessible =  $\Psi_{//}$ -accessible ( $\Psi$ : finite limits)
- $\kappa$ -accessible =  $\Psi_{//}$ -accessible ( $\Psi$ :  $\kappa$ -limits)
- generalized variety =  $\Psi_{//}$ -accessible ( $\Psi$ : finite products) (Adámek and Rosický 2001)

## Theorem

If  $\Psi$  satisfies a “nice” condition and  $\mathbf{A}$ :  $\Psi$ -cocomplete, then

$\mathbf{A} \xrightarrow{F} \mathbf{B}$  is  $\Psi_{//}$ -weighty  $\Leftrightarrow \mathbf{A} \xrightarrow{F} \mathbf{B}$  is  $\Psi$ -cocontinuous

## Definition (only for today)

$\mathbf{X}$  is **locally  $\Psi$ -presentable**  $\stackrel{\text{def}}{\Leftrightarrow}$  it is a  $\Psi_{//}$ -ind-completion of Cauchy cpl  $\wedge$   $\Psi$ -cocpl small cat.

## Theorem (Duality for the locally $\Psi$ -presentable context)

If  $\Psi$  satisfies a “nice” condition,

$$\begin{array}{ccc} \mathcal{C}o\mathcal{H}_{\Psi}^{\text{co}} & \simeq_{\text{bi}} & \mathcal{L}p_{\Psi}^{\text{op}} \\ \cap & & \cap \\ ( \mathcal{C}a\mathcal{U}_{\Psi//}^{\text{co}} & \simeq_{\text{bi}} & \mathcal{A}cc_{\Psi//}^{\text{op}} ) \end{array}$$

The 2-category  $\mathcal{C}o\mathcal{H}_{\Psi}$ :

- 0-cell  $\dots$  Cauchy cpl  $\wedge$   $\Psi$ -cocpl small cat
- 1-cell  $\dots$   $\Psi$ -cocontinuous functor
- 2-cell  $\dots$  natural transformation

The 2-category  $\mathcal{L}p_{\Psi}$ :

- 0-cell  $\dots$  locally  $\Psi$ -presentable category
- 1-cell  $\dots$   $\Psi_{//}$ -cocts right adjoint functor
- 2-cell  $\dots$  natural transformation

This subsumes *Gabriel–Ulmer duality* ( $\Psi = \text{fin.lim}$ ), *Adamek–Lawvere–Rosický duality* ( $\Psi = \text{fin.products}$ ).

# Compositions

## Definition

$$\textcircled{1} \quad \begin{array}{ccccccc} A'_0 & \xrightarrow{-\vec{u}_1} & A'_1 & \xrightarrow{-\vec{u}_2} & \dots & \xrightarrow{-\vec{u}_n} & A'_n \\ f_0 \downarrow & \alpha_1 f_1 \downarrow & \alpha_2 & \dots & \alpha_n & \downarrow f_n & \\ A_0 & \xrightarrow{-\vec{v}_1} & A_1 & \xrightarrow{-\vec{v}_2} & \dots & \xrightarrow{-\vec{v}_n} & A_n \end{array} \text{ is opcartesian}$$

$$\begin{array}{ccc} \begin{array}{c} A'_0 \xrightarrow{-\vec{u}_1} A'_1 \xrightarrow{-\vec{u}_2} \dots \xrightarrow{-\vec{u}_n} A'_n \\ f_0 \downarrow \\ A_0 \end{array} & \xrightarrow{\forall \beta} & \begin{array}{c} A'_0 \xrightarrow{-\vec{u}_1} A'_1 \xrightarrow{-\vec{u}_2} \dots \xrightarrow{-\vec{u}_n} A'_n \\ \downarrow f_n \\ A_n \end{array} \\ \downarrow \forall g & & \downarrow \forall h \\ \forall X & \xrightarrow{\forall w} & \forall Y \end{array} = \begin{array}{ccc} \begin{array}{c} A'_0 \xrightarrow{-\vec{u}_1} A'_1 \xrightarrow{-\vec{u}_2} \dots \xrightarrow{-\vec{u}_n} A'_n \\ f_0 \downarrow \\ A_0 \end{array} & \xrightarrow{\exists ! \bar{\beta}} & \begin{array}{c} A'_0 \xrightarrow{-\vec{u}_1} A'_1 \xrightarrow{-\vec{u}_2} \dots \xrightarrow{-\vec{u}_n} A'_n \\ \downarrow f_n \\ A_n \end{array} \\ \downarrow g & & \downarrow h \\ X & \xrightarrow{w} & Y \end{array}$$

$$\textcircled{2} \quad \begin{array}{ccc} A_0 & \xrightarrow{-\vec{u}} & A_n \\ \parallel & \alpha & \parallel \\ A_0 & \xrightarrow{-\vec{v}} & A_n \end{array} \text{ is composing} \quad \Leftrightarrow \quad \begin{array}{ccccccc} X & \xrightarrow{-\vec{p}} & A_0 & \xrightarrow{-\vec{u}} & A_n & \xrightarrow{-\vec{q}} & Y \\ \parallel & \parallel & \parallel & \alpha & \parallel & \parallel & \parallel \\ X & \xrightarrow{-\vec{p}} & A_0 & \xrightarrow{-\vec{v}} & A_n & \xrightarrow{-\vec{q}} & Y \end{array} \text{ is opcartesian. } (\forall X, Y, \vec{p}, \vec{q})$$

# Compositions

## Example in $\mathcal{V}\text{-Prof}$

$$\textcircled{1} \quad \begin{array}{ccc} \mathbf{A} & \xrightarrow{P} & \mathbf{B} & \xrightarrow{Q} & \mathbf{C} \\ \parallel & & \text{comp} & & \parallel \\ \mathbf{A} & \xrightarrow{P \odot Q} & & & \mathbf{C} \end{array} \quad (P \odot Q)(a, c) := \int^{b \in \mathbf{B}} P(a, b) \otimes Q(b, c) \quad \text{in } \mathcal{V}$$

(Suppose that the above coend is preserved by  $X \otimes -, - \otimes Y$ .)

$$\textcircled{2} \quad \begin{array}{ccc} \mathbf{A} & \xrightarrow{\text{Id}_A} & \mathbf{A} \\ \swarrow & \text{cart} & \searrow \\ & \mathbf{A} & \end{array} \quad \text{Id}_A(a, a') := \mathbf{A}(a, a') \quad \text{in } V$$

By universality,  $\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ A & & A \\ \swarrow & & \searrow \\ & A & \end{array} = \begin{array}{ccc} & A & \\ \swarrow & \exists! \alpha & \searrow \\ A & \xrightarrow{\text{cart}} & A \\ \swarrow & & \searrow \\ & A & \end{array} \rightsquigarrow \alpha \text{ becomes composing.}$

# Ext/Lift

## Definition

$$\textcircled{1} \quad \begin{array}{ccc} A & \xrightarrow{\vec{u}} & B \xrightarrow{p} C \\ f \downarrow & \alpha & \parallel \\ X & \xrightarrow{v} & C \end{array} \text{ is extending} \quad (\text{We say that } p \text{ is an extension of } (\vec{u}, f, v).)$$

$$\stackrel{\text{def}}{\Leftrightarrow} \begin{array}{ccc} A & \xrightarrow{\vec{u}} & B \xrightarrow{\vec{q}} Y \\ f \downarrow & \beta & \downarrow g \\ X & \xrightarrow{v} & C \end{array} = \begin{array}{ccc} A & \xrightarrow{\vec{u}} & B \xrightarrow{\vec{q}} Y \\ \parallel & \parallel & \parallel \exists! \bar{\beta} \downarrow g \\ A & \xrightarrow{\vec{u}} & B \xrightarrow{p} C \\ f \downarrow & \alpha & \parallel \\ X & \xrightarrow{v} & C \end{array}$$

$$\textcircled{2} \quad \begin{array}{ccc} C & \xrightarrow{p} & B \xrightarrow{\vec{u}} A \\ \parallel & \alpha & \downarrow f \\ C & \xrightarrow{v} & X \end{array} \text{ is lifting} \quad (\text{the dual notion of extension})$$

## Example in $\mathcal{V}$ -Prof

$$\begin{array}{ccc}
 \textcircled{1} & \begin{array}{ccccc}
 \mathbf{A} & \xrightarrow{P} & \mathbf{B} & \xrightarrow{P \triangleright^F Q} & \mathbf{C} \\
 \downarrow F & & \text{ext} & & \parallel \\
 \mathbf{X} & \xrightarrow{Q} & & & \mathbf{C}
 \end{array} & (P \triangleright^F Q)(b, c) := \int_{a \in \mathbf{A}} P(a, b) \triangleright Q(Fa, c) \quad \text{in } \mathcal{V}
 \end{array}$$

$$\begin{array}{ccc}
 \textcircled{2} & \begin{array}{ccccc}
 \mathbf{C} & \xrightarrow{Q^F \blacktriangleleft P} & \mathbf{B} & \xrightarrow{P} & \mathbf{A} \\
 \parallel & & \text{lift} & & \downarrow F \\
 \mathbf{C} & \xrightarrow{Q} & & & \mathbf{X}
 \end{array} & (Q^F \blacktriangleleft P)(c, b) := \int_{a \in \mathbf{A}} Q(c, Fa) \blacktriangleleft P(b, a) \quad \text{in } \mathcal{V}
 \end{array}$$

Here,  $\mathcal{V} \xrightleftharpoons[X \triangleright -]{X \otimes -} \mathcal{V}, \quad \mathcal{V} \xrightleftharpoons[-\blacktriangleleft X]{-\otimes X} \mathcal{V}.$

# Lan/Ran

## Definition (Koudenburg 2022)

$$\begin{array}{c}
 \textcircled{1} \quad \begin{array}{c} A \xrightarrow{u} B \\ f \downarrow \quad \swarrow \alpha \\ X \end{array} \quad \text{is a \textbf{lan-cell}} \quad \begin{array}{c} \text{def} \\ \Leftrightarrow \end{array} \quad \begin{array}{c} A \xrightarrow{u} B \xrightarrow{\vec{v}} Y \\ f \downarrow \quad \swarrow \beta \\ X \end{array} = \begin{array}{c} A \xrightarrow{u} B \xrightarrow{\vec{v}} Y \\ f \downarrow \quad \swarrow \alpha \quad \exists! \bar{\beta} \\ X \end{array}
 \end{array}$$

(We say that  $\alpha$  exhibits  $l$  as a *left Kan extension* of  $f$  along  $u$ .)

$$\begin{array}{c}
 \textcircled{2} \quad \begin{array}{c} B \xrightarrow{u} A \\ \searrow \alpha \\ X \end{array} \quad \text{is a \textbf{ran-cell}} \quad \begin{array}{c} \text{(the dual notion of lan-cells)} \\ \downarrow f \\ X \end{array}
 \end{array}$$

## Lemma

$$\begin{array}{c} A \xrightarrow{u} B \\ f \downarrow \quad \swarrow \alpha \\ X \end{array} \quad \text{is a lan-cell} \quad \Leftrightarrow \quad \begin{array}{c} A \xrightarrow{u} B \xrightarrow{l_*} X \\ f \downarrow \quad \swarrow \alpha \quad \text{cart} \\ X \end{array} \quad \text{is extending}$$

## Example in $\mathcal{V}\text{-}\mathbb{P}\text{rof}$

$$\textcircled{1} \quad \begin{array}{ccc} \mathbf{A} & \xrightarrow{W} & \mathbf{1} \\ F \downarrow & \text{lan} \swarrow L & \\ \mathbf{X} & & \end{array} \quad \Leftrightarrow \quad L* \cong \operatorname{Colim}_{a \in \mathbf{A}}^{W a} F a. \quad (W\text{-weighted colimit of } F)$$

$$\textcircled{2} \quad \begin{array}{ccc} \mathbf{A} & \xrightarrow{W} & \mathbf{B} \\ F \downarrow & \text{lan} \swarrow L & \\ \mathbf{X} & & \end{array} \quad \Leftrightarrow \quad \forall b \in \mathbf{B}, \quad Lb \cong \operatorname{Colim}_{a \in \mathbf{A}}^{W(a,b)} F a. \quad (W(-, b)\text{-weighted colimit of } F)$$

$$\textcircled{3} \quad \begin{array}{ccc} \mathbf{A} & \xrightarrow{G_*} & \mathbf{B} \\ F \downarrow & \text{lan} \swarrow L & \\ \mathbf{X} & & \end{array} \quad \Leftrightarrow \quad L \text{ is a } \underline{\text{pointwise}} \text{ left Kan extension of } F \text{ along } G.$$

Lan(ran)-cells subsume pointwise Kan extensions and weighted (co)limits.



# Doctrine

## Definition

A class  $\Phi$  of left weights is a **left doctrine** (or,  $\Phi$  is *saturated*)

def  
 $\Leftrightarrow$

- $\text{Id}_A \in \Phi \ (\forall A)$ ;
- $\varphi, \varphi' \in \Phi \implies \varphi \odot \varphi' \in \Phi$ ;
- $f^* \in \Phi \ (\forall f)$ .

$\Phi^*$ : the smallest left doctrine containing  $\Phi$

In  $\mathcal{V}\text{-}\mathbb{P}\text{rof}$ ,

- $\mathbf{A} \xrightarrow{\psi} \mathbf{B} \in \Phi^* \iff \psi(-, \forall b)$  lies in the itelated closure of  $\{\text{rep}\} \subset [\mathbf{A}^{\text{op}}, \mathcal{V}]$  under  $\Phi$ -colimits.
- Thus,  $\Phi^*$  is the “*saturation*” of  $\Phi$ .

## Remark

In an arbitrary virtual equipment,

- $\Phi$ -cocomplete  $\iff \Phi^*$ -cocomplete
- $\Phi$ -cocontinuous  $\iff \Phi^*$ -cocontinuous

# Commutation

$$\begin{array}{c}
 A_1 \xrightarrow{\varphi_1} B_1 \text{ preserves an extension} \quad \begin{array}{ccccc}
 A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{p} & A_1 \\
 f \downarrow & & \text{ext} & & \parallel \\
 X & \xrightarrow{u} & & & A_1
 \end{array}
 \end{array}$$
  

$$\stackrel{\text{def}}{\Leftrightarrow} \exists \alpha, \beta, \gamma \text{ s.t.} \quad \begin{array}{ccccc}
 A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{p} & A_1 & \xrightarrow{\varphi_1} & B_1 \\
 f \downarrow & & \text{ext} & & \parallel & & \parallel \\
 X & \xrightarrow{u} & & & A_1 & \xrightarrow{\varphi_1} & B_1 \\
 \parallel & & \alpha: \text{comp} & & \parallel & & \parallel \\
 X & \xrightarrow{\quad} & & & B_1 & & B_1
 \end{array} = \begin{array}{ccccc}
 A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{p} & A_1 & \xrightarrow{\varphi_1} & B_1 \\
 \parallel & & \parallel & & \beta: \text{comp} & & \parallel \\
 A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\quad} & B_1 & & B_1 \\
 f \downarrow & & \gamma: \text{ext} & & \parallel & & \parallel \\
 X & \xrightarrow{\quad} & & & B_1 & & B_1
 \end{array}$$

## Definition

- A pair  $(\varphi_0, \varphi_1)$  of left weights **commutes**  $(\varphi_0 \parallel \varphi_1)$

$$\stackrel{\text{def}}{\Leftrightarrow} A_1 \xrightarrow{\varphi_1} B_1 \text{ preserves } \begin{array}{ccccc}
 A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\quad} & A_1 \\
 \forall f \downarrow & & \text{ext} & & \parallel \\
 X & \xrightarrow{\quad} & & & A_1 \\
 & & \forall u & &
 \end{array}$$

- A pair  $(\varphi_0, \varphi_1)$  of l.w. **weakly commutes**  $(\varphi_0 / \varphi_1)$

$$\stackrel{\text{def}}{\Leftrightarrow} A_1 \xrightarrow{\varphi_1} B_1 \text{ preserves } \begin{array}{ccccc}
 A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\quad} & A_1 \\
 \searrow \forall f & & \text{ext} & & \parallel \\
 & & A_1 & &
 \end{array}$$

# Commutation

In  $\mathcal{V}\text{-}\mathbb{P}\text{rof}$ ,

- $(\mathbf{A} \xrightarrow{\varphi} \mathbf{B}) \parallel (\mathbf{C} \xrightarrow{\psi} \mathbf{D}) \Leftrightarrow \varphi\text{-limits and } \psi\text{-colimits commute in } \mathcal{V}.$   
 $\Leftrightarrow [\mathbf{C}, \mathcal{V}] \xrightarrow{\text{Colim}^{\psi(-,d)}} \mathcal{V} \text{ preserves } \varphi\text{-limits}.$
- $(\mathbf{A} \xrightarrow{\varphi} \mathbf{B}) / (\mathbf{C} \xrightarrow{\psi} \mathbf{D}) \Leftrightarrow [\mathbf{C}, \mathcal{V}] \xrightarrow{\text{Colim}^{\psi(-,d)}} \mathcal{V} \text{ preserves } \varphi\text{-limits of representables}.$

## Notation

$\Phi$ : a class of left weights.  $\Phi_{\parallel}$  and  $\Phi_{/}$  denote the classes of left weights defined by the following:

$$\Phi_{\parallel} \ni \varphi' \stackrel{\text{def}}{\Leftrightarrow} \varphi \parallel \varphi' \text{ for all } \varphi \in \Phi;$$

$$\Phi_{/} \ni \varphi' \stackrel{\text{def}}{\Leftrightarrow} \varphi / \varphi' \text{ for all } \varphi \in \Phi.$$

## Remark

$\Phi_{\parallel}$  and  $\Phi_{/}$  become left dogmas.

# Soundness

Definition (Adámek, Borceux, et al. 2002)

A class  $\Phi$  of left weights is **sound**  $\stackrel{\text{def}}{\iff} \Phi_{//} = \Phi_{/}$

$\Phi$ : sound  $\rightsquigarrow$  Theory of  $\Phi_{//}(=\Phi_{/})$ -accessible categories behaves well.

## Example in Set-Prof

A class  $\text{Fin} = \{\text{left weights of finite (co)limits}\}$  is sound.

Then,  $\text{Fin}_{//} = \text{Fin}_{/} = \{\text{l.w. of filtered colim}\}$ .

# Adjunctions of weights

## Definition

A **(horizontal) adjunction**  $(\psi \dashv \varphi)$  consists of:

- Horizontal arrows  $Y \xrightarrow{\psi} X, X \xrightarrow{\varphi} Y$ ;

- Cells 
$$\begin{array}{ccc} & Y & \\ \eta \swarrow & & \searrow \eta \\ Y & \xrightarrow{\psi \circ \varphi} & Y \end{array} \quad \begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & X \\ & \searrow \varepsilon & & \swarrow \varepsilon & \\ & X & & X & \end{array}$$
 s.t. the *zigzag identities* hold.

## Theorem

$\psi \dashv \varphi$ : hor.adj. Then,  $\psi$ : a right weight  $\Leftrightarrow \varphi$ : a left weight.

## Definition

A left weight  $X \xrightarrow{\varphi} Y$  is **left-absolute**  $\stackrel{\text{def}}{\Leftrightarrow}$  “ $\varphi$ -colimits are always absolute.”

## Theorem (Street 1983)

In  $\mathcal{V}\text{-Prof}$ ,  $X \xrightarrow{\varphi} Y$  has a left adjoint  $\Leftrightarrow \varphi$  is left-absolute.

# Street's characterization in a virtual equipment

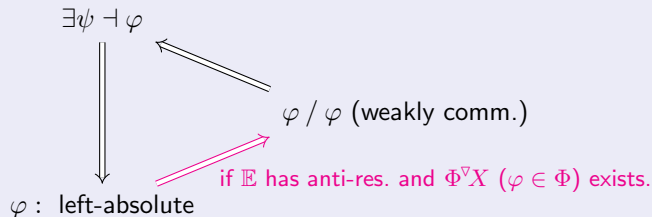
## Definition

$\mathbb{E}$  has **anti-restrictions**  $\stackrel{\text{def}}{\iff}$  For every  $X \xrightarrow{u} Y$ ,

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ & \exists \text{cart} \swarrow \searrow & \\ & \exists \downarrow \quad \downarrow \exists & \\ & \exists Z & \end{array}$$

## Theorem

$X \xrightarrow{\varphi} Y$ : a left weight



# Characterization of ind-morphisms

$\Phi$ : a class of left weights.

## Definition

$X \xrightarrow{f} Y$  is  **$\Phi$ -atomic**  $\stackrel{\text{def}}{\Leftrightarrow} \forall A \xrightarrow{\varphi} \forall B \text{ in } \Phi, A \xrightarrow{g} Y, \exists \alpha, \beta \text{ s.t.}$

$$\begin{array}{ccc}
 X & \xrightarrow{Y(f,g)} A & \xrightarrow{\varphi} B \\
 & \searrow f & \downarrow g \\
 & & Y
 \end{array}
 \quad
 \begin{array}{c}
 \text{cart} \quad \text{lan} \\
 \downarrow \quad \swarrow
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{Y(f,g)} A & \xrightarrow{\varphi} B \\
 \parallel & \alpha: \text{comp} & \parallel \\
 X & \xrightarrow{\quad} B & \\
 \searrow f & \beta: \text{cart} & \swarrow \\
 & Y &
 \end{array}
 \quad \text{whenever lan exists.}$$

In  $\mathcal{V}\text{-}\mathbb{P}\text{rof}$ ,

$\mathbf{X} \xrightarrow{F} \mathbf{Y}$  is  $\Phi$ -atomic  $\Leftrightarrow \forall x \in \mathbf{X}, \mathbf{Y}(Fx, -): \mathbf{Y} \rightarrow \mathcal{V}$  is  $\Phi$ -cocontinuous.

# Characterization of ind-morphisms

$\Phi$ : a class of left weights.

## Theorem

- $X$  is  $\Phi$ -cocomplete;
- $k$  is  $\Phi$ -atomic and *fully faithful*;
- For any  $Y \xrightarrow{f} X$ , there exist  $B, B \xrightarrow{\varphi} Y$  in  $\Phi$ ,  $B \xrightarrow{g} A$ , and a lan-cell:

$A \xrightarrow{k} X$  is a  $\Phi$ -ind-morphism.  $\Leftrightarrow$

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & Y \\ g \downarrow & \text{lan} & \nearrow f \\ A & & \\ k \downarrow & & \\ X & & \end{array}$$

The 3rd condition says that “every  $x \in X$  is a  $\Phi$ -colimit of  $\Phi$ -atomic objects.”



# The functors in detail

$$\begin{array}{ccc} \mathbb{E}_\Phi' & \subseteq & \mathbb{E}_\Phi \\ & \searrow \Phi^\nabla & \uparrow U \\ & & \mathbb{Q}\Phi^\nabla \end{array}$$

$$\begin{array}{c} A \\ \eta_A \downarrow \\ U\Phi^\nabla A \end{array} := \begin{array}{c} A \\ a \downarrow \\ \Phi^\nabla A \end{array} \quad (\text{a } \Phi\text{-ind-morphism in } \mathbb{E})$$

fullsub  $\mathbb{E}_\Phi' := \{A \mid \Phi^\nabla A \text{ exists}\} \subseteq \mathbb{E}_\Phi$ .

## Definition of $\Phi^\nabla$

- $A \mapsto \Phi^\nabla A$

- $\begin{array}{c} A \\ f \downarrow \\ B \end{array} \mapsto \begin{array}{ccc} A & \xrightarrow{a_*} & \Phi^\nabla A \\ f \downarrow & \text{lan} & \\ B & & \\ b \downarrow & \swarrow \Phi^\nabla f & \\ \Phi^\nabla B & & \end{array} \quad \text{in } \mathbb{E}$

- $A \xrightarrow{u} B \mapsto \begin{array}{ccc} A & \xrightarrow{u} & B \xrightarrow{b_*} \Phi^\nabla B \\ a \downarrow & \text{lan} & \\ \Phi^\nabla A & & \end{array} \quad \text{in } \mathbb{E}$

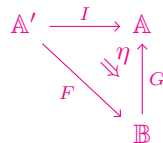
## Definition of $U$

$$\begin{array}{ccc} X \xleftarrow{u} Y & & X \xrightarrow{u^*} Y \\ f \downarrow & \downarrow g & f \downarrow \quad \downarrow g \\ Z \xleftarrow{v} W & \mapsto & Z \xrightarrow{v^*} W \end{array} \quad \text{in } \mathbb{E}$$

# Relative companied biadjoints

We fix the following data:

- pseudo-double categories  $\mathbb{A}'$ ,  $\mathbb{A}$ , and  $\mathbb{B}$ ;
- “pseudo-double functors”  $\mathbb{A}' \xrightarrow{I} \mathbb{A}$ ,  $\mathbb{A}' \xrightarrow{F} \mathbb{B}$ , and  $\mathbb{B} \xrightarrow{G} \mathbb{A}$ ;
- a pseudo-vertical trans  $I \Rightarrow GF$  whose components have companions.



(HTrans): The “horizontal naturality” of  $\eta$

$$\left( \begin{smallmatrix} F \\ G \end{smallmatrix} \right): \begin{array}{c} FA \\ f \downarrow \\ B \end{array} \parallel \begin{array}{c} IA \\ \hat{f} \downarrow \\ GB \end{array} \quad \left( \begin{smallmatrix} F \\ G \end{smallmatrix} \right): \frac{FA \xrightarrow{u} B}{IA \xrightarrow{\hat{u}} GB}$$

$$\left( \begin{smallmatrix} F \\ G \end{smallmatrix} \begin{smallmatrix} G \\ G \end{smallmatrix} \right): \begin{array}{ccc} FA & \xrightarrow{u} & B_0 \\ f \downarrow & \alpha & \downarrow g \\ B_1 & \xrightarrow{v} & B_2 \end{array} \parallel \begin{array}{ccc} IA & \xrightarrow{\hat{u}} & GB_0 \\ \hat{f} \downarrow & \hat{\alpha} & \downarrow Gg \\ GB_1 & \xrightarrow{Gv} & GB_2 \end{array}$$

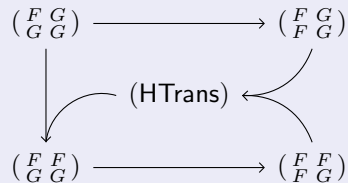
$$\left( \begin{smallmatrix} F \\ F \end{smallmatrix} \begin{smallmatrix} F \\ G \end{smallmatrix} \right): \begin{array}{ccc} FA_0 & \xrightarrow{Fu} & FA_1 \\ Ff \downarrow & \alpha & \downarrow g \\ FA_2 & \xrightarrow{v} & B \end{array} \parallel \begin{array}{ccc} IA_0 & \xrightarrow{Iu} & IA_1 \\ If \downarrow & \hat{\alpha} & \downarrow \hat{g} \\ IA_2 & \xrightarrow{\hat{v}} & GB \end{array}$$

$$\left( \begin{smallmatrix} F \\ F \end{smallmatrix} \begin{smallmatrix} G \\ G \end{smallmatrix} \right): \begin{array}{ccc} FA_0 & \xrightarrow{u} & B_0 \\ Ff \downarrow & \alpha & \downarrow g \\ FA_1 & \xrightarrow{v} & B_1 \end{array} \parallel \begin{array}{ccc} IA_0 & \xrightarrow{\hat{u}} & GB_0 \\ If \downarrow & \hat{\alpha} & \downarrow Gg \\ IA_1 & \xrightarrow{\hat{v}} & GB_1 \end{array}$$

$$\left( \begin{smallmatrix} F \\ G \end{smallmatrix} \begin{smallmatrix} F \\ G \end{smallmatrix} \right): \begin{array}{ccc} FA_0 & \xrightarrow{Fu} & FA_1 \\ f \downarrow & \alpha & \downarrow g \\ B_0 & \xrightarrow{v} & B_1 \end{array} \parallel \begin{array}{ccc} IA_0 & \xrightarrow{Iu} & IA_1 \\ \hat{f} \downarrow & \hat{\alpha} & \downarrow \hat{g} \\ GB_0 & \xrightarrow{Gv} & GB_1 \end{array}$$

## Relationship among the 7 axioms

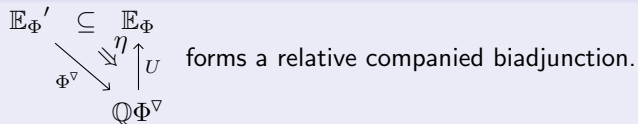
Under  $(\begin{smallmatrix} F \\ G \end{smallmatrix})$  and  $(\begin{smallmatrix} F & G \\ & G \end{smallmatrix})$ , implications of the following directions hold.



## Definition



## Theorem



## Nerves and realizations

## Theorem (Companion theorem)

$$\begin{array}{ccc}
 \mathbb{A}' & \xrightarrow{I} & \mathbb{A} \\
 & \searrow F & \uparrow \eta \\
 & & \mathbb{B}
 \end{array}
 \quad G : \text{rel.comp-biadj} \quad \Rightarrow$$

$$\begin{array}{l}
 \textcircled{1} \quad \begin{array}{c} FA \\ f \downarrow \\ B \end{array} \text{ has a companion} \quad \Leftrightarrow \quad \begin{array}{c} IA \\ \hat{f} \downarrow \\ GB \end{array} \text{ has a companion.} \\
 \textcircled{2} \quad FA \xrightarrow{u} B \text{ is a companion} \quad \Leftrightarrow \quad IA \xrightarrow{\hat{u}} GB \text{ is a companion.}
 \end{array}$$

## Corollary

① 
$$\begin{array}{ccc} A & \xrightarrow{a_*} & \Phi^\nabla A \\ f \downarrow \text{lan} & \swarrow l & \\ \underline{E} & & \end{array}$$
  
 $\Phi\text{-cocpl}$

Then,  $f_* \in \Phi \iff l$  has a right adjoint.

② 
$$\begin{array}{ccc} A & \xrightarrow[\vdash]{\varphi \in \Phi} & \underline{E} \\ a \downarrow & \swarrow \text{lan}^{\Phi\text{-cocpl}} & \\ \Phi^\nabla A & & \end{array}$$
 Then,  $\varphi$  is a companion  $\Leftrightarrow r$  has a left adjoint.