Formal accessibility in a virtual equipment

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← Today's slide

Definition

The *free cocompletion* of $\mathbf{A}$ under filtered colimits
$\cdots$ fullsub $\mathbf{Ind}(\mathbf{A}) := \{\text{fil.colim of repr}\} \subseteq \mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$
($=$ ind-completion of $\mathbf{A}$.)

Definition

The *free cocompletion* of $\mathbf{A}$ under $\underline{\Phi\text{-colimits}}$
$\cdots$ fullsub $\mathbf{Ind}_{\Phi}(\mathbf{A}) := \{\underline{\Phi\text{-colim}} \text{ of repr}\} \subseteq \mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$
($=$ "Φ-ind-completion" of $\mathbf{A}$.)

Definition

$X \in \mathbf{X}$ is finitely presentable (f.p.)
$\overset{\text{def}}{\Leftrightarrow}$ $\mathbf{X}(X, -)$ preserves filtered colimits.

Definition

$X \in \mathbf{X}$ is Φ-atomic
$\overset{\text{def}}{\Leftrightarrow}$ $\mathbf{X}(X, -)$ preserves $\underline{\Phi\text{-colimits}}$.

Fact

TFAE for a category $\mathbf{X}$:

1. $\mathbf{X}$ has filtered colimits, and every $X \in \mathbf{X}$ is a filtered colimit of f.p.objects.
2. $\mathbf{X} \simeq \mathbf{Ind}(\mathbf{A})$ ($\exists \mathbf{A}$).

$\Updownarrow$def (if we ignore "size.")

$\mathbf{X}$ is finitely accessible.

Fact

TFAE for a category $\mathbf{X}$:

1. $\mathbf{X}$ has $\underline{\Phi\text{-colimits}}$, and every $X \in \mathbf{X}$ is a $\underline{\Phi\text{-colimit}}$ of $\underline{\Phi\text{-atomic obj}}$.
2. $\mathbf{X} \simeq \mathbf{Ind}_{\Phi}(\mathbf{A})$ ($\exists \mathbf{A}$).

$\Updownarrow$def (if we ignore "size.")

$\mathbf{X}$ is Φ-accessible. (Φ: a "shape" of colim)

Duality

Φ: a shape of colim.

Definition (only for today)

A functor $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ is Φ-weighty
$\overset{\text{def}}{\Leftrightarrow}$ (Pointwise) left Kan extensions along F are given by Φ-colimits.

Theorem (Duality in the Φ-accessible context)

There is a biequivalence of 2-categories:

$$\mathscr{C}au_{\Phi}^{\mathrm{co}} \quad \simeq_{\mathsf{bi}} \quad \mathscr{A}cc_{\Phi}^{\mathrm{op}}$$

The 2-category $\mathscr{C}au_{\Phi}$:

- 0-cell $\cdots$ Cauchy complete small category
- 1-cell $\cdots$ Φ-weighty functor
- 2-cell $\cdots$ natural transformation

The 2-category $\mathscr{A}cc_{\Phi}$:

- 0-cell $\cdots$ Φ-accessible category
- 1-cell $\cdots$ Φ-cocontinuous right adjoint functor
- 2-cell $\cdots$ natural transformation

This is a "Φ-modified" version of *Makkai–Paré duality* (Makkai and Paré 1989).
This duality has recently been generalized to the enriched context (Tendas 2023).

Commutation of limits and colimits

Commutation in $\mathbf{Set}$

Φ: a "shape" of colim, Ψ: a "shape" of lim.

In $\mathbf{Set}$,

Φ-colimits	$=$ colim commuting with	Ψ-limits
filtered colimits		finite limits
κ-filtered colimits		κ-limits
sifted colimits		finite products
connected colimits		terminal
coproducts of filtered colimits		finite connected limits
absolute colimits		small limits
small colimits		"nothing"

$\Psi_{/\!/}$: the "shape" of colim commuting with Ψ-lim in $\mathbf{Set}$.

- finitely accessible $=$ $\Psi_{/\!/}$-accessible (Ψ: finite limits)
- κ-accessible $=$ $\Psi_{/\!/}$-accessible (Ψ: κ-limits)
- generalized variety $=$ $\Psi_{/\!/}$-accessible (Ψ: finite products) (Adámek and Rosický 2001)

Theorem

If Ψ satisfies a “nice” condition and $\mathbf{A}$: Ψ-cocomplete, then

$\mathbf{A} \xrightarrow{F} \mathbf{B}$ is $\Psi_{/\!/}$-weighty $\quad\Leftrightarrow\quad$ $\mathbf{A} \xrightarrow{F} \mathbf{B}$ is Ψ-cocontinuous

Definition (only for today)

$\mathbf{X}$ is locally Ψ-presentable $\overset{\text{def}}{\Leftrightarrow}$ it is a $\Psi_{/\!/}$-ind-completion of Cauchy cpl $\wedge$ Ψ-cocpl small cat.

Theorem (Duality for the locally Ψ-presentable context)

If Ψ satisfies a “nice” condition,

$$\begin{array}{ccc} \mathscr{C}oth_{\Psi}^{\mathrm{co}} & \simeq_{\text{bi}} & \mathscr{L}p_{\Psi}^{\mathrm{op}} \\ \cap & & \cap \\ (\ \mathscr{C}au_{\Psi_{/\!/}}^{\mathrm{co}} & \simeq_{\text{bi}} & \mathscr{A}cc_{\Psi_{/\!/}}^{\mathrm{op}}\ ) \end{array}$$

The 2-category $\mathscr{C}oth_{\Psi}$:

- 0-cell $\cdots$ Cauchy cpl $\wedge$ Ψ-cocpl small cat
- 1-cell $\cdots$ Ψ-cocontinuous functor
- 2-cell $\cdots$ natural transformation

The 2-category $\mathscr{L}p_{\Psi}$:

- 0-cell $\cdots$ locally Ψ-presentable category
- 1-cell $\cdots$ $\Psi_{/\!/}$-cocts right adjoint functor
- 2-cell $\cdots$ natural transformation

This subsumes *Gabriel–Ulmer duality* (Ψ=fin.lim), *Adamek–Lawvere–Rosický duality* (Ψ=fin.products).

Goal

($\mathscr{V}$-enriched) accessibility

- duality
- ind-completion
- Cauchy completeness
- commutation of lim and colim

$=$ Accessibility in $\mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$

The *virtual equipment* $\mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$:

- $\mathscr{V}$-enriched categories
- $\mathscr{V}$-functors
- $\mathscr{V}$-profunctors

Formal accessibility in a virtual equipment

$$\mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \xrightarrow{\text{generalize}} \mathbb{E} \text{ (an arbitrary virtual equipment)}$$

This extends the "accessible notion" to other category-theoretic contexts:

- bicategory-enriched categories
- fibered (or indexed) categories
- internal categories
- something that is no longer categories

Related work: formal accessibility in a 2-category with a "KZ context" (Di Liberti and Loregian 2023)

Why virtual equipments?

2-categories are suitable for capturing:

- ✓ ordinary limit and colimits,
- ✓ adjunctions,
- ✓ monads,
- ✓ Kan extensions and lifts.

2-categories are not suitable for capturing interactions of functors and profunctors: $\rightsquigarrow$ *virtual equipments*

- × weighted limits and colimits,
- × presheaves,
- × cocompletions,
- × pointwise Kan extensions,
- × Cauchy completeness,
- × commutation of weights.

Main features

In our formalization,

- We do not use *opposite categories*.
  - $\rightsquigarrow$ categories enriched by a non-symmetric monoidal category or a bicategory
- We do not require neither *smallness* of categories nor *composition* of arbitrary profunctors.
  - $\rightsquigarrow$ overcoming the *size matters*
- We do not demand "(co)completeness" for the universe, but rather for each weight.
  - $\rightsquigarrow$ enrichment by a monoidal category that is neither (co)complete nor closed.

Profunctors

Definition

$\mathbf{A}, \mathbf{B}$: $\mathscr{V}$-categories
A $\mathscr{V}$-profunctor $\mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$ $\cdots$ a functor $\mathbf{A}^{\mathrm{op}} \otimes \mathbf{B} \longrightarrow \mathscr{V}$

$\mathscr{V}$-functors can be composed:
$$\begin{array}{c} \mathbf{A} \\ F\big\downarrow \\ \mathbf{B} \\ G\big\downarrow \\ \mathbf{C} \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \mathbf{A} \\ F{\,;\,}G\Big\downarrow \\ \mathbf{C} \end{array}$$

If $\mathbf{B}$ is small, $\mathscr{V}$-profunctors can be composed $\mathbf{A} \overset{P}{\nrightarrow} \mathbf{B} \overset{Q}{\nrightarrow} \mathbf{C} \rightsquigarrow \mathbf{A} \xrightarrow{P \odot Q} \mathbf{C}$

In general, can not $\mathbf{A} \overset{P}{\nrightarrow} \mathbf{B} \overset{Q}{\nrightarrow} \mathbf{C} \not\rightsquigarrow \mathbf{A} \xrightarrow{P \odot Q} \mathbf{C}$

Even if $P \odot Q$ does not exist, $\mathscr{V}$-nat.trans $P \odot Q \Rightarrow R$ can be considered:

A $\mathscr{V}$-nat.trans $P \odot Q \Rightarrow R$
$=$ a family $\{P(A,B) \otimes Q(B,C) \to R(A,C) \quad \text{in } \mathscr{V}\}_{A,B,C}$ that is nat in A, B and extra-nat in B.
($\mathscr{V}$-forms)

The augmented virtual double category $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

- $\mathscr{V}$-categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$;
- $\mathscr{V}$-functors $\begin{tikzcd} \mathbf{A} \arrow[d, "F"'] \\ \mathbf{B} \end{tikzcd}$ $, \ldots$ and their compositions and identities;
- $\mathscr{V}$-profunctors $\begin{tikzcd} \mathbf{A} \arrow[r, "P", "\shortmid"{marking}] & \mathbf{B} \end{tikzcd}$ $, \ldots$;
- $\binom{n}{1}$-$\mathscr{V}$-forms

$$\begin{tikzcd} \mathbf{A}_0 \arrow[r, "P_1", "\shortmid"{marking}] \arrow[d, "F"'] & \mathbf{A}_1 \arrow[r, "P_2", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "P_n", "\shortmid"{marking}] & \mathbf{A}_\mathbf{n} \arrow[d, "G"] \\ \mathbf{B} \arrow[rrr, "Q"', "\shortmid"{marking}] & & & \mathbf{C} \end{tikzcd} = \{P_1(A_0, A_1) \otimes \cdots \otimes P_n(A_{n-1}, A_n) \to Q(FA_0, GA_n)\},$$

$\binom{n}{0}$-$\mathscr{V}$-forms

$$\begin{tikzcd} \mathbf{A}_0 \arrow[r, "P_1", "\shortmid"{marking}] \arrow[dr, "F"'] & \cdots \arrow[r, "P_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \mathbf{A}_\mathbf{n} \arrow[dl, "G"] \\ & \mathbf{B} & \end{tikzcd} = \{P_1(A_0, A_1) \otimes \cdots \otimes P_n(A_{n-1}, A_n) \to \mathbf{B}(FA_0, GA_n)\},$$

- and their compositions

$$\begin{tikzcd} \mathbf{A}_0 \arrow[r, dashed, "\vec{P}_1", "\shortmid"{marking}] \arrow[d, "F_0"'] \arrow[dr, phantom, "\alpha_1"] & \mathbf{A}_1 \arrow[r, dashed, "\vec{P}_2", "\shortmid"{marking}] \arrow[d, "F_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{P}_n", "\shortmid"{marking}] \arrow[d, phantom, "\cdots \quad \alpha_n"] & \mathbf{A}_n \arrow[d, "F_n"] \\ \mathbf{B}_0 \arrow[r, dotted, "Q_1"', "\shortmid"{marking}] \arrow[d, "G"'] & \mathbf{B}_1 \arrow[r, dotted, "Q_2"', "\shortmid"{marking}] \arrow[d, phantom, "\beta"] & \cdots \arrow[r, dotted, "Q_n"', "\shortmid"{marking}] & \mathbf{B}_n \arrow[d, "H"] \\ \mathbf{C} \arrow[rrr, dotted, "R"', "\shortmid"{marking}] & & & \mathbf{D} \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathbf{A}_0 \arrow[r, "\vec{P}_1", "\shortmid"{marking}] \arrow[d, "F_0 ⨟ G"'] & \mathbf{A}_1 \arrow[r, "\vec{P}_2", "\shortmid"{marking}] \arrow[d, phantom, "\vec{\alpha} ⨟ \beta"] & \cdots \arrow[r, "\vec{P}_n", "\shortmid"{marking}] & \mathbf{A}_n \arrow[d, "F_n ⨟ H"] \\ \mathbf{C} \arrow[rrr, dotted, "R"', "\shortmid"{marking}] & & & \mathbf{D} \end{tikzcd}$$

An augmented virtual double category $\mathbb{X}$

- objects $A, B, C, \ldots$;
- vertical arrows $\begin{array}{c} A \\ f\big\downarrow \\ B \end{array}$, . . . and their compositions and identities;
- horizontal arrows $A \xrightarrow{p} B$, . . . ;
- cells: $\binom{n}{1}$-cells
$$\begin{tikzcd} A_0 \arrow[r, "p_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "p_2", "\shortmid"{marking}] & \cdots \arrow[r, "p_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\ B \arrow[rrr, "q"', "\shortmid"{marking}] \arrow[rrr, phantom, "\alpha", yshift=4ex] & & & C \end{tikzcd}, \ldots$$
$\binom{n}{0}$-cells
$$\begin{tikzcd} A_0 \arrow[r, "p_1", "\shortmid"{marking}] \arrow[dr, "f"'] & \cdots \arrow[r, "p_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & A_n \arrow[dl, "g"] \\ & B & \end{tikzcd}, \ldots$$
- and their compositions:
$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{p}_1", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{p}_2", "\shortmid"{marking}] \arrow[d, "f_1"] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{p}_n", "\shortmid"{marking}] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\ B_0 \arrow[r, dotted, "q_1", "\shortmid"{marking}] \arrow[d, "g"'] & B_1 \arrow[r, dotted, "q_2", "\shortmid"{marking}] & \cdots \arrow[r, dotted, "q_n", "\shortmid"{marking}] & B_n \arrow[d, "h"] \\ C \arrow[rrr, dotted, "r"', "\shortmid"{marking}] \arrow[rrr, phantom, "\beta", yshift=4ex] & & & D \end{tikzcd} \rightsquigarrow \begin{tikzcd} A_0 \arrow[r, dashed, "\vec{p}_1", "\shortmid"{marking}] \arrow[d, "f_0 ⨾ g"'] & A_1 \arrow[r, dashed, "\vec{p}_2", "\shortmid"{marking}] & \cdots \arrow[r, dashed, "\vec{p}_n", "\shortmid"{marking}] & A_n \arrow[d, "f_n ⨾ h"] \\ C \arrow[rrr, dotted, "r"', "\shortmid"{marking}] \arrow[rrr, phantom, "\vec{\alpha} ⨾ \beta", yshift=4ex] & & & D \end{tikzcd}$$

  and identity-cells:
$$\begin{tikzcd} X \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & Y \arrow[d, equal] \\ X \arrow[r, "p"', "\shortmid"{marking}] & Y \end{tikzcd}$$

Definition

$$
\begin{tikzcd}
A \arrow[r, "p", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "u"', "|"] & Y
\end{tikzcd}
\text{ is cartesian } \overset{\text{def}}{\Longleftrightarrow}
\begin{tikzcd}
\forall A' \arrow[r, dashed, "\forall \vec{q}", "|"] \arrow[d, "\forall h"'] & \forall B' \arrow[d, "\forall k"] \\
A \arrow[r, phantom, "\forall \beta"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "u"', "|"] & Y
\end{tikzcd}
=
\begin{tikzcd}
A' \arrow[r, dashed, "\vec{q}", "|"] \arrow[d, "h"'] \arrow[dr, phantom, "\exists! \bar{\beta}"] & B' \arrow[d, "k"] \\
A \arrow[r, "p"', "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "u"', "|"] & Y
\end{tikzcd}
$$

Notation

$$
\begin{tikzcd}
A \arrow[r, "u(f,g)", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{cart}"] & B \arrow[d, "g"] \\
X \arrow[r, "u"', "|"] & Y
\end{tikzcd}
$$

(the restriction of u along f, g)

$$
\begin{tikzcd}
A \arrow[rr, "X(f,g)", "|"] \arrow[dr, "f"'] & \text{cart} & B \arrow[dl, "g"] \\
& X &
\end{tikzcd}
$$

(the restriction on X along f, g)

$$
\begin{tikzcd}
A \arrow[rr, "f_*", "|"] \arrow[dr, "f"'] & \text{cart} & B \arrow[dl, equal] \\
& B &
\end{tikzcd}
$$

(the companion of f)

$$
\begin{tikzcd}
B \arrow[rr, "f^*", "|"] \arrow[dr, equal] & \text{cart} & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}
$$

(the conjoint of f)

$$
\begin{tikzcd}
X \arrow[rr, equal, "\mathsf{Id}_X", "|"] \arrow[dr, equal] & \text{cart} & X \arrow[dl, equal] \\
& X &
\end{tikzcd}
$$

(the unit on X)

Virtual equipments

Definition (Cruttwell and Shulman 2010; Koudenburg 2020)

A virtual equipment = an augmented virtual double category s.t.
$$\begin{tikzcd} A \arrow[r, "\exists p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\exists\text{cart}"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"', "\shortmid"{marking}] & Y \end{tikzcd}$$

Example

virtual equipment	object	vert.arrow	hor.arrow
$\mathscr{V}$-$\mathbb{P}\mathrm{rof}$ ($\mathscr{V}$: a monoidal cat)	$\mathscr{V}$-enriched cat	$\mathscr{V}$-functor	$\mathscr{V}$-profunctor
$\mathscr{W}$-$\mathbb{P}\mathrm{rof}$ ($\mathscr{W}$: a bicategory)	$\mathscr{W}$-enriched cat	$\mathscr{W}$-functor	$\mathscr{W}$-profunctor
$\mathbb{P}\mathrm{rof}(\mathbf{C})$ ($\mathbf{C}$: cat with p.b.)	$\mathbf{C}$-internal cat	$\mathbf{C}$-internal functor	$\mathbf{C}$-internal profunctor

and so on.

From now on, we fix a virtual equipment $\mathbb{E}$. (e.g. $\mathbb{E} := \mathscr{V}$-$\mathbb{P}\mathrm{rof}$)

Compositions

Definition

1. $$\begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "v_1"'] & A_1 \arrow[r, dotted, "v_2"'] & \cdots \arrow[r, dotted, "v_n"'] & A_n \end{tikzcd}$$ is opcartesian

$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0"'] & A'_1 \arrow[r, dashed, "\vec{u}_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[d, "\forall g"'] \arrow[rrr, phantom, "\forall\beta"] & & & A_n \arrow[d, "\forall h"] \\ \forall X \arrow[rrr, dotted, "\forall w"'] & & & \forall Y \end{tikzcd} = \begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "v_1"'] \arrow[d, "g"'] \arrow[drrr, phantom, "\exists!\bar{\beta}"] & A_1 \arrow[r, dotted, "v_2"'] & \cdots \arrow[r, dotted, "v_n"'] & A_n \arrow[d, "h"] \\ X \arrow[rrr, dotted, "w"'] & & & Y \end{tikzcd}$$

2. $$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & A_n \arrow[d, equal] \\ A_0 \arrow[r, "v"'] & A_n \end{tikzcd}$$ is composing $\overset{\text{def}}{\Leftrightarrow}$ $$\begin{tikzcd} X \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "\|"] & A_0 \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & A_n \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "\|"] & Y \arrow[d, equal] \\ X \arrow[r, dashed, "\vec{p}"'] & A_0 \arrow[r, "v"'] & A_n \arrow[r, dashed, "\vec{q}"'] & Y \end{tikzcd}$$ is opcartesian. $(\forall X, Y, \vec{p}, \vec{q})$

Compositions

Example in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

1. $$\begin{tikzcd} \mathbf{A} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\text{comp}"] & \mathbf{B} \arrow[r, "Q", "\shortmid"{marking}] & \mathbf{C} \arrow[d, equal] \\ \mathbf{A} \arrow[rr, "P \odot Q"', "\shortmid"{marking}] & & \mathbf{C} \end{tikzcd} \qquad (P \odot Q)(a, c) := \int^{b \in \mathbf{B}} P(a, b) \otimes Q(b, c) \quad \text{in } \mathscr{V}$$

   (Suppose that the above coend is preserved by $X \otimes -, - \otimes Y$.)

2. $$\begin{tikzcd} \mathbf{A} \arrow[rr, equal, "\mathrm{Id}_A", "\shortmid"{marking}] \arrow[dr, equal] & \arrow[d, phantom, "\text{cart}"] & \mathbf{A} \arrow[dl, equal] \\ & \mathbf{A} & \end{tikzcd} \qquad \mathrm{Id}_A(a, a') := \mathbf{A}(a, a') \quad \text{in } V$$

By universality, $$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] & \\ A \arrow[dr, equal] \arrow[rr, phantom, "="] & & A \arrow[dl, equal] \\ & A & \end{tikzcd} = \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\exists!\alpha"] & \\ A \arrow[rr, equal, "\shortmid"{marking}] \arrow[dr, equal] & \arrow[d, phantom, "\text{cart}"] & A \arrow[dl, equal] \\ & A & \end{tikzcd} \quad \rightsquigarrow \quad \alpha \text{ becomes composing.}$$

Ext/Lift

Definition

1. $$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha"] & B \arrow[r, "p"] & C \arrow[d, equal] \\ X \arrow[rr, dotted, "v"'] & & C \end{tikzcd}$$ is extending (We say that p is an *extension* of $(\vec{u}, f, v)$.)

$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\beta"] & B \arrow[r, dashed, "\vec{q}"] & Y \arrow[d, "g"] \\ X \arrow[rr, dotted, "v"'] & & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "\exists! \bar{\beta}"] & Y \arrow[d, "g"] \\ A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha"] & B \arrow[r, "p"] & C \arrow[d, equal] \\ X \arrow[rr, dotted, "v"'] & & C \end{tikzcd}$$

2. $$\begin{tikzcd} C \arrow[r, "p"] \arrow[d, equal] \arrow[drr, phantom, "\alpha"] & B \arrow[r, dashed, "\vec{u}"] & A \arrow[d, "f"] \\ C \arrow[rr, dotted, "v"'] & & X \end{tikzcd}$$ is lifting (the dual notion of extension)

Ext/Lift

Example in $\mathscr{V}$-$\mathbb{P}$rof

1.
$$\begin{tikzcd} \mathbf{A} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "F"'] & \mathbf{B} \arrow[r, "P \triangleright^F Q", "\shortmid"{marking}] \arrow[d, phantom, "\text{ext}"] & \mathbf{C} \arrow[d, equal] \\ \mathbf{X} \arrow[rr, "Q"', "\shortmid"{marking}] & & \mathbf{C} \end{tikzcd} \qquad (P \triangleright^F Q)(b,c) := \int_{a \in \mathbf{A}} P(a,b) \triangleright Q(Fa,c) \quad \text{in } \mathscr{V}$$

2.
$$\begin{tikzcd} \mathbf{C} \arrow[r, "Q^F \blacktriangleleft P", "\shortmid"{marking}] \arrow[d, equal] & \mathbf{B} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, phantom, "\text{lift}"] & \mathbf{A} \arrow[d, "F"] \\ \mathbf{C} \arrow[rr, "Q"', "\shortmid"{marking}] & & \mathbf{X} \end{tikzcd} \qquad (Q^F \blacktriangleleft P)(c,b) := \int_{a \in \mathbf{A}} Q(c,Fa) \blacktriangleleft P(b,a) \quad \text{in } \mathscr{V}$$

Here, $\begin{tikzcd} \mathscr{V} \arrow[r, shift left, "X \otimes -"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, shift left, "X \triangleright -"] \end{tikzcd}$, $\qquad \begin{tikzcd} \mathscr{V} \arrow[r, shift left, "- \otimes X"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, shift left, "- \blacktriangleleft X"] \end{tikzcd}$.

Lan/Ran

Definition (Koudenburg 2022)

1.
$$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[r, phantom, shift right=4, "\alpha"] & B \arrow[dl, "l"] \\ X & \end{tikzcd}$$
is a lan-cell $\overset{\text{def}}{\Leftrightarrow}$
$$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, dashed, "\vec{v}", "\shortmid"{marking}] \arrow[d, phantom, "\beta"] & Y \arrow[dll, "g"] \\ X & & \end{tikzcd} = \begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[r, phantom, shift right=4, "\alpha"] & B \arrow[r, dashed, "\vec{v}", "\shortmid"{marking}] \arrow[dl, "l"'] \arrow[r, phantom, shift right=4, "\exists! \bar{\beta}"] & Y \arrow[dll, "g"] \\ X & & \end{tikzcd}$$
(We say that α exhibits l as a *left Kan extension* of f along u.)

2.
$$\begin{tikzcd} B \arrow[r, "u", "\shortmid"{marking}] \arrow[dr, "r"'] \arrow[r, phantom, shift right=4, "\alpha"] & A \arrow[d, "f"] \\ & X \end{tikzcd}$$
is a ran-cell (the dual notion of lan-cells)

Lemma

$$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[r, phantom, shift right=4, "\alpha"] & B \arrow[dl, "l"] \\ X & \end{tikzcd}$$
is a lan-cell $\Leftrightarrow$
$$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[r, phantom, shift right=4, "\alpha"] & B \arrow[r, "l_*", "\shortmid"{marking}] \arrow[dl, "l"'] \arrow[r, phantom, shift right=4, "\text{cart}"] & X \arrow[dll, equal] \\ X & & \end{tikzcd}$$
is extending

Lan/Ran

Example in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

1. $$\begin{tikzcd} \mathbf{A} \arrow[r, "W", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\mathsf{lan}"] & \mathbf{1} \arrow[dl, "L"] \\ \mathbf{X} & \end{tikzcd} \quad \Leftrightarrow \quad L* \cong \operatorname*{Colim}_{a \in \mathbf{A}}^{Wa} Fa.$$ (W-weighted colimit of F)

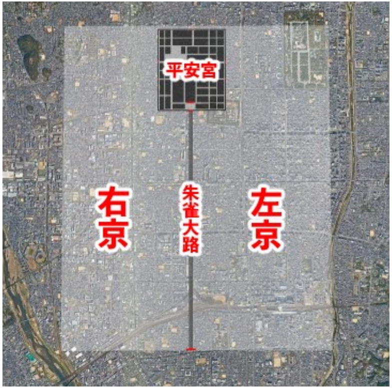
2. $$\begin{tikzcd} \mathbf{A} \arrow[r, "W", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\mathsf{lan}"] & \mathbf{B} \arrow[dl, "L"] \\ \mathbf{X} & \end{tikzcd} \quad \Leftrightarrow \quad \forall b \in \mathbf{B},\ \ Lb \cong \operatorname*{Colim}_{a \in \mathbf{A}}^{W(a,b)} Fa.$$ ($W(-,b)$-weighted colimit of F)

3. $$\begin{tikzcd} \mathbf{A} \arrow[r, "G_*", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\mathsf{lan}"] & \mathbf{B} \arrow[dl, "L"] \\ \mathbf{X} & \end{tikzcd} \quad \Leftrightarrow$$ L is a <u>pointwise</u> left Kan extension of F along G.

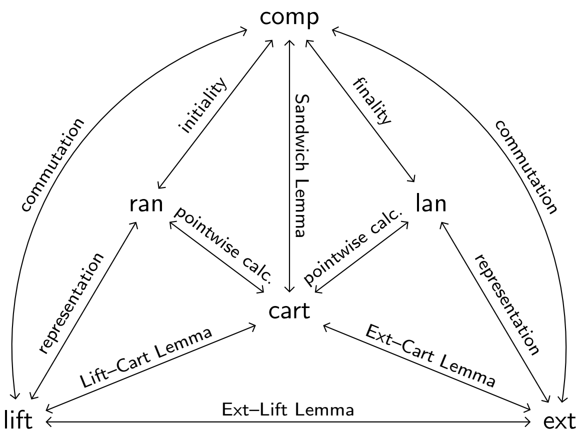
Lan(ran)-cells subsume <u>pointwise</u> Kan extensions and weighted (co)limits.

Why are left Kan extensions on the "right"?

$$\begin{tikzcd} D \arrow[r, "v", "\shortmid"{marking}] \arrow[dr, "r"'] & A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dl, phantom, "\text{ran}"] \arrow[dr, phantom, "\text{lan}"] & C \arrow[dl, "l"] \\ & X & \end{tikzcd}$$



Techniques in a virtual equipment



Formal category theory in a virtual equipment
= A *puzzle* to be solved using some lemmas and relationships as above.

Weights

Definition

$X \overset{u}{\longrightarrow\!\!\!\!\!\!\!+\;\;\,} Y$ is a left weight $\overset{\text{def}}{\Leftrightarrow}$

(LW1) For any $W \xrightarrow{v} X$, the composite $v \odot u$ exists.

$$\begin{tikzcd} W \arrow[r, "v", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\text{comp}"] & X \arrow[r, "u", "\shortmid"{marking}] & Y \arrow[d, equal] \\ W \arrow[rr, "v \odot u"', "\shortmid"{marking}] & & Y \end{tikzcd}$$

(LW2) For any $X \xrightarrow{f} Z$ and $Z \overset{v}{\cdots\!\!\!\to} W$, $u \rhd^f v$ exists.

$$\begin{tikzcd} X \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\text{ext}"] & Y \arrow[r, "u \rhd^f v", "\shortmid"{marking}] & W \arrow[d, equal] \\ Z \arrow[rr, dotted, "v"', "\shortmid"{marking}] & & W \end{tikzcd}$$

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

When $\mathscr{V}$ is itself a $\mathscr{V}$-category,

$\mathbf{A} \overset{\varphi}{\nrightarrow} \mathbf{B}$ is a left weight $\Leftrightarrow$ $\mathscr{V}$ has $\varphi(-, b)$-weighted limits and colimits $(\forall b \in \mathbf{B})$.

Given a left weight $X \overset{u}{\nrightarrow} Y$, we regard X as a "diagram," and u as "weights parametrized by Y."

(Co)completeness and (co)continuity

Definition

Φ: a class of left weights

1. X is Φ-cocomplete $\overset{\text{def}}{\Leftrightarrow}$
$$\begin{tikzcd} A \arrow[r, "\varphi", "\mapsto"'] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{lan}"] & B \arrow[dl, "\exists"] \\ X & \end{tikzcd} \quad (\forall \varphi \in \Phi,\ \forall f)$$

2. $\begin{array}{c} X \\ g\downarrow \\ Y \end{array}$ is Φ-cocontinuous $\overset{\text{def}}{\Leftrightarrow}$
$$\begin{tikzcd} A \arrow[r, "\varphi", "\mapsto"'] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{lan}"] & B \arrow[dl] \\ X \arrow[d, "g"'] & \\ Y & \end{tikzcd}$$
is a lan-cell $(\forall \varphi \in \Phi,\ \forall f)$.

A class Φ of left weights plays a role as a “shape” of colimits.

Dogmas

Definition

A class Φ of left weights is a left dogma (or, Φ is *saturated*) $\overset{\text{def}}{\Leftrightarrow}$

- $\mathsf{Id}_A \in \Phi$ $(\forall A)$;
- $\varphi, \varphi' \in \Phi \implies \varphi \odot \varphi' \in \Phi$;
- $f^* \in \Phi$ $(\forall f)$.

Φ^*: the smallest left dogma containing Φ

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

- $\mathbf{A} \overset{\psi}{\nrightarrow} \mathbf{B} \in \Phi^* \quad \Leftrightarrow \quad \psi(-, \forall b)$ lies in the itelated closure of $\{\mathrm{rep}\} \subset [\mathbf{A}^{\mathrm{op}}, \mathscr{V}]$ under Φ-colimits.
- Thus, Φ^* is the "*saturation*" of Φ.

Remark

In an arbitrary virtual equipment,

- Φ-cocomplete $\quad \Leftrightarrow \quad$ Φ^*-cocomplete
- Φ-cocontinuous $\quad \Leftrightarrow \quad$ Φ^*-cocontinuous

Commutation

$A_1 \overset{\varphi_1}{\nrightarrow} B_1$ preserves an extension
$$\begin{tikzcd} A_0 \arrow[r, "\varphi_0", "|"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\text{ext}"] & B_0 \arrow[r, "p", "|"{marking}] & A_1 \arrow[d, equal] \\ X \arrow[rr, dotted, "u"', "|"{marking}] & & A_1 \end{tikzcd}$$

$\overset{\text{def}}{\Leftrightarrow}$ $\exists \alpha, \beta, \gamma$ s.t.
$$\begin{tikzcd} A_0 \arrow[r, "\varphi_0", "|"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\text{ext}"] & B_0 \arrow[r, "p", "|"{marking}] & A_1 \arrow[r, "\varphi_1", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & B_1 \arrow[d, equal] \\ X \arrow[rr, dotted, "u"', "|"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\alpha:\ \text{comp}"] & & A_1 \arrow[r, "\varphi_1", "|"{marking}] & B_1 \arrow[d, equal] \\ X \arrow[rrr, "|"{marking}] & & & B_1 \end{tikzcd} = \begin{tikzcd} A_0 \arrow[r, "\varphi_0", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & B_0 \arrow[r, "p", "|"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\beta:\ \text{comp}"] & A_1 \arrow[r, "\varphi_1", "|"{marking}] & B_1 \arrow[d, equal] \\ A_0 \arrow[r, "\varphi_0", "|"{marking}] \arrow[d, "f"'] \arrow[drrr, phantom, "\gamma:\ \text{ext}"] & B_0 \arrow[rr, "|"{marking}] & & B_1 \arrow[d, equal] \\ X \arrow[rrr, "|"{marking}] & & & B_1 \end{tikzcd}$$

Definition

- A pair (φ_0, φ_1) of left weights commutes ($\varphi_0 \mathbin{/\!\!/} \varphi_1$) $\overset{\text{def}}{\Leftrightarrow}$ $A_1 \overset{\varphi_1}{\nrightarrow} B_1$ *preserves*
$$\begin{tikzcd} A_0 \arrow[r, "\varphi_0", "|"{marking}] \arrow[d, "\forall f"'] \arrow[drr, phantom, "\text{ext}"] & B_0 \arrow[r, "|"{marking}] & A_1 \arrow[d, equal] \\ X \arrow[rr, dotted, "\forall u"', "|"{marking}] & & A_1 \end{tikzcd}$$

- A pair (φ_0, φ_1) of l.w. weakly commutes (φ_0 / φ_1) $\overset{\text{def}}{\Leftrightarrow}$ $A_1 \overset{\varphi_1}{\nrightarrow} B_1$ *preserves*
$$\begin{tikzcd} A_0 \arrow[r, "\varphi_0", "|"{marking}] \arrow[dr, "\forall f"'] & B_0 \arrow[r, "|"{marking}] \arrow[d, phantom, "\text{ext}"] & A_1 \arrow[dl, equal] \\ & A_1 & \end{tikzcd}$$

Commutation

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

- $(\mathbf{A} \overset{\varphi}{\nrightarrow} \mathbf{B}) \mathbin{/\!/} (\mathbf{C} \overset{\psi}{\nrightarrow} \mathbf{D}) \iff$ φ-limits and ψ-colimits commute in $\mathscr{V}$.
  $\iff [\mathbf{C}, \mathscr{V}] \xrightarrow{\mathrm{Colim}^{\psi(-,d)}} \mathscr{V}$ preserves φ-limits.
- $(\mathbf{A} \overset{\varphi}{\nrightarrow} \mathbf{B}) \mathbin{/} (\mathbf{C} \overset{\psi}{\nrightarrow} \mathbf{D}) \iff [\mathbf{C}, \mathscr{V}] \xrightarrow{\mathrm{Colim}^{\psi(-,d)}} \mathscr{V}$ preserves φ-limits of representables.

Notation

Φ: a class of left weights. $\Phi_{/\!/}$ and $\Phi_{/}$ denote the classes of left weights defined by the following:

$$\Phi_{/\!/} \ni \varphi' \overset{\text{def}}{\iff} \varphi \mathbin{/\!/} \varphi' \text{ for all } \varphi \in \Phi;$$

$$\Phi_{/} \ni \varphi' \overset{\text{def}}{\iff} \varphi \mathbin{/} \varphi' \text{ for all } \varphi \in \Phi.$$

Remark

$\Phi_{/\!/}$ and $\Phi_{/}$ become left dogmas.

Soundness

Definition (Adámek, Borceux, et al. 2002)

A class Φ of left weights is sound $\overset{\text{def}}{\Leftrightarrow}$ $\Phi_{/\!/} = \Phi_{/}$

Φ: sound $\rightsquigarrow$ Theory of $\Phi_{/\!/}(=\Phi_{/})$-accessible categories behaves well.

Example in $\mathbf{Set}$-$\mathbb{P}\mathrm{rof}$

A class $\mathrm{Fin} = \{$left weights of finite (co)limits$\}$ is sound.
Then, $\mathrm{Fin}_{/\!/} = \mathrm{Fin}_{/} = \{$l.w. of filtered colim$\}$.

1. The ordinary accessibility
2. Virtual equipments
3. Formal category theory in a virtual equipment
4. Classes of weights
5. Ind-completions

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

$$\begin{array}{c}\mathbf{A}\\ {\scriptstyle よ}\downarrow\\ \mathbf{Ind}_\Phi(\mathbf{A})\end{array} = \{\varphi\colon \mathbf{A}^{\mathrm{op}} \to \mathscr{V} \ \text{ in } \Phi\} \quad : \text{ the free cocompletion of } \mathbf{A} \text{ under } \Phi.$$

Then,

$$\begin{array}{c}\mathbf{B}\\ {\scriptstyle F}\downarrow\\ \mathbf{Ind}_\Phi(\mathbf{A})\end{array} \quad \Bigg\| \quad \mathbf{A}^{\mathrm{op}} \otimes \mathbf{B} \xrightarrow{F} \mathscr{V} \text{ s.t. } F(-, \forall b) \in \Phi \quad \Bigg\| \quad \mathbf{A} \overset{F}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbf{B} \quad \text{in } \Phi$$

Definition

Φ: a left dogma on $\mathbb{E}$.

$\begin{array}{c}A\\ {\scriptstyle k}\downarrow\\ X\end{array}$ is a Φ-ind-morphism $\overset{\text{def}}{\Leftrightarrow}$ k yields the adj equiv: $\mathbf{Hom}_{\mathbb{E}}\left(\begin{smallmatrix}B\\X\end{smallmatrix}\right) \underset{\mathrm{Lan}_{-}k}{\overset{X(k,-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathbf{Hom}_\Phi(A, B)$. $(\forall B \in \mathbb{E})$

$\Leftrightarrow$

- $A \overset{k_*}{\nrightarrow} X$ belongs to Φ.
- $$\begin{tikzcd} A \arrow[r, "k_*", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\mathrm{cart}"] & X \arrow[dl, equal] \\ X & \end{tikzcd}$$ is also a lan-cell.
- Every $A \overset{\varphi}{\nrightarrow} \cdot$ in Φ is a restriction of k and some f. $$\begin{tikzcd} A \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\mathrm{cart}"] & \cdot \arrow[dl, "f"] \\ X & \end{tikzcd}$$

Remark

Φ-ind-morphisms are a Φ-modified version of *Yoneda morphisms* in the sense of (Koudenburg 2022).

Ψ: a right dogma $\rightsquigarrow$ Ψ-pro-morphisms (the dual notion of ind-morphisms)

Remark

$A \longrightarrow X_i$ $(i = 0, 1)$: Φ-ind-morphisms $\Longrightarrow$ $X_0 \simeq X_1$ (equiv in the vertical 2-category)

Notation

$A \longrightarrow \Phi^{\triangledown} A$: a Φ-ind-morphism, $A \longrightarrow \Psi^{\blacktriangledown} A$: a Ψ-pro-morphism.

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

- $\mathbf{A} \xrightarrow{\text{よ}} \{\mathbf{A} \overset{\varphi}{\nrightarrow} \mathbf{1} \text{ in } \Phi\}$ $(\subseteq [\mathbf{A}^{\mathrm{op}}, \mathscr{V}])$ is a Φ-ind-morphism. $\rightsquigarrow$ Φ-*cocompletion*
- $\mathbf{A} \xrightarrow{\text{よ}^{\mathrm{op}}} \{\mathbf{1} \overset{\psi}{\nrightarrow} \mathbf{A} \text{ in } \Psi\}$ $(\subseteq [\mathbf{A}, \mathscr{V}]^{\mathrm{op}})$ is a Ψ-pro-morphism. $\rightsquigarrow$ Ψ-*completion*

Characterization of ind-morphisms

Φ: a left dogma.

Definition

$X \xrightarrow{f} Y$ is **Φ-atomic** $\overset{\text{def}}{\Leftrightarrow}$ $\forall A \overset{\forall\varphi}{\nrightarrow} \forall B$ in Φ, $A \xrightarrow{\forall g} Y$, $\exists \alpha, \beta$ s.t.

$$
\begin{tikzcd}
X \arrow[r, "Y(f{,}g)", "|" marking] \arrow[dr, "f"'] & A \arrow[r, "\varphi", "|" marking] \arrow[d, "g"', "\text{cart}"', "\text{lan}"] & B \arrow[dl] \\
& Y &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "Y(f{,}g)", "|" marking] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "\alpha\colon \text{comp}"] & A \arrow[r, "\varphi", "|" marking] & B \arrow[d, equal] \\
X \arrow[rr, "|" marking] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=4ex, "\beta\colon \text{cart}"] & & B \arrow[dl] \\
& Y &
\end{tikzcd}
$$

whenever lan exists.

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

$\mathbf{X} \xrightarrow{F} \mathbf{Y}$ is Φ-atomic $\Leftrightarrow$ $\forall x \in \mathbf{X}$, $\mathbf{Y}(Fx, -)\colon \mathbf{Y} \to \mathscr{V}$ is Φ-cocontinuous.

Definition

$X \xrightarrow{f} Y$ is **fully faithful** $\overset{\text{def}}{\Leftrightarrow}$ The canonical cell
$\begin{tikzcd}
X \arrow[rr, equal, "|" marking] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=3ex, "="] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$
is cartesian.

Characterization of ind-morphisms

Φ: a left dogma.

Theorem

$A \xrightarrow{k} X$ is a Φ-ind-morphism. $\Leftrightarrow$

- X is Φ-cocomplete;
- k is Φ-atomic and fully faithful;
- For any $Y \xrightarrow{f} X$, there exist B, $B \overset{\varphi}{\nrightarrow} Y$ in Φ, $B \xrightarrow{g} A$, and a lan-cell:

$$\begin{tikzcd} B \arrow[r, "\varphi", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\mathrm{lan}"] & Y \arrow[ddl, "f"] \\ A \arrow[d, "k"'] & \\ X & \end{tikzcd}$$

The 3rd condition says that "every $x \in X$ is a Φ-colimit of Φ-atomic objects."

The "functor" $A \mapsto \Phi^{\nabla}A$

Question

- Does the assignment $A \mapsto \Phi^{\nabla}A$ yields a "functor" ?
- What are the domain and codomain of Φ^{∇} ?
- What is the universality of Φ^{∇} ?

Φ^∇ behaves like a left adjoint

Observation 1

$$\frac{A \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} UB \quad \text{in } \Phi}{\Phi^\nabla A \overset{\hat\varphi}{\longleftarrow} B} \quad \text{(by def. of } \Phi\text{-ind-mor.)}$$

$\rightsquigarrow \quad \Phi^\nabla \dashv U\ ? \qquad (U : B \mapsto B)$

Observation 2

$$\begin{array}{c|c} A & \Phi^\nabla A \\ f\downarrow & \downarrow \hat f \text{: } \Phi\text{-cocts} \\ UB & B \end{array} \quad (\Phi^\nabla \text{ is a "}\Phi\text{-cocompletion."})$$

Definition

Φ: a left dogma on $\mathbb{E}$.

The pseudo-double category $\mathbb{E}_\Phi$:

- object $\cdots$ the same as $\mathbb{E}$
- vert.arrow $\cdots$ the same as $\mathbb{E}$
- hor.arrow $\cdots$ hor.arrow in Φ
- cell $\cdots$ the same as $\mathbb{E}$

fullsub $\mathbb{E}_\Phi{}' := \{A \mid \Phi^\nabla A \text{ exists}\} \subseteq \mathbb{E}_\Phi$.

The (strict) dbl cat $\mathbb{Q}\Phi^\nabla$ (*quinted const.*):

- object $\cdots$ Φ-cocomplete object in $\mathbb{E}$
- vert.arrow $\cdots$ Φ-cocts vert.arrow in $\mathbb{E}$
- hor.arrow $X \looparrowright Y$ $\cdots$ vert.arrow $X \leftarrow Y$
- cell $\begin{array}{ccc} X & \overset{u}{\looparrowright} & Y \\ f\downarrow & \alpha & \downarrow g \\ Z & \underset{v}{\looparrowright} & W \end{array}$ $\cdots$ $\begin{array}{ccccc} & & Y & & \\ & \swarrow^{u} & & \searrow^{g} & \\ X & & \alpha & & W \\ & \searrow_{f} & & \swarrow_{v} & \\ & & Z & & \end{array}$ in $\mathbb{E}$

$$\begin{tikzcd}
{\mathbb{E}_\Phi}' \arrow[dr, "\Phi^\nabla"'] & \subseteq & \mathbb{E}_\Phi \\
& \arrow[ur, phantom, "\eta\,\Searrow"] & \mathbb{Q}\Phi^\nabla \arrow[u, "U"']
\end{tikzcd}$$

$$\begin{array}{c} A \\ \eta_A\big\downarrow \\ U\Phi^\nabla A \end{array} \quad := \quad \begin{array}{c} A \\ a\big\downarrow \\ \Phi^\nabla A \end{array}$$ (a Φ-ind-morphism in $\mathbb{E}$)

Definition of Φ^∇

- $A \quad \mapsto \quad \Phi^\nabla A$

- $\begin{array}{c} A \\ f\big\downarrow \\ B \end{array} \quad \mapsto$
$$\begin{tikzcd}
A \arrow[r, "a_*", "\shortmid"{marking}] \arrow[d, "f"'] & \Phi^\nabla A \arrow[ddl, "\Phi^\nabla f"] \\
B \arrow[d, "b"'] \arrow[ur, phantom, "\mathrm{lan}"] & \\
\Phi^\nabla B &
\end{tikzcd}$$
in $\mathbb{E}$

- $A \overset{u}{\nrightarrow} B \quad \mapsto$
$$\begin{tikzcd}
A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "a"'] & B \arrow[r, "b_*", "\shortmid"{marking}] \arrow[dl, phantom, "\mathrm{lan}"] & \Phi^\nabla B \arrow[dll, "\Phi^\nabla u"] \\
\Phi^\nabla A & &
\end{tikzcd}$$
in $\mathbb{E}$

Definition of U

$$\begin{tikzcd}
X \arrow[d, "f"'] & Y \arrow[l, "u"', "\shortmid"{marking}] \arrow[d, "g"] \\
Z & W \arrow[l, "v", "\shortmid"{marking}]
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
X \arrow[r, "u^*", "\shortmid"{marking}] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
Z \arrow[r, "v^*"', "\shortmid"{marking}] & W
\end{tikzcd}$$
in $\mathbb{E}$

Relative company-biadjoints

We fix the following data:

- pseudo-double categories $\mathbb{A}'$, $\mathbb{A}$, and $\mathbb{B}$;
- "pseudo-double functors" $\mathbb{A}' \xrightarrow{I} \mathbb{A}$, $\mathbb{A}' \xrightarrow{F} \mathbb{B}$, and $\mathbb{B} \xrightarrow{G} \mathbb{A}$;
- a pseudo-vertical trans $I \overset{\eta}{\Rightarrow} GF$ whose components have companions.

$$\begin{tikzcd}
\mathbb{A}' \arrow[r, "I"] \arrow[dr, "F"'] & \mathbb{A} \arrow[d, phantom, "\Downarrow\eta" near start, shift right=4] \\
& \mathbb{B} \arrow[u, "G"']
\end{tikzcd}$$

(HTrans): The companions η_{A*} of each components of η yields a horizontal trans $I \overset{\eta_*}{\Rightarrow} GF$.

$$\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right): \quad \begin{tikzcd} FA \arrow[d, "f"'] \\ B \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} IA \arrow[d, "\hat{f}"'] \\ GB \end{tikzcd} \qquad \left(\begin{smallmatrix} F & G \end{smallmatrix}\right): \quad \frac{FA \overset{u}{\nrightarrow} B}{IA \overset{\hat{u}}{\nrightarrow} GB}$$

$$\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right): \quad \begin{tikzcd} FA \arrow[r, "u", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B_0 \arrow[d, "g"] \\ B_1 \arrow[r, "v"', "|" marking] & B_2 \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} IA \arrow[r, "\hat{u}", "|" marking] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\hat{\alpha}"] & GB_0 \arrow[d, "Gg"] \\ GB_1 \arrow[r, "Gv"', "|" marking] & GB_2 \end{tikzcd}$$

$$\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right): \quad \begin{tikzcd} FA_0 \arrow[r, "Fu", "|" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "\alpha"] & FA_1 \arrow[d, "g"] \\ FA_2 \arrow[r, "v"', "|" marking] & B \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} IA_0 \arrow[r, "Iu", "|" marking] \arrow[d, "If"'] \arrow[dr, phantom, "\hat{\alpha}"] & IA_1 \arrow[d, "\hat{g}"] \\ IA_2 \arrow[r, "\hat{v}"', "|" marking] & GB \end{tikzcd}$$

$$\left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right): \quad \begin{tikzcd} FA_0 \arrow[r, "u", "|" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "\alpha"] & B_0 \arrow[d, "g"] \\ FA_1 \arrow[r, "v"', "|" marking] & B_1 \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} IA_0 \arrow[r, "\hat{u}", "|" marking] \arrow[d, "If"'] \arrow[dr, phantom, "\hat{\alpha}"] & GB_0 \arrow[d, "Gg"] \\ IA_1 \arrow[r, "\hat{v}"', "|" marking] & GB_1 \end{tikzcd}$$

$$\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right): \quad \begin{tikzcd} FA_0 \arrow[r, "Fu", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & FA_1 \arrow[d, "g"] \\ B_0 \arrow[r, "v"', "|" marking] & B_1 \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} IA_0 \arrow[r, "Iu", "|" marking] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\hat{\alpha}"] & IA_1 \arrow[d, "\hat{g}"] \\ GB_0 \arrow[r, "Gv"', "|" marking] & GB_1 \end{tikzcd}$$

Relationship among the 7 axioms

Under $\left(\begin{smallmatrix}F\\G\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}F & G\end{smallmatrix}\right)$, implications of the following directions hold.

$$\begin{tikzcd}
\left(\begin{smallmatrix}F & G\\G & G\end{smallmatrix}\right) \arrow[rr] \arrow[dd] & & \left(\begin{smallmatrix}F & G\\F & G\end{smallmatrix}\right) \arrow[dl, bend left] \\
& \text{(HTrans)} \arrow[dl, bend right] & \\
\left(\begin{smallmatrix}F & F\\G & G\end{smallmatrix}\right) \arrow[rr] & & \left(\begin{smallmatrix}F & F\\F & G\end{smallmatrix}\right) \arrow[ul, bend right]
\end{tikzcd}$$

Definition

$$\begin{tikzcd}
\mathbb{A}' \arrow[r, "I"] \arrow[dr, "F"'] & \mathbb{A} \arrow[d, phantom, "\Swarrow\eta" near start, shift right=3ex] \\
& \mathbb{B} \arrow[u, "G"']
\end{tikzcd}$$

forms an I-relative company-biadjunction $\overset{\text{def}}{\Leftrightarrow}$ $\left(\begin{smallmatrix}F\\G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F & G\end{smallmatrix}\right)$, (HTrans), and $\left(\begin{smallmatrix}F & G\\G & G\end{smallmatrix}\right)$ hold.

$\Leftrightarrow$ All the 7 axioms hold.

Theorem

$$\begin{tikzcd}
{\mathbb{E}_{\Phi}}' \arrow[r, phantom, "\subseteq"] \arrow[dr, "\Phi^{\nabla}"'] & \mathbb{E}_{\Phi} \arrow[d, phantom, "\Swarrow\eta" near start, shift right=3ex] \\
& \mathbb{Q}\Phi^{\nabla} \arrow[u, "U"']
\end{tikzcd}$$

forms a relative company-biadjunction.

Nerves and realizations

Theorem (Companion theorem)

$$\begin{tikzcd} \mathbb{A}' \arrow[r, "I"] \arrow[dr, "F"'] & \mathbb{A} \arrow[d, phantom, "\overset{\eta}{\Downarrow}" near start] \\ & \mathbb{B} \arrow[u, "G"'] \end{tikzcd} : \text{rel.comp-biadj} \implies$$

1. $\begin{array}{c} FA \\ f\downarrow \\ B \end{array}$ has a companion $\Leftrightarrow$ $\begin{array}{c} IA \\ \hat{f}\downarrow \\ GB \end{array}$ has a companion.
2. $FA \overset{u}{\nrightarrow} B$ is a companion $\Leftrightarrow$ $IA \overset{\hat{u}}{\nrightarrow} GB$ is a companion.

Corollary

1. $\begin{tikzcd} A \arrow[r, "a_*", "\mid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\text{lan}" near start] & \Phi^\nabla A \arrow[dl, "l"] \\ \underset{\Phi\text{-cocpl}}{\underline{E}} & \end{tikzcd}$ Then, $f_* \in \Phi \quad \Leftrightarrow \quad l$ has a right adjoint.

2. $\begin{tikzcd} A \arrow[r, "\varphi\in\Phi", "\mid" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\text{lan}" near start] & \underset{\Phi\text{-cocpl}}{\underline{E}} \arrow[dl, "r"] \\ \Phi^\nabla A & \end{tikzcd}$ Then, φ is a companion $\quad \Leftrightarrow \quad r$ has a left adjoint.

Ongoing works

- Restricting the relative company-biadj $\begin{tikzcd} {\mathbb{E}_{\Phi}}' \arrow[dr, "\Phi^{\nabla}"'] & \subseteq & \mathbb{E}_{\Phi} \\ & \mathbb{Q}\Phi^{\nabla} \arrow[ur, phantom, "\Searrow\eta"] \arrow[ur, "U"'] & \end{tikzcd}$ to *Cauchy-cpl* obj

  $\rightsquigarrow$ We would get a "duality" subsuming existing dualities.

- Developing theory of $\Phi_{/\!\!/}$-accessible objects for a sound class Φ
  $\rightsquigarrow$ formal theory of locally presentable categories
- Exploring virtual equipments $\mathbb{E}$ that provide interesting duality
- Removing the condition about the existence of units from $\mathbb{E}$
  $\rightsquigarrow$ non-locally-small categories
- Double categories where vertical arrows can be composited only up to isomorphism
- Clarifying the relationship between composing cells and bi-virtual double categories
- Developing technical lemmas for "universal" cells (e.g. Sandwich Lemma)
- Solving more puzzles!

Thank you!

Today's slide

My homepage

Hoshino's homepage

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Adjunctions of weights

Definition

A (horizontal) adjunction $(\psi \dashv \varphi)$ consists of:

- Horizontal arrows $Y \overset{\psi}{\nrightarrow} X$, $X \overset{\varphi}{\nrightarrow} Y$;
- Cells
$$\begin{tikzcd} & Y \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\ Y \arrow[ur, equal] \arrow[rr, "\psi\odot\varphi"', "|"] & {} & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi", "|"'] \arrow[dr, equal] & Y \arrow[r, "\psi", "|"'] \arrow[d, phantom, "\varepsilon"] & X \arrow[dl, equal] \\ & X & \end{tikzcd}$$
s.t. the *zigzag identities* hold.

Theorem

$\psi \dashv \varphi$: hor.adj. Then, ψ: a right weight $\Leftrightarrow$ φ: a left weight.

Definition

A left weight $X \overset{\varphi}{\nrightarrow} Y$ is left-absolute $\overset{\text{def}}{\Leftrightarrow}$ "φ-colimits are always absolute."

Theorem (Street 1983)

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$, $X \overset{\varphi}{\nrightarrow} Y$ has a left adjoint $\Leftrightarrow$ φ is left-absolute.

Street's characterization in a virtual equipment

Definition

$\mathbb{E}$ has anti-restrictions $\overset{\text{def}}{\Leftrightarrow}$ For every $X \overset{u}{\nrightarrow} Y$,

$$\begin{tikzcd} X \arrow[rr, "u", "\shortmid"{marking}] \arrow[dr, "\exists"'] \arrow[rr, phantom, shift right=4ex, "\exists\text{cart}"] & & Y \arrow[dl, "\exists"] \\ & \exists Z & \end{tikzcd}$$

Theorem

$X \overset{\varphi}{\nrightarrow} Y$: a left weight

$$\begin{tikzcd}[column sep=small] \exists\psi \dashv \varphi \arrow[d, Rightarrow] & \\ \varphi : \text{ left-absolute} \arrow[r, Rightarrow, "\text{if } \mathbb{E} \text{ has anti-res. and } \Phi^{\triangledown}X\ (\varphi \in \Phi) \text{ exists.}"'] & \varphi\,/\,\varphi \text{ (weakly comm.)} \arrow[ul, Rightarrow] \end{tikzcd}$$