Monads, partial algebras, and partial logic

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Abstract

Goal

(classical) universal algebra = study of algebras on $\mathbf{Set}$

$\downarrow$generalize

relativized universal algebra = study of algebras on $\mathscr{A}$

($\mathscr{A}$: a locally presentable category)

In this generalization,

classical algebra		relativized algebra
finite-arity	$\rightsquigarrow$	*κ-arity* (κ: a regular cardinal)
total function	$\rightsquigarrow$	*partial function*
equation	$\rightsquigarrow$	*Horn sequent*

Our relativized algebra will be written in (infinitary) partial Horn logic [PV07].

Finally, we will give a *generalized Birkhoff's variety theorem*.

Partial function

A small category $\mathscr{C}$ consists of...

- a set $\cdots$ $\mathrm{ob}\mathscr{C}$ ("objects"),
- a set $\cdots$ $\mathrm{mor}\mathscr{C}$ ("morphisms"),
- a function $\cdots$ $\mathrm{id}\colon \mathrm{ob}\mathscr{C} \to \mathrm{mor}\mathscr{C}$ ("identities"),
- a function $\cdots$ $\mathrm{d}\colon \mathrm{mor}\mathscr{C} \to \mathrm{ob}\mathscr{C}$ ("domain"),
- a function $\cdots$ $\mathrm{c}\colon \mathrm{mor}\mathscr{C} \to \mathrm{ob}\mathscr{C}$ ("codomain"), and
- a partial function $\cdots$ $\circ\colon \mathrm{mor}\mathscr{C} \times \mathrm{mor}\mathscr{C} \rightharpoonup \mathrm{mor}\mathscr{C}$ ("composition").

- We can define "the theory of small categories" as a partial Horn theory.
- Partial Horn theory = a logical theory which can deal with partial functions (and relations).

Multi-sorted signature

Definition

S: the set of sorts. κ: an infinite regular cardinal.
An S-sorted (κ-ary) signature Σ consists of:

- function symbols $f, f', f'', \ldots$
- relation symbols $R, R', R'', \ldots$
- arity of each function symbol $f\colon \sqcap_{i<\alpha} s_i \to s,\ f'\colon \sqcap \cdots$
- arity of each relation symbol $R\colon \sqcap_{j<\beta} s_j,\ R'\colon \sqcap \cdots$

where $\alpha, \beta < \kappa$ and $s_i, s_j, s \in S$.

From now on, we fix κ.

Partial Horn theory

Σ: an S-sorted signature.

- A term $\tau ::= x \mid f(\tau_i)_{i<\alpha}$;
- A Horn formula $\varphi ::= \top \mid \bigwedge_{i<\alpha} \varphi_i \mid \tau = \tau' \mid R(\tau_i)_{i<\alpha}$;
- A context $\cdots$ $\vec{x} = (x_i)_{i<\alpha}$ (a family of distinct variables).

Here, $\alpha < \kappa$. The notation $\vec{x}.\varphi$ [resp. $\vec{x}.\tau$] means that all variables of φ [τ] are in the context $\vec{x}$. (*Horn formula [term]-in-context*)

Definition

1. A Horn sequent over Σ is an expression of the form

$$\varphi \vdash_{\vec{x}} \psi \qquad (\text{“}\varphi \text{ implies } \psi\text{”})$$

   (φ, ψ are Horn formulas over Σ in the same context $\vec{x}$.)
2. A partial Horn theory $\mathbb{T}$ over Σ is a set of Horn sequents over Σ.

Horn vs partial Horn

What is the difference between ordinary Horn theory and partial Horn theory?
$\rightsquigarrow$ It lies in the concept of models.

	(ordinary) Horn theory	partial Horn theory
Axiom	Horn sequent $\varphi \vdash_{\vec{x}} \psi$	Horn sequent $\varphi \vdash_{\vec{x}} \psi$
Interpretation of func.symb.	total map $M_{\vec{s}} \xrightarrow{[\![f]\!]_M} M_s$	partial map $M_{\vec{s}} \xrightarrow{[\![f]\!]_M} M_s$
Interpretation of rel.symb.	subset $[\![R]\!]_M \subseteq M_{\vec{s}}$	subset $[\![R]\!]_M \subseteq M_{\vec{s}}$
Validity of φ	"φ holds."	"All terms in φ are defined and φ holds."
Validity of $\varphi \vdash_{\vec{x}} \psi$	"If φ holds then ψ holds."	"If all terms in φ are defined and φ holds, then all terms in ψ are defined and ψ holds."

Especially,

An equation $\tau = \tau$ holds iff the value of the partial map $[\![\tau]\!]_M$ is defined.

So, we will use the abbreviation $\tau\!\downarrow$ for $\tau = \tau$.

Homomorphisms

Definition

$\mathbb{T}$: a partial Horn theory. M, N: (partial) models of $\mathbb{T}$.

A homomorphism $M \xrightarrow{h} N$ is a (total) map $M \xrightarrow{h} N$ such that for each f, R:

- if $\vec{a} \in [\![R]\!]_M$, then $h(\vec{a}) \in [\![R]\!]_N$;
- if $[\![f]\!]_M(\vec{a})$ is defined, then so is $[\![f]\!]_N(h(\vec{a}))$ and

$$h([\![f]\!]_M(\vec{a})) = [\![f]\!]_N(h(\vec{a})).$$

Notation

$\mathbb{T}$: a partial Horn theory.

$\mathbf{PMod}\,\mathbb{T}$: the category of (partial) models of $\mathbb{T}$ and homomorphisms.

Example: small categories

Example (small categories)

We can define the partial Horn theory $\mathbb{T}_{\mathrm{cat}}$ of small categories as follows:
The $S := \{\mathrm{ob}, \mathrm{mor}\}$-sorted signature Σ_{cat} consists of:

$$\mathrm{id}\colon \mathrm{ob} \to \mathrm{mor}, \quad \mathrm{d}\colon \mathrm{mor} \to \mathrm{ob}, \quad \mathrm{c}\colon \mathrm{mor} \to \mathrm{ob}, \quad \circ\colon \mathrm{mor} \sqcap \mathrm{mor} \to \mathrm{mor}.$$

The partial Horn theory $\mathbb{T}_{\mathrm{cat}}$ over Σ_{cat} consists of:

$$\top \xmapsto{x\,:\,\mathrm{ob}} \mathrm{id}(x)\downarrow, \qquad (\text{ id is total. })$$

$$\top \xmapsto{f\,:\,\mathrm{mor}} \mathrm{d}(f)\downarrow \wedge \mathrm{c}(f)\downarrow, \qquad (\text{ d and c are total. })$$

$$\begin{aligned} \mathrm{d}(g) = \mathrm{c}(f) &\xmapsto{g, f\,:\,\mathrm{mor}} (g \circ f)\downarrow, \\ (g \circ f)\downarrow &\xmapsto{g, f\,:\,\mathrm{mor}} \mathrm{d}(g) = \mathrm{c}(f), \end{aligned} \qquad (\ g \circ f \text{ is defined iff } \mathrm{d}(g) = \mathrm{c}(f).\ )$$

and so on.
$\rightsquigarrow$ We have $\mathbf{PMod}\,\mathbb{T}_{\mathrm{cat}} \cong \mathbf{Cat}$.

Example: posets

Example (posets)

We present the partial Horn theory $\mathbb{T}_{\mathrm{pos}}$ of posets. Let $S := \{*\}$, $\Sigma_{\mathrm{pos}} := \{\leq : * \sqcap *\}$. The partial Horn theory $\mathbb{T}_{\mathrm{pos}}$ over Σ_{pos} consists of:

$$\top \xmapsto{\;x\;} x \leq x, \quad x \leq y \wedge y \leq x \xmapsto{\;x,y\;} x = y, \quad x \leq y \wedge y \leq z \xmapsto{\;x,y,z\;} x \leq z.$$

Then, we have $\mathbf{PMod}\,\mathbb{T}_{\mathrm{pos}} \cong \mathbf{Pos}$.

Example: groups or empty

Example

$$S := \{*\}, \quad \Sigma_{\mathrm{grp}_\varnothing} := \left\{ \begin{matrix} e \colon () \to *, \quad ()^{-1} \colon * \to *; \\ \cdot : * \sqcap * \to *. \end{matrix} \right\}$$

The partial Horn theory $\mathbb{T}_{\mathrm{grp}_\varnothing}$ over $\Sigma_{\mathrm{grp}_\varnothing}$ consists of:

$$\top \vdash^{x,y} (x \cdot y)\downarrow, \qquad \top \vdash^{x} x^{-1}\downarrow;$$
$$\top \vdash^{x,y,z} (x \cdot y) \cdot z = x \cdot (y \cdot z);$$
$$\top \vdash^{x} e \cdot x = x \wedge x \cdot e = x;$$
$$\top \vdash^{x} x^{-1} \cdot x = e \wedge x \cdot x^{-1} = e.$$

Then, a $\mathbb{T}_{\mathrm{grp}_\varnothing}$-model is either a group or the empty set.

Inference rules

$$\frac{}{\varphi \vdash_{\vec{x}} \varphi}\ (\mathsf{Id}) \qquad \frac{\varphi \vdash_{\vec{x}} \psi \quad \psi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \chi}\ (\mathsf{Cut})$$

$$\frac{\varphi \vdash_{\vec{x}} \psi}{\varphi(\vec{\tau}/\vec{x}) \wedge \bigwedge_{i<\alpha} \tau_i{\downarrow} \vdash_{\vec{y}} \psi(\vec{\tau}/\vec{x})}\ (\mathsf{Subst}) \qquad \frac{}{\top \vdash_{\vec{x}} x_i{\downarrow}}\ (\mathsf{Refl})$$

$$\frac{}{\varphi \wedge \bigwedge_{i<\alpha} x_i = y_i \vdash_{\vec{z}} \varphi(\vec{y}/\vec{x})}\ (\mathsf{Eq}) \qquad \frac{}{R(\tau_i)_{i<\alpha} \vdash_{\vec{x}} \tau_j{\downarrow}}\ (\mathsf{SRel})$$

$$\frac{}{\tau = \sigma \vdash_{\vec{x}} \tau{\downarrow} \wedge \sigma{\downarrow}}\ (\mathsf{SEq}) \qquad \frac{}{f(\tau_i)_{i<\alpha} \vdash_{\vec{x}} \tau_j{\downarrow}}\ (\mathsf{SFun})$$

$$\frac{}{\bigwedge_{i<\alpha} \varphi_i \vdash_{\vec{x}} \varphi_j}\ (\mathsf{EConj}) \qquad \frac{(\varphi \vdash_{\vec{x}} \psi_i)_{i<\alpha}}{\varphi \vdash_{\vec{x}} \bigwedge_{i<\alpha} \psi_i}\ (\mathsf{IConj})$$

Derivation

Definition

A derivation from $\mathbb{T}$ is a well-founded rooted tree of Horn sequents such that:

- for every node, the number of its children is less than κ;
- for every node, the pair of its children and itself exhibits an inference rule or axiom of $\mathbb{T}$.

$$\dfrac{\dfrac{\vdots}{\varphi_0 \xmapsto{\vec{x}_0} \psi_0} \quad \dfrac{\vdots}{\varphi_1 \xmapsto{\vec{x}_1} \psi_1} \quad \dots \quad \dfrac{\vdots}{\varphi_i \xmapsto{\vec{x}_i} \psi_i} \quad \dots \quad (i < \alpha)}{\varphi \xmapsto{\vec{x}} \psi}$$

Here $\alpha < \kappa$.

Definition

A Horn sequent $\varphi \xmapsto{\vec{x}} \psi$ is called a PHL_κ-theorem of $\mathbb{T}$ and written as

$$\mathbb{T} \vdash (\varphi \xmapsto{\vec{x}} \psi)$$

if there exists a derivation of $\varphi \xmapsto{\vec{x}} \psi$ from $\mathbb{T}$.

Cut rule

Lemma (The cut rule)

$$\dfrac{(\varphi_i \overset{\vec{x}}{\vdash} \psi_i)_{i<\alpha} \qquad \chi \wedge \bigwedge_{i<\alpha} \psi_i \overset{\vec{x}}{\vdash} \theta}{\chi \wedge \bigwedge_{i<\alpha} \varphi_i \overset{\vec{x}}{\vdash} \theta}$$

Proof.

$$\dfrac{\dfrac{}{\chi \wedge \bigwedge_{i<\alpha} \varphi_i \overset{\vec{x}}{\vdash} \varphi_j}\,(\mathrm{EConj}) \qquad \varphi_j \overset{\vec{x}}{\vdash} \psi_j}{\chi \wedge \bigwedge_{i<\alpha} \varphi_i \overset{\vec{x}}{\vdash} \psi_j}\,(\mathrm{Cut})$$

$$\dfrac{\dfrac{\dfrac{(\chi \wedge \bigwedge_{i<\alpha} \varphi_i \overset{\vec{x}}{\vdash} \psi_j)_{j<\alpha}}{\chi \wedge \bigwedge_{i<\alpha} \varphi_i \overset{\vec{x}}{\vdash} \bigwedge_{j<\alpha} \psi_j}\,(\mathrm{IConj}) \qquad \dfrac{}{\chi \wedge \bigwedge_{i<\alpha} \varphi_i \overset{\vec{x}}{\vdash} \chi}\,(\mathrm{EConj})}{\chi \wedge \bigwedge_{i<\alpha} \varphi_i \overset{\vec{x}}{\vdash} \chi \wedge \bigwedge_{i<\alpha} \psi_i}\,(\mathrm{IConj}) \qquad \chi \wedge \bigwedge_{i<\alpha} \psi_i \overset{\vec{x}}{\vdash} \theta}{\chi \wedge \bigwedge_{i<\alpha} \varphi_i \overset{\vec{x}}{\vdash} \theta}\,(\mathrm{Cut})$$

□

$\mathbb{T}$-terms

Definition

1. A term $\vec{x}.\tau$ is called a $\mathbb{T}$-term generated by $\vec{x}.\varphi$ if

$$\mathbb{T} \vdash (\varphi \vdash\!\xrightarrow{\vec{x}} \tau\downarrow).$$

2. $\vec{x}.\tau$, $\vec{x}.\tau'$: $\mathbb{T}$-terms generated by $\vec{x}.\varphi$ of the same type.

$$\vec{x}.\tau \approx_{\mathbb{T}} \vec{x}.\tau' \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathbb{T} \vdash (\varphi \vdash\!\xrightarrow{\vec{x}} \tau = \tau')$$

Notation

$(S, \Sigma, \mathbb{T})$: a partial Horn theory.

$\mathbb{T}\text{-Term}(\vec{x}.\varphi)$: the S-sorted set of all $\mathbb{T}$-terms generated by $\vec{x}.\varphi$.

$\langle \vec{x}.\varphi \rangle_{\mathbb{T}} := \mathbb{T}\text{-Term}(\vec{x}.\varphi)/\approx_{\mathbb{T}}$

Proposition

$\langle \vec{x}.\varphi \rangle_{\mathbb{T}}$ becomes a (partial) $\mathbb{T}$-model.

Completeness theorem

Lemma

For any Horn formula $\vec{y}.\psi$,

$$[\![\vec{y}.\psi]\!]_{\langle\vec{x}.\varphi\rangle_{\mathbb{T}}} \ni ([\vec{x}.\tau_j])_j \quad \Leftrightarrow \quad \mathbb{T} \vdash \left(\varphi \xmapsto{\vec{x}} \psi(\vec{\tau}/\vec{y})\right).$$

Completeness theorem of PHL

$\mathbb{T}$: a partial Horn theory. For any Horn sequent $\varphi \xmapsto{\vec{x}} \psi$, TFAE:

1. $\mathbb{T} \vdash (\varphi \xmapsto{\vec{x}} \psi)$.
2. $\mathbb{T} \vDash (\varphi \xmapsto{\vec{x}} \psi)$, i.e., for every $M \in \mathbf{PMod}\,\mathbb{T}$, $M \vDash (\varphi \xmapsto{\vec{x}} \psi)$.

Proof.

[2 $\Longrightarrow$ 1] By assumption, we particularly get $\langle\vec{x}.\varphi\rangle_{\mathbb{T}} \vDash (\varphi \xmapsto{\vec{x}} \psi)$.
By Lemma, $\mathbb{T} \vdash (\varphi \xmapsto{\vec{x}} \varphi(\vec{\tau}/\vec{x}))$ implies $\mathbb{T} \vdash (\varphi \xmapsto{\vec{x}} \psi(\vec{\tau}/\vec{x})) \quad (\forall\vec{\tau})$.
Taking $\vec{\tau} \mapsto \vec{x}$, we have $\mathbb{T} \vdash (\varphi \xmapsto{\vec{x}} \psi)$. $\square$

Multi-sorted algebras

Definition

An S-sorted algebraic theory (also called an equational theory) consists of:

- Ω: a set of operators;
- arity $\omega \colon s_1 \sqcap \cdots \sqcap s_n \to s$ for each $\omega \in \Omega$;
- E: a set of equations.

Here $s_i, s \in S$.

Example (Chain complexes)

$$S := \mathbb{Z}, \qquad \Omega := \left\{ \begin{aligned} 0_n \colon () \to n, &\quad -_n \colon n \to n; \\ +_n \colon (n,n) \to n, &\quad d_n \colon n \to n+1. \end{aligned} \right\}$$

$$E := \{ 0_n +_n x \overset{x:n}{=} x, \quad \text{and so on.} \}$$

Finitary monads

Definition

A monad on a category $\mathscr{C}$ consists of:

- a functor $T : \mathscr{C} \to \mathscr{C}$,
- a natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow T$,
- a natural transformation $\mu : T^2 (:= T \circ T) \Rightarrow T$

such that the following commute.

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

Definition

A monad (T, η, μ) is finitary when the functor T preserves filtered colimits.

Linton's theorem

Theorem ([Lin69])

There is an equivalence
$$\mathbf{Th}^S \simeq \mathbf{Mnd}_{\mathrm{f}}(\mathbf{Set}^S).$$

Here,

$\mathbf{Th}^S$: the category of S-sorted algebraic theories,

$\mathbf{Mnd}_{\mathrm{f}}(\mathbf{Set}^S)$: the category of finitary monads on $\mathbf{Set}^S$.

S-sorted algebraic theory $=$ finitary monad on $\mathbf{Set}^S$

$\downarrow$ generalize

??? $\mathbb{S}$-relative algebraic theory $=$ κ-ary monad on $\mathbf{PMod}\,\mathbb{S}$

($\mathbb{S}$: a κ-ary partial Horn theory)

Relative algebraic theories

$(S, \Sigma, \mathbb{S})$: a κ-ary partial Horn theory

Definition ([Kaw23a; Kaw24])

An $\mathbb{S}$-relative algebraic theory consists of:

- Ω: a set of partial operators,
- E: a set of Horn sequents over $\Sigma + \Omega$

such that

- For each operator $\omega \in \Omega$, its domain is defined by a Horn formula φ_ω over Σ. (arity)
- For each sequent in E, its precondition is over Σ. (must contain no operator in Ω)

Definition ([Kaw23a; Kaw24])

(Ω, E): an $\mathbb{S}$-relative alg.theory.
An (Ω, E)-algebra $\mathbb{A}$ is an $\mathbb{S}$-model A with interpretation of all $\omega \in \Omega$ that satisfies all axioms in E.

$$\prod_{i<\alpha} A_{s_i} \supseteq [\![\vec{x}.\varphi_\omega]\!]_A \xrightarrow{[\![\omega]\!]_{\mathbb{A}}} A_s$$

Via theory morphism

Definition

A theory morphism (translation) $(S, \Sigma, \mathbb{T}) \xrightarrow{\rho} (S', \Sigma', \mathbb{T}')$ consists of the following assignments:

- $S \ni s \quad \mapsto \quad s^\rho \in S'$;
- $f\colon \sqcap_{i<\alpha} s_i \to s$ in $\Sigma \quad \mapsto \quad$ a term $\vec{x}^\rho . f^\rho$ of type s^ρ over Σ';
- $R\colon \sqcap_{i<\alpha} s_i$ in $\Sigma \quad \mapsto \quad$ a Horn formula $\vec{x}^\rho . R^\rho$ in Σ'

s.t. for every $\varphi \vdash_{\vec{x}} \psi$ in $\mathbb{T}$, $\mathbb{T}' \vdash (\varphi^\rho \vdash_{\vec{x}^\rho} \psi^\rho)$.
$(\vec{x}^\rho = (x_i^\rho : s_i^\rho)_{i<\alpha})$

Remark

An $(S, \Sigma, \mathbb{S})$-relative algebraic theory is precisely an extension $(S, \Sigma, \mathbb{S}) \xrightarrow{\rho} (S, \Sigma + \Omega, \mathbb{S} + E)$ such that

- Ω contains no relation symbol;
- E consists of:
  - $\omega(\vec{x}){\downarrow} \vdash_{\vec{x}} \varphi_\omega$ (φ_ω: over Σ) for each $\omega \in \Omega$;
  - Horn sequents $\varphi \vdash_{\vec{x}} \psi$ (φ: over Σ, ψ: over $\Sigma + \Omega$).

Ordinary algebra vs relative algebra

$(S, \Sigma, \mathbb{S})$: a partial Horn theory.

	S-sorted algebraic theory (Ω, E)	$\mathbb{S}$-relative algebraic theory (Ω, E)
Base category	$\mathbf{Set}^S$	$\mathbf{PMod}\,\mathbb{S}$
Operator	$(s_i)_i \xrightarrow{\omega} s$	$(x_i{:}s_i)_i.\ \underset{\text{over } \Sigma}{\underline{\varphi}} \xrightarrow{\omega} s$
Axiom	equation $\tau = \tau'$	$\underset{\text{over } \Sigma}{\underline{\varphi}} \vdash^{\vec{x}} \psi$

$$\begin{tikzcd} \mathbf{Alg}(\Omega, E) \arrow[d, bend left, "U"] \\ \mathbf{Set}^S \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Alg}(\Omega, E) \arrow[d, bend left, "U"] \\ \mathbf{PMod}\,\mathbb{S} \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

In $\mathbb{S}$-relative algebraic theory ...

- Each operator $\omega \in \Omega$ needs not be total, but its domain must be defined by "$\mathbb{S}$'s language."
- We can use (Horn) implications as axioms, but its precondition must not contain any operator $\omega \in \Omega$. Preconditions must be written in "$\mathbb{S}$'s language."

Examples

Example (Small categories)

The finitary partial Horn theory $\mathbb{S}_{\mathrm{quiv}}$ for quivers is given by:

$$S_{\mathrm{quiv}} := \{e, v\}, \quad \Sigma_{\mathrm{quiv}} := \{s, t \colon e \to v\}, \quad \mathbb{S}_{\mathrm{quiv}} := \{\top \xvdash{f:e} s(f)\downarrow \wedge t(f)\downarrow\}.$$

We define an $\mathbb{S}_{\mathrm{quiv}}$-relative finitary algebraic theory (Ω, E) as follows:

Ω :

	arity	type
$\circ$	$(g,f{:}e).s(g) = t(f)$	e
id	$(x{:}v).\top$	e

$$E := \left\{ \begin{array}{c} \top \xvdash{x:v} s(\mathrm{id}(x)) = x \wedge t(\mathrm{id}(x)) = x, \\ s(g) = t(f) \xvdash{g,f:e} s(g \circ f) = s(f) \wedge t(g \circ f) = t(g), \\ \top \xvdash{f:e} f \circ \mathrm{id}(s(f)) = f \wedge \mathrm{id}(t(f)) \circ f = f, \\ s(h) = t(g) \wedge s(g) = t(f) \xvdash{h,g,f:e} (h \circ g) \circ f = h \circ (g \circ f) \end{array} \right\}$$

Then, we have $\mathbf{Alg}(\Omega, E) \cong \mathbf{Cat}$.

Example (ω-cpos)

Let $\mathbb{S}_{\mathrm{pos}}$ be the partial Horn theory of posets. In what follows, we regard $\mathbb{S}_{\mathrm{pos}}$ as an $\aleph_1$-ary partial Horn theory. We present an $\mathbb{S}_{\mathrm{pos}}$-relative $\aleph_1$-ary algebraic theory (Ω, E) for *ω-cpos*. Let $\Omega := \{\sup\}$ with

$$\mathrm{ar}(\sup) := (x_n)_{n<\omega}. \bigwedge_{n<\omega} x_n \leq x_{n+1}, \quad \mathrm{type}(\sup) := *.$$

The set E is defined by the following:

$$E := \left\{ \begin{array}{c} \displaystyle\bigwedge_{n<\omega} x_n \leq x_{n+1} \overset{(x_n)_{n<\omega}}{\vdash\!\!\!-\!\!\!-\!\!\!-} \bigwedge_{n<\omega} x_n \leq \sup(\vec{x}); \\ \displaystyle\bigwedge_{n<\omega} x_n \leq x_{n+1} \wedge \bigwedge_{n<\omega} x_n \leq y \overset{(x_n)_{n<\omega}, y}{\vdash\!\!\!-\!\!\!-\!\!\!-} \sup(\vec{x}) \leq y \end{array} \right\}$$

Then, an (Ω, E)-algebra is precisely an *ω-cpo*, i.e., a poset where every ω-chain has a supremum.

A generalized Linton's theorem

Theorem ([Kaw23a; Kaw24])

There is an equivalence

$$\mathbf{Th}^{\mathbb{S}}_{\kappa} \simeq \mathbf{Mnd}_{\kappa}(\mathbf{PMod}\,\mathbb{S}).$$

Here,

$\mathbf{Th}^{\mathbb{S}}$: the category of $\mathbb{S}$-relative (κ-ary) algebraic theories,

$\mathbf{Mnd}_{\kappa}(\mathbf{PMod}\,\mathbb{S})$: the category of κ-ary monads on $\mathbf{PMod}\,\mathbb{S}$.

$\uparrow$ generalize

Recall (Linton's theorem)

$$\mathbf{Th}^{S} \simeq \mathbf{Mnd}_{\aleph_0}(\mathbf{Set}^{S}).$$

S-sorted algebraic theory $= (S, \varnothing, \varnothing)$-relative algebraic theory

Birkhoff's variety theorem

Birkhoff's variety theorem [Bir35]

(Ω, E): a single-sorted algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable by equations.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under *products*, *subobjects*, and *quotients*.

closed under products: $A_i \in \mathscr{E} \implies \prod_i A_i \in \mathscr{E}$.

closed under subobjects: $B \subseteq A$: sub, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

closed under quotients: $A \twoheadrightarrow B$: surj, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

Birkhoff-type theorem for partial Horn logic

Theorem ([Kaw23a; Kaw24])

$\mathbb{S} \xrightarrow{\rho} \mathbb{T}$: a theory morphism between (κ-ary) partial Horn theories.
For every $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$, TFAE:

1. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is definable by (κ-ary) Horn sequents in the form $\varphi^\rho \xmapsto{\vec{x}^\rho} \psi$.
2. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$*-closed subobjects*, U^ρ*-retracts*, and κ*-filtered colimits*.

Remark

1. For example, $\mathbb{T}_{\mathrm{pos}}$-closed sub $=$ embedding.
2. $\mathbf{PMod}\,\mathbb{S} \xleftarrow{U^\rho} \mathbf{PMod}\,\mathbb{T}$ is the ρ-translation functor.

   q in $\mathbf{PMod}\,\mathbb{T}$ is a *U^ρ-retraction* $\quad\overset{\text{def}}{\Leftrightarrow}\quad U^\rho(q)$ is a retraction.

Taking ρ to be $(S, \varnothing, \varnothing) \to (S, \Sigma, \mathbb{T})$:

Corollary

$\mathbb{T}$: a (κ-ary) partial Horn theory. For every $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$, TFAE:

1. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is definable by (κ-ary) Horn formulas.
2. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$-*closed subobjects*, *surjections*, and κ-*filtered colimits*.

Taking ρ to be $\mathbb{T} \xrightarrow{\mathrm{id}} \mathbb{T}$:

Corollary

$\mathbb{T}$: a (κ-ary) partial Horn theory. For every $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$, TFAE:

1. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is definable by (κ-ary) Horn sequents.
2. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$-*closed subobjects*, and κ-*filtered colimits*.

Axiomatizability of groups

$$S := \{*\},\ \Sigma_{\mathrm{mon}} := \{e\colon () \to *,\ \cdot\colon * \sqcap * \to *\},$$

$$\mathbb{T}_{\mathrm{mon}} := \left\{ \begin{array}{c} \top \vdash^{} e{\downarrow}, \quad \top \vdash^{x,y} x \cdot y{\downarrow}, \\ \top \vdash^{x,y,z} (x \cdot y) \cdot z = x \cdot (y \cdot z), \\ \top \vdash^{x} x \cdot e = x = e \cdot x \end{array} \right\}$$

Then, we have $\mathbf{PMod}\,\mathbb{T}_{\mathrm{mon}} \cong \mathbf{Mon}$.

The inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ in $\mathbf{Mon}$ is $\mathbb{T}_{\mathrm{mon}}$-closed.

$\{\text{groups}\} \subseteq \mathbf{PMod}\,\mathbb{T}_{\mathrm{mon}}$ is not closed under $\mathbb{T}_{\mathrm{mon}}$-closed sub.

$\downarrow$

$\{\text{groups}\} \subseteq \mathbf{PMod}\,\mathbb{T}_{\mathrm{mon}}$ is not axiomatizable.

$$\Sigma'_{\mathrm{mon}} := \Sigma_{\mathrm{mon}} + \{\bullet^{-1}\colon * \to *\},$$

$$\mathbb{T}'_{\mathrm{mon}} := \mathbb{T}_{\mathrm{mon}} + \left\{ \begin{array}{c} x^{-1}{\downarrow} \vdash^{x} x^{-1} \cdot x = e = x \cdot x^{-1}, \\ x \cdot y = e = y \cdot x \vdash^{x,y} x^{-1} = y \end{array} \right\}$$

Then, we have $\mathbf{PMod}\,\mathbb{T}'_{\mathrm{mon}} \cong \mathbf{Mon}$.

The inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ in $\mathbf{Mon}$ is not $\mathbb{T}'_{\mathrm{mon}}$-closed.

$\{\text{groups}\} \subseteq \mathbf{PMod}\,\mathbb{T}'_{\mathrm{mon}}$ is closed under $\mathbb{T}'_{\mathrm{mon}}$-closed sub.

$\downarrow$

$\{\text{groups}\} \subseteq \mathbf{PMod}\,\mathbb{T}'_{\mathrm{mon}}$ is axiomatizable.

Birkhoff's theorem for relativized algebra

Taking ρ to be a relative algebraic theory:

Corollary

(Ω, E): an $\mathbb{S}$-relative algebraic theory. For every $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$, TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under *products*, $\mathbb{S}$-*closed subobjects*, U-*retracts*, and κ-*filtered colimits*.

single-sorted alg. ($(1, \varnothing, \varnothing)$-rel.alg.)		$\mathbb{S}$-relative alg.
products	$\rightsquigarrow$	products
subobjects	$\rightsquigarrow$	$\mathbb{S}$-*closed subobjects*
quotients	$\rightsquigarrow$	U-*retracts*
	$\rightsquigarrow$	κ-*filtered colimits* (new)

The filtered colimit elimination problem

Question

Why can the closure property under filtered colimits be eliminated in the case of single-sorted algebra?

Answer

Because the category $\mathbf{Set}$ satisfies a "noetherian" condition.

Definition ([Kaw23b])

A category $\mathscr{A}$ satisfies the ascending chain condition (ACC) if it has no chain $A_0 \to A_1 \to A_2 \to \cdots$ of objects such that there is no morphism $A_n \leftarrow A_{n+1}$ for all n.

Filtered colimit elimination

Recall

$\mathbb{S} \xrightarrow{\rho} \mathbb{T}$: a theory morphism. TFAE:

1. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is definable by (κ-ary) Horn sequents in the form $\varphi^\rho \vdash_{\vec{x}^\rho} \psi$.
2. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$*-closed subobjects*, U^ρ*-retracts*, and κ*-filtered colimits*.

$\downarrow$ Assuming that $\mathbf{PMod}\,\mathbb{S}$ satisfies ACC

Theorem ([Kaw23b; Kaw24])

also equivalent to the following:

3. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$*-closed subobjects*, and (U^ρ, κ)*-local retracts*.

TFAE:

1. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is definable by (κ-ary) Horn formulas.
2. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$-*closed subobjects*, *surjections*, and κ-*filtered colimits*.

$\downarrow$ Assuming that $\mathbf{Set}^S$ satisfies ACC

also equivalent to the following:

3. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$-*closed subobjects*, and *surjections*.

TFAE:

1. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is definable by (κ-ary) Horn sequents.
2. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$-*closed subobjects*, and κ-*filtered colimits*.

$\downarrow$ Assuming that $\mathbf{PMod}\,\mathbb{T}$ satisfies ACC

also equivalent to the following:

3. $\mathscr{E} \subseteq \mathbf{PMod}\,\mathbb{T}$ is closed under *products*, $\mathbb{T}$-*closed subobjects*, and κ-*local retracts*.

Recall

(Ω, E): an $\mathbb{S}$-relative algebraic theory. TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under *products*, $\mathbb{S}$*-closed subobjects*, U*-retracts*, and κ*-filtered colimits*.

$\downarrow$ Assuming that $\mathbf{PMod}\,\mathbb{S}$ satisfies ACC

Theorem ([Kaw23b; Kaw24])

also equivalent to the following:

3. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under *products*, $\mathbb{S}$*-closed subobjects*, and (U, κ)*-local retracts*.

This subsumes Birkhoff-type theorems for the following algebras:

- Finite-sorted algebra $= (n, \varnothing, \varnothing)$-relative algebra,
- Ordered algebra $= \mathbb{T}_{\mathrm{pos}}$-relative algebra,
- Metric algebra, etc.

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