DOUBLE CATEGORIES OF PROFUNCTORS (DRAFT)

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Abstract. We give an axiomatization of virtual double categories of enriched profunctors.

Contents

1. Introductions

Remark 1.1. For clarity, let us declare the sizes of the categories we treat. We fix three Grothendieck universes $\mathscr{U}_0 \subseteq \mathscr{U}_1 \in \mathscr{U}_2$. Elements in $\mathscr{U}_0$ are called *small*, elements in $\mathscr{U}_1$ are called *large*, elements in $\mathscr{U}_2$ are called *huge*. Arbitrary sets (not necessarily in $\mathscr{U}_0$ nor $\mathscr{U}_1$ nor $\mathscr{U}_2$) are called *classes*. ♦

2. Preliminaries

2.1. Augmented virtual double categories.

2.1.1. *The 2-category of augmented virtual double categories.*

Definition 2.1 ([Kou20]). An augmented virtual double category (AVDC) $\mathbb{L}$ consists of the following data:

- A class $\mathrm{Ob}\mathbb{L}$, whose elements are called *objects* in $\mathbb{L}$. We write $A \in \mathbb{L}$ to mean $A \in \mathrm{Ob}\mathbb{L}$.
- For $A, B \in \mathbb{L}$, a class $\mathrm{Hom}_{\mathbb{L}}\binom{A}{B}$, whose elements are called *vertical arrows* from A to B in $\mathbb{L}$. The objects and the vertical arrows are supposed to form a category $\mathbf{V}\mathbb{L}$, the

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vertical category of $\mathbb{L}$. We write id_A for the identity on an object $A \in \mathbb{L}$. The composite of $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{V}\mathbb{L}$ is denoted by $f \mathbin{⨾} g$. Vertical arrows are often written vertically:

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[d, equal, "\mathsf{id}_A"] & \text{in } \mathbb{L} \\
B & A &
\end{tikzcd}$$

- For $A, B \in \mathbb{L}$, a class $\mathrm{Hom}_{\mathbb{L}}(A, B)$, whose elements are called *horizontal arrows* from A to B in $\mathbb{L}$. A horizontal arrow is denoted by $\xrightarrow{\shortmid}$ and is often written horizontally. A path of horizontal arrows $A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} \cdots \xrightarrow{u_n} A_n$ is called a *horizontal path* of length n and is often denoted by $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$. We write $A \dashrightarrow B$ for a horizontal path of length 0 or 1.
- A class $\mathsf{Cell}_{\mathbb{L}}\left(f \overset{\vec{u}}{\underset{v}{\,}} g\right)$, whose elements are called *cells*, for each "boundary" formed by horizontal arrows and vertical arrows in the following way:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] & A_n \arrow[d, "g"] & \text{in } \mathbb{L}. \\
B \arrow[r, dotted, "v"'] & C &
\end{tikzcd}$$

Cells where v is of length 1 [resp. 0] are called *unary* [resp. *nullary*].
- Two kinds of special cells:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel_u"] & B \arrow[d, equal] & A \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "f"] \arrow[d, phantom, "=_f"] \\
A \arrow[r, "u"'] & B & B
\end{tikzcd}$$

These cells are called *identity cells*. For a horizontal path $A \overset{u}{\dashrightarrow} A$ of length 0, we also write $\shortparallel_u$ for $=_{\mathsf{id}_A}$.
- For cells $\alpha_1, \ldots, \alpha_n, \beta$ on the left below, a cell $\vec{\alpha} \mathbin{⨾} \beta$, of the following form:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] & & A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0 ⨾ g"'] & A_1 \arrow[r, dashed, "\vec{u}_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] & A_n \arrow[d, "f_n ⨾ h"] \\
B_0 \arrow[r, dotted, "v_1"] \arrow[d, "g"'] & B_1 \arrow[r, dotted, "v_2"] & \cdots \arrow[r, dotted, "v_n"] \arrow[d, phantom, "\beta"] & B_n \arrow[d, "h"] & \longmapsto & C \arrow[rrr, dotted, "w"'] & & \vec{\alpha} ⨾ \beta & D \\
C \arrow[rrr, dotted, "w"'] & & & D & & & & &
\end{tikzcd}$$

The composition defined by the assignments $(\alpha_1, \ldots, \alpha_n, \beta) \mapsto \vec{\alpha} \mathbin{⨾} \beta$ is required to satisfy a suitable associative law and a unit law with identity cells. See [Kou20] for more detail. ◆

Notation 2.2. Let $\alpha_1, \ldots, \alpha_n$ be cells in an AVDC of the following form:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, dotted, "v_1"'] & B_1 \arrow[r, dotted, "v_2"'] & \cdots \arrow[r, dotted, "v_n"'] & B_n
\end{tikzcd} \tag{1}$$

When the composite path $\vec{v}$ of $v_1, \cdots, v_n$ has length ≤ 1, we use the same notation (1) for the composite of the following cells:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, dotted, "v_1"] \arrow[d, equal] & B_1 \arrow[r, dotted, "v_2"] \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[r, dotted, "v_n"] & B_n \arrow[d, equal] \\
B_0 \arrow[rrr, dotted, "\vec{v}"'] & & & B_n
\end{tikzcd}$$

For example, we can get the following cell by the composition:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[dr, "f_0"'] \arrow[dr, phantom, "\alpha_1", shift left=3] & A_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"] \arrow[r, phantom, "\alpha_2"', shift right=3] & A_2 \arrow[dl, equal] \arrow[d, "f_3"] \arrow[dl, phantom, "\alpha_3"', shift right=3] \\
& A_2 \arrow[r, dotted, "v_3"'] & B_3
\end{tikzcd} \tag{2}$$

Note that the cell (2) coincides with another composite of the following cells.

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[dr, "f_0"'] \arrow[dr, phantom, "\alpha_1", shift left=3] & A_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"] \arrow[r, phantom, "\alpha_2"', shift right=3] & A_2 \arrow[dl, equal] \\
& A_2 \arrow[dl, equal] \arrow[dr, "f_3"] \arrow[d, phantom, "\alpha_3"] & \\
A_2 \arrow[rr, dotted, "v_3"'] & & B_3
\end{tikzcd}$$

◆

Notation 2.3. Let $\mathbb{L}$ be an AVDC. We write $\mathcal{V}\mathbb{L}$ for the 2-category defined as follows: The underlying category is $\mathbf{V}\mathbb{L}$; 2-cells are cells whose top and bottom boundaries are of length 0. The 2-category $\mathcal{V}\mathbb{L}$ is called the *vertical 2-category* of $\mathbb{L}$. ◆

Example 2.4. The AVDC $\mathbb{R}\mathrm{el}$ is defined as follows:

- An object is a (large) set.
- A vertical arrow is a map.
- A horizontal arrow $X \longrightarrow Y$ is a relation $R \subseteq X \times Y$.
- $\mathbb{R}\mathrm{el}$ has at most one cell for every boundary. A unary cell on the left below exists if and only if, for any $x_0 \in X_0, \ldots, x_n \in X_n$, the conjunction of $(x_0, x_1) \in R_1, \ldots, (x_{n-1}, x_n) \in R_n$ implies $(f(x_0), g(x_n)) \in S$. A nullary cell on the right below exists if and only if, for any $x_0 \in X_0, \ldots, x_n \in X_n$, the conjunction of $(x_0, x_1) \in R_1, \ldots, (x_{n-1}, x_n) \in R_n$ implies $f(x_0) = g(x_n)$.

$$\begin{tikzcd}
X_0 \arrow[r, dashed, "\vec{R}"] \arrow[d, "f"'] \arrow[dr, phantom, "\cdot"] & X_n \arrow[d, "g"] \\
Y \arrow[r, "S"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[rr, dashed, "\vec{R}"] \arrow[dr, "f"'] & \cdot & X_n \arrow[dl, "g"] \\
& Y &
\end{tikzcd}
\qquad \text{in } \mathbb{R}\mathrm{el}$$

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Definition 2.5 ([Kou20])**.** Let $\mathbb{K}$ and $\mathbb{L}$ be AVDCs. An *augmented virtual double (AVD)-functor* $\mathbb{K} \xrightarrow{F} \mathbb{L}$ consists of:

- a functor $F\colon \mathbf{V}\mathbb{K} \to \mathbf{V}\mathbb{L}$;

- assignments to horizontal arrows

$$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] & B \end{tikzcd} \quad \text{in } \mathbb{K} \quad \mapsto \quad \begin{tikzcd} FA \arrow[r, "Fu", "\shortmid"{marking}] & FB \end{tikzcd} \quad \text{in } \mathbb{L};$$

- assignments to cells

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "v"', "\shortmid"{marking}] & Y \end{tikzcd} \quad \text{in } \mathbb{K} \quad \mapsto \quad \begin{tikzcd} FA \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[d, "Fg"] \\ FX \arrow[r, dotted, "Fv"', "\shortmid"{marking}] & FY \end{tikzcd} \quad \text{in } \mathbb{L}$$

satisfying the following:

- For any composable cells

$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}_1", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{u}_2", "\shortmid"{marking}] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n", "\shortmid"{marking}] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\ B_0 \arrow[r, dotted, "v_1"', "\shortmid"{marking}] \arrow[d, "g"'] & B_1 \arrow[r, dotted, "v_2"', "\shortmid"{marking}] \arrow[d, phantom, "\beta"] & \cdots \arrow[r, dotted, "v_n"', "\shortmid"{marking}] & B_n \arrow[d, "h"] \\ X \arrow[rrr, dotted, "w"', "\shortmid"{marking}] & & & Y \end{tikzcd} = \begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}_1", "\shortmid"{marking}] \arrow[d, "f_0"'] & A_1 \arrow[r, dashed, "\vec{u}_2", "\shortmid"{marking}] & \cdots \arrow[r, dashed, "\vec{u}_n", "\shortmid"{marking}] & A_n \arrow[d, "f_n"] \\ B_0 \arrow[d, "g"'] & & \vec{\alpha}\fatsemi\beta & B_n \arrow[d, "h"] \\ X \arrow[rrr, dotted, "w"', "\shortmid"{marking}] & & & Y \end{tikzcd} \quad \text{in } \mathbb{K},$$

  the equality $F\vec{\alpha}\fatsemi F\beta = F(\vec{\alpha}\fatsemi\beta)$ holds.

$$\begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}_1", "\shortmid"{marking}] \arrow[d, "Ff_0"'] \arrow[dr, phantom, "F\alpha_1"] & FA_1 \arrow[r, dashed, "F\vec{u}_2", "\shortmid"{marking}] \arrow[d, "Ff_1"'] \arrow[dr, phantom, "F\alpha_2"] & \cdots \arrow[r, dashed, "F\vec{u}_n", "\shortmid"{marking}] \arrow[dr, phantom, "F\alpha_n"] & FA_n \arrow[d, "Ff_n"] \\ FB_0 \arrow[r, dotted, "Fv_1"', "\shortmid"{marking}] \arrow[d, "Fg"'] & FB_1 \arrow[r, dotted, "Fv_2"', "\shortmid"{marking}] \arrow[d, phantom, "F\beta"] & \cdots \arrow[r, dotted, "Fv_n"', "\shortmid"{marking}] & FB_n \arrow[d, "Fh"] \\ FX \arrow[rrr, dotted, "Fw"', "\shortmid"{marking}] & & & FY \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}_1", "\shortmid"{marking}] \arrow[d, "Ff_0"'] & FA_1 \arrow[r, dashed, "F\vec{u}_2", "\shortmid"{marking}] & \cdots \arrow[r, dashed, "F\vec{u}_n", "\shortmid"{marking}] & FA_n \arrow[d, "Ff_n"] \\ FB_0 \arrow[d, "Fg"'] & & F(\vec{\alpha}\fatsemi\beta) & FB_n \arrow[d, "Fh"] \\ FX \arrow[rrr, dotted, "Fw"', "\shortmid"{marking}] & & & FY \end{tikzcd} \quad \text{in } \mathbb{L}$$

- For any $A \xrightarrow{u} B$ in $\mathbb{K}$, the equality $F\|_u = \|_{Fu}$ holds.

$$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\|_u"] & B \arrow[d, equal] \\ A \arrow[r, "u"', "\shortmid"{marking}] & B \end{tikzcd} \mapsto \begin{tikzcd} FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F\|_u"] & FB \arrow[d, equal] \\ FA \arrow[r, "Fu"', "\shortmid"{marking}] & FB \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\|_{Fu}"] & FB \arrow[d, equal] \\ FA \arrow[r, "Fu"', "\shortmid"{marking}] & FB \end{tikzcd}$$

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$, the equality $F{=}_f = {=}_{Ff}$ holds.

$$\begin{tikzcd} A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "{=}_f"] \\ B \end{tikzcd} \mapsto \begin{tikzcd} FA \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "F{=}_f"] \\ FB \end{tikzcd} = \begin{tikzcd} FA \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "{=}_{Ff}"] \\ FB \end{tikzcd}$$

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Definition 2.6 ([Kou20])**.** Let $F, G\colon \mathbb{K} \to \mathbb{L}$ be AVD-functors between AVDCs. An *AVD-transformation* $F \overset{\rho}{\Longrightarrow} G$ consists of:

- for each $A \in \mathbb{K}$, a vertical arrow $\begin{tikzcd} FA \arrow[d, "\rho_A"'] \\ GA \end{tikzcd}$ in $\mathbb{L}$;

- for each $A \xrightarrow{u} B$ in $\mathbb{K}$, a cell $$\begin{tikzcd} FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "\rho_A"'] \arrow[dr, phantom, "\rho_u"] & FB \arrow[d, "\rho_B"] \\ GA \arrow[r, "Gu"', "\shortmid"{marking}] & GB \end{tikzcd}$$ in $\mathbb{L}$

satisfying the following:

- ρ yields a natural transformation $\begin{tikzcd} \mathbf{V}\mathbb{K} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\rho"] & \mathbf{V}\mathbb{L} \end{tikzcd}$, i.e., for any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$\begin{tikzcd} & FA \arrow[dl, "\rho_A"'] \arrow[dr, "Ff"] & \\ GA \arrow[dr, "Gf"'] & = & FB \arrow[dl, "\rho_B"] \\ & GB & \end{tikzcd}$$ in $\mathbb{L}$.

- For any unary cell

$$\begin{tikzcd} A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "u_2", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\ X \arrow[rrr, "v"', "\shortmid"{marking}] & & & Y \end{tikzcd}$$ in $\mathbb{K}$,

the following equality holds.

$$\begin{tikzcd} FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[d, "\rho_{A_0}"'] \arrow[dr, phantom, "\rho_{u_1}"] & FA_1 \arrow[r, "Fu_2", "\shortmid"{marking}] \arrow[d, "\rho_{A_1}"'] \arrow[dr, phantom, "\rho_{u_2}"] & \cdots \arrow[r, "Fu_n", "\shortmid"{marking}] \arrow[dr, phantom, "\rho_{u_n}"] & FA_n \arrow[d, "\rho_{A_n}"] \\ GA_0 \arrow[r, "Gu_1"', "\shortmid"{marking}] \arrow[d, "Gf"'] & GA_1 \arrow[r, "Gu_2"', "\shortmid"{marking}] \arrow[d, phantom, "G\alpha"] & \cdots \arrow[r, "Gu_n"', "\shortmid"{marking}] & GA_n \arrow[d, "Gg"] \\ GX \arrow[rrr, "Gv"', "\shortmid"{marking}] & & & GY \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[d, "Ff"'] & FA_1 \arrow[r, "Fu_2", "\shortmid"{marking}] \arrow[d, phantom, "F\alpha"] & \cdots \arrow[r, "Fu_n", "\shortmid"{marking}] & FA_n \arrow[d, "Fg"] \\ FX \arrow[rrr, "Fv"', "\shortmid"{marking}] \arrow[d, "\rho_X"'] & \arrow[d, phantom, "\rho_v"] & & FY \arrow[d, "\rho_Y"] \\ GX \arrow[rrr, "Gv"', "\shortmid"{marking}] & & & GY \end{tikzcd}$$

- For any nullary cell

$$\begin{tikzcd} A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[dr, "f"'] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & A_n \arrow[dl, "g"] \\ & X & \end{tikzcd}$$ in $\mathbb{K}$,

the following equality holds.

$$\begin{tikzcd} FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[d, "\rho_{A_0}"'] \arrow[dr, phantom, "\rho_{u_1}"] & \cdots \arrow[r, "Fu_n", "\shortmid"{marking}] \arrow[dr, phantom, "\rho_{u_n}"] & FA_n \arrow[d, "\rho_{A_n}"] \\ GA_0 \arrow[r, "Gu_1"', "\shortmid"{marking}] \arrow[dr, "Gf"'] & \cdots \arrow[r, "Gu_n"', "\shortmid"{marking}] \arrow[d, phantom, "G\alpha"] & GA_n \arrow[dl, "Gg"] \\ & GX & \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[dr, "Ff"'] & \cdots \arrow[r, "Fu_n", "\shortmid"{marking}] \arrow[d, phantom, "F\alpha"] & FA_n \arrow[dl, "Fg"] \\ & FX \arrow[d, bend right, "\rho_X"'] \arrow[d, bend left, "\rho_X"] \arrow[d, phantom, "="] & \\ & GX & \end{tikzcd}$$

◆

Notation 2.7. The huge AVDCs, AVD-functors, and AVD-transformations form a 2-category [Kou20], denoted by $\mathcal{AVDC}$. ◆

Definition 2.8. Let $\mathbb{L}$ be an AVDC. A *full sub-AVDC* of $\mathbb{L}$ is an AVDC whose class of objects is a subclass of $\mathrm{Ob}\mathbb{L}$ and whose "local" classes of vertical arrows, horizontal arrows, and cells are identical to those of $\mathbb{L}$. Additionally, all compositions and identities in the full sub-AVDC are inherited directly from $\mathbb{L}$. ◆

Notation 2.9. For an AVDC $\mathbb{L}$, let $\mathbf{V}^{\leq 1}\mathbb{L}$ denote a category defined as follows:

- An object is a horizontal path $A^0 \overset{A}{\dashrightarrow} A^1$ in $\mathbb{L}$ of length ≤ 1.
- A morphism from $A^0 \overset{A}{\dashrightarrow} A^1$ to $B^0 \overset{B}{\dashrightarrow} B^1$ is a tuple $(\alpha^0, \alpha^1, \alpha)$ of the following form:

$$\begin{tikzcd}
A^0 \arrow[r, "A", "|"{marking}, dotted] \arrow[d, "\alpha^0"'] \arrow[dr, phantom, "\alpha"] & A^1 \arrow[d, "\alpha^1"] \\
B^0 \arrow[r, "B"', "|"{marking}, dotted] & B^1
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

We write $\mathbf{V}^1\mathbb{L}$ for the full subcategory of $\mathbf{V}^{\leq 1}\mathbb{L}$ consisting of paths of length 1, i.e., horizontal arrows. ◆

Definition 2.10 (Vertically invertible cells)**.** Let $\mathbb{L}$ be an AVDC. Isomorphisms in the category $\mathbf{V}\mathbb{L}$ are called *invertible* vertical arrows. Isomorphisms in the category $\mathbf{V}^{\leq 1}\mathbb{L}$ are called *vertically invertible* cells and are often denoted by the symbol "v.inv" as follows:

$$\begin{tikzcd}
\cdot \arrow[r, "|"{marking}, dotted] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{v.inv}"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "|"{marking}, dotted] & \cdot
\end{tikzcd} \quad \text{in } \mathbb{L}$$

For a vertically invertible cell of the above form, the vertical arrows f and g become automatically invertible. ◆

Theorem 2.11 ([Kou20, 3.8. Proposition])**.** An AVD-functor $F \colon \mathbb{K} \to \mathbb{L}$ is a part of an equivalence in the 2-category $\mathcal{AVDC}$ if and only if it satisfies the following conditions:

- The assignments $\alpha \mapsto F\alpha$ induce bijections $\mathsf{Cell}_{\mathbb{K}}\left(f \ \substack{\vec{u} \\ \\ v} \ g \right) \cong \mathsf{Cell}_{\mathbb{L}}\left(Ff \ \substack{F\vec{u} \\ \\ Fv} \ Fg \right)$;
- The assignments $f \mapsto Ff$ induce bijections $\mathrm{Hom}_{\mathbb{K}}\left(\substack{A \\ B} \right) \cong \mathrm{Hom}_{\mathbb{L}}\left(\substack{FA \\ FB} \right)$;
- We can simultaneously make the following choices:
  - for each $A \in \mathbb{L}$, an object $A' \in \mathbb{K}$ and an invertible vertical arrow $FA' \xrightarrow{\varepsilon_A} A$ in $\mathbb{L}$;
  - for each $A \overset{u}{\longrightarrow} B$ in $\mathbb{L}$, a horizontal arrow $A' \overset{u'}{\longrightarrow} B'$ in $\mathbb{K}$ and a vertically invertible cell

$$\begin{tikzcd}
FA' \arrow[r, "Fu'", "|"{marking}] \arrow[d, "\varepsilon_A"'] \arrow[dr, phantom, "\mathsf{v.inv}"] & FB' \arrow[d, "\varepsilon_B"] \\
A \arrow[r, "u"', "|"{marking}] & B
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

2.1.2. *Cartesian cells.*

Definition 2.12 (Cartesian cells)**.** A cell

$$\begin{tikzcd}
X^0 \arrow[r, "X", "|"{marking}, dotted] \arrow[d, "\alpha^0"'] \arrow[dr, phantom, "\alpha"] & X^1 \arrow[d, "\alpha^1"] \\
Y^0 \arrow[r, "Y"', "|"{marking}, dotted] & Y^1
\end{tikzcd} \tag{3}$$

in an AVDC is called *cartesian* if it satisfies the following condition: Suppose that we are given a horizontal path $A \overset{\vec{u}}{\dashrightarrow} B$, vertical arrows $A \xrightarrow{f} X^0$ and $B \xrightarrow{g} X^1$, and a cell β on the right below; then there uniquely exists a cell γ satisfying the following equation.

$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[rd, phantom, "\gamma"] & B \arrow[d, "g"] & A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X^0 \arrow[r, dotted, "X", "\shortmid"{marking}] \arrow[d, "\alpha^0"'] \arrow[rd, phantom, "\alpha"] & X^1 \arrow[d, "\alpha^1"] \arrow[r, phantom, "="] & X^0 \arrow[r, phantom, "\beta"] \arrow[d, "\alpha^0"'] & X^1 \arrow[d, "\alpha^1"] \\
Y^0 \arrow[r, dotted, "Y"', "\shortmid"{marking}] & Y^1 & Y^0 \arrow[r, dotted, "Y"', "\shortmid"{marking}] & Y^1
\end{tikzcd}$$

We will use a symbol "cart" to represent a cartesian cell:

$$\begin{tikzcd}
\cdot \arrow[r, dotted, "\shortmid"{marking}] \arrow[d] \arrow[rd, phantom, "\mathsf{cart}"] & \cdot \arrow[d] \\
\cdot \arrow[r, dotted, "\shortmid"{marking}] & \cdot
\end{tikzcd}$$

♦

Proposition 2.13. Let α be a cell of the form (3) in an AVDC, and suppose that α^0 and α^1 are invertible. Then, the cell α is cartesian if and only if it is vertically invertible. In particular, every vertically invertible cell is cartesian.

Proof. Straightforward. □

Definition 2.14 (Restrictions)**.** Suppose that we are given a cartesian cell in an AVDC of the following form:

$$\begin{tikzcd}
\cdot \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[rd, phantom, "\mathsf{cart}"] & \cdot \arrow[d, "g"] \\
X \arrow[r, dotted, "u"', "\shortmid"{marking}] & Y
\end{tikzcd}$$

(i) Since the horizontal arrow p is unique up to vertically invertible cell, we call p the *restriction* of u along f and g and write $u(f,g)$ for it. When u has length 0 (hence $X = Y$), we also write $X(f,g)$ for p. To emphasize that u has length 1 [resp. 0], we sometimes call $u(f,g)$ the *unary restriction* [resp. *nullary restriction*].

$$\begin{tikzcd}
\cdot \arrow[r, "u(f{,}g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[rd, phantom, "\mathsf{cart}"] & \cdot \arrow[d, "g"] \\
X \arrow[r, dotted, "u"', "\shortmid"{marking}] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[rr, "X(f{,}g)", "\shortmid"{marking}] \arrow[rd, "f"'] & \mathsf{cart} & \cdot \arrow[ld, "g"] \\
& X &
\end{tikzcd}$$

(ii) When $g = \mathsf{id}$ and u has length 0, we call p the *companion* of f and write f_* for it. When $f = \mathsf{id}$ and u has length 0, we call p the *conjoint* of g and write g^* for it. We write $f_\dagger$ and $g^\dagger$ for the associated cartesian cells as follows:

$$\begin{tikzcd}
\cdot \arrow[rr, "f_*", "\shortmid"{marking}] \arrow[rd, "f"'] & f_\dagger & X \arrow[ld, equal] \\
& X &
\end{tikzcd} : \mathsf{cart}
\qquad
\begin{tikzcd}
X \arrow[rr, "g^*", "\shortmid"{marking}] \arrow[rd, equal] & g^\dagger & \cdot \arrow[ld, "g"] \\
& X &
\end{tikzcd} : \mathsf{cart}$$

(iii) When $f = g = \mathsf{id}$ and u has length 0, we call p the *horizontal unit* on X and write U_X for it. We write υ_X for the associated cartesian cell, which is vertically invertible, as follows:

$$\begin{tikzcd}
X \arrow[rr, "\mathsf{U}_X", "\shortmid"{marking}] \arrow[rd, equal] & \upsilon_X & X \arrow[ld, equal] \\
& X &
\end{tikzcd} : \mathsf{cart}$$

♦

Definition 2.15. Let $\mathbb{L}$ be an AVDC. We say $\mathbb{L}$ *has restrictions* [resp. *unary restrictions*] if the restriction $u(f,g)$ exists for any f, g, and u of length ≤ 1 [resp. length 1]. We say $\mathbb{L}$ *has companions* [resp. *conjoints*] if the companion f_* [resp. conjoint f^*] exists for any f. We say $\mathbb{L}$ *has horizontal units* if the horizontal unit U_X exists for any X. We refer to such $\mathbb{L}$ as an AVDC with restrictions, companions, etc. ♦

Remark 2.16. An AVDC with horizontal units, called a *unital AVDC* in [Kou20], can be identified with a *unital virtual double category* by the 2-equivalence as in [Kou20, 10.1. Theorem]. When we regard an AVDC with horizontal units as a unital virtual double category (VDC), the AVD-functors between them correspond to the *normal* virtual double (VD)-functors [CS10]. An AVDC with unary restrictions is called an *augmented virtual equipment*, and AVDC with restrictions is called a *unital virtual equipment* in [Kou20]. The latter can be identified with a *virtual equipment* [CS10] by the 2-equivalence. ♦

Proposition 2.17 ([Kou20, 5.4. Lemma])**.** Let $A \xrightarrow{f} X$ be a vertical arrow in an AVDC. Then, the following data correspond bijectively to each other:

(i) A pair (p, ε) of a horizontal arrow $A \xrightarrow{p} X$ and a cartesian cell

$$\begin{tikzcd} A \arrow[rr, "p", "\shortmid"{marking}] \arrow[dr, "f"'] & \arrow[d, phantom, "\varepsilon"] & X \arrow[dl, equal] \\ & X & \end{tikzcd} \quad : \mathsf{cart},$$

which gives a companion of f.

(ii) A tuple (p, η, ε) of a horizontal arrow $A \xrightarrow{p} X$ and cells η, ε satisfying the following equations:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, equal, "\eta"] \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] & X \\ X \arrow[ur, equal, "\varepsilon"] & \end{tikzcd} = \begin{tikzcd} A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ X \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"] & X \\ A \arrow[ur, equal, "\eta"'] \arrow[r, "p"', "\shortmid"{marking}] & X \arrow[ur, equal, "\varepsilon"] & \end{tikzcd} = \begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[d, equal] \\ A \arrow[r, "p"', "\shortmid"{marking}] & X \end{tikzcd}$$

Corollary 2.18 ([Kou20, 5.5. Corollary])**.** Companions, conjoints, horizontal units are preserved by any AVD-functor.

Definition 2.19. Let $A \overset{\vec{u}}{\dashrightarrow} B$ be a horizontal path in an AVDC $\mathbb{L}$. Let $\mathbf{C}$ be a category, and let $F \colon \mathbf{C} \to \mathbf{V}^{\leq 1}\mathbb{L}$ be a functor. A *cone* over F with the vertex $\vec{u}$ is a family of cells α_c for $c \in \mathbf{C}$ satisfying the following equality for any morphism $c \xrightarrow{s} d$ in $\mathbf{C}$:

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\alpha_c^0"'] \arrow[dr, phantom, "\alpha_c"] & B \arrow[d, "\alpha_c^1"] \\ F^0 c \arrow[r, dotted, "Fc", "\shortmid"{marking}] \arrow[d, "F^0 s"'] \arrow[dr, phantom, "Fs"] & F^1 c \arrow[d, "F^1 s"] \\ F^0 d \arrow[r, dotted, "Fd"', "\shortmid"{marking}] & F^1 d \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\alpha_d^0"'] \arrow[dr, phantom, "\alpha_d"] & B \arrow[d, "\alpha_d^1"] \\ F^0 d \arrow[r, dotted, "Fd"', "\shortmid"{marking}] & F^1 d \end{tikzcd} \quad \text{in } \mathbb{L}.$$

♦

Definition 2.20 (Jointly cartesian cells)**.** Let $\mathbb{L}$ be an AVDC, let $\mathbf{C}$ be a category, and let $F\colon \mathbf{C} \to \mathbf{V}^{\leq 1}\mathbb{L}$ be a functor. A cone over F

$$\begin{tikzcd}
X^0 \arrow[r, dotted, "X", "\shortmid"{marking}] \arrow[d, "\alpha^0_c"'] \arrow[dr, phantom, "\alpha_c"] & X^1 \arrow[d, "\alpha^1_c"] \\
F^0c \arrow[r, dotted, "Fc"', "\shortmid"{marking}] & F^1c
\end{tikzcd} \quad \text{in } \mathbb{L} \quad (c \in \mathbf{C})$$

is called *jointly cartesian* in $\mathbb{L}$ if it satisfies the following condition: Suppose that we are given a horizontal path $A \overset{\vec{u}}{\dashrightarrow} B$, vertical arrows $A \xrightarrow{f} X^0$ and $B \xrightarrow{g} X^1$, and a cone β over F on the right below; then there uniquely exists a cell γ satisfying the following equality for any $c \in \mathbf{C}$.

$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\gamma"] & B \arrow[d, "g"] \\
X^0 \arrow[r, dotted, "X", "\shortmid"{marking}] \arrow[d, "\alpha^0_c"'] \arrow[dr, phantom, "\alpha_c"] & X^1 \arrow[d, "\alpha^1_c"] \\
F^0c \arrow[r, dotted, "Fc"', "\shortmid"{marking}] & F^1c
\end{tikzcd} = \begin{tikzcd}
A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X^0 \arrow[r, phantom, "\beta_c"] \arrow[d, "\alpha^0_c"'] & X^1 \arrow[d, "\alpha^1_c"] \\
F^0c \arrow[r, dotted, "Fc"', "\shortmid"{marking}] & F^1c
\end{tikzcd} \quad \text{in } \mathbb{L}$$

◆

Definition 2.21 (Cocartesian cells)**.** A cell

$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\
A \arrow[r, dotted, "v"', "\shortmid"{marking}] & B
\end{tikzcd} \tag{4}$$

in an AVDC is called *cocartesian* if the following assignment induces a bijection $\mathsf{Cell}\left({}_{f}\,{}^{\vec{p}v\vec{q}}_{w}\,{}_{g}\right) \cong \mathsf{Cell}\left({}_{f}\,{}^{\vec{p}\vec{u}\vec{q}}_{w}\,{}_{g}\right)$ for any $f, g, \vec{p}, \vec{q}, w$:

$$\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] & A \arrow[r, dotted, "v", "\shortmid"{marking}] & B \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & \cdot \arrow[d, "g"] \\
\cdot \arrow[rrr, dotted, "w"', "\shortmid"{marking}] & & & \cdot
\arrow[from=1-2, to=2-3, phantom, "\cdot"]
\end{tikzcd} \mapsto \begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] & A \arrow[r, dotted, "v", "\shortmid"{marking}] & B \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & \cdot \arrow[d, "g"] \\
\cdot \arrow[rrr, dotted, "w"', "\shortmid"{marking}] & & & \cdot
\arrow[from=2-2, to=3-3, phantom, "\cdot"]
\end{tikzcd}$$

The cell α is called *VD-cocartesian* if it induces the above bijection only for w of length 1. ◆

Remark 2.22. We can also consider cocartesian cells with an arbitrary boundary rather than identity vertical arrows. See [Kou20, Section 7] for details. ◆

Definition 2.23. Let $\mathbb{L}$ be an AVDC. An object $A \in \mathbb{L}$ is called *VD-composable* in $\mathbb{L}$ if:

- For any horizontal arrows $\cdot \xrightarrow{u_1} A \xrightarrow{u_2} \cdot$ in $\mathbb{L}$, there exists a VD-cocartesian cell of the following form:

$$\begin{tikzcd}
\cdot \arrow[r, "u_1", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "u_2", "\shortmid"{marking}] & \cdot \arrow[d, equal] \\
\cdot \arrow[rr, "\shortmid"{marking}] & & \cdot
\arrow[from=1-1, to=2-3, phantom, "\cdot"]
\end{tikzcd} \quad \text{in } \mathbb{L}; \tag{5}$$

- There exists a VD-cocartesian cell

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\cdot"] & \\ A \arrow[rr, "|" marking] & & A \end{tikzcd} \quad \text{in } \mathbb{L}. \tag{6}$$

◆

Notation 2.24. Let $\mathbb{L}$ be an AVDC. Then, all of the VD-composable objects yield a bicategory $\mathcal{H}\mathbb{L}$, called the *horizontal bicategory* of $\mathbb{L}$, where 1-cells are horizontal arrows and compositions and identities are defined by the VD-cocartesian cells (5) and (6). ◆

2.1.3. *Connection with virtual double categories.*

Definition 2.25. An AVDC is called *diminished* if all nullary cells are vertical identities, that is $=_f$ for some vertical morphism f. ◆

Notation 2.26. Let $\mathbb{L}$ be an AVDC. We write $\mathbb{L}^\flat$ for the diminished AVDC obtained by removing all nullary cells, except for identities, from $\mathbb{L}$. ◆

Remark 2.27. A diminished AVDC is the essentially same concept as a *VDC* [CS10], which is also called fc-*multicategories* [Lei99; Lei02; Lei04] and is originally introduced in [Bur71]. Indeed, the AVD-functors between diminished AVDCs correspond to the VD-functors between VDCs. In addition, a diminished AVDC where all objects are VD-composable is the essentially same concept as a *pseudo double category*. See [CS10, 5.2. Theorem] or [DPP06, 2.8. Proposition] for details. ◆

Remark 2.28. We now have two ways to regard unital VDCs as AVDCs. The first one is to regard as diminished AVDCs, where the AVD-functors between them correspond to the VD-functors. The second one is to regard as AVDCs with horizontal units, where the AVD-functors between them correspond to the normal VD-functors. Depending on which VD-functors are considered, we will use both ways. ◆

Notation 2.29. Given a bicategory $\mathcal{W}$, we can obtain a diminished AVDC $\mathbb{H}\mathcal{W}$ as follows. The vertical category $\mathbf{V}(\mathbb{H}\mathcal{W})$ is the discrete category of objects in $\mathcal{W}$. A horizontal arrow in $\mathbb{H}\mathcal{W}$ is a 1-cell in $\mathcal{W}$. A cell from $\vec{f}$ to g in $\mathbb{H}\mathcal{W}$ is a 2-cell from $\odot\vec{f}$ to g in $\mathcal{W}$:

$$\begin{tikzcd} c \arrow[r, dashed, "\vec{f}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & c' \arrow[d, equal] \\ c \arrow[r, "g"', "|" marking] & c' \end{tikzcd} \quad \text{in } \mathbb{H}\mathcal{W} \quad \Bigg\| \quad \begin{tikzcd} c \arrow[r, bend left, "\odot\vec{f}"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & c' \end{tikzcd} \quad \text{in } \mathcal{W}$$

Here, $\odot\vec{f}$ denotes the composition of $\vec{f}$ in $\mathcal{W}$. ◆

Theorem 2.30. For bicategories $\mathcal{W}$ and $\mathcal{W}'$, there is a bijective correspondence between the lax-functors $\mathcal{W} \to \mathcal{W}'$ and the AVD-functors $\mathbb{H}\mathcal{W} \to \mathbb{H}\mathcal{W}'$.

Proof. See [CS10, 3.5. Example]. □

We recall the $\mathbb{M}\mathrm{od}$-construction from [Lei99; Lei04; CS10], which is a construction of a VDC "$\mathbb{M}\mathrm{od}(\mathbb{K})$" from a VDC $\mathbb{K}$. Since the resulting VDC is always unital and normal VD-functors from itself are often considered, we redefine "$\mathbb{M}\mathrm{od}(\mathbb{K})$" as an AVDC with horizontal units, which is also considered in [Kou20].

Definition 2.31 ([Lei99; Lei04; CS10; Kou20])**.** Let $\mathbb{K}$ be an AVDC. The AVDC $\mathbb{M}\mathrm{od}(\mathbb{K})$ is defined as follows:

- An object is a *monoid*, which consists of the following data $A := (A^0, A^1, A^e, A^m)$:

$$\begin{tikzcd}
& A^0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "A^e"] & \\
A^0 \arrow[rr, "A^1"', "|"{marking}] & {} & A^0
\end{tikzcd}
\qquad
\begin{tikzcd}
A^0 \arrow[r, "A^1", "|"{marking}] \arrow[d, equal] & A^0 \arrow[r, "A^1", "|"{marking}] \arrow[d, phantom, "A^m"] & A^0 \arrow[d, equal] \\
A^0 \arrow[rr, "A^1"', "|"{marking}] & {} & A^0
\end{tikzcd}
\quad \text{in } \mathbb{K}.$$

The data (A^0, A^1, A^e, A^m) are required to satisfy a monoid-like axiom. The cells A^e and A^m are called the *unit* and the *multiplication* of the monoid A, respectively.

- A vertical arrow $A \xrightarrow{f} B$ consists of the following data (f^0, f^1):

$$\begin{tikzcd}
A^0 \arrow[r, "A^1", "|"{marking}] \arrow[d, "f^0"'] \arrow[dr, phantom, "f^1"] & A^0 \arrow[d, "f^0"] \\
B^0 \arrow[r, "B^1"', "|"{marking}] & B^0
\end{tikzcd}
\quad \text{in } \mathbb{K},$$

which is required to be compatible with units and multiplications.

- A horizontal arrow $A \xrightarrow{M} B$, called *(bi)module*, consists of the following data (M^1, M^l, M^r):

$$\begin{tikzcd}
A^0 \arrow[r, "A^1", "|"{marking}] \arrow[d, equal] & A^0 \arrow[r, "M^1", "|"{marking}] \arrow[d, phantom, "M^l"] & B^0 \arrow[d, equal] \\
A^0 \arrow[rr, "M^1"', "|"{marking}] & {} & B^0
\end{tikzcd}
\qquad
\begin{tikzcd}
A^0 \arrow[r, "M^1", "|"{marking}] \arrow[d, equal] & B^0 \arrow[r, "B^1", "|"{marking}] \arrow[d, phantom, "M^r"] & B^0 \arrow[d, equal] \\
A^0 \arrow[rr, "M^1"', "|"{marking}] & {} & B^0
\end{tikzcd}
\quad \text{in } \mathbb{K},$$

which is required to satisfy a module-like axiom.

- A unary cell α in $\mathbb{M}\mathrm{od}(\mathbb{K})$ on the left below is a cell in $\mathbb{K}$ on the right below

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{M}", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A_n \arrow[d, "g"] \\
B \arrow[r, "N"', "|"{marking}] & C
\end{tikzcd}
\quad \text{in } \mathbb{M}\mathrm{od}(\mathbb{K})
\qquad
\begin{tikzcd}
A_0^0 \arrow[r, "M_1^1", "|"{marking}] \arrow[d, "f^0"'] & \cdots \arrow[r, "M_n^1", "|"{marking}] \arrow[d, phantom, "\alpha"] & A_n^0 \arrow[d, "g^0"] \\
B^0 \arrow[rr, "N^1"', "|"{marking}] & {} & C^0
\end{tikzcd}
\quad \text{in } \mathbb{K}$$

such that, for each $0 \leq i \leq n$, two canonical ways to fill the following boundary give the same cell in $\mathbb{K}$:

$$\begin{tikzcd}
A_0^0 \arrow[r, dashed, "(M_j^1)_{0<j\leq i}", "|"{marking}] \arrow[d, "f^0"'] & A_i^0 \arrow[r, "A_i^1", "|"{marking}] & A_i^0 \arrow[r, dashed, "(M_j^1)_{i<j\leq n}", "|"{marking}] & A_n^0 \arrow[d, "g^0"] \\
B^0 \arrow[rrr, "N^1"', "|"{marking}] & & & C^0
\end{tikzcd}
\quad \text{in } \mathbb{K}.$$

- A nullary cell β in $\mathbb{M}\mathrm{od}(\mathbb{K})$ on the left below is a cell in $\mathbb{K}$ on the right below

$$\begin{tikzcd}
A_0 \arrow[rr, dashed, "\vec{M}", "|"{marking}] \arrow[dr, "f"'] & \beta & A_n \arrow[dl, "g"] \\
& B &
\end{tikzcd}
\quad \text{in } \mathbb{M}\mathrm{od}(\mathbb{K})
\qquad
\begin{tikzcd}
A_0^0 \arrow[r, "M_1^1", "|"{marking}] \arrow[d, "f^0"'] & \cdots \arrow[r, "M_n^1", "|"{marking}] \arrow[d, phantom, "\beta"] & A_n^0 \arrow[d, "g^0"] \\
B^0 \arrow[rr, "B^1"', "|"{marking}] & {} & B^0
\end{tikzcd}
\quad \text{in } \mathbb{K}$$

such that, for each $0 \leq i \leq n$, two canonical ways to fill the following boundary give the same cell in $\mathbb{K}$:

$$\begin{tikzcd}
A_0^0 \arrow[r, dashed, "|" marking, "(M_j^1)_{0<j\leq i}"] \arrow[d, "f^0"'] & A_i^0 \arrow[r, "|" marking, "A_i^1"] & A_i^0 \arrow[r, dashed, "|" marking, "(M_j^1)_{i<j\leq n}"] & A_n^0 \arrow[d, "g^0"] \\
B^0 \arrow[rrr, "|" marking, "B^1"'] & & & B^0
\end{tikzcd} \quad \text{in } \mathbb{K}.$$

♦

Remark 2.32. In the construction of $\mathbb{M}\mathrm{od}(\mathbb{K})$, no nullary cell in $\mathbb{K}$ is used except for identities. In particular, we have $\mathbb{M}\mathrm{od}(\mathbb{K}) = \mathbb{M}\mathrm{od}(\mathbb{K}^\flat)$. ♦

Theorem 2.33 ([CS10])**.** Let $\mathbb{L}$ be an AVDC with horizontal units and let $\mathbb{K}$ be an AVDC. Then, the following data correspond to each other up to isomorphism:

(i) An AVD-functor $\mathbb{L} \to \mathbb{M}\mathrm{od}(\mathbb{K})$.
(ii) An AVD-functor $\mathbb{L}^\flat \to \mathbb{K}$.

Proof. An AVD-functor $\mathbb{L}^\flat \to \mathbb{K}$ is nothing but an VD-functor $\mathbb{L}^\flat \to \mathbb{K}^\flat$. By the universal property of the $\mathbb{M}\mathrm{od}$-construction [CS10, 5.14. Proposition], it corresponds to a *normal* VD-functor $\mathbb{L}^\flat \to \mathbb{M}\mathrm{od}(\mathbb{K}^\flat)^\flat$ in the sense of [CS10]. Since $\mathbb{M}\mathrm{od}(\mathbb{K}^\flat) = \mathbb{M}\mathrm{od}(\mathbb{K})$ and since both $\mathbb{L}$ and $\mathbb{M}\mathrm{od}(\mathbb{K})$ have horizontal units, it also corresponds to an AVD-functor $\mathbb{L} \to \mathbb{M}\mathrm{od}(\mathbb{K})$. ☐

Notation 2.34. For an AVDC $\mathbb{K}$ with horizontal units, we write $U \colon \mathbb{K} \to \mathbb{M}\mathrm{od}(\mathbb{K})$ for the AVD-functor corresponding to the inclusion $\mathbb{K}^\flat \to \mathbb{K}$. Since U locally induces bijections on the classes of vertical arrows, horizontal arrows, and cells, we can regard $\mathbb{K}$ as a full sub-AVDC of $\mathbb{M}\mathrm{od}(\mathbb{K})$ by U. ♦

Proposition 2.35 ([CS10])**.** Let $\mathbb{K}$ be an AVDC.

(i) $\mathbb{M}\mathrm{od}(\mathbb{K})$ has horizontal units.
(ii) If $\mathbb{K}$ has unary restrictions, then $\mathbb{M}\mathrm{od}(\mathbb{K})$ has restrictions.

Proof.

(i) By [CS10, 5.5. Proposition], the diminished AVDC $\mathbb{M}\mathrm{od}(\mathbb{K})^\flat$ has *units* as a VDC. Those units automatically become horizontal units in $\mathbb{M}\mathrm{od}(\mathbb{K})$ since all nullary cells are inherited from them.
(ii) By [CS10, 7.4. Proposition], unary restrictions in $\mathbb{K}$ give those in $\mathbb{M}\mathrm{od}(\mathbb{K})$. ☐

2.1.4. *Horizontally indiscrete shapes.*

Definition 2.36. An AVDC $\mathbb{K}$ is called *horizontally discrete* if:

- It has no horizontal arrows.
- $\mathbb{K}$ is diminished. ♦

Definition 2.37. An AVDC $\mathbb{K}$ is called *horizontally indiscrete* if:

- For any objects $A, B \in \mathbb{K}$, there is a unique horizontal arrow from A to B, denoted by $A \xrightarrow{!_{AB}} B$.
- For any $A_0, A_1, \dots, A_n, X, Y \in \mathbb{K}$ ($n \geq 0$) and any vertical morphisms $A_0 \xrightarrow{f} X$, $A_n \xrightarrow{g} Y$ in $\mathbb{K}$, there is a unique cell of the following form:

$$\begin{tikzcd}
A_0 \arrow[r, "|" marking, "!_{A_0A_1}"] \arrow[d, "f"'] & A_1 \arrow[r, "|" marking, "!_{A_1A_2}"] & \cdots \arrow[r, "|" marking, "!_{A_{n-1}A_n}"] \arrow[d, phantom, "!"] & A_n \arrow[d, "g"] \\
X \arrow[rrr, "|" marking, "!_{XY}"'] & & & Y
\end{tikzcd} \quad \text{in } \mathbb{K}.$$

- $\mathbb{K}$ is diminished. ♦

Notation 2.38. Let $\mathbf{C}$ be a category. Let $\mathbb{I}\mathbf{C}$ [resp. $\mathbb{D}\mathbf{C}$] denote a horizontally indiscrete [resp. discrete] AVDC uniquely determined by $\mathbf{V}(\mathbb{I}\mathbf{C}) = \mathbf{C}$ [resp. $\mathbf{V}(\mathbb{D}\mathbf{C}) = \mathbf{C}$]. Note that this construction enumerates all horizontally (in)discrete AVDCs. ♦

Notation 2.39. For a large set S, we write $\mathbb{I}$S [resp. $\mathbb{D}$S] for the horizontally indiscrete [resp. discrete] large AVDC of Notation 2.38 obtained from the discrete category S. ♦

Remark 2.40. A monoid in an AVDC $\mathbb{L}$ is the same thing as an AVD-functor $\mathbb{I}1 \to \mathbb{L}$, where 1 denote the singleton. ♦

Definition 2.41. A cell

$$\begin{tikzcd}
A_0 \arrow[r, dotted, "u", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha"] & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[r, dotted, "v"', "\shortmid"{marking}] & B_1
\end{tikzcd}$$

in an AVDC is called *split* if there are data $(p_0, p_1, q_0, q_1, \beta_0, \beta_1, \gamma, \delta_0, \delta_1, \sigma, \eta_0, \eta_1)$ of the following forms:

$$\begin{tikzcd}
& A_0 \arrow[dl, equal] \arrow[d, "f_0"] \arrow[dl, phantom, "\beta_0"' near end, shift right=3] \\
A_0 \arrow[r, dotted, "p_0"', "\shortmid"{marking}] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[dr, equal] \arrow[d, "f_1"'] \arrow[dr, phantom, "\beta_1" near end, shift left=3] & \\
B_1 \arrow[r, dotted, "p_1"', "\shortmid"{marking}] & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[r, dotted, "p_0", "\shortmid"{marking}] \arrow[d, equal] & B_0 \arrow[r, dotted, "v", "\shortmid"{marking}] \arrow[d, phantom, "\gamma", shift left=9] & B_1 \arrow[r, dotted, "p_1", "\shortmid"{marking}] & A_1 \arrow[d, equal] \\
A_0 \arrow[rrr, dotted, "u"', "\shortmid"{marking}] & & & A_1
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \arrow[r, dotted, "p_0", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\delta_0"] & B_0 \arrow[d, equal] \\
B_0 \arrow[r, dotted, "q_0"', "\shortmid"{marking}] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, dotted, "p_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\delta_1"] & A_1 \arrow[d, "f_1"] \\
B_1 \arrow[r, dotted, "q_1"', "\shortmid"{marking}] & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0 \arrow[r, dotted, "q_0", "\shortmid"{marking}] \arrow[d, equal] & B_0 \arrow[r, dotted, "v", "\shortmid"{marking}] \arrow[d, phantom, "\sigma", shift left=9] & B_1 \arrow[r, dotted, "q_1", "\shortmid"{marking}] & B_1 \arrow[d, equal] \\
B_0 \arrow[rrr, dotted, "v"', "\shortmid"{marking}] & & & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
& B_0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_0"] & \\
B_0 \arrow[rr, dotted, "q_0"', "\shortmid"{marking}] & & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
& B_1 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_1"] & \\
B_1 \arrow[rr, dotted, "q_1"', "\shortmid"{marking}] & & B_1
\end{tikzcd}$$

These are required to satisfy the following equations:

$$\begin{tikzcd}
& A_0 \arrow[dl, equal] \arrow[r, dotted, "u", "\shortmid"{marking}] \arrow[d, "f_0"] \arrow[dr, phantom, "\alpha"] \arrow[dl, phantom, "\beta_0"' near end, shift right=3] & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] \arrow[dr, phantom, "\beta_1" near end, shift left=3] & \\
A_0 \arrow[r, dotted, "p_0"', "\shortmid"{marking}] \arrow[d, equal] & B_0 \arrow[r, dotted, "v"', "\shortmid"{marking}] \arrow[dr, phantom, "\gamma"] & B_1 \arrow[r, dotted, "p_1"', "\shortmid"{marking}] & A_1 \arrow[d, equal] \\
A_0 \arrow[rrr, dotted, "u"', "\shortmid"{marking}] & & & A_1
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, dotted, "u", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & A_1 \arrow[d, equal] \\
A_0 \arrow[r, dotted, "u"', "\shortmid"{marking}] & A_1
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \arrow[r, dotted, "p_0", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\delta_0"] & B_0 \arrow[r, dotted, "v", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & B_1 \arrow[r, dotted, "p_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\delta_1"] & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[r, dotted, "q_0"', "\shortmid"{marking}] \arrow[d, equal] & B_0 \arrow[r, dotted, "v"', "\shortmid"{marking}] \arrow[d, phantom, "\sigma", shift left=9] & B_1 \arrow[r, dotted, "q_1"', "\shortmid"{marking}] & B_1 \arrow[d, equal] \\
B_0 \arrow[rrr, dotted, "v"', "\shortmid"{marking}] & & & B_1
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, dotted, "p_0", "\shortmid"{marking}] \arrow[d, equal] & B_0 \arrow[r, dotted, "v", "\shortmid"{marking}] \arrow[d, phantom, "\gamma", shift left=9] & B_1 \arrow[r, dotted, "p_1", "\shortmid"{marking}] & A_1 \arrow[d, equal] \\
A_0 \arrow[rrr, dotted, "u"', "\shortmid"{marking}] \arrow[d, "f_0"'] & & & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[rrr, dotted, "v"', "\shortmid"{marking}] \arrow[urrr, phantom, "\alpha"] & & & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
& A_0 \arrow[dl, equal] \arrow[d, "f_0"] \arrow[dl, phantom, shift right=3, "\beta_0"] \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, "f_0"'] \arrow[dr, phantom, "\delta_0"] & B_0 \arrow[d, equal] \\
B_0 \arrow[r, dotted, "q_0"'] & B_0
\end{tikzcd}
=
\begin{tikzcd}
& A_0 \arrow[d, bend right, "f_0"'] \arrow[d, bend left, "f_0"] \arrow[d, phantom, "="] & \\
& B_0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_0"] & \\
B_0 \arrow[rr, dotted, "q_0"'] & & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_1 \arrow[d, bend right, "f_1"'] \arrow[d, bend left, "f_1"] \arrow[d, phantom, "="] & \\
& B_1 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_1"] & \\
B_1 \arrow[rr, dotted, "q_1"'] & & B_1
\end{tikzcd}
=
\begin{tikzcd}
A_1 \arrow[d, "f_1"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\beta_1"] & \\
B_1 \arrow[r, dotted, "p_1"'] \arrow[d, equal] \arrow[dr, phantom, "\delta_1"] & A_1 \arrow[d, "f_1"] \\
B_1 \arrow[r, dotted, "q_1"'] & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
& B_0 \arrow[r, dotted, "v"] \arrow[dl, equal] \arrow[d, equal] \arrow[dr, phantom, "="] \arrow[dl, phantom, shift right=3, "\eta_0"] & B_1 \arrow[d, equal] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\eta_1"] & \\
B_0 \arrow[r, dotted, "q_0"'] \arrow[d, equal] & B_0 \arrow[r, dotted, "v"'] \arrow[d, phantom, shift left=6, "\sigma"] & B_1 \arrow[r, dotted, "q_1"'] & B_1 \arrow[d, equal] \\
B_0 \arrow[rrr, dotted, "v"'] & & & B_1
\end{tikzcd}
=
\begin{tikzcd}
B_0 \arrow[r, dotted, "v"] \arrow[d, equal] \arrow[dr, phantom, "="] & B_1 \arrow[d, equal] \\
B_0 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}$$

◆

Lemma 2.42. Every split cell is cartesian. In particular, every split cell is *absolutely cartesian*; that is, it is a cartesian cell preserved by any AVD-functor.

Proof. Let α be a split cell as in Definition 2.41. Take an arbitrary cell θ on the left below:

$$\begin{tikzcd}
X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] & X_1 \arrow[d, "x_1"] \\
A_0 \arrow[d, "f_0"'] \arrow[r, phantom, "\theta"] & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}
=
\begin{tikzcd}
X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] \arrow[dr, phantom, "\bar{\theta}"] & X_1 \arrow[d, "x_1"] \\
A_0 \arrow[r, dotted, "u"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha"] & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}
\tag{7}$$

If there exists a cell $\bar{\theta}$ satisfying the above equation, then $\bar{\theta}$ must be given by the following:

$$\bar{\theta} =
\begin{tikzcd}
& X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] \arrow[dr, phantom, "\bar{\theta}"] & X_1 \arrow[d, "x_1"] & \\
& A_0 \arrow[r, dotted, "u"] \arrow[dl, equal] \arrow[d, "f_0"] \arrow[dr, phantom, "\alpha"] \arrow[dl, phantom, shift right=3, "\beta_0"] & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\beta_1"] & \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, equal] & B_0 \arrow[r, dotted, "v"'] \arrow[d, phantom, shift left=6, "\gamma"] & B_1 \arrow[r, dotted, "p_1"'] & A_1 \arrow[d, equal] \\
A_0 \arrow[rrr, dotted, "u"'] & & & A_1
\end{tikzcd}
=
\begin{tikzcd}
& X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] & X_1 \arrow[d, "x_1"] & \\
& A_0 \arrow[r, phantom, "\theta"] \arrow[dl, equal] \arrow[d, "f_0"] \arrow[dl, phantom, shift right=3, "\beta_0"] & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\beta_1"] & \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, equal] & B_0 \arrow[r, dotted, "v"'] \arrow[d, phantom, shift left=6, "\gamma"] & B_1 \arrow[r, dotted, "p_1"'] & A_1 \arrow[d, equal] \\
A_0 \arrow[rrr, dotted, "u"'] & & & A_1
\end{tikzcd}$$

Conversely, let us define $\bar{\theta}$ by the above equation. Then, the following calculation shows that $\bar{\theta}$ satisfies the desired equation (7):

$$\begin{tikzcd}
& X_0 \arrow[rr, dashed, "\vec{w}", "\shortmid" marking] \arrow[d, "x_0"'] & & X_1 \arrow[d, "x_1"] & \\
& A_0 \arrow[dl, equal] \arrow[d, "f_0"] \arrow[rr, phantom, "\theta"] & & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] & \\
A_0 \arrow[r, dotted, "p_0"', "\shortmid" marking] \arrow[d, equal] \arrow[ur, phantom, "\beta_0"] & B_0 \arrow[rr, dotted, "v"', "\shortmid" marking] & \arrow[d, phantom, "\gamma"] & B_1 \arrow[r, dotted, "p_1"', "\shortmid" marking] & A_1 \arrow[d, equal] \arrow[ul, phantom, "\beta_1"'] \\
A_0 \arrow[rrrr, dotted, "u"', "\shortmid" marking] \arrow[d, "f_0"'] & & \arrow[d, phantom, "\alpha"] & & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[rrrr, dotted, "v"', "\shortmid" marking] & & & & B_1
\end{tikzcd}
=
\begin{tikzcd}
& X_0 \arrow[rr, dashed, "\vec{w}", "\shortmid" marking] \arrow[d, "x_0"'] & & X_1 \arrow[d, "x_1"] & \\
& A_0 \arrow[dl, equal] \arrow[d, "f_0"] \arrow[rr, phantom, "\theta"] & & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] & \\
A_0 \arrow[r, dotted, "p_0"', "\shortmid" marking] \arrow[d, "f_0"'] \arrow[ur, phantom, "\beta_0"] \arrow[dr, phantom, "\delta_0"] & B_0 \arrow[rr, dotted, "v"', "\shortmid" marking] \arrow[d, equal] & \arrow[d, phantom, "\shortparallel"] & B_1 \arrow[r, dotted, "p_1"', "\shortmid" marking] \arrow[d, equal] & A_1 \arrow[d, "f_1"] \arrow[ul, phantom, "\beta_1"'] \arrow[dl, phantom, "\delta_1"'] \\
B_0 \arrow[r, dotted, "q_0"', "\shortmid" marking] \arrow[d, equal] & B_0 \arrow[rr, dotted, "v"', "\shortmid" marking] & \arrow[d, phantom, "\sigma"] & B_1 \arrow[r, dotted, "q_1"', "\shortmid" marking] & B_1 \arrow[d, equal] \\
B_0 \arrow[rrrr, dotted, "v"', "\shortmid" marking] & & & & B_1
\end{tikzcd}$$

$$=
\begin{tikzcd}
& X_0 \arrow[rr, dashed, "\vec{w}", "\shortmid" marking] \arrow[d, "x_0"'] & & X_1 \arrow[d, "x_1"] & \\
& A_0 \arrow[d, "f_0"'] \arrow[rr, phantom, "\theta"] & & A_1 \arrow[d, "f_1"] & \\
& B_0 \arrow[rr, dotted, "v", "\shortmid" marking] \arrow[dl, equal] \arrow[d, equal] & \arrow[d, phantom, "\shortparallel"] & B_1 \arrow[d, equal] \arrow[dr, equal] & \\
B_0 \arrow[r, dotted, "q_0"', "\shortmid" marking] \arrow[d, equal] \arrow[ur, phantom, "\eta_0"'] & B_0 \arrow[rr, dotted, "v"', "\shortmid" marking] & \arrow[d, phantom, "\sigma"] & B_1 \arrow[r, dotted, "q_1"', "\shortmid" marking] & B_1 \arrow[d, equal] \arrow[ul, phantom, "\eta_1"] \\
B_0 \arrow[rrrr, dotted, "v"', "\shortmid" marking] & & & & B_1
\end{tikzcd}
= \theta.$$

This shows that α is cartesian. □

Corollary 2.43. Let $\mathbb{K}$ be a horizontally indiscrete AVDC. Then, every cell of the following form is absolutely cartesian.

$$\begin{tikzcd}
A \arrow[r, "!_{AB}", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "!_{fg}"] & B \arrow[d, "g"] \\
X \arrow[r, "!_{XY}"', "\shortmid" marking] & Y
\end{tikzcd}
\quad \text{in } \mathbb{K}.$$

Proof. By the horizontal indiscreteness, it immediately follows that the cell $!_{fg}$ is split. Then, Lemma 2.42 shows that it is absolutely cartesian. □

2.2. **Categories enriched by a virtual double category.**

In this subsection, we will recall the notion of enriched categories over VDC from [Lei99; Lei02]. We first define the diminished AVDC of *matrices*, whose special case is described in [Lei04, Example 5.1.9].

Definition 2.44. Let $\mathbb{X}$ be an AVDC. By an $\mathbb{X}$-*colored large set*, we mean a large set A equipped with a map $A \xrightarrow{|\cdot|_A} \mathrm{Ob}\mathbb{X}$. ♦

Definition 2.45. Let $\mathbb{X}$ be an AVDC. Let A and B be $\mathbb{X}$-colored large sets. A *morphism of families* F from A to B consists of:

- For $x \in A$, an element $F^0 x \in B$;

- For $x \in A$, a vertical arrow $|x|_A \xrightarrow{F^1 x} |F^0 x|_B$ in $\mathbb{X}$. ◆

Definition 2.46. Let $\mathbb{X}$ be an AVDC. Let A and B be $\mathbb{X}$-colored large sets. An $(A \times B)$*-matrix* M over $\mathbb{X}$ is defined to be a family of horizontal arrows $|x|_A \xrightarrow{M(x,y)} |y|_B$ in $\mathbb{X}$ for $x \in A$ and $y \in B$. ◆

Definition 2.47. Let $\mathbb{X}$ be an AVDC. The (diminished) AVDC $\mathbb{M}\mathrm{at}(\mathbb{X})$ of matrices over $\mathbb{X}$ is defined as follows: its objects are $\mathbb{X}$-colored large sets, its vertical arrows are morphisms of families, its horizontal arrows $A \longrightarrow B$ are $(A \times B)$-matrices over $\mathbb{X}$, and a cell of the form

$$\begin{tikzcd}
A_0 \arrow[r, "M_1"] \arrow[d, "F"'] & A_1 \arrow[r, "M_2"] & \cdots \arrow[r, "M_n"] & A_n \arrow[d, "G"] \\
B \arrow[rrr, "N"'] & & \alpha & C
\end{tikzcd} \quad \text{in } \mathbb{M}\mathrm{at}(\mathbb{X})$$

consists of a family of cells

$$\begin{tikzcd}
{|x_0|_{A_0}} \arrow[r, "M_1(x_0{,}x_1)"] \arrow[d, "F^1 x_0"'] & {|x_1|_{A_1}} \arrow[r, "M_2(x_1{,}x_2)"] & \cdots \arrow[r, "M_n(x_{n-1}{,}x_n)"] & {|x_n|_{A_n}} \arrow[d, "G^1 x_n"] \\
{|F^0 x_0|_B} \arrow[rrr, "N(F^0 x_0{,}G^0 x_n)"'] & & \alpha_{x_0,x_1,\dots,x_n} & {|G^0 x_n|_C}
\end{tikzcd} \quad \text{in } \mathbb{X},$$

one for each tuple of $x_0 \in A_0, x_1 \in A_1, \dots, x_n \in A_n$. ◆

Remark 2.48. In the above definition of $\mathbb{M}\mathrm{at}(\mathbb{X})$, we do not use any nullary cell in $\mathbb{X}$, hence $\mathbb{M}\mathrm{at}(\mathbb{X}) = \mathbb{M}\mathrm{at}(\mathbb{X}^\flat)$. ◆

Remark 2.49. The vertical category $\mathbf{V}\mathbb{M}\mathrm{at}(\mathbb{X})$ is isomorphic to $\mathbf{Fam}(\mathbf{V}\mathbb{X})$, known as the category of *families* or the coproduct cocompletion of $\mathbf{V}\mathbb{X}$. ◆

Example 2.50. Let $\mathcal{V}$ be a monoidal category. Regarding $\mathcal{V}$ as a single-object bicategory, we have a diminished AVDC $\mathbb{M}\mathrm{at}(\mathbb{H}\mathcal{V})$, which is also denoted by $\mathcal{V}$-$\mathbb{M}\mathrm{at}$, whose objects are (large) sets, whose vertical arrows are maps, and whose horizontal arrows $X \longrightarrow Y$ are families $(M(x,y))_{x \in X, y \in Y}$ of objects in $\mathcal{V}$. When $\mathcal{V}$ is the two element chain, we have $\mathcal{V}$-$\mathbb{M}\mathrm{at} \cong \mathbb{R}\mathrm{el}^\flat$. ◆

Proposition 2.51. If an AVDC $\mathbb{X}$ has all unary restrictions, $\mathbb{M}\mathrm{at}(\mathbb{X})$ also has them.

Proof. Suppose that we are given the following data:

$$\begin{tikzcd}
A' \arrow[d, "F"'] & B' \arrow[d, "G"] \\
A \arrow[r, "N"'] & B
\end{tikzcd} \quad \text{in } \mathbb{M}\mathrm{at}(\mathbb{X}).$$

For $x \in A'$ and $y \in B'$, let $N(F,G)(x,y)$ denote the following horizontal arrow:

$$\begin{tikzcd}
{|x|} \arrow[r, "N(F{,}G)(x{,}y)"] \arrow[d, "F^1 x"'] \arrow[dr, phantom, "\mathsf{cart}"] & {|y|} \arrow[d, "G^1 y"] \\
{|F^0 x|} \arrow[r, "N(F^0 x{,}G^0 y)"'] & {|G^0 y|}
\end{tikzcd} \quad \text{in } \mathbb{X}.$$

Then, the matrix $N(F,G)$ over $\mathbb{X}$ gives a desired restriction. □

Definition 2.52 (Enrichment over a virtual double category)**.** Let $\mathbb{X}$ be an AVDC. The *AVDC of* $\mathbb{X}$*-enriched profunctors*, denoted by $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$, is defined to be $\mathbb{M}\mathrm{od}(\mathbb{M}\mathrm{at}(\mathbb{X}))$. Objects in $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ are called $\mathbb{X}$*-enriched (large) categories*, vertical arrows are called $\mathbb{X}$*-functors*, and horizontal arrows are called $\mathbb{X}$*-profunctors*. Note that $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ has restrictions whenever $\mathbb{X}$ has all unary restrictions, which follows from Proposition 2.51. ◆

Remark 2.53. Our $\mathbb{X}$-enriched categories, $\mathbb{X}$-functors, and $\mathbb{X}$-profunctors coincide with Leinster's [Lei99; Lei02]. For a bicategory $\mathcal{W}$, the AVDC $(\mathbb{H}\mathcal{W})$-$\mathbb{P}\mathrm{rof}$ recovers the classical notion of enrichment over a bicategory, which includes ordinary enrichment over a monoidal category as a special case. Indeed, the vertical 2-category $\mathcal{V}((\mathbb{H}\mathcal{W})\text{-}\mathbb{P}\mathrm{rof})$ is isomorphic to the 2-category of $\mathcal{W}$-enriched categories and $\mathcal{W}$-functors defined by Walters [Wal82]. Moreover, the horizontal bicategory $\mathcal{H}((\mathbb{H}\mathcal{W})\text{-}\mathbb{P}\mathrm{rof})$ of VD-composable objects coincides with the bicategory of sufficiently small $\mathcal{W}$-enriched categories and $\mathcal{W}$-profunctors, sometimes called $\mathcal{W}$*-modules*. ◆

We now unpack the definition.

Remark 2.54. Let $\mathbb{X}$ be an AVDC. An $\mathbb{X}$-enriched (large) category $\mathbf{A}$ consists of:

- (*Colored objects*) An $\mathbb{X}$-colored large set $\mathrm{Ob}\mathbf{A}$. For $x \in \mathrm{Ob}\mathbf{A}$, its color is denoted by $|x|_{\mathbf{A}}$ or simply $|x|$. When $|x| = c$, we call x an *object colored with* c.
- (*Hom-horizontal arrows*) For $x, y \in \mathrm{Ob}\mathbf{A}$, a horizontal arrow $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathbb{X}$.
- (*Compositions*) For $x, y, z \in \mathrm{Ob}\mathbf{A}$, a cell $\mu_{x,y,z}$ of the following form:

$$\begin{tikzcd} {|x|} \arrow[r, "\mathbf{A}(x,y)", "\shortmid"{marking}] \arrow[d, equal] & {|y|} \arrow[r, "\mathbf{A}(y,z)", "\shortmid"{marking}] \arrow[d, phantom, "\mu_{x,y,z}"] & {|z|} \arrow[d, equal] \\ {|x|} \arrow[rr, "\mathbf{A}(x,z)"', "\shortmid"{marking}] & & {|z|} \end{tikzcd} \quad \text{in } \mathbb{X}.$$

- (*Identities*) For each $x \in \mathrm{Ob}\mathbf{A}$, a cell η_x of the following form:

$$\begin{tikzcd} & {|x|} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_x"] & \\ {|x|} \arrow[rr, "\mathbf{A}(x,x)"', "\shortmid"{marking}] & {} & {|x|} \end{tikzcd} \quad \text{in } \mathbb{X}.$$

The above data are required to satisfy suitable axioms. ◆

Proposition 2.55. Let $\mathbb{X}$ be an AVDC. Then, an $\mathbb{X}$-enriched (large) category is the same as the following data:

- A (large) set S;
- An AVD-functor $\mathbb{I}\mathrm{S} \to \mathbb{X}$.

Proof. Let $\mathbf{A}$ be an $\mathbb{X}$-enriched large category. Then, the following assignments yield an AVD-functor $\mathbb{I}\mathrm{Ob}\mathbf{A} \to \mathbb{X}$:

$$x \mapsto |x|_{\mathbf{A}}, \qquad x \xrightarrow{!_{xy}} y \quad \mapsto \quad |x| \xrightarrow{\mathbf{A}(x,y)} |y|,$$

$$\begin{tikzcd} & x \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "!"] & \\ x \arrow[rr, "!_{xx}"', "\shortmid"{marking}] & {} & x \end{tikzcd} \mapsto \begin{tikzcd} & {|x|} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_x"] & \\ {|x|} \arrow[rr, "\mathbf{A}(x,x)"', "\shortmid"{marking}] & {} & {|x|} \end{tikzcd} \quad \begin{tikzcd} x \arrow[r, "!_{xy}", "\shortmid"{marking}] \arrow[d, equal] & y \arrow[r, "!_{yz}", "\shortmid"{marking}] \arrow[d, phantom, "!"] & z \arrow[d, equal] \\ x \arrow[rr, "!_{xz}"', "\shortmid"{marking}] & & z \end{tikzcd} \mapsto \begin{tikzcd} {|x|} \arrow[r, "\mathbf{A}(x,y)", "\shortmid"{marking}] \arrow[d, equal] & {|y|} \arrow[r, "\mathbf{A}(y,z)", "\shortmid"{marking}] \arrow[d, phantom, "\mu_{x,y,z}"] & {|z|} \arrow[d, equal] \\ {|x|} \arrow[rr, "\mathbf{A}(x,z)"', "\shortmid"{marking}] & & {|z|} \end{tikzcd}$$

Furthermore, we can reconstruct $\mathbb{A}$ from the AVD-functor $\mathbb{I}\mathrm{Ob}\mathbf{A} \to \mathbb{X}$. □

Notation 2.56. Let $\mathbb{X}$ be an AVDC. For $c \in \mathbb{X}$, let Yc denote the $\mathbb{X}$-colored set $Yc := \{*\}$ containing a unique element $*$ colored with c. It easily follows that all of Yc yields the full sub-AVDC of $\mathbb{M}\mathrm{at}(\mathbb{X})$ isomorphic to $\mathbb{X}^\flat$. We write $Y\colon \mathbb{X}^\flat \to \mathbb{M}\mathrm{at}(\mathbb{X})$ for the corresponding AVD-functor. ◆

Notation 2.57. Let $\mathbb{X}$ be an AVDC with horizontal units. We write $Z\colon \mathbb{X} \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ for an AVD-functor corresponding to $Y\colon \mathbb{X}^\flat \to \mathbb{M}\mathrm{at}(\mathbb{X})$ by Theorem 2.33. We write $\mathbf{Z}_c$ for the $\mathbb{X}$-enriched category assigned to each $c \in \mathbb{X}$ by Z. ◆

Lemma 2.58. Let $\mathbb{X}$ be an AVDC with horizontal units, and let $c \in \mathbb{X}$. Then, the unit cell associated with the monoid $\mathbf{Z}_c$ is VD-cocartesian in $\mathbb{M}\mathrm{at}(\mathbb{X})$.

Proof. Let

$$\begin{tikzcd} & c \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "v_c^{-1}"] & \\ c \arrow[rr, "\mathsf{U}_c"', "|" marking] & {} & c \end{tikzcd} \quad \text{in } \mathbb{X}$$

be the vertically invertible (cocartesian) cell associated with the horizontal unit U_c of c. In the diminished AVDC $\mathbb{X}^\flat$, the cell v_c^{-1} is no longer cocartesian but VD-cocartesian. Moreover, we see at once that the VD-cocartesian cell v_c^{-1} is preserved by the AVD-functor $Y\colon \mathbb{X}^\flat \to \mathbb{M}\mathrm{at}(\mathbb{X})$. Thus, the monoid structure of $\mathbf{Z}_c$ is induced by the VD-cocartesian cell Yv_c^{-1}. ☐

Definition 2.59. Let $\mathbf{A}$ be an $\mathbb{X}$-enriched category. A *semiobject* of $\mathbf{A}$ colored with $c \in \mathbb{X}$ is a pair $x = (x^0, x^1)$ of an object $x^0 \in \mathrm{Ob}\mathbf{A}$ and a vertical arrow $c \xrightarrow{x^1} |x^0|$ in $\mathbb{X}$. ◆

We call $\mathbf{Z}_c$ the *semiobject classifier* because it classifies the semiobjects colored with c in the following sense:

Theorem 2.60. Let $\mathbb{X}$ be an AVDC with horizontal units, and let $c \in \mathbb{X}$. Then, there is a bijective correspondence between the $\mathbb{X}$-functors $\mathbf{Z}_c \to \mathbf{A}$ and the semiobjects of $\mathbf{A}$ colored with c.

Proof. By Lemma 2.58, a monoid homomorphism $\mathbf{Z}_c \to \mathbf{A}$ is simply a vertical morphism $Yc \to \mathrm{Ob}\mathbf{A}$ in $\mathbb{M}\mathrm{at}(\mathbb{X})$. Thus, we get the desired bijective correspondence. ☐

Theorem 2.61. For an AVDC $\mathbb{X}$ with horizontal units, the AVD-functor $Z\colon \mathbb{X} \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ makes $\mathbb{X}$ into a full sub-AVDC of $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$.

Proof. This follows from Lemma 2.58. ☐

3. COLIMITS IN AUGMENTED VIRTUAL DOUBLE CATEGORIES

3.1. Cocones, modules, and modulations.

To give a notion of "colimits" in an AVDC, we consider "cocones" for each of the three directions: left, right, and down. The "cocones" for the down direction are called *vertical cocones*, and the "cocones" for the left and right directions are called left and right *modules*, respectively. In addition, we also consider several types of morphisms between them, called *modulations*. The terms "module" and "modulations" come from the essentially same concept in [Par11].

Definition 3.1 (Vertical cocones)**.** Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. A *vertical cocone* l (from F) consists of:

- an object $L \in \mathbb{L}$ (the *vertex* of l);
- for each $A \in \mathbb{K}$, a vertical arrow $\begin{tikzcd} FA \arrow[d, "l_A"'] \\ L \end{tikzcd}$ in $\mathbb{L}$;

- for each $A \xrightarrow{u} B$ in $\mathbb{K}$, a cell $$\begin{tikzcd} FA \arrow[rr, "Fu", "\shortmid"{marking}] \arrow[dr, "l_A"'] & l_u & FB \arrow[dl, "l_B"] \\ & L & \end{tikzcd}$$ in $\mathbb{L}$

satisfying the following conditions:

- For any vertical arrow $A \xrightarrow{f} B$ in $\mathbb{K}$, $(Ff) \fatsemi l_B = l_A$;
- For any cell

$$\begin{tikzcd} A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "u_2", "\shortmid"{marking}] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\ X \arrow[rrr, dotted, "v"', "\shortmid"{marking}] & \alpha & & Y \end{tikzcd} \quad \text{in } \mathbb{K},$$

$$\begin{tikzcd} FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Ff"'] & F\alpha & FA_n \arrow[d, "Fg"] \\ FX \arrow[rr, dotted, "Fv", "\shortmid"{marking}] \arrow[dr, "l_X"'] & l_v & FY \arrow[dl, "l_Y"] \\ & L & \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[dr, "l_{A_0}"'] & l_{\vec{u}} & FA_n \arrow[dl, "l_{A_n}"] \\ & L & \end{tikzcd} \quad \text{in } \mathbb{L}.$$

Here $l_{\vec{u}}$ denotes the composite of the following cells:

$$\begin{tikzcd} FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[ddrr, "l_{A_0}"'] & FA_1 \arrow[r, "Fu_2", "\shortmid"{marking}] \arrow[ddr, "l_{A_1}"] & \cdots \arrow[r, "Fu_{n-1}", "\shortmid"{marking}] & FA_{n-1} \arrow[r, "Fu_n", "\shortmid"{marking}] \arrow[ddl, "l_{A_{n-1}}"'] & FA_n \arrow[ddll, "l_{A_n}"] \\ & l_{u_1} & \cdots & l_{u_n} & \\ & & L & & \end{tikzcd} \quad \text{in } \mathbb{L}.$$

When $\vec{u}$ is length 0, the cell $l_{\vec{u}}$ is defined to be the identity. ♦

Definition 3.2. A vertical cocone l is called *strong* if l_u is cartesian for any horizontal arrow u. ♦

Definition 3.3 (Modules)**.** Let $F \colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. A *left F-module* m consists of:

- an object $M \in \mathbb{L}$ (the *vertex* of m);
- for each $A \in \mathbb{K}$, a horizontal arrow $FA \xrightarrow{m_A} M$ in $\mathbb{L}$;
- for each $A \xrightarrow{f} B$ in $\mathbb{K}$, a cartesian cell

$$\begin{tikzcd} FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"'] & M \arrow[d, equal] \\ FB \arrow[r, "m_B"', "\shortmid"{marking}] \arrow[ur, phantom, "m_f \colon \mathsf{cart}"] & M \end{tikzcd} \quad \text{in } \mathbb{L};$$

- for each $A \xrightarrow{u} B$ in $\mathbb{K}$, a cell

$$\begin{tikzcd} FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "m_B", "\shortmid"{marking}] & M \arrow[d, equal] \\ FA \arrow[rr, "m_A"', "\shortmid"{marking}] & m_u & M \end{tikzcd} \quad \text{in } \mathbb{L}$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "m_f"] & M \arrow[d, equal] \\
FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, "Fg"'] \arrow[dr, phantom, "m_g"] & M \arrow[d, equal] \\
FC \arrow[r, "m_C"', "\shortmid"{marking}] & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"'] & M \arrow[dd, equal] \\
FB \arrow[d, "Fg"'] \arrow[r, phantom, "m_{f\fatsemi g}"] & {} \\
FC \arrow[r, "m_C"', "\shortmid"{marking}] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

- For any $A \in \mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, equal, "F\mathsf{id}_A"'] \arrow[dr, phantom, "m_{\mathsf{id}_A}"] & M \arrow[d, equal] \\
FA \arrow[r, "m_A"', "\shortmid"{marking}] & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & M \arrow[d, equal] \\
FA \arrow[r, "m_A"', "\shortmid"{marking}] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

- For any cell

$$\begin{tikzcd}
A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "u_2", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\
X \arrow[rrr, dotted, "v"', "\shortmid"{marking}] & & & Y
\end{tikzcd}
\quad \text{in } \mathbb{K},$$

$$\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FA_n \arrow[r, "m_{A_n}", "\shortmid"{marking}] \arrow[d, "Fg"'] \arrow[dr, phantom, "m_g"] & M \arrow[d, equal] \\
FX \arrow[r, dotted, "Fv"', "\shortmid"{marking}] \arrow[d, equal] & FY \arrow[r, "m_Y"', "\shortmid"{marking}] \arrow[d, phantom, "m_v"] & M \arrow[d, equal] \\
FX \arrow[rr, "m_X"', "\shortmid"{marking}] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, equal] & FA_n \arrow[r, "m_{A_n}", "\shortmid"{marking}] \arrow[d, phantom, "m_{\vec{u}}"] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "m_{A_0}"', "\shortmid"{marking}] \arrow[d, "Ff"'] & {} \arrow[d, phantom, "m_f"] & M \arrow[d, equal] \\
FX \arrow[rr, "m_X"', "\shortmid"{marking}] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Here, $m_{\vec{u}}$ denotes the composition of the following cells:

$$\begin{tikzcd}
FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & FA_1 \arrow[r, "Fu_2", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & \cdots \arrow[r, "Fu_{n-1}", "\shortmid"{marking}] \arrow[d, phantom, "\cdots \; ="] & FA_{n-1} \arrow[r, "Fu_n", "\shortmid"{marking}] \arrow[d, equal] & FA_n \arrow[r, "m_{A_n}", "\shortmid"{marking}] \arrow[d, phantom, "m_{u_n}"] & M \arrow[d, equal] \\
FA_0 \arrow[r, "Fu_1"', "\shortmid"{marking}] \arrow[d, equal] & FA_1 \arrow[r, "Fu_2"', "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "Fu_{n-1}"', "\shortmid"{marking}] & FA_{n-1} \arrow[rr, "m_{A_{n-1}}"', "\shortmid"{marking}] & & M \arrow[d, equal] \\
\vdots \arrow[d, equal] & \vdots \arrow[d, equal] & & & & \vdots \arrow[d, equal] \\
FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[d, equal] & FA_1 \arrow[rrrr, "m_{A_1}", "\shortmid"{marking}] & & {} \arrow[d, phantom, "m_{u_1}"] & & M \arrow[d, equal] \\
FA_0 \arrow[rrrrr, "m_{A_0}"', "\shortmid"{marking}] & & & & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

◆

Remark 3.4. *Right F-modules* are also defined as the horizontal dual of the left F-modules. ◆

Notation 3.5. A vertical cocone from F with a vertex L is denoted by a double arrow $F \Rightarrow L$. A (left) F-module with a vertex M is denoted by a slashed double arrow $\begin{tikzcd} F \arrow[r, Rightarrow, "\shortmid"{marking}] & M \end{tikzcd}$. ◆

Definition 3.6. Let $F \colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Let m, m' be left F-modules whose vertices are $M, M' \in \mathbb{L}$, respectively. Consider $\begin{tikzcd} M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M'' \arrow[r, "j"] & M' \end{tikzcd}$ in $\mathbb{L}$. A *modulation (of type 0)* ρ, denoted by

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "m", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\rho"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M'' \arrow[d, "j"] \\
F \arrow[rr, Rightarrow, "m'"', "\shortmid"{marking}] & & M'
\end{tikzcd} \tag{8}$$

consists of:

- for each $A \in \mathbb{K}$, a cell

$$\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\rho_A"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M'' \arrow[d, "j"] \\
FA \arrow[rr, "m'_A"', "\shortmid"{marking}] & & M'
\end{tikzcd} \quad \text{in } \mathbb{L}$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[rd, phantom, "m_f"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\parallel"] & M'' \arrow[d, equal] \\
FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\rho_B"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M'' \arrow[d, "j"] \\
FB \arrow[rr, "m'_B"', "\shortmid"{marking}] & & M'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\rho_A"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M'' \arrow[d, "j"] \\
FA \arrow[rr, "m'_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[rrd, phantom, "m'_f"] & & M' \arrow[d, equal] \\
FB \arrow[rr, "m'_B"', "\shortmid"{marking}] & & M'
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

- For any $A \overset{u}{\shortmid\!\!\longrightarrow} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "m_u"] & FB \arrow[r, "m_B", "\shortmid"{marking}] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\parallel"] & M'' \arrow[d, equal] \\
FA \arrow[rr, "m_A"', "\shortmid"{marking}] \arrow[d, equal] \arrow[rrrd, phantom, "\rho_A"] & & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M'' \arrow[d, "j"] \\
FA \arrow[rrr, "m'_A"', "\shortmid"{marking}] & & & M'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\parallel"] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\rho_B"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M'' \arrow[d, "j"] \\
FA \arrow[r, "Fu"', "\shortmid"{marking}] \arrow[d, equal] \arrow[rrrd, phantom, "m'_u"] & FB \arrow[rr, "m'_B"', "\shortmid"{marking}] & & M' \arrow[d, equal] \\
FA \arrow[rrr, "m'_A"', "\shortmid"{marking}] & & & M'
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

◆

Notation 3.7. For a functor $F\colon \mathbb{K} \to \mathbb{L}$ between AVDCs and $M \in \mathbb{L}$, let $\mathbf{Mdl}(F, M)$ denote the category of left F-modules with the vertex M and special modulations (of type 0) where the length of $\vec{p}$ is 0 and j is the identity. We write $\mathbf{Mdl}(M, F)$ for the category of right F-modules with the vertex M. ◆

Remark 3.8. A modulation (of type 0) $\rho\colon m \to m'$ in $\mathbf{Mdl}(F, M)$ is called *invertible* if every components ρ_A is an iso-cell. The invertible modulations (of type 0) are the same as the isomorphisms in $\mathbf{Mdl}(F, M)$. ◆

Definition 3.9. Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Let $F \xRightarrow{l} L \in \mathbb{L}$ be a vertical cocone and let $F \overset{m}{\Longrightarrow} M \in \mathbb{L}$ be a left F-module. Consider $M \overset{\vec{p}}{\dashrightarrow} M'$, $M' \xrightarrow{j} L'$, and $L \begin{tikzcd}[cramped] {} \arrow[r, dotted, "q", "\shortmid"{marking}] & {} \end{tikzcd} L'$ in $\mathbb{L}$. A *modulation (of type 1)* σ, denoted by

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "m", "\shortmid"{marking}] \arrow[d, Rightarrow, "l"'] \arrow[drr, phantom, "\sigma"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd}$$

consists of:

- for each $A \in \mathbb{K}$, a cell

$$\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[drr, phantom, "\sigma_A"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd} \quad \text{in } \mathbb{L}$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "m_f"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M' \arrow[d, equal] \\
FB \arrow[r, "m_B"', "\shortmid"{marking}] \arrow[d, "l_B"'] \arrow[drr, phantom, "\sigma_B"] & M \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[drr, phantom, "\sigma_A"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

- For any $A \overset{u}{\longrightarrow} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, phantom, "m_u"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M' \arrow[d, equal] \\
FA \arrow[rr, "m_A"', "\shortmid"{marking}] \arrow[d, "l_A"'] & & M \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rrr, dotted, "q"', "\shortmid"{marking}] \arrow[urr, phantom, "\sigma_A"] & & & L'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[dr, phantom, "l_u"] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[dl, "l_B"] \arrow[drr, phantom, "\sigma_B"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rrr, dotted, "q"', "\shortmid"{marking}] & & & L'
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

◆

Definition 3.10. Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Let $F \overset{l}{\Longrightarrow} L \in \mathbb{L}$ and $F \overset{l'}{\Longrightarrow} L' \in \mathbb{L}$ be vertical cocones. Consider $L \begin{tikzcd}[cramped] {} \arrow[r, dotted, "q", "\shortmid"{marking}] & {} \end{tikzcd} L'$ in $\mathbb{L}$. A *modulation (of type 2)* τ, denoted by

$$\begin{tikzcd}
& F \arrow[dl, Rightarrow, "l"'] \arrow[dr, Rightarrow, "l'"] \arrow[d, phantom, "\tau"] & \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & {} & L'
\end{tikzcd}$$

consists of:

- for each $A \in \mathbb{K}$, a cell

$$\begin{tikzcd}
& FA \arrow[dl, "l_A"'] \arrow[dr, "l'_A"] \arrow[d, phantom, "\tau_A"] & \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & {} & L'
\end{tikzcd} \quad \text{in } \mathbb{L}$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
& FA \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "="] & \\
& FB \arrow[dl, "l_B"'] \arrow[dr, "l'_B"] \arrow[d, phantom, "\tau_B"] & \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[dl, "l_A"'] \arrow[dr, "l'_A"] \arrow[d, phantom, "\tau_A"] & \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

- For any $A \overset{u}{\longrightarrow} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[dr, "l'_A"'] \arrow[dr, phantom, "l'_u", near start, shift left=3] \arrow[d, phantom, "\tau_A", shift left=4, near end] & FB \arrow[d, "l'_B"] \\
L \arrow[r, dotted, "q"', "\shortmid"{marking}] & L'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "l_A"'] & FB \arrow[d, "l'_B"] \arrow[dl, "l_B"] \arrow[dl, phantom, "l_u", near start, shift right=3] \arrow[d, phantom, "\tau_B", shift right=4, near end] \\
L \arrow[r, dotted, "q"', "\shortmid"{marking}] & L'
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

♦

Notation 3.11. Let $\mathbf{Cone}(\substack{F\\L})$ denote the category of vertical cocones from F with a vertex L and special modulations (of type 2) where the length of q is 0. ♦

Definition 3.12. Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Let $N \overset{n}{\Longrightarrow} F \overset{m}{\Longrightarrow} M$ be a right F-module and a left F-module, respectively. Consider $N' \overset{\vec{q}}{\dashrightarrow} N$, $M \overset{\vec{p}}{\dashrightarrow} M'$, $N' \xrightarrow{j} N''$, $M' \xrightarrow{i} M''$, and $N'' \overset{r}{\dashrightarrow} M''$ in $\mathbb{L}$. A *modulation (of type 3)* ω, denoted by

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "j"'] & N \arrow[r, Rightarrow, "n", "\shortmid"{marking}] & F \arrow[r, Rightarrow, "m", "\shortmid"{marking}] \arrow[d, phantom, "\omega"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "r"', "\shortmid"{marking}] & & & & M''
\end{tikzcd}$$

consists of:

- for each $A \in \mathbb{K}$, a cell

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "j"'] & N \arrow[r, "n_A", "\shortmid"{marking}] & FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, phantom, "\omega_A"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "r"', "\shortmid"{marking}] & & & & M''
\end{tikzcd}$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & N \arrow[r, "n_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "n_f"] & FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"] \arrow[dr, phantom, "m_f"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "j"'] & N \arrow[r, "n_B", "\shortmid"{marking}] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, phantom, "\omega_B"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "r"', "\shortmid"{marking}] & & & & M''
\end{tikzcd}
= \quad \omega_A \quad \text{in } \mathbb{L}.$$

- For any $A \xrightarrow{u} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & N \arrow[r, "n_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "m_u"] & FB \arrow[r, "m_B", "\shortmid"{marking}] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "j"'] \arrow[drrrrr, phantom, "\omega_A"] & N \arrow[r, "n_A", "\shortmid"{marking}] & FA \arrow[rr, "m_A", "\shortmid"{marking}] & & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "i"] \\
N'' \arrow[rrrrr, dotted, "r"', "\shortmid"{marking}] & & & & & M''
\end{tikzcd}$$

$$= \begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & N \arrow[r, "n_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "n_u"] & FA \arrow[r, "Fu", "\shortmid"{marking}] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "j"'] \arrow[drrrrr, phantom, "\omega_B"] & N \arrow[rr, "n_B", "\shortmid"{marking}] & & FB \arrow[r, "m_B", "\shortmid"{marking}] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "i"] \\
N'' \arrow[rrrrr, dotted, "r"', "\shortmid"{marking}] & & & & & M''
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

◆

Construction 3.13. Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs and let $L \in \mathbb{L}$. Let $F \xRightarrow{\xi} \Xi \in \mathbb{L}$ be a vertical cocone. For a vertical arrow $\Xi \xrightarrow{k} L$ in $\mathbb{L}$, we have a vertical cone $F \xRightarrow{\xi \fatsemi k} L$ as follows:

- For any $A \in \mathbb{K}$,

$$\begin{tikzcd}
& FA \arrow[dl, "\xi_A"'] \arrow[dd, "(\xi\fatsemi k)_A"] \\
\Xi \arrow[r, phantom, "=:"] \arrow[dr, "k"'] & {} \\
& L
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

- For any $A \xrightarrow{u} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[rr, "Fu", "\shortmid"{marking}] \arrow[dr, "\xi_A"'] & \xi_u & FB \arrow[dl, "\xi_B"] \\
& \Xi \arrow[d, bend right, "k"'] \arrow[d, bend left, "k"] \arrow[d, phantom, "="] & \\
& L &
\end{tikzcd} =: \begin{tikzcd}
FA \arrow[rr, "Fu", "\shortmid"{marking}] \arrow[dr, "(\xi\fatsemi k)_A"'] & (\xi\fatsemi k)_u & FB \arrow[dl, "(\xi\fatsemi k)_B"] \\
& L &
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

Furthermore, the assignment $k \mapsto \xi\fatsemi k$ extends to a functor $\mathbf{Hom}_{\mathbb{L}}(\substack{\Xi \\ L}) \xrightarrow{\xi\fatsemi -} \mathbf{Cone}(\substack{F \\ L})$. ◆

Definition 3.14. A vertical arrow $A \xrightarrow{f} B$ in an AVDC is called *left-pullable* if every horizontal arrow $B \xrightarrow{p} \cdot$ has its restriction $p(f, \mathsf{id})$ along f:

$$\begin{tikzcd}
A \arrow[r, "{p(f,\mathsf{id})}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d, equal] \\
B \arrow[r, "p"', "\shortmid"{marking}] & \cdot
\end{tikzcd}$$

Right-pullable vertical arrows are also defined in the horizontally dual way. Left-pullable and right-pullable vertical arrows are simply called *pullable*. ◆

Construction 3.15. Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs and let $L \in \mathbb{L}$. Let ξ be a vertical cocone from F to $\Xi \in \mathbb{L}$. Assume that ξ_A is left-pullable for any $A \in \mathbb{K}$. Then, depending on a choice of cartesian cells

$$\begin{tikzcd}
FA \arrow[r, "p(\xi_A,\mathsf{id})", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\tilde{p}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\
\Xi \arrow[r, "p"', "\shortmid"{marking}] & L
\end{tikzcd} \quad \text{in } \mathbb{L}$$

for each horizontal arrow p, the following assignments yield a functor $\mathbf{Hom}_{\mathbb{L}}(\Xi, L) \xrightarrow{\xi_*-} \mathbf{Mdl}(F, L)$ between categories.

- For each $\Xi \xrightarrow{p} L$ in $\mathbb{L}$, a left F-module $\xi_* p$ with the vertex L is defined as follows:
  - For each $A \in \mathbb{K}$, $(\xi_* p)_A := p(\xi_A, \mathsf{id})$.
  - For each $A \xrightarrow{f} B$ in $\mathbb{K}$, $(\xi_* p)_f$ is a unique cell such that

$$\begin{tikzcd}
FA \arrow[r, "(\xi_*p)_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "(\xi_*p)_f"] & L \arrow[d, equal] \\
FB \arrow[r, "(\xi_*p)_B", "\shortmid"{marking}] \arrow[d, "\xi_B"'] \arrow[dr, phantom, "\tilde{p}_B\colon \mathsf{cart}"] & L \arrow[d, equal] \\
\Xi \arrow[r, "p"', "\shortmid"{marking}] & L
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "(\xi_*p)_A", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\tilde{p}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\
\Xi \arrow[r, "p"', "\shortmid"{marking}] & L
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

  - For each $A \xrightarrow{u} B$ in $\mathbb{K}$, $(\xi_* p)_u$ is a unique cell such that

$$\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "(\xi_*p)_B", "\shortmid"{marking}] \arrow[d, phantom, "(\xi_*p)_u"] & L \arrow[d, equal] \\
FA \arrow[rr, "(\xi_*p)_A", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \arrow[d, phantom, "\tilde{p}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\
\Xi \arrow[rr, "p"', "\shortmid"{marking}] & & L
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[r, phantom, shift right=4, "\xi_u"] & FB \arrow[r, "(\xi_*p)_B", "\shortmid"{marking}] \arrow[dl, "\xi_B"] \arrow[d, phantom, "\tilde{p}_B\colon \mathsf{cart}"] & L \arrow[d, equal] \\
\Xi \arrow[rr, "p"', "\shortmid"{marking}] & \, & L
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

- For each cell

$$\begin{tikzcd}
\Xi \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\delta"] & L \arrow[d, equal] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & L
\end{tikzcd} \quad \text{in } \mathbb{L},$$

  a modulation $\xi_*\delta\colon \xi_* p \to \xi_* q$ is defined as follows:
  - For each $A \in \mathbb{K}$, $(\xi_*\delta)_A$ is a unique cell such that

$$\begin{tikzcd}
FA \arrow[r, "(\xi_*p)_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "(\xi_*\delta)_A"] & L \arrow[d, equal] \\
FA \arrow[r, "(\xi_*q)_A", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\tilde{q}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & L
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "(\xi_*p)_A", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\tilde{p}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\
\Xi \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\delta"] & L \arrow[d, equal] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & L
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

◆

Notation 3.16. In Construction 3.15, the cartesian cells $(\tilde{p}_A)_{A\in\mathbb{K}}$ yields a modulation of type 1 below. We write $\xi_\dagger p$ for such modulation.

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_* p", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] \arrow[dr, phantom, "\xi_\dagger p"] & L \arrow[d, equal] \\
\Xi \arrow[r, "p"', "\shortmid"{marking}] & L
\end{tikzcd}$$

◆

Remark 3.17. By an argument similar to Construction 3.15, we can show that every vertical cocone $F \xRightarrow{l} L$ induces a left F-module $F \xRightarrow{l_*} L$ whenever the companions l_{A*} $(A \in \mathbb{K})$ exist. ◆

Notation 3.18. In Construction 3.15, if we alternatively assume that the restriction $q(\mathsf{id}_L, \xi_A)$ exists for any horizontal arrow $L \xrightarrow{q} \Xi$ in $\mathbb{L}$ and for any $A \in \mathbb{K}$, then we can construct in the same way a functor $\mathbf{Hom}_{\mathbb{L}}(L, \Xi) \xrightarrow{-\xi^*} \mathbf{Mdl}(L, F)$, which sends q to a right F-module $q\xi^*$. As well as Notation 3.16, we can get a modulation of type 1, denoted by $q\xi^\dagger$, of the following form:

$$\begin{tikzcd}
L \arrow[r, Rightarrow, "q\xi^*", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "q\xi^\dagger"] & F \arrow[d, Rightarrow, "\xi"] \\
L \arrow[r, "q"', "\shortmid"{marking}] & \Xi
\end{tikzcd}$$

◆

Remark 3.19. We have defined the modulations of the following types:

$$\begin{tikzcd}
\cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] & \cdot \arrow[r, Rightarrow] \arrow[d, phantom, "\text{type } 0"] & F \arrow[d, equal] \\
\cdot \arrow[rr, Rightarrow, "\shortmid"{marking}] & & F
\end{tikzcd}
\quad
\begin{tikzcd}
F \arrow[r, Rightarrow] \arrow[d, equal] & \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, phantom, "\text{type } 0"] & \cdot \arrow[d] \\
F \arrow[rr, Rightarrow, "\shortmid"{marking}] & & \cdot
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] & \cdot \arrow[r, Rightarrow] \arrow[d, phantom, "\text{type } 1"] & F \arrow[d, Rightarrow] \\
\cdot \arrow[rr, dotted, "\shortmid"{marking}] & & \cdot
\end{tikzcd}
\quad
\begin{tikzcd}
F \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, phantom, "\text{type } 1"] & \cdot \arrow[d] \\
\cdot \arrow[rr, dotted, "\shortmid"{marking}] & & \cdot
\end{tikzcd}$$

$$\begin{tikzcd}
& F \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] \arrow[d, phantom, "\text{type } 2"] & \\
\cdot \arrow[rr, dotted, "\shortmid"{marking}] & {} & \cdot
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] & \cdot \arrow[r, Rightarrow] & F \arrow[r, Rightarrow] \arrow[d, phantom, "\text{type } 3"] & \cdot \arrow[r, dashed, "\shortmid"{marking}] & \cdot \arrow[d] \\
\cdot \arrow[rrrr, dotted, "\shortmid"{marking}] & & & & \cdot
\end{tikzcd}$$

We may consider another type of "modulation." For example:

$$\begin{tikzcd}
& F \arrow[dl, equal] \arrow[dr, Rightarrow] & \\
F \arrow[rr, Rightarrow, "\shortmid"{marking}] & & \cdot
\end{tikzcd}$$

In the paper, we will only treat "modulations" whose bottom boundary is a horizontal path with length ≤ 1 or a module inherited from the functors ξ_*-, $-\xi^*$. Furthermore, such "modulations," which include the type 0, are attributed to one of the types 1, 2, or 3 by the universal property of restrictions. ◆

3.2. Final functors.

Definition 3.20. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ be an AVD-functor between AVDCs. For a path $A \overset{\vec{u}}{\dashrightarrow} B$ in $\mathbb{K}$, we define a category $\mathbf{S}\left(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi}\right)$ as follows:

- An object in $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is a tuple $(X^0, X^1, X, \varphi^0, \varphi^1, \varphi)$ of the following form:

$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\varphi^0"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "\varphi^1"] \\
\Phi X^0 \arrow[r, dotted, "\Phi X"', "\shortmid"{marking}] & \Phi X^1
\end{tikzcd} \quad \text{in } \mathbb{K}. \tag{9}$$

  We also write (X, φ) for such a object $(X^0, X^1, X, \varphi^0, \varphi^1, \varphi)$.

- A morphism $(X, \varphi) \xrightarrow{\theta} (Y, \psi)$ in $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is a tuple $(\theta^0, \theta^1, \theta)$ such that

$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\varphi^0"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "\varphi^1"] \\
\Phi X^0 \arrow[r, dotted, "\Phi X"', "\shortmid"{marking}] \arrow[d, "\Phi\theta^0"'] \arrow[dr, phantom, "\Phi\theta"] & \Phi X^1 \arrow[d, "\Phi\theta^1"] \\
\Phi Y^0 \arrow[r, dotted, "\Phi Y"', "\shortmid"{marking}] & \Phi Y^1
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\psi^0"'] \arrow[dr, phantom, "\psi"] & B \arrow[d, "\psi^1"] \\
\Phi Y^0 \arrow[r, dotted, "\Phi Y"', "\shortmid"{marking}] & \Phi Y^1
\end{tikzcd}
\quad \text{in } \mathbb{K}.$$

The assignments $(X, \varphi) \mapsto (X^i, \varphi^i)$ $(i = 0, 1)$ yield two functors to the comma categories: $(-)^0 \colon \mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi}) \to A/(\mathbf{V}\Phi)$ and $(-)^1 \colon \mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi}) \to B/(\mathbf{V}\Phi)$.

♦

Definition 3.21. For a category $\mathbf{C}$, we write $\pi_1\mathbf{C}$ for the strict localization of $\mathbf{C}$ by all morphisms. The groupoid $\pi_1\mathbf{C}$ is called the *fundamental groupoid* of $\mathbf{C}$. A category $\mathbf{C}$ is called *simply connected* if the fundamental groupoid $\pi_1\mathbf{C}$ has at most one morphism between any two objects.

♦

Definition 3.22. An AVD-functor $\Phi \colon \mathbb{J} \to \mathbb{K}$ between AVDCs is called *final* if:

- For every object $A \in \mathbb{K}$, the comma category $A/(\mathbf{V}\Phi)$ is simply connected.
- For every horizontal path $\vec{u}$ in $\mathbb{K}$, the category $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is connected.
- For every horizontal path $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$ in $\mathbb{K}$, there exist data of the following form:

$$\begin{tikzcd}
A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[d, "p_0"'] \arrow[dr, phantom, "\varphi_1"] & A_1 \arrow[r, "u_2", "\shortmid"{marking}] \arrow[d, "p_1"] \arrow[dr, phantom, "\varphi_2"] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] \arrow[dr, phantom, "\varphi_n"] & A_n \arrow[d, "p_n"] \\
\Phi X_0 \arrow[r, dotted, "\Phi v_1", "\shortmid"{marking}] \arrow[d, "\Phi f"'] & \Phi X_1 \arrow[r, dotted, "\Phi v_2", "\shortmid"{marking}] \arrow[d, phantom, "\Phi\theta"] & \cdots \arrow[r, dotted, "\Phi v_n", "\shortmid"{marking}] & \Phi X_n \arrow[d, "\Phi g"] \\
\Phi Y \arrow[rrr, dotted, "\Phi w"', "\shortmid"{marking}] & & & \Phi Z
\end{tikzcd} \quad \text{in } \mathbb{K}. \tag{10}$$

♦

Lemma 3.23. Let $\Phi \colon \mathbb{J} \to \mathbb{K}$ be a final AVD-functor between AVDCs. Then, for every $A \in \mathbb{K}$, the comma category $A/(\mathbf{V}\Phi)$ is connected (and simply connected).

Proof. This follows from that $A/(\mathbf{V}\Phi)$ is a retract of the category $\mathbf{S}(\genfrac{}{}{0pt}{}{A}{\Phi})$ for any $A \in \mathbb{K}$. □

Proposition 3.24. The following are equivalent for a functor $\Phi \colon \mathbf{C} \to \mathbf{D}$ between categories:

(i) For every object $d \in \mathbf{D}$, the comma category d/Φ is connected and simply connected.

(ii) The induced AVD-functor $\mathbb{I}\mathbf{C} \xrightarrow{\mathbb{I}\Phi} \mathbb{I}\mathbf{D}$ is final.

Proof. [(ii) $\Longrightarrow$ (i)] This follows from Lemma 3.23.

[(i) $\Longrightarrow$ (ii)] The first and third conditions for finality are trivial. We will show the second condition. Let $a \overset{\vec{u}}{\dashrightarrow} b$ in $\mathbb{I}\mathbf{D}$ be a path of horizontal arrows. The following shows that every object (x, φ) in $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\mathbb{I}\Phi})$ on the left below is connected with an object such that the length of X is 1 in (9):

$$\begin{tikzcd}
a \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\varphi^0"'] \arrow[dr, phantom, "\varphi"] & b \arrow[d, "\varphi^1"] & & & \\
\Phi x^0 \arrow[r, dotted, "\Phi x", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Phi!"] & \Phi x^1 \arrow[d, equal] & = & a \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\varphi^0"'] \arrow[dr, phantom, "!"] & b \arrow[d, "\varphi^1"] & \text{in } \mathbb{I}\mathbf{D} \\
\Phi x^0 \arrow[r, "\shortmid"{marking}, "\Phi!"'] & \Phi x^1 & & \Phi x^0 \arrow[r, "\shortmid"{marking}, "\Phi!"'] & \Phi x^1 &
\end{tikzcd}$$

The full subcategory of $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ consists of objects where X has the length 1 in (9) is isomorphic to a product $a/\Phi \times b/\Phi$ of comma categories, which are connected by the assumption. Therefore, $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is connected. ☐

Notation 3.25. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ and $F\colon \mathbb{K} \to \mathbb{L}$ be AVD-functors between AVDCs. Then, a vertical cocone l from F yields a vertical cocone from $F\Phi$, denoted by l_Φ, in a natural way. We also use such a notation for modules and modulations. ♦

Theorem 3.26. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ be a final AVD-functor. Then, the following hold for any AVD-functor $F\colon \mathbb{K} \to \mathbb{L}$.

(i) The assignment $l \mapsto l_\Phi$ yields isomorphisms of categories

$$-_\Phi\colon \mathbf{Cone}(\genfrac{}{}{0pt}{}{F}{L}) \xrightarrow{\cong} \mathbf{Cone}(\genfrac{}{}{0pt}{}{F\Phi}{L}) \qquad (L \in \mathbb{L}).$$

(ii) Assume that the following additional condition: for any $A \in \mathbb{K}$ there exists an object $(X, p) \in A/(\mathbf{V}\Phi)$ such that Fp is left-pullable in $\mathbb{L}$. Then, the assignment $m \mapsto m_\Phi$ yields equivalences of categories

$$-_\Phi\colon \mathbf{Mdl}(F, M) \xrightarrow{\simeq} \mathbf{Mdl}(F\Phi, M) \qquad (M \in \mathbb{L}).$$

(iii) The assignment $\rho \mapsto \rho_\Phi$ yields bijections among the classes of modulations of the same type.

Proof. We first show (iii) for modulations of type 1. Let σ be a modulation of type 1 exhibited by the following:

$$\begin{tikzcd}
F\Phi \arrow[r, Rightarrow, "m_\Phi", "\shortmid"{marking}] \arrow[d, Rightarrow, "l_\Phi"'] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, phantom, "\sigma"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "\shortmid"{marking}, "q"'] & & L'
\end{tikzcd}$$

Here, m is a left F-module, and l is a vertical cocone from F. We have to construct a modulation $\mathfrak{s}$ such that $\mathfrak{s}_\Phi = \sigma$. For each $A \in \mathbb{K}$, let us take a vertical arrow $A \xrightarrow{a} \Phi X$ in $\mathbb{K}$ by using the ordinary finality of $\mathbf{V}\Phi$ and define $\mathfrak{s}_A$ as the following cell:

$$\mathfrak{s}_A := \begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Fa"'] \arrow[dr, phantom, "m_a"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M' \arrow[d, equal] \\
F\Phi X \arrow[r, "\shortmid"{marking}, "m_{\Phi X}"'] \arrow[d, "l_{\Phi X}"'] & \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, phantom, "\sigma_X"] & \cdot \arrow[d, "j"] \\
L \arrow[rr, dotted, "\shortmid"{marking}, "q"'] & & L'
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

By using the ordinary finality of $\mathbf{V}\Phi$ again, we can show that the cells $\mathfrak{s}_A$ are independent of the choice of $A \xrightarrow{a} \Phi X$. Then, it easily follows that the cells $\mathfrak{s}$ form a desired modulation $\mathfrak{s}$.

The uniqueness of $\mathfrak{s}$ is trivial. The same way works in the case of modulations of the other types.

We next show (i). Since the functor $-_\Phi\colon \mathbf{Cone}(\genfrac{}{}{0pt}{}{F}{L}) \longrightarrow \mathbf{Cone}(\genfrac{}{}{0pt}{}{F\Phi}{L})$ is fully faithful by (iii), it suffices to show that the functor $-_\Phi$ is bijective on objects. Let l be a vertical cocone from $F\Phi$ to L. Since $A/(\mathbf{V}\Phi)$ is connected for each $A \in \mathbb{K}$, we can define $\mathfrak{l}_A$ as $(Fp)\fatsemi l_X$ independently of the choice of $A \xrightarrow{p} \Phi X$ in $\mathbb{K}$. Since $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is connected for $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$ in $\mathbb{K}$, we can also define a cell $\mathfrak{l}_{\vec{u}}$ as follows independently of the choice of an object $(X,\varphi) \in \mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$:

$$\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid" marking] \arrow[dr, "\mathfrak{l}_{A_0}"'] & \mathfrak{l}_{\vec{u}} & FA_n \arrow[dl, "\mathfrak{l}_{A_n}"] \\
& L &
\end{tikzcd}
:=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid" marking] \arrow[d, "F\varphi^0"'] & F\varphi & FA_n \arrow[d, "F\varphi^1"] \\
F\Phi X^0 \arrow[rr, dotted, "F\Phi X", "\shortmid" marking] \arrow[dr, "l_{X^0}"'] & l_X & F\Phi X^1 \arrow[dl, "l_{X^1}"] \\
& L &
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Taking data $(\vec{X}, Y, Z, \vec{p}, f, g, \vec{v}, w, \vec{\varphi}, \theta)$ as in (10), we can show that the cell $\mathfrak{l}_{\vec{u}}$ is a composite of the cells $(\mathfrak{l}_{u_1}, \ldots, \mathfrak{l}_{u_n})$:

$$\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid" marking] \arrow[dr, "\mathfrak{l}_{A_0}"'] & \mathfrak{l}_{\vec{u}} & FA_n \arrow[dl, "\mathfrak{l}_{A_n}"] \\
& L &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid" marking] \arrow[d, "Fp_0"'] & F\vec{\varphi} & FA_n \arrow[d, "Fp_n"] \\
F\Phi X_0 \arrow[rr, dashed, "F\Phi\vec{v}", "\shortmid" marking] \arrow[d, "F\Phi f"'] & F\Phi\theta & F\Phi X_n \arrow[d, "F\Phi g"] \\
F\Phi Y \arrow[rr, dotted, "F\Phi w", "\shortmid" marking] \arrow[dr, "l_Y"'] & l_w & F\Phi Z \arrow[dl, "l_Z"] \\
& L &
\end{tikzcd}$$

$$= \begin{tikzcd}
FA_0 \arrow[rr, "Fu_1", "\shortmid" marking] \arrow[dd, "Fp_0"'] & & FA_1 \arrow[rr, "Fu_2", "\shortmid" marking] \arrow[dd, "Fp_1"] & & \cdots \arrow[rr, "Fu_{n-1}", "\shortmid" marking] & & FA_{n-1} \arrow[rr, "Fu_n", "\shortmid" marking] \arrow[dd, "Fp_{n-1}"] & & FA_n \arrow[dd, "Fp_n"] \\
& F\varphi_1 & & F\varphi_2 & & F\varphi_{n-1} & & F\varphi_n & \\
F\Phi X_0 \arrow[rr, dotted, "F\Phi v_1", "\shortmid" marking] \arrow[ddrrrr, bend right, "l_{X_0}"'] & & F\Phi X_1 \arrow[rr, dotted, "F\Phi v_2", "\shortmid" marking] \arrow[ddrr, "l_{X_1}"] & & \cdots \arrow[rr, dotted, "F\Phi v_{n-1}", "\shortmid" marking] & & F\Phi X_{n-1} \arrow[rr, dotted, "F\Phi v_n", "\shortmid" marking] \arrow[ddll, "l_{X_{n-1}}"'] & & F\Phi X_n \arrow[ddllll, bend left, "l_{X_n}"] \\
& l_{v_1} & & & & & & l_{v_n} & \\
& & & & L & & & &
\end{tikzcd}$$

$$= \begin{tikzcd}
FA_0 \arrow[rr, "Fu_1", "\shortmid" marking] \arrow[ddrrrr, bend right, "\mathfrak{l}_{A_0}"'] & & FA_1 \arrow[rr, "Fu_2", "\shortmid" marking] \arrow[ddrr, "\mathfrak{l}_{A_1}"] & & \cdots \arrow[rr, "Fu_{n-1}", "\shortmid" marking] & & FA_{n-1} \arrow[rr, "Fu_n", "\shortmid" marking] \arrow[ddll, "\mathfrak{l}_{A_{n-1}}"'] & & FA_n \arrow[ddllll, bend left, "\mathfrak{l}_{A_n}"] \\
& \mathfrak{l}_{u_1} & & & & & & \mathfrak{l}_{u_n} & \\
& & & & L & & & &
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

To show that $\mathfrak{l}$ is a vertical cocone, take an arbitrary cell

$$\begin{tikzcd}
A_0 \arrow[rr, dashed, "\vec{u}", "\shortmid" marking] \arrow[d, "b"'] & \alpha & A_n \arrow[d, "c"] \\
B \arrow[rr, dotted, "v"', "\shortmid" marking] & & C
\end{tikzcd}
\quad \text{in } \mathbb{K}. \tag{11}$$

Taking an object $(Z, \chi) \in \mathbf{S}\binom{v}{\Phi}$, we have the following:

$$\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}"] \arrow[d, "Fb"'] & F\alpha & FA_n \arrow[d, "Fc"] \\
FB \arrow[rr, dotted, "Fv"] \arrow[dr, "\mathfrak{l}_B"'] & \mathfrak{l}_v & FC \arrow[dl, "\mathfrak{l}_C"] \\
& L &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}"] \arrow[d, "Fb"'] & F\alpha & FA_n \arrow[d, "Fc"] \\
FB \arrow[rr, dotted, "Fv"] \arrow[d, "F\chi^0"'] & F\chi & FC \arrow[d, "F\chi^1"] \\
F\Phi Z^0 \arrow[rr, dotted, "F\Phi Z"] \arrow[dr, "l_{Z^0}"'] & l_Z & F\Phi Z^1 \arrow[dl, "l_{Z^1}"] \\
& L &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}"] \arrow[dr, "\mathfrak{l}_{A_0}"'] & \mathfrak{l}_{\vec{u}} & FA_n \arrow[dl, "\mathfrak{l}_{A_n}"] \\
& L &
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Therefore, $\mathfrak{l}$ becomes a vertical cocone.

We next show (ii) under the additional assumption of left-pullability. Since the functor $-_\Phi \colon \mathbf{Mdl}(F, M) \longrightarrow \mathbf{Mdl}(F\Phi, M)$ is fully faithful by (iii), it suffices to show that the functor $-_\Phi$ is essentially surjective. Let m be a left $F\Phi$-module with a vertex M. Consider a functor $G_A \colon A/(\mathbf{V}\Phi) \to \mathbf{V}^1\mathbb{L}$ defined by the following assignment:

$$\begin{tikzcd}
A \arrow[d, "p"] \\
\Phi X
\end{tikzcd}
\quad \text{in } \mathbb{K} \qquad \mapsto \qquad F\Phi X \xrightarrow{m_X} M \quad \text{in } \mathbb{L}.$$

By the assumption, there are an object $A \overset{p_0}{\to} \Phi X_0$ in $A/(\mathbf{V}\Phi)$ and a restriction, denoted by $\mathfrak{m}_A$, of the following form:

$$\begin{tikzcd}
FA \arrow[r, "\mathfrak{m}_A"] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "\mathsf{cart}"] & M \arrow[d, equal] \\
F\Phi X_0 \arrow[r, "m_{X_0}"'] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}. \tag{12}$$

Since $A/(\mathbf{V}\Phi)$ is connected and simply connected, the above cell (12) uniquely extends to a cone over G_A of the following form:

$$\begin{tikzcd}
FA \arrow[r, "\mathfrak{m}_A"] \arrow[d, "Fp"'] \arrow[dr, phantom, "\rho^p_X : \mathsf{cart}"] & M \arrow[d, equal] \\
F\Phi X \arrow[r, "m_X"'] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}, \text{ where } (X, p) \in A/(\mathbf{V}\Phi). \tag{13}$$

Note that ρ^p_X automatically becomes cartesian since the cell (12) $(=\rho^{p_0}_{X_0})$ is cartesian. Since $A/(\mathbf{V}\Phi)$ is connected, the cone (13) over G_A becomes jointly cartesian. Furthermore, since $\mathbf{S}\binom{\vec{u}}{\Phi}$ is connected for $A \overset{\vec{u}}{\dashrightarrow} B$ in $\mathbb{K}$, a cone over $\mathbf{S}\binom{\vec{u}}{\Phi} \xrightarrow{(-)^0} A/(\mathbf{V}\Phi) \xrightarrow{G_A} \mathbf{V}^1\mathbb{L}$ obtained by pre-composing $(-)^0$ with the cone (13) also becomes jointly cartesian.

Let $A \overset{f}{\to} B$ be a vertical arrow in $\mathbb{K}$. Then, the assignment to $(X, p) \in B/(\mathbf{V}\Phi)$, the cell $\rho^{f \fatsemi p}_X$ gives a cone over G_B. Using the joint cartesianness of "ρ," we have a unique cell $\mathfrak{m}_f$ satisfying the following for any $(X, p) \in B/(\mathbf{V}\Phi)$:

$$\begin{tikzcd}
FA \arrow[r, "\mathfrak{m}_A"] \arrow[d, "Ff"'] & M \arrow[d, equal] \\
FB \arrow[d, "Fp"'] \arrow[r, phantom, "\rho^{f \fatsemi p}_X"] & M \arrow[d, equal] \\
F\Phi X \arrow[r, "m_X"'] & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\mathfrak{m}_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\mathfrak{m}_f"] & M \arrow[d, equal] \\
FB \arrow[r, "\mathfrak{m}_B"] \arrow[d, "Fp"'] \arrow[dr, phantom, "\rho^p_X"] & M \arrow[d, equal] \\
F\Phi X \arrow[r, "m_X"'] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

It easily follows that the assignment $f \mapsto \mathfrak{m}_f$ is functorial.

Let $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$ be a horizontal path in $\mathbb{K}$. Then, the assignment to $(X, \varphi) \in \mathbf{S}\left(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi}\right)$, a cell on the left below gives a cone over $\mathbf{S}\left(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi}\right) \xrightarrow{(-)^0} A_0/(\mathbf{V}\Phi) \xrightarrow{G_{A_0}} \mathbf{V}^1\mathbb{L}$. Using the joint cartesianness of "ρ," we have a unique cell, denoted by $\mathfrak{m}_{\vec{u}}$, such that the following holds for every object $(X, \varphi) \in \mathbf{S}\left(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi}\right)$:

$$
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "F\varphi^0"'] \arrow[dr, phantom, "F\varphi"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] \arrow[d, "F\varphi^1"'] \arrow[dr, phantom, "\rho^{\varphi^1}_{X^1}"] & M \arrow[d, equal] \\
F\Phi X^0 \arrow[r, dotted, "F\Phi X"'] \arrow[d, equal] \arrow[drr, phantom, "m_X"] & F\Phi X^1 \arrow[r, "m_{X^1}"'] & M \arrow[d, equal] \\
F\Phi X^0 \arrow[rr, "m_{X^0}"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_{\vec{u}}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}"] \arrow[d, "F\varphi^0"'] \arrow[drr, phantom, "\rho^{\varphi^0}_{X^0}"] & & M \arrow[d, equal] \\
F\Phi X^0 \arrow[rr, "m_{X^0}"'] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

Taking data $(\vec{X}, Y, Z, \vec{p}, f, g, \vec{v}, w, \vec{\varphi}, \theta)$ as in (10), we can decompose the cell $\mathfrak{m}_{\vec{u}}$ into the cells $(\mathfrak{m}_{u_1}, \ldots, \mathfrak{m}_{u_n})$ as follows:

$$
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F\vec{\varphi}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] \arrow[d, "Fp_n"'] & M \arrow[ddd, equal] \\
F\Phi X_0 \arrow[r, dashed, "F\Phi\vec{v}"] \arrow[d, "F\Phi f"'] \arrow[dr, phantom, "F\Phi\theta"] & F\Phi X_n \arrow[d, "F\Phi g"'] \arrow[r, phantom, "\rho^{p_n ⨟ \Phi g}_{Z}"] & \\
F\Phi Y \arrow[r, dotted, "F\Phi w"'] \arrow[d, equal] \arrow[drr, phantom, "m_w"] & F\Phi Z \arrow[r, "m_Z"'] & M \arrow[d, equal] \\
F\Phi Y \arrow[rr, "m_Y"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F\vec{\varphi}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] \arrow[d, "Fp_n"'] \arrow[dr, phantom, "\rho^{p_n}_{Z}"] & M \arrow[d, equal] \\
F\Phi X_0 \arrow[r, dashed, "F\Phi\vec{v}"] \arrow[d, "F\Phi f"'] \arrow[dr, phantom, "F\Phi\theta"] & F\Phi X_n \arrow[r, "m_{X_n}"] \arrow[d, "F\Phi g"'] \arrow[dr, phantom, "m_g"] & M \arrow[d, equal] \\
F\Phi Y \arrow[r, dotted, "F\Phi w"'] \arrow[d, equal] \arrow[drr, phantom, "m_w"] & F\Phi Z \arrow[r, "m_Z"'] & M \arrow[d, equal] \\
F\Phi Y \arrow[rr, "m_Y"'] & & M
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F\vec{\varphi}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] \arrow[d, "Fp_n"'] \arrow[dr, phantom, "\rho^{p_n}_{Z}"] & M \arrow[d, equal] \\
F\Phi X_0 \arrow[r, dashed, "F\Phi\vec{v}"] \arrow[d, equal] \arrow[drr, phantom, "m_{\vec{v}}"] & F\Phi X_n \arrow[r, "m_{X_n}"] & M \arrow[d, equal] \\
F\Phi X_0 \arrow[rr, "m_{X_0}"] \arrow[d, "F\Phi f"'] \arrow[drr, phantom, "m_f"] & & M \arrow[d, equal] \\
F\Phi Y \arrow[rr, "m_Y"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F(u_1,\ldots,u_{n-1})"] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & FA_{n-1} \arrow[r, "Fu_n"] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_{u_n}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] & M \arrow[d, equal] \\
FA_0 \arrow[r, dashed] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F(\varphi_1,\ldots,\varphi_{n-1})"] & FA_{n-1} \arrow[rr, "\mathfrak{m}_{A_{n-1}}"] \arrow[d, "Fp_{n-1}"] \arrow[drr, phantom, "\rho^{p_{n-1}}_{X_{n-1}}"] & & M \arrow[d, equal] \\
F\Phi X_0 \arrow[r, dashed, "F\Phi(v_1,\ldots,v_{n-1})"] \arrow[d, equal] \arrow[drrr, phantom, "m_{(v_1,\ldots,v_{n-1})}"] & F\Phi X_{n-1} \arrow[rr, "m_{X_{n-1}}"] & & M \arrow[d, equal] \\
F\Phi X_0 \arrow[rrr, "m_{X_0}"] \arrow[d, "F\Phi f"'] \arrow[drrr, phantom, "m_f"] & & & M \arrow[d, equal] \\
F\Phi Y \arrow[rrr, "m_Y"'] & & & M
\end{tikzcd}
$$

$$= \cdots = \begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "(\mathfrak{m}_{u_1},\dots,\mathfrak{m}_{u_n})"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}", "\shortmid"{marking}] \arrow[d, "Fp_0"'] \arrow[drr, phantom, "\rho^{p_0}_{X_0}"] & & M \arrow[d, equal] \\
F\Phi X_0 \arrow[rr, "m_{X_0}", "\shortmid"{marking}] \arrow[d, "F\Phi f"'] \arrow[drr, phantom, "m_f"] & & M \arrow[d, equal] \\
F\Phi Y \arrow[rr, "m_Y"', "\shortmid"{marking}] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "(\mathfrak{m}_{u_1},\dots,\mathfrak{m}_{u_n})"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}", "\shortmid"{marking}] \arrow[d, "Fp_0"'] \arrow[ddrr, phantom, "\rho_Y^{p_0 \fatsemi \Phi f} : \mathsf{cart}"] & & M \arrow[dd, equal] \\
F\Phi X_0 \arrow[d, "F\Phi f"'] & & \\
F\Phi Y \arrow[rr, "m_Y"', "\shortmid"{marking}] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

To show that $\mathfrak{m}$ is a left F-module, let us take an arbitrary cell α in $\mathbb{K}$ as in (11). Taking an object $(Y, \psi) \in \mathbf{S}\binom{v}{\Phi}$, we have the following:

$$\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Fb"'] \arrow[dr, phantom, "F\alpha"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] \arrow[d, "Fc"] \arrow[dr, phantom, "\mathfrak{m}_c"] & M \arrow[d, equal] \\
FB \arrow[r, dotted, "Fv", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_v"] & FC \arrow[r, "\mathfrak{m}_C", "\shortmid"{marking}] & M \arrow[d, equal] \\
FB \arrow[rr, "\mathfrak{m}_B", "\shortmid"{marking}] \arrow[d, "F\psi^0"'] \arrow[drr, phantom, "\rho^{\psi^0}_{Y^0} : \mathsf{cart}"] & & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"', "\shortmid"{marking}] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Fb"'] \arrow[dr, phantom, "F\alpha"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] \arrow[d, "Fc"] \arrow[dr, phantom, "\mathfrak{m}_c"] & M \arrow[d, equal] \\
FB \arrow[r, dotted, "Fv", "\shortmid"{marking}] \arrow[d, "F\psi^0"'] \arrow[dr, phantom, "F\psi"] & FC \arrow[r, "\mathfrak{m}_C", "\shortmid"{marking}] \arrow[d, "F\psi^1"] \arrow[dr, phantom, "\rho^{\psi^1}_{Y^1}"] & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[r, dotted, "F\Phi Y", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "m_Y"] & F\Phi Y^1 \arrow[r, "m_{Y^1}", "\shortmid"{marking}] & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"', "\shortmid"{marking}] & & M
\end{tikzcd}$$

$$=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Fb"'] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] \arrow[d, "Fc"] \arrow[ddr, phantom, "\rho^{c \fatsemi \psi^1}_{Y^1}"] & M \arrow[dd, equal] \\
FB \arrow[r, phantom, "F(\alpha \fatsemi \psi)"] \arrow[d, "F\psi^0"'] & FC \arrow[d, "F\psi^1"] & \\
F\Phi Y^0 \arrow[r, dotted, "F\Phi Y", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "m_Y"] & F\Phi Y^1 \arrow[r, "m_{Y^1}", "\shortmid"{marking}] & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"', "\shortmid"{marking}] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_{\vec{u}}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}", "\shortmid"{marking}] \arrow[d, "Fb"'] & & M \arrow[dd, equal] \\
FB \arrow[rr, phantom, "\rho^{b \fatsemi \psi^0}_{Y^0}"] \arrow[d, "F\psi^0"'] & & \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"', "\shortmid"{marking}] & & M
\end{tikzcd}$$

$$=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_{\vec{u}}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}", "\shortmid"{marking}] \arrow[d, "Fb"'] \arrow[drr, phantom, "\mathfrak{m}_b"] & & M \arrow[d, equal] \\
FB \arrow[rr, "\mathfrak{m}_B", "\shortmid"{marking}] \arrow[d, "F\psi^0"'] \arrow[drr, phantom, "\rho^{\psi^0}_{Y^0} : \mathsf{cart}"] & & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"', "\shortmid"{marking}] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L},$$

which shows that $\mathfrak{m}$ becomes a left F-module. We can easily verify that the cells ρ^{id}_X for $X \in \mathbb{J}$ form an invertible modulation $\mathfrak{m}_\Phi \cong m$ of type 0, which finishes the proof. ☐

Example 3.27. Let $\mathbb{J}$ be the AVDC consisting of two objects $0, 1$ and a unique horizontal arrow $0 \nrightarrow 1$. Let $\mathbb{K}$ be an AVDC defined by the following:

- $\mathbb{K}$ has just two objects $0, 1$;

- $\mathbb{K}$ has no non-trivial vertical arrow;
- $\mathbb{K}$ has just three horizontal arrows $0 \twoheadrightarrow 0 \twoheadrightarrow 1 \twoheadrightarrow 1$;
- For any boundary for cells, which includes nullary one, $\mathbb{K}$ has a unique cell filling it.

Then, the inclusion $\mathbb{J} \to \mathbb{K}$ gives a final AVD-functor. An AVD-functor $F\colon \mathbb{K} \to \mathbb{L}$ is the same as a choice of a horizontal arrow $F0 \twoheadrightarrow F1$ and horizontal units on $F0$ and $F1$. By Theorem 3.26, we can ignore the horizontal units when we regard F as a diagram for vertical cocones, modules, and modulations. ♦

3.3. **Versatile colimits.** In this subsection, we fix an AVD-functor $F\colon \mathbb{K} \to \mathbb{L}$ between AVDCs and a vertical cocone ξ from F to $\Xi \in \mathbb{L}$.

Definition 3.28. We consider the following conditions for ξ:

(V) The canonical functor $\mathbf{Hom}_{\mathbb{L}}\left(\begin{smallmatrix}\Xi\\ L\end{smallmatrix}\right) \xrightarrow{\xi \fatsemi -} \mathbf{Cone}\left(\begin{smallmatrix}F\\ L\end{smallmatrix}\right)$ of Construction 3.13 is bijective on objects for any $L \in \mathbb{L}$.

(H-L) ξ_A is left-pullable for any $A \in \mathbb{K}$, and the canonical functor $\mathbf{Hom}_{\mathbb{L}}(\Xi, L) \xrightarrow{\xi_* -} \mathbf{Mdl}(F, L)$ of Construction 3.15 is essentially surjective for any $L \in \mathbb{L}$.

(H-R) The horizontal dual of (H-L) holds.

(M0-L) ξ_A is left-pullable for any $A \in \mathbb{K}$, and the following hold: Take $M, M' \in \mathbb{L}$ and $\Xi \xrightarrow{p} M, \Xi \xrightarrow{p'} M'$ in $\mathbb{L}$ arbitrarily. Then, for any modulation ρ of type 0

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_* p", "\shortmid"{marking}] \arrow[d, equal] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, phantom, "\rho"] & M'' \arrow[d, "j"] \\
F \arrow[rr, Rightarrow, "\xi_* p'"', "\shortmid"{marking}] & & M',
\end{tikzcd}$$

There exists a unique cell $\hat{\rho}$ such that

$$\begin{tikzcd}
FA \arrow[r, "(\xi_* p)_A", "\shortmid"{marking}] \arrow[d, equal] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, phantom, "\rho_A"] & M'' \arrow[d, "j"] \\
FA \arrow[rr, "(\xi_* p')_A"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \arrow[d, phantom, "(\xi_\dagger p')_A\colon \mathsf{cart}"] & M' \arrow[d, equal] \\
\Xi \arrow[rr, "p'"', "\shortmid"{marking}] & & M'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "(\xi_* p)_A", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger p)_A\colon \mathsf{cart}"] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & M'' \arrow[d, equal] \\
\Xi \arrow[r, "p"', "\shortmid"{marking}] \arrow[d, equal] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, phantom, "\hat{\rho}"] & M'' \arrow[d, "j"] \\
\Xi \arrow[rr, "p'"', "\shortmid"{marking}] & & M'
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (\text{for any } A \in \mathbb{K}).$$

(M0-R) The horizontal dual of (M0-L) holds.

(M1-L) ξ_A is left-pullable for any $A \in \mathbb{K}$, and the following hold: Take $L, M \in \mathbb{L}$ and $\Xi \xrightarrow{k} L, \Xi \xrightarrow{p} M$ in $\mathbb{L}$ arbitrarily. Then, for any modulation σ of type 1

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_* p", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi \fatsemi k"'] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, phantom, "\sigma"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "r"', "\shortmid"{marking}] & & L',
\end{tikzcd}$$

there exists a unique cell $\hat{\sigma}$ such that

$$\begin{tikzcd}
FA \arrow[r, "(\xi_*p)_A"] \arrow[d, "(\xi\fatsemi k)_A"'] & M \arrow[r, dashed, "\vec{q}"] \arrow[d, phantom, "\sigma_A"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "r"'] & & L'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "(\xi_*p)_A"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger p)_A\colon \mathsf{cart}"] & M \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
\Xi \arrow[r, "p"'] \arrow[d, "k"'] & M \arrow[r, dashed, "\vec{q}"'] \arrow[d, phantom, "\hat{\sigma}"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "r"'] & & L'
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (\text{for any } A \in \mathbb{K}).$$

(M1-R) The horizontal dual of (M1-L) holds.

(M2) Take $L, L' \in \mathbb{L}$ and $\Xi \xrightarrow{k} L, \Xi \xrightarrow{k'} L'$ in $\mathbb{L}$ arbitrarily. Then, for any modulation τ of type 2

$$\begin{tikzcd}
& F \arrow[dl, Rightarrow, "\xi\fatsemi k"'] \arrow[dr, Rightarrow, "\xi\fatsemi k'"] \arrow[d, phantom, "\tau"] & \\
L \arrow[rr, dotted, "q"'] & & L',
\end{tikzcd}$$

there exists a unique cell $\hat{\tau}$ such that

$$\begin{tikzcd}
& FA \arrow[dl, "(\xi\fatsemi k)_A"'] \arrow[dr, "(\xi\fatsemi k')_A"] \arrow[d, phantom, "\tau_A"] & \\
L \arrow[rr, dotted, "q"'] & & L'
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[d, bend right, "\xi_A"'] \arrow[d, bend left, "\xi_A"] \arrow[d, phantom, "="] & \\
& \Xi \arrow[dl, "k"'] \arrow[dr, "k'"] \arrow[d, phantom, "\hat{\tau}"] & \\
L \arrow[rr, dotted, "q"'] & & L'
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (\text{for any } A \in \mathbb{K}).$$

(M3) ξ_A is pullable for any $A \in \mathbb{K}$, and the following hold: Take $N, M \in \mathbb{L}$ and $N \xrightarrow{t} \Xi \xrightarrow{s} M$ in $\mathbb{L}$ arbitrarily. Then, for any modulation ω of type 3

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, "j"'] & N \arrow[r, Rightarrow, "t\xi^*"] & F \arrow[r, Rightarrow, "\xi_*s"] \arrow[d, phantom, "\omega"] & M \arrow[r, dashed, "\vec{p}"] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "r"'] & & & & M''
\end{tikzcd}$$

there exists a unique cell $\hat{\omega}$ such that

$$\omega_A =
\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & N \arrow[r, "(t\xi^*)_A"] \arrow[d, equal] \arrow[dr, phantom, "(t\xi^\dagger)_A\colon \mathsf{cart}"] & FA \arrow[r, "(\xi_*s)_A"] \arrow[d, "\xi_A"] \arrow[dr, phantom, "(\xi_\dagger s)_A\colon \mathsf{cart}"] & M \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, "j"'] & N \arrow[r, "t"] & \Xi \arrow[r, "s"] \arrow[d, phantom, "\hat{\omega}"] & M \arrow[r, dashed, "\vec{p}"] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "r"'] & & & & M''
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (\text{for any } A \in \mathbb{K}).$$

♦

Remark 3.29. The above conditions are independent of the construction of the functors ξ_*- and $-\xi^*$. In particular, the condition (H-L) can be rephrased as follows:

(H-L)' ξ_A is left-pullable for any $A \in \mathbb{K}$. Furthermore, for any left F-module $m\colon F \Rightarrow L$, there exist a horizontal arrow $\Xi \overset{p}{\nrightarrow} L$ in $\mathbb{L}$ and a modulation σ of type 1

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "m", "|" marking] \arrow[d, Rightarrow, "\xi"'] \arrow[dr, phantom, "\sigma"] & L \arrow[d, equal] \\
\Xi \arrow[r, "p"', "|" marking] & L
\end{tikzcd}$$

such that every component σ_A ($A \in \mathbb{K}$) is cartesian. ♦

Proposition 3.30.

- (i) (M2) implies that the functor $\mathbf{Hom}_{\mathbb{L}}\left(\substack{\Xi \\ L}\right) \xrightarrow{\xi \fatsemi -} \mathbf{Cone}\left(\substack{F \\ L}\right)$ is fully faithful for any $L \in \mathbb{L}$.
- (ii) (M0-L) implies that the functor $\mathbf{Hom}_{\mathbb{L}}(\Xi, L) \xrightarrow{\xi_* -} \mathbf{Mdl}(F, L)$ is fully faithful for any $L \in \mathbb{L}$.

Proof. This follows from the fact that morphisms between vertical cocones or modules are a special case of modulations of type 2 or 0. ☐

Proposition 3.31.

- (i) (M1-L) implies (M0-L).
- (ii) If $\mathbb{L}$ has horizontal units and every vertical arrow is left-pullable in $\mathbb{L}$, then (M1-L) and (M0-L) are equivalent.

Proof. Using the universal property of restrictions, we can establish a bijection between the modulations of type 1 and the modulations of type 0. ☐

Proposition 3.32.

- (i) If $\mathbb{L}$ has companions, then (M1-L) implies (M2).
- (ii) If $\mathbb{L}$ has conjoints, then (M3) implies (M1-L).

Proof.

- (i) Suppose (M1-L) and that $\mathbb{L}$ has companions, in particular, horizontal units. Consider the canonical cells associated with the companions ξ_{A*}:

$$\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "|" marking] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\cdot"] & \Xi \arrow[dl, equal] & & FA \arrow[d, "\xi_A"] \\
\Xi & & FA \arrow[r, "\xi_{A*}"', "|" marking] \arrow[ur, equal] \arrow[ur, phantom, "\cdot"' near end] & \Xi
\end{tikzcd} \quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}). \qquad (14)$$

Let ξ_* denote the left F-module given by the companions ξ_{A*}. Then, we have bijective correspondences among the following data:

$$\begin{tikzcd}
 & F \arrow[dl, Rightarrow, "\xi"'] \arrow[dr, Rightarrow, "\xi"] & \\
\Xi \arrow[d, "k"'] & \tau & \Xi \arrow[d, "k'"] \\
L \arrow[rr, dotted, "q"', "|" marking] & & L'
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_*", "|" marking] \arrow[d, Rightarrow, "\xi"'] & \Xi \arrow[dd, "k'"] \\
\Xi \arrow[d, "k"'] \arrow[r, phantom, "\sigma"] & {} \\
L \arrow[r, dotted, "q"', "|" marking] & L'
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
\Xi \arrow[r, equal, "|" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\hat{\sigma}"] & \Xi \arrow[d, "k'"] \\
L \arrow[r, dotted, "q"', "|" marking] & L'
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
 & \Xi \arrow[dl, "k"'] \arrow[dr, "k'"] \arrow[d, phantom, "\hat{\tau}"] & \\
L \arrow[rr, dotted, "q"', "|" marking] & {} & L'
\end{tikzcd}$$

Here, the first correspondence is given by component-wise pasting with the cells (14). The second one is given by (M1-L). The third one is given by the universal property of horizontal units. Therefore (M2) follows.

(ii) Suppose (M3) and that $\mathbb{L}$ has conjoints. Then, we have bijective correspondences among the following data:

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_* p", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & M' \arrow[dd, "j"] \\
\Xi \arrow[d, "k"'] & \sigma & \\
L \arrow[rr, dotted, "r"', "\shortmid"{marking}] & & L'
\end{tikzcd}
\quad\Big\|\quad
\begin{tikzcd}
L \arrow[r, Rightarrow, "k^*\xi^*", "\shortmid"{marking}] \arrow[d, equal] & F \arrow[r, Rightarrow, "\xi_* p", "\shortmid"{marking}] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rrr, dotted, "r"', "\shortmid"{marking}] & \omega & & L'
\end{tikzcd}$$

$$\Big\|\quad
\begin{tikzcd}
L \arrow[r, "k^*", "\shortmid"{marking}] \arrow[d, equal] & \Xi \arrow[r, "p", "\shortmid"{marking}] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rrr, dotted, "r"', "\shortmid"{marking}] & \hat{\omega} & & L'
\end{tikzcd}
\quad\Big\|\quad
\begin{tikzcd}
\Xi \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "k"'] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "r"', "\shortmid"{marking}] & \hat{\sigma} & L'
\end{tikzcd}$$

The first correspondence is given by component-wise pasting with the canonical cells associated with the conjoints $\xi_A ⨾ k^* = (k^*\xi^*)_A$. The second one is given by (M3). The third one is given by pasting with the canonical cell associated with the conjoint k^*. Therefore (M1-L) follows. ☐

Definition 3.33 (Versatile colimits)**.** ξ is called a *versatile colimit* of F if it satisfies the conditions (V)(H-L)(H-R)(M1-L)(M1-R)(M2)(M3). ◆

Corollary 3.34. When $\mathbb{L}$ has companions and conjoints, ξ becomes a versatile colimit if and only if it satisfies (V)(H-L)(H-R)(M3).

Proof. This follows from Proposition 3.32. ☐

Corollary 3.35. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ be a final AVD-functor. Suppose that Ff is pullable in $\mathbb{L}$ for any vertical arrow f in $\mathbb{K}$. Then, ξ_Φ is a versatile colimit of $F\Phi$ if and only if ξ is a versatile colimit of F.

Proof. This follows from Theorem 3.26. ☐

Theorem 3.36 (Unitality theorem)**.** Suppose (H-L)(M1-L)(M2) and that ξ_A has a companion for every $A \in \mathbb{K}$. Then, Ξ has a horizontal unit.

Proof. Let ξ_* denote the left F-module given by the companions ξ_{A*}. Then, the canonical cartesian cells $\xi_{A\dagger}$ on the right below form a modulation $\xi_\dagger$ of type 1 on the left below:

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_*", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] & \Xi \\
\Xi \arrow[ur, equal] \arrow[u, phantom, "\xi_\dagger" near end, shift left=4ex] &
\end{tikzcd}
\quad\Big\|\quad
\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \\
\Xi \arrow[ur, equal] \arrow[u, phantom, "\xi_{A\dagger}" near end, shift left=4ex] &
\end{tikzcd}
\ : \mathsf{cart} \quad \text{in } \mathbb{L} \quad (A \in \mathbb{K})$$

By (H-L), we have a horizontal arrow $\Xi \xrightarrow{u} \Xi$ in $\mathbb{L}$ and a modulation $\xi_\dagger u$ of type 1 whose components are cartesian:

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_*", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] \arrow[dr, phantom, "\xi_\dagger u"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi
\end{tikzcd}
\quad\Big\|\quad
\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\mathsf{cart}"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K})$$

By (M1-L), there is a unique cell ε corresponding to the modulation $\xi_\dagger$. The cell ε is uniquely determined by the following equations:

$$\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger u)_A"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon", pos=0.3] & \Xi \arrow[dl, equal] \\
\Xi &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \arrow[dl, equal, "\xi_{A\dagger}"'] \\
\Xi &
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}).$$

Let us consider a modulation τ of type 2 given by the following:

$$\begin{tikzcd}
& F \arrow[dl, Rightarrow, "\xi"'] \arrow[dr, Rightarrow, "\xi"] & \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] \arrow[ur, phantom, "\tau", pos=0.1, shift left=3ex] & & \Xi
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
& FA \arrow[dl, equal] \arrow[dr, "\xi_A"] \arrow[d, phantom, "\delta_A"] & \\
FA \arrow[rr, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[drr, phantom, "(\xi_\dagger u)_A"] & & \Xi \arrow[d, equal] \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] & & \Xi
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}),$$

where δ_A denote the canonical cell associated with the companion ξ_{A*}. By (M2), there is a unique cell η corresponding to τ. The cell η is uniquely determined by the following equations:

$$\begin{tikzcd}
& FA \arrow[d, bend right, "\xi_A"'] \arrow[d, bend left, "\xi_A"] \arrow[d, phantom, "="] & \\
& \Xi \arrow[dl, equal] \arrow[dr, equal] & \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] \arrow[rr, phantom, "\eta", shift left=4ex] & & \Xi
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[dl, equal] \arrow[dr, "\xi_A"] \arrow[d, phantom, "\delta_A"] & \\
FA \arrow[rr, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[drr, phantom, "(\xi_\dagger u)_A"] & & \Xi \arrow[d, equal] \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] & & \Xi
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}).$$

Then, (M1-L)(M2) and the following calculations conclude that u becomes a horizontal unit on Ξ:

$$\begin{tikzcd}
FA \arrow[rr, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[drr, phantom, "(\xi_\dagger u)_A"] & & \Xi \arrow[d, equal] \\
\Xi \arrow[rr, "u", "\shortmid"{marking}] \arrow[ddr, equal] & & \Xi \arrow[ddl, equal] \\
& \varepsilon & \\
& \Xi \arrow[ddl, equal] \arrow[ddr, equal] & \\
& \eta & \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] & & \Xi
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \arrow[dl, equal, "\xi_{A\dagger}"'] \\
& \Xi \arrow[d, equal] \arrow[dl, equal, "\eta"] & \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi &
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \arrow[dl, equal, "\xi_{A\dagger}"'] \\
FA \arrow[ur, equal, "\delta_A"'] \arrow[r, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger u)_A"] & \Xi \arrow[d, equal] & \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger u)_A"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi
\end{tikzcd}$$

$$\begin{tikzcd}
& FA \arrow[d, bend right, "\xi_A"'] \arrow[d, bend left, "\xi_A"] \arrow[d, phantom, "="] & \\
& \Xi \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, phantom, "\eta", pos=0.3] \arrow[dd, phantom, "\varepsilon", pos=0.75] & \\
\Xi \arrow[rr, "u", "\shortmid"{marking}] \arrow[dr, equal] & & \Xi \arrow[dl, equal] \\
& \Xi &
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[d, "\xi_A"] \\
FA \arrow[ur, equal, "\delta_A"'] \arrow[r, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger u)_A"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u", "\shortmid"{marking}] \arrow[d, equal] & \Xi \arrow[dl, equal, "\varepsilon"'] \\
\Xi &
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[d, "\xi_A"] \\
FA \arrow[ur, equal, "\delta_A"'] \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \arrow[dl, equal, "\xi_{A\dagger}"'] \\
\Xi &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[d, bend right, "\xi_A"'] \arrow[d, bend left, "\xi_A"] \arrow[d, phantom, "="] \\
\Xi
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

□

Example 3.37 (Versatile coproducts)**.** Consider the diminished AVDC $\mathbb{R}\mathrm{el}^\flat$ of relations. Let $(X,Y)\colon \mathbb{D}2 \to \mathbb{R}\mathrm{el}$ be an AVD-functor determined by two (large) sets $X, Y \in \mathbb{R}\mathrm{el}$, where 2 denotes the two-element set. Then, the disjoint union $X+Y$ gives a versatile colimit of (X,Y), which is an example of a *versatile coproduct* (Definition 4.3). ◆

Example 3.38. A *collage*, also called *cograph*, of a profunctor $\mathbf{A} \xrightarrow{P} \mathbf{B}$ between categories is the category $\mathbf{X}$ whose class of objects is the disjoint union of $\mathrm{Ob}\mathbf{A}$ and $\mathrm{Ob}\mathbf{B}$ and where

$$\mathbf{X}(x,y) := \begin{cases} \mathbf{A}(x,y) & \text{if } x, y \in \mathbf{A}; \\ \mathbf{B}(x,y) & \text{if } x, y \in \mathbf{B}; \\ P(x,y) & \text{if } x \in \mathbf{A},\ y \in \mathbf{B}; \\ \varnothing & \text{if } x \in \mathbf{B},\ y \in \mathbf{A}. \end{cases}$$

Let $\mathbb{J}$ denote the AVDC consisting of just two objects $0, 1$ and a unique horizontal arrow $0 \nrightarrow 1$. Let **Set** and **SET** denote the categories of small sets and large sets, respectively. If the categories $\mathbf{A}$ and $\mathbf{B}$ are large and the profunctor P is locally large, then $\mathbf{X}$ gives a versatile colimit of P, where P is regarded as an AVD-functor from $\mathbb{J}$ to $\mathbf{SET}$-$\mathbb{P}\mathrm{rof}$, the AVDC of large categories. When the profunctor P is locally small, $\mathbf{X}$ still gives a versatile colimit in $(\mathbf{Set}, \mathbf{SET})$-$\mathbb{P}\mathrm{rof}$, the AVDC of large categories and locally small profunctors [Kou20, 2.6. Example], which gives an example of a versatile colimit with no horizontal unit. ◆

3.4. **The case of horizontally indiscrete shapes.** In this subsection, we study versatile colimits in the special case when the shape is horizontally indiscrete. Let us fix an AVD-functor $F\colon \mathbb{K} \to \mathbb{L}$ from a horizontally indiscrete AVDC $\mathbb{K}$.

Proposition 3.39. A vertical cocone from F with a vertex $L \in \mathbb{L}$ is equivalent to the following data:

- For each object $A \in \mathbb{K}$, a vertical arrow $FA \xrightarrow{l_A} L$ in $\mathbb{L}$.
- For objects $A, B \in \mathbb{K}$, a cell l_{AB} of the following form:

$$\begin{tikzcd}
FA \arrow[rr, "F!_{AB}", "\shortmid"{marking}] \arrow[dr, "l_A"'] & \arrow[d, phantom, "l_{AB}"] & FB \arrow[dl, "l_B"] \\
& L &
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

These are required to satisfy the following:

- For $A \xrightarrow{f} B$ in $\mathbb{K}$, the cell

$$\begin{tikzcd}
& FA \arrow[dl, "Ff"'] \arrow[d, equal] \arrow[dl, phantom, shift right=3, "F!"] \\
FB \arrow[r, "F!_{BA}", "\shortmid"{marking}] \arrow[dr, "l_B"'] \arrow[dr, phantom, shift left=3, "l_{BA}"] & FA \arrow[d, "l_A"] \\
& L
\end{tikzcd}$$

becomes identity.

- For $A, B, C \in \mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "F!_{BC}", "\shortmid"{marking}] \arrow[d, phantom, "F!"] & FC \arrow[d, equal] \\
FA \arrow[rr, "F!_{AC}", "\shortmid"{marking}] \arrow[dr, "l_A"'] & {} \arrow[d, phantom, "l_{AC}"] & FC \arrow[dl, "l_C"] \\
& L &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[dr, "l_A"'] \arrow[r, phantom, shift right=4, "l_{AB}"] & FB \arrow[r, "F!_{BC}", "\shortmid"{marking}] \arrow[d, "l_B"] \arrow[r, phantom, shift right=4, "l_{BC}"] & FC \arrow[dl, "l_C"] \\
& L &
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Proof. By the first condition for the identities $A \xrightarrow{\mathsf{id}_A} A$ in $\mathbb{K}$, the second condition is extended for horizontal paths in $\mathbb{K}$ of arbitrary length rather than length 2. Then, we have

$$\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{!}", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F!"] & FA_n \arrow[d, "Fg"] \\
FB \arrow[r, "F!_{BC}", "\shortmid"{marking}] \arrow[dr, phantom, "l_{BC}"] \arrow[d, "l_B"'] & FC \arrow[dl, "l_C"] \\
L &
\end{tikzcd}
=
\begin{tikzcd}
& FA_0 \arrow[r, dashed, "F\vec{!}", "\shortmid"{marking}] \arrow[dl, equal] \arrow[d, "Ff"'] \arrow[dr, phantom, "F!"] & FA_n \arrow[dr, equal] \arrow[d, "Fg"] & \\
FA_0 \arrow[r, "F!_{A_0B}", "\shortmid"{marking}] \arrow[ur, phantom, "F!"] \arrow[drr, "l_{A_0}"'] \arrow[dr, phantom, "l_{A_0B}"] & FB \arrow[r, "F!_{BC}", "\shortmid"{marking}] \arrow[dr, "l_B"] \arrow[r, phantom, shift right=4, "l_{BC}"] & FC \arrow[r, "F!_{CA_n}", "\shortmid"{marking}] \arrow[d, "l_C"] \arrow[ur, phantom, "F!"] \arrow[r, phantom, shift right=4, "l_{CA_n}"] & FA_n \arrow[dl, "l_{A_n}"] \\
& & L &
\end{tikzcd}$$

$$=
\begin{tikzcd}
& FA_0 \arrow[rr, dashed, "F\vec{!}", "\shortmid"{marking}] \arrow[dl, equal] \arrow[d, "Ff"'] \arrow[drr, phantom, "F!"] & & FA_n \arrow[dr, equal] \arrow[d, "Fg"] & \\
FA_0 \arrow[r, "F!_{A_0B}", "\shortmid"{marking}] \arrow[ur, phantom, "F!"] \arrow[d, equal] & FB \arrow[rr, "F!_{BC}", "\shortmid"{marking}] & & FC \arrow[r, "F!_{CA_n}", "\shortmid"{marking}] \arrow[ur, phantom, "F!"] & FA_n \arrow[d, equal] \\
FA_0 \arrow[rrrr, "F!_{A_0A_n}", "\shortmid"{marking}] \arrow[drr, "l_{A_0}"'] \arrow[urrrr, phantom, "F!"] & & {} \arrow[d, phantom, "l_{A_0A_n}"] & & FA_n \arrow[dll, "l_{A_n}"] \\
& & L & &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{!}", "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[d, phantom, "F!"] & FA_n \arrow[d, equal] \\
FA_0 \arrow[rr, "F!_{A_0A_n}", "\shortmid"{marking}] \arrow[dr, "l_{A_0}"'] & {} \arrow[d, phantom, "l_{A_0A_n}"] & FA_n \arrow[dl, "l_{A_n}"] \\
& L &
\end{tikzcd}$$

$$=
\begin{tikzcd}
FA_0 \arrow[r, "F_{A_0A_1}", "\shortmid"{marking}] \arrow[drr, bend right, "l_{A_0}"'] \arrow[r, phantom, shift right=4, "l_{A_0A_1}"] & \cdots \arrow[r, "F_{A_{n-1}A_n}", "\shortmid"{marking}] \arrow[r, phantom, shift right=4, "\ldots\ l_{A_{n-1}A_n}"] & FA_n \arrow[dl, bend left, "l_{A_n}"] \\
& L &
\end{tikzcd}
\quad \text{in } \mathbb{L},$$

which finishes the proof. □

Proposition 3.40. A left F-module with a vertex $M \in \mathbb{L}$ is equivalent to the following data:

- For each object $A \in \mathbb{K}$, a horizontal arrow $FA \xrightarrow{m_A} M$ in $\mathbb{L}$.
- For objects $A, B \in \mathbb{K}$, a cell m_{AB} of the following form:

$$\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, phantom, "m_{AB}"] & M \arrow[d, equal] \\
FA \arrow[rr, "m_A"', "\shortmid"{marking}] & {} & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

These are required to satisfy the following:

- For each $A \in \mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[rr, "m_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, equal, "F!"'] & \arrow[dr, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FA \arrow[r, "F!_{AA}"', "\shortmid"{marking}] \arrow[d, equal] & FA \arrow[r, "m_A"', "\shortmid"{marking}] \arrow[d, phantom, "m_{AA}"] & M \arrow[d, equal] \\
FA \arrow[rr, "m_A"', "\shortmid"{marking}] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FA \arrow[r, "m_A"', "\shortmid"{marking}] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

- For $A, B, C \in \mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[d, phantom, "F!"] \arrow[r, "F!_{BC}", "\shortmid"{marking}] & FC \arrow[r, "m_C", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FA \arrow[rr, "F!_{AC}"', "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "m_{AC}"] & FC \arrow[r, "m_C"', "\shortmid"{marking}] & M \arrow[d, equal] \\
FA \arrow[rrr, "m_A"', "\shortmid"{marking}] & & & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & FB \arrow[r, "F!_{BC}", "\shortmid"{marking}] \arrow[d, equal] & FC \arrow[d, phantom, "m_{BC}"] \arrow[r, "m_C", "\shortmid"{marking}] & M \arrow[d, equal] \\
FA \arrow[r, "F!_{AB}"', "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[rr, "m_B"', "\shortmid"{marking}] \arrow[d, phantom, "m_{AB}"] & & M \arrow[d, equal] \\
FA \arrow[rrr, "m_A"', "\shortmid"{marking}] & & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Proof. We have to show that the above data (m_A, m_{AB}) uniquely extend to a left F-module. If such an extension exists, for each vertical arrow f in $\mathbb{K}$, the cell m_f must be defined as follows:

$$\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "m_f"] & M \arrow[d, equal] \\
FB \arrow[r, "m_B"', "\shortmid"{marking}] & M
\end{tikzcd}
:=
\begin{tikzcd}
FA \arrow[rr, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, equal, "F!"'] & \arrow[dr, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FB \arrow[r, "F!_{BA}"', "\shortmid"{marking}] \arrow[d, equal] & FA \arrow[r, "m_A"', "\shortmid"{marking}] \arrow[d, phantom, "m_{BA}"] & M \arrow[d, equal] \\
FB \arrow[rr, "m_B"', "\shortmid"{marking}] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Let define several cells in $\mathbb{L}$ as follows:

$$\beta_0 := \begin{tikzcd}
& FA \arrow[d, "Ff"] \\
FA \arrow[ur, equal] \arrow[r, "F!_{AB}"', "\shortmid"{marking}] \arrow[ur, phantom, "F!"', near end] & FB
\end{tikzcd}
\qquad
\delta_0 := \begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F!"] & FB \arrow[d, equal] \\
FB \arrow[r, "F!_{BB}"', "\shortmid"{marking}] & FB
\end{tikzcd}
\qquad
\eta_0 := \begin{tikzcd}
& FB \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "F!"] & \\
FB \arrow[rr, "F!_{BB}"', "\shortmid"{marking}] & & FB
\end{tikzcd}$$

$$\gamma := m_{AB} \qquad \sigma := m_{BB} \qquad \beta_1 = \delta_1 = \eta_1 := \begin{tikzcd}
M \arrow[d, equal, bend right] \arrow[d, equal, bend left] \arrow[d, phantom, "="] \\
M
\end{tikzcd}$$

Since the above cells make m_f split, m_f becomes cartesian by Lemma 2.42. Then, we can easily verify that the data (m_A, m_{AB}, m_f) actually give a left F-module. ☐

Proposition 3.41. When the shape $\mathbb{K}$ of the diagram AVD-functor F is horizontally indiscrete, the axiom of modulations for vertical arrows in $\mathbb{K}$ automatically follows from the axiom for horizontal arrows in $\mathbb{K}$.

Proof. This follows from Propositions 3.39 and 3.40. ☐

Theorem 3.42 (Strongness theorem)**.** Let $F \colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs, and let $\mathbb{K}$ be horizontally indiscrete. Suppose that we are given a vertical cocone ξ from F to a vertex $\Xi \in \mathbb{L}$ that satisfies the conditions (H-L)(M1-L). Then, ξ_A has a conjunction for every $A \in \mathbb{K}$, and ξ becomes strong.

Proof. Fix $K \in \mathbb{K}$. Let us define a left F-module m with the vertex FK as follows:

- For each $A \in \mathbb{K}$, $m_A := F!_{AK} \colon FA \nrightarrow FK$ in $\mathbb{L}$.
- For $A, B \in \mathbb{K}$, m_{AB} is defined as the following cell:

$$\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "F!_{BK}", "\shortmid"{marking}] \arrow[d, phantom, "F!_{ABK}"] & FK \arrow[d, equal] & \text{in } \mathbb{L}. \\
FA \arrow[rr, "F!_{AK}"', "\shortmid"{marking}] & & FK &
\end{tikzcd}$$

Here, $!_{ABK}$ is a unique cell in $\mathbb{K}$.

By (H-L), we have a horizontal arrow $\Xi \xrightarrow{q} FK$ in $\mathbb{L}$ and a modulation $\xi_\dagger q$ of type 1 whose components are cartesian as follows:

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "m", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] \arrow[dr, phantom, "\xi_\dagger q"] & FK \arrow[d, "\xi_K"] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
FA \arrow[r, "m_A = F!_{AK}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger q)_A \mathsf{cart}"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}).$$

We can define a modulation σ of type 1 by $\sigma_A := \xi_{AK}$:

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "m", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] \arrow[d, phantom, shift left=4ex, "\sigma"] & FK \arrow[dl, "\xi_K"] \\
\Xi &
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
FA \arrow[r, "F!_{AK}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[d, phantom, shift left=4ex, "\xi_{AK}"] & FK \arrow[dl, "\xi_K"] \\
\Xi &
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}).$$

By (M1-L), we have a cell ε corresponding to the modulation σ:

$$\begin{tikzcd}
\Xi \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] \arrow[d, phantom, shift left=4ex, "\varepsilon"] & FK \arrow[dl, "\xi_K"] & \text{in } \mathbb{L}. \\
\Xi & &
\end{tikzcd}$$

Now, we shall show that ε is cartesian. Equivalently, we shall show that q is a conjunction of ξ_K. To show that, let us consider the following cell η:

$$\begin{tikzcd}
& FK \arrow[dl, "\xi_K"'] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\
\Xi \arrow[rr, "q"', "\shortmid"{marking}] & \, & FK
\end{tikzcd}
:=
\begin{tikzcd}
& FK \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "F!"] & \\
FK \arrow[rr, "m_K = F!_{KK}"', "\shortmid"{marking}] \arrow[d, "\xi_K"'] & \, \arrow[d, phantom, "(\xi_\dagger q)_K"] & FK \arrow[d, equal] \\
\Xi \arrow[rr, "q"', "\shortmid"{marking}] & \, & FK
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Then, one of the triangle identities can be shown as follows:

$$\begin{tikzcd}
& FK \arrow[dl, "\xi_K"'] \arrow[d, equal] \\
\Xi \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] \arrow[ur, phantom, "\eta"] & FK \arrow[dl, "\xi_K"] \\
\Xi \arrow[ur, phantom, "\varepsilon"] &
\end{tikzcd}
=
\begin{tikzcd}
& FK \arrow[dl, equal] \arrow[d, equal] \\
FK \arrow[r, "F!_{KK}"', "\shortmid"{marking}] \arrow[d, "\xi_K"'] \arrow[ur, phantom, "F!"] \arrow[dr, phantom, "(\xi_\dagger q)_K"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] & FK \arrow[dl, "\xi_K"] \\
\Xi \arrow[ur, phantom, "\varepsilon"] &
\end{tikzcd}
=
\begin{tikzcd}
& FK \arrow[dl, equal] \arrow[d, equal] \\
FK \arrow[r, "F!_{KK}"', "\shortmid"{marking}] \arrow[d, "\xi_K"'] \arrow[ur, phantom, "F!"] & FK \arrow[dl, "\xi_K"] \\
\Xi \arrow[ur, phantom, "\xi_{!_{KK}}"] &
\end{tikzcd}
\overset{(\xi)}{=}
\begin{tikzcd}
FK \arrow[d, bend right=50, "\xi_K"'] \arrow[d, bend left=50, "\xi_K"] \arrow[d, phantom, "="] \\
\Xi
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

We next prove the other triangle identity. The following calculation shows that a cell $q \to q$, which appears in the triangle identity, is sent to the identity modulation on $m = \xi_* q$ by the functor $\xi_*-\colon \mathbf{Hom}_{\mathbb{L}}(\Xi, FK) \longrightarrow \mathbf{Mdl}(F, FK)$:

$$\begin{tikzcd}
FA \arrow[r, "m_A=F!_{AK}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger q)_A"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] & FK \arrow[d, equal] \arrow[dl, "\xi_K", "\varepsilon"', "\eta"{below}] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AK}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & FK \arrow[d, equal] \arrow[dl, "\xi_K", "\xi_{AK}"', "\eta"{below}] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AK}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & FK \arrow[d, equal] \arrow[dr, equal] & \\
FA \arrow[r, "F!_{AK}", "\shortmid"{marking}] \arrow[dr, "\xi_A"'] \arrow[dr, phantom, "\xi_{AK}", near end, shift left=2ex] & FK \arrow[r, "F!", "F!_{KK}"', "\shortmid"{marking}] \arrow[d, "\xi_K"] \arrow[dr, phantom, "(\xi_\dagger q)_K"] & FK \arrow[d, equal] \\
& \Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}$$

$$\overset{(\xi_\dagger q)}{=}
\begin{tikzcd}
FA \arrow[r, "F!_{AK}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & FK \arrow[d, equal] \arrow[dr, equal] & \\
FA \arrow[r, "F!_{AK}", "\shortmid"{marking}] \arrow[d, equal] & FK \arrow[r, "F!", "F!_{KK}"', "\shortmid"{marking}] \arrow[d, phantom, "F!"] & FK \arrow[d, equal] \\
FA \arrow[rr, "F!_{AK}", "\shortmid"{marking}] \arrow[dr, "\xi_A"'] & \arrow[d, phantom, "(\xi_\dagger q)_A"] & FK \arrow[d, equal] \\
& \Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AK}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger q)_A"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Since the functor ξ_*- is fully faithful, we have

$$\begin{tikzcd}
\Xi \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] & FK \arrow[d, equal] \arrow[dl, "\xi_K", "\varepsilon"', "\eta"{below}] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}
=
\begin{tikzcd}
\Xi \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q"', "\shortmid"{marking}] & FK
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Thus $q = {\xi_K}^*$, and the cell ε is cartesian.

Consequently, we have the following for any $A \in \mathbb{K}$:

$$\begin{tikzcd}
FA \arrow[r, "F!_{AK}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & FK \arrow[dl, "\xi_K", "\xi_{AK}"'] \\
\Xi &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A=F!_{AK}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger q)_A\colon \mathsf{cart}"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] & FK \arrow[dl, "\xi_K", "\varepsilon\colon \mathsf{cart}"'] \\
\Xi &
\end{tikzcd}
\colon \mathsf{cart} \quad \text{in } \mathbb{L}.$$

This proves that ξ_{AK} is cartesian. □

Corollary 3.43. Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs, and let $\mathbb{K}$ be horizontally indiscrete. Then, a vertex of a vertical cocone ξ from F has a horizontal unit in $\mathbb{L}$ if ξ satisfies the conditions (H-L)(H-R)(M1-L)(M1-R)(M2).

Proof. Combine the strongness theorem (Theorem 3.42) and the horizontal dual of the unitality theorem (Theorem 3.36). □

Example 3.44 (Versatile collapses)**.** Let $A := (A^0 \xrightarrow{A^1} A^0, A^e, A^m)$ be a monoid in an AVDC $\mathbb{K}$. Suppose that A^0 has a horizontal unit in $\mathbb{K}$. Let UA^0 denote the monoid in $\mathbb{K}$ induced by the horizontal unit on A^0, let $UA^0 \xrightarrow{UA^1} UA^0$ denote the module in $\mathbb{K}$ induced by A^1, and let UA^e and UA^m denote the cells in $\mathbb{M}\mathrm{od}(\mathbb{K})$ induced by A^e and A^m, respectively. Now, we have a monoid $UA := (UA^0, UA^1, UA^e, UA^m)$ in $\mathbb{M}\mathrm{od}(\mathbb{K})$ and the corresponding AVD-functor $F\colon \mathbb{I}1 \to \mathbb{M}\mathrm{od}(\mathbb{K})$, where 1 denotes the singleton. Then, the monoid A gives a versatile colimit of F, which is strong. This is an example of a *versatile collapse* (Definition 4.3). ◆

Example 3.45. Consider the AVDC $\mathbb{R}\mathrm{el}$ (with horizontal units) of relations. Let $R \subseteq X \times X$ be an equivalence relation on a (large) set X. Since a monoid in $\mathbb{R}\mathrm{el}$ is simply a (large) preordered set, we have an AVD-functor $F \colon \mathbb{I}1 \to \mathbb{R}\mathrm{el}$ corresponding to R. Then, the quotient set X/R becomes a versatile colimit (collapse) of F. However, such a versatile colimit does not exist in general unless the relation R is symmetric. ♦

4. AXIOMATIZATION OF DOUBLE CATEGORIES OF PROFUNCTORS

4.1. The formal construction of enriched categories.

Remark 4.1. Let $\mathbb{X}$ be an AVDC with horizontal units, and let $\mathbf{A}$ be an $\mathbb{X}$-enriched large category. We now regard $\mathbf{A}$ as an AVD-functor $\mathbf{A} \colon \mathbb{I}(\mathrm{Ob}\mathbf{A}) \to \mathbb{X}$ as in Proposition 2.55, where $\mathrm{Ob}\mathbf{A}$ denotes the large set of objects in $\mathbf{A}$. Then, we obtain an AVD-functor $F_{\mathbf{A}} \colon \mathbb{I}(\mathrm{Ob}\mathbf{A}) \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ by post-composing with the embedding Z as in Notation 2.57:

$$\begin{tikzcd}
\mathbb{I}(\mathrm{Ob}\mathbf{A}) \arrow[r, "\mathbf{A}"] \arrow[dr, "F_{\mathbf{A}}"'] & \mathbb{X} \arrow[d, "Z"] \\
 & \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}
\end{tikzcd}$$

♦

Theorem 4.2. Let $\mathbb{X}$ be an AVDC with horizontal units. Then, every $\mathbb{X}$-enriched large category $\mathbf{A}$ is a versatile colimit of the AVD-functor $F_{\mathbf{A}} \colon \mathbb{I}(\mathrm{Ob}\mathbf{A}) \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ in Remark 4.1.

Proof. This is a special case of the construction in the proof of Lemma 4.5 and Theorem 4.6. ☐

Definition 4.3.

(i) A *(large) versatile coproduct* is a versatile colimit of an AVD-functor from $\mathbb{D}\mathrm{S}$ for some (large) set S.

(ii) A *versatile collapse* is a versatile colimit of an AVD-functor from $\mathbb{I}1$, where 1 denotes the singleton.

(iii) A *(large) versatile collage* is a versatile colimit of an AVD-functor from $\mathbb{I}\mathrm{S}$ for some (large) set S. ♦

Remark 4.4. The term "collapse" has been used for similar concepts in a virtual equipment: For a monoid M in a virtual equipment, a vertical cocone from M satisfying (V) is called a "collapse" in [Sch15]; The same term is also used in [AM24] for a vertical cocone from a monoid satisfying a stronger condition, which coincides with our term "versatile collapse." ♦

Lemma 4.5. For any AVDC $\mathbb{X}$, $\mathbb{M}\mathrm{at}(\mathbb{X})$ has all large versatile coproducts.

Proof. Let $(A_i)_{i\in\mathrm{S}}$ be $\mathbb{X}$-colored large sets indexed by a large set S. Let Ξ be a (large) disjoint union of $(A_i)_{i\in\mathrm{S}}$, and let $A_i \xrightarrow{\xi_i} \Xi$ denote the coprojections. We write $(i;x)$ for an element of Ξ, where $x \in A_i$, and define its color by $|(i;x)| := |x|$.

We have to show that Ξ is a versatile coproduct of $(A_i)_{i\in\mathrm{S}}$. The condition (V) follows clearly by the construction. Since the vertical arrow part of $\xi_i(x)$ for each $x \in A_i$ is the identity, ξ_i is pullable in $\mathbb{M}\mathrm{at}(\mathbb{X})$. The remaining conditions (H-L)(H-R)(M1-L)(M1-R)(M2)(M3) follow directly from the structure of Ξ as a disjoint union. ☐

Theorem 4.6. Let $\mathbb{K}$ be an AVDC, and let $\mathbf{C}$ be a category. If $\mathbb{K}$ has versatile colimits of any AVD-functors $\mathbb{D}\mathbf{C} \to \mathbb{K}$, then $\mathbb{M}\mathrm{od}(\mathbb{K})$ has versatile colimits of any AVD-functors $\mathbb{I}\mathbf{C} \to \mathbb{K}$.

Proof. Let $A \colon \mathbb{I}\mathbf{C} \to \mathbb{M}\mathrm{od}(\mathbb{K})$ be an AVD-functor. Now, A assigns to each object $i \in \mathbf{C}$, a monoid $A_i = (A_i^0 \xrightarrow{A_i^1} A_i^0, A_i^e, A_i^m)$ in $\mathbb{K}$, where A_i^e is the unit and A_i^m is the multiplication, and A also assigns to each pair (i,j) of $i,j \in \mathbf{C}$, a bimodule $A_{ij} = (A_i^0 \xrightarrow{A_{ij}^1} A_j^0, A_{ij}^l, A_{ij}^r)$ in $\mathbb{K}$, where A_{ij}^l and A_{ij}^r are the left action and the right action, respectively.

Let $F\colon \mathbb{I}\mathbf{C} \longrightarrow \mathbb{K}$ denote an AVD-functor given by post-composing A with the forgetful functor $\mathbb{M}\mathrm{od}(\mathbb{K})^\flat \longrightarrow \mathbb{K}$. Let $G\colon \mathbb{D}\mathbf{C} \longrightarrow \mathbb{K}$ denote an AVD-functor given by pre-composing F with the inclusion $\mathbb{D}\mathbf{C} \longrightarrow \mathbb{I}\mathbf{C}$. Let us take a versatile colimit $A_i^0 \xrightarrow{\xi_i^0} \Xi^0$ in $\mathbb{K}$ of G. By (M1-R) and (M1-L), there exist, for each $i \in \mathbf{C}$, two horizontal arrows $A_i^0 \overset{\mathfrak{q}_i}{\nrightarrow} \Xi^0 \overset{\mathfrak{p}_i}{\nrightarrow} A_i^0$ in $\mathbb{K}$ and modulations $\mathfrak{q}_i\xi^{0\dagger}$ and ${\xi^0}_\dagger\mathfrak{p}_i$ of type 1 whose components are cartesian:

$$\begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "(\mathfrak{q}_i\xi^{0\dagger})_j\colon \text{cart}"] & A_j^0 \arrow[r, "A_{ji}^1", "\shortmid"{marking}] \arrow[d, dashed, "\xi_j^0"'] \arrow[dr, phantom, "({\xi^0}_\dagger\mathfrak{p}_i)_j\colon \text{cart}"] & A_i^0 \arrow[d, equal] \\
A_i^0 \arrow[r, "\mathfrak{q}_i"', "\shortmid"{marking}] & \Xi^0 \arrow[r, "\mathfrak{p}_i"', "\shortmid"{marking}] & A_i^0
\end{tikzcd} \quad \text{in } \mathbb{K} \quad (i,j \in \mathbf{C}).$$

By (M0-R) for Ξ^0, there exist, for each $i,j \in \mathbf{C}$, a unique cell $\mathfrak{q}_{ij}$ in $\mathbb{K}$ corresponding to a modulation of type 0 on the right side below:

$$\begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, equal] & A_j^0 \arrow[r, "\mathfrak{q}_j", "\shortmid"{marking}] \arrow[d, phantom, "\mathfrak{q}_{ij}"] & \Xi^0 \arrow[d, equal] \\
A_i^0 \arrow[rr, "\mathfrak{q}_i"', "\shortmid"{marking}] & & \Xi^0
\end{tikzcd} \quad \text{in } \mathbb{K} \quad \Bigg\| \quad \begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, equal] & A_j^0 \arrow[r, "A_{jk}^1", "\shortmid"{marking}] \arrow[d, phantom, "A!"] & A_k^0 \arrow[d, equal] \\
A_i^0 \arrow[rr, "A_{ik}^1"', "\shortmid"{marking}] & & A_k^0
\end{tikzcd} \quad \text{in } \mathbb{K} \quad (k \in \mathbf{C})$$

Then, $(\mathfrak{q}_i, \mathfrak{q}_{ij})$ uniquely extends to a left F-module $\mathfrak{q}$ by Proposition 3.40 and (M0-R). In particular, $\mathfrak{q}$ is also a left G-module. Thus, by (H-L) for Ξ^0, we obtain a unique horizontal arrow Ξ^1 in $\mathbb{K}$ and a modulation ${\xi^0}_\dagger\Xi^1$ of type 1 whose components are cartesian:

$$\begin{tikzcd}
A_i^0 \arrow[r, "\mathfrak{q}_i", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "({\xi^0}_\dagger\Xi^1)_i\colon \text{cart}"] & \Xi^0 \arrow[d, equal] \\
\Xi^0 \arrow[r, "\Xi^1"', "\shortmid"{marking}] & \Xi^0
\end{tikzcd} \quad \text{in } \mathbb{K} \quad (i \in \mathbf{C}).$$

In the same way, we can construct a right F-module $\mathfrak{p} = (\mathfrak{p}_i, \mathfrak{p}_{ij})$, a horizontal arrow $\Xi^{1\prime}$, and a modulation $\Xi^{1\prime}\xi^{0\dagger}$ of type 1 whose components are cartesian. By replacing $\mathfrak{p}_i$ appropriately, we can assume $\Xi^1 = \Xi^{1\prime}$ without loss of generality. We now have cartesian cells as follows:

$$\begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "\text{cart}"] & A_j^0 \arrow[d, "\xi_j^0"] \\
\Xi^0 \arrow[r, "\Xi^1"', "\shortmid"{marking}] & \Xi^0
\end{tikzcd} = \begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "(\mathfrak{q}_i\xi^{0\dagger})_j\colon \text{cart}"] & A_j^0 \arrow[d, "\xi_j^0"] \\
A_i^0 \arrow[r, "\mathfrak{q}_i", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "({\xi^0}_\dagger\Xi^1)_i\colon \text{cart}"] & \Xi^0 \arrow[d, equal] \\
\Xi^0 \arrow[r, "\Xi^1"', "\shortmid"{marking}] & \Xi^0
\end{tikzcd} = \begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "({\xi^0}_\dagger\mathfrak{p}_j)_i\colon \text{cart}"] & A_j^0 \arrow[d, equal] \\
\Xi^0 \arrow[r, "\mathfrak{p}_j", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "(\Xi^1\xi^{0\dagger})_j\colon \text{cart}"] & A_j^0 \arrow[d, "\xi_j^0"] \\
\Xi^0 \arrow[r, "\Xi^1"', "\shortmid"{marking}] & \Xi^0
\end{tikzcd} \quad \text{in } \mathbb{K} \quad (i,j \in \mathbf{C}). \tag{15}$$

By (M2), we have a unique cell Ξ^e below:

$$\begin{tikzcd}
& A_i^0 \arrow[d, shift right=2, "\xi_i^0"'] \arrow[d, shift left=2, "\xi_i^0"] \arrow[d, phantom, "="] & \\
& \Xi^0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\Xi^e"] & \\
\Xi^0 \arrow[rr, "\Xi^1"', "|" marking] & & \Xi^0
\end{tikzcd}
=
\begin{tikzcd}
& A_i^0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "A!"] & \\
A_i^0 \arrow[rr, "A_{ii}^1"', "|" marking] \arrow[d, "\xi_i^0"'] & \text{cart} & A_i^0 \arrow[d, "\xi_i^0"] \\
\Xi^0 \arrow[rr, "\Xi^1"', "|" marking] & & \Xi^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad (i \in \mathbf{C}).$$

By (M0-L), (M0-R), and (M3), we have a unique cell Ξ^m below:

$$\begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1", "|" marking] \arrow[d, "\xi_i^0"'] & A_j^0 \arrow[r, "A_{jk}^1", "|" marking] \arrow[d, "\xi_j^0"'] \arrow[l, phantom, "\text{cart}"] \arrow[r, phantom, "\text{cart}"', shift right=6] & A_k^0 \arrow[d, "\xi_k^0"] \\
\Xi^0 \arrow[r, "\Xi^1"', "|" marking] \arrow[d, equal] & \Xi^0 \arrow[r, "\Xi^1"', "|" marking] \arrow[d, phantom, "\Xi^m"] & \Xi^0 \arrow[d, equal] \\
\Xi^0 \arrow[rr, "\Xi^1"', "|" marking] & & \Xi^0
\end{tikzcd}
=
\begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1", "|" marking] \arrow[d, equal] & A_j^0 \arrow[r, "A_{jk}^1", "|" marking] \arrow[d, phantom, "A!"] & A_k^0 \arrow[d, equal] \\
A_i^0 \arrow[rr, "A_{ik}^1"', "|" marking] \arrow[d, "\xi_i^0"'] & \text{cart} & A_k^0 \arrow[d, "\xi_k^0"] \\
\Xi^0 \arrow[rr, "\Xi^1"', "|" marking] & & \Xi^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad (i, j, k \in \mathbf{C}).$$

Using the functoriality of A and the universal property of versatile colimits, we can verify that $(\Xi^0, \Xi^1, \Xi^e, \Xi^m)$ becomes a monoid Ξ in $\mathbb{K}$.

By the naturality axiom of cells in $\mathbb{M}\mathrm{od}(\mathbb{K})$, the following two composites of cells coincide:

$$\begin{tikzcd}
& A_i^0 \arrow[rr, "A_i^1", "|" marking] \arrow[dl, equal] & \shortparallel & A_i^0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "A!"] & \\
A_i^0 \arrow[rr, "A_i^1"', "|" marking] \arrow[d, equal] & & A_i^0 \arrow[rr, "A_{ii}^1"', "|" marking] \arrow[d, phantom, "A_{ii}^l"] & & A_i^0 \arrow[d, equal] \\
A_i^0 \arrow[rrrr, "A_{ii}^1"', "|" marking] & & & & A_i^0
\end{tikzcd}
=
\begin{tikzcd}
& A_i^0 \arrow[rr, "A_i^1", "|" marking] \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "A!"] & \shortparallel & A_i^0 \arrow[dr, equal] & \\
A_i^0 \arrow[rr, "A_{ii}^1"', "|" marking] \arrow[d, equal] & & A_i^0 \arrow[rr, "A_i^1"', "|" marking] \arrow[d, phantom, "A_{ii}^r"] & & A_i^0 \arrow[d, equal] \\
A_i^0 \arrow[rrrr, "A_{ii}^1"', "|" marking] & & & & A_i^0
\end{tikzcd}
\quad \text{in } \mathbb{K}.$$

Let ξ_i^1 be a cell obtained by the vertical composition of the above cell and the cell (15) with $i = j$. Then, we can verify that (ξ_i^0, ξ_i^1) becomes a vertical arrow $A_i \xrightarrow{\xi_i} \Xi$ in $\mathbb{M}\mathrm{od}(\mathbb{K})$ for each $i \in \mathbf{C}$.

For objects $i, j \in \mathbf{C}$, the cell (15) yields a cartesian cell ξ_{ij} in $\mathbb{M}\mathrm{od}(\mathbb{K})$ of the following form:

$$\begin{tikzcd}
A_i \arrow[rr, "A_{ij}", "|" marking] \arrow[dr, "\xi_i"'] & \xi_{ij} & A_j \arrow[dl, "\xi_j"] \\
& \Xi &
\end{tikzcd}
\quad : \text{cart} \quad \text{in } \mathbb{M}\mathrm{od}(\mathbb{K}).$$

Then, the data $(\xi_i, \xi_{ij})_{i,j}$ yield a vertical cocone ξ from A with the vertex $\Xi \in \mathbb{M}\mathrm{od}(\mathbb{K})$.

We should show that ξ is a versatile colimit of A. Let us begin with the verification of (V) for ξ. Let $l = (l_i, l_{ij})_{i,j}$ be a vertical cocone from A with a vertex $L \in \mathbb{M}\mathrm{od}(\mathbb{K})$. By (V) for the versatile colimit Ξ^0, there is a unique vertical arrow $\Xi^0 \xrightarrow{k^0} L^0$ in $\mathbb{K}$ such that, for all i, $\xi_i^0 \fatsemi k^0 = l_i^0$. By (M1-L) and (M1-R) for the versatile colimit Ξ^0, there is a unique cell k^1 as follows:

$$\begin{tikzcd}
A_i^0 \arrow[r, "A^1_{ij}", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "\xi_{ij}\colon \mathsf{cart}"] & A_j^0 \arrow[d, "\xi_j^0"] \\
\Xi^0 \arrow[r, "\Xi^1", "\shortmid"{marking}] \arrow[d, "k^0"'] \arrow[dr, phantom, "k^1"] & \Xi^0 \arrow[d, "k^0"] \\
L^0 \arrow[r, "L^1"', "\shortmid"{marking}] & L^0
\end{tikzcd}
=
\begin{tikzcd}
A_i^0 \arrow[r, "A^1_{ij}", "\shortmid"{marking}] \arrow[d, "l_i^0"'] \arrow[dr, phantom, "l_{ij}"] & A_j^0 \arrow[d, "l_j^0"] \\
L^0 \arrow[r, "L^1"', "\shortmid"{marking}] & L^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad (i, j \in \mathbf{C}).$$

Using (M2)(M1-L)(M1-R)(M3) for Ξ^0, we can verify that (k^0, k^1) becomes a vertical arrow $\Xi \xrightarrow{k} L$ in $\mathbb{M}\mathrm{od}(\mathbb{K})$ and that it is a unique one satisfying $\xi \fatsemi k = l$.

We next show (H-L) for ξ. Since ξ_i^0 are pullable in $\mathbb{K}$ and since $\mathbb{M}\mathrm{od}(\mathbb{K})$ inherits restrictions from $\mathbb{K}^\flat$ [CS10, 7.4], ξ_i become pullable in $\mathbb{M}\mathrm{od}(\mathbb{K})$. Let $m = (m_i, m_{ij})_{i,j}$ be a left A-module with a vertex $M \in \mathbb{M}\mathrm{od}(\mathbb{K})$. By (H-L) for the versatile colimit Ξ^0, there are horizontal arrow p^1 and cartesian cells σ_i in $\mathbb{K}$ being a modulation of type 1:

$$\begin{tikzcd}
A_i^0 \arrow[r, "m_i^1", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "\sigma_i\colon \mathsf{cart}"] & M^0 \arrow[d, equal] \\
\Xi^0 \arrow[r, "p^1"', "\shortmid"{marking}] & M^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad (i \in \mathbf{C}).$$

By (M0-L) and (M3) for Ξ^0, there exists a unique cell p^l in $\mathbb{K}$ satisfying the following:

$$\begin{tikzcd}
A_i^0 \arrow[r, "A^1_{ij}", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "\xi_{ij}"] & A_j^0 \arrow[r, "m_j^1", "\shortmid"{marking}] \arrow[d, "\xi_j^0"] \arrow[dr, phantom, "\sigma_j"] & M^0 \arrow[d, equal] \\
\Xi^0 \arrow[r, "\Xi^1", "\shortmid"{marking}] \arrow[d, equal] & \Xi^0 \arrow[r, "p^1", "\shortmid"{marking}] \arrow[d, phantom, "p^l"] & M^0 \arrow[d, equal] \\
\Xi^0 \arrow[rr, "p^1"', "\shortmid"{marking}] & & M^0
\end{tikzcd}
=
\begin{tikzcd}
A_i^0 \arrow[r, "A^1_{ij}", "\shortmid"{marking}] \arrow[d, equal] & A_j^0 \arrow[r, "m_j^1", "\shortmid"{marking}] \arrow[d, phantom, "m_{ij}"] & M^0 \arrow[d, equal] \\
A_i^0 \arrow[rr, "m_i^1", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] & \arrow[d, phantom, "\sigma_i"] & M^0 \arrow[d, equal] \\
\Xi^0 \arrow[rr, "p^1"', "\shortmid"{marking}] & & M^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad (i, j \in \mathbf{C})$$

By (M0-L) for Ξ^0, there exists a unique cell p^r in $\mathbb{K}$ corresponding to a modulation of type 0 on the right below:

$$\begin{tikzcd}
\Xi^0 \arrow[r, "p^1", "\shortmid"{marking}] \arrow[d, equal] & M^0 \arrow[r, "M^1", "\shortmid"{marking}] \arrow[d, phantom, "p^r"] & M^0 \arrow[d, equal] \\
\Xi^0 \arrow[rr, "p^1"', "\shortmid"{marking}] & & M^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad \Bigg\| \quad
\begin{tikzcd}
A_i^0 \arrow[r, "m_i^1", "\shortmid"{marking}] \arrow[d, equal] & M^0 \arrow[r, "M^1", "\shortmid"{marking}] \arrow[d, phantom, "m_i^r"] & M^0 \arrow[d, equal] \\
A_i^0 \arrow[rr, "m_i^1"', "\shortmid"{marking}] & & M^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad (i \in \mathbf{C})$$

Then, $p := (p^1, p^l, p^r)$ and the cells σ_i form a horizontal arrow and cells in $\mathbb{M}\mathrm{od}(\mathbb{K})$. Then, we can verify that the cells σ_i become a modulation (of type 1), which shows (H-L) for ξ. The horizontal dual (H-R) also follows similarly. The rest conditions (M1-L)(M1-R)(M2)(M3) for ξ follow from those for Ξ^0 directly. ☐

Corollary 4.7. For any AVDC $\mathbb{K}$, $\mathbb{M}\mathrm{od}(\mathbb{K})$ has all versatile collapses.

Proof. Since versatile colimits for the shape $\mathbb{D}1$ are trivial, this follows directly from Theorem 4.6. ☐

Corollary 4.8. For any AVDC $\mathbb{X}$, $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ has all large versatile collages.

Proof. Combine Lemma 4.5 and Theorem 4.6. ☐

4.2. **Density.**

4.2.1. *A general case.*

Definition 4.9. Let $\mathbb{L}$ be an AVDC. An object $A \in \mathbb{L}$ is called *collage-atomic* [resp. *coproduct-atomic*; *collapse-atomic*] if, for any large versatile collage [resp. coproduct; collapse] $\Xi \in \mathbb{L}$ of $F \colon \mathbb{I}\mathsf{S} \to \mathbb{L}$ [resp. $\mathbb{D}\mathsf{S} \to \mathbb{L}$; $\mathbb{I}1 \to \mathbb{L}$], every vertical arrow $A \xrightarrow{f} \Xi$ in $\mathbb{L}$ uniquely factors through a unique coprojection $Fc \xrightarrow{\xi_c} \Xi$:

$$\begin{tikzcd} & A \arrow[ld, "\exists!"'] \arrow[dd, "f"] \\ Fc \arrow[r, phantom, "="] \arrow[rd, "\xi_c"'] & {} \\ & \Xi \end{tikzcd} \quad \text{in } \mathbb{L} \quad (\exists! c \in \mathsf{S}).$$

◆

Proposition 4.10. Let $\mathbb{X}$ be an AVDC with horizontal units. An $\mathbb{X}$-enriched large category is collage-atomic in $\mathbb{X}$-$\mathbb{P}$rof if and only if it is vertically isomorphic to a semi-object classifier $\mathbf{Z}_c$ for some $c \in \mathbb{X}$.

Proof. Take a versatile collage Ξ of an AVD-functor $A \colon \mathbb{I}\mathsf{S} \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$. By the proof of Theorem 4.6, the forgetful AVD-functor $G \colon \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}^\flat \to \mathbb{M}\mathrm{at}(\mathbb{X})$ sends Ξ to a versatile coproduct of $(G\mathbf{A}_i)_{i \in \mathsf{S}}$. Thus, we obtain the following bijections:

$$\mathrm{Hom}_{\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}}\left(\begin{smallmatrix}\mathbf{Z}_c\\ \Xi\end{smallmatrix}\right) \cong \mathrm{Hom}_{\mathbb{M}\mathrm{at}(\mathbb{X})}\left(\begin{smallmatrix}Y_c\\ G\Xi\end{smallmatrix}\right) \cong \coprod_{i \in \mathsf{S}} \mathrm{Hom}_{\mathbb{M}\mathrm{at}(\mathbb{X})}\left(\begin{smallmatrix}Y_c\\ G\mathbf{A}_i\end{smallmatrix}\right) \cong \coprod_{i \in \mathsf{S}} \mathrm{Hom}_{\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}}\left(\begin{smallmatrix}\mathbf{Z}_c\\ \mathbf{A}_i\end{smallmatrix}\right)$$

This shows that any semi-object classifier $\mathbf{Z}_c$ is collage-atomic in $\mathbb{X}$-$\mathbb{P}$rof.

To prove the converse direction, take a collage-atomic $\mathbb{X}$-enriched large category $\mathbf{A}$ arbitrarily. By Theorem 4.2, $\mathbf{A}$ can be regarded as a large versatile collage of semi-object classifiers. Since $\mathbf{A}$ is collage-atomic, the identity vertical arrow on $\mathbf{A}$ factors through some coprojection $\mathbf{Z}_c \xrightarrow{x} \mathbf{A}$:

$$\begin{tikzcd} & \mathbf{A} \arrow[ld, "\exists! K"'] \arrow[dd, equal] \\ \mathbf{Z}_c \arrow[r, phantom, "="] \arrow[rd, "x"'] & {} \\ & \mathbf{A} \end{tikzcd} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}.$$

Since $\mathbf{Z}_c$ is also collage-atomic, the vertical arrow x must uniquely factor through itself. Thus we have $x \fatsemi K = \mathsf{id}$ and $\mathbf{A} \cong \mathbf{Z}_c$. ☐

A similar proof to Proposition 4.10 works for the following propositions:

Proposition 4.11. Let $\mathbb{K}$ be an AVDC with horizontal units. Then, $A \in \mathbb{M}\mathrm{od}(\mathbb{K})$ is collapse-atomic if and only if it is vertically isomorphic to Uc for some $c \in \mathbb{X}$.

Proposition 4.12. Let $\mathbb{X}$ be an AVDC. Then, $A \in \mathbb{M}\mathrm{at}(\mathbb{X})$ is coproduct-atomic if and only if it is vertically isomorphic to Yc for some $c \in \mathbb{X}$.

Definition 4.13. Let $\mathbb{L}$ be an AVDC. A full sub-AVDC $\mathbb{X} \subseteq \mathbb{L}$ is called *collage-dense* [resp. *coproduct-dense*; *collapse-dense*] if it satisfies following:

- Every object in $\mathbb{X}$ is collage-atomic [resp. coproduct-atomic; collapse-atomic] in $\mathbb{L}$.
- Every object in $\mathbb{L}$ can be written as a large versatile collage [resp. a large versatile coproduct; a versatile collapse] of objects from $\mathbb{X}$. ◆

Remark 4.14. Collage-dense full sub-AVDCs are called *Cauchy generator* in the bicategorical setting [Str04]. ◆

Proposition 4.15. Let $\mathbb{X}$ be an AVDC.

(i) If $\mathbb{X}$ has horizontal units, the full sub-AVDC given by $\mathbb{X} \xhookrightarrow{Z} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ is collage-dense.

(ii) The full sub-AVDC given by $\mathbb{X} \xhookrightarrow{Y} \mathbb{M}\mathrm{at}(\mathbb{X})$ is coproduct-dense.

(iii) If $\mathbb{X}$ has horizontal units, the full sub-AVDC given by $\mathbb{X} \xhookrightarrow{U} \mathbb{M}\mathrm{od}(\mathbb{X})$ is collapse-dense.

4.2.2. *The case of virtual equipments.*

Notation 4.16. Let $\mathbb{L}$ be an AVDC, and let $\mathbb{X} \subseteq \mathbb{L}$ be a full sub-AVDC. For an object $L \in \mathbb{L}$, let $\mathbf{V}\mathbb{X}/L$ denote a category defined as follows:

- An object is a pair (X, x) of an object $X \in \mathbb{X}$ and a vertical arrow $X \xrightarrow{x} L$ in $\mathbb{L}$.
- A morphism $(X, x) \to (X', x')$ is a vertical arrow $X \xrightarrow{f} X'$ in $\mathbb{L}$ such that $f ⨾ x' = x$.

Given $(X, x) \in \mathbf{V}\mathbb{X}/L$, we write Dx for X and identify x with $(Dx, x) \in \mathbf{V}\mathbb{X}/L$. ◆

Definition 4.17. Let $\mathbf{C}$ be a category. An object $m \in \mathbf{C}$ is called *maximal* if every parallel morphisms $m \rightrightarrows \cdot$ have a common retraction. Let $\mathbf{Max}(\mathbf{C}) \subseteq \mathbf{C}$ denote the full subcategory of all maximal objects in a category $\mathbf{C}$. ◆

Remark 4.18. The category $\mathbf{Max}(\mathbf{C})$ always becomes a simply connected groupoid. That is, $\mathbf{Max}(\mathbf{C})$ has at most one morphism between any two objects, and such a morphism is an isomorphism. ◆

Definition 4.19. A category $\mathbf{C}$ is called *C-discrete* if:

- The isomorphism classes of $\mathbf{Max}(\mathbf{C})$ form a large set;
- The inclusion functor $\mathbf{Max}(\mathbf{C}) \hookrightarrow \mathbf{C}$ is final. ◆

Lemma 4.20. The following are equivalent for a category $\mathbf{C}$:

(i) $\mathbf{C}$ is C-discrete.
(ii) There is a final functor $\mathrm{S} \to \mathbf{C}$ from a large discrete category S.
(iii) There is a large set S of objects in $\mathbf{C}$ such that any object in $\mathbf{C}$ has a unique morphism from itself whose codomain lies in S.

Moreover, if these conditions are satisfied, the large set S above becomes isomorphic to a skeleton of $\mathbf{Max}(\mathbf{C})$.

Proof. [(i) $\Longrightarrow$ (ii)] Let S be a skeleton of $\mathbf{Max}(\mathbf{C})$. Since $\mathbf{Max}(\mathbf{C})$ is a simply connected groupoid, the inclusion functor $\mathrm{S} \hookrightarrow \mathbf{Max}(\mathbf{C})$ is final. Since finality is closed under composition, the composite of the inclusions $\mathrm{S} \hookrightarrow \mathbf{Max}(\mathbf{C}) \hookrightarrow \mathbf{C}$ gives a desired final functor.

[(ii) $\Longrightarrow$ (iii)] Let $\Phi \colon \mathrm{S} \to \mathbf{C}$ be a final functor from a large discrete category. By the finality, Φ becomes injective on objects. Then, the image of Φ gives a desired class of objects in $\mathbf{C}$.

[(iii) $\Longrightarrow$ (i)] Let $\mathrm{S} \subseteq \mathrm{Ob}\mathbf{C}$ be the large set in the condition (iii). Let $s \in \mathrm{S}$, and let $f, g \colon s \rightrightarrows c$ be morphisms in $\mathbf{C}$. By the assumption, there is a morphism $h \colon c \to s'$ such that $s' \in \mathrm{S}$. By the uniqueness, we have $f ⨾ h = \mathsf{id} = g ⨾ h$, which shows that s is maximal in $\mathbf{C}$. Thus, the inclusion $\mathrm{S} \hookrightarrow \mathbf{C}$ factors through $\mathbf{Max}(\mathbf{C}) \subseteq \mathbf{C}$, where S is regarded as a large discrete category. Since $\mathrm{S} \hookrightarrow \mathbf{C}$ is final and the inclusion $\mathbf{Max}(\mathbf{C}) \hookrightarrow \mathbf{C}$ is full, the functor $\mathrm{S} \to \mathbf{Max}(\mathbf{C})$ becomes final, hence $\mathbf{Max}(\mathbf{C}) \hookrightarrow \mathbf{C}$ is final. Furthermore, S gives a large skeleton of $\mathbf{Max}(\mathbf{C})$. ☐

Definition 4.21. Let $\mathbb{E}$ be an AVDC with restrictions. Let $\mathbb{X} \subseteq \mathbb{E}$ be a full sub-AVDC. Fix an object $E \in \mathbb{E}$.

(i) We define an AVD-functor $K_E \colon \mathbb{I}(\mathbf{V}\mathbb{X}/E) \to \mathbb{X}$ as follows:

- For $x \in \mathbf{V}\mathbb{X}/E$, $K_E(x) := Dx$.
- For $x, y \in \mathbf{V}\mathbb{X}/E$, $K_E(!_{xy}) := E(x, y)$.

$$\begin{tikzcd}
Dx \arrow[rr, "K_E(!_{xy})", "\shortmid"{marking}] \arrow[ddr, "x"'] & & Dy \arrow[ddl, "y"] \\
& \text{cart} & \\
& E &
\end{tikzcd} \qquad \text{in } \mathbb{E}. \tag{16}$$

- For $x_0, \dots, x_n \in \mathbf{V}\mathbb{X}/E$ and $x_0 \xrightarrow{f} y, x_n \xrightarrow{g} z$ in $\mathbf{V}\mathbb{X}/E$, the assignment to the unique cell ! in $\mathbb{I}(\mathbf{V}\mathbb{X}/E)$ is defined using the universality of the restrictions:

$$\begin{tikzcd}
Dx_0 \arrow[r, "K_E(!_{x_0x_1})", "\shortmid"{marking}] \arrow[dd, "f"'] & \cdots \arrow[r, "K_E(!_{x_{n-1}x_n})", "\shortmid"{marking}] & Dx_n \arrow[dd, "g"] \\
& K_E(!) & \\
Dy \arrow[rr, "K_E(!_{yz})", "\shortmid"{marking}] \arrow[ddr, "y"'] & & Dz \arrow[ddl, "z"] \\
& \text{cart} & \\
& E &
\end{tikzcd}
=
\begin{tikzcd}
Dx_0 \arrow[r, "K_E(!)", "\shortmid"{marking}] \arrow[ddrr, "x_0"'] & Dx_1 \arrow[r, "K_E(!)", "\shortmid"{marking}] \arrow[ddr, "x_1"] & \cdots \arrow[r, "K_E(!)", "\shortmid"{marking}] & Dx_{n-1} \arrow[r, "K_E(!)", "\shortmid"{marking}] \arrow[ddl, "x_{n-1}"'] & Dx_n \arrow[ddll, "x_n"] \\
& \text{cart} & \cdots & \text{cart} & \\
& & E & &
\end{tikzcd} \qquad \text{in } \mathbb{E}.$$

(ii) Furthermore, the cartesian cells (16) yield a vertical cocone $K_E \Rightarrow E$, which is denoted by κ_E. ♦

Theorem 4.22 (The density theorem)**.** Let $\mathbb{E}$ be an AVDC with restrictions. For a full sub-AVDC $\mathbb{X} \subseteq \mathbb{E}$ whose objects are collage-atomic in $\mathbb{E}$, the following are equivalent:

(i) $\mathbb{X} \subseteq \mathbb{E}$ is collage-dense.
(ii) For every $E \in \mathbb{E}$, the vertical cocone κ_E of Definition 4.21 is a versatile colimit and the category $\mathbf{V}\mathbb{X}/E$ is C-discrete.

Proof. [(ii) $\implies$ (i)] Since $\mathbf{V}\mathbb{X}/E$ is C-discrete, there is a final functor $\Phi \colon \mathrm{S} \to \mathbf{V}\mathbb{X}/E$ from a large discrete category S. By Proposition 3.24, Φ induces a final AVD-functor $\mathbb{I}\Phi \colon \mathbb{I}\mathrm{S} \to \mathbb{I}(\mathbf{V}\mathbb{X}/E)$. Then, Theorem 3.26 makes $(\kappa_E)_{\mathbb{I}\Phi}$ be a versatile collage.

[(i) $\implies$ (ii)] Fix $E \in \mathbb{E}$. Let S be a large set, and let $F \colon \mathbb{I}\mathrm{S} \to \mathbb{L}$ be an AVD-functor such that $Fi \in \mathbb{X}$ for any $i \in \mathrm{S}$. Let ξ be a vertical cocone that exhibits E as a versatile colimit of F. Then, the following assignment yields a functor $\Phi \colon \mathrm{S} \to \mathbf{V}\mathbb{X}/E$:

$$i \in \mathrm{S} \quad \overset{\Phi}{\longmapsto} \quad \begin{tikzcd} Fi \arrow[d, "\xi_i"] \\ E \end{tikzcd} \quad \text{in } \mathbf{V}\mathbb{X}/E.$$

By the definition of collage-atomic objects, the functor Φ becomes final, hence $\mathbf{V}\mathbb{X}/E$ is C-discrete. By virtue of the strongness theorem (Theorem 3.42), we have an invertible AVD-transformation of the following form:

$$\begin{tikzcd}
\mathbb{I}\mathrm{S} \arrow[rr, "F"] \arrow[dr, "\mathbb{I}\Phi"'] & \Downarrow\cong & \mathbb{E} \\
& \mathbb{I}(\mathbf{V}\mathbb{X}/E) \arrow[ur, "K_E"'] &
\end{tikzcd} \qquad \text{in } \mathcal{AVDC}.$$

By Proposition 3.24, the induced AVD functor $\mathbb{I}\Phi$ is final. Then, Theorem 3.26 implies that the canonical vertical cocone κ_L becomes a versatile colimit. ☐

4.3. Characterization theorems.

Construction 4.23 (Nerve construction)**.** Let $\mathbb{X} \subseteq \mathbb{L}$ be a full sub-AVDC of an AVDC. Suppose that the following conditions hold for every $L \in \mathbb{L}$:

- The category $\mathbf{V}\mathbb{X}/L$ is C-discrete;
- $\mathbf{Max}(\mathbf{V}\mathbb{X}/L)$ has a skeleton whose elements are pullable in $\mathbb{L}$.

Then, we can construct an AVD-functor $N\colon \mathbb{L}^\flat \to \mathbb{M}\mathrm{at}(\mathbb{X})$ as follows:

(i) Fix $L \in \mathbb{L}$. We choose a skeleton S_L of $\mathbf{Max}(\mathbf{V}\mathbb{X}/L)$ whose elements are pullable in $\mathbb{L}$ and define $NL := \mathrm{S}_L$. For $x \in NL$, its color is defined by $|x| := Dx$.

(ii) For a vertical arrow $A \xrightarrow{f} B$ in $\mathbb{L}$, we write Nf for a morphism $NA \longrightarrow NB$ defined as follows: Let $x \in NA$; since $\mathbf{V}\mathbb{X}/B$ is C-discrete, the vertical arrow $x \mathbin{⨾} f$ uniquely factors through a unique $(Nf)^0x \in NB$:

$$\begin{tikzcd}
& {|x|} \arrow[dl, "x"'] \arrow[dr, "(Nf)^1x"] & \\
A \arrow[dr, "f"'] & = & {|y|} \arrow[dl, "(Nf)^0x"] \\
& B &
\end{tikzcd} \quad \text{in } \mathbb{L},$$

which gives a morphism $x \mapsto (Nf)x$.

(iii) For a horizontal arrow $A \overset{u}{\nrightarrow} B$ in $\mathbb{L}$, we write Nu for a matrix $NA \nrightarrow NB$ over $\mathbb{X}$ defined as follows: For $x \in NA$ and $y \in NB$, the horizontal arrow $(Nu)(x,y)$ is defined as a restriction:

$$\begin{tikzcd}
{|x|} \arrow[r, "(Nu)(x{,}y)", "\shortmid"{marking}] \arrow[d, "x"'] \arrow[dr, phantom, "\mathsf{cart}"] & {|y|} \arrow[d, "y"] \\
A \arrow[r, "u"', "\shortmid"{marking}] & B
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

(iv) For a cell

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A_n \arrow[d, "g"] \\
B \arrow[r, "v"', "\shortmid"{marking}] & C
\end{tikzcd} \quad \text{in } \mathbb{L},$$

we write $N\alpha$ for a cell in $\mathbb{M}\mathrm{at}(\mathbb{X})$ defined by the following:

$$\begin{tikzcd}
{|x_0|} \arrow[r, "Nu_1(x_0{,}x_1)", "\shortmid"{marking}] \arrow[d, equal] & {|x_1|} \arrow[r, "Nu_2(x_1{,}x_2)", "\shortmid"{marking}] \arrow[d, phantom, "(N\alpha)_{x_0x_1\dots x_n}"] & \cdots \arrow[r, "N_{u_n}(x_{n-1}{,}x_n)", "\shortmid"{marking}] & {|x_n|} \arrow[d, equal] \\
{|(Nf)^0x_0|} \arrow[rrr, "Nv((Nf)^0x_0{,}(Ng)^0x_n)"', "\shortmid"{marking}] \arrow[d, "(Nf)^0x_0"'] & & & {|(Ng)^0x_n|} \arrow[d, "(Ng)^0x_n"] \\
B \arrow[rrr, "v"', "\shortmid"{marking}] & & \mathsf{cart} & C
\end{tikzcd}$$

$$= \begin{tikzcd}
{|x_0|} \arrow[r, "Nu_1(x_0{,}x_1)", "\shortmid"{marking}] \arrow[d, "x_0"'] \arrow[dr, phantom, "\mathsf{cart}"] & {|x_1|} \arrow[r, "Nu_2(x_1{,}x_2)", "\shortmid"{marking}] \arrow[d, "x_1"'] \arrow[dr, phantom, "\mathsf{cart}"] & \cdots \arrow[r, "N_{u_n}(x_{n-1}{,}x_n)", "\shortmid"{marking}] \arrow[dr, phantom, "\cdots\ \mathsf{cart}"] & {|x_n|} \arrow[d, "x_n"] \\
A_0 \arrow[r, "u_1"', "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "u_2"', "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "u_n"', "\shortmid"{marking}] & A_n \arrow[d, "g"] \\
B \arrow[rrr, "v"', "\shortmid"{marking}] & & & C
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

♦

Theorem 4.24. The following are equivalent for an AVDC $\mathbb{L}$:

(i) $\mathbb{L}$ is equivalent to $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ for some AVDC $\mathbb{X}$ with horizontal units.
(ii) $\mathbb{L}$ has large versatile collages and a collage-dense full sub-AVDC.

Proof. [(i) $\Longrightarrow$ (ii)] This follows from Corollary 4.8 and Proposition 4.15.

[(ii) $\Longrightarrow$ (i)] In what follows, we write I for the inclusion AVD-functor $\mathbb{X} \hookrightarrow \mathbb{L}$. We first show that the conditions of Construction 4.23 are satisfied for every $L \in \mathbb{L}$. By the collage-density, there are a large set S_L, an AVD-functor $F_L \colon \mathbb{I}\mathrm{S}_L \to \mathbb{L}$ factoring through $\mathbb{X}$, and a vertical cocone ξ^L exhibiting L as a versatile colimit of F_L. Then, by the collage-atomicity, the assignment $s \mapsto \xi^L_s$ yields a final functor $\mathrm{S}_L \to \mathbf{V}\mathbb{X}/L$, which implies C-discreteness. Moreover, the large set $\mathrm{S}_L \cong \{\xi^L_s \mid s \in \mathrm{S}_L\}$ gives a skeleton of $\mathbf{Max}(\mathbf{V}\mathbb{X}/L)$ whose elements are pullable in $\mathbb{L}$. Thus, we obtain the AVD-functor $N \colon \mathbb{L}^\flat \to \mathbb{M}\mathrm{at}(\mathbb{X})$ of Construction 4.23. By Corollary 3.43, $\mathbb{L}$ has all horizontal units, hence we have the AVD-functor $\mathscr{N} \colon \mathbb{L} \to \mathbb{M}\mathrm{od}(\mathbb{M}\mathrm{at}(\mathbb{X})) = \mathbb{X}$-$\mathbb{P}\mathrm{rof}$ corresponding to N.

Let $L \in \mathbb{L}$. By the bijection $\mathrm{S}_L \cong \{\xi^L_s \mid s \in \mathrm{S}_L\}$, the $\mathbb{X}$-enriched large category $\mathbf{N}L := \mathscr{N}(L)$ can be regarded as an AVD-functor of the following form:

$$\mathbb{I}\mathrm{S}_L \xrightarrow{\mathbf{N}L} \mathbb{X} \xhookrightarrow{I} \mathbb{L}.$$

For $s, t \in \mathrm{S}_L$, $I \circ \mathbf{N}L$ sends the unique horizontal arrow $!_{st}$ in $\mathbb{I}\mathrm{S}_L$ to the following restriction:

$$\begin{tikzcd}
F_L s \arrow[r, "\mathbf{N}L(\xi^L_s, \xi^L_t)", "\shortmid" marking] \arrow[d, "\xi^L_s"'] \arrow[dr, phantom, "\mathsf{cart}"] & F_L t \arrow[d, "\xi^L_t"] \\
L \arrow[r, "\mathsf{U}_L"', "\shortmid" marking] & L
\end{tikzcd} \quad \text{in } \mathbb{L},$$

where U_L denotes the horizontal unit on L. Then, by the strongness theorem (Theorem 3.42), $I \circ \mathbf{N}L$ becomes isomorphic to F_L. In what follows, we will regard $F_L = I \circ \mathbf{N}L$.

To show that $\mathscr{N}$ is an equivalence, we will use Theorem 2.11. Let $A, B \in \mathbb{L}$. Since A is a versatile collage of F_A, by (V), the vertical arrows $A \longrightarrow B$ in $\mathbb{L}$ bijectively correspond to the vertical cocones from F_A with the vertex B. By the collage-atomicity and $F_A = \mathbf{N}A$, those vertical cocones correspond to the $\mathbb{X}$-functors $\mathbf{N}A \longrightarrow \mathbf{N}B$.

Take arbitrary data on the left below:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}", "\shortmid" marking] \arrow[d, "f"'] & A_n \arrow[d, "g"] \\
B \arrow[r, dotted, "v"', "\shortmid" marking] & C
\end{tikzcd} \quad \text{in } \mathbb{L} \qquad
\begin{tikzcd}
\mathbf{N}A_0 \arrow[r, dashed, "\mathscr{N}\vec{u}", "\shortmid" marking] \arrow[d, "\mathscr{N}f"'] & \mathbf{N}A_n \arrow[d, "\mathscr{N}g"] \\
\mathbf{N}B \arrow[r, dotted, "\mathscr{N}v"', "\shortmid" marking] & \mathbf{N}C
\end{tikzcd} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \tag{17}$$

Using (M1-L)(M1-R)(M2)(M3) for the versatile collages A_i of F_{A_i}, we can straightforwardly show that the cells fitting into the left of (17) correspond to the cells fitting into the right of (17).

Take $\mathbf{A} \in \mathbb{X}$-$\mathbb{P}\mathrm{rof}$ arbitrarily. Regarding $\mathbf{A}$ as an AVD-functor, we can take a versatile collage ζ with a vertex $Z \in \mathbb{L}$ from the following AVD-functor:

$$\mathbb{I}\mathrm{Ob}\mathbf{A} \xrightarrow{\mathbf{A}} \mathbb{X} \xhookrightarrow{I} \mathbb{L}.$$

Let $s \in \mathrm{S}_Z$. Since $F_Z s \in \mathbb{L}$ is collage-atomic, the vertical arrow ξ^Z_s uniquely factors through $\zeta_{Q^0 s}$ for a unique object $Q^0 s \in \mathbf{A}$:

$$\begin{tikzcd}
& F_Z s \arrow[ld, "Q^1 s"'] \arrow[dd, "\xi^Z_s"] \\
{|Q^0 s|_{\mathbf{A}}} \arrow[rd, "\zeta_{Q^0 s}"'] \arrow[r, phantom, "="] & {} \\
& Z
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

By the strongness theorem (Theorem 3.42) and the universal property of restrictions, there is a unique cell Q_{st} for $s, t \in \mathrm{S}_Z$ as follows:

$$\begin{tikzcd}
F_Z s \arrow[rr, "F_Z(!_{st})", "\shortmid"{marking}] \arrow[d, "Q^1 s"'] \arrow[rrd, phantom, "Q_{st}"] & & F_Z t \arrow[d, "Q^1 t"] & & & & \\
{|Q^0 s|} \arrow[rr, "\mathbf{A}(Q^0 s, Q^0 t)", "\shortmid"{marking}] \arrow[rd, "\zeta_{Q^0 s}"'] \arrow[rrd, phantom, "\zeta_{Q^0 s Q^0 t}"] & & {|Q^0 t|} \arrow[ld, "\zeta_{Q^0 t}"] \arrow[r, phantom, "="] & F_Z s \arrow[rr, "F_Z(!_{st})", "\shortmid"{marking}] \arrow[rd, "\xi^Z_s"'] \arrow[rrd, phantom, "\xi^Z_{st}"] & & F_Z t \arrow[ld, "\xi^Z_t"] \\
& Z & & & Z &
\end{tikzcd} \quad \text{in } \mathbb{L},$$

which gives an invertible $\mathbb{X}$-functor $Q\colon \mathbf{N}Z \xrightarrow{\cong} \mathbf{A}$.

Let $Q\colon \mathbf{N}Z \xrightarrow{\cong} \mathbf{A}$ and $R\colon \mathbf{N}W \xrightarrow{\cong} \mathbf{B}$ be the invertible $\mathbb{X}$-functors constructed above for $\mathbf{A}, \mathbf{B} \in \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$. Let $\mathbf{A} \xrightarrow{P} \mathbf{B}$ be an $\mathbb{X}$-profunctor. Then, by (H-L) for Z and (H-R) for W, we obtain a horizontal arrow $Z \xrightarrow{p} W$ in $\mathbb{L}$ and a vertically invertible cell of the following form:

$$\begin{tikzcd}
\mathbf{N}Z \arrow[r, "\mathscr{N}p", "\shortmid"{marking}] \arrow[d, "Q"', "\cong"] \arrow[rd, phantom, "\mathsf{v.inv}"] & \mathbf{N}W \arrow[d, "R", "\cong"'] \\
\mathbf{A} \arrow[r, "P"', "\shortmid"{marking}] & \mathbf{B}
\end{tikzcd} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}.$$

Then, we conclude that the AVD-functor $\mathscr{N}\colon \mathbb{L} \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ becomes an equivalence. ☐

We can also prove the following theorems in a similar way to Theorem 4.24:

Theorem 4.25. The following are equivalent for an AVDC $\mathbb{L}$:

- (i) $\mathbb{L}$ is equivalent to $\mathbb{M}\mathrm{at}(\mathbb{X})$ for some AVDC $\mathbb{X}$.
- (ii) $\mathbb{L}$ is diminished and has large versatile coproducts and a coproduct-dense full sub-AVDC.

Theorem 4.26. The following are equivalent for an AVDC $\mathbb{L}$:

- (i) $\mathbb{L}$ is equivalent to $\mathbb{M}\mathrm{od}(\mathbb{K})$ for some AVDC $\mathbb{K}$ with horizontal units.
- (ii) $\mathbb{L}$ has versatile collapses and a collapse-dense full sub-AVDC.

4.4. **Closedness under slicing.** In this subsection, we prove that the AVDCs of profunctors are closed under "slicing" as a direct consequence of our characterization theorems. We first generalize to AVDCs the notion of slice double categories [Par11], which has been denoted by the double slash "$/\!\!/$."

Definition 4.27. Let $\mathbb{L}$ be an AVDC, and let $L \in \mathbb{L}$. The *slice* AVDC, denoted by $\mathbb{L}/L$, is the AVDC defined by the following:

- The vertical category is $\mathbf{V}\mathbb{L}/L$;

- A horizontal arrow $x \xrightarrow{u} y$ in $\mathbb{L}/L$ is a pair (Du, u) of a horizontal arrow Du and a cell u
$$\begin{tikzcd} Dx \arrow[rr, "Du", "\shortmid"{marking}] \arrow[dr, "x"'] & & Dy \arrow[dl, "y"] \\ & L \arrow[u, phantom, "u"] & \end{tikzcd} \quad \text{in } \mathbb{L};$$

- A cell $\alpha \in \mathsf{Cell}_{\mathbb{L}/L}\big(f \overset{\vec{u}}{\underset{v}{}} g \big)$ is a cell in $\mathbb{L}$ satisfting the following:
$$\begin{tikzcd} Dx_0 \arrow[r, "Du_1", "\shortmid"{marking}] \arrow[d, "f"'] & \cdots \arrow[r, "Du_n", "\shortmid"{marking}] & Dx_n \arrow[d, "g"] \\ Dy \arrow[rr, dotted, "Dv", "\shortmid"{marking}] \arrow[dr, "y"'] & \alpha & Dz \arrow[dl, "z"] \\ & L \arrow[u, phantom, "v"] & \end{tikzcd} = \begin{tikzcd} Dx_0 \arrow[r, "Du_1", "\shortmid"{marking}] \arrow[dr, "x_0"'] & \cdots \arrow[r, "Du_n", "\shortmid"{marking}] & Dx_n \arrow[dl, "x_n"] \\ & L \arrow[u, phantom, "u_1 \cdots u_n"] & \end{tikzcd} \quad \text{in } \mathbb{L}$$

.

We write $D_L \colon \mathbb{L}/L \to \mathbb{L}$ for the canonical AVD-functor defined as $x \mapsto Dx$. For a full sub-AVDC $\mathbb{X} \subseteq \mathbb{L}$, we write $\mathbb{X}/L \subseteq \mathbb{L}/L$ for the full sub-AVDC consisting of objects $x \in \mathbb{L}/L$ such that $Dx \in \mathbb{X}$. ◆

Lemma 4.28. Let $F \colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Then, a vertical cocone from F with a vertex $L \in \mathbb{L}$ is the same thing as an AVD-functor $\mathbb{K} \to \mathbb{L}/L$ where the post-composite with $D_L \colon \mathbb{L}/L \to \mathbb{L}$ is F.

$$\begin{tikzcd} \mathbb{K} \arrow[r] \arrow[dr, "F"'] & \mathbb{L}/L \arrow[d, "D_L"] \\ & \mathbb{L} \end{tikzcd}$$

Lemma 4.29. Let $\mathbb{L}$ be an AVDC, and let $L \in \mathbb{L}$. Let $G \colon \mathbb{K} \to \mathbb{L}/L$ be an AVD-functor from an AVDC. Suppose that we are given a versatile colimit ξ of $D_L G$ with a vertex $\Xi \in \mathbb{L}$. Then, there is a versatile colimit of G, which is sent to ξ by D_L.

Proof. Let l denote the vertical cocone from $D_L G$ associated with G, and let $L \in \mathbb{L}$ be its vertex. By (V) for the versatile colimit ξ, we obtain the canonical vertical arrow $\Xi \xrightarrow{k} L$ in $\mathbb{L}$. Then, the AVD-functor $H \colon \mathbb{K} \to \mathbb{L}/\Xi$ corresponding to ξ makes the following diagram commute:

$$\begin{tikzcd} \mathbb{K} \arrow[r, "H"] \arrow[drr, "G"'] & \mathbb{L}/\Xi \cong & (\mathbb{L}/L)/k \arrow[d, "D_k"] \\ & & \mathbb{L}/L \end{tikzcd}$$

This gives a vertical cocone from G with the vertex k, which becomes a versatile colimit of G straightforwardly. ☐

Lemma 4.30. Let $\mathbb{X} \subseteq \mathbb{L}$ be a collage-dense [resp. collapse-dense] full sub-AVDC of an AVDC, and let $L \in \mathbb{L}$. Then, $\mathbb{X}/L \subseteq \mathbb{L}/L$ also becomes collage-dense [resp. collapse-dense].

Proof. This follows from Lemma 4.29 directly. ☐

By the characterization theorems (Theorems 4.24 and 4.26), we now have the following:

Corollary 4.31. Let $\mathbb{X}$ be an AVDC with horizontal units.

(i) For an $\mathbb{X}$-enriched category $\mathbf{A}$, there is an equivalence $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}/\mathbf{A} \simeq (\mathbb{X}/\mathbf{A})\text{-}\mathbb{P}\mathrm{rof}$ in $\mathcal{AVDC}$.

(ii) For a monoid M in $\mathbb{X}$, there is an equivalence $\mathbb{M}\mathrm{od}(\mathbb{X})/M \simeq \mathbb{M}\mathrm{od}(\mathbb{X}/M)$ in $\mathcal{AVDC}$.

Remark 4.32. Corollary 4.31(i) is a double categorical refinement of the result in [FL24], which treats the (strict) slice 2-category of the 2-category of enriched categories and functors over a bicategory. ♦

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