On the decomposition of a strong epimorphism into regular epimorphisms

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← Today's slides

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1. Strong and regular epimorphisms

2. The decomposition number

3. Partial Horn theories

4. Main results

Strong and regular epimorphisms

Strong epimorphisms = morphisms having the left lifting property w.r.t. every monomorphisms.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\text{strong epi}"'] & \cdot \arrow[d, hook, "\text{mono}"] \\
\cdot \arrow[r] \arrow[ur, dotted, "\exists!"] & \cdot
\end{tikzcd}$$

Regular epimorphisms = morphisms being the coequalizer of some parallel pair of morphisms.

$$\begin{tikzcd}
\cdot \arrow[r, bend left] \arrow[r, bend right] & \cdot \arrow[r, two heads, "\text{regular epi}"] & \cdot
\end{tikzcd}$$

Theorem ([Gabriel and Ulmer 1971])

In a locally presentable category,

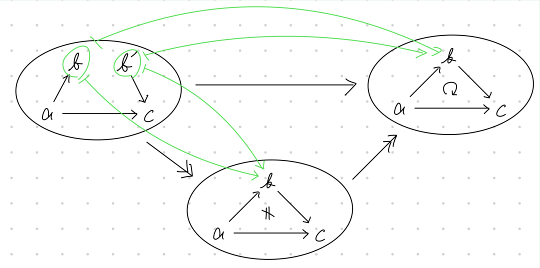
strong epis = transfinite composites of *regular epis*

Example

Cat: the category of small categories.

$$\begin{tikzcd}
A_0 \arrow[rr, "p"] \arrow[dr, two heads, "\text{reg. epi}"'] & & A_2 \\
& A_1 \arrow[ur, two heads, "\text{reg. epi}"'] &
\end{tikzcd}$$

in **Cat**

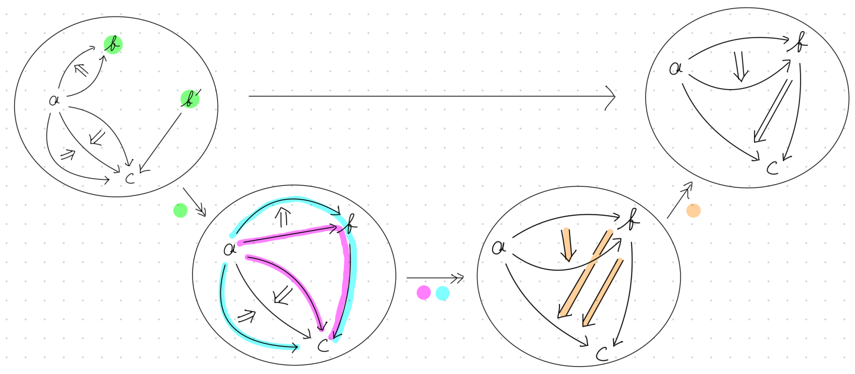


Example

$\mathbf{2Cat}$: the category of small 2-categories.

$$\begin{tikzcd}
A_0 \arrow[rrr, "p"] \arrow[dr, two heads, "\text{reg. epi}"'] & & & A_3 \\
& A_1 \arrow[r, two heads, "\text{reg. epi}"'] & A_2 \arrow[ur, two heads, "\text{reg. epi}"'] &
\end{tikzcd}$$

in $\mathbf{2Cat}$



Actually...

Fact I

The length of the regular epi chains in the previous slides can NOT be shorter.

Fact II

1. In $\mathbf{Cat}$, every strong epimorphism is decomposed into two regular epimorphisms.
2. In $\mathbf{2Cat}$, every strong epimorphism is decomposed into three regular epimorphisms.

How to prove?

Definition

A regular decomposition (of length α) of $A \xrightarrow{p} X$ in $\mathscr{C}$ is a cocts. functor D s.t.

- $$\begin{tikzcd} \mathbb{1} \arrow[d, "\ulcorner \alpha \urcorner"'] \arrow[dr, "\ulcorner p \urcorner"] & \\ \mathbb{\alpha}+1 \arrow[r, dotted, "D"'] & A/\mathscr{C} \end{tikzcd}$$ commutes in $\mathbf{CAT}$;

  ($\mathbb{1}$: the terminal, $\mathbb{\alpha}+1 := \{0 < 1 < \cdots < \alpha\}$)

- $D_{\beta,\beta+1}$ is a regular epimorphism for any $0 \leq \beta < \alpha$.

$$\begin{tikzcd} A \arrow[rrr, "p"] \arrow[d, "D0"', "\cong"] \arrow[dr, "D1"] \arrow[drrr, "D\alpha"] \arrow[drr, phantom, "\cdots"] & & & X \arrow[d, equal] \\ \cdot \arrow[r, two heads, "D_{0,1}"'] & \cdot \arrow[r, two heads, "D_{1,2}"'] & \cdots \arrow[r, two heads] & \cdot \end{tikzcd}$$ in $\mathscr{C}$

The decomposition number

Definition

$\mathscr{A}$: a locally presentable category.

1. The decomposition number $\delta(f)$ of $A \xrightarrow{f} B$ in $\mathscr{A}$ is the smallest ordinal number α s.t. $f = {}^{\exists}\underset{\text{mono}}{\underline{m}} \circ {}^{\exists}p$ with a reg.decomp. of length α of p.

$$\begin{tikzcd}
A \arrow[rrrr, "f"] \arrow[d, equal] \arrow[drrrr, "p"] & & & & B \\
A_0 \arrow[r, two heads] & A_1 \arrow[r, two heads] & A_2 \arrow[r, two heads] & \cdots & A_\alpha \arrow[u, hook, "m"']
\end{tikzcd}$$

2. $\delta(\mathscr{A}) := \min\{\alpha \mid \delta(f) < \alpha$ for every f in $\mathscr{A}\}$.

Theorem ([Gabriel and Ulmer 1971])

$\mathscr{A}$: a locally λ-presentable category.

$\Longrightarrow$ $\forall f$ in $\mathscr{A}$, $\delta(f) \leq \lambda$. Therefore, $\delta(\mathscr{A}) \leq \lambda + 1$.

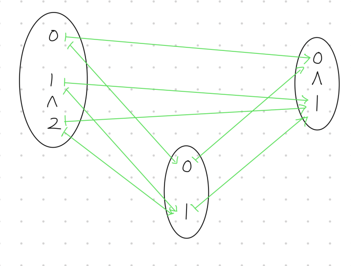
The decomposition number

Example

Pos: the category of posets.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "\text{reg. epi}"'] & & B \\
& X \arrow[ur, hook, "\text{mono}"'] &
\end{tikzcd}$$

in **Pos**



In this case, $\delta(f) = 1$ and $\delta(\mathbf{Pos}) = 2$.

The small object argument

$\mathscr{A}$: locally λ-presentable category.

$$\begin{tikzcd}
A \arrow[r, two heads, "q_{0,1}"] \arrow[d, "f"'] & A_1 \arrow[r, two heads, "q_{1,2}"] \arrow[dl, "f_1"'] & A_2 \arrow[dll, "f_2"'] & \cdots & A_\omega \arrow[r, two heads, "q_{\omega,\omega+1}"] \arrow[dllll, "f_\omega"'] & A_{\omega+1} \arrow[dlllll, "f_{\omega+1}"'] & \cdots & A_\lambda \arrow[dlllllll, hook, "f_\lambda"] & & \text{in } \mathscr{A} \\
B & & & & & & & & &
\end{tikzcd}$$

A_1: the *coimage* of f (=: the coequalizer of the kernel pair of f)

A_2: the *coimage* of f_1

A_ω: the colimit of the chain $(A_n)_{n<\omega}$

$A_{\omega+1}$: the *coimage* of f_ω

At least f_λ becomes monic. Let $\sigma(f)$ denote the smallest ordinal number α s.t. f_α is monic.

Corollary

$$\delta(f) \leq \sigma(f)$$

Theorem

In a locally presentable category, $$\delta(f) = \sigma(f).$$

Proof.

For simplicity, we assume $\delta(f) = n < \omega$.

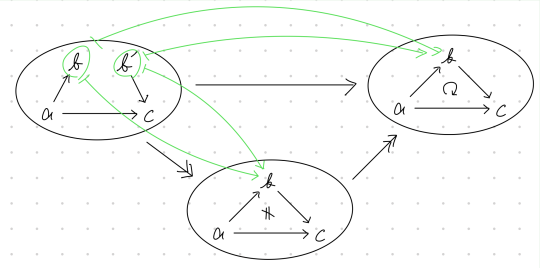
$$\begin{tikzcd}
X_0 \arrow[rrrr, bend left, "f"] \arrow[r, two heads, "p_1"] \arrow[d, two heads, "q_{0,1}"'] & X_1 \arrow[r, two heads, "p_2"] \arrow[dl, dotted, "\text{epi}"'] \arrow[d, phantom, "\text{p.o.}"] & X_2 \arrow[r, two heads, "p_3"] \arrow[dl, "\text{epi}"'] \arrow[d, phantom, "\text{p.o.}"] & \cdots \arrow[r, two heads, "p_n"] \arrow[d, phantom, "\text{p.o.}"] & X_n \arrow[d, equal] \arrow[dl, equal, "\cong"'] \\
A_1 \arrow[r, two heads] \arrow[rrr, bend right=10, "f_1"'] \arrow[rrrr, bend right=20, "f_1"] \arrow[d, two heads, "q_{1,2}"'] & \cdot \arrow[r, two heads] \arrow[dl, "\text{epi}"'] \arrow[d, phantom, "\text{p.o.}"] & \cdots \arrow[r, two heads] \arrow[d, phantom, "\text{p.o.}"] & \cdot \arrow[r, dotted, "\cong"] \arrow[dl, equal] & X_n \\
A_2 \arrow[r, two heads] \arrow[rr, bend right=10, "f_2"'] \arrow[d, two heads, "q_{2,3}"'] & \cdots \arrow[r, two heads] & \cdot \arrow[dl, equal] & & \\
\vdots \arrow[d, two heads, "q_{n-1,n}"'] & \ddots \arrow[dl, equal] & & & \\
A_n & & & &
\end{tikzcd}$$

Thus, we have $\sigma(f) \leq n$. □

Example (recall)

$$\begin{tikzcd}
A_0 \arrow[rr, "p"] \arrow[dr, two heads, "\text{reg. epi}"'] & & A_2 \\
& A_1 \arrow[ur, two heads, "\text{reg. epi}"'] &
\end{tikzcd}$$

in **Cat**

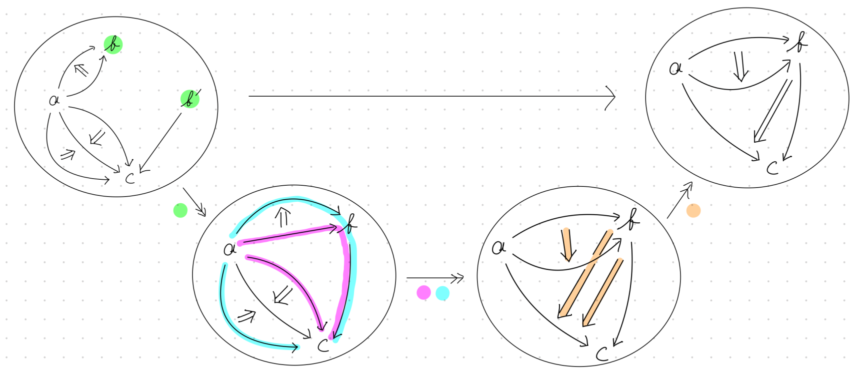


$$\implies \quad \delta(p) = \sigma(p) = 2.$$

Example (recall)

$$\begin{tikzcd}
A_0 \arrow[rrr, "p"] \arrow[dr, two heads, "\text{reg. epi}"'] & & & A_3 \\
& A_1 \arrow[r, two heads, "\text{reg. epi}"'] & A_2 \arrow[ur, two heads, "\text{reg. epi}"'] &
\end{tikzcd}$$

in **2Cat**



$\Longrightarrow \quad \delta(p) = \sigma(p) = 3.$

Milestones

✓

Fact I (recall)

The regular epi chains in our examples can NOT be shorter.

Fact II (recall)

1. In $\mathbf{Cat}$, every strong epimorphism is decomposed into <u>two</u> regular epimorphisms.
2. In $\mathbf{2Cat}$, every strong epimorphism is decomposed into <u>three</u> regular epimorphisms.

Partial Horn theories

Σ: an S-sorted (λ-ary) signature.

- A term $\tau ::= x \mid f(\tau_i)_{i<\alpha}$;
- A (λ-ary) Horn formula $\varphi ::= \top \mid \bigwedge_{i<\alpha} \varphi_i \mid \tau = \tau' \mid R(\tau_i)_{i<\alpha}$;
- A (λ-ary) context $\cdots$ $\vec{x} = (x_i)_{i<\alpha}$ (a family of distinct variables);
- $\vec{x}.\tau$: a *term-in-context*, i.e., all variables of τ are in the context $\vec{x}$;
- $\vec{x}.\varphi$: a *Horn formula-in-context*, i.e., all variables of φ are in the context $\vec{x}$.

Here, $\alpha < \lambda$.

Definition

1. A (λ-ary) Horn sequent over Σ is an expression of the form

$$\varphi \overset{\vec{x}}{\longmapsto} \psi \qquad (\text{“}\varphi \text{ implies } \psi\text{”})$$

   (φ, ψ are λ-ary Horn formulas over Σ in the same λ-ary context $\vec{x}$.)

2. A (λ-ary) partial Horn theory $\mathbb{T}$ over Σ is a set of (λ-ary) Horn sequents over Σ.

Horn vs partial Horn

What is the difference between ordinary Horn theory and partial Horn theory?
⇝ It lies in the concept of models.

	(ordinary) Horn theory	partial Horn theory
Axiom	Horn sequent $\varphi \overset{\vec{x}}{\vdash} \psi$	Horn sequent $\varphi \overset{\vec{x}}{\vdash} \psi$
Interpretation of func.symb.	total map $M_{\vec{s}} \xrightarrow{[\![f]\!]_M} M_s$	partial map $M_{\vec{s}} \xrightharpoonup{[\![f]\!]_M} M_s$
Interpretation of rel.symb.	subset $[\![R]\!]_M \subseteq M_{\vec{s}}$	subset $[\![R]\!]_M \subseteq M_{\vec{s}}$
Validity of φ	"φ holds."	"All terms in φ are defined and φ holds."
Validity of $\varphi \overset{\vec{x}}{\vdash} \psi$	"If φ holds then ψ holds."	"If all terms in φ are defined and φ holds, then all terms in ψ are defined and ψ holds."

Especially,

An equation $\tau = \tau$ holds iff the value of the partial map $[\![\tau]\!]_M$ is defined.

So, we will use the abbreviation $\tau{\downarrow}$ for $\tau = \tau$.

Categories of partial models

Notation

$\mathbb{T}$: a partial Horn theory.

$\mathbf{PMod}\,\mathbb{T}$: the category of (partial) models of $\mathbb{T}$.

Fact

A category $\mathscr{A}$ is locally λ-presentable $\iff$ $\mathscr{A} \simeq \mathbf{PMod}\,\mathbb{T}$ for some λ-ary partial Horn theory $\mathbb{T}$.

Example: small categories

Example (small categories)

The $S := \{\mathrm{ob}, \mathrm{mor}\}$-sorted signature Σ_{cat} consists of:

$$\mathrm{id}\colon \mathrm{ob} \to \mathrm{mor}, \quad \mathrm{d}\colon \mathrm{mor} \to \mathrm{ob}, \quad \mathrm{c}\colon \mathrm{mor} \to \mathrm{ob}, \quad \circ\colon \mathrm{mor} \sqcap \mathrm{mor} \to \mathrm{mor}.$$

The partial Horn theory $\mathbb{T}_{\mathrm{cat}}$ over Σ_{cat} consists of:

$$\top \xmapsto{x:\mathrm{ob}} \mathrm{id}(x)\downarrow, \qquad (\text{ id is total. })$$

$$\top \xmapsto{f:\mathrm{mor}} \mathrm{d}(f)\downarrow \wedge \mathrm{c}(f)\downarrow, \qquad (\text{ d and c are total. })$$

$$(g \circ f)\downarrow \xmapsto{g,f:\mathrm{mor}} \mathrm{d}(g) = \mathrm{c}(f), \qquad (\ g \circ f \text{ is defined iff } \mathrm{d}(g) = \mathrm{c}(f). \ )$$

and so on.
⇝ We have $\mathbf{PMod}\,\mathbb{T}_{\mathrm{cat}} \cong \mathbf{Cat}$.

Example: small 2-categories

Example (small 2-categories)

There is an $S := \{0, 1, 2\}$-sorted signature Σ_{2cat} and a finitary PHT $\mathbb{T}_{\mathrm{2cat}}$ over Σ_{2cat} s.t.

$$\mathbf{PMod}\,\mathbb{T}_{\mathrm{2cat}} \cong \mathbf{2Cat}.$$

Example: posets

Example (posets)

Let $S := \{*\}$, $\Sigma_{\mathrm{pos}} := \{\leq : * \sqcap *\}$.
The partial Horn theory $\mathbb{T}_{\mathrm{pos}}$ over Σ_{pos} consists of:

$$\top \xrightarrow{x} x \leq x, \quad x \leq y \wedge y \leq x \xrightarrow{x, y} x = y, \quad x \leq y \wedge y \leq z \xrightarrow{x, y, z} x \leq z.$$

Then, we have $\mathbf{PMod}\,\mathbb{T}_{\mathrm{pos}} \cong \mathbf{Pos}$.

Gauges

Definition

$\mathbb{T}$: a λ-ary PHT.

A gauge (of length α) for $\mathbb{T}$ is an assignment to each term $\vec{x}.\tau$ in a λ-ary context, of the following data:

- an ordinal number $\sharp(\vec{x}.\tau) < \alpha$;
- a set $\mathrm{Def}(\vec{x}.\tau)$ of pairs (σ^0, σ^1) of terms in the context $\vec{x}$

such that, for every $\vec{x}.\tau$,

- $\mathbb{T} \vDash \left(\tau = \tau \overset{\vec{x}}{\longmapsto\!\!\dashv} \bigwedge_{(\sigma^0,\sigma^1)\in\mathrm{Def}(\vec{x}.\tau)} \sigma^0 = \sigma^1\right)$;
- $\forall(\sigma^0, \sigma^1) \in \mathrm{Def}(\vec{x}.\tau).\ \ \sharp(\vec{x}.\sigma^0), \sharp(\vec{x}.\sigma^1) < \sharp(\vec{x}.\tau)$.

Theorem

$\mathbb{T}$: a λ-ary PHT with a gauge of length α.

$\implies \delta(f) \leq \alpha$ ($\forall f$ in $\mathbf{PMod}\,\mathbb{T}$), hence $\delta(\mathbf{PMod}\,\mathbb{T}) \leq \alpha + 1$.

How to construct a gauge?

Definition (depth)

$\mathbb{T}$: a λ-ary partial Horn theory.

- Let $\vec{x}$ be a λ-ary context.

$$\mathrm{Term}_1(\vec{x}) := \{\vec{x}.\tau \mid \mathbb{T} \vDash (\tau\!\downarrow \xmapsto{\vec{x}} \top)\}.$$

$$\mathrm{Term}_{\beta+1}(\vec{x}) := \mathrm{Term}_\beta(\vec{x}) \cup$$

$$\left\{ \vec{x}.\tau \;\middle|\; \exists E \subseteq \mathrm{Term}_\beta(\vec{x})^2 \text{ s.t. } \mathbb{T} \vDash (\tau\!\downarrow \xmapsto{\vec{x}} \bigwedge_{(\sigma^0,\sigma^1)\in E} \sigma^0 = \sigma^1) \right\}.$$

$$\mathrm{Term}_{\sup\beta}(\vec{x}) := \bigcup_{\beta} \mathrm{Term}_\beta(\vec{x}).$$

- dep$(\vec{x})$ $:= \min\{\alpha \mid \mathrm{Term}_\alpha(\vec{x}) = \mathrm{Term}_{\alpha+1}(\vec{x})\}$.
- dep$(\mathbb{T})$ $:= \min\{\alpha \mid \forall \vec{x}$: λ-ary. dep$(\vec{x}) < \alpha\}$ (the depth of $\mathbb{T}$).

Lemma

If every $\vec{x}.\tau$ belongs to $\mathrm{Term}_\alpha(\vec{x})$ for some α ($\overset{\text{def}}{\Leftrightarrow}$: $\mathbb{T}$ is essentially algebraic)
$\Longrightarrow$ $\mathbb{T}$ has a gauge of length "dep$(\mathbb{T}) - 1$."

Theorem

$\mathbb{T}$: essentially algebraic $\implies \delta(\mathbf{PMod}\,\mathbb{T}) \leq \begin{cases} \mathsf{dep}(\mathbb{T}) & \text{if } \mathsf{dep}(\mathbb{T})\text{: a successor} \\ \mathsf{dep}(\mathbb{T})+1 & \text{else} \end{cases}$

Example

$$\delta(\mathbf{Pos}) \leq \mathsf{dep}(\mathbb{T}_{\mathrm{pos}}) = 2;$$
$$\delta(\mathbf{Cat}) \leq \mathsf{dep}(\mathbb{T}_{\mathrm{cat}}) = 3;$$
$$\delta(\mathbf{2Cat}) \leq \mathsf{dep}(\mathbb{T}_{\mathrm{2cat}}) = 4.$$

Therefore,

$$\delta(\mathbf{Pos}) = 2;$$
$$\delta(\mathbf{Cat}) = 3;$$
$$\delta(\mathbf{2Cat}) = 4.$$

Milestones

✓

Fact I (recall)

The regular epi chains in our examples can NOT be shorter.

✓

Fact II (recall)

1. In $\mathbf{Cat}$, every strong epimorphism is decomposed into <u>two</u> regular epimorphisms.
2. In $\mathbf{2Cat}$, every strong epimorphism is decomposed into <u>three</u> regular epimorphisms.

The decay number

Definition

$\mathbb{T}$: a λ-ary partial Horn theory.

- L: a set of terms in a common context.

$$\mathsf{eq}(L) := \left(\bigwedge_{\substack{\tau, \tau' \in L \\ \text{with the same sort}}} \tau = \tau' \right).$$

- $\vec{x}$: a λ-ary context.

$$\mathsf{dec}(\vec{x}) := \min \left\{ \alpha \,\middle|\, \mathbb{T} \vDash \left(\mathsf{eq}(\mathrm{Term}_\alpha(\vec{x})) \vdash^{\vec{x}} \mathsf{eq}(\mathrm{Term}_{\alpha+1}(\vec{x})) \right) \right\}.$$

- $\mathsf{dec}(\mathbb{T}) := \min\{\alpha \mid \forall \vec{x}\colon \lambda\text{-ary}.\ \mathsf{dec}(\vec{x}) < \alpha\}$ (the decay number of $\mathbb{T}$).

Remark

$$\mathsf{dec}(\vec{x}) \leq \mathsf{dep}(\vec{x}),\ \text{hence } \mathsf{dec}(\mathbb{T}) \leq \mathsf{dep}(\mathbb{T}).$$

Proposition

For $\langle \vec{x}.\top \rangle \xrightarrow{\ !\ } 1$ in $\mathbf{PMod}\,\mathbb{T}$, $\delta(!) = \mathsf{dec}(\vec{x})$.

Example

Let $\mathbb{T}$ be the single-sorted finitary PHT defined as follows:

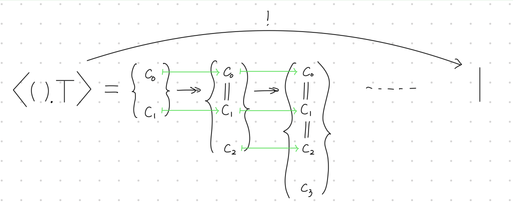
$$\Sigma := \{\ c_n\colon \text{constants}\ \ (\text{for } n \geq 0)\ \},$$

$$\mathbb{T} := \left\{ \begin{array}{c} \top \vdash\!\!\relbar\!\!\relbar\ c_0 = c_0 \\ c_0 = c_n \vdash\!\!\relbar\!\!\dashv c_{n+1} = c_{n+1}\ \ (\text{for } n \geq 0) \end{array} \right\}.$$

Then,

$$\mathrm{Term}_1() = \{c_0, c_1\}, \quad \mathrm{Term}_2() = \{c_0, c_1, c_2\}, \quad \mathrm{Term}_3() = \{c_0, c_1, c_2, c_3\}, \ldots$$

$$\mathsf{dec}() = \mathsf{dep}() = \omega.$$



in $\mathbf{PMod}\,\mathbb{T}$.

Corollary

$$\mathsf{dec}(\mathbb{T}) \leq \delta(\mathbf{PMod}\,\mathbb{T}).$$

Theorem (summary)

1. If $\mathbb{T}$ is essentially algebraic,

$$\mathsf{dec}(\mathbb{T}) \leq \delta(\mathbf{PMod}\,\mathbb{T}) \leq \begin{cases} \mathsf{dep}(\mathbb{T}) & \text{if } \mathsf{dep}(\mathbb{T})\text{: a successor} \\ \mathsf{dep}(\mathbb{T}) + 1 & \text{else} \end{cases}$$

2. If $\mathbb{T}$: ess.alg., $\mathsf{dec}(\mathbb{T}) = \mathsf{dep}(\mathbb{T})$, and it is a successor, then

$$\delta(\mathbf{PMod}\,\mathbb{T}) = \mathsf{dep}(\mathbb{T}).$$

Thank you!

Today's slides

References I

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Future directions

1. Can we replace "$=$" with an arbitrary relation symbol R? (e.g. *coinserters* in $\mathbf{Pos}$ rather than regular epis)
2. Is there a locally finitely presentable category $\mathscr{A}$ s.t. $\delta(\mathscr{A}) = \omega$? (We already have examples s.t. $\delta(\mathscr{A}) = 1, 2, 3, 4, \ldots$ and $\omega + 1$.)
3. Is there a better way to determine $\delta(\mathscr{A})$ completely?
4. Is there any connection with other logical theories (rather than partial Horn theories)? (e.g. *generalized algebraic theories (GAT)*, *essentially algebraic theories*, etc.)

Motivation

In abstract algebra (or universal algebra), the homomorphism theorem is fundamental. Categorically, it can be treated by *regular categories*.

Recall

In a regular category,

- Every morphism can be decomposed into a *regular epimorphism* and a *monomorphism*.
- Such a decomposition is always given in the "canonical" way: taking a quotient by the *kernel pair*.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\
& A/\mathrm{Ker}\, f \arrow[ur, hook] &
\end{tikzcd}$$

- The class of regular epimorphisms is stable under pullbacks.

Motivation

Example

The regular categories include various categories considered in classical universal algebra: groups, monoids, etc.

The above examples are captured by the following general fact:

Fact

Monadic categories over $\mathbf{Set}$ are regular.

Motivation

There are several directions to generalize classical universal algebra "syntactically." For example:

Enriching	$\mathscr{V}$-enriched λ-ary monadic categories over $\mathscr{V}$ [Rosický and Tendas 2024].
Relativizing	($\mathbf{Set}$-enriched) λ-ary monadic categories over a locally λ-presentable category [Kawase 2024].
Enr. & Rel.	$\mathscr{V}$-enriched λ-ary monadic categories over a locally λ-presentable $\mathscr{V}$-category [Rosický 2021].

$$\begin{tikzcd}
 & \text{Enr. \& Rel.} & \\
\text{Enr.} \arrow[ur, squiggly, "\text{relativize}"] & & \text{Rel.} \arrow[ul, squiggly, "\text{enrich}"'] \\
 & \text{Classical univ. alg.} \arrow[ul, squiggly, "\text{enrich}"'] \arrow[ur, squiggly, "\text{relativize}"] &
\end{tikzcd}$$

Motivation

A problem

Monadic categories over a locally presentable category are NOT regular in general, even when the base category is regular.

Example

$\mathbf{Cat}$, the category of small categories, are finitary monadic over $\mathbf{Quiv}$, the category of quivers (=directed graphs). However, $\mathbf{Cat}$ is not regular even if $\mathbf{Quiv}$ is regular.

Representing models

$\mathbb{T}$: a λ-ary partial Horn theory.

Construction

$\vec{x}.\varphi$: a $\kappa(\geq \lambda)$-ary Horn formula (in a κ-ary context).

- A term $\vec{x}.\tau$ is defined under $\vec{x}.\varphi$ $\overset{\text{def}}{\Leftrightarrow}$ $\varphi \vdash^{\vec{x}} \tau\downarrow$ can be derived from $\mathbb{T}$.
(written $\mathbb{T} \vDash (\varphi \vdash^{\vec{x}} \tau\downarrow)$)
- The following gives an equivalence relation on the terms defined under $\vec{x}.\varphi$:

$$\tau \sim \tau' \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathbb{T} \vDash (\varphi \vdash^{\vec{x}} \tau = \tau').$$

- Quotienting all of the terms defined under $\vec{x}.\varphi$ by $\sim$, we obtain a $\mathbb{T}$-model $\langle \vec{x}.\varphi \rangle_{\mathbb{T}}$, called the representing $\mathbb{T}$-model.

Fact

1. For every $\mathbb{T}$-model M, $[\![\vec{x}.\varphi]\!]_M \cong \mathbf{PMod}\,\mathbb{T}(\langle \vec{x}.\varphi \rangle_{\mathbb{T}}, M)$.
2. A $\mathbb{T}$-model M is $\kappa(\geq \lambda)$-presentable $\iff$ $M \cong \langle \vec{x}.\varphi \rangle_{\mathbb{T}}$ for some κ-ary Horn formula $\vec{x}.\varphi$.