

Universal algebra over locally presentable categories

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Categories in Tokyo 1st



← Today's slides

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1 Relativization of universal algebra

2 Birkhoff's variety theorem

3 Filtered colimit elimination

Relativization via monads

Theorem ([Lin69])

There is an equivalence

$$\mathbf{Th}^S \simeq \mathbf{Mnd}_f(\mathbf{Set}^S).$$

Here,

\mathbf{Th}^S : the category of S -sorted equational theories,

$\mathbf{Mnd}_f(\mathbf{Set}^S)$: the category of finitary monads on \mathbf{Set}^S .

S -sorted equational theory = finitary monad on \mathbf{Set}^S

↓ generalize

???

\mathcal{A} -relative algebraic theory = κ -ary monad on \mathcal{A}
(\mathcal{A} : a locally κ -presentable category)

Relative algebraic theories

Informal definition [Kaw23]

\mathcal{A} : a (locally presentable) category

An \mathcal{A} -relative algebraic theory consists of:

- a set Ω of partial operators;
- a set E of implications $\dots (\underbrace{\text{YYY}}_{\text{postcondition}} \text{ whenever } \underbrace{\text{XXX}}_{\text{precondition}})$

such that

- For each operator $\omega \in \Omega$, its domain must be defined by “ \mathcal{A} ’s language.”
- For each implication in E , its precondition must be written in “ \mathcal{A} ’s language.”

\mathbf{Set}^S -relative algebraic theories = S -sorted equational theories

A generalized Linton theorem

Theorem ([Kaw23; Kaw24])

For a locally κ -presentable category \mathcal{A} , there is an equivalence

$$\mathbf{Th}_{\kappa}^{\mathcal{A}} \simeq \mathbf{Mnd}_{\kappa}(\mathcal{A}).$$

Here,

$\mathbf{Th}_{\kappa}^{\mathcal{A}}$: the category of \mathcal{A} -relative (κ -ary) algebraic theories,

$\mathbf{Mnd}_{\kappa}(\mathcal{A})$: the category of κ -ary monads on \mathcal{A} .

Example: small categories

Example

A **small category** consists of:

- a base quiver $\text{mor}\mathcal{C} \xrightleftharpoons[c]{d} \text{ob}\mathcal{C}$;
- a total operator $\text{id}: \text{ob}\mathcal{C} \rightarrow \text{mor}\mathcal{C}$;
- a **partial** operator $\circ: \text{mor}\mathcal{C} \times \text{mor}\mathcal{C} \rightarrow \text{mor}\mathcal{C}$ such that

$$g \circ f \text{ is defined iff } d(g) = c(f)$$

which satisfy the following:

- $d(\text{id}(x)) = x$ and $c(\text{id}(x)) = x$;
- $d(g \circ f) = d(f)$ and $c(g \circ f) = c(g)$ whenever $d(g) = c(f)$;
- $f \circ \text{id}(d(f)) = f$ and $\text{id}(c(f)) \circ f = f$;
- $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $d(h) = c(g)$ and $d(g) = c(f)$.

Small categories are algebras over quivers.

Further examples

Example

algebras over \sim		
small categories	\rightsquigarrow	quivers
UDO semirings	\rightsquigarrow	posets
partial Boolean algebras	\rightsquigarrow	graphs
monoid-graded rings	\rightsquigarrow	monoid-graded sets
generalized complete metric spaces	\rightsquigarrow	generalized metric spaces
Banach spaces	\rightsquigarrow	pointed metric spaces

A technical remark

Definition ([PV07])

- (κ -ary) **partial Horn theory** \cdots a logical theory based on multi-sorts, partial functions, relations, and (partial) Horn implications.
- **Mod \mathbb{S}** \cdots the category of models of a partial Horn theory \mathbb{S} .

Theorem ([PV07])

TFAE for a category \mathcal{A} :

- 1 \mathcal{A} is locally κ -presentable.
- 2 $\mathcal{A} \simeq \mathbf{Mod} \mathbb{S}$ for some κ -ary partial Horn theory \mathbb{S} .

We actually define **\mathbb{S} -relative algebraic theories** for partial Horn theories \mathbb{S} .

$$\rightsquigarrow \mathcal{A}\text{-rel. alg. theory} = \mathbb{S}\text{-rel. alg. theory where } \mathcal{A} \simeq \mathbf{Mod} \mathbb{S}.$$

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Birkhoff's variety theorem

Birkhoff's variety theorem [Bir35]

(Ω, E) : a single-sorted algebraic theory. $\mathcal{C} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.

TFAE:

- ① $\mathcal{C} \subseteq \mathbf{Alg}(\Omega, E)$ is definable by equations.
- ② $\mathcal{C} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, subobjects, and quotients.

closed under products: $A_i \in \mathcal{C} \implies \prod_i A_i \in \mathcal{C}$.

closed under subobjects: $B \subseteq A$: sub , $A \in \mathcal{C} \implies B \in \mathcal{C}$.

closed under quotients: $A \twoheadrightarrow B$: surj , $A \in \mathcal{C} \implies B \in \mathcal{C}$.

A generalized Birkhoff's theorem

Theorem ([Kaw23; Kaw24])

(Ω, E) : an \mathcal{A} -relative (κ -ary) algebraic theory. $\mathcal{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
TFAE:

- ① $\mathcal{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
- ② $\mathcal{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, closed subobjects, (U, κ) -pure quotients, and κ -filtered colimits.

single-sorted alg. (Set-relative alg.)		\mathcal{A} -relative alg.
products	\rightsquigarrow	products
subobjects	\rightsquigarrow	<i>closed subobjects</i>
quotients	\rightsquigarrow	<i>(U, κ)-pure quotients</i>
	\rightsquigarrow	<i>κ-filtered colimits (new)</i>

What are closed subobjects and (U, κ) -pure quotients?

\mathcal{A} ... a locally κ -presentable category
 (Ω, E) ... an \mathcal{A} -rel. alg. theory
 $\mathbf{Alg}(\Omega, E) \xrightarrow{U} \mathcal{A}$... the forgetful functor

Informal definition

① A subalg. $B \subseteq A$ in $\mathbf{Alg}(\Omega, E)$ is **closed** if:

- ▶ For every relation R in “the language of \mathcal{A} ,”

$$R(\vec{b}) \text{ holds in } UA \quad \Rightarrow \quad R(\vec{b}) \text{ holds in } UB.$$

② $A \xrightarrow{p} B$ in $\mathbf{Alg}(\Omega, E)$ is a **(U, κ) -pure quotient** if:

- ▶ For every κ -ary formula in “the language of \mathcal{A} ,”

$$\varphi(\vec{b}) \text{ holds in } UB \quad \Rightarrow \quad \exists \vec{a} \xrightarrow{Up} \vec{b} \text{ s.t. } \varphi(\vec{a}) \text{ holds in } UA.$$

Example

Pos ... the category of posets.

(Ω, \emptyset) ... a **Pos**-rel. alg. theory defined by
 $\Omega := \{\ominus\}, \quad x \ominus y \text{ is defined iff } x \geq y.$

$\mathbf{Alg}(\Omega, \emptyset) \xrightarrow{U} \mathbf{Pos}$... the forgetful functor.

In $\mathbf{Alg}(\Omega, \emptyset)$, under $x \ominus y := x - y$ in \mathbb{N} ,

• $\{0 \leq 2 \leq 3\} \subseteq \{0 < 1 < 2 < \dots\}$... subalgebra, but **not** closed.

• $\{0 \leq 2 \leq 4\} \subseteq \{0 < 1 < 2 < \dots\}$... closed subalgebra.

• $\left\{ \begin{array}{cc} 0 & 0 \\ \wedge & \\ 1 & 1 \\ & \wedge \\ & 2 \end{array} \right\} \rightarrow \left\{ \begin{array}{c} 0 \\ \wedge \\ 1 \\ \wedge \\ 2 \end{array} \right\}$... surjection, but **not** a (U, \aleph_0) -pure quotient.

• $\left\{ \begin{array}{cccc} 0 & 0 & 0 & \dots \\ & \wedge & \wedge & \\ & 1 & 1 & \dots \\ & & \wedge & \\ & & 2 & \dots \\ & & & \ddots \end{array} \right\} \rightarrow \left\{ \begin{array}{c} 0 \\ \wedge \\ 1 \\ \wedge \\ 2 \\ \wedge \\ \vdots \end{array} \right\}$... (U, \aleph_0) -pure quotient,
 but **not** (U, \aleph_1) -pure quotient.

Filtered colimits are necessary

Example ($\mathbf{Set}^{\mathbb{N}}$ -relative algebra [ARV12])

$$\mathcal{E} := \{1\} \cup \{A \in \mathbf{Set}^{\mathbb{N}} \mid \exists m \in \mathbb{N}. A_m = \emptyset\}.$$

$\mathcal{E} \subseteq \mathbf{Set}^{\mathbb{N}}$ is closed under...

- ✓ products
- ✓ closed subobjects = sort-wise injections
- ✓ pure quotients = sort-wise surjections
- ✗ **filtered colimits**

$$\begin{array}{rcll} A_0 & := & (\emptyset, & \emptyset, & \emptyset, & \emptyset, & \dots) & \in \mathcal{E} \\ \cap & & \cap & \cap & \cap & \cap & & \\ A_1 & := & (2, & \emptyset, & \emptyset, & \emptyset, & \dots) & \in \mathcal{E} \\ \cap & & \cap & \cap & \cap & \cap & & \\ A_2 & := & (2, & 2, & \emptyset, & \emptyset, & \dots) & \in \mathcal{E} \\ \cap & & \cap & \cap & \cap & \cap & & \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \\ \operatorname{Colim}_{n \in \omega} A_n & = & (2, & 2, & 2, & 2, & \dots) & \notin \mathcal{E} \end{array}$$

Example (\mathbf{Set}_ω -relative algebra [Kaw25])

\mathbf{Set}_ω ... the category of sets with countably many constants $(c_n)_n$.

$$\mathcal{E} := \{1\} \cup \{A \in \mathbf{Set}_\omega \mid \exists i, j \text{ s.t. } c_i \neq c_j \text{ in } A\}.$$

$\mathcal{E} \subseteq \mathbf{Set}_\omega$ is closed under...

- ✓ products
- ✓ closed subobjects = subalgebras
- ✓ pure quotients = surjections that do not merge any constants
- ✗ **filtered colimits**

$$\begin{array}{rclcl}
 A_0 & := & \{c_0, & c_1, & c_2, & c_3, & \dots, & \infty\} & \in \mathcal{E} \\
 \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
 A_1 & := & \{c_0 = c_1, & c_2, & c_3, & \dots, & \infty\} & \in \mathcal{E} \\
 \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
 A_2 & := & \{c_0 = c_1 = c_2, & c_3, & \dots, & \infty\} & \in \mathcal{E} \\
 \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
 \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \\
 \operatorname{Colim}_{n \in \omega} A_n & := & \{c_0 = c_1 = c_2 = c_3 = \dots, & \infty\} & \notin \mathcal{E}
 \end{array}$$

The filtered colimit elimination problem

However, filtered colimits are **not** required for **Set**-rel. alg. in Birkhoff's theorem.

Question

Why can filtered colimits be eliminated in the case of **Set**-relative algebras?

Answer

The category **Set** satisfies a “noetherian” condition.

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A noetherian condition for categories

Definition ([Kaw25])

A category \mathcal{A} satisfies the **ascending chain condition (ACC)** if it has no chain $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots$ of objects such that there is no morphism $A_n \leftarrow A_{n+1}$ for all n .

Example

Set satisfies ACC.

Proof.

Let $A_0 \rightarrow A_1 \rightarrow \cdots$ be an ω -chain of sets. If there is no map $A_0 \leftarrow A_1$, then $A_0 = \emptyset$ and $A_1 \neq \emptyset$. Thus, a map $A_1 \leftarrow A_2$ exists. □

More generally,

Proposition

\mathbf{Set}^S satisfies ACC \Leftrightarrow the set S is finite.

Filtered colimit elimination

Recall Theorem ([Kaw25; Kaw24])

(Ω, E) : an \mathcal{A} -relative (κ -ary) algebraic theory. $\mathcal{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.

If \mathcal{A} satisfies ACC,

TFAE:

- ① $\mathcal{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
- ② $\mathcal{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, closed subobjects, (U, κ) -pure quotients, and κ -filtered colimits. ~~and κ -filtered colimits.~~

Some applications of filtered colimit elimination

Corollary

- **Set** satisfies ACC.
 - ↪ fil.colim.elim. holds for **single-sorted alg.**
 - ↪ The classical Birkhoff theorem [Bir35]
- **Setⁿ** satisfied ACC.
 - ↪ fil.colim.elim. holds for **finite-sorted alg.**
 - ↪ This subsumes a result in [ARV12].
- **Pos** satisfied ACC.
 - ↪ fil.colim.elim. holds for **ordered alg.**
 - ↪ This subsumes a result in [Blo76].
- **Met_∞**, the category of generalized metric spaces, satisfied ACC.
 - ↪ fil.colim.elim. holds for **metric alg.**
 - ↪ This subsumes a result in [Hin16].

Filtered colimit elimination: sketch of proof

fullsub $\mathcal{E} \subseteq \mathbf{Alg}(\Omega, E)$: closed under products, closed sub, (U, κ) -pure quo.
(A_J) $_{J \in \mathbb{I}}$: a κ -filtered diagram s.t. $A_J \in \mathcal{E}$.

For each $J \in \mathbb{I}$, we can construct a “nice” wide sub-diagram $J \in \mathbb{I}_J \subseteq \mathbb{I}$.

$$\begin{array}{ccccc} \prod_{I \in \mathbb{I}} A_I & & & & \\ \uparrow m_J & \nwarrow m & & \xrightarrow{\text{Colim}_{J \in \mathbb{I}} \pi_J} & \text{Colim}_{J \in \mathbb{I}} A_J \\ & \text{Colim}_{J \in \mathbb{I}} \text{Lim}_{I \in \mathbb{I}_J} A_I & & & \text{in } \mathbf{Alg}(\Omega, E) \\ & \nearrow & & \nearrow & \\ \text{Lim}_{I \in \mathbb{I}_J} A_I & \xrightarrow{\pi_J} & A_J & & \end{array}$$

$\leadsto \mathcal{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under κ -filtered colimits.

□

Weak converse

Theorem ([Kaw25])

\mathcal{A} : a l.f.p. category. Assume that, for every fullsub. of \mathcal{A} , closure under filtered colimits follows from the others: **P**(products), **S**(closed sub), **H**(\aleph_0 -pure quo).

Then,

- 1 The fullsub $\mathcal{A}_{\text{fp},c} := \{\text{finitely presentable connected objs}\} \subseteq \mathcal{A}$ satisfies ACC.
- 2 If $\emptyset \xrightarrow{!} 1$ in \mathcal{A} is strongly monic, the fullsub $\mathcal{A}_{\text{fp}} := \{\text{finitely presentable objs}\} \subseteq \mathcal{A}$ satisfies ACC.

Sketch of proof:

Let $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots$ in \mathcal{A}_{fp} . Consider

$$\mathcal{C} := \{X \mid \exists n. X \xrightarrow{\exists} A_n\} \subseteq \mathcal{A}.$$

Using finite presentability, its **HSP**-closure can be computed as

$$\mathbf{HSP}(\mathcal{C}) = \mathbf{S}(1) \cup \mathbf{H}(\mathcal{C}).$$

Since $A_n + A_n \in \mathcal{C}$ ($\forall n$), $B := \operatorname{Colim}_{n \in \omega} (A_n + A_n) \in \mathbf{HSP}(\mathcal{C})$.

The additional conditions ensure that $B \notin \mathbf{S}(1)$. $\therefore B \in \mathbf{H}(\mathcal{C})$

$$\begin{aligned}
 &A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \text{ in } \mathcal{A}_{\text{fp}} \\
 &\mathcal{E} := \{X \mid \exists n. X \xrightarrow{\exists} A_n\} \subseteq \mathcal{A} \\
 &B := \operatorname{Colim}_{n \in \omega} (A_n + A_n) \in \mathbf{H}(\mathcal{E})
 \end{aligned}$$

Thus, we have:

$$\begin{array}{ccccc}
 & & & X \xrightarrow{\in \mathcal{E}} A_N \\
 & \nearrow & & \downarrow \text{pure quo} \\
 A_n & \xrightarrow{\text{coproj}} & A_n + A_n & \xrightarrow{\text{coproj}} & B
 \end{array}
 \quad (\forall n \geq N)$$

Thus, $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots$ eventually “stabilizes.”



Open problems

Open problem 1

Is there any locally presentable category that satisfies filtered colimit elimination but not ACC?

More precisely, is there any κ -ary partial Horn theory \mathbb{S} for some κ that satisfies the following conditions?

- Every full subcategory of $\mathbf{Mod}\,\mathbb{S}$ is closed under κ -filtered colimits whenever it is closed under products, \mathbb{S} -closed subobjects, and κ -pure quotients.
- The category $\mathbf{Mod}\,\mathbb{S}$ does not satisfy ACC.

The next one is weaker than 1 and independent of partial Horn theories:

Open problem 2

Is there any locally finitely presentable category that does not satisfy ACC but satisfies it for the full subcategory of finitely presentable objects?

Thank you!



Today's slides



My homepage

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