

# On the decomposition of a strong epimorphism into regular epimorphisms

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← Today's slides

1 Strong and regular epimorphisms

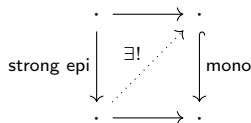
2 The decomposition number

3 Partial Horn theories

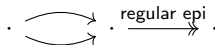
4 Main results

# Strong and regular epimorphisms

**Strong epimorphisms** = morphisms having the left lifting property w.r.t. every monomorphisms.



**Regular epimorphisms** = morphisms being the coequalizer of some parallel pair of morphisms.



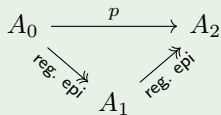
## Theorem ([Gabriel and Ulmer 1971])

In a locally presentable category,

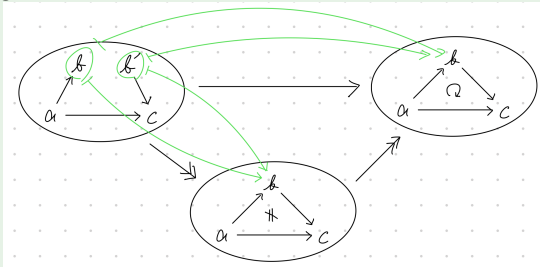
$$\text{strong epis} = \text{transfinite composites of regular epis}$$

## Example

**Cat**: the category of small categories.



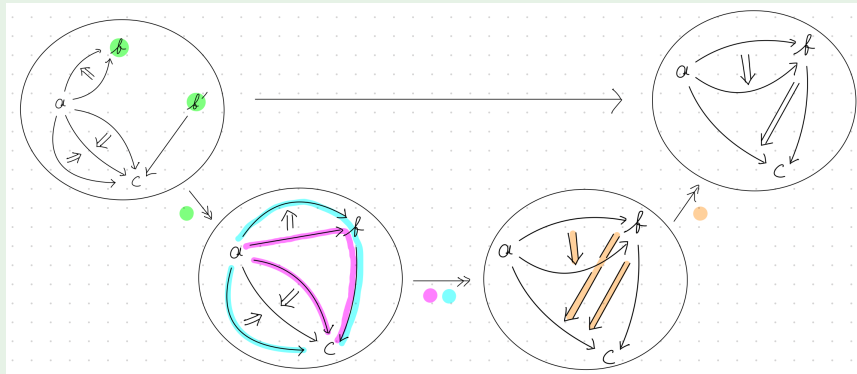
in **Cat**



## Example

**2Cat**: the category of small 2-categories.

$$\begin{array}{ccccc}
 A_0 & \xrightarrow{p} & A_3 & & \\
 \searrow \text{reg. epi} & & \nearrow \text{reg. epi} & & \\
 & A_1 \xrightarrow{\text{reg. epi}} A_2 & & & 
 \end{array}
 \quad \text{in } \mathbf{2Cat}$$



Actually...

### Fact I

The length of the regular epi chains in the previous slides can NOT be shorter.

### Fact II

- 1 In **Cat**, every strong epimorphism is decomposed into two regular epimorphisms.
- 2 In **2Cat**, every strong epimorphism is decomposed into three regular epimorphisms.

How to prove?

1 Strong and regular epimorphisms

2 The decomposition number

3 Partial Horn theories

4 Main results

## Definition

A **decomposition (of length  $\alpha$ )** of  $A \xrightarrow{p} X$  in  $\mathcal{C}$  is a cocontinuous functor  $D$  such that the following commutes:

$$\begin{array}{ccc} \mathbb{1} & & \\ \lceil \alpha \rceil \downarrow & \searrow \lceil p \rceil & \\ \mathbb{Q} + 1 & \xrightarrow{D} & A/\mathcal{C} \end{array} \quad \text{in } \mathbf{CAT}.$$

Here,  $\mathbb{1}$  denotes the terminal category, and  $\mathbb{Q} + 1$  denotes the category obtained by regarding the ordinal number  $\alpha + 1$  as a poset  $\{0 < 1 < \dots < \alpha\}$ .

$$\begin{array}{ccc} A & \xrightarrow{p} & X \\ D0 \downarrow \cong & \searrow D1 & \searrow D\alpha \\ \cdot & \xrightarrow{D_{01}} \cdot \xrightarrow{D_{12}} \dots & \xrightarrow{\quad} \cdot \end{array} \quad \begin{array}{c} \\ \\ \parallel \end{array} \quad \text{in } \mathcal{C}$$

## Definition

A decomposition  $D$  (of length  $\alpha$ ) is called **regular** if  $D_{\beta, \beta+1}$  is a regular epimorphism for any  $0 \leq \beta < \alpha$ .



# The decomposition number

## Definition

$\mathcal{A}$ : a locally presentable category.

- ① The **decomposition number**  $\delta(f)$  of  $A \xrightarrow{f} B$  in  $\mathcal{A}$  is the smallest ordinal number  $\alpha$  s.t. there is a factorization  $f = m \circ p$  by a morphism  $p$  with its regular decomposition of length  $\alpha$  and by a monomorphism  $m$ .

$$\begin{array}{ccccc} A & \xrightarrow{f} & B \\ \downarrow & \searrow p & \uparrow m \\ A_1 & \longrightarrow \twoheadrightarrow A_2 \longrightarrow \twoheadrightarrow \dots & A_\alpha \end{array}$$

- ②  $\delta(\mathcal{A}) := \min\{\alpha \mid \delta(f) < \alpha \text{ for every } f \text{ in } \mathcal{A}\}.$

## Theorem ([Gabriel and Ulmer 1971])

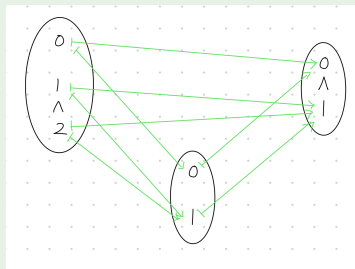
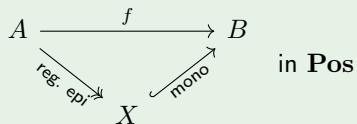
$\mathcal{A}$ : a locally  $\lambda$ -presentable category.

$\implies$  For every morphism  $f$  in  $\mathcal{A}$ ,  $\delta(f) \leq \lambda$ . Therefore,  $\delta(\mathcal{A}) \leq \lambda + 1$ .

# The decomposition number

## Example

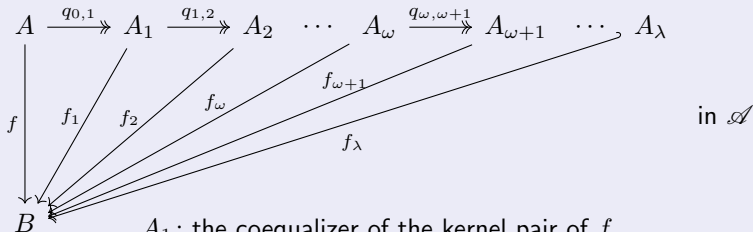
**Pos**: the category of posets.



In this case,  $\delta(f) = 1$  and  $\delta(\mathbf{Pos}) = 2$ .

## The small object argument

$\mathcal{A}$ : locally  $\lambda$ -presentable category.



$A_1$ : the coequalizer of the kernel pair of  $f$

$A_2$ : the coequalizer of the kernel pair of  $f_1$

$A_\omega$ : the colimit of the chain  $(A_n)_{n < \omega}$

$A_{\omega+1}$ : the coequalizer of the kernel pair of  $f_\omega$

At least  $f_\lambda$  becomes monic. Let  $\sigma(f)$  denote the smallest ordinal number  $\alpha$  s.t.  $f_\alpha$  is monic.

## Corollary

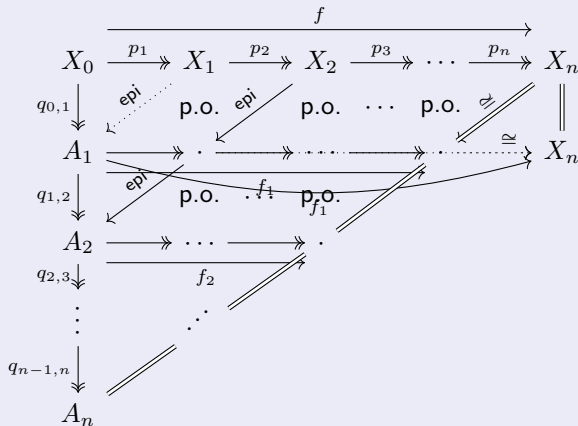
$$\delta(f) \leq \sigma(f)$$

## Theorem

In a locally presentable category,  $\delta(f) = \sigma(f)$ .

## Proof.

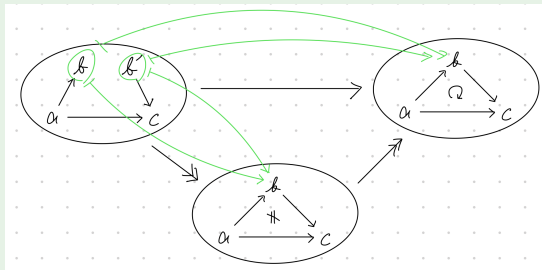
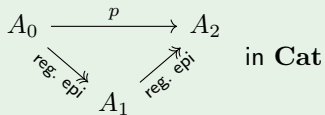
For simplicity, we assume  $\delta(f) = n < \omega$ .



Thus, we have  $\sigma(f) \leq n$ .

□

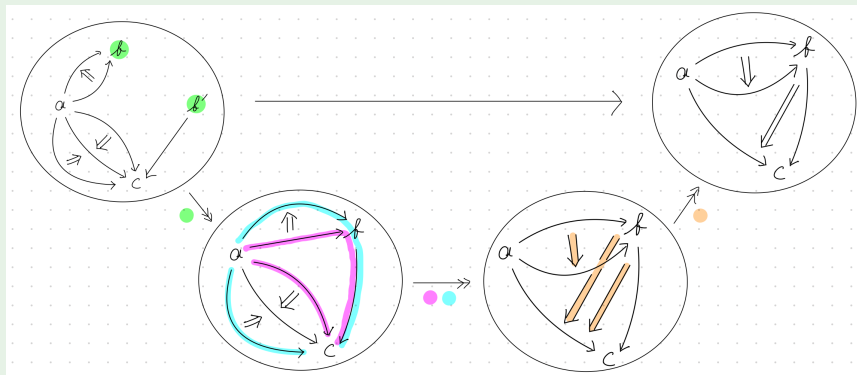
## Example (recall)



$$\implies \delta(p) = \sigma(p) = 2.$$

## Example (recall)

$$\begin{array}{ccccc}
 A_0 & \xrightarrow{p} & A_3 & & \\
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 & A_1 \xrightarrow{\text{reg. epi}} A_2 & & & 
 \end{array}
 \quad \text{in } \mathbf{2Cat}$$



$$\Rightarrow \quad \delta(p) = \sigma(p) = 3.$$

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# Multi-sorted signature

## Definition

$S$ : the set of *sorts*.  $\lambda$ : an infinite regular cardinal.

An  **$S$ -sorted ( $\lambda$ -ary) signature**  $\Sigma$  consists of:

- function symbols  $f, f', f'', \dots$
- relation symbols  $R, R', R'', \dots$
- arity of each function symbol  $f: \prod_{i < \alpha} s_i \rightarrow s$ ,
- arity of each relation symbol  $R: \prod_{j < \beta} s_j$ ,

where  $\alpha, \beta < \lambda$  and  $s_i, s_j, s \in S$ .



# Partial Horn theory

$\Sigma$ : an  $S$ -sorted signature.

- A **term**  $\tau ::= x \mid f(\tau_i)_{i < \alpha}$ ;
- A ( $\lambda$ -ary) **Horn formula**  $\varphi ::= \top \mid \bigwedge_{i < \alpha} \varphi_i \mid \tau = \tau' \mid R(\tau_i)_{i < \alpha}$ ;
- A ( $\lambda$ -ary) **context**  $\cdots \vec{x} = (x_i)_{i < \alpha}$  (a family of distinct variables).

Here,  $\alpha < \lambda$ . The notation  $\vec{x}.\varphi$  (resp.  $\vec{x}.\tau$ ) means that all variables of  $\varphi$  (resp.  $\tau$ ) are in the context  $\vec{x}$ . (*Horn formula (resp. term)-in-context*)

## Definition

- ① A ( $\lambda$ -ary) **Horn sequent** over  $\Sigma$  is an expression of the form

$$\varphi \vdash_{\vec{x}} \psi \quad (\text{"}\varphi \text{ implies } \psi\text{"})$$

( $\varphi, \psi$  are  $\lambda$ -ary Horn formulas over  $\Sigma$  in the same  $\lambda$ -ary context  $\vec{x}$ .)

- ② A ( $\lambda$ -ary) **partial Horn theory**  $\mathbb{T}$  over  $\Sigma$  is a set of ( $\lambda$ -ary) Horn sequents over  $\Sigma$ .

# Horn vs partial Horn

What is the difference between ordinary Horn theory and partial Horn theory?

↪ It lies in the concept of models.

	(ordinary) Horn theory	partial Horn theory
Axiom	Horn sequent $\varphi \vdash_{\vec{x}} \psi$	Horn sequent $\varphi \vdash_{\vec{x}} \psi$
Interpretation of func.symb.	total map $M_{\vec{s}} \xrightarrow{[f]_M} M_s$	<b>partial</b> map $M_{\vec{s}} \xrightarrow{[f]_M} M_s$
Interpretation of rel.symb.	subset $\llbracket R \rrbracket_M \subseteq M_{\vec{s}}$	subset $\llbracket R \rrbracket_M \subseteq M_{\vec{s}}$
Validity of $\varphi$	" $\varphi$ holds."	" <b>All terms in <math>\varphi</math> are defined</b> and $\varphi$ holds."
Validity of $\varphi \vdash_{\vec{x}} \psi$	"If $\varphi$ holds then $\psi$ holds."	"If <b>all terms in <math>\varphi</math> are defined</b> and $\varphi$ holds, then <b>all terms in <math>\psi</math> are defined</b> and $\psi$ holds."

Especially,

An equation  $\tau = \tau$  holds iff the value of the partial map  $\llbracket \tau \rrbracket_M$  is defined.

So, we will use the abbreviation  $\tau \downarrow$  for  $\tau = \tau$ .

# Categories of partial models

## Notation

$\mathbb{T}$ : a partial Horn theory.

**PMo**d  $\mathbb{T}$  : the category of (partial) models of  $\mathbb{T}$ .

## Fact

A category  $\mathcal{A}$  is locally  $\lambda$ -presentable  $\iff \mathcal{A} \simeq \mathbf{PMo}d \mathbb{T}$  for some  $\lambda$ -ary partial Horn theory  $\mathbb{T}$ .

# Example: small categories

## Example (small categories)

We can define the partial Horn theory  $\mathbb{T}_{\text{cat}}$  of small categories as follows:  
The  $S := \{\text{ob}, \text{mor}\}$ -sorted signature  $\Sigma_{\text{cat}}$  consists of:

$$\text{id} : \text{ob} \rightarrow \text{mor}, \quad \text{d} : \text{mor} \rightarrow \text{ob}, \quad \text{c} : \text{mor} \rightarrow \text{ob}, \quad \circ : \text{mor} \sqcap \text{mor} \rightarrow \text{mor}.$$

The partial Horn theory  $\mathbb{T}_{\text{cat}}$  over  $\Sigma_{\text{cat}}$  consists of:

$$\top \vdash \frac{x:\text{ob}}{} \text{id}(x) \downarrow, \quad (\text{id is total.})$$

$$\top \vdash \frac{f:\text{mor}}{} \text{d}(f) \downarrow \wedge \text{c}(f) \downarrow, \quad (\text{d and c are total.})$$

$$\begin{aligned} \text{d}(g) = \text{c}(f) &\vdash \frac{g, f:\text{mor}}{} (g \circ f) \downarrow, \\ (g \circ f) \downarrow &\vdash \frac{g, f:\text{mor}}{} \text{d}(g) = \text{c}(f), \end{aligned} \quad (g \circ f \text{ is defined iff } \text{d}(g) = \text{c}(f).)$$

and so on.

$\rightsquigarrow$  We have  $\mathbf{PMod} \mathbb{T}_{\text{cat}} \cong \mathbf{Cat}$ .

## Example: small 2-categories

### Example (small 2-categories)

There is an  $S := \{0, 1, 2\}$ -sorted signature  $\Sigma_{2\text{cat}}$  and a finitary PHT  $\mathbb{T}_{2\text{cat}}$  over  $\Sigma_{2\text{cat}}$  s.t.

$$\mathbf{PMod} \, \mathbb{T}_{2\text{cat}} \cong \mathbf{2Cat}.$$

## Example: posets

### Example (posets)

We present the partial Horn theory  $\mathbb{T}_{\text{pos}}$  of posets. Let  $S := \{*\}$ ,  $\Sigma_{\text{pos}} := \{\leq : * \sqcap *\}$ . The partial Horn theory  $\mathbb{T}_{\text{pos}}$  over  $\Sigma_{\text{pos}}$  consists of:

$$\top \vdash^x x \leq x, \quad x \leq y \wedge y \leq x \vdash^{x,y} x = y, \quad x \leq y \wedge y \leq z \vdash^{x,y,z} x \leq z.$$

Then, we have  $\mathbf{PMod} \mathbb{T}_{\text{pos}} \cong \mathbf{Pos}$ .

# Representing models

$\mathbb{T}$ : a  $\lambda$ -ary partial Horn theory.

## Construction

$\vec{x}.\varphi$ : a  $\kappa(\geq \lambda)$ -ary Horn formula (in a  $\kappa$ -ary context).

- A term  $\vec{x}.\tau$  is **defined under  $\vec{x}.\varphi$**   $\stackrel{\text{def}}{\iff} \varphi \vdash_{\vec{x}} \tau \downarrow$  can be derived from  $\mathbb{T}$ .  
(written  $\mathbb{T} \models (\varphi \vdash_{\vec{x}} \tau \downarrow)$ )
- The following gives an equivalence relation on the terms defined under  $\vec{x}.\varphi$ :

$$\tau \sim \tau' \quad \stackrel{\text{def}}{\iff} \quad \mathbb{T} \models (\varphi \vdash_{\vec{x}} \tau = \tau').$$

- Quotienting all of the terms defined under  $\vec{x}.\varphi$  by  $\sim$ , we obtain a  $\mathbb{T}$ -model  $\langle \vec{x}.\varphi \rangle_{\mathbb{T}}$ , called the **representing  $\mathbb{T}$ -model**.

## Fact

- 1 For every  $\mathbb{T}$ -model  $M$ ,  $\llbracket \vec{x}.\varphi \rrbracket_M \cong \mathbf{PMod} \mathbb{T}(\langle \vec{x}.\varphi \rangle_{\mathbb{T}}, M)$ .
- 2 A  $\mathbb{T}$ -model  $M$  is  $\kappa(\geq \lambda)$ -presentable  $\iff M \cong \langle \vec{x}.\varphi \rangle_{\mathbb{T}}$  for some  $\kappa$ -ary Horn formula  $\vec{x}.\varphi$ .

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# Gauges

## Definition

$\mathbb{T}$ : a  $\lambda$ -ary PHT.

A **gauge** (of length  $\alpha$ ) for  $\mathbb{T}$  is an assignment to each term  $\vec{x}.\tau$  in a  $\lambda$ -ary context, of the following data:

- an ordinal number  $\sharp(\vec{x}.\tau) < \alpha$ ;
- a set  $\text{Def}(\vec{x}.\tau)$  of pairs  $(\sigma^0, \sigma^1)$  of terms in the context  $\vec{x}$

such that, for every  $\vec{x}.\tau$ ,

- $\mathbb{T} \models \left( \tau = \tau \xrightarrow{\vec{x}} \bigwedge_{(\sigma^0, \sigma^1) \in \text{Def}(\vec{x}.\tau)} \sigma^0 = \sigma^1 \right)$ ;
- $\forall (\sigma^0, \sigma^1) \in \text{Def}(\vec{x}.\tau). \sharp(\vec{x}.\sigma^0), \sharp(\vec{x}.\sigma^1) < \sharp(\vec{x}.\tau).$

## Theorem

$\mathbb{T}$ : a  $\lambda$ -ary PHT with a gauge of length  $\alpha$ .

$\implies \delta(f) \leq \alpha$  ( $\forall f$  in  $\mathbf{PMod} \mathbb{T}$ ), hence  $\delta(\mathbf{PMod} \mathbb{T}) \leq \alpha + 1$ .

# How to construct a gauge?

## Definition (Depth)

$\mathbb{T}$ : a  $\lambda$ -ary partial Horn theory.

- Let  $\vec{x}$  be a  $\lambda$ -ary context.

$$\text{Term}_1(\vec{x}) := \{\vec{x}.\tau \mid \mathbb{T} \models (\tau \downarrow \vdash^{\vec{x}} \top)\}.$$

$$\text{Term}_{\beta+1}(\vec{x}) := \text{Term}_{\beta}(\vec{x}) \cup$$

$$\left\{ \vec{x}.\tau \mid \exists E \subseteq \text{Term}_{\beta}(\vec{x})^2 \text{ s.t. } \mathbb{T} \models (\tau \downarrow \vdash^{\vec{x}} \bigwedge_{(\sigma^0, \sigma^1) \in E} \sigma^0 = \sigma^1) \right\}.$$

$$\text{Term}_{\sup \beta}(\vec{x}) := \bigcup_{\beta} \text{Term}_{\beta}(\vec{x}).$$

- $\text{dep}(\vec{x}) := \min\{\alpha \mid \text{Term}_{\alpha}(\vec{x}) = \text{Term}_{\alpha+1}(\vec{x})\}.$
- $\text{dep}(\mathbb{T}) := \min\{\alpha \mid \forall \vec{x}: \lambda\text{-ary. } \text{dep}(\vec{x}) < \alpha\}$  (the **depth** of  $\mathbb{T}$ ).

## Lemma

If every  $\vec{x}.\tau$  belongs to  $\text{Term}_{\alpha}(\vec{x})$  for some  $\alpha$  ( $\stackrel{\text{def}}{\Leftrightarrow}$ :  $\mathbb{T}$  is **essentially algebraic**)  
 $\implies \mathbb{T}$  has a gauge of length “ $\text{dep}(\mathbb{T}) - 1$ .”

## Theorem

$$\mathbb{T}: \text{essentially algebraic} \implies \delta(\mathbf{PMod} \mathbb{T}) \leq \begin{cases} \text{dep}(\mathbb{T}) & \text{if } \text{dep}(\mathbb{T}): \text{a successor} \\ \text{dep}(\mathbb{T}) + 1 & \text{else} \end{cases}$$

## Example

$$\delta(\mathbf{Pos}) \leq \text{dep}(\mathbb{T}_{\text{pos}}) = 2;$$

$$\delta(\mathbf{Cat}) \leq \text{dep}(\mathbb{T}_{\text{cat}}) = 3;$$

$$\delta(\mathbf{2Cat}) \leq \text{dep}(\mathbb{T}_{\text{2cat}}) = 4.$$

Therefore,

$$\delta(\mathbf{Pos}) = 2;$$

$$\delta(\mathbf{Cat}) = 3;$$

$$\delta(\mathbf{2Cat}) = 4.$$

# The decay number

## Definition

$\mathbb{T}$ : a  $\lambda$ -ary partial Horn theory.

- $L$ : a set of terms in a common context.

$$\text{eq}(L) := \left( \bigwedge_{\tau, \tau' \in L} \tau = \tau' \right).$$

- $\vec{x}$ : a  $\lambda$ -ary context.

$$\text{dec}(\vec{x}) := \min \left\{ \alpha \mid \mathbb{T} \models \left( \text{eq}(\text{Term}_\alpha(\vec{x})) \stackrel{\vec{x}}{\vdash} \text{eq}(\text{Term}_{\alpha+1}(\vec{x})) \right) \right\}.$$

- $\text{dec}(\mathbb{T}) := \min \{ \alpha \mid \forall \vec{x}: \lambda\text{-ary. } \text{dec}(\vec{x}) < \alpha \}$  (the **decay number** of  $\mathbb{T}$ ).

## Remark

$$\text{dec}(\vec{x}) \leq \text{dep}(\vec{x}), \text{ hence } \text{dec}(\mathbb{T}) \leq \text{dep}(\mathbb{T}).$$

## Proposition

For  $\langle \vec{x}. \top \rangle \xrightarrow{!} 1$  in  $\mathbf{PMod} \mathbb{T}$ ,  $\delta(!) = \text{dec}(\vec{x})$ .

## Example

Let  $\mathbb{T}$  be the single-sorted finitary PHT defined as follows:

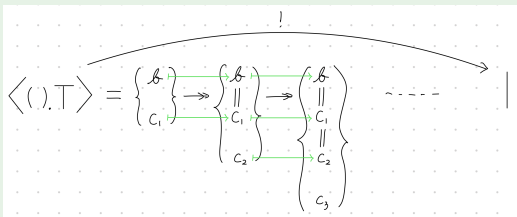
$$\Sigma := \{ b, c_n : \text{constants (for } n \geq 1) \},$$

$$\mathbb{T} := \left\{ \begin{array}{l} \top \vdash b = b \wedge c_1 = c_1 \\ b = c_n \vdash c_{n+1} = c_{n+1} \text{ (for } n \geq 1) \end{array} \right\}.$$

Then,

$$\text{Term}_1() = \{b, c_1\}, \quad \text{Term}_2() = \{b, c_1, c_2\}, \quad \text{Term}_3() = \{b, c_1, c_2, c_3\}, \dots$$

$$\text{dec}() = \text{dep}() = \omega.$$



in  $\mathbf{PMod} \mathbb{T}$ .

## Corollary

$$\text{dec}(\mathbb{T}) \leq \delta(\mathbf{PMod} \mathbb{T}).$$

## Theorem (summary)

- ① If  $\mathbb{T}$  is essentially algebraic,

$$\text{dec}(\mathbb{T}) \leq \delta(\mathbf{PMod} \mathbb{T}) \leq \begin{cases} \text{dep}(\mathbb{T}) & \text{if } \text{dep}(\mathbb{T}): \text{ a successor} \\ \text{dep}(\mathbb{T}) + 1 & \text{else} \end{cases}$$

- ② If  $\text{dec}(\mathbb{T}) = \text{dep}(\mathbb{T})$  and it is a successor additionally, then

$$\delta(\mathbf{PMod} \mathbb{T}) = \text{dep}(\mathbb{T}).$$

Thank you!



Today's slides

# References I



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# Future directions

- ① Can we replace “=” with an arbitrary relation symbol  $R$ ? (e.g. *coinserters* in **Pos** rather than regular epis)
- ② Is there a locally finitely presentable category  $\mathcal{A}$  s.t.  $\delta(\mathcal{A}) = \omega$ ? (We already have examples s.t.  $\delta(\mathcal{A}) = 1, 2, 3, 4, \dots$  and  $\omega + 1$ .)
- ③ Is there a better way to determine  $\delta(\mathcal{A})$  completely?
- ④ Is there any connection with other logical theories (rather than partial Horn theories)? (e.g. *generalized algebraic theories (GAT)*, *essentially algebraic theories*, etc.)

# Motivation

In abstract algebra (or universal algebra), the homomorphism theorem is fundamental. Categorically, it can be treated by *regular categories*.

## Recall

In a regular category,

- Every morphism can be decomposed into a *regular epimorphism* and a *monomorphism*.
- Such a decomposition is always given in the “canonical” way: taking a quotient by the *kernel pair*.

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & \searrow & \nearrow \\ & A/\text{Ker } f & \end{array}$$

- The class of regular epimorphisms is stable under pullbacks.

# Motivation

## Example

The regular categories include various categories considered in classical universal algebra: groups, monoids, etc.

The above examples are captured by the following general fact:

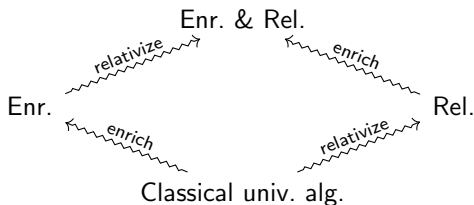
## Fact

Monadic categories over **Set** are regular.

# Motivation

There are several directions to generalize classical universal algebra “syntactically.” For example:

- Enriching**  $\mathcal{V}$ -enriched  $\lambda$ -ary monadic categories over  $\mathcal{V}$  [Rosický and Tendas 2024].
- Relativizing** (**Set**-enriched)  $\lambda$ -ary monadic categories over a locally  $\lambda$ -presentable category [Kawase 2024].
- Enr. & Rel.**  $\mathcal{V}$ -enriched  $\lambda$ -ary monadic categories over a locally  $\lambda$ -presentable  $\mathcal{V}$ -category [Rosický 2021].



# Motivation

## A problem

Monadic categories over a locally presentable category are NOT regular in general, even when the base category is regular.

## Example

**Cat**, the category of small categories, are finitary monadic over **Quiv**, the category of quivers (=directed graphs). However, **Cat** is not regular even if **Quiv** is regular.