Universal algebra over locally presentable categories

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← Today's slides

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Relativization via monads

Theorem ([Lin69])

There is an equivalence

$$\mathbf{Th}^S \simeq \mathbf{Mnd}_{\mathrm{f}}(\mathbf{Set}^S).$$

Here,

$$\mathbf{Th}^S\colon \text{ the category of } S\text{-sorted equational theories,}$$
$$\mathbf{Mnd}_{\mathrm{f}}(\mathbf{Set}^S)\colon \text{ the category of finitary monads on } \mathbf{Set}^S.$$

S-sorted equational theory $=$ finitary monad on $\mathbf{Set}^S$

$\downarrow$ generalize

??? $\mathscr{A}$-relative algebraic theory $=$ κ-ary monad on $\mathscr{A}$

($\mathscr{A}$: a locally κ-presentable category)

Relative algebraic theories

Informal definition [Kaw23]

$\mathscr{A}$: a (locally presentable) category

An $\mathscr{A}$-relative algebraic theory consists of:

- a set Ω of partial operators;
- a set E of implications $\cdots\left(\underbrace{\underline{\text{YYY}}}_{\text{postcondition}} \text{ whenever } \underbrace{\underline{\text{XXX}}}_{\text{precondition}}\right)$

such that

- For each operator $\omega \in \Omega$, its domain must be defined by "$\mathscr{A}$'s language."
- For each implication in E, its precondition must be written in "$\mathscr{A}$'s language."

$\mathbf{Set}^S$-relative algebraic theories $=$ S-sorted equational theories

A generalized Linton theorem

Theorem ([Kaw23; Kaw24])

For a locally κ-presentable category $\mathscr{A}$, there is an equivalence

$$\mathbf{Th}_\kappa^{\mathscr{A}} \simeq \mathbf{Mnd}_\kappa(\mathscr{A}).$$

Here,

$\mathbf{Th}_\kappa^{\mathscr{A}}$: the category of $\mathscr{A}$-relative (κ-ary) algebraic theories,

$\mathbf{Mnd}_\kappa(\mathscr{A})$: the category of κ-ary monads on $\mathscr{A}$.

Example: small categories

Example

A small category consists of:

- a base quiver $\mathrm{mor}\mathscr{C} \overset{\mathrm{d}}{\underset{\mathrm{c}}{\rightrightarrows}} \mathrm{ob}\mathscr{C}$;
- a total operator $\mathrm{id}\colon \mathrm{ob}\mathscr{C} \to \mathrm{mor}\mathscr{C}$;
- a partial operator $\circ\colon \mathrm{mor}\mathscr{C} \times \mathrm{mor}\mathscr{C} \to \mathrm{mor}\mathscr{C}$ such that

$$g \circ f \text{ is defined iff } \mathrm{d}(g) = \mathrm{c}(f)$$

which satisfy the following:

- $\mathrm{d}(\mathrm{id}(x)) = x$ and $\mathrm{c}(\mathrm{id}(x)) = x$;
- $\mathrm{d}(g \circ f) = \mathrm{d}(f)$ and $\mathrm{c}(g \circ f) = \mathrm{c}(g)$ whenever $\mathrm{d}(g) = \mathrm{c}(f)$;
- $f \circ \mathrm{id}(\mathrm{d}(f)) = f$ and $\mathrm{id}(\mathrm{c}(f)) \circ f = f$;
- $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $\mathrm{d}(h) = \mathrm{c}(g)$ and $\mathrm{d}(g) = \mathrm{c}(f)$.

Small categories are algebras over quivers.

Further examples

Example

		algebras over $\sim$
small categories	$\rightsquigarrow$	quivers
UDO semirings	$\rightsquigarrow$	posets
partial Boolean algebras	$\rightsquigarrow$	graphs
monoid-graded rings	$\rightsquigarrow$	monoid-graded sets
generalized complete metric spaces	$\rightsquigarrow$	generalized metric spaces
Banach spaces	$\rightsquigarrow$	pointed metric spaces

A technical remark

Definition ([PV07])

- (κ-ary) partial Horn theory $\cdots$ a logical theory based on multi-sorts, partial functions, relations, and (partial) Horn implications.
- $\mathbf{Mod}\,\mathbb{S}$ $\cdots$ the category of models of a partial Horn theory $\mathbb{S}$.

Theorem ([PV07])

TFAE for a category $\mathscr{A}$:

1. $\mathscr{A}$ is locally κ-presentable.
2. $\mathscr{A} \simeq \mathbf{Mod}\,\mathbb{S}$ for some κ-ary partial Horn theory $\mathbb{S}$.

We actually define $\mathbb{S}$-relative algebraic theories for partial Horn theories $\mathbb{S}$.

$$\rightsquigarrow \quad \mathscr{A}\text{-rel. alg. theory} \;=\; \mathbb{S}\text{-rel. alg. theory where } \mathscr{A} \simeq \mathbf{Mod}\,\mathbb{S}.$$

Birkhoff's variety theorem

Birkhoff's variety theorem [Bir35]

(Ω, E): a single-sorted algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable by equations.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, subobjects, and quotients.

closed under products: $A_i \in \mathscr{E} \implies \prod_i A_i \in \mathscr{E}$.

closed under subobjects: $B \subseteq A$: sub, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

closed under quotients: $A \twoheadrightarrow B$: surj, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

A generalized Birkhoff's theorem

Theorem ([Kaw23; Kaw24])

(Ω, E): an $\mathscr{A}$-relative (κ-ary) algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under <u>products</u>, <u>closed subobjects</u>, <u>(U, κ)-pure quotients</u>, and <u>κ-filtered colimits</u>.

single-sorted alg. ($\mathbf{Set}$-relative alg.)		$\mathscr{A}$-relative alg.
products	$\rightsquigarrow$	products
subobjects	$\rightsquigarrow$	*closed subobjects*
quotients	$\rightsquigarrow$	(U, κ)-*pure quotients*
	$\rightsquigarrow$	κ-*filtered colimits* (new)

What are closed subobjects and (U,κ)-pure quotients?

$$\begin{array}{rcl} \mathscr{A} & \cdots & \text{a locally } \kappa\text{-presentable category} \\ (\Omega, E) & \cdots & \text{an } \mathscr{A}\text{-rel. alg. theory} \\ \mathbf{Alg}(\Omega, E) \xrightarrow{U} \mathscr{A} & \cdots & \text{the forgetful functor} \end{array}$$

Informal definition

1. A subalg. $B \subseteq A$ in $\mathbf{Alg}(\Omega, E)$ is closed if:
   - For every relation R in "the language of $\mathscr{A}$,"

$$R(\vec{b}) \text{ holds in } UA \quad \Rightarrow \quad R(\vec{b}) \text{ holds in } UB.$$

2. $A \xrightarrow{p} B$ in $\mathbf{Alg}(\Omega, E)$ is a (U,κ)-pure quotient if:
   - For every κ-ary formula in "the language of $\mathscr{A}$,"

$$\varphi(\vec{b}) \text{ holds in } UB \quad \Rightarrow \quad \exists \vec{a} \xmapsto{Up} \vec{b} \text{ s.t. } \varphi(\vec{a}) \text{ holds in } UA.$$

Example

$$
\begin{array}{rcl}
\mathbf{Pos} & \cdots & \text{the category of posets.} \\
(\Omega, \varnothing) & \cdots & \text{a } \mathbf{Pos}\text{-rel. alg. theory defined by} \\
 & & \qquad \Omega := \{\ominus\}, \quad x \ominus y \text{ is defined iff } x \geq y. \\
\mathbf{Alg}(\Omega, \varnothing) \xrightarrow{U} \mathbf{Pos} & \cdots & \text{the forgetful functor.}
\end{array}
$$

In $\mathbf{Alg}(\Omega, \varnothing)$, under $x \ominus y := x - y$ in $\mathbb{N}$,

- $\{0 \leq 2 \quad 3\} \subseteq \{0 < 1 < 2 < \cdots\}$ $\cdots$ subalgebla, but not closed.
- $\{0 \leq 2 \leq 4\} \subseteq \{0 < 1 < 2 < \cdots\}$ $\cdots$ closed subalgebla.
- $\left\{\begin{array}{cc} 0 & 0 \\ \wedge & \\ 1 & 1 \\ & \wedge \\ & 2 \end{array}\right\} \to \left\{\begin{array}{c} 0 \\ \wedge \\ 1 \\ \wedge \\ 2 \end{array}\right\}$ $\cdots$ surjection, but not a $(U, \aleph_0)$-pure quotient.
- $\left\{\begin{array}{cccc} 0 & 0 & 0 & \cdots \\ & \wedge & \wedge & \\ & 1 & 1 & \cdots \\ & & \wedge & \\ & & 2 & \cdots \\ & & & \\ & & & \ddots \end{array}\right\} \to \left\{\begin{array}{c} 0 \\ \wedge \\ 1 \\ \wedge \\ 2 \\ \wedge \\ \vdots \end{array}\right\}$ $\cdots$ $(U, \aleph_0)$-pure quotient, but not $(U, \aleph_1)$-pure quotient.

Filtered colimits are necessary

Example ($\mathbf{Set}^{\mathbb{N}}$-relative algebra [ARV12])

$$\mathscr{E} := \{1\} \cup \{A \in \mathbf{Set}^{\mathbb{N}} \mid \exists m \in \mathbb{N}.\ A_m = \varnothing\}.$$

$\mathscr{E} \subseteq \mathbf{Set}^{\mathbb{N}}$ is closed under...

- ✓ products
- ✓ closed subobjects = sort-wise injections
- ✓ pure quotients = sort-wise surjections
- × filtered colimits

$$\begin{array}{ccccccccc}
A_0 & := & (\varnothing, & \varnothing, & \varnothing, & \varnothing, & \cdots &) & \in \mathscr{E} \\
\cap & & \cap & \cap & \cap & \cap & & & \\
A_1 & := & (2, & \varnothing, & \varnothing, & \varnothing, & \cdots &) & \in \mathscr{E} \\
\cap & & \cap & \cap & \cap & \cap & & & \\
A_2 & := & (2, & 2, & \varnothing, & \varnothing, & \cdots &) & \in \mathscr{E} \\
\cap & & \cap & \cap & \cap & \cap & & & \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & & \\
\operatorname*{Colim}_{n \in \omega} A_n & = & (2, & 2, & 2, & 2, & \cdots &) & \notin \mathscr{E}
\end{array}$$

Example ($\mathbf{Set}_\omega$-relative algebra [Kaw25])

$\mathbf{Set}_\omega \cdots$ the category of <u>sets with countably many constants $(c_n)_n$</u>.

$$\mathscr{E} := \{1\} \cup \{A \in \mathbf{Set}_\omega \mid \exists i, j \text{ s.t. } c_i \neq c_j \text{ in } A\}.$$

$\mathscr{E} \subseteq \mathbf{Set}_\omega$ is closed under...

- ✓ products
- ✓ closed subobjects = subalgebras
- ✓ pure quotients = <u>surjections that do not merge any constants</u>
- × filtered colimits

$$\begin{array}{ccccccccc}
A_0 & := & \{c_0, & c_1, & c_2, & c_3, & \cdots, & \infty\} & \in \mathscr{E} \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
A_1 & := & \{c_0 = & c_1, & c_2, & c_3, & \cdots, & \infty\} & \in \mathscr{E} \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
A_2 & := & \{c_0 = & c_1 = & c_2, & c_3, & \cdots, & \infty\} & \in \mathscr{E} \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \\
\operatorname*{Colim}_{n \in \omega} A_n & := & \{c_0 = & c_1 = & c_2 = & c_3 = & \cdots, & \infty\} & \notin \mathscr{E}
\end{array}$$

The filtered colimit elimination problem

However, filtered colimits are not required for $\mathbf{Set}$-rel. alg. in Birkhoff's theorem.

Question

Why can filtered colimits be eliminated in the case of $\mathbf{Set}$-relative algebras?

Answer

The category $\mathbf{Set}$ satisfies a “noetherian” condition.

A noetherian condition for categories

Definition ([Kaw25])

A category $\mathscr{A}$ satisfies the ascending chain condition (ACC) if it has no chain $A_0 \to A_1 \to A_2 \to \cdots$ of objects such that there is no morphism $A_n \leftarrow A_{n+1}$ for all n.

Example

$\mathbf{Set}$ satisfies ACC.

Proof.

Let $A_0 \to A_1 \to \cdots$ be an ω-chain of sets.If there is no map $A_0 \leftarrow A_1$, then $A_0 = \varnothing$ and $A_1 \neq \varnothing$.Thus, a map $A_1 \leftarrow A_2$ exists. $\square$

More generally,

Proposition

$$\mathbf{Set}^S \text{ satisfies ACC } \Leftrightarrow \text{ the set } S \text{ is finite.}$$

Filtered colimit elimination

Theorem ([Kaw25; Kaw24])

(Ω, E): an $\mathscr{A}$-relative (κ-ary) algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
If $\mathscr{A}$ satisfies ACC,
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under <u>products</u>, <u>closed subobjects</u>, <u>(U, κ)-pure quotients</u>, ~~and κ-filtered colimits~~.

Some applications of filtered colimit elimination

Corollary

- $\mathbf{Set}$ satisfies ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for single-sorted alg.
  - $\rightsquigarrow$ The classical Birkhoff theorem [Bir35]
- $\mathbf{Set}^n$ satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for finite-sorted alg.
  - $\rightsquigarrow$ This subsumes a result in [ARV12].
- $\mathbf{Pos}$ satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for ordered alg.
  - $\rightsquigarrow$ This subsumes a result in [Blo76].
- $\mathbf{Met}_\infty$, the category of generalized metric spaces, satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for metric alg.
  - $\rightsquigarrow$ This subsumes a result in [Hin16].

Filtered colimit elimination: sketch of proof

fullsub $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: closed under products, closed sub, (U, κ)-pure quo.
$(A_J)_{J\in\mathbb{I}}$: a κ-filtered diagram s.t. $A_J \in \mathscr{E}$.

For each $J \in \mathbb{I}$, we can construct a “nice” wide sub-diagram $J \in \mathbb{I}_J \subseteq \mathbb{I}$.

$$\begin{tikzcd}
\prod_{I\in\mathbb{I}} A_I & & & \\
& \operatorname*{Colim}_{J\in\mathbb{I}} \operatorname*{Lim}_{I\in\mathbb{I}_J} A_I \arrow[ul, hook', "m"'] \arrow[rr, "\operatorname*{Colim}_{J\in\mathbb{I}} \pi_J"] & & \operatorname*{Colim}_{J\in\mathbb{I}} A_J \\
\operatorname*{Lim}_{I\in\mathbb{I}_J} A_I \arrow[uu, hook, "m_J"] \arrow[ur, hook] \arrow[rr, "\pi_J"'] & & A_J \arrow[ur] &
\end{tikzcd}$$

in $\mathbf{Alg}(\Omega, E)$

$\rightsquigarrow$ $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under κ-filtered colimits. $\square$

Weak converse

Theorem ([Kaw25])

$\mathscr{A}$: a l.f.p. category. Assume that, for every fullsub. of $\mathscr{A}$, closure under filtered colimits follows from the others: $\mathbf{P}$(products), $\mathbf{S}$(closed sub), $\mathbf{H}$($\aleph_0$-pure quo). Then,

1. The fullsub $\mathscr{A}_{\mathrm{fp,c}} := \{$finitely presentable connected objs$\} \subseteq \mathscr{A}$ satisfies ACC.
2. If $\varnothing \xrightarrow{!} 1$ in $\mathscr{A}$ is strongly monic, the fullsub $\mathscr{A}_{\mathrm{fp}} := \{$finitely presentable objs$\} \subseteq \mathscr{A}$ satisfies ACC.

Sketch of proof:
Let $A_0 \to A_1 \to A_2 \to \cdots$ in $\mathscr{A}_{\mathrm{fp}}$. Consider

$$\mathscr{E} := \{X \mid \exists n.\ X \xrightarrow{\exists} A_n\} \quad \subseteq \mathscr{A}.$$

Using finite presentability, its $\mathbf{HSP}$-closure can be computed as

$$\mathbf{HSP}(\mathscr{E}) = \mathbf{S}(1) \cup \mathbf{H}(\mathscr{E}).$$

Since $A_n + A_n \in \mathscr{E}$ $(\forall n)$, $B := \operatorname*{Colim}_{n\in\omega}(A_n + A_n) \ \in \mathbf{HSP}(\mathscr{E})$.
The additional conditions ensure that $B \notin \mathbf{S}(1)$. $\quad \therefore B \in \mathbf{H}(\mathscr{E})$

$$A_0 \to A_1 \to A_2 \to \cdots \quad \text{in } \mathscr{A}_{\mathrm{fp}}$$

$$\mathscr{E} := \{X \mid \exists n.\ X \xrightarrow{\exists} A_n\} \quad \subseteq \mathscr{A}$$

$$B := \operatorname*{Colim}_{n \in \omega}(A_n + A_n) \ \in \mathbf{H}(\mathscr{E})$$

Thus, we have:

$$\begin{tikzcd}
 & & \overset{\in \mathscr{E}}{X} \arrow[r] \arrow[d, "\text{pure quo}"] & A_N \\
A_n \arrow[urr] \arrow[r, "\text{coproj}"'] & A_n + A_n \arrow[r, "\text{coproj}"'] & B &
\end{tikzcd} \qquad (\forall n \geq N)$$

Thus, $A_0 \to A_1 \to A_2 \to \cdots$ eventually "stabilizes." □

Open problems

Open problem 1

Is there any locally presentable category that satisfies filtered colimit elimination but not ACC?

More precisely, is there any κ-ary partial Horn theory $\mathbb{S}$ for some κ that satisfies the following conditions?

- Every full subcategory of $\mathbf{Mod}\,\mathbb{S}$ is closed under κ-filtered colimits whenever it is closed under products, $\mathbb{S}$-closed subobjects, and κ-pure quotients.
- The category $\mathbf{Mod}\,\mathbb{S}$ does not satisfy ACC.

The next one is weaker than 1 and independent of partial Horn theories:

Open problem 2

Is there any locally finitely presentable category that does not satisfy ACC but satisfies it for the full subcategory of finitely presentable objects?

Thank you!

Today's slides

My homepage

References I

[AR04] J. Adámek and J. Rosický. "On pure quotients and pure subobjects". In: *Czechoslovak Math. J.* 54(129).3 (2004), pp. 623–636.

[AR94] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*. Vol. 189. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1994.

[ARV12] J. Adámek, J. Rosický, and E. M. Vitale. "Birkhoff's variety theorem in many sorts". In: *Algebra Universalis* 68.1-2 (2012), pp. 39–42.

[Adá+21] J. Adámek et al. "Finitary monads on the category of posets". In: *Math. Structures Comput. Sci.* 31.7 (2021), pp. 799–821.

[BH12] B. van den Berg and C. Heunen. "Noncommutativity as a colimit". In: *Appl. Categ. Structures* 20.4 (2012), pp. 393–414.

[BMW12] C. Berger, P.-A. Melliès, and M. Weber. "Monads with arities and their associated theories". In: *J. Pure Appl. Algebra* 216.8-9 (2012), pp. 2029–2048.

[Bir35] G. Birkhoff. "On the structure of abstract algebras". In: *Math. Proc. Cambridge Philos. Soc.* 31.4 (1935), pp. 433–454.

[Blo76] S. L. Bloom. "Varieties of ordered algebras". In: *J. Comput. System Sci.* 13.2 (1976), pp. 200–212.

[BG19] J. Bourke and R. Garner. "Monads and theories". In: *Adv. Math.* 351 (2019), pp. 1024–1071.

[CV98] A. Carboni and E. M. Vitale. "Regular and exact completions". In: *J. Pure Appl. Algebra* 125.1-3 (1998), pp. 79–116.

[Gol03] J. S. Golan. *Semirings and Affine Equations over Them: Theory and Applications*. Vol. 556. Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 2003, pp. xiv+241.

References II

[Hin16] W. Hino. *Varieties of metric and quantitative algebras*. 2016. arXiv: 1612.06054 [cs.LO].

[Kaw23] Y. Kawase. *Birkhoff's variety theorem for relative algebraic theories*. 2023. arXiv: 2304.04382 [math.CT].

[Kaw24] Y. Kawase. *Relativized universal algebra via partial Horn logic*. 2024. arXiv: 2403.19661 [math.CT].

[Kaw25] Y. Kawase. "Filtered colimit elimination from Birkhoff's variety theorem". In: *J. Pure Appl. Algebra* 229.1 (2025), Paper No. 107794, 25.

[Lin69] F. E. J. Linton. "An outline of functorial semantics". In: *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*. Springer, Berlin, 1969, pp. 7–52.

[LP23] R. B. B. Lucyshyn-Wright and J. Parker. "Diagrammatic presentations of enriched monads and varieties for a subcategory of arities". In: *Appl. Categ. Structures* 31.5 (2023), Paper No. 40, 39.

[PV07] E. Palmgren and S. J. Vickers. "Partial Horn logic and cartesian categories". In: *Ann. Pure Appl. Logic* 145.3 (2007), pp. 314–353.

[Ros21] J. Rosický. "Metric monads". In: *Math. Structures Comput. Sci.* 31.5 (2021), pp. 535–552.

[Wea95] N. Weaver. "Quasi-varieties of metric algebras". In: *Algebra Universalis* 33.1 (1995), pp. 1–9.