DOUBLE CATEGORIES OF PROFUNCTORS (DRAFT)

YUTO KAWASE

Abstract. We give an axiomatization of virtual double categories of enriched profunctors.

Contents

1. Introductions 1
2. Preliminaries 1
2.1. Augmented virtual double categories 1
2.2. Categories enriched over a virtual double category 17
3. Colimits in augmented virtual double categories 20
3.1. Cocones, modules, and modulations 20
3.2. Final functors 28
3.3. Versatile colimits 35
3.4. The case of loosewise virtual double (VD)-indiscrete shapes 40
4. Axiomatization of double categories of profunctors 45
4.1. The formal construction of enriched categories 45
4.2. Density 49
4.3. Characterization theorems 52
4.4. Closedness under slicing 54
References 56

1. Introductions

Remark 1.1. For clarity, let us declare the sizes of the categories we treat. We fix three Grothendieck universes $\mathscr{U}_0 \in \mathscr{U}_1 \in \mathscr{U}_2$. Elements in $\mathscr{U}_0$ are called ***small***, elements in $\mathscr{U}_1$ are called ***large***, elements in $\mathscr{U}_2$ are called ***huge***. Arbitrary sets (not necessarily in $\mathscr{U}_0$ nor $\mathscr{U}_1$ nor $\mathscr{U}_2$) are called ***classes***. ♦

2. Preliminaries

2.1. Augmented virtual double categories.

2.1.1. *The 2-category of augmented virtual double categories.*

Definition 2.1 ([Kou20])**.** An augmented virtual double category (AVDC) $\mathbb{L}$ consists of the following data:

- A class $\mathrm{Ob}\mathbb{L}$, whose elements are called ***objects*** in $\mathbb{L}$. We write $A \in \mathbb{L}$ to mean $A \in \mathrm{Ob}\mathbb{L}$.

Date: February 23, 2025.
The author would like to thank Hayato Nasu for providing the idea for the proof of the Strongness Theorem and for suggesting the term "versatile colimits."

- For $A, B \in \mathbb{L}$, a class $\mathrm{Hom}_{\mathbb{L}}\binom{A}{B}$, whose elements are called ***tight arrows*** from A to B in $\mathbb{L}$. The objects and the tight arrows are supposed to form a category $\mathbf{T}\mathbb{L}$, which is called the ***tight category*** of $\mathbb{L}$. We write id_A for the identity on an object $A \in \mathbb{L}$. The composite of $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{T}\mathbb{L}$ is denoted by $f ⨾ g$. Tight arrows are often written vertically:

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, equal, "\mathsf{id}_A"] \\ A \end{tikzcd} \quad \text{in } \mathbb{L}$$

- For $A, B \in \mathbb{L}$, a class $\mathrm{Hom}_{\mathbb{L}}(A, B)$, whose elements are called ***loose arrows*** from A to B in $\mathbb{L}$. A loose arrow is denoted by $\to\!\!\!\!\!+$ and is often written loosely. A path of loose arrows $A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} \cdots \xrightarrow{u_n} A_n$ is called a ***loose path*** of length n and is often denoted by a dashed arrow $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$. A loose path v of length 0 or 1 is denoted by a dotted arrow $A \overset{v}{\cdots\rightarrow} B$. Note that $A = B$ is required when the loose path v is of length 0.
- A class $\mathrm{Cell}_{\mathbb{L}}\left(f \, {}^{\vec{u}}_{v} \, g \right)$, whose elements are called ***cells***, for each "boundary" formed by loose arrows and tight arrows in the following way:

$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] & A_n \arrow[d, "g"] \\ B \arrow[r, dotted, "v"'] & C \end{tikzcd} \quad \text{in } \mathbb{L}.$$

Cells where v is of length 1 (resp. 0) are called ***unicoary*** (resp. ***nullcoary***).
- Two kinds of special cells:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel_u"] & B \arrow[d, equal] \\ A \arrow[r, "u"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "=_f"] \\ B \end{tikzcd} \quad \text{in } \mathbb{L}$$

The cells $\shortparallel_u$ on the left are called ***loose identity cells***. The cells $=_f$ on the right are called ***tight identity cells***.
- For cells $\alpha_1, \dots, \alpha_n, \beta$ on the left below, a cell $\vec{\alpha} ⨾ \beta$ of the following form:

$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\ B_0 \arrow[r, dotted, "v_1"] \arrow[d, "g"'] & B_1 \arrow[r, dotted, "v_2"] & \cdots \arrow[r, dotted, "v_n"] & B_n \arrow[d, "h"] \\ C \arrow[rrr, dotted, "w"'] \arrow[urrr, phantom, "\beta"] & & & D \end{tikzcd} \mapsto \begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0 ⨾ g"'] & A_1 \arrow[r, dashed, "\vec{u}_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] & A_n \arrow[d, "f_n ⨾ h"] \\ C \arrow[rrr, dotted, "w"'] \arrow[urrr, phantom, "\vec{\alpha} ⨾ \beta"] & & & D \end{tikzcd}$$

The composition defined by the assignments $(\alpha_1, \dots, \alpha_n, \beta) \mapsto \vec{\alpha} ⨾ \beta$ is required to satisfy a suitable associative law and a unit law with identity cells. See [Kou20] for more detail. ♦

Notation 2.2. Let $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$ be a loose path of length n in an AVDC. We extend the notation for the loose identity cells as follows:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel_{\vec{u}}"] & A_n \arrow[d, equal] \\
A_0 \arrow[r, dashed, "\vec{u}"', "\shortmid"{marking}] & A_n
\end{tikzcd} \tag{1}$$

When $n \geq 1$, the notation (1) means the path $(\shortparallel_{u_1}, \dots, \shortparallel_{u_n})$ of loose identity cells. When $n = 0$, the notation (1) means the tight identity cell $=_{\mathsf{id}_{A_0}}$, where $A_0 = A_n$. ♦

Notation 2.3. Let $\alpha_1, \dots, \alpha_n$ be cells in an AVDC of the following form:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}_1", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{u}_2", "\shortmid"{marking}] \arrow[d, "f_1"] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n", "\shortmid"{marking}] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, dotted, "v_1"', "\shortmid"{marking}] & B_1 \arrow[r, dotted, "v_2"', "\shortmid"{marking}] & \cdots \arrow[r, dotted, "v_n"', "\shortmid"{marking}] & B_n
\end{tikzcd} \tag{2}$$

When the composite path $\vec{v}$ of $v_1, \cdots, v_n$ is of length ≤ 1, we use the same notation (2) for the composite of the following cells:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}_1", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{u}_2", "\shortmid"{marking}] \arrow[d, "f_1"] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n", "\shortmid"{marking}] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, dotted, "v_1", "\shortmid"{marking}] \arrow[d, equal] & B_1 \arrow[r, dotted, "v_2", "\shortmid"{marking}] \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[r, dotted, "v_n", "\shortmid"{marking}] & B_n \arrow[d, equal] \\
B_0 \arrow[rrr, dotted, "\vec{v}"', "\shortmid"{marking}] & & & B_n
\end{tikzcd}$$

For example, the following exhibits a cell given by the composition:

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}_1", "\shortmid"{marking}] \arrow[dr, "f_0"'] & A_1 \arrow[r, dashed, "\vec{u}_2", "\shortmid"{marking}] \arrow[d, "f_1"] & A_2 \arrow[dl, equal] \arrow[d, "f_3"] \\
& A_2 \arrow[r, dotted, "v_3"', "\shortmid"{marking}] & B_3
\end{tikzcd} \tag{3}$$

Note that the cell (3) coincides with another composite of the following cells.

$$\begin{tikzcd}
A_0 \arrow[rr, dashed, "\vec{u}_1", "\shortmid"{marking}] \arrow[drr, "f_0"'] & & A_1 \arrow[rr, dashed, "\vec{u}_2", "\shortmid"{marking}] \arrow[d, "f_1"] & & A_2 \arrow[dll, equal] \\
& & A_2 \arrow[dl, equal] \arrow[dr, "f_3"] \arrow[d, phantom, "\alpha_3"] & & \\
& A_2 \arrow[rr, dotted, "v_3"', "\shortmid"{marking}] & {} & B_3 &
\end{tikzcd}$$

♦

Notation 2.4. Let $\mathbb{L}$ be an AVDC. We write $\mathcal{T}\mathbb{L}$ for the 2-category defined as follows: The underlying category is $\mathbf{T}\mathbb{L}$; 2-cells are cells whose top and bottom boundaries are of length 0. The 2-category $\mathcal{T}\mathbb{L}$ is called the ***tight 2-category*** of $\mathbb{L}$. ♦

Example 2.5. The AVDC $\mathbb{R}\mathrm{el}$ is defined as follows:

- An object is a (large) set.
- A tight arrow is a map.
- A loose arrow $X \longrightarrow Y$ is a relation $R \subseteq X \times Y$.

- $\mathbb{R}\mathrm{el}$ has at most one cell for every boundary. A unicoary cell on the left below exists if and only if, for any $x_0 \in X_0, \ldots, x_n \in X_n$, the conjunction of $(x_0, x_1) \in R_1, \ldots, (x_{n-1}, x_n) \in R_n$ implies $(f(x_0), g(x_n)) \in S$. A nullcoary cell on the right below exists if and only if, for any $x_0 \in X_0, \ldots, x_n \in X_n$, the conjunction of $(x_0, x_1) \in R_1, \ldots, (x_{n-1}, x_n) \in R_n$ implies $f(x_0) = g(x_n)$.

$$\begin{tikzcd} X_0 \arrow[r, dashed, "\vec{R}"] \arrow[d, "f"'] \arrow[dr, phantom, "\cdot"] & X_n \arrow[d, "g"] \\ Y \arrow[r, "S"'] & Z \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[rr, dashed, "\vec{R}"] \arrow[dr, "f"'] & \cdot & X_n \arrow[dl, "g"] \\ & Y & \end{tikzcd} \quad \text{in } \mathbb{R}\mathrm{el}$$ ♦

Definition 2.6 ([Kou20])**.** Let $\mathbb{K}$ and $\mathbb{L}$ be AVDCs. An ***augmented virtual double (AVD)-functor*** $\mathbb{K} \xrightarrow{F} \mathbb{L}$ consists of:

- A functor $F\colon \mathbf{T}\mathbb{K} \to \mathbf{T}\mathbb{L}$.
- Assignments to loose arrows
$$A \xrightarrow{u} B \quad \text{in } \mathbb{K} \quad \mapsto \quad FA \xrightarrow{Fu} FB \quad \text{in } \mathbb{L}.$$
In what follows, we extend the assignments from loose arrows to loose paths. Specifically, $F\vec{u} = F(u_1, \ldots, u_n) := (Fu_1, \ldots, Fu_n)$.
- Assignments to cells
$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "v"'] & Y \end{tikzcd} \text{ in } \mathbb{K} \quad \mapsto \quad \begin{tikzcd} FA \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[d, "Fg"] \\ FX \arrow[r, dotted, "Fv"'] & FY \end{tikzcd} \text{ in } \mathbb{L}.$$

These are required to satisfy the following:

- For any composable cells
$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{u}_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\ B_0 \arrow[r, dotted, "v_1"'] \arrow[d, "g"'] & B_1 \arrow[r, dotted, "v_2"'] \arrow[d, phantom, "\beta"] & \cdots \arrow[r, dotted, "v_n"'] & B_n \arrow[d, "h"] \\ X \arrow[rrr, dotted, "w"'] & & & Y \end{tikzcd} = \begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}_1"] \arrow[dd, "f_0"'] & A_1 \arrow[r, dashed, "\vec{u}_2"] & \cdots \arrow[r, dashed, "\vec{u}_n"] & A_n \arrow[dd, "f_n"] \\ & & & \\ B_0 \arrow[rrr, phantom, "\vec{\alpha}\mathbin{;}\beta"] \arrow[d, "g"'] & & & B_n \arrow[d, "h"] \\ X \arrow[rrr, dotted, "w"'] & & & Y \end{tikzcd} \text{ in } \mathbb{K},$$
the equality $F\vec{\alpha}\mathbin{;}F\beta = F(\vec{\alpha}\mathbin{;}\beta)$ holds.

$$\begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}_1"] \arrow[d, "Ff_0"'] \arrow[dr, phantom, "F\alpha_1"] & FA_1 \arrow[r, dashed, "F\vec{u}_2"] \arrow[d, "Ff_1"'] \arrow[dr, phantom, "F\alpha_2"] & \cdots \arrow[r, dashed, "F\vec{u}_n"] \arrow[dr, phantom, "F\alpha_n"] & FA_n \arrow[d, "Ff_n"] \\ FB_0 \arrow[r, dotted, "Fv_1"'] \arrow[d, "Fg"'] & FB_1 \arrow[r, dotted, "Fv_2"'] \arrow[d, phantom, "F\beta"] & \cdots \arrow[r, dotted, "Fv_n"'] & FB_n \arrow[d, "Fh"] \\ FX \arrow[rrr, dotted, "Fw"'] & & & FY \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}_1"] \arrow[d, "Ff_0"'] & FA_1 \arrow[r, dashed, "F\vec{u}_2"] & \cdots \arrow[r, dashed, "F\vec{u}_n"] & FA_n \arrow[d, "Ff_n"] \\ FB_0 \arrow[rrr, phantom, "F(\vec{\alpha}\mathbin{;}\beta)"] \arrow[d, "Fg"'] & & & FB_n \arrow[d, "Fh"] \\ FX \arrow[rrr, dotted, "Fw"'] & & & FY \end{tikzcd} \text{ in } \mathbb{L}$$

- For any $A \xrightarrow{u} B$ in $\mathbb{K}$, the equality $F\shortparallel_u = \shortparallel_{Fu}$ holds.
$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel_u"] & B \arrow[d, equal] \\ A \arrow[r, "u"'] & B \end{tikzcd} \mapsto \begin{tikzcd} FA \arrow[r, "Fu"] \arrow[d, equal] \arrow[dr, phantom, "F\shortparallel_u"] & FB \arrow[d, equal] \\ FA \arrow[r, "Fu"'] & FB \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "Fu"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel_{Fu}"] & FB \arrow[d, equal] \\ FA \arrow[r, "Fu"'] & FB \end{tikzcd}$$

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$, the equality $F\mathord{=}_f = \mathord{=}_{Ff}$ holds.

$$\begin{tikzcd} A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "=_f"] \\ B \end{tikzcd} \mapsto \begin{tikzcd} FA \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "F=_f"] \\ FB \end{tikzcd} = \begin{tikzcd} FA \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "=_{Ff}"] \\ FB \end{tikzcd}$$

♦

Definition 2.7 ([Kou20])**.** Let $F, G\colon \mathbb{K} \to \mathbb{L}$ be AVD-functors between AVDCs. A ***tight*** ***AVD-transformation*** $F \overset{\rho}{\Longrightarrow} G$ consists of:

- for each $A \in \mathbb{K}$, a tight arrow $\begin{tikzcd} FA \arrow[d, "\rho_A"'] \\ GA \end{tikzcd}$ in $\mathbb{L}$;
- for each $A \overset{u}{\longrightarrow} B$ in $\mathbb{K}$, a cell $\begin{tikzcd} FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "\rho_A"'] \arrow[dr, phantom, "\rho_u"] & FB \arrow[d, "\rho_B"] \\ GA \arrow[r, "Gu"', "\shortmid"{marking}] & GB \end{tikzcd}$ in $\mathbb{L}$

satisfying the following:

- ρ yields a natural transformation $\mathbf{T}\mathbb{K} \begin{tikzcd} {} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\rho"] & {} \end{tikzcd} \mathbf{T}\mathbb{L}$, i.e., for any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$\begin{tikzcd} & FA \arrow[dl, "\rho_A"'] \arrow[dr, "Ff"] & \\ GA \arrow[dr, "Gf"'] & = & FB \arrow[dl, "\rho_B"] \\ & GB & \end{tikzcd} \quad \text{in } \mathbb{L}.$$

- For any unicoary cell

$$\begin{tikzcd} A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "u_2", "\shortmid"{marking}] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\ X \arrow[rrr, "v"', "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "\alpha"] & & & Y \end{tikzcd} \quad \text{in } \mathbb{K},$$

  the following equality holds.

$$\begin{tikzcd} FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[d, "\rho_{A_0}"'] \arrow[dr, phantom, "\rho_{u_1}"] & FA_1 \arrow[r, "Fu_2", "\shortmid"{marking}] \arrow[d, "\rho_{A_1}"'] \arrow[dr, phantom, "\rho_{u_2}"] & \cdots \arrow[r, "Fu_n", "\shortmid"{marking}] \arrow[dr, phantom, "\rho_{u_n}"] & FA_n \arrow[d, "\rho_{A_n}"] \\ GA_0 \arrow[r, "Gu_1"', "\shortmid"{marking}] \arrow[d, "Gf"'] & GA_1 \arrow[r, "Gu_2"', "\shortmid"{marking}] & \cdots \arrow[r, "Gu_n"', "\shortmid"{marking}] & GA_n \arrow[d, "Gg"] \\ GX \arrow[rrr, "Gv"', "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "G\alpha"] & & & GY \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[r, "Fu_1", "\shortmid"{marking}] \arrow[d, "Ff"'] & FA_1 \arrow[r, "Fu_2", "\shortmid"{marking}] & \cdots \arrow[r, "Fu_n", "\shortmid"{marking}] & FA_n \arrow[d, "Fg"] \\ FX \arrow[rrr, "Fv"', "\shortmid"{marking}] \arrow[d, "\rho_X"'] \arrow[rrr, phantom, shift left=4ex, "F\alpha"] & & & FY \arrow[d, "\rho_Y"] \\ GX \arrow[rrr, "Gv"', "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "\rho_v"] & & & GY \end{tikzcd}$$

- For any nullcoary cell

$$\begin{tikzcd} A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[dr, "f"'] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & A_n \arrow[dl, "g"] \\ & X & \end{tikzcd} \quad \text{in } \mathbb{K},$$

the following equality holds.

$$
\begin{tikzcd}
FA_0 \arrow[r, "Fu_1"] \arrow[d, "\rho_{A_0}"'] & \cdots \arrow[r, "Fu_n"] \arrow[d, phantom, "\rho_{u_1}\quad\rho_{u_n}"] & FA_n \arrow[d, "\rho_{A_n}"] \\
GA_0 \arrow[r, "Gu_1"'] \arrow[dr, "Gf"'] & \cdots \arrow[r, "Gu_n"'] \arrow[d, phantom, "G\alpha"] & GA_n \arrow[dl, "Gg"] \\
& GX &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, "Fu_1"] \arrow[dr, "Ff"'] & \cdots \arrow[r, "Fu_n"] \arrow[d, phantom, "F\alpha"] & FA_n \arrow[dl, "Fg"] \\
& FX \arrow[d, bend right, "\rho_X"'] \arrow[d, bend left, "\rho_X"] \arrow[d, phantom, "="] & \\
& GX &
\end{tikzcd}
$$

♦

Notation 2.8. The huge AVDCs, AVD-functors, and tight AVD-transformations form a 2-category [Kou20], which is denoted by $\mathcal{AVDC}$. ♦

Definition 2.9. Let $\mathbb{L}$ be an AVDC. A ***full sub-AVDC*** of $\mathbb{L}$ is an AVDC whose class of objects is a subclass of $\mathrm{Ob}\mathbb{L}$ and whose "local" classes of tight arrows, loose arrows, and cells are identical to those of $\mathbb{L}$. Additionally, all compositions and identities in the full sub-AVDC are required to be inherited directly from $\mathbb{L}$. ♦

The following is convenient to treat virtual-double-categorical concepts in the augmented-virtual-double-categorical setting.

Definition 2.10. An AVDC is called ***diminished*** if all nullcoary cells are tight identity cells, that is, $=_f$ for some tight morphism f. ♦

Notation 2.11. Let $\mathbb{L}$ be an AVDC. We write $\mathbb{L}^\flat$ for the diminished AVDC obtained by removing all nullcoary cells, except for tight identity cells, from $\mathbb{L}$. ♦

Remark 2.12. A diminished AVDC is the essentially same concept as a ***virtual double category (VDC)*** [CS10], which is also called fc-***multicategories*** [Lei99; Lei02; Lei04] and is originally introduced in [Bur71]. Indeed, the AVD-functors between diminished AVDCs correspond to the VD-functors between VDCs. ♦

2.1.2. *Equivalences in the 2-category $\mathcal{AVDC}$.*

Notation 2.13. For an AVDC $\mathbb{L}$, let $\mathbf{T}^{\leq 1}\mathbb{L}$ denote a category defined as follows:

- An object is a loose path $A^0 \overset{A}{\dashrightarrow} A^1$ in $\mathbb{L}$ of length ≤ 1.
- A morphism from $A^0 \overset{A}{\dashrightarrow} A^1$ to $B^0 \overset{B}{\dashrightarrow} B^1$ is a tuple $(\alpha^0, \alpha^1, \alpha)$ of the following form:

$$
\begin{tikzcd}
A^0 \arrow[r, dashed, "A"] \arrow[d, "\alpha^0"'] \arrow[dr, phantom, "\alpha"] & A^1 \arrow[d, "\alpha^1"] \\
B^0 \arrow[r, dashed, "B"'] & B^1
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

We write $\mathbf{T}^1\mathbb{L}$ for the full subcategory of $\mathbf{T}^{\leq 1}\mathbb{L}$ consisting of paths of length 1, i.e., loose arrows. ♦

Definition 2.14 (Loosewise invertible cells)**.** Let $\mathbb{L}$ be an AVDC. Isomorphisms in the category $\mathbf{T}\mathbb{L}$ are called ***invertible tight arrows***. Isomorphisms in the category $\mathbf{T}^{\leq 1}\mathbb{L}$ are called ***loosewise invertible cells*** and are often denoted by the symbol "$\cong$" as follows:

$$
\begin{tikzcd}
\cdot \arrow[r, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, dashed] & \cdot
\end{tikzcd}
\quad \text{in } \mathbb{L}
$$

For a loosewise invertible cell of the above form, the tight arrows f and g automatically become invertible. ♦

Theorem 2.15 ([Kou20, 3.8. Proposition])**.** An AVD-functor $F\colon \mathbb{K} \to \mathbb{L}$ is a part of an equivalence in the 2-category $\mathcal{AVDC}$ if and only if it satisfies the following conditions:

- The assignments $\alpha \mapsto F\alpha$ induce bijections $\mathrm{Cell}_{\mathbb{K}}\left(f \overset{\vec{u}}{\underset{v}{}} g \right) \cong \mathrm{Cell}_{\mathbb{L}}\left(Ff \overset{F\vec{u}}{\underset{Fv}{}} Fg \right)$;
- The assignments $f \mapsto Ff$ induce bijections $\mathrm{Hom}_{\mathbb{K}}\left(\begin{smallmatrix} A \\ B \end{smallmatrix} \right) \cong \mathrm{Hom}_{\mathbb{L}}\left(\begin{smallmatrix} FA \\ FB \end{smallmatrix} \right)$;
- We can simultaneously make the following choices:
  - for each $A \in \mathbb{L}$, an object $A' \in \mathbb{K}$ and an invertible tight arrow $FA' \xrightarrow{\varepsilon_A} A$ in $\mathbb{L}$;
  - for each $A \xrightarrow{u} B$ in $\mathbb{L}$, a loose arrow $A' \xrightarrow{u'} B'$ in $\mathbb{K}$ and a loosewise invertible cell

$$\begin{tikzcd} FA' \arrow[r, "Fu'"] \arrow[d, "\varepsilon_A"'] \arrow[dr, phantom, "\cong"] & FB' \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \quad \text{in } \mathbb{L}.$$

2.1.3. *Cartesian cells.*

Definition 2.16 (Cartesian cells)**.** A cell

$$\begin{tikzcd} X^0 \arrow[r, dotted, "X"] \arrow[d, "\alpha^0"'] \arrow[dr, phantom, "\alpha"] & X^1 \arrow[d, "\alpha^1"] \\ Y^0 \arrow[r, dotted, "Y"'] & Y^1 \end{tikzcd} \tag{4}$$

in an AVDC is called ***cartesian*** if it satisfies the following condition: Suppose that we are given a loose path $A \overset{\vec{u}}{\dashrightarrow} B$, tight arrows $A \xrightarrow{f} X^0$ and $B \xrightarrow{g} X^1$, and a cell β on the right below; then there uniquely exists a cell γ satisfying the following equation.

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\gamma"] & B \arrow[d, "g"] \\ X^0 \arrow[r, dotted, "X"] \arrow[d, "\alpha^0"'] \arrow[dr, phantom, "\alpha"] & X^1 \arrow[d, "\alpha^1"] \\ Y^0 \arrow[r, dotted, "Y"'] & Y^1 \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[ddr, phantom, "\beta"] & B \arrow[d, "g"] \\ X^0 \arrow[d, "\alpha^0"'] & X^1 \arrow[d, "\alpha^1"] \\ Y^0 \arrow[r, dotted, "Y"'] & Y^1 \end{tikzcd}$$

We will use a symbol "cart" to represent a cartesian cell:

$$\begin{tikzcd} \cdot \arrow[r, dotted] \arrow[d] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d] \\ \cdot \arrow[r, dotted] & \cdot \end{tikzcd}$$ ♦

Proposition 2.17. Let α be a cell of the form (4) in an AVDC, and suppose that α^0 and α^1 are invertible. Then, the cell α is cartesian if and only if it is loosewise invertible. In particular, every loosewise invertible cell is cartesian.

Proof. Straightforward. □

Definition 2.18 (Restrictions)**.** Suppose that we are given a cartesian cell in an AVDC of the following form:

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d, "g"] \\ X \arrow[r, dotted, "u"'] & Y \end{tikzcd}$$

(i) Since the loose arrow p is unique up to loosewise invertible cell, we call p the ***restriction*** of u along f and g and write $u(f,g)$ for it. When u is of length 0 (hence $X = Y$), we also write $X(f,g)$ for p. To emphasize that u is of length 1 (resp. 0), we sometimes call $u(f,g)$ the ***unicoary restriction*** (resp. ***nullcoary restriction***).

$$\begin{tikzcd} \cdot \arrow[r, "u(f,g)"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d, "g"] \\ X \arrow[r, dotted, "u"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[rr, "X(f,g)"] \arrow[dr, "f"'] & \arrow[d, phantom, "\mathsf{cart}"] & \cdot \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

(ii) When $g = \mathsf{id}$ and u is of length 0, we call p the ***companion*** of f and write f_* for it. When $f = \mathsf{id}$ and u is of length 0, we call p the ***conjoint*** of g and write g^* for it. We write $f_\dagger$ and $g^\dagger$ for the associated cartesian cells as follows:

$$\begin{tikzcd} \cdot \arrow[rr, "f_*"] \arrow[dr, "f"'] & \arrow[d, phantom, "f_\dagger"] & X \arrow[dl, equal] \\ & X & \end{tikzcd} : \mathsf{cart} \qquad \begin{tikzcd} X \arrow[rr, "g^*"] \arrow[dr, equal] & \arrow[d, phantom, "g^\dagger"] & \cdot \arrow[dl, "g"] \\ & X & \end{tikzcd} : \mathsf{cart}$$

(iii) When $f = g = \mathsf{id}$ and u is of length 0, we call p the ***loose unit*** on X and write U_X for it. Note that the associated cartesian cell is loosewise invertible automatically:

$$\begin{tikzcd} X \arrow[rr, "\mathsf{U}_X"] \arrow[dr, equal] & \arrow[d, phantom, "\cong"] & X \arrow[dl, equal] \\ & X & \end{tikzcd} : \mathsf{cart}$$

♦

Definition 2.19. Let $\mathbb{L}$ be an AVDC. We say $\mathbb{L}$ ***has restrictions*** (resp. ***unicoary restrictions***) if the restriction $u(f,g)$ exists for any f, g, and u of length ≤ 1 (resp. length 1). We say $\mathbb{L}$ ***has companions*** (resp. ***conjoints***) if the companion f_* (resp. conjoint f^*) exists for any f. We say $\mathbb{L}$ ***has loose units*** if the loose unit U_X exists for any X. We refer to such $\mathbb{L}$ as an AVDC with restrictions, companions, etc. ♦

Proposition 2.20 ([Kou20, 5.4. Lemma])**.** Let $A \xrightarrow{f} X$ be a tight arrow in an AVDC. Then, the following data correspond bijectively to each other:

(i) A pair (p, ε) of a loose arrow $A \xrightarrow{p} X$ and a cartesian cell

$$\begin{tikzcd} A \arrow[rr, "p"] \arrow[dr, "f"'] & \arrow[d, phantom, "\varepsilon"] & X \arrow[dl, equal] \\ & X & \end{tikzcd} : \mathsf{cart},$$

which gives a companion of f.

(ii) A tuple (p, η, ε) of a loose arrow $A \xrightarrow{p} X$ and cells η, ε satisfying the following equations:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, equal] \arrow[r, "p"] \arrow[d, "f"'] \arrow[ur, phantom, "\eta"] & X \arrow[dl, equal] \arrow[dl, phantom, "\varepsilon"] \\ X & \end{tikzcd} = \begin{tikzcd} A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ X \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, "p"] \arrow[d, "f"] \arrow[dr, phantom, "\varepsilon"] & X \\ A \arrow[ur, equal] \arrow[r, "p"'] \arrow[ur, phantom, "\eta"] & X \arrow[ur, equal] & \end{tikzcd} = \begin{tikzcd} A \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[d, equal] \\ A \arrow[r, "p"'] & X \end{tikzcd}$$

Corollary 2.21 ([Kou20, 5.5. Corollary])**.** Companions, conjoints, and loose units are preserved by any AVD-functor.

Remark 2.22. An AVDC with loose units, called a ***unital AVDC*** in [Kou20], can be identified with a ***unital VDC*** in the sense of [CS10]. When we regard an AVDC with loose units as a unital VDC, the AVD-functors between them correspond to the ***normal*** VD-functors [CS10]. Indeed, there is a 2-equivalence [Kou20, 10.1. Theorem]:

$$\mathcal{UAVDC} \simeq \mathcal{UVDC}\mathrm{n}. \tag{5}$$

Here, $\mathcal{UAVDC}$ denotes the 2-category of (huge) unital AVDCs and AVD-functors, and $\mathcal{UVDC}$n denotes the 2-category of (huge) unital VDCs and normal VD-functors.

An AVDC with unicoary restrictions is called an ***augmented virtual equipment***, and AVDC with restrictions is called a ***unital virtual equipment*** in [Kou20]. The latter can be identified with a ***virtual equipment*** [CS10] by the 2-equivalence (5). ♦

Remark 2.23. We now have two ways to regard unital VDCs as AVDCs. The first one is to regard as diminished AVDCs, where the AVD-functors between them correspond to the VD-functors. The second one is to regard as AVDCs with loose units, where the AVD-functors between them correspond to the normal VD-functors. Depending on which types of VD-functors are considered, we will use both ways. ♦

We now present a slight generalization of cartesian cells. While this may seem somewhat technical, we introduce it here since it will be used later.

Definition 2.24. Let $A \overset{\vec{u}}{\dashrightarrow} B$ be a loose path in an AVDC $\mathbb{L}$. Let $\mathbf{C}$ be a category, and let $F\colon \mathbf{C} \to \mathbf{T}^{\leq 1}\mathbb{L}$ be a functor. A ***cone*** over F with the vertex $\vec{u}$ is a family of cells α_c for $c \in \mathbf{C}$ satisfying the following equality for any morphism $c \xrightarrow{s} d$ in $\mathbf{C}$:

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, "\alpha^0_c"'] \arrow[dr, phantom, "\alpha_c"] & B \arrow[d, "\alpha^1_c"] \\ F^0c \arrow[r, dotted, "Fc"] \arrow[d, "F^0s"'] \arrow[dr, phantom, "Fs"] & F^1c \arrow[d, "F^1s"] \\ F^0d \arrow[r, dotted, "Fd"'] & F^1d \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, "\alpha^0_d"'] \arrow[dr, phantom, "\alpha_d"] & B \arrow[d, "\alpha^1_d"] \\ F^0d \arrow[r, dotted, "Fd"'] & F^1d \end{tikzcd} \quad \text{in } \mathbb{L}.$$

♦

Definition 2.25 (Jointly cartesian cells)**.** Let $\mathbb{L}$ be an AVDC, let $\mathbf{C}$ be a category, and let $F\colon \mathbf{C} \to \mathbf{T}^{\leq 1}\mathbb{L}$ be a functor. A cone over F

$$\begin{tikzcd} X^0 \arrow[r, dotted, "X"] \arrow[d, "\alpha^0_c"'] \arrow[dr, phantom, "\alpha_c"] & X^1 \arrow[d, "\alpha^1_c"] \\ F^0c \arrow[r, dotted, "Fc"'] & F^1c \end{tikzcd} \quad \text{in } \mathbb{L} \quad (c \in \mathbf{C})$$

is called ***jointly cartesian*** in $\mathbb{L}$ if it satisfies the following condition: Suppose that we are given a loose path $A \overset{\vec{u}}{\dashrightarrow} B$, tight arrows $A \xrightarrow{f} X^0$ and $B \xrightarrow{g} X^1$, and a cone β over F on the right below; then there uniquely exists a cell γ satisfying the following equality for any $c \in \mathbf{C}$.

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\gamma"] & B \arrow[d, "g"] \\
X^0 \arrow[r, dotted, "X"] \arrow[d, "\alpha^0_c"'] \arrow[dr, phantom, "\alpha_c"] & X^1 \arrow[d, "\alpha^1_c"] \\
F^0c \arrow[r, dotted, "Fc"'] & F^1c
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X^0 \arrow[r, phantom, "\beta_c"] \arrow[d, "\alpha^0_c"'] & X^1 \arrow[d, "\alpha^1_c"] \\
F^0c \arrow[r, dotted, "Fc"'] & F^1c
\end{tikzcd}
\quad \text{in } \mathbb{L}
$$

♦

2.1.4. *Cocartesian cells.*

Definition 2.26 (Cocartesian cells)**.** A cell

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\
A \arrow[r, dotted, "v"'] & B
\end{tikzcd}
\tag{6}
$$

in an AVDC is called ***cocartesian*** if the following assignment induces a bijection $\mathrm{Cell}\big({}_f\,{}^{\vec{p}v\vec{q}}_{w}\,{}_g\big) \cong \mathrm{Cell}\big({}_f\,{}^{\vec{p}\vec{u}\vec{q}}_{w}\,{}_g\big)$ for any $f, g, \vec{p}, \vec{q}, w$:

$$
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] & A \arrow[r, dotted, "v"] \arrow[d, phantom, "\cdot"] & B \arrow[r, dashed, "\vec{q}"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[rrr, dotted, "w"'] & & & \cdot
\end{tikzcd}
\mapsto
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] & A \arrow[r, dotted, "v"] \arrow[d, phantom, "\cdot"] & B \arrow[r, dashed, "\vec{q}"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[rrr, dotted, "w"'] & & & \cdot
\end{tikzcd}
$$

The cell α is called ***VD-cocartesian*** if it induces the above bijection only for w of length 1. Cocartesian cells and VD-cocartesian cells are often denoted by the symbol "cocart" and "VD.cocart," respectively:

$$
\begin{tikzcd}
\cdot \arrow[r, dashed] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cocart}"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, dotted] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, dashed] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{VD.cocart}"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, dotted] & \cdot
\end{tikzcd}
$$

♦

Remark 2.27. We can also consider cocartesian cells with an arbitrary boundary rather than identity tight arrows. See [Kou20, Section 7] for details. ♦

Remark 2.28. The VD-cocartesian cells recover the concept of "cocartesian cells in VDCs" introduced in [CS10], where a different term "opcartesian" is used. Indeed, VD-cocartesian cells in a diminished AVDC are nothing but opcartesian cells, in the sense of [CS10], in the corresponding VDC. ♦

Definition 2.29. Let $\mathbb{L}$ be an AVDC, and let $X \in \mathbb{L}$. A loose arrow u in a VD-cocartesian cell of the following form is called the ***loose VD-unit*** on X.

$$
\begin{tikzcd}
& X \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\mathsf{VD.cocart}"] & \\
X \arrow[rr, "u"'] & {} & X
\end{tikzcd}
\quad \text{in } \mathbb{L}.
\tag{7}
$$

Note that the loose VD-unit on X is, if it exists, unique up to loosewise invertible cell. ♦

Remark 2.30. If the cell (7) is cocartesian rather than VD-cocartesian, the loose cell u in (7) becomes the loose unit on X. Indeed, every cocartesian cell of the form (7) is loosewise invertible. Thus, the loose VD-units are a weaker concept than the loose units. Clearly, loose VD-units in diminished AVDCs are the same concept as (loose) "units" in VDCs in the sense of [CS10]. ♦

Definition 2.31. Let $\mathbb{L}$ be an AVDC. An object $A \in \mathbb{L}$ is called ***VD-composable*** in $\mathbb{L}$ if:

- For any loose arrows $\cdot \xrightarrow{u_1} A \xrightarrow{u_2} \cdot$ in $\mathbb{L}$, there exists a VD-cocartesian cell of the following form:

$$\begin{tikzcd} \cdot \arrow[r, "u_1"] \arrow[d, equal] & A \arrow[r, "u_2"] \arrow[d, phantom, "\text{VD.cocart}"] & \cdot \arrow[d, equal] \\ \cdot \arrow[rr] & & \cdot \end{tikzcd} \quad \text{in } \mathbb{L}; \tag{8}$$

- A has the loose VD-unit. That is, there is a VD-cocartesian cell of the following form:

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\text{VD.cocart}"] & \\ A \arrow[rr] & & A \end{tikzcd} \quad \text{in } \mathbb{L}. \tag{9}$$

♦

Notation 2.32. Let $\mathbb{L}$ be an AVDC. Then, all of the VD-composable objects yield a bicategory $\mathcal{L}\mathbb{L}$, called the ***loose bicategory*** of $\mathbb{L}$, where 1-cells are loose arrows and compositions and identities are defined by the VD-cocartesian cells (8) and (9). ♦

Remark 2.33. A diminished AVDC where all objects are VD-composable is the essentially same concept as a ***pseudo double category***. See [CS10, 5.2. Theorem] or [DPP06, 2.8. Proposition] for details. ♦

Notation 2.34. Given a bicategory $\mathcal{W}$, we can obtain a diminished AVDC $\mathbb{V}\mathcal{W}$ as follows. The tight category $\mathbf{T}(\mathbb{V}\mathcal{W})$ is the discrete category of objects in $\mathcal{W}$. A loose arrow in $\mathbb{V}\mathcal{W}$ is a 1-cell in $\mathcal{W}$. A cell from $\vec{f}$ to g in $\mathbb{V}\mathcal{W}$ is a 2-cell from $\odot\vec{f}$ to g in $\mathcal{W}$:

$$\begin{tikzcd} c \arrow[r, dashed, "\vec{f}"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & c' \arrow[d, equal] \\ c \arrow[r, "g"'] & c' \end{tikzcd} \quad \text{in } \mathbb{V}\mathcal{W} \qquad \Bigg\| \qquad \begin{tikzcd} c \arrow[r, bend left, "\odot\vec{f}"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & c' \end{tikzcd} \quad \text{in } \mathcal{W}$$

Here, $\odot\vec{f}$ denotes the composite of $\vec{f}$ in $\mathcal{W}$. ♦

Theorem 2.35. For bicategories $\mathcal{W}$ and $\mathcal{W}'$, the lax-functors $\mathcal{W} \to \mathcal{W}'$ are the same thing as the AVD-functors $\mathbb{V}\mathcal{W} \to \mathbb{V}\mathcal{W}'$. Moreover, the pseudo-functors $\mathcal{W} \to \mathcal{W}'$ are the same thing as the AVD-functors that preserve all VD-cocartesian cells.

Proof. See [CS10, 3.5. Example]. □

2.1.5. *The* $\mathbb{M}\mathrm{od}$*-construction.* We recall the $\mathbb{M}\mathrm{od}$-construction from [Lei99; Lei04; CS10], which is a construction of a VDC "$\mathbb{M}\mathrm{od}(\mathbb{K})$" from a VDC $\mathbb{K}$. Since the resulting VDCs are always unital and normal VD-functors between them are often considered, we redefine "$\mathbb{M}\mathrm{od}(\mathbb{K})$" as an AVDC with loose units. Such a redefinition is also considered in [Kou20].

Definition 2.36 ([Lei99; Lei04; CS10; Kou20])**.** Let $\mathbb{K}$ be an AVDC. The AVDC $\mathbb{M}\mathrm{od}(\mathbb{K})$ is defined as follows:

- An object is a ***monoid***, which consists of the following data $A := (A^0, A^1, A^e, A^m)$:

$$\begin{tikzcd} & A^0 \arrow[d, phantom, "A^e"] & \\ A^0 \arrow[ur, equal] \arrow[rr, "A^1"', "\shortmid"{marking}] & & A^0 \arrow[ul, equal] \end{tikzcd} \qquad \begin{tikzcd} A^0 \arrow[r, "A^1", "\shortmid"{marking}] \arrow[d, equal] & A^0 \arrow[r, "A^1", "\shortmid"{marking}] \arrow[d, phantom, "A^m"] & A^0 \arrow[d, equal] \\ A^0 \arrow[rr, "A^1"', "\shortmid"{marking}] & & A^0 \end{tikzcd} \quad \text{in } \mathbb{K}.$$

The data (A^0, A^1, A^e, A^m) are required to satisfy a monoid-like axiom. The cells A^e and A^m are called the ***unit*** and the ***multiplication*** of the monoid A, respectively.

- A tight arrow $A \xrightarrow{f} B$ is called a ***monoid homomorphism***. It consists of the following data (f^0, f^1):

$$\begin{tikzcd} A^0 \arrow[r, "A^1", "\shortmid"{marking}] \arrow[d, "f^0"'] \arrow[dr, phantom, "f^1"] & A^0 \arrow[d, "f^0"] \\ B^0 \arrow[r, "B^1"', "\shortmid"{marking}] & B^0 \end{tikzcd} \quad \text{in } \mathbb{K}$$

that is required to be compatible with units and multiplications.

- A loose arrow $A \overset{M}{\relbar\joinrel\mapstochar\joinrel\rightarrow} B$ is called a ***(bi)module***. It consists of the following data (M^1, M^l, M^r):

$$\begin{tikzcd} A^0 \arrow[r, "A^1", "\shortmid"{marking}] \arrow[d, equal] & A^0 \arrow[r, "M^1", "\shortmid"{marking}] \arrow[d, phantom, "M^l"] & B^0 \arrow[d, equal] \\ A^0 \arrow[rr, "M^1"', "\shortmid"{marking}] & & B^0 \end{tikzcd} \qquad \begin{tikzcd} A^0 \arrow[r, "M^1", "\shortmid"{marking}] \arrow[d, equal] & B^0 \arrow[r, "B^1", "\shortmid"{marking}] \arrow[d, phantom, "M^r"] & B^0 \arrow[d, equal] \\ A^0 \arrow[rr, "M^1"', "\shortmid"{marking}] & & B^0 \end{tikzcd} \quad \text{in } \mathbb{K}$$

that is required to satisfy a module-like axiom.

- A unicoary cell α in $\mathbb{M}\mathrm{od}(\mathbb{K})$ on the left below is a cell in $\mathbb{K}$ on the right below

$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{M}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A_n \arrow[d, "g"] \\ B \arrow[r, "N"', "\shortmid"{marking}] & C \end{tikzcd} \quad \text{in } \mathbb{M}\mathrm{od}(\mathbb{K}) \qquad \begin{tikzcd} A_0^0 \arrow[r, "M_1^1", "\shortmid"{marking}] \arrow[d, "f^0"'] & \cdots \arrow[r, "M_n^1", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & A_n^0 \arrow[d, "g^0"] \\ B^0 \arrow[rr, "N^1"', "\shortmid"{marking}] & & C^0 \end{tikzcd} \quad \text{in } \mathbb{K}$$

such that, for each $0 \leq i \leq n$, two canonical ways to fill the following boundary give the same cell in $\mathbb{K}$:

$$\begin{tikzcd} A_0^0 \arrow[r, dashed, "(M_j^1)_{0<j\leq i}", "\shortmid"{marking}] \arrow[d, "f^0"'] & A_i^0 \arrow[r, "A_i^1", "\shortmid"{marking}] & A_i^0 \arrow[r, dashed, "(M_j^1)_{i<j\leq n}", "\shortmid"{marking}] & A_n^0 \arrow[d, "g^0"] \\ B^0 \arrow[rrr, "N^1"', "\shortmid"{marking}] & & & C^0 \end{tikzcd} \quad \text{in } \mathbb{K}.$$

- A nullcoary cell β in $\mathbb{M}\mathrm{od}(\mathbb{K})$ on the left below is a cell in $\mathbb{K}$ on the right below

$$\begin{tikzcd} A_0 \arrow[rr, dashed, "\vec{M}", "\shortmid"{marking}] \arrow[dr, "f"'] & \beta & A_n \arrow[dl, "g"] \\ & B & \end{tikzcd} \quad \text{in } \mathbb{M}\mathrm{od}(\mathbb{K}) \qquad \begin{tikzcd} A_0^0 \arrow[r, "M_1^1", "\shortmid"{marking}] \arrow[d, "f^0"'] & \cdots \arrow[r, "M_n^1", "\shortmid"{marking}] \arrow[d, phantom, "\beta"] & A_n^0 \arrow[d, "g^0"] \\ B^0 \arrow[rr, "B^1"', "\shortmid"{marking}] & & B^0 \end{tikzcd} \quad \text{in } \mathbb{K}$$

such that, for each $0 \le i \le n$, two canonical ways to fill the following boundary give the same cell in $\mathbb{K}$:

$$\begin{tikzcd} A^0_0 \arrow[r, dashed, "(M^1_j)_{0<j\le i}"] \arrow[d, "f^0"'] & A^0_i \arrow[r, "A^1_i"] & A^0_i \arrow[r, dashed, "(M^1_j)_{i<j\le n}"] & A^0_n \arrow[d, "g^0"] \\ B^0 \arrow[rrr, "B^1"'] & & & B^0 \end{tikzcd} \quad \text{in } \mathbb{K}.$$

♦

Remark 2.37. In the construction of $\mathbb{M}\mathrm{od}(\mathbb{K})$, no nullcoary cell in $\mathbb{K}$ is used except for identities. In particular, we have $\mathbb{M}\mathrm{od}(\mathbb{K}) = \mathbb{M}\mathrm{od}(\mathbb{K}^\flat)$. ♦

Theorem 2.38 ([CS10]). Let $\mathbb{L}$ be an AVDC with loose units and let $\mathbb{K}$ be an AVDC. Then, the following data correspond to each other up to isomorphism:

(i) An AVD-functor $\mathbb{L} \to \mathbb{M}\mathrm{od}(\mathbb{K})$.
(ii) An AVD-functor $\mathbb{L}^\flat \to \mathbb{K}$.

Proof. An AVD-functor $\mathbb{L}^\flat \to \mathbb{K}$ is nothing but a VD-functor $\mathbb{L}^\flat \to \mathbb{K}^\flat$. By the universal property of the $\mathbb{M}\mathrm{od}$-construction [CS10, 5.14. Proposition], it corresponds to a normal VD-functor $\mathbb{L}^\flat \to \mathbb{M}\mathrm{od}(\mathbb{K}^\flat)^\flat$ in the sense of [CS10]. Since $\mathbb{M}\mathrm{od}(\mathbb{K}^\flat) = \mathbb{M}\mathrm{od}(\mathbb{K})$ and since both $\mathbb{L}$ and $\mathbb{M}\mathrm{od}(\mathbb{K})$ have loose units, it also corresponds to an AVD-functor $\mathbb{L} \to \mathbb{M}\mathrm{od}(\mathbb{K})$. □

Notation 2.39. For an AVDC $\mathbb{K}$ with loose units, we write $U\colon \mathbb{K} \to \mathbb{M}\mathrm{od}(\mathbb{K})$ for the AVD-functor corresponding to the inclusion $\mathbb{K}^\flat \to \mathbb{K}$. Since U locally induces bijections on the classes of tight arrows, loose arrows, and cells, we can regard $\mathbb{K}$ as a full sub-AVDC of $\mathbb{M}\mathrm{od}(\mathbb{K})$ by U. ♦

Proposition 2.40 ([CS10]). Let $\mathbb{K}$ be an AVDC.

(i) $\mathbb{M}\mathrm{od}(\mathbb{K})$ has loose units.
(ii) If $\mathbb{K}$ has unicoary restrictions, then $\mathbb{M}\mathrm{od}(\mathbb{K})$ has restrictions.

Proof.

(i) By [CS10, 5.5. Proposition], the diminished AVDC $\mathbb{M}\mathrm{od}(\mathbb{K})^\flat$ has loose VD-units. Those units automatically become loose units in $\mathbb{M}\mathrm{od}(\mathbb{K})$ since all nullcoary cells are inherited from them.
(ii) By [CS10, 7.4. Proposition], unicoary restrictions in $\mathbb{K}$ give those in $\mathbb{M}\mathrm{od}(\mathbb{K})$. □

2.1.6. *Loosewise indiscreteness.*

Definition 2.41. An AVDC $\mathbb{K}$ is called ***loosewise discrete*** if:

- It has no loose arrows.
- It has no cells except for tight identity cells ♦

Definition 2.42. An AVDC $\mathbb{K}$ is called ***loosewise indiscrete*** if:

- For any objects $A, B \in \mathbb{K}$, there is a unique loose arrow from A to B, denoted by $A \xrightarrow{!_{AB}} B$.
- For any boundary for cells, there is a unique cell filling it. ♦

Definition 2.43. An AVDC $\mathbb{K}$ is called ***loosewise VD-indiscrete*** if:

- For any objects $A, B \in \mathbb{K}$, there is a unique loose arrow from A to B, denoted by $A \xrightarrow{!_{AB}} B$.

- For any $A_0, A_1, \ldots, A_n, X, Y \in \mathbb{K}$ $(n \geq 0)$ and any tight arrows $A_0 \xrightarrow{f} X, A_n \xrightarrow{g} Y$ in $\mathbb{K}$, there is a unique cell of the following form:

$$\begin{tikzcd} A_0 \arrow[r, "!_{A_0A_1}"] \arrow[d, "f"'] & A_1 \arrow[r, "!_{A_1A_2}"] & \cdots \arrow[r, "!_{A_{n-1}A_n}"] & A_n \arrow[d, "g"] \\ X \arrow[rrr, "!_{XY}"'] \arrow[rrr, phantom, shift left=4, "!"] & & & Y \end{tikzcd} \quad \text{in } \mathbb{K}.$$

- $\mathbb{K}$ is diminished. ♦

Notation 2.44. Let $\mathbf{C}$ be a category. Let $\mathbb{D}\mathbf{C}$ (resp. $\mathbb{I}\mathbf{C}$; $\mathbb{I}^\flat\mathbf{C}$) denote a loosewise discrete (resp. indiscrete; VD-indiscrete) AVDC uniquely determined by $\mathbf{T}(\mathbb{D}\mathbf{C}) = \mathbf{C}$ (resp. $\mathbf{T}(\mathbb{I}\mathbf{C}) = \mathbf{C}$; $\mathbf{T}(\mathbb{I}^\flat\mathbf{C}) = \mathbf{C}$). Then, $\mathbb{I}^\flat\mathbf{C} = (\mathbb{I}\mathbf{C})^\flat$ follows immediately. Note that every loosewise discrete (resp. indescrete; VD-indiscrete) AVDC is of the form $\mathbb{D}\mathbf{C}$ (resp. $\mathbb{I}\mathbf{C}$; $\mathbb{I}^\flat\mathbf{C}$) for some $\mathbf{C}$. ♦

Notation 2.45. For a large set S, we write $\mathbb{D}\mathrm{S}$ (resp. $\mathbb{I}\mathrm{S}$; $\mathbb{I}^\flat\mathrm{S}$) for the loosewise discrete (resp. indiscrete; VD-indiscrete) large AVDC of Notation 2.44 obtained from the discrete category S. ♦

Remark 2.46. Let 1 denote the singleton, and let $\mathbb{L}$ be an AVDC.

(i) An AVD-functor $\mathbb{D}1 \to \mathbb{L}$ is the same thing as an object in $\mathbb{L}$.
(ii) An AVD-functor $\mathbb{I}1 \to \mathbb{L}$ is the same thing as an object with a chosen loose unit in $\mathbb{L}$.
(iii) An AVD-functor $\mathbb{I}^\flat 1 \to \mathbb{L}$ is the same thing as a monoid in $\mathbb{L}$. ♦

Surprisingly, almost all cells in a loosewise (VD-)indiscrete AVDC become cartesian for a diagrammatic reason. To show this, we introduce a special type of "absolutely" cartesian cells.

Definition 2.47. A cell

$$\begin{tikzcd} A_0 \arrow[r, dotted, "u"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha"] & A_1 \arrow[d, "f_1"] \\ B_0 \arrow[r, dotted, "v"'] & B_1 \end{tikzcd}$$

in an AVDC is called ***split*** if there are data $(p_0, p_1, q_0, q_1, \beta_0, \beta_1, \gamma, \delta_0, \delta_1, \sigma, \eta_0, \eta_1)$ of the following forms:

$$\begin{tikzcd} & A_0 \arrow[dl, equal] \arrow[d, "f_0"] \arrow[dl, phantom, shift right=3, "\beta_0"] \\ A_0 \arrow[r, dotted, "p_0"'] & B_0 \end{tikzcd} \quad \begin{tikzcd} A_1 \arrow[d, "f_1"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\beta_1"] & \\ B_1 \arrow[r, dotted, "p_1"'] & A_1 \end{tikzcd} \quad \begin{tikzcd} A_0 \arrow[r, dotted, "p_0"] \arrow[d, equal] & B_0 \arrow[r, dotted, "v"] \arrow[d, phantom, "\gamma"] & B_1 \arrow[r, dotted, "p_1"] & A_1 \arrow[d, equal] \\ A_0 \arrow[rrr, dotted, "u"'] & & & A_1 \end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[r, dotted, "p_0"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\delta_0"] & B_0 \arrow[d, equal] \\ B_0 \arrow[r, dotted, "q_0"'] & B_0 \end{tikzcd} \quad \begin{tikzcd} B_1 \arrow[r, dotted, "p_1"] \arrow[d, equal] \arrow[dr, phantom, "\delta_1"] & A_1 \arrow[d, "f_1"] \\ B_1 \arrow[r, dotted, "q_1"'] & B_1 \end{tikzcd} \quad \begin{tikzcd} B_0 \arrow[r, dotted, "q_0"] \arrow[d, equal] & B_0 \arrow[r, dotted, "v"] \arrow[d, phantom, "\sigma"] & B_1 \arrow[r, dotted, "q_1"] & B_1 \arrow[d, equal] \\ B_0 \arrow[rrr, dotted, "v"'] & & & B_1 \end{tikzcd}$$

$$\begin{tikzcd} & B_0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_0"] & \\ B_0 \arrow[rr, dotted, "q_0"'] & {} & B_0 \end{tikzcd} \quad \begin{tikzcd} & B_1 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_1"] & \\ B_1 \arrow[rr, dotted, "q_1"'] & {} & B_1 \end{tikzcd}$$

These are required to satisfy the following equations:

$$\begin{tikzcd}
& A_0 \arrow[r, dotted, "u"] \arrow[d, "f_0"] \arrow[dl, equal] \arrow[dr, phantom, "\alpha"] & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] & \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, equal] \arrow[ur, phantom, "\beta_0"] & B_0 \arrow[r, dotted, "v"'] & B_1 \arrow[r, dotted, "p_1"'] & A_1 \arrow[d, equal] \arrow[ul, phantom, "\beta_1"] \\
A_0 \arrow[rrr, dotted, "u"'] \arrow[rrru, phantom, "\gamma"] & & & A_1
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, dotted, "u"] \arrow[d, equal] \arrow[dr, phantom, "="] & A_1 \arrow[d, equal] \\
A_0 \arrow[r, dotted, "u"'] & A_1
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \arrow[r, dotted, "p_0"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\delta_0"] & B_0 \arrow[r, dotted, "v"] \arrow[d, equal] \arrow[dr, phantom, "="] & B_1 \arrow[r, dotted, "p_1"] \arrow[d, equal] \arrow[dr, phantom, "\delta_1"] & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[r, dotted, "q_0"'] \arrow[d, equal] \arrow[drrr, phantom, "\sigma"] & B_0 \arrow[r, dotted, "v"'] & B_1 \arrow[r, dotted, "q_1"'] & B_1 \arrow[d, equal] \\
B_0 \arrow[rrr, dotted, "v"'] & & & B_1
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, dotted, "p_0"] \arrow[d, equal] \arrow[drrr, phantom, "\gamma"] & B_0 \arrow[r, dotted, "v"] & B_1 \arrow[r, dotted, "p_1"] & A_1 \arrow[d, equal] \\
A_0 \arrow[rrr, dotted, "u"'] \arrow[d, "f_0"'] \arrow[drrr, phantom, "\alpha"] & & & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[rrr, dotted, "v"'] & & & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
& A_0 \arrow[d, "f_0"] \arrow[dl, equal] \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, "f_0"'] \arrow[dr, phantom, "\delta_0"] \arrow[ur, phantom, "\beta_0"] & B_0 \arrow[d, equal] \\
B_0 \arrow[r, dotted, "q_0"'] & B_0
\end{tikzcd}
=
\begin{tikzcd}
& A_0 \arrow[d, bend right, "f_0"'] \arrow[d, bend left, "f_0"] \arrow[d, phantom, "="] & \\
& B_0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_0"] & \\
B_0 \arrow[rr, dotted, "q_0"'] & {} & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_1 \arrow[d, bend right, "f_1"'] \arrow[d, bend left, "f_1"] \arrow[d, phantom, "="] & \\
& B_1 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_1"] & \\
B_1 \arrow[rr, dotted, "q_1"'] & {} & B_1
\end{tikzcd}
=
\begin{tikzcd}
A_1 \arrow[d, "f_1"'] \arrow[dr, equal] & \\
B_1 \arrow[r, dotted, "p_1"'] \arrow[d, equal] \arrow[dr, phantom, "\delta_1"] \arrow[ur, phantom, "\beta_1"] & A_1 \arrow[d, "f_1"] \\
B_1 \arrow[r, dotted, "q_1"'] & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
& B_0 \arrow[r, dotted, "v"] \arrow[d, equal] \arrow[dl, equal] \arrow[dr, phantom, "="] & B_1 \arrow[d, equal] \arrow[dr, equal] & \\
B_0 \arrow[r, dotted, "q_0"'] \arrow[d, equal] \arrow[ur, phantom, "\eta_0"] & B_0 \arrow[r, dotted, "v"'] & B_1 \arrow[r, dotted, "q_1"'] & B_1 \arrow[d, equal] \arrow[ul, phantom, "\eta_1"] \\
B_0 \arrow[rrr, dotted, "v"'] \arrow[rrru, phantom, "\sigma"] & & & B_1
\end{tikzcd}
=
\begin{tikzcd}
B_0 \arrow[r, dotted, "v"] \arrow[d, equal] \arrow[dr, phantom, "="] & B_1 \arrow[d, equal] \\
B_0 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}$$

♦

Lemma 2.48. Every split cell is cartesian. In particular, every split cell is ***absolutely cartesian***; that is, it is a cartesian cell preserved by any AVD-functor.

Proof. Let α be a split cell as in Definition 2.47. Take an arbitrary cell θ on the left below:

$$\begin{tikzcd}
X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] & X_1 \arrow[d, "x_1"] \\
A_0 \arrow[d, "f_0"'] \arrow[r, phantom, "\theta"] & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}
=
\begin{tikzcd}
X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] \arrow[dr, phantom, "\bar{\theta}"] & X_1 \arrow[d, "x_1"] \\
A_0 \arrow[r, dotted, "u"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha"] & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}
\tag{10}$$

If there exists a cell $\bar\theta$ satisfying the above equation, then $\bar\theta$ must be given by the following:

$$
\bar\theta =
\begin{tikzcd}
 & X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] \arrow[dr, phantom, "\bar\theta"] & X_1 \arrow[d, "x_1"] & \\
 & A_0 \arrow[r, dotted, "u"] \arrow[d, "f_0"] \arrow[dl, equal] \arrow[dr, phantom, "\alpha"] & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] & \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, equal] \arrow[ur, phantom, "\beta_0"] \arrow[drrr, phantom, "\gamma"] & B_0 \arrow[r, dotted, "v"'] & B_1 \arrow[r, dotted, "p_1"'] & A_1 \arrow[d, equal] \arrow[ul, phantom, "\beta_1"'] \\
A_0 \arrow[rrr, dotted, "u"'] & & & A_1
\end{tikzcd}
=
\begin{tikzcd}
 & X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] & X_1 \arrow[d, "x_1"] & \\
 & A_0 \arrow[d, "f_0"] \arrow[dl, equal] \arrow[r, phantom, "\theta"] & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] & \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, equal] \arrow[ur, phantom, "\beta_0"] \arrow[drrr, phantom, "\gamma"] & B_0 \arrow[r, dotted, "v"'] & B_1 \arrow[r, dotted, "p_1"'] & A_1 \arrow[d, equal] \arrow[ul, phantom, "\beta_1"'] \\
A_0 \arrow[rrr, dotted, "u"'] & & & A_1
\end{tikzcd}
$$

Conversely, let us define $\bar\theta$ by the above equation. Then, the following calculation shows that $\bar\theta$ satisfies the desired equation (10):

$$
\begin{tikzcd}
 & X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] & X_1 \arrow[d, "x_1"] & \\
 & A_0 \arrow[d, "f_0"] \arrow[dl, equal] \arrow[r, phantom, "\theta"] & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] & \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, equal] \arrow[ur, phantom, "\beta_0"] \arrow[drrr, phantom, "\gamma"] & B_0 \arrow[r, dotted, "v"'] & B_1 \arrow[r, dotted, "p_1"'] & A_1 \arrow[d, equal] \arrow[ul, phantom, "\beta_1"'] \\
A_0 \arrow[rrr, dotted, "u"'] \arrow[d, "f_0"'] \arrow[drrr, phantom, "\alpha"] & & & A_1 \arrow[d, "f_1"] \\
B_0 \arrow[rrr, dotted, "v"'] & & & B_1
\end{tikzcd}
=
\begin{tikzcd}
 & X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] & X_1 \arrow[d, "x_1"] & \\
 & A_0 \arrow[d, "f_0"] \arrow[dl, equal] \arrow[r, phantom, "\theta"] & A_1 \arrow[d, "f_1"'] \arrow[dr, equal] & \\
A_0 \arrow[r, dotted, "p_0"'] \arrow[d, "f_0"'] \arrow[ur, phantom, "\beta_0"] \arrow[dr, phantom, "\delta_0"] & B_0 \arrow[r, dotted, "v"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B_1 \arrow[r, dotted, "p_1"'] \arrow[d, equal] \arrow[dr, phantom, "\delta_1"] & A_1 \arrow[d, "f_1"] \arrow[ul, phantom, "\beta_1"'] \\
B_0 \arrow[r, dotted, "q_0"'] \arrow[d, equal] \arrow[drrr, phantom, "\sigma"] & B_0 \arrow[r, dotted, "v"'] & B_1 \arrow[r, dotted, "q_1"'] & B_1 \arrow[d, equal] \\
B_0 \arrow[rrr, dotted, "v"'] & & & B_1
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
 & X_0 \arrow[r, dashed, "\vec{w}"] \arrow[d, "x_0"'] & X_1 \arrow[d, "x_1"] & \\
 & A_0 \arrow[d, "f_0"'] \arrow[r, phantom, "\theta"] & A_1 \arrow[d, "f_1"] & \\
 & B_0 \arrow[r, dotted, "v"] \arrow[d, equal] \arrow[dl, equal] \arrow[dr, phantom, "="] & B_1 \arrow[d, equal] \arrow[dr, equal] & \\
B_0 \arrow[r, dotted, "q_0"'] \arrow[d, equal] \arrow[ur, phantom, "\eta_0"] \arrow[drrr, phantom, "\sigma"] & B_0 \arrow[r, dotted, "v"'] & B_1 \arrow[r, dotted, "q_1"'] & B_1 \arrow[d, equal] \arrow[ul, phantom, "\eta_1"'] \\
B_0 \arrow[rrr, dotted, "v"'] & & & B_1
\end{tikzcd}
= \theta.
$$

This shows that α is cartesian. $\square$

Corollary 2.49. Let $\mathbb{K}$ be a loosewise indiscrete or VD-indiscrete AVDC. Then, every cell of the following form is absolutely cartesian.

$$
\begin{tikzcd}
A \arrow[r, "!_{AB}"] \arrow[d, "f"'] \arrow[dr, phantom, "!_{fg}"] & B \arrow[d, "g"] \\
X \arrow[r, "!_{XY}"'] & Y
\end{tikzcd}
\quad \text{in } \mathbb{K}.
$$

Proof. By the loosewise (VD-)indiscreteness, it immediately follows that the cell $!_{fg}$ is split. Then, Lemma 2.48 shows that it is absolutely cartesian. □

2.2. **Categories enriched over a virtual double category.** In this subsection, we will recall the notion of enriched categories over a VDC from [Lei99; Lei02]. We first define the diminished AVDC of ***matrices***, whose special case is described in [Lei04, Example 5.1.9].

Definition 2.50. Let $\mathbb{X}$ be an AVDC. By an ***$\mathbb{X}$-colored large set***, we mean a large set A equipped with a map $A \xrightarrow{|\cdot|_A} \mathrm{Ob}\mathbb{X}$. ♦

Definition 2.51. Let $\mathbb{X}$ be an AVDC. Let A and B be $\mathbb{X}$-colored large sets. A ***morphism of families*** F from A to B consists of:

- For $x \in A$, an element $F^0x \in B$;
- For $x \in A$, a tight arrow $|x|_A \xrightarrow{F^1x} |F^0x|_B$ in $\mathbb{X}$. ♦

Definition 2.52. Let $\mathbb{X}$ be an AVDC. Let A and B be $\mathbb{X}$-colored large sets. An ***$(A \times B)$-matrix*** M over $\mathbb{X}$ is defined to be a family of loose arrows $|x|_A \xrightarrow{M(x,y)} |y|_B$ in $\mathbb{X}$ for $x \in A$ and $y \in B$. ♦

Definition 2.53. Let $\mathbb{X}$ be an AVDC. The (diminished) AVDC $\mathbb{X}$-$\mathbb{M}\mathrm{at}$ of matrices over $\mathbb{X}$ is defined as follows: its objects are $\mathbb{X}$-colored large sets, its tight arrows are morphisms of families, its loose arrows $A \longrightarrow B$ are $(A \times B)$-matrices over $\mathbb{X}$, and a cell of the form

$$\begin{tikzcd} A_0 \arrow[r, "M_1"] \arrow[d, "F"'] & A_1 \arrow[r, "M_2"] & \cdots \arrow[r, "M_n"] & A_n \arrow[d, "G"] \\ B \arrow[rrr, "N"'] \arrow[rrr, phantom, shift left=4ex, "\alpha"] & & & C \end{tikzcd} \quad \text{in } \mathbb{X}\text{-}\mathbb{M}\mathrm{at}$$

consists of a family of cells

$$\begin{tikzcd} {|x_0|_{A_0}} \arrow[r, "M_1(x_0,x_1)"] \arrow[d, "F^1x_0"'] & {|x_1|_{A_1}} \arrow[r, "M_2(x_1,x_2)"] & \cdots \arrow[r, "M_n(x_{n-1},x_n)"] & {|x_n|_{A_n}} \arrow[d, "G^1x_n"] \\ {|F^0x_0|_B} \arrow[rrr, "N(F^0x_0, G^0x_n)"'] \arrow[rrr, phantom, shift left=4ex, "\alpha_{x_0,x_1,\dots,x_n}"] & & & {|G^0x_n|_C} \end{tikzcd} \quad \text{in } \mathbb{X},$$

one for each tuple of $x_0 \in A_0, x_1 \in A_1, \dots, x_n \in A_n$. ♦

Remark 2.54. In the above definition of $\mathbb{X}$-$\mathbb{M}\mathrm{at}$, we do not use any nullcoary cell in $\mathbb{X}$, hence $\mathbb{X}$-$\mathbb{M}\mathrm{at} = \mathbb{X}^\flat$-$\mathbb{M}\mathrm{at}$. ♦

Remark 2.55. The tight category $\mathbf{T}(\mathbb{X}\text{-}\mathbb{M}\mathrm{at})$ is isomorphic to $\mathbf{Fam}(\mathbf{T}\mathbb{X})$, known as the category of ***families*** or the coproduct cocompletion of $\mathbf{T}\mathbb{X}$. ♦

Example 2.56. Let $\mathcal{V}$ be a monoidal category. Regarding $\mathcal{V}$ as a single-object bicategory, we have a diminished AVDC $(\mathbb{V}\mathcal{V})$-$\mathbb{M}\mathrm{at}$, which is also denoted by $\mathcal{V}$-$\mathbb{M}\mathrm{at}$, whose objects are (large) sets, whose tight arrows are maps, and whose loose arrows $X \longrightarrow Y$ are families $(M(x,y))_{x\in X, y\in Y}$ of objects in $\mathcal{V}$. When $\mathcal{V}$ is the two element chain, we have $\mathcal{V}$-$\mathbb{M}\mathrm{at} \cong \mathbb{R}\mathrm{el}^\flat$. ♦

Proposition 2.57. If an AVDC $\mathbb{X}$ has all unicoary restrictions, so does $\mathbb{X}$-$\mathbb{M}\mathrm{at}$.

Proof. Suppose that we are given the following data:

$$\begin{tikzcd} A' \arrow[d, "F"'] & B' \arrow[d, "G"] \\ A \arrow[r, "N"'] & B \end{tikzcd} \quad \text{in } \mathbb{X}\text{-}\mathbb{M}\mathrm{at}.$$

For $x \in A'$ and $y \in B'$, let $N(F,G)(x,y)$ denote the following loose arrow:

$$\begin{tikzcd} {|x|} \arrow[r, "N(F,G)(x,y)"] \arrow[d, "F^1 x"'] \arrow[dr, phantom, "\mathsf{cart}"] & {|y|} \arrow[d, "G^1 y"] \\ {|F^0 x|} \arrow[r, "N(F^0 x, G^0 y)"'] & {|G^0 y|} \end{tikzcd} \quad \text{in } \mathbb{X}.$$

Then, the matrix $N(F,G)$ over $\mathbb{X}$ gives the desired restriction. □

Definition 2.58 (Enrichment over a virtual double category)**.** Let $\mathbb{X}$ be an AVDC. The ***AVDC of $\mathbb{X}$-enriched profunctors***, denoted by $\mathbb{X}$-$\mathbb{P}$rof, is defined to be $\mathbb{M}$od($\mathbb{X}$-$\mathbb{M}$at). Objects in $\mathbb{X}$-$\mathbb{P}$rof are called ***$\mathbb{X}$-enriched (large) categories***, tight arrows are called ***$\mathbb{X}$-functors***, and loose arrows are called ***$\mathbb{X}$-profunctors***. Note that $\mathbb{X}$-$\mathbb{P}$rof has restrictions whenever $\mathbb{X}$ has all unicoary restrictions, which follows from Proposition 2.57. ♦

Remark 2.59. Our $\mathbb{X}$-enriched categories, $\mathbb{X}$-functors, and $\mathbb{X}$-profunctors coincide with Leinster's [Lei99; Lei02]. For a bicategory $\mathcal{W}$, the AVDC $(\mathbb{V}\mathcal{W})$-$\mathbb{P}$rof recovers the classical notion of enrichment over a bicategory, which includes ordinary enrichment over a monoidal category as a special case. Indeed, the tight 2-category $\mathcal{T}((\mathbb{V}\mathcal{W})\text{-}\mathbb{P}\mathrm{rof})$ is isomorphic to the 2-category of $\mathcal{W}$-enriched categories and $\mathcal{W}$-functors defined by Walters [Wal82]. Moreover, the loose bicategory $\mathcal{L}((\mathbb{V}\mathcal{W})\text{-}\mathbb{P}\mathrm{rof})$ of VD-composable objects coincides with the bicategory of sufficiently small $\mathcal{W}$-enriched categories and $\mathcal{W}$-profunctors, sometimes called ***$\mathcal{W}$-modules***. ♦

We now unpack the definition.

Remark 2.60. Let $\mathbb{X}$ be an AVDC. An $\mathbb{X}$-enriched (large) category $\mathbf{A}$ consists of:

- (***Colored objects***) An $\mathbb{X}$-colored large set $\mathrm{Ob}\mathbf{A}$. For $x \in \mathrm{Ob}\mathbf{A}$, its color is denoted by $|x|_{\mathbf{A}}$ or simply $|x|$. When $|x| = c$, we call x an ***object colored with*** c.
- (***Hom-loose arrows***) For $x, y \in \mathrm{Ob}\mathbf{A}$, a loose arrow $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathbb{X}$.
- (***Compositions***) For $x, y, z \in \mathrm{Ob}\mathbf{A}$, a cell $\mu_{x,y,z}$ of the following form:

$$\begin{tikzcd} {|x|} \arrow[r, "\mathbf{A}(x,y)"] \arrow[d, equal] & {|y|} \arrow[r, "\mathbf{A}(y,z)"] \arrow[d, phantom, "\mu_{x,y,z}"] & {|z|} \arrow[d, equal] \\ {|x|} \arrow[rr, "\mathbf{A}(x,z)"'] & & {|z|} \end{tikzcd} \quad \text{in } \mathbb{X}.$$

- (***Identities***) For each $x \in \mathrm{Ob}\mathbf{A}$, a cell η_x of the following form:

$$\begin{tikzcd} & {|x|} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_x"] & \\ {|x|} \arrow[rr, "\mathbf{A}(x,x)"'] & {} & {|x|} \end{tikzcd} \quad \text{in } \mathbb{X}.$$

The above data are required to satisfy suitable axioms. ♦

Proposition 2.61. Let $\mathbb{X}$ be an AVDC. Then, an $\mathbb{X}$-enriched (large) category is the same as the following data:

- A (large) set S;
- An AVD-functor $\mathbb{I}^\flat \mathrm{S} \to \mathbb{X}$.

Proof. Let $\mathbf{A}$ be an $\mathbb{X}$-enriched large category. Then, the following assignments yield an AVD-functor $\mathbb{I}^\flat \mathrm{Ob}\mathbf{A} \to \mathbb{X}$:

$$x \mapsto |x|_{\mathbf{A}}, \qquad x \xrightarrow{!_{xy}} y \quad \mapsto \quad |x| \xrightarrow{\mathbf{A}(x,y)} |y|,$$

$$
\begin{tikzcd}[column sep=small]
& x \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "!"] & \\
x \arrow[rr, "!_{xx}"'] & {} & x
\end{tikzcd}
\mapsto
\begin{tikzcd}[column sep=small]
& {|x|} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_x"] & \\
{|x|} \arrow[rr, "\mathbf{A}(x,x)"'] & {} & {|x|}
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "!_{xy}"] \arrow[d, equal] & y \arrow[r, "!_{yz}"] \arrow[d, phantom, "!"] & z \arrow[d, equal] \\
x \arrow[rr, "!_{xz}"'] & {} & z
\end{tikzcd}
\mapsto
\begin{tikzcd}
{|x|} \arrow[r, "\mathbf{A}(x,y)"] \arrow[d, equal] & {|y|} \arrow[r, "\mathbf{A}(y,z)"] \arrow[d, phantom, "\mu_{x,y,z}"] & {|z|} \arrow[d, equal] \\
{|x|} \arrow[rr, "\mathbf{A}(x,z)"'] & {} & {|z|}
\end{tikzcd}
$$

Furthermore, we can reconstruct $\mathbb{A}$ from the AVD-functor $\mathbb{I}^\flat\mathrm{Ob}\mathbf{A} \to \mathbb{X}$. □

Notation 2.62. Let $\mathbb{X}$ be an AVDC. For $c \in \mathbb{X}$, let Yc denote the $\mathbb{X}$-colored set $Yc := \{*\}$ containing a unique element $*$ colored with c. It easily follows that all of Yc form the full sub-AVDC of $\mathbb{X}$-$\mathbb{M}\mathrm{at}$ isomorphic to $\mathbb{X}^\flat$. We write $Y\colon \mathbb{X}^\flat \to \mathbb{X}$-$\mathbb{M}\mathrm{at}$ for the corresponding AVD-functor. ♦

Notation 2.63. Let $\mathbb{X}$ be an AVDC with loose units. We write $Z\colon \mathbb{X} \to \mathbb{X}$-$\mathbb{P}\mathrm{rof}$ for an AVD-functor corresponding to $Y\colon \mathbb{X}^\flat \to \mathbb{X}$-$\mathbb{M}\mathrm{at}$ by Theorem 2.38. We write $\mathbf{Z}_c$ for the $\mathbb{X}$-enriched category assigned to each $c \in \mathbb{X}$ by Z. ♦

Lemma 2.64. Let $\mathbb{X}$ be an AVDC with loose units, and let $c \in \mathbb{X}$. Then, the unit cell associated with the monoid $\mathbf{Z}_c$ is VD-cocartesian in $\mathbb{X}$-$\mathbb{M}\mathrm{at}$.

Proof. Let

$$
\begin{tikzcd}[column sep=small]
& c \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\gamma"] & \\
c \arrow[rr, "\mathsf{U}_c"'] & {} & c
\end{tikzcd}
\quad \text{in } \mathbb{X} \tag{11}
$$

be the loosewise invertible (cocartesian) cell associated with the loose unit U_c of c. In the diminished AVDC $\mathbb{X}^\flat$, the cell γ is no longer cocartesian but VD-cocartesian. Moreover, we see at once that the VD-cocartesian cell γ is preserved by the AVD-functor $Y\colon \mathbb{X}^\flat \to \mathbb{X}$-$\mathbb{M}\mathrm{at}$. Thus, the monoid structure of $\mathbf{Z}_c$ is induced by the VD-cocartesian cell $Y\gamma$. □

Definition 2.65. Let $\mathbf{A}$ be an $\mathbb{X}$-enriched category. A ***semiobject*** in $\mathbf{A}$ colored with $c \in \mathbb{X}$ is a pair $x = (x^0, x^1)$ of an object $x^0 \in \mathrm{Ob}\mathbf{A}$ and a tight arrow $c \xrightarrow{x^1} |x^0|$ in $\mathbb{X}$. ♦

We call $\mathbf{Z}_c$ the ***semiobject classifier*** because it classifies the semiobjects colored with c in the following sense:

Theorem 2.66. Let $\mathbb{X}$ be an AVDC with loose units, and let $c \in \mathbb{X}$. Then, there is a bijective correspondence between the $\mathbb{X}$-functors $\mathbf{Z}_c \to \mathbf{A}$ and the semiobjects in $\mathbf{A}$ colored with c.

Proof. By Lemma 2.64, a monoid homomorphism $\mathbf{Z}_c \to \mathbf{A}$ is simply a tight arrow $Yc \to \mathrm{Ob}\mathbf{A}$ in $\mathbb{X}$-$\mathbb{M}\mathrm{at}$. Indeed, a monoid homomorphism $\mathbf{Z}_c \xrightarrow{(f^0,f^1)} \mathbf{A}$ must be compatible with units as follows:

$$
\begin{tikzcd}[column sep=small]
& Y_c \arrow[d, "f^0"', bend right] \arrow[d, "f^0", bend left] \arrow[d, phantom, "="] & \\
& A^0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "A^e"] & \\
A^0 \arrow[rr, "A^1"'] & {} & A^0
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& Yc \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\text{VD.cocart}"] & \\
Yc \arrow[rr, "Y\mathsf{U}_c"] \arrow[d, "f^0"'] & {} \arrow[d, phantom, "f^1"] & Yc \arrow[d, "f^0"] \\
A^0 \arrow[rr, "A^1"'] & {} & A^0
\end{tikzcd}
\quad \text{in } \mathbb{X}\text{-}\mathbb{M}\mathrm{at}.
$$

Here, $\mathbf{A}$ is regarded as a monoid $(\mathrm{Ob}\mathbf{A} = A^0, A^1, A^e, A^m)$ in $\mathbb{X}$-$\mathbb{M}\mathrm{at}$. By the universal property of the VD-cocartesian cell, f^1 can be reconstructed uniquely from f^0. Since the compatibility of f^1 with multiplications is automatically satisfied, the monoid homomorphism (f^0, f^1) is the same thing as a tight arrow f^0. Since f^0 is simply a choice of a semiobject in $\mathbf{A}$ colored with c, this finishes the proof. □

Theorem 2.67. For an AVDC $\mathbb{X}$ with loose units, the AVD-functor $Z\colon \mathbb{X} \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ makes $\mathbb{X}$ into a full sub-AVDC of $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$.

Proof. Let c, d be objects in $\mathbb{X}$. By Theorem 2.66, the $\mathbb{X}$-functors $\mathbf{Z}_c \to \mathbf{Z}_d$ are the same thing as the tight arrows $c \to d$ in $\mathbb{X}$. The same is true for loose arrows. Indeed, an $\mathbb{X}$-profunctor $\mathbf{Z}_c \xrightarrow{(P^1, P^l, P^r)} \mathbf{Z}_d$ must be compatible with the unit of $\mathbf{Z}_c$ for example:

$$\begin{tikzcd}
 & Y_c \arrow[rr, "P^1"] \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\mathrm{VD.cocart}"] & & Y_d \arrow[dr, equal] & \\
Y_c \arrow[rr, "Y\mathrm{U}_c"'] \arrow[d, equal] & {} & Y_c \arrow[rr, "P^1"'] \arrow[d, phantom, "P^l"] & & Y_d \arrow[d, equal] \\
Y_c \arrow[rrrr, "P^1"'] & & {} & & Y_d
\end{tikzcd}
=
\begin{tikzcd}
Y_c \arrow[r, "P^1"] \arrow[d, equal] \arrow[dr, phantom, "="] & Y_d \arrow[d, equal] \\
Y_c \arrow[r, "P^1"'] & Y_d
\end{tikzcd}
\quad \text{in } \mathbb{X}\text{-}\mathbb{M}\mathrm{at}.$$

By the universal property of the VD-cocartesian cell, P^l can be reconstructed uniquely from P^1, and so does P^r. Since the compatibility with the multiplications of $\mathbf{Z}_c$ and $\mathbf{Z}_d$ is automatically satisfied, the $\mathbb{X}$-profunctor (P^1, P^l, P^r) is the same thing as a loose arrow P^1. Since $Y\colon X^\flat \to \mathbb{X}\text{-}\mathbb{M}\mathrm{at}$ is a full inclusion, the loose arrow P^1 is simply a loose arrow $c \twoheadrightarrow d$ in $\mathbb{X}$.

Similarly, we can establish between $\mathbb{X}$ and $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$, a bijective correspondence of unicoary cells. Furthermore, since both $\mathbb{X}$ and $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ have loose units, the same is true also for nullcoary cells. This finishes the proof. $\square$

3. Colimits in augmented virtual double categories

3.1. **Cocones, modules, and modulations.** To give a notion of "colimits" in an AVDC, we consider "cocones" for each of the three directions: left, right, and downward. The "cocones" for the downward direction are called ***tight cocones***, and the "cocones" for the left and right directions are called left and right ***modules***, respectively. In addition, we also consider several types of morphisms between them, called ***modulations***. The terms "module" and "modulations" come from the essentially same concept in [Par11].

Definition 3.1 (Tight cocones)**.** Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. A ***tight cocone*** l (from F) consists of:

- an object $L \in \mathbb{L}$ (the ***vertex*** of l);
- for each $A \in \mathbb{K}$, a tight arrow $\begin{tikzcd} FA \arrow[d, "l_A"'] \\ L \end{tikzcd}$ in $\mathbb{L}$;
- for each $A \xrightarrow{u} B$ in $\mathbb{K}$, a cell $\begin{tikzcd} FA \arrow[rr, "Fu"] \arrow[dr, "l_A"'] & \arrow[d, phantom, "l_u"] & FB \arrow[dl, "l_B"] \\ & L & \end{tikzcd}$ in $\mathbb{L}$

satisfying the following conditions:

- For any tight arrow $A \xrightarrow{f} B$ in $\mathbb{K}$, $(Ff) ⨾ l_B = l_A$;
- For any cell

$$\begin{tikzcd}
A_0 \arrow[r, "u_1"] \arrow[d, "f"'] & A_1 \arrow[r, "u_2"] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "u_n"] & A_n \arrow[d, "g"] \\
X \arrow[rrr, dotted, "v"'] & & & Y
\end{tikzcd}
\quad \text{in } \mathbb{K},$$

$$
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}"] \arrow[d, "Ff"'] \arrow[drr, phantom, "F\alpha"] & & FA_n \arrow[d, "Fg"] \\
FX \arrow[rr, dotted, "Fv"] \arrow[dr, "l_X"'] \arrow[rr, phantom, "l_v", shift right=3ex] & & FY \arrow[dl, "l_Y"] \\
& L &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}"] \arrow[dr, "l_{A_0}"'] \arrow[rr, phantom, "l_{\vec{u}}", shift right=3ex] & & FA_n \arrow[dl, "l_{A_n}"] \\
& L &
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

Here $l_{\vec{u}}$ denotes the composite of the following cells:

$$
\begin{tikzcd}
FA_0 \arrow[r, "Fu_1"] \arrow[drr, "l_{A_0}"'] & FA_1 \arrow[r, "Fu_2"] \arrow[dr, "l_{A_1}"] & \cdots \arrow[r, "Fu_{n-1}"] & FA_{n-1} \arrow[r, "Fu_n"] \arrow[dl, "l_{A_{n-1}}"'] & FA_n \arrow[dll, "l_{A_n}"] \\
& l_{u_1} & L \arrow[u, phantom, "\cdots"] & l_{u_n} &
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

When $\vec{u}$ is length 0, the cell $l_{\vec{u}}$ is defined to be the identity. ♦

Definition 3.2. A tight cocone l is called ***strong*** if l_u is cartesian for any loose arrow u. ♦

Definition 3.3 (Left/right modules)**.** Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. A ***left*** F***-module*** m consists of:

- an object $M \in \mathbb{L}$ (the ***vertex*** of m);
- for each $A \in \mathbb{K}$, a loose arrow $FA \xrightarrow{m_A} M$ in $\mathbb{L}$;
- for each $A \xrightarrow{f} B$ in $\mathbb{K}$, a cartesian cell

$$
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "m_f\colon \mathsf{cart}"] & M \arrow[d, equal] \\
FB \arrow[r, "m_B"'] & M
\end{tikzcd}
\quad \text{in } \mathbb{L};
$$

- for each $A \xrightarrow{u} B$ in $\mathbb{K}$, a cell

$$
\begin{tikzcd}
FA \arrow[r, "Fu"] \arrow[d, equal] & FB \arrow[r, "m_B"] \arrow[d, phantom, "m_u"] & M \arrow[d, equal] \\
FA \arrow[rr, "m_A"'] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}
$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "m_f"] & M \arrow[d, equal] \\
FB \arrow[r, "m_B"] \arrow[d, "Fg"'] \arrow[dr, phantom, "m_g"] & M \arrow[d, equal] \\
FC \arrow[r, "m_C"'] & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, "Ff"'] & M \arrow[dd, equal] \\
FB \arrow[r, phantom, "m_{f\fatsemi g}"] \arrow[d, "Fg"'] & \\
FC \arrow[r, "m_C"'] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

- For any $A \in \mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, equal, "F\mathsf{id}_A"'] \arrow[dr, phantom, "m_{\mathsf{id}_A}"] & M \arrow[d, equal] \\
FA \arrow[r, "m_A"'] & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FA \arrow[r, "m_A"'] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

- For any cell

$$\begin{array}{ccc} A_0 \overset{u_1}{\nrightarrow} A_1 \overset{u_2}{\nrightarrow} \cdots \overset{u_n}{\nrightarrow} A_n & & \\ f\downarrow \quad \alpha \quad \downarrow g & \text{in } \mathbb{K}, & \\ X \xrightarrow[v]{\quad\quad} Y & & \end{array}$$

$$\begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FA_n \arrow[r, "m_{A_n}"] \arrow[d, "Fg"] \arrow[dr, phantom, "m_g"] & M \arrow[d, equal] \\ FX \arrow[r, dotted, "Fv"'] \arrow[d, equal] \arrow[drr, phantom, "m_v"] & FY \arrow[r, "m_Y"'] & M \arrow[d, equal] \\ FX \arrow[rr, "m_X"'] & & M \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, equal] \arrow[drr, phantom, "m_{\vec{u}}"] & FA_n \arrow[r, "m_{A_n}"] & M \arrow[d, equal] \\ FA_0 \arrow[rr, "m_{A_0}"'] \arrow[d, "Ff"'] \arrow[drr, phantom, "m_f"] & & M \arrow[d, equal] \\ FX \arrow[rr, "m_X"'] & & M \end{tikzcd} \quad \text{in } \mathbb{L}.$$

Here, $m_{\vec{u}}$ denotes the composition of the following cells:

$$\begin{tikzcd} FA_0 \arrow[r, "Fu_1"] \arrow[d, equal] \arrow[dr, phantom, "="] & FA_1 \arrow[r, "Fu_2"] \arrow[d, equal] \arrow[dr, phantom, "= \cdots ="] & \cdots \arrow[r, "Fu_{n-1}"] & FA_{n-1} \arrow[r, "Fu_n"] \arrow[d, equal] \arrow[drr, phantom, "m_{u_n}"] & FA_n \arrow[r, "m_{A_n}"] & M \arrow[d, equal] \\ FA_0 \arrow[r, "Fu_1"'] \arrow[d, equal] & FA_1 \arrow[r, "Fu_2"'] \arrow[d, equal] & \cdots \arrow[r, "Fu_{n-1}"'] & FA_{n-1} \arrow[rr, "m_{A_{n-1}}"'] & & M \arrow[d, equal] \\ \vdots \arrow[d, equal] & \vdots \arrow[d, equal] & & & & \vdots \arrow[d, equal] \\ FA_0 \arrow[r, "Fu_1"] \arrow[d, equal] \arrow[drrrrr, phantom, "m_{u_1}"] & FA_1 \arrow[rrrr, "m_{A_1}"] & & & & M \arrow[d, equal] \\ FA_0 \arrow[rrrrr, "m_{A_0}"'] & & & & & M \end{tikzcd} \quad \text{in } \mathbb{L}.$$

Moreover, ***right*** F***-modules*** are also defined as the loosewise dual of the left F-modules. ♦

Notation 3.4. A tight cocone from F with a vertex L is denoted by a double arrow $F \Rightarrow L$. A left (resp. right) F-module with a vertex M is denoted by a slashed double arrow $F \Longrightarrow M$ (resp. $M \Longrightarrow F$). ♦

Definition 3.5 (Modulations of type 0)**.** Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Let m, m' be left F-modules whose vertices are $M, M' \in \mathbb{L}$, respectively. Consider $M \overset{\vec{p}}{\dashrightarrow} M'' \xrightarrow{j} M'$ in $\mathbb{L}$. A ***modulation (of type 0)*** ρ, denoted by

$$\begin{tikzcd} F \arrow[r, Rightarrow, "m"] \arrow[d, equal] \arrow[drr, phantom, "\rho"] & M \arrow[r, dashed, "\vec{p}"] & M'' \arrow[d, "j"] \\ F \arrow[rr, Rightarrow, "m'"'] & & M' \end{tikzcd} \tag{12}$$

consists of:

- for each $A \in \mathbb{K}$, a cell

$$\begin{tikzcd} FA \arrow[r, "m_A"] \arrow[d, equal] \arrow[drr, phantom, "\rho_A"] & M \arrow[r, dashed, "\vec{p}"] & M'' \arrow[d, "j"] \\ FA \arrow[rr, "m'_A"'] & & M' \end{tikzcd} \quad \text{in } \mathbb{L}$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "m_f"] & M \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M'' \arrow[d, equal] \\
FB \arrow[r, "m_B"] \arrow[d, equal] \arrow[drr, phantom, "\rho_B"] & M \arrow[r, dashed, "\vec{p}"] & M'' \arrow[d, "j"] \\
FB \arrow[rr, "m'_B"'] & & M'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, equal] \arrow[drr, phantom, "\rho_A"] & M \arrow[r, dashed, "\vec{p}"] & M'' \arrow[d, "j"] \\
FA \arrow[rr, "m'_A"] \arrow[d, "Ff"'] \arrow[drr, phantom, "m'_f"] & & M' \arrow[d, equal] \\
FB \arrow[rr, "m'_B"'] & & M'
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

- For any $A \xrightarrow{u} B$ in $\mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[r, "Fu"] \arrow[d, equal] \arrow[drr, phantom, "m_u"] & FB \arrow[r, "m_B"] & M \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M'' \arrow[d, equal] \\
FA \arrow[rr, "m_A"'] \arrow[d, equal] \arrow[drrr, phantom, "\rho_A"] & & M \arrow[r, dashed, "\vec{p}"] & M'' \arrow[d, "j"] \\
FA \arrow[rrr, "m'_A"'] & & & M'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fu"] \arrow[d, equal] \arrow[dr, phantom, "="] & FB \arrow[r, "m_B"] \arrow[d, equal] \arrow[drr, phantom, "\rho_B"] & M \arrow[r, dashed, "\vec{p}"] & M'' \arrow[d, "j"] \\
FA \arrow[r, "Fu"'] \arrow[d, equal] \arrow[drrr, phantom, "m'_u"] & FB \arrow[rr, "m'_B"'] & & M' \arrow[d, equal] \\
FA \arrow[rrr, "m'_A"'] & & & M'
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

♦

Notation 3.6. For a functor $F\colon \mathbb{K} \to \mathbb{L}$ between AVDCs and $M \in \mathbb{L}$, let $\mathbf{Mdl}(F, M)$ denote the category of left F-modules with the vertex M and special modulations (of type 0) where the length of $\vec{p}$ is 0 and j is the identity. We write $\mathbf{Mdl}(M, F)$ for the category of right F-modules with the vertex M. ♦

Remark 3.7. A modulation (of type 0) $\rho\colon m \to m'$ in $\mathbf{Mdl}(F, M)$ is called ***invertible*** if every component ρ_A is loosewise invertible. The invertible modulations (of type 0) are the same thing as the isomorphisms in $\mathbf{Mdl}(F, M)$. ♦

Definition 3.8 (Modulations of type 1)**.** Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Let $F \overset{l}{\Longrightarrow} L \in \mathbb{L}$ be a tight cocone and let $F \overset{m}{\Longrightarrow} M \in \mathbb{L}$ be a left F-module. Consider $M \overset{\vec{p}}{\dashrightarrow} M'$, $M' \xrightarrow{j} L'$, and $L \overset{q}{\dashrightarrow} L'$ in $\mathbb{L}$. A ***modulation (of type 1)*** σ, denoted by

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "m"] \arrow[d, Rightarrow, "l"'] \arrow[drr, phantom, "\sigma"] & M \arrow[r, dashed, "\vec{p}"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"'] & & L'
\end{tikzcd}
$$

consists of:

- for each $A \in \mathbb{K}$, a cell

$$
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, "l_A"'] \arrow[drr, phantom, "\sigma_A"] & M \arrow[r, dashed, "\vec{p}"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"'] & & L'
\end{tikzcd}
\quad \text{in } \mathbb{L}
$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "m_f"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M' \arrow[d, equal] \\
FB \arrow[r, "m_B"', "\shortmid"{marking}] \arrow[d, "l_B"'] \arrow[drr, phantom, "\sigma_B"] & M \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[drr, phantom, "\sigma_A"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

- For any $A \overset{u}{\longrightarrow} B$ in $\mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "m_u"] & FB \arrow[r, "m_B", "\shortmid"{marking}] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M' \arrow[d, equal] \\
FA \arrow[rr, "m_A"', "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[drrr, phantom, "\sigma_A"] & & M \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rrr, dotted, "q"', "\shortmid"{marking}] & & & L'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[dr, phantom, "l_u"] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[dl, "l_B"] \arrow[drr, phantom, "\sigma_B"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rrr, dotted, "q"', "\shortmid"{marking}] & & & L'
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

♦

Remark 3.9. Suppose that, in the situation of Definition 3.8, we are alternatively given a right F-module $M \overset{m}{\Longrightarrow} F$, loose paths $M' \overset{\vec{p}}{\dashrightarrow} M$ and $L' \overset{q}{\dashrightarrow} L$ in $\mathbb{L}$. Then, we can also define the loosewise dual concept, which is called modulations of type 1 as well and is denoted by

$$
\begin{tikzcd}
M' \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "j"'] \arrow[drr, phantom, "\sigma"] & M \arrow[r, Rightarrow, "m", "\shortmid"{marking}] & F \arrow[d, Rightarrow, "l"] \\
L' \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L
\end{tikzcd}
$$

♦

Definition 3.10 (Modulations of type 2)**.** Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Let $F \overset{l}{\Longrightarrow} L \in \mathbb{L}$ and $F \overset{l'}{\Longrightarrow} L' \in \mathbb{L}$ be tight cocones. Consider $L \overset{q}{\dashrightarrow} L'$ in $\mathbb{L}$. A ***modulation (of type 2)*** τ, denoted by

$$
\begin{tikzcd}
& F \arrow[dl, Rightarrow, "l"'] \arrow[dr, Rightarrow, "l'"] \arrow[d, phantom, "\tau"] & \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & {} & L'
\end{tikzcd}
$$

consists of:

- for each $A \in \mathbb{K}$, a cell

$$
\begin{tikzcd}
& FA \arrow[dl, "l_A"'] \arrow[dr, "l'_A"] \arrow[d, phantom, "\tau_A"] & \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & {} & L'
\end{tikzcd}
\quad \text{in } \mathbb{L}
$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
 & FA \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "="] & \\
 & FB \arrow[dl, "l_B"'] \arrow[dr, "l'_B"] \arrow[d, phantom, "\tau_B"] & \\
L \arrow[rr, dotted, "\shortmid"{marking}, "q"'] & & L'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & FA \arrow[dl, "l_A"'] \arrow[dr, "l'_A"] \arrow[d, phantom, "\tau_A"] & \\
L \arrow[rr, dotted, "\shortmid"{marking}, "q"'] & {} & L'
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

- For any $A \xrightarrow{u} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[dr, "l'_A"{description}] \arrow[dr, phantom, shift left=3, "l'_u"] \arrow[dr, phantom, shift right=3, "\tau_A"] & FB \arrow[d, "l'_B"] \\
L \arrow[r, dotted, "\shortmid"{marking}, "q"'] & L'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "l_A"'] \arrow[dr, phantom, shift right=3, "l_u"] & FB \arrow[d, "l'_B"] \arrow[dl, "l_B"{description}] \arrow[dl, phantom, shift left=3, "\tau_B"] \\
L \arrow[r, dotted, "\shortmid"{marking}, "q"'] & L'
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

♦

Notation 3.11. Let $\mathbf{Cone}(\substack{F\\L})$ denote the category of tight cocones from F with a vertex L and special modulations (of type 2) where the length of q is 0. ♦

Definition 3.12 (Modulations of type 3)**.** Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Let $N \overset{n}{\Longrightarrow} F \overset{m}{\Longrightarrow} M$ be a right F-module and a left F-module, respectively. Consider $N' \overset{\vec{q}}{\dashrightarrow} N$, $M \overset{\vec{p}}{\dashrightarrow} M'$, $N' \xrightarrow{j} N''$, $M' \xrightarrow{i} M''$, and $N'' \overset{r}{\dashrightarrow} M''$ in $\mathbb{L}$. A ***modulation (of type 3)*** ω, denoted by

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "j"'] & N \arrow[r, Rightarrow, "n", "\shortmid"{marking}] & F \arrow[r, Rightarrow, "m", "\shortmid"{marking}] \arrow[d, phantom, "\omega"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "\shortmid"{marking}, "r"'] & & & & M''
\end{tikzcd}$$

consists of:

- for each $A \in \mathbb{K}$, a cell

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "j"'] & N \arrow[r, "n_A", "\shortmid"{marking}] & FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, phantom, "\omega_A"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "\shortmid"{marking}, "r"'] & & & & M''
\end{tikzcd}$$

satisfying the following conditions:

- For any $A \xrightarrow{f} B$ in $\mathbb{K}$,

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & N \arrow[r, "n_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "n_f"] & FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Ff"] \arrow[dr, phantom, "m_f"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
N' \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "j"'] & N \arrow[r, "n_B", "\shortmid"{marking}] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, phantom, "\omega_B"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "\shortmid"{marking}, "r"'] & & & & M''
\end{tikzcd}
\quad = \quad \omega_A \quad \text{in } \mathbb{L}.$$

- For any $A \xrightarrow{u} B$ in $\mathbb{K}$,

$$
\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & N \arrow[r, "n_A"] \arrow[d, equal] \arrow[dr, phantom, "="] & FA \arrow[r, "Fu"] \arrow[d, equal] \arrow[drr, phantom, "m_u"] & FB \arrow[r, "m_B"] & M \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, "j"'] \arrow[drrrrr, phantom, "\omega_A"] & N \arrow[r, "n_A"] & FA \arrow[rr, "m_A"] & & M \arrow[r, dashed, "\vec{p}"] & M' \arrow[d, "i"] \\
N'' \arrow[rrrrr, dotted, "r"'] & & & & & M''
\end{tikzcd}
$$

$$
= \begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & N \arrow[r, "n_A"] \arrow[d, equal] \arrow[drr, phantom, "n_u"] & FA \arrow[r, "Fu"] & FB \arrow[r, "m_B"] \arrow[d, equal] \arrow[dr, phantom, "="] & M \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, "j"'] \arrow[drrrrr, phantom, "\omega_B"] & N \arrow[rr, "n_B"] & & FB \arrow[r, "m_B"] & M \arrow[r, dashed, "\vec{p}"] & M' \arrow[d, "i"] \\
N'' \arrow[rrrrr, dotted, "r"'] & & & & & M''
\end{tikzcd} \quad \text{in } \mathbb{L}.
$$

♦

Construction 3.13. Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs and let $L \in \mathbb{L}$. Let $F \xRightarrow{\xi} \Xi \in \mathbb{L}$ be a tight cocone. For a tight arrow $\Xi \xrightarrow{k} L$ in $\mathbb{L}$, we have a tight cone $F \xRightarrow{\xi ⨾ k} L$ as follows:

- For any $A \in \mathbb{K}$,

$$
\begin{tikzcd}
 & FA \arrow[dl, "\xi_A"'] \arrow[dd, "(\xi ⨾ k)_A"] \\
\Xi \arrow[r, phantom, "=:"] \arrow[dr, "k"'] & {} \\
 & L
\end{tikzcd} \quad \text{in } \mathbb{L}.
$$

- For any $A \xrightarrow{u} B$ in $\mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[rr, "Fu"] \arrow[dr, "\xi_A"'] \arrow[rr, phantom, shift right=4ex, "\xi_u"] & & FB \arrow[dl, "\xi_B"] \\
 & \Xi \arrow[d, bend right, "k"'] \arrow[d, bend left, "k"] \arrow[d, phantom, "="] & \\
 & L &
\end{tikzcd}
=:
\begin{tikzcd}
FA \arrow[rr, "Fu"] \arrow[dr, "(\xi ⨾ k)_A"'] \arrow[rr, phantom, shift right=4ex, "(\xi ⨾ k)_u"] & & FB \arrow[dl, "(\xi ⨾ k)_B"] \\
 & L &
\end{tikzcd} \quad \text{in } \mathbb{L}.
$$

Furthermore, the assignment $k \mapsto \xi ⨾ k$ extends to a functor $\mathbf{Hom}_{\mathbb{L}}(\substack{\Xi \\ L}) \xrightarrow{\xi ⨾ -} \mathbf{Cone}(\substack{F \\ L})$. ♦

Definition 3.14. A tight arrow $A \xrightarrow{f} B$ in an AVDC is called ***left-pulling*** if every loose arrow $B \xrightarrow{p} \cdot$ has its restriction $p(f, \mathsf{id})$ along f:

$$
\begin{tikzcd}
A \arrow[r, "p(f{,}\mathsf{id})"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d, equal] \\
B \arrow[r, "p"'] & \cdot
\end{tikzcd}
$$

Moreover, ***right-pulling*** tight arrows are also defined in the loosewise dual way. Left-pulling and right-pulling tight arrows are simply called ***pulling***. ♦

Construction 3.15. Let $F\colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs and let $L \in \mathbb{L}$. Let ξ be a tight cocone from F to $\Xi \in \mathbb{L}$. Assume that ξ_A is left-pulling for any $A \in \mathbb{K}$. Then, depending on a choice of cartesian cells

$$\begin{tikzcd} FA \arrow[r, "p(\xi_A, \mathsf{id})"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\tilde{p}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\ \Xi \arrow[r, "p"'] & L \end{tikzcd} \quad \text{in } \mathbb{L}$$

for each loose arrow p, the following assignments yield a functor $\mathbf{Hom}_{\mathbb{L}}(\Xi, L) \xrightarrow{\xi_* -} \mathbf{Mdl}(F, L)$ between categories.

- For each $\Xi \xrightarrow{p} L$ in $\mathbb{L}$, a left F-module $\xi_* p$ with the vertex L is defined as follows:
  - For each $A \in \mathbb{K}$, $(\xi_* p)_A := p(\xi_A, \mathsf{id})$.
  - For each $A \xrightarrow{f} B$ in $\mathbb{K}$, $(\xi_* p)_f$ is a unique cell such that

$$\begin{tikzcd} FA \arrow[r, "(\xi_* p)_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "(\xi_* p)_f"] & L \arrow[d, equal] \\ FB \arrow[r, "(\xi_* p)_B"] \arrow[d, "\xi_B"'] \arrow[dr, phantom, "\tilde{p}_B\colon \mathsf{cart}"] & L \arrow[d, equal] \\ \Xi \arrow[r, "p"'] & L \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "(\xi_* p)_A"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\tilde{p}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\ \Xi \arrow[r, "p"'] & L \end{tikzcd} \quad \text{in } \mathbb{L}.$$

  - For each $A \xrightarrow{u} B$ in $\mathbb{K}$, $(\xi_* p)_u$ is a unique cell such that

$$\begin{tikzcd} FA \arrow[r, "Fu"] \arrow[d, equal] & FB \arrow[r, "(\xi_* p)_B"] \arrow[d, phantom, "(\xi_* p)_u"] & L \arrow[d, equal] \\ FA \arrow[rr, "(\xi_* p)_A"] \arrow[d, "\xi_A"'] & \arrow[d, phantom, "\tilde{p}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\ \Xi \arrow[rr, "p"'] & & L \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "Fu"] \arrow[d, "\xi_A"'] \arrow[d, phantom, shift left=4, "\xi_u"] & FB \arrow[r, "(\xi_* p)_B"] \arrow[dl, "\xi_B"] \arrow[dr, phantom, "\tilde{p}_B\colon \mathsf{cart}"] & L \arrow[d, equal] \\ \Xi \arrow[rr, "p"'] & & L \end{tikzcd} \quad \text{in } \mathbb{L}.$$

- For each cell

$$\begin{tikzcd} \Xi \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "\delta"] & L \arrow[d, equal] \\ \Xi \arrow[r, "q"'] & L \end{tikzcd} \quad \text{in } \mathbb{L},$$

  a modulation $\xi_*\delta\colon \xi_* p \to \xi_* q$ is defined as follows:
  - For each $A \in \mathbb{K}$, $(\xi_*\delta)_A$ is a unique cell such that

$$\begin{tikzcd} FA \arrow[r, "(\xi_* p)_A"] \arrow[d, equal] \arrow[dr, phantom, "(\xi_*\delta)_A"] & L \arrow[d, equal] \\ FA \arrow[r, "(\xi_* q)_A"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\tilde{q}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\ \Xi \arrow[r, "q"'] & L \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "(\xi_* p)_A"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\tilde{p}_A\colon \mathsf{cart}"] & L \arrow[d, equal] \\ \Xi \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "\delta"] & L \arrow[d, equal] \\ \Xi \arrow[r, "q"'] & L \end{tikzcd} \quad \text{in } \mathbb{L}.$$

♦

Notation 3.16. In Construction 3.15, the cartesian cells $(\tilde{p}_A)_{A\in\mathbb{K}}$ yields a modulation of type 1 below. We write $\xi_\dagger p$ for such modulation.

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_* p", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] \arrow[dr, phantom, "\xi_\dagger p"] & L \arrow[d, equal] \\
\Xi \arrow[r, "p"', "\shortmid"{marking}] & L
\end{tikzcd}$$

♦

Remark 3.17. By an argument similar to Construction 3.15, we can show that every tight cocone $F \overset{l}{\Longrightarrow} L$ induces a left F-module $F \overset{l_*}{\Longrightarrow} L$ whenever the companions l_{A*} $(A \in \mathbb{K})$ exist. ♦

Notation 3.18. In Construction 3.15, if we alternatively assume that the restriction $q(\mathsf{id}_L, \xi_A)$ exists for any loose arrow $L \overset{q}{\longrightarrow} \Xi$ in $\mathbb{L}$ and for any $A \in \mathbb{K}$, then we can construct in the same way a functor $\mathbf{Hom}_{\mathbb{L}}(L, \Xi) \xrightarrow{-\xi^*} \mathbf{Mdl}(L, F)$, which sends q to a right F-module $q\xi^*$. As well as Notation 3.16, we can get a modulation of type 1, denoted by $q\xi^\dagger$, of the following form:

$$\begin{tikzcd}
L \arrow[r, Rightarrow, "q\xi^*", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "q\xi^\dagger"] & F \arrow[d, Rightarrow, "\xi"] \\
L \arrow[r, "q"', "\shortmid"{marking}] & \Xi
\end{tikzcd}$$

♦

Remark 3.19. We have defined the modulations of the following types:

$$\begin{tikzcd}
\cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] & \cdot \arrow[r, Rightarrow] \arrow[d, phantom, "\text{type } 0"] & F \arrow[d, equal] & F \arrow[r, Rightarrow] \arrow[d, equal] & \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, phantom, "\text{type } 0"] & \cdot \arrow[d] \\
\cdot \arrow[rr, Rightarrow, "\shortmid"{marking}] & & F & F \arrow[rr, Rightarrow, "\shortmid"{marking}] & & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] & \cdot \arrow[r, Rightarrow] \arrow[d, phantom, "\text{type } 1"] & F \arrow[d, Rightarrow] & F \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, phantom, "\text{type } 1"] & \cdot \arrow[d] \\
\cdot \arrow[rr, dotted, "\shortmid"{marking}] & & \cdot & \cdot \arrow[rr, dotted, "\shortmid"{marking}] & & \cdot
\end{tikzcd}$$

$$\begin{tikzcd}
& F \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] \arrow[d, phantom, "\text{type } 2"] & \\
\cdot \arrow[rr, dotted, "\shortmid"{marking}] & {} & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] & \cdot \arrow[r, Rightarrow] & F \arrow[r, Rightarrow] \arrow[d, phantom, "\text{type } 3"] & \cdot \arrow[r, dashed, "\shortmid"{marking}] & \cdot \arrow[d] \\
\cdot \arrow[rrrr, dotted, "\shortmid"{marking}] & & {} & & \cdot
\end{tikzcd}$$

We may consider another type of "modulation." For example:

$$\begin{tikzcd}
& F \arrow[dl, equal] \arrow[dr, Rightarrow] & \\
F \arrow[rr, Rightarrow, "\shortmid"{marking}] & & \cdot
\end{tikzcd}$$

In the paper, we will only treat "modulations" whose bottom boundary is a loose path with length ≤ 1 or a module inherited from the functors $\xi_*-, -\xi^*$. Furthermore, such "modulations," which include the type 0, are reduced to one of the types 1, 2, or 3 by the universal property of restrictions. ♦

3.2. **Final functors.**

Definition 3.20. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ be an AVD-functor between AVDCs. For a path $A \overset{\vec{u}}{\dashrightarrow} B$ in $\mathbb{K}$, we define a category $\mathbf{S}(\frac{\vec{u}}{\Phi})$ as follows:

- An object in $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is a tuple $(X^0, X^1, X, \varphi^0, \varphi^1, \varphi)$ of the following form:

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\varphi^0"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "\varphi^1"] \\ \Phi X^0 \arrow[r, dotted, "\Phi X"', "\shortmid"{marking}] & \Phi X^1 \end{tikzcd} \quad \text{in } \mathbb{K}. \tag{13}$$

We also write (X, φ) for such a object $(X^0, X^1, X, \varphi^0, \varphi^1, \varphi)$.

- A morphism $(X, \varphi) \xrightarrow{\theta} (Y, \psi)$ in $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is a tuple $(\theta^0, \theta^1, \theta)$ such that

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\varphi^0"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "\varphi^1"] \\ \Phi X^0 \arrow[r, dotted, "\Phi X"', "\shortmid"{marking}] \arrow[d, "\Phi\theta^0"'] \arrow[dr, phantom, "\Phi\theta"] & \Phi X^1 \arrow[d, "\Phi\theta^1"] \\ \Phi Y^0 \arrow[r, dotted, "\Phi Y"', "\shortmid"{marking}] & \Phi Y^1 \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "\psi^0"'] \arrow[dr, phantom, "\psi"] & B \arrow[d, "\psi^1"] \\ \Phi Y^0 \arrow[r, dotted, "\Phi Y"', "\shortmid"{marking}] & \Phi Y^1 \end{tikzcd} \quad \text{in } \mathbb{K}.$$

When $A = B$ and $\vec{u}$ is of length 0, the category $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is also denoted by $\mathbf{S}(\genfrac{}{}{0pt}{}{A}{\Phi})$. ♦

Remark 3.21. In the situation of Definition 3.20, the assignments $(X, \varphi) \mapsto (X^i, \varphi^i)$ $(i = 0, 1)$ yield two functors to the comma categories: $(-)^0 \colon \mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi}) \to A/(\mathbf{T}\Phi)$ and $(-)^1 \colon \mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi}) \to B/(\mathbf{T}\Phi)$. If $A = B$ and $\vec{u}$ is of length 0, both functors $(-)^0$ and $(-)^1$ has a common section:

$$\begin{tikzcd} & A/(\mathbf{T}\Phi) \arrow[dl, equal] \arrow[d] \arrow[dr, equal] & \\ A/(\mathbf{T}\Phi) & \mathbf{S}(\genfrac{}{}{0pt}{}{A}{\Phi}) \arrow[l, "(-)^0"] \arrow[r, "(-)^1"'] & A/(\mathbf{T}\Phi) \end{tikzcd}$$

Indeed, the assignment

$$\begin{tikzcd} A \arrow[d, "p"'] \\ \Phi X \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[d, bend right, "p"'] \arrow[d, bend left, "p"] \arrow[d, phantom, "="] \\ \Phi X \end{tikzcd}$$

gives such a common section $A/(\mathbf{T}\Phi) \to \mathbf{S}(\genfrac{}{}{0pt}{}{A}{\Phi})$. ♦

As in [Par90], we use the following terminology:

Definition 3.22. For a category $\mathbf{C}$, we write $\pi_1\mathbf{C}$ for the strict localization of $\mathbf{C}$ by all morphisms. The groupoid $\pi_1\mathbf{C}$ is called the ***fundamental groupoid*** of $\mathbf{C}$. A category $\mathbf{C}$ is called ***simply connected*** if the fundamental groupoid $\pi_1\mathbf{C}$ has at most one morphism between any two objects. ♦

Definition 3.23. An AVD-functor $\Phi \colon \mathbb{J} \to \mathbb{K}$ between AVDCs is called ***final*** if:

- For every object $A \in \mathbb{K}$, the comma category $A/(\mathbf{T}\Phi)$ is simply connected.
- For every loose path $\vec{u}$ in $\mathbb{K}$, the category $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is connected.
- For every loose path $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$ in $\mathbb{K}$, there exist data of the following form:

$$\begin{tikzcd} A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[d, "p_0"'] \arrow[dr, phantom, "\varphi_1"] & A_1 \arrow[r, "u_2", "\shortmid"{marking}] \arrow[d, "p_1"] \arrow[dr, phantom, "\varphi_2"] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] \arrow[dr, phantom, "\varphi_n"] & A_n \arrow[d, "p_n"] \\ \Phi X_0 \arrow[r, dotted, "\Phi v_1", "\shortmid"{marking}] \arrow[d, "\Phi f"'] & \Phi X_1 \arrow[r, dotted, "\Phi v_2", "\shortmid"{marking}] \arrow[dr, phantom, "\Phi\theta"] & \cdots \arrow[r, dotted, "\Phi v_n", "\shortmid"{marking}] & \Phi X_n \arrow[d, "\Phi g"] \\ \Phi Y \arrow[rrr, dotted, "\Phi w"', "\shortmid"{marking}] & & & \Phi Z \end{tikzcd} \quad \text{in } \mathbb{K}. \tag{14}$$

♦

Example 3.24. For a large set S, the inclusion AVD-functor $\mathbb{I}^\flat\mathrm{S} \to \mathbb{I}\mathrm{S}$ is always final. On the other hand, the inclusion $\mathbb{I}^\flat\mathbf{C} \to \mathbb{I}\mathbf{C}$ for a category $\mathbf{C}$ is not necessarily final due to the lack of simple connectedness of the coslice categories $c/\mathbf{C}$. ♦

Lemma 3.25. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ be a final AVD-functor between AVDCs. Then, for every $A \in \mathbb{K}$, the comma category $A/(\mathbf{T}\Phi)$ is connected (and simply connected).

Proof. This follows from that $A/(\mathbf{T}\Phi)$ is a retract of the category $\mathbf{S}(\genfrac{}{}{0pt}{}{A}{\Phi})$ for any $A \in \mathbb{K}$ (Remark 3.21). □

Proposition 3.26. The following are equivalent for a functor $\Phi\colon \mathbf{C} \to \mathbf{D}$ between categories:

(i) For every object $d \in \mathbf{D}$, the comma category d/Φ is connected and simply connected.

(ii) The induced AVD-functor $\mathbb{I}^\flat\mathbf{C} \xrightarrow{\mathbb{I}^\flat\Phi} \mathbb{I}^\flat\mathbf{D}$ is final.

Proof. [(ii) $\Longrightarrow$ (i)] This follows from Lemma 3.25.

[(i) $\Longrightarrow$ (ii)] The first and third conditions for finality are trivial. We will show the second condition. Let $a \overset{\vec{u}}{\dashrightarrow} b$ in $\mathbb{I}^\flat\mathbf{D}$ be a path of loose arrows. The following shows that every object (x,φ) in $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\mathbb{I}^\flat\Phi})$ on the left below is connected with an object such that X is of length 1 in (13):

$$\begin{tikzcd}
a \arrow[r, dashed, "\vec{u}"] \arrow[d, "\varphi^0"'] \arrow[dr, phantom, "\varphi"] & b \arrow[d, "\varphi^1"] \\
\Phi x^0 \arrow[r, dotted, "\Phi x"] \arrow[d, equal] \arrow[dr, phantom, "\Phi!"] & \Phi x^1 \arrow[d, equal] \\
\Phi x^0 \arrow[r, "\Phi!"'] & \Phi x^1
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, dashed, "\vec{u}"] \arrow[d, "\varphi^0"'] \arrow[dr, phantom, "!"] & b \arrow[d, "\varphi^1"] \\
\Phi x^0 \arrow[r, "\Phi!"'] & \Phi x^1
\end{tikzcd}
\quad \text{in } \mathbb{I}^\flat\mathbf{D}$$

The full subcategory of $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ consists of objects where X has the length 1 in (13) is isomorphic to a product $a/\Phi \times b/\Phi$ of comma categories, which are connected by the assumption. Therefore, $\mathbf{S}(\genfrac{}{}{0pt}{}{\vec{u}}{\Phi})$ is connected. □

Notation 3.27. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ and $F\colon \mathbb{K} \to \mathbb{L}$ be AVD-functors between AVDCs. Then, a tight cocone l from F yields a tight cocone from $F\Phi$, denoted by l_Φ, in a natural way. We also use such a notation for modules and modulations. ♦

Theorem 3.28. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ be a final AVD-functor. Then, the following hold for any AVD-functor $F\colon \mathbb{K} \to \mathbb{L}$.

(i) The assignment $l \mapsto l_\Phi$ yields isomorphisms of categories

$$-_\Phi\colon \mathbf{Cone}(\genfrac{}{}{0pt}{}{F}{L}) \xrightarrow{\cong} \mathbf{Cone}(\genfrac{}{}{0pt}{}{F\Phi}{L}) \qquad (L \in \mathbb{L}).$$

(ii) Assume that the following additional condition: for any $A \in \mathbb{K}$ there exists an object $(X,p) \in A/(\mathbf{T}\Phi)$ such that Fp is left-pulling in $\mathbb{L}$. Then, the assignment $m \mapsto m_\Phi$ yields equivalences of categories

$$-_\Phi\colon \mathbf{Mdl}(F,M) \xrightarrow{\simeq} \mathbf{Mdl}(F\Phi,M) \qquad (M \in \mathbb{L}).$$

(iii) The assignment $\rho \mapsto \rho_\Phi$ yields bijections among the classes of modulations of the same type.

Proof. We first show (iii) for modulations of type 1. Let σ be a modulation of type 1 exhibited by the following:

$$\begin{tikzcd}
F\Phi \arrow[r, Rightarrow, "m_\Phi", "\shortmid"{marking}] \arrow[d, Rightarrow, "l_\Phi"'] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, phantom, "\sigma"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd}$$

Here, m is a left F-module, and l is a tight cocone from F. We have to construct a modulation $\mathfrak{s}$ such that $\mathfrak{s}_\Phi = \sigma$. For each $A \in \mathbb{K}$, let us take a tight arrow $A \xrightarrow{a} \Phi X$ in $\mathbb{K}$ by using the ordinary finality of $\mathbf{T}\Phi$ and define $\mathfrak{s}_A$ as the following cell:

$$\mathfrak{s}_A := \begin{tikzcd}
FA \arrow[r, "m_A", "\shortmid"{marking}] \arrow[d, "Fa"'] \arrow[dr, phantom, "m_a"] & M \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
F\Phi X \arrow[r, "m_{\Phi X}"', "\shortmid"{marking}] \arrow[d, "l_{\Phi X}"'] & M \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] \arrow[d, phantom, "\sigma_X"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "q"', "\shortmid"{marking}] & & L'
\end{tikzcd} \quad \text{in } \mathbb{L}.$$

By using the ordinary finality of $\mathbf{T}\Phi$ again, we can show that the cells $\mathfrak{s}_A$ are independent of the choice of $A \xrightarrow{a} \Phi X$. Then, from the independence of $\mathfrak{s}_A$ and the second condition in the definition of finality, it easily follows that the cells $\mathfrak{s}$ form a desired modulation $\mathfrak{s}$. The uniqueness of $\mathfrak{s}$ is trivial. The same argument works in the case of modulations of the other types.

We next show (i). Since the functor $-_\Phi \colon \mathbf{Cone}\left(\begin{smallmatrix}F\\L\end{smallmatrix}\right) \longrightarrow \mathbf{Cone}\left(\begin{smallmatrix}F\Phi\\L\end{smallmatrix}\right)$ is fully faithful by (iii), it suffices to show that the functor $-_\Phi$ is bijective on objects. Let l be a tight cocone from $F\Phi$ to L. Since $A/(\mathbf{T}\Phi)$ is connected for each $A \in \mathbb{K}$, we can define $\mathfrak{l}_A$ as $(Fp)\fatsemi l_X$ independently of the choice of $A \xrightarrow{p} \Phi X$ in $\mathbb{K}$. Since $\mathbf{S}\left(\begin{smallmatrix}\vec{u}\\\Phi\end{smallmatrix}\right)$ is connected for $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$ in $\mathbb{K}$, we can also define a cell $\mathfrak{l}_{\vec{u}}$ as follows independently of the choice of an object $(X, \varphi) \in \mathbf{S}\left(\begin{smallmatrix}\vec{u}\\\Phi\end{smallmatrix}\right)$:

$$\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[dr, "\mathfrak{l}_{A_0}"'] & \mathfrak{l}_{\vec{u}} & FA_n \arrow[dl, "\mathfrak{l}_{A_n}"] \\
& L &
\end{tikzcd}
:=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "F\varphi^0"'] & F\varphi & FA_n \arrow[d, "F\varphi^1"] \\
F\Phi X^0 \arrow[rr, dotted, "F\Phi X", "\shortmid"{marking}] \arrow[dr, "l_{X^0}"'] & l_X & F\Phi X^1 \arrow[dl, "l_{X^1}"] \\
& L &
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

Taking data $(\vec{X}, Y, Z, \vec{p}, f, g, \vec{v}, w, \vec{\varphi}, \theta)$ as in (14), we can show that the cell $\mathfrak{l}_{\vec{u}}$ is a composite of the cells $(\mathfrak{l}_{u_1}, \dots, \mathfrak{l}_{u_n})$:

$$\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[dr, "\mathfrak{l}_{A_0}"'] & \mathfrak{l}_{\vec{u}} & FA_n \arrow[dl, "\mathfrak{l}_{A_n}"] \\
& L &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Fp_0"'] & F\vec{\varphi} & FA_n \arrow[d, "Fp_n"] \\
F\Phi X_0 \arrow[rr, dashed, "F\Phi\vec{v}", "\shortmid"{marking}] \arrow[d, "F\Phi f"'] & F\Phi\theta & F\Phi X_n \arrow[d, "F\Phi g"] \\
F\Phi Y \arrow[rr, dotted, "F\Phi w", "\shortmid"{marking}] \arrow[dr, "l_Y"'] & l_w & F\Phi Z \arrow[dl, "l_Z"] \\
& L &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & FA_0 \arrow[r, "Fu_1"] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F\varphi_1"] & FA_1 \arrow[r, "Fu_2"] \arrow[d, "Fp_1"] \arrow[dr, phantom, "F\varphi_2"] & \cdots \arrow[r, "Fu_{n-1}"] \arrow[dr, phantom, "F\varphi_{n-1}"] & FA_{n-1} \arrow[r, "Fu_n"] \arrow[d, "Fp_{n-1}"] \arrow[dr, phantom, "F\varphi_n"] & FA_n \arrow[d, "Fp_n"] \\
= & F\Phi X_0 \arrow[r, dotted, "F\Phi v_1"] \arrow[ddrr, bend right, "l_{X_0}"'] \arrow[drr, phantom, "l_{v_1}"] & F\Phi X_1 \arrow[r, dotted, "F\Phi v_2"] \arrow[dd, "l_{X_1}"] & \cdots \arrow[r, dotted, "F\Phi v_{n-1}"] & F\Phi X_{n-1} \arrow[r, dotted, "F\Phi v_n"] \arrow[ddl, "l_{X_{n-1}}"'] \arrow[d, phantom, "l_{v_n}"] & F\Phi X_n \arrow[ddll, bend left, "l_{X_n}"] \\
 & & & & & \\
 & & & L & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & FA_0 \arrow[r, "Fu_1"] \arrow[ddrr, bend right, "\mathfrak{l}_{A_0}"'] \arrow[drr, phantom, "\mathfrak{l}_{u_1}"] & FA_1 \arrow[r, "Fu_2"] \arrow[dd, "\mathfrak{l}_{A_1}"] & \cdots \arrow[r, "Fu_{n-1}"] & FA_{n-1} \arrow[r, "Fu_n"] \arrow[ddl, "\mathfrak{l}_{A_{n-1}}"'] \arrow[d, phantom, "\mathfrak{l}_{u_n}"] & FA_n \arrow[ddll, bend left, "\mathfrak{l}_{A_n}"] \\
= & & & & & & \text{in } \mathbb{L}. \\
 & & & L & &
\end{tikzcd}
$$

To show that $\mathfrak{l}$ is a tight cocone, take an arbitrary cell

$$
\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{u}"] \arrow[d, "b"'] \arrow[dr, phantom, "\alpha"] & A_n \arrow[d, "c"] \\
B \arrow[r, dotted, "v"'] & C
\end{tikzcd}
\quad \text{in } \mathbb{K}. \tag{15}
$$

Taking an object $(Z, \chi) \in \mathbf{S}(\begin{smallmatrix} v \\ \Phi \end{smallmatrix})$, we have the following:

$$
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fb"'] \arrow[dr, phantom, "F\alpha"] & FA_n \arrow[d, "Fc"] \\
FB \arrow[r, dotted, "Fv"] \arrow[dr, "\mathfrak{l}_B"'] \arrow[r, phantom, shift right=4, "\mathfrak{l}_v"] & FC \arrow[dl, "\mathfrak{l}_C"] \\
 L &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fb"'] \arrow[dr, phantom, "F\alpha"] & FA_n \arrow[d, "Fc"] \\
FB \arrow[r, dotted, "Fv"] \arrow[d, "F\chi^0"'] \arrow[dr, phantom, "F\chi"] & FC \arrow[d, "F\chi^1"] \\
F\Phi Z^0 \arrow[r, dotted, "F\Phi Z"] \arrow[dr, "l_{Z^0}"'] \arrow[r, phantom, shift right=4, "l_Z"] & F\Phi Z^1 \arrow[dl, "l_{Z^1}"] \\
 L &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[dr, "\mathfrak{l}_{A_0}"'] \arrow[r, phantom, shift right=4, "\mathfrak{l}_{\vec{u}}"] & FA_n \arrow[dl, "\mathfrak{l}_{A_n}"] \\
 L &
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

Therefore, $\mathfrak{l}$ becomes a tight cocone.

We next show (ii) under the additional assumption of left-pullability. Since the functor $-_\Phi \colon \mathbf{Mdl}(F, M) \to \mathbf{Mdl}(F\Phi, M)$ is fully faithful by (iii), it suffices to show that the functor $-_\Phi$ is essentially surjective. Let m be a left $F\Phi$-module with a vertex M. Consider a functor $G_A \colon A/(\mathbf{T}\Phi) \to \mathbf{T}^1\mathbb{L}$ defined by the following assignment:

$$
\begin{array}{c} A \\ \downarrow p \\ \Phi X \end{array} \quad \text{in } \mathbb{K} \qquad \mapsto \qquad F\Phi X \xrightarrow{m_X} M \quad \text{in } \mathbb{L}.
$$

Note that G_A can be decomposed into two functors $A/(\mathbf{T}\Phi) \to \mathbf{T}\mathbb{J} \xrightarrow{m_{(-)}} \mathbf{T}^1\mathbb{L}$, where the first one is the forgetful functor and the second one is induced by the left module m. By the assumption, there are an object $A \xrightarrow{p_0} \Phi X_0$ in $A/(\mathbf{T}\Phi)$ and a restriction, denoted by $\mathfrak{m}_A$, of the following form:

$$
\begin{tikzcd}
FA \arrow[r, "\mathfrak{m}_A"] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "\mathsf{cart}"] & M \arrow[d, equal] \\
F\Phi X_0 \arrow[r, "m_{X_0}"'] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}. \tag{16}
$$

Since $A/(\mathbf{T}\Phi)$ is connected and simply connected, the above cell (16) uniquely extends to a cone over G_A of the following form:

$$\begin{tikzcd} FA \arrow[r, "\mathfrak{m}_A", "\shortmid"{marking}] \arrow[d, "Fp"'] \arrow[dr, phantom, "\rho^p_X : \mathsf{cart}"] & M \arrow[d, equal] \\ F\Phi X \arrow[r, "m_X"', "\shortmid"{marking}] & M \end{tikzcd} \quad \text{in } \mathbb{L}, \text{ where } (X,p) \in A/(\mathbf{T}\Phi). \tag{17}$$

Note that ρ^p_X automatically becomes cartesian since the cell (16) ($=\rho^{p_0}_{X_0}$) is cartesian. Since $A/(\mathbf{T}\Phi)$ is connected, the cone (17) over G_A becomes jointly cartesian. Furthermore, since $\mathbf{S}(\frac{\vec{u}}{\Phi})$ is connected for $A \overset{\vec{u}}{\dashrightarrow} B$ in $\mathbb{K}$, a cone over $\mathbf{S}(\frac{\vec{u}}{\Phi}) \xrightarrow{(-)^0} A/(\mathbf{T}\Phi) \xrightarrow{G_A} \mathbf{T}^1\mathbb{L}$ obtained by pre-composing $(-)^0$ with the cone (17) also becomes jointly cartesian.

Let $A \xrightarrow{f} B$ be a tight arrow in $\mathbb{K}$. Then, the assignment to $(X,p) \in B/(\mathbf{T}\Phi)$, the cell $\rho^{f ⨾ p}_X$ gives a cone over G_B. Using the joint cartesianness of "ρ," we have a unique cell $\mathfrak{m}_f$ satisfying the following for any $(X,p) \in B/(\mathbf{T}\Phi)$:

$$\begin{tikzcd} FA \arrow[r, "\mathfrak{m}_A", "\shortmid"{marking}] \arrow[d, "Ff"'] & M \arrow[d, equal] \\ FB \arrow[d, "Fp"'] \arrow[r, phantom, "\rho^{f ⨾ p}_X"] & M \arrow[d, equal] \\ F\Phi X \arrow[r, "m_X"', "\shortmid"{marking}] & M \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\mathfrak{m}_A", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "\mathfrak{m}_f"] & M \arrow[d, equal] \\ FB \arrow[r, "\mathfrak{m}_B", "\shortmid"{marking}] \arrow[d, "Fp"'] \arrow[dr, phantom, "\rho^p_X"] & M \arrow[d, equal] \\ F\Phi X \arrow[r, "m_X"', "\shortmid"{marking}] & M \end{tikzcd} \quad \text{in } \mathbb{L}.$$

It easily follows that the assignment $f \mapsto \mathfrak{m}_f$ is functorial.

Let $A_0 \overset{\vec{u}}{\dashrightarrow} A_n$ be a loose path in $\mathbb{K}$. Then, the assignment to $(X,\varphi) \in \mathbf{S}(\frac{\vec{u}}{\Phi})$, a cell on the left below gives a cone over $\mathbf{S}(\frac{\vec{u}}{\Phi}) \xrightarrow{(-)^0} A_0/(\mathbf{T}\Phi) \xrightarrow{G_{A_0}} \mathbf{T}^1\mathbb{L}$. Using the joint cartesianness of "ρ," we have a unique cell, denoted by $\mathfrak{m}_{\vec{u}}$, such that the following holds for every object $(X,\varphi) \in \mathbf{S}(\frac{\vec{u}}{\Phi})$:

$$\begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "F\varphi^0"'] \arrow[dr, phantom, "F\varphi"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] \arrow[d, "F\varphi^1"] \arrow[dr, phantom, "\rho^{\varphi^1}_{X^1}"] & M \arrow[d, equal] \\ F\Phi X^0 \arrow[r, dotted, "F\Phi X"', "\shortmid"{marking}] \arrow[d, equal] & F\Phi X^1 \arrow[r, "m_{X^1}"', "\shortmid"{marking}] \arrow[d, phantom, "m_X"] & M \arrow[d, equal] \\ F\Phi X^0 \arrow[rr, "m_{X^0}"', "\shortmid"{marking}] & & M \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, equal] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] \arrow[d, phantom, "\mathfrak{m}_{\vec{u}}"] & M \arrow[d, equal] \\ FA_0 \arrow[rr, "\mathfrak{m}_{A_0}", "\shortmid"{marking}] \arrow[d, "F\varphi^0"'] & \arrow[d, phantom, "\rho^{\varphi^0}_{X^0}"] & M \arrow[d, equal] \\ F\Phi X^0 \arrow[rr, "m_{X^0}"', "\shortmid"{marking}] & & M \end{tikzcd} \quad \text{in } \mathbb{L}.$$

Taking data $(\vec{X}, Y, Z, \vec{p}, f, g, \vec{v}, w, \vec{\varphi}, \theta)$ as in (14), we can decompose the cell $\mathfrak{m}_{\vec{u}}$ into the cells $(\mathfrak{m}_{u_1}, \ldots, \mathfrak{m}_{u_n})$ as follows:

$$\begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F\vec{\varphi}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] \arrow[d, "Fp_n"] & M \arrow[ddd, equal] \\ F\Phi X_0 \arrow[r, dashed, "F\Phi\vec{v}", "\shortmid"{marking}] \arrow[d, "F\Phi f"'] \arrow[dr, phantom, "F\Phi\theta"] & F\Phi X_n \arrow[d, "F\Phi g"] \arrow[r, phantom, "\rho^{p_n ⨾ \Phi g}_Z"] & \\ F\Phi Y \arrow[r, dotted, "F\Phi w"', "\shortmid"{marking}] \arrow[d, equal] & F\Phi Z \arrow[r, "m_Z"', "\shortmid"{marking}] \arrow[d, phantom, "m_w"] & M \arrow[d, equal] \\ F\Phi Y \arrow[rr, "m_Y"', "\shortmid"{marking}] & & M \end{tikzcd} = \begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F\vec{\varphi}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}", "\shortmid"{marking}] \arrow[d, "Fp_n"] \arrow[dr, phantom, "\rho^{p_n}_Z"] & M \arrow[d, equal] \\ F\Phi X_0 \arrow[r, dashed, "F\Phi\vec{v}", "\shortmid"{marking}] \arrow[d, "F\Phi f"'] \arrow[dr, phantom, "F\Phi\theta"] & F\Phi X_n \arrow[r, "m_{X_n}", "\shortmid"{marking}] \arrow[d, "F\Phi g"] \arrow[dr, phantom, "m_g"] & M \arrow[d, equal] \\ F\Phi Y \arrow[r, dotted, "F\Phi w"', "\shortmid"{marking}] \arrow[d, equal] & F\Phi Z \arrow[r, "m_Z"', "\shortmid"{marking}] \arrow[d, phantom, "m_w"] & M \arrow[d, equal] \\ F\Phi Y \arrow[rr, "m_Y"', "\shortmid"{marking}] & & M \end{tikzcd}$$

$$
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F\vec{\varphi}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] \arrow[d, "Fp_n"'] \arrow[dr, phantom, "\rho^{p_n}_Z"] & M \arrow[d, equal] \\
F\Phi X_0 \arrow[r, dashed, "F\Phi\vec{v}"] \arrow[d, equal] \arrow[drr, phantom, "m_{\vec{v}}"] & F\Phi X_n \arrow[r, "m_{X_n}"] & M \arrow[d, equal] \\
F\Phi X_0 \arrow[rr, "m_{X_0}"] \arrow[d, "F\Phi f"'] \arrow[drr, phantom, "m_f"] & & M \arrow[d, equal] \\
F\Phi Y \arrow[rr, "m_Y"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F(u_1,\dots,u_{n-1})"] \arrow[d, equal] \arrow[dr, phantom, "="] & FA_{n-1} \arrow[r, "Fu_n"] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_{u_n}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] & M \arrow[d, equal] \\
FA_0 \arrow[r, dashed] \arrow[d, "Fp_0"'] \arrow[dr, phantom, "F(\varphi_1,\dots,\varphi_{n-1})"] & FA_{n-1} \arrow[rr, "\mathfrak{m}_{A_{n-1}}"] \arrow[d, "Fp_{n-1}"] \arrow[drr, phantom, "\rho^{p_{n-1}}_{X_{n-1}}"] & & M \arrow[d, equal] \\
F\Phi X_0 \arrow[r, dashed, "F\Phi(v_1,\dots,v_{n-1})"] \arrow[d, equal] \arrow[drrr, phantom, "m_{(v_1,\dots,v_{n-1})}"] & F\Phi X_{n-1} \arrow[rr, "m_{X_{n-1}}"] & & M \arrow[d, equal] \\
F\Phi X_0 \arrow[rrr, "m_{X_0}"] \arrow[d, "F\Phi f"'] \arrow[drrr, phantom, "m_f"] & & & M \arrow[d, equal] \\
F\Phi Y \arrow[rrr, "m_Y"'] & & & M
\end{tikzcd}
$$

$$
= \cdots =
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, equal] \arrow[drr, phantom, "(\mathfrak{m}_{u_1},\dots,\mathfrak{m}_{u_n})"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}"] \arrow[d, "Fp_0"'] \arrow[drr, phantom, "\rho^{p_0}_{X_0}"] & & M \arrow[d, equal] \\
F\Phi X_0 \arrow[rr, "m_{X_0}"] \arrow[d, "F\Phi f"'] \arrow[drr, phantom, "m_f"] & & M \arrow[d, equal] \\
F\Phi Y \arrow[rr, "m_Y"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, equal] \arrow[drr, phantom, "(\mathfrak{m}_{u_1},\dots,\mathfrak{m}_{u_n})"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}"] \arrow[d, "Fp_0"'] & & M \arrow[dd, equal] \\
F\Phi X_0 \arrow[d, "F\Phi f"'] \arrow[r, phantom, "\rho^{p_0 \fatsemi \Phi f}_Y : \mathsf{cart}"] & {} & \\
F\Phi Y \arrow[rr, "m_Y"'] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

To show that $\mathfrak{m}$ is a left F-module, let us take an arbitrary cell α in $\mathbb{K}$ as in (15). Taking an object $(Y, \psi) \in \mathbf{S}(\,{}^{v}_{\Phi}\,)$, we have the following:

$$
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fb"'] \arrow[dr, phantom, "F\alpha"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] \arrow[d, "Fc"] \arrow[dr, phantom, "\mathfrak{m}_c"] & M \arrow[d, equal] \\
FB \arrow[r, dotted, "Fv"] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_v"] & FC \arrow[r, "\mathfrak{m}_C"] & M \arrow[d, equal] \\
FB \arrow[rr, "\mathfrak{m}_B"] \arrow[d, "F\psi^0"'] \arrow[drr, phantom, "\rho^{\psi^0}_{Y^0} : \mathsf{cart}"] & & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fb"'] \arrow[dr, phantom, "F\alpha"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] \arrow[d, "Fc"] \arrow[dr, phantom, "\mathfrak{m}_c"] & M \arrow[d, equal] \\
FB \arrow[r, dotted, "Fv"] \arrow[d, "F\psi^0"'] \arrow[dr, phantom, "F\psi"] & FC \arrow[r, "\mathfrak{m}_C"] \arrow[d, "F\psi^1"] \arrow[dr, phantom, "\rho^{\psi^1}_{Y^1}"] & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[r, dotted, "F\Phi Y"] \arrow[d, equal] \arrow[drr, phantom, "m_Y"] & F\Phi Y^1 \arrow[r, "m_{Y^1}"] & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"'] & & M
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "Fb"'] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] \arrow[d, "Fc"] & M \arrow[ddd, equal] \\
FB \arrow[r, phantom, "F(\alpha \fatsemi \psi)"] \arrow[d, "F\psi^0"'] & FC \arrow[d, "F\psi^1"] \arrow[r, phantom, "\rho^{c \fatsemi \psi^1}_{Y^1}"] & {} \\
F\Phi Y^0 \arrow[r, dotted, "F\Phi Y"] \arrow[d, equal] \arrow[drr, phantom, "m_Y"] & F\Phi Y^1 \arrow[r, "m_{Y^1}"] & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_{\vec{u}}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}"] \arrow[d, "Fb"'] & & M \arrow[dd, equal] \\
FB \arrow[d, "F\psi^0"'] \arrow[r, phantom, "\rho^{b \fatsemi \psi^0}_{Y^0}"] & {} & \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"'] & & M
\end{tikzcd}
$$

$$
= \begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, equal] \arrow[drr, phantom, "\mathfrak{m}_{\vec{u}}"] & FA_n \arrow[r, "\mathfrak{m}_{A_n}"] & M \arrow[d, equal] \\
FA_0 \arrow[rr, "\mathfrak{m}_{A_0}"] \arrow[d, "Fb"'] \arrow[drr, phantom, "\mathfrak{m}_b"] & & M \arrow[d, equal] \\
FB \arrow[rr, "\mathfrak{m}_B"] \arrow[d, "F\psi^0"'] \arrow[drr, phantom, "\rho^{\psi^0}_{Y^0} : \mathsf{cart}"] & & M \arrow[d, equal] \\
F\Phi Y^0 \arrow[rr, "m_{Y^0}"'] & & M
\end{tikzcd} \quad \text{in } \mathbb{L},
$$

which shows that $\mathfrak{m}$ becomes a left F-module. We can easily verify that the cells ρ^{id}_X for $X \in \mathbb{J}$ form an invertible modulation $\mathfrak{m}_\Phi \cong m$ of type 0, which finishes the proof. □

Example 3.29. Let $\mathbb{J}$ be the AVDC consisting of two objects $0, 1$ and a unique loose arrow $0 \to 1$. Let $\mathbb{K}$ be an AVDC defined by the following:

- $\mathbb{K}$ has just two objects $0, 1$;
- $\mathbb{K}$ has no non-trivial tight arrow;
- $\mathbb{K}$ has just three loose arrows $0 \to 0 \to 1 \to 1$;
- For any boundary for cells, which includes nullcoary one, $\mathbb{K}$ has a unique cell filling it.

Then, the inclusion $\mathbb{J} \to \mathbb{K}$ gives a final AVD-functor. An AVD-functor $F \colon \mathbb{K} \to \mathbb{L}$ is the same thing as a choice of a loose arrow $F0 \to F1$ and loose units on $F0$ and $F1$. By Theorem 3.28, we can ignore the loose units when we regard F as a diagram for tight cocones, modules, and modulations. ♦

3.3. **Versatile colimits.** In this subsection, we fix an AVD-functor $F \colon \mathbb{K} \to \mathbb{L}$ between AVDCs and a tight cocone ξ from F to $\Xi \in \mathbb{L}$.

Definition 3.30. We consider the following conditions for ξ:

(T) The canonical functor $\mathbf{Hom}_{\mathbb{L}}(\frac{\Xi}{L}) \xrightarrow{\xi \circ -} \mathbf{Cone}(\frac{F}{L})$ of Construction 3.13 is bijective on objects for any $L \in \mathbb{L}$.

(L-l) ξ_A is left-pulling for any $A \in \mathbb{K}$, and the canonical functor $\mathbf{Hom}_{\mathbb{L}}(\Xi, L) \xrightarrow{\xi_* -} \mathbf{Mdl}(F, L)$ of Construction 3.15 is essentially surjective for any $L \in \mathbb{L}$.

(L-r) The loosewise dual of (L-l) holds.

(M0-l) ξ_A is left-pulling for any $A \in \mathbb{K}$, and the following hold: Take $M, M' \in \mathbb{L}$ and $\Xi \xrightarrow{p} M, \Xi \xrightarrow{p'} M'$ in $\mathbb{L}$ arbitrarily. Then, for any modulation ρ of type 0

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_* p"] \arrow[d, equal] \arrow[drr, phantom, "\rho"] & M \arrow[r, dashed, "\vec{q}"] & M'' \arrow[d, "j"] \\
F \arrow[rr, Rightarrow, "\xi_* p'"'] & & M',
\end{tikzcd}
$$

There exists a unique cell $\hat{\rho}$ such that

$$
\begin{tikzcd}
FA \arrow[r, "(\xi_* p)_A"] \arrow[d, equal] \arrow[drr, phantom, "\rho_A"] & M \arrow[r, dashed, "\vec{q}"] & M'' \arrow[d, "j"] \\
FA \arrow[rr, "(\xi_* p')_A"'] \arrow[d, "\xi_A"'] \arrow[drr, phantom, "(\xi_\dagger p')_A : \mathsf{cart}"] & & M' \arrow[d, equal] \\
\Xi \arrow[rr, "p'"'] & & M'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "(\xi_* p)_A"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger p)_A : \mathsf{cart}"] & M \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M'' \arrow[d, equal] \\
\Xi \arrow[r, "p"'] \arrow[d, equal] \arrow[drr, phantom, "\hat{\rho}"] & M \arrow[r, dashed, "\vec{q}"] & M'' \arrow[d, "j"] \\
\Xi \arrow[rr, "p'"'] & & M'
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (\text{for any } A \in \mathbb{K}).
$$

(M0-r) The loosewise dual of (M0-l) holds.

(M1-l) ξ_A is left-pulling for any $A \in \mathbb{K}$, and the following hold: Take $L, M \in \mathbb{L}$ and $\Xi \xrightarrow{k} L, \Xi \xrightarrow{p} M$ in $\mathbb{L}$ arbitrarily. Then, for any modulation σ of type 1

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_* p"] \arrow[d, Rightarrow, "\xi \fatsemi k"'] & M \arrow[r, dashed, "\vec{q}"] \arrow[d, phantom, "\sigma"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "r"'] & & L',
\end{tikzcd}$$

there exists a unique cell $\hat{\sigma}$ such that

$$\begin{tikzcd}
FA \arrow[r, "(\xi_* p)_A"] \arrow[d, "(\xi \fatsemi k)_A"'] & M \arrow[r, dashed, "\vec{q}"] \arrow[d, phantom, "\sigma_A"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "r"'] & & L'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "(\xi_* p)_A"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger p)_A\colon \mathsf{cart}"] & M \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
\Xi \arrow[r, "p"'] \arrow[d, "k"'] & M \arrow[r, dashed, "\vec{q}"'] \arrow[d, phantom, "\hat{\sigma}"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "r"'] & & L'
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (\text{for any } A \in \mathbb{K}).$$

(M1-r) The loosewise dual of (M1-l) holds.

(M2) Take $L, L' \in \mathbb{L}$ and $\Xi \xrightarrow{k} L, \Xi \xrightarrow{k'} L'$ in $\mathbb{L}$ arbitrarily. Then, for any modulation τ of type 2

$$\begin{tikzcd}
& F \arrow[dl, Rightarrow, "\xi \fatsemi k"'] \arrow[dr, Rightarrow, "\xi \fatsemi k'"] \arrow[d, phantom, "\tau"] & \\
L \arrow[rr, dotted, "q"'] & {} & L',
\end{tikzcd}$$

there exists a unique cell $\hat{\tau}$ such that

$$\begin{tikzcd}
& FA \arrow[dl, "(\xi \fatsemi k)_A"'] \arrow[dr, "(\xi \fatsemi k')_A"] \arrow[d, phantom, "\tau_A"] & \\
L \arrow[rr, dotted, "q"'] & {} & L'
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[d, bend right, "\xi_A"'] \arrow[d, bend left, "\xi_A"] \arrow[d, phantom, "="] & \\
& \Xi \arrow[dl, "k"'] \arrow[dr, "k'"] \arrow[d, phantom, "\hat{\tau}"] & \\
L \arrow[rr, dotted, "q"'] & {} & L'
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (\text{for any } A \in \mathbb{K}).$$

(M3) ξ_A is pulling for any $A \in \mathbb{K}$, and the following hold: Take $N, M \in \mathbb{L}$ and $N \xrightarrow{t} \Xi \xrightarrow{s} M$ in $\mathbb{L}$ arbitrarily. Then, for any modulation ω of type 3

$$\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, "j"'] & N \arrow[r, Rightarrow, "t\xi^*"] & F \arrow[r, Rightarrow, "\xi_* s"] \arrow[d, phantom, "\omega"] & M \arrow[r, dashed, "\vec{p}"] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "r"'] & & {} & & M''
\end{tikzcd}$$

there exists a unique cell $\hat{\omega}$ such that

$$\omega_A =
\begin{tikzcd}
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & N \arrow[r, "(t\xi^*)_A"] \arrow[d, equal] \arrow[dr, phantom, "(t\xi^\dagger)_A\colon \mathsf{cart}"] & FA \arrow[r, "(\xi_* s)_A"] \arrow[d, "\xi_A"] \arrow[dr, phantom, "(\xi_\dagger s)_A\colon \mathsf{cart}"] & M \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "="] & M' \arrow[d, equal] \\
N' \arrow[r, dashed, "\vec{q}"] \arrow[d, "j"'] & N \arrow[r, "t"] & \Xi \arrow[r, "s"] \arrow[d, phantom, "\hat{\omega}"] & M \arrow[r, dashed, "\vec{p}"] & M' \arrow[d, "i"] \\
N'' \arrow[rrrr, dotted, "r"'] & & {} & & M''
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (\text{for any } A \in \mathbb{K}).$$

♦

Remark 3.31. The above conditions are independent of the construction of the functors ξ_*- and $-\xi^*$. In particular, the condition (L-l) can be rephrased as follows:

(L-l)' ξ_A is left-pulling for any $A \in \mathbb{K}$. Furthermore, for any left F-module $m\colon F \Rightarrow L$, there exist a loose arrow $\Xi \xrightarrow{p} L$ in $\mathbb{L}$ and a modulation σ of type 1

$$\begin{tikzcd} F \arrow[r, Rightarrow, "m"] \arrow[d, Rightarrow, "\xi"'] \arrow[dr, phantom, "\sigma"] & L \arrow[d, equal] \\ \Xi \arrow[r, "p"'] & L \end{tikzcd}$$

such that every component σ_A $(A \in \mathbb{K})$ is cartesian. ♦

Proposition 3.32.

(i) (M2) implies that the functor $\mathbf{Hom}_{\mathbb{L}}(\begin{smallmatrix}\Xi\\ L\end{smallmatrix}) \xrightarrow{\xi ⨾ -} \mathbf{Cone}(\begin{smallmatrix}F\\ L\end{smallmatrix})$ is fully faithful for any $L \in \mathbb{L}$.

(ii) (M0-l) implies that the functor $\mathbf{Hom}_{\mathbb{L}}(\Xi, L) \xrightarrow{\xi_* -} \mathbf{Mdl}(F, L)$ is fully faithful for any $L \in \mathbb{L}$.

Proof. This follows from the fact that morphisms between tight cocones or modules are a special case of modulations of type 2 or 0. □

Proposition 3.33.

(i) (M1-l) implies (M0-l).

(ii) If $\mathbb{L}$ has loose units and every tight arrow is left-pulling in $\mathbb{L}$, then (M1-l) and (M0-l) are equivalent.

Proof. Using the universal property of restrictions, we can establish a bijection between the modulations of type 1 and the modulations of type 0. □

Proposition 3.34.

(i) If $\mathbb{L}$ has companions, then (M1-l) implies (M2).

(ii) If $\mathbb{L}$ has conjoints, then (M3) implies (M1-l).

Proof.

(i) Suppose (M1-l) and that $\mathbb{L}$ has companions, in particular, loose units. Consider the canonical cells associated with the companions ξ_{A*}:

$$\begin{tikzcd} FA \arrow[r, "\xi_{A*}"] \arrow[d, "\xi_A"'] & \Xi \arrow[dl, equal] \\ \Xi & \end{tikzcd} \qquad \begin{tikzcd} & FA \arrow[d, "\xi_A"] \arrow[dl, equal] \\ FA \arrow[r, "\xi_{A*}"'] & \Xi \end{tikzcd} \quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}). \tag{18}$$

Let ξ_* denote the left F-module given by the companions ξ_{A*}. Then, we have bijective correspondences among the following data:

$$\begin{tikzcd} & F \arrow[dl, Rightarrow, "\xi"'] \arrow[dr, Rightarrow, "\xi"] & \\ \Xi \arrow[d, "k"'] & \tau & \Xi \arrow[d, "k'"] \\ L \arrow[rr, dotted, "q"'] & & L' \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} F \arrow[r, Rightarrow, "\xi_*"] \arrow[d, Rightarrow, "\xi"'] & \Xi \arrow[dd, "k'"] \\ \Xi \arrow[d, "k"'] \arrow[r, phantom, "\sigma"] & \\ L \arrow[r, dotted, "q"'] & L' \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} \Xi \arrow[r, equal] \arrow[d, "k"'] \arrow[dr, phantom, "\hat{\sigma}"] & \Xi \arrow[d, "k'"] \\ L \arrow[r, dotted, "q"'] & L' \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} & \Xi \arrow[dl, "k"'] \arrow[dr, "k'"] & \\ L \arrow[rr, dotted, "q"'] & \hat{\tau} & L' \end{tikzcd}$$

Here, the first correspondence is given by component-wise pasting with the cells (18). The second one is given by (M1-l). The third one is given by the universal property of loose units. Therefore (M2) follows.

(ii) Suppose (M3) and that $\mathbb{L}$ has conjoints. Then, we have bijective correspondences among the following data:

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_* p"] \arrow[d, Rightarrow, "\xi"'] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & M' \arrow[dd, "j"] \\
\Xi \arrow[d, "k"'] & \sigma & \\
L \arrow[rr, dotted, "r"', "\shortmid"{marking}] & & L'
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
L \arrow[r, Rightarrow, "k^*\xi^*", "\shortmid"{marking}] \arrow[d, equal] & F \arrow[r, Rightarrow, "\xi_* p", "\shortmid"{marking}] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rrr, dotted, "r"', "\shortmid"{marking}] & & \omega & L'
\end{tikzcd}
$$

$$
\Bigg\|\quad
\begin{tikzcd}
L \arrow[r, "k^*", "\shortmid"{marking}] \arrow[d, equal] & \Xi \arrow[r, "p", "\shortmid"{marking}] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] & M' \arrow[d, "j"] \\
L \arrow[rrr, dotted, "r"', "\shortmid"{marking}] & & \hat{\omega} & L'
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
\Xi \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "k"'] & M \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, phantom, "\hat{\sigma}"] & M' \arrow[d, "j"] \\
L \arrow[rr, dotted, "r"', "\shortmid"{marking}] & & L'
\end{tikzcd}
$$

The first correspondence is given by component-wise pasting with the canonical cells associated with the conjoints $\xi_A ⨾ k^* = (k^*\xi^*)_A$. The second one is given by (M3). The third one is given by pasting with the canonical cell associated with the conjoint k^*. Therefore (M1-l) follows. $\square$

Definition 3.35 (Versatile colimits)**.** ξ is called a ***versatile colimit*** of F if it satisfies the conditions (T)(L-l)(L-r)(M1-l)(M1-r)(M2)(M3). ♦

Corollary 3.36. When $\mathbb{L}$ has companions and conjoints, ξ becomes a versatile colimit if and only if it satisfies (T)(L-l)(L-r)(M3).

Proof. This follows from Proposition 3.34. $\square$

Corollary 3.37. Let $\Phi\colon \mathbb{J} \to \mathbb{K}$ be a final AVD-functor. Suppose that Ff is pulling in $\mathbb{L}$ for any tight arrow f in $\mathbb{K}$. Then, ξ_Φ is a versatile colimit of $F\Phi$ if and only if ξ is a versatile colimit of F.

Proof. This follows from Theorem 3.28. $\square$

Theorem 3.38 (Unitality theorem)**.** Suppose (L-l)(M1-l)(M2) and that ξ_A has a companion for every $A \in \mathbb{K}$. Then, Ξ has a loose unit.

Proof. Let ξ_* denote the left F-module given by the companions ξ_{A*}. Then, the canonical cartesian cells $\xi_{A\dagger}$ on the right below form a modulation $\xi_\dagger$ of type 1 on the left below:

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_*", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] & \Xi \arrow[dl, phantom, "\xi_\dagger"] \\
\Xi \arrow[ur, equal] &
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \arrow[dl, phantom, "\xi_{A\dagger}"] \\
\Xi \arrow[ur, equal] &
\end{tikzcd}
: \mathsf{cart} \quad \text{in } \mathbb{L} \quad (A \in \mathbb{K})
$$

By (L-l), we have a loose arrow $\Xi \xrightarrow{u} \Xi$ in $\mathbb{L}$ and a modulation $\xi_\dagger u$ of type 1 whose components are cartesian:

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "\xi_*", "\shortmid"{marking}] \arrow[d, Rightarrow, "\xi"'] \arrow[dr, phantom, "\xi_\dagger u"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\mathsf{cart}"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K})
$$

By (M1-l), there is a unique cell ε corresponding to the modulation $\xi_\dagger$. The cell ε is uniquely determined by the following equations:

$$\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger u)_A"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \Xi \\
\Xi \arrow[ur, equal] &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \\
\Xi \arrow[ur, equal, "\xi_{A\dagger}"] &
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}).$$

Let us consider a modulation τ of type 2 given by the following:

$$\begin{tikzcd}
& F \arrow[dl, Rightarrow, "\xi"'] \arrow[dr, Rightarrow, "\xi"] \arrow[d, phantom, "\tau"] & \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] & {} & \Xi
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
& FA \arrow[dl, equal] \arrow[dr, "\xi_A"] \arrow[d, phantom, "\delta_A"] & \\
FA \arrow[rr, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[drr, phantom, "(\xi_\dagger u)_A"] & {} & \Xi \arrow[d, equal] \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] & & \Xi
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}),$$

where δ_A denote the canonical cell associated with the companion ξ_{A*}. By (M2), there is a unique cell η corresponding to τ. The cell η is uniquely determined by the following equations:

$$\begin{tikzcd}
& FA \arrow[d, bend right, "\xi_A"'] \arrow[d, bend left, "\xi_A"] \arrow[d, phantom, "="] & \\
& \Xi \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] & {} & \Xi
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[dl, equal] \arrow[dr, "\xi_A"] \arrow[d, phantom, "\delta_A"] & \\
FA \arrow[rr, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[drr, phantom, "(\xi_\dagger u)_A"] & {} & \Xi \arrow[d, equal] \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] & & \Xi
\end{tikzcd}
\quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}).$$

Then, (M1-l)(M2) and the following calculations conclude that u becomes a loose unit on Ξ:

$$\begin{tikzcd}
FA \arrow[rr, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[drr, phantom, "(\xi_\dagger u)_A"] & & \Xi \arrow[d, equal] \\
\Xi \arrow[rr, "u", "\shortmid"{marking}] \arrow[dr, equal] & & \Xi \arrow[dl, equal] \\
& \Xi \arrow[u, phantom, "\varepsilon"] \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\
\Xi \arrow[rr, "u"', "\shortmid"{marking}] & {} & \Xi
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \\
& \Xi \arrow[ur, equal, "\xi_{A\dagger}"] \arrow[d, equal] & \\
\Xi \arrow[ur, equal, "\eta"] \arrow[r, "u"', "\shortmid"{marking}] & \Xi &
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \\
FA \arrow[ur, equal, "\delta_A"'] \arrow[r, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger u)_A"] & \Xi \arrow[ur, equal, "\xi_{A\dagger}"] \arrow[d, equal] & \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\xi_{A*}", "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger u)_A"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u"', "\shortmid"{marking}] & \Xi
\end{tikzcd}$$

$$\begin{tikzcd}
& FA \arrow[d, bend right, "\xi_A"'] \arrow[d, bend left, "\xi_A"] \arrow[d, phantom, "="] & \\
& \Xi \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\
\Xi \arrow[rr, "u", "\shortmid"{marking}] \arrow[dr, equal] & {} & \Xi \arrow[dl, equal] \\
& \Xi \arrow[u, phantom, "\varepsilon"] &
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[d, "\xi_A"] \\
FA \arrow[ur, equal, "\delta_A"'] \arrow[r, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger u)_A"] & \Xi \arrow[d, equal] \\
\Xi \arrow[r, "u", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \Xi \\
\Xi \arrow[ur, equal] &
\end{tikzcd}
=
\begin{tikzcd}
& FA \arrow[d, "\xi_A"] \\
FA \arrow[ur, equal, "\delta_A"'] \arrow[r, "\xi_{A*}"', "\shortmid"{marking}] \arrow[d, "\xi_A"'] & \Xi \\
\Xi \arrow[ur, equal, "\xi_{A\dagger}"] &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[d, bend right, "\xi_A"'] \arrow[d, bend left, "\xi_A"] \arrow[d, phantom, "="] \\
\Xi
\end{tikzcd}
\quad \text{in } \mathbb{L}.$$

$\square$

Example 3.39 (Versatile coproducts)**.** Consider the diminished AVDC $\mathbb{R}\mathrm{el}^\flat$ of relations. Let $(X,Y)\colon \mathbb{D}2 \to \mathbb{R}\mathrm{el}$ be an AVD-functor determined by two (large) sets $X, Y \in \mathbb{R}\mathrm{el}$, where 2 denotes the two-element set. Then, the disjoint union $X+Y$ gives a versatile colimit of (X,Y). This is an example of a ***versatile coproduct*** defined later (Definition 4.3). ⧫

Example 3.40. A ***collage***, also called ***cograph***, of a profunctor $\mathbf{A} \xrightarrow{P} \mathbf{B}$ between categories is the category $\mathbf{X}$ whose class of objects is the disjoint union of $\mathrm{Ob}\mathbf{A}$ and $\mathrm{Ob}\mathbf{B}$ and where

$$\mathbf{X}(x,y) := \begin{cases} \mathbf{A}(x,y) & \text{if } x,y \in \mathbf{A}; \\ \mathbf{B}(x,y) & \text{if } x,y \in \mathbf{B}; \\ P(x,y) & \text{if } x \in \mathbf{A},\ y \in \mathbf{B}; \\ \varnothing & \text{if } x \in \mathbf{B},\ y \in \mathbf{A}. \end{cases}$$

Let $\mathbb{J}$ denote the AVDC consisting of just two objects $0, 1$ and a unique loose arrow $0 \nrightarrow 1$. Let **Set** and **SET** denote the categories of small sets and large sets, respectively. If the categories $\mathbf{A}$ and $\mathbf{B}$ are large and the profunctor P is locally large, then $\mathbf{X}$ gives a versatile colimit of P, where P is regarded as an AVD-functor from $\mathbb{J}$ to $\mathbf{SET}$-$\mathbb{P}\mathrm{rof}$, the AVDC of large categories. When the profunctor P is locally small, $\mathbf{X}$ still gives a versatile colimit in $(\mathbf{Set}, \mathbf{SET})$-$\mathbb{P}\mathrm{rof}$, the AVDC of large categories and locally small profunctors [Kou20, 2.6. Example]. This gives an example of a versatile colimit with no loose unit. ⧫

3.4. **The case of loosewise VD-indiscrete shapes.** In this subsection, we study versatile colimits in the special case when the shape is loosewise VD-indiscrete. Let us fix an AVD-functor $F\colon \mathbb{K} \to \mathbb{L}$ from a loosewise VD-indiscrete AVDC $\mathbb{K}$.

Proposition 3.41. A tight cocone from F with a vertex $L \in \mathbb{L}$ is equivalent to the following data:

- For each object $A \in \mathbb{K}$, a tight arrow $FA \xrightarrow{l_A} L$ in $\mathbb{L}$.
- For objects $A, B \in \mathbb{K}$, a cell l_{AB} of the following form:

$$\begin{tikzcd} FA \arrow[rr, "F!_{AB}", "\shortmid"{marking}] \arrow[dr, "l_A"'] & \arrow[d, phantom, "l_{AB}"] & FB \arrow[dl, "l_B"] \\ & L & \end{tikzcd} \quad \text{in } \mathbb{L}.$$

These are required to satisfy the following:

- For $A \xrightarrow{f} B$ in $\mathbb{K}$, the cell

$$\begin{tikzcd} & FA \arrow[dl, "Ff"'] \arrow[d, equal] \\ FB \arrow[r, "F!_{BA}", "\shortmid"{marking}] \arrow[dr, "l_B"'] \arrow[ur, phantom, "F!"] & FA \arrow[d, "l_A"] \\ & L \arrow[u, phantom, "l_{BA}"] \end{tikzcd}$$

  becomes identity.

- For $A, B, C \in \mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "F!_{BC}", "\shortmid"{marking}] \arrow[d, phantom, "F!"] & FC \arrow[d, equal] \\
FA \arrow[rr, "F!_{AC}", "\shortmid"{marking}] \arrow[dr, "l_A"'] & {} \arrow[d, phantom, "l_{AC}"] & FC \arrow[dl, "l_C"] \\
& L &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[dr, "l_A"'] \arrow[r, phantom, shift right=4, "l_{AB}"] & FB \arrow[r, "F!_{BC}", "\shortmid"{marking}] \arrow[d, "l_B"] \arrow[r, phantom, shift right=4, "l_{BC}"] & FC \arrow[dl, "l_C"] \\
& L &
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

Proof. By the first condition for the identities $A \xrightarrow{\mathsf{id}_A} A$ in $\mathbb{K}$, the second condition is extended for loose paths in $\mathbb{K}$ of arbitrary length rather than length 2. Then, we have

$$
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "\vec{F!}", "\shortmid"{marking}] \arrow[d, "Ff"'] & {} \arrow[d, phantom, "F!"] & FA_n \arrow[d, "Fg"] \\
FB \arrow[rr, "F!_{BC}", "\shortmid"{marking}] \arrow[dr, "l_B"'] & {} \arrow[d, phantom, "l_{BC}"] & FC \arrow[dl, "l_C"] \\
& L &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& & FA_0 \arrow[rr, dashed, "\vec{F!}", "\shortmid"{marking}] \arrow[dll, equal] \arrow[d, "Ff"] \arrow[dl, phantom, "F!"] & \arrow[d, phantom, "F!"] & FA_n \arrow[drr, equal] \arrow[d, "Fg"'] \arrow[dr, phantom, "F!"] & & \\
FA_0 \arrow[rr, "F!_{A_0B}", "\shortmid"{marking}] \arrow[drrr, "l_{A_0}"'] & & FB \arrow[rr, "F!_{BC}", "\shortmid"{marking}] \arrow[dr, "l_B"] \arrow[l, phantom, shift right=4, "l_{A_0B}"] & {} \arrow[d, phantom, "l_{BC}"] & FC \arrow[rr, "F!_{CA_n}", "\shortmid"{marking}] \arrow[dl, "l_C"'] \arrow[r, phantom, shift right=4, "l_{CA_n}"] & & FA_n \arrow[dlll, "l_{A_n}"] \\
& & & L & & &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}[column sep=small]
& & FA_0 \arrow[rr, dashed, "\vec{F!}", "\shortmid"{marking}] \arrow[dll, equal] \arrow[d, "Ff"] \arrow[dl, phantom, "F!"] & \arrow[d, phantom, "F!"] & FA_n \arrow[drr, equal] \arrow[d, "Fg"'] \arrow[dr, phantom, "F!"] & & \\
FA_0 \arrow[rr, "F!_{A_0B}", "\shortmid"{marking}] \arrow[d, equal] & & FB \arrow[rr, "F!_{BC}", "\shortmid"{marking}] & {} \arrow[d, phantom, "F!"] & FC \arrow[rr, "F!_{CA_n}", "\shortmid"{marking}] & & FA_n \arrow[d, equal] \\
FA_0 \arrow[rrrrrr, "F!_{A_0A_n}", "\shortmid"{marking}] \arrow[drrr, "l_{A_0}"'] & & & {} \arrow[d, phantom, "l_{A_0A_n}"] & & & FA_n \arrow[dlll, "l_{A_n}"] \\
& & & L & & &
\end{tikzcd}
=
\begin{tikzcd}
FA_0 \arrow[rr, dashed, "\vec{F!}", "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[d, phantom, "F!"] & FA_n \arrow[d, equal] \\
FA_0 \arrow[rr, "F!_{A_0A_n}", "\shortmid"{marking}] \arrow[dr, "l_{A_0}"'] & {} \arrow[d, phantom, "l_{A_0A_n}"] & FA_n \arrow[dl, "l_{A_n}"] \\
& L &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
FA_0 \arrow[r, "F_{A_0A_1}", "\shortmid"{marking}] \arrow[drr, bend right, "l_{A_0}"'] \arrow[r, phantom, shift right=5, "l_{A_0A_1}"] & \cdots \arrow[r, "F_{A_{n-1}A_n}", "\shortmid"{marking}] \arrow[d, phantom, "\cdots"] & FA_n \arrow[dl, bend left, "l_{A_n}"] \arrow[l, phantom, shift left=5, "l_{A_{n-1}A_n}"] \\
& L &
\end{tikzcd}
\quad \text{in } \mathbb{L},
$$

which finishes the proof. ∎

Proposition 3.42. A left F-module with a vertex $M \in \mathbb{L}$ is equivalent to the following data:

- For each object $A \in \mathbb{K}$, a loose arrow $FA \xrightarrow{m_A} M$ in $\mathbb{L}$.
- For objects $A, B \in \mathbb{K}$, a cell m_{AB} of the following form:

$$
\begin{tikzcd}
FA \arrow[r, "F!_{AB}", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "m_B", "\shortmid"{marking}] \arrow[d, phantom, "m_{AB}"] & M \arrow[d, equal] \\
FA \arrow[rr, "m_A"', "\shortmid"{marking}] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

These are required to satisfy the following:

- For each $A \in \mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[rr, "m_A"] \arrow[d, equal] \arrow[dr, equal] & & M \arrow[d, equal] \\
FA \arrow[r, "F!_{AA}"'] \arrow[d, equal] \arrow[ur, phantom, "F!"] & FA \arrow[r, "m_A"'] \arrow[ur, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FA \arrow[rr, "m_A"'] \arrow[urr, phantom, "m_{AA}"] & & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FA \arrow[r, "m_A"'] & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

- For $A, B, C \in \mathbb{K}$,

$$
\begin{tikzcd}
FA \arrow[r, "F!_{AB}"] \arrow[d, equal] & FB \arrow[r, "F!_{BC}"] \arrow[d, phantom, "F!"] & FC \arrow[r, "m_C"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FA \arrow[rr, "F!_{AC}"'] \arrow[d, equal] & & FC \arrow[r, "m_C"'] & M \arrow[d, equal] \\
FA \arrow[rrr, "m_A"'] \arrow[urr, phantom, "m_{AC}"] & & & M
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AB}"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & FB \arrow[r, "F!_{BC}"] \arrow[d, equal] & FC \arrow[r, "m_C"] \arrow[d, phantom, "m_{BC}"] & M \arrow[d, equal] \\
FA \arrow[r, "F!_{AB}"'] \arrow[d, equal] & FB \arrow[rr, "m_B"'] \arrow[d, phantom, "m_{AB}"] & & M \arrow[d, equal] \\
FA \arrow[rrr, "m_A"'] & & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

Proof. We have to show that the above data (m_A, m_{AB}) uniquely extend to a left F-module. If such an extension exists, for each tight arrow f in $\mathbb{K}$, the cell m_f must be defined as follows:

$$
\begin{tikzcd}
FA \arrow[r, "m_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "m_f"] & M \arrow[d, equal] \\
FB \arrow[r, "m_B"'] & M
\end{tikzcd}
:=
\begin{tikzcd}
FA \arrow[rr, "m_A"] \arrow[d, "Ff"'] \arrow[dr, equal] & & M \arrow[d, equal] \\
FB \arrow[r, "F!_{BA}"'] \arrow[d, equal] \arrow[ur, phantom, "F!"] & FA \arrow[r, "m_A"'] \arrow[ur, phantom, "\shortparallel"] & M \arrow[d, equal] \\
FB \arrow[rr, "m_B"'] \arrow[urr, phantom, "m_{BA}"] & & M
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

Let define several cells in $\mathbb{L}$ as follows:

$$
\beta_0 :=
\begin{tikzcd}
& FA \arrow[d, "Ff"] \arrow[dl, equal] \\
FA \arrow[r, "F!_{AB}"'] \arrow[ur, phantom, "F!"] & FB
\end{tikzcd}
\qquad
\delta_0 :=
\begin{tikzcd}
FA \arrow[r, "F!_{AB}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "F!"] & FB \arrow[d, equal] \\
FB \arrow[r, "F!_{BB}"'] & FB
\end{tikzcd}
\qquad
\eta_0 :=
\begin{tikzcd}
& FB \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "F!"] & \\
FB \arrow[rr, "F!_{BB}"'] & & FB
\end{tikzcd}
$$

$$
\gamma := m_{AB} \qquad \sigma := m_{BB} \qquad \beta_1 = \delta_1 = \eta_1 :=
\begin{tikzcd}
M \arrow[d, equal, bend right] \arrow[d, equal, bend left] \arrow[d, phantom, "="] \\
M
\end{tikzcd}
$$

Since the above cells make m_f split, m_f becomes cartesian by Lemma 2.48. Then, we can easily verify that the data (m_A, m_{AB}, m_f) actually give a left F-module. $\square$

Proposition 3.43. When the shape $\mathbb{K}$ of the diagram AVD-functor F is loosewise VD-indiscrete, the axiom of modulations for tight arrows in $\mathbb{K}$ automatically follows from the axiom for loose arrows in $\mathbb{K}$.

Proof. This follows from Propositions 3.41 and 3.42. $\square$

Theorem 3.44 (Strongness theorem)**.** Let $F \colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs, and let $\mathbb{K}$ be either loosewise indiscrete or loosewise VD-indiscrete. Suppose that we are given a tight cocone ξ from F to a vertex $\Xi \in \mathbb{L}$ that satisfies the conditions (L-l)(M1-l). Then, ξ_A has a conjunction for every $A \in \mathbb{K}$, and ξ becomes strong.

Proof. Fix $K \in \mathbb{K}$. Let us define a left F-module m with the vertex FK as follows:

- For each $A \in \mathbb{K}$, $m_A := F!_{AK} \colon FA \to FK$ in $\mathbb{L}$.
- For $A, B \in \mathbb{K}$, m_{AB} is defined as the following cell:

$$\begin{tikzcd} FA \arrow[r, "F!_{AB}"] \arrow[d, equal] & FB \arrow[r, "F!_{BK}"] \arrow[d, phantom, "F!_{ABK}"] & FK \arrow[d, equal] \\ FA \arrow[rr, "F!_{AK}"'] & & FK \end{tikzcd} \quad \text{in } \mathbb{L}.$$

  Here, $!_{ABK}$ is a unique cell in $\mathbb{K}$.

By (L-l), we have a loose arrow $\Xi \xrightarrow{q} FK$ in $\mathbb{L}$ and a modulation $\xi_\dagger q$ of type 1 whose components are cartesian as follows:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "m"] \arrow[d, Rightarrow, "\xi"'] \arrow[dr, phantom, "\xi_\dagger q"] & FK \arrow[d, "\xi_K"] \\ \Xi \arrow[r, "q"'] & FK \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} FA \arrow[r, "m_A = F!_{AK}"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger q)_A \mathsf{cart}"] & FK \arrow[d, equal] \\ \Xi \arrow[r, "q"'] & FK \end{tikzcd} \quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}).$$

We can define a modulation σ of type 1 by $\sigma_A := \xi_{AK}$:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "m"] \arrow[d, Rightarrow, "\xi"'] & FK \arrow[dl, "\xi_K"] \arrow[dl, phantom, bend right, "\sigma"'] \\ \Xi \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} FA \arrow[r, "F!_{AK}"] \arrow[d, "\xi_A"'] & FK \arrow[dl, "\xi_K"] \arrow[dl, phantom, bend right, "\xi_{AK}"'] \\ \Xi \end{tikzcd} \quad \text{in } \mathbb{L} \quad (A \in \mathbb{K}).$$

By (M1-l), we have a cell ε corresponding to the modulation σ:

$$\begin{tikzcd} \Xi \arrow[r, "q"] \arrow[d, equal] & FK \arrow[dl, "\xi_K"] \arrow[dl, phantom, bend right, "\varepsilon"'] \\ \Xi \end{tikzcd} \quad \text{in } \mathbb{L}.$$

Now, we shall show that ε is cartesian. Equivalently, we shall show that q is a conjunction of ξ_K. To show that, let us consider the following cell η:

$$\begin{tikzcd} & FK \arrow[dl, "\xi_K"'] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\ \Xi \arrow[rr, "q"'] & & FK \end{tikzcd} \quad := \quad \begin{tikzcd} & FK \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "F!"] & \\ FK \arrow[rr, "m_K = F!_{KK}"'] \arrow[d, "\xi_K"'] & \arrow[d, phantom, "(\xi_\dagger q)_K"] & FK \arrow[d, equal] \\ \Xi \arrow[rr, "q"'] & & FK \end{tikzcd} \quad \text{in } \mathbb{L}.$$

Then, one of the triangle identities can be shown as follows:

$$\begin{tikzcd} & FK \arrow[dl, "\xi_K"'] \arrow[d, equal] \arrow[dl, phantom, bend left, "\eta"] \\ \Xi \arrow[r, "q"] \arrow[d, equal] & FK \arrow[dl, "\xi_K"] \arrow[dl, phantom, bend right, "\varepsilon"'] \\ \Xi \end{tikzcd} = \begin{tikzcd} & FK \arrow[dl, equal] \arrow[d, equal] \arrow[dl, phantom, bend left, "F!"] \\ FK \arrow[r, "F!_{KK}"'] \arrow[d, "\xi_K"'] \arrow[dr, phantom, "(\xi_\dagger q)_K"] & FK \arrow[d, equal] \\ \Xi \arrow[r, "q"] \arrow[d, equal] & FK \arrow[dl, "\xi_K"] \arrow[dl, phantom, bend right, "\varepsilon"'] \\ \Xi \end{tikzcd} = \begin{tikzcd} & FK \arrow[dl, equal] \arrow[d, equal] \arrow[dl, phantom, bend left, "F!"] \\ FK \arrow[r, "F!_{KK}"'] \arrow[d, "\xi_K"'] & FK \arrow[dl, "\xi_K"] \arrow[dl, phantom, bend right, "\xi!_{KK}"'] \\ \Xi \end{tikzcd} \overset{(\xi)}{=} \begin{tikzcd} FK \arrow[d, bend right, "\xi_K"'] \arrow[d, bend left, "\xi_K"] \arrow[d, phantom, "="] \\ \Xi \end{tikzcd} \quad \text{in } \mathbb{L}.$$

We next prove the other triangle identity. The following calculation shows that a cell $q \to q$, which appears in the triangle identity, is sent to the identity modulation on $m = \xi_* q$ by the functor $\xi_* - \colon \mathbf{Hom}_{\mathbb{L}}(\Xi, FK) \longrightarrow \mathbf{Mdl}(F, FK)$:

$$
\begin{tikzcd}
FA \arrow[r, "m_A = F!_{AK}"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger q)_A"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & FK \arrow[d, equal] \arrow[dl, "\xi_K"'] \\
\Xi \arrow[r, "q"'] \arrow[ur, phantom, "\eta"] & FK
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AK}"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\xi_{AK}"] & FK \arrow[d, equal] \arrow[dl, "\xi_K"'] \\
\Xi \arrow[r, "q"'] \arrow[ur, phantom, "\eta"] & FK
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AK}"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & FK \arrow[d, equal] \arrow[dr, equal] \\
FA \arrow[r, "F!_{AK}"] \arrow[dr, "\xi_A"'] & FK \arrow[r, "F!_{KK}"'] \arrow[u, phantom, "F!"'] \arrow[d, "\xi_K"] \arrow[dr, phantom, "(\xi_\dagger q)_K"] \arrow[l, phantom, "\xi_{AK}"] & FK \arrow[d, equal] \\
& \Xi \arrow[r, "q"'] & FK
\end{tikzcd}
$$

$$
\overset{(\xi_\dagger q)}{=}
\begin{tikzcd}
FA \arrow[r, "F!_{AK}"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & FK \arrow[d, equal] \arrow[dr, equal] \\
FA \arrow[r, "F!_{AK}"] \arrow[d, equal] & FK \arrow[r, "F!_{KK}"'] \arrow[u, phantom, "F!"'] \arrow[d, phantom, "F!"] & FK \arrow[d, equal] \\
FA \arrow[rr, "F!_{AK}"] \arrow[dr, "\xi_A"'] & \arrow[r, phantom, "(\xi_\dagger q)_A"] & FK \arrow[d, equal] \\
& \Xi \arrow[r, "q"'] & FK
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F!_{AK}"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger q)_A"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q"'] & FK
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

Since the functor $\xi_* -$ is fully faithful, we have

$$
\begin{tikzcd}
\Xi \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & FK \arrow[d, equal] \arrow[dl, "\xi_K"'] \\
\Xi \arrow[r, "q"'] \arrow[ur, phantom, "\eta"] & FK
\end{tikzcd}
=
\begin{tikzcd}
\Xi \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q"'] & FK
\end{tikzcd}
\quad \text{in } \mathbb{L}.
$$

Thus $q = {\xi_K}^*$, and the cell ε is cartesian.

Consequently, we have the following for any $A \in \mathbb{K}$:

$$
\begin{tikzcd}
FA \arrow[r, "F!_{AK}"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\xi_{AK}"] & FK \arrow[dl, "\xi_K"] \\
\Xi &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "m_A = F!_{AK}"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "(\xi_\dagger q)_A \colon \mathsf{cart}"] & FK \arrow[d, equal] \\
\Xi \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon \colon \mathsf{cart}"] & FK \arrow[dl, "\xi_K"] \\
\Xi &
\end{tikzcd}
\ \colon \mathsf{cart} \quad \text{in } \mathbb{L}.
$$

This proves that ξ_{AK} is cartesian. $\square$

Corollary 3.45. Let $F \colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs, and let $\mathbb{K}$ be loosewise VD-indiscrete. Then, a vertex of a tight cocone ξ from F has a loose unit in $\mathbb{L}$ if ξ satisfies the conditions (L-l)(L-r)(M1-l)(M1-r)(M2).

Proof. Combine the strongness theorem (Theorem 3.44) and the loosewise dual of the unitality theorem (Theorem 3.38). $\square$

Example 3.46 (Versatile collapses)**.** Let $A := (A^0 \xrightarrow{A^1} A^0, A^e, A^m)$ be a monoid in an AVDC $\mathbb{K}$. Suppose that A^0 has a loose unit in $\mathbb{K}$. Let UA^0 denote the monoid in $\mathbb{K}$ induced by the loose unit on A^0, let $UA^0 \xrightarrow{UA^1} UA^0$ denote the module in $\mathbb{K}$ induced by A^1, and let UA^e and UA^m denote the cells in $\mathbb{M}\mathrm{od}(\mathbb{K})$ induced by A^e and A^m, respectively. Now, we have a monoid $UA := (UA^0, UA^1, UA^e, UA^m)$ in $\mathbb{M}\mathrm{od}(\mathbb{K})$ and the corresponding AVD-functor $F \colon \mathbb{I}^\flat 1 \to \mathbb{M}\mathrm{od}(\mathbb{K})$, where 1 denotes the singleton. Then, the monoid A gives a versatile colimit of F, which is strong. This is an example of a ***versatile collapse*** (Definition 4.3). ♦

Example 3.47. Consider the AVDC $\mathbb{R}\mathrm{el}$ (with loose units) of relations. Let $R \subseteq X \times X$ be an equivalence relation on a (large) set X. Since a monoid in $\mathbb{R}\mathrm{el}$ is simply a (large) preordered set, we have an AVD-functor $F\colon \mathbb{I}^\flat 1 \to \mathbb{R}\mathrm{el}$ corresponding to R. Then, the quotient set X/R becomes a versatile colimit (collapse) of F. However, such a versatile colimit does not exist in general unless the relation R is symmetric. ♦

4. Axiomatization of double categories of profunctors

4.1. The formal construction of enriched categories.

Remark 4.1. Let $\mathbb{X}$ be an AVDC with loose units, and let $\mathbf{A}$ be an $\mathbb{X}$-enriched large category. We now regard $\mathbf{A}$ as an AVD-functor $\mathbf{A}\colon \mathbb{I}^\flat(\mathrm{Ob}\mathbf{A}) \to \mathbb{X}$ as in Proposition 2.61, where $\mathrm{Ob}\mathbf{A}$ denotes the large set of objects in $\mathbf{A}$. Then, we obtain an AVD-functor $F_{\mathbf{A}}\colon \mathbb{I}^\flat(\mathrm{Ob}\mathbf{A}) \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ by post-composing with the embedding Z as in Notation 2.63:

$$\begin{tikzcd} \mathbb{I}^\flat(\mathrm{Ob}\mathbf{A}) \arrow[r, "\mathbf{A}"] \arrow[dr, "F_{\mathbf{A}}"'] & \mathbb{X} \arrow[d, "Z"] \\ & \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \end{tikzcd}$$

♦

Theorem 4.2. Let $\mathbb{X}$ be an AVDC with loose units. Then, every $\mathbb{X}$-enriched large category $\mathbf{A}$ is a versatile colimit of the AVD-functor $F_{\mathbf{A}}\colon \mathbb{I}^\flat(\mathrm{Ob}\mathbf{A}) \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ in Remark 4.1.

Proof. This is a special case of the construction in the proof of Lemma 4.5 and Theorem 4.6. □

Definition 4.3.
- (i) A ***(large) versatile coproduct*** is a versatile colimit of an AVD-functor from $\mathbb{D}\mathrm{S}$ for some (large) set S.
- (ii) A ***versatile collapse*** is a versatile colimit of an AVD-functor from $\mathbb{I}^\flat 1$, where 1 denotes the singleton.
- (iii) A ***(large) versatile collage*** is a versatile colimit of an AVD-functor from $\mathbb{I}^\flat\mathrm{S}$ for some (large) set S. ♦

Remark 4.4. The term "collapse" has been used for similar concepts in a virtual equipment: For a monoid M in a virtual equipment, a tight cocone from M satisfying (T) is called a "collapse" in [Sch15]; The same term is also used in [AM24] for a tight cocone from a monoid satisfying a stronger condition, which coincides with our term "versatile collapse." ♦

Lemma 4.5. For any AVDC $\mathbb{X}$, $\mathbb{X}$-$\mathbb{M}\mathrm{at}$ has all large versatile coproducts.

Proof. Let $(A_i)_{i\in\mathrm{S}}$ be $\mathbb{X}$-colored large sets indexed by a large set S. Let Ξ be a (large) disjoint union of $(A_i)_{i\in\mathrm{S}}$, and let $A_i \xrightarrow{\xi_i} \Xi$ denote the coprojections. We write $(i;x)$ for an element of Ξ, where $x \in A_i$, and define its color by $|(i;x)| := |x|$.

We have to show that Ξ is a versatile coproduct of $(A_i)_{i\in\mathrm{S}}$. The condition (T) follows clearly by the construction. Since the tight arrow part of $\xi_i(x)$ for each $x \in A_i$ is the identity, ξ_i is pulling in $\mathbb{X}$-$\mathbb{M}\mathrm{at}$. The remaining conditions (L-l)(L-r)(M1-l)(M1-r)(M2)(M3) follow directly from the structure of Ξ as a disjoint union. □

Theorem 4.6. Let $\mathbb{K}$ be an AVDC, and let $\mathbf{C}$ be a category. If $\mathbb{K}$ has versatile colimits of any AVD-functors $\mathbb{D}\mathbf{C} \to \mathbb{K}$, then $\mathbb{M}\mathrm{od}(\mathbb{K})$ has versatile colimits of any AVD-functors $\mathbb{I}^\flat\mathbf{C} \to \mathbb{K}$.

Proof. Let $A\colon \mathbb{I}^\flat\mathbf{C} \to \mathbb{M}\mathrm{od}(\mathbb{K})$ be an AVD-functor. Now, A assigns to each object $i \in \mathbf{C}$, a monoid $A_i = (A_i^0 \xrightarrow{A_i^1} A_i^0, A_i^e, A_i^m)$ in $\mathbb{K}$, where A_i^e is the unit and A_i^m is the multiplication, and A also assigns to each pair (i,j) of $i,j\in\mathbf{C}$, a bimodule $A_{ij}=(A^0_i\xrightarrow{A^1_{ij}}A^0_j, A^l_{ij}, A^r_{ij})$ in $\mathbb{K}$, where A^l_{ij} and A^r_{ij} are the left action and the right action, respectively.

Let $F\colon \mathbb{I}^\flat\mathbf{C}\longrightarrow\mathbb{K}$ denote an AVD-functor given by post-composing A with the forgetful functor $\mathbb{M}\mathrm{od}(\mathbb{K})^\flat\longrightarrow\mathbb{K}$. Let $G\colon\mathbb{D}\mathbf{C}\longrightarrow\mathbb{K}$ denote an AVD-functor given by pre-composing F with the inclusion $\mathbb{D}\mathbf{C}\longrightarrow\mathbb{I}^\flat\mathbf{C}$. Let us take a versatile colimit $A^0_i\xrightarrow{\xi^0_i}\Xi^0$ in $\mathbb{K}$ of G. By (M1-r) and (M1-l), there exist, for each $i\in\mathbf{C}$, two loose arrows $A^0_i\overset{\mathfrak{q}_i}{\nrightarrow}\Xi^0\overset{\mathfrak{p}_i}{\nrightarrow}A^0_i$ in $\mathbb{K}$ and modulations $\mathfrak{q}_i{\xi^0}^\dagger$ and ${\xi^0}_\dagger\mathfrak{p}_i$ of type 1 whose components are cartesian:

$$\begin{tikzcd}
A^0_i \arrow[r, "A^1_{ij}"] \arrow[d, equal] \arrow[dr, phantom, "(\mathfrak{q}_i{\xi^0}^\dagger)_j\colon \mathsf{cart}"] & A^0_j \arrow[r, "A^1_{ji}"] \arrow[d, dashed, "\xi^0_j"] \arrow[dr, phantom, "({\xi^0}_\dagger\mathfrak{p}_i)_j\colon \mathsf{cart}"] & A^0_i \arrow[d, equal] \\
A^0_i \arrow[r, "\mathfrak{q}_i"'] & \Xi^0 \arrow[r, "\mathfrak{p}_i"'] & A^0_i
\end{tikzcd} \quad \text{in } \mathbb{K} \quad (i,j\in\mathbf{C}).$$

By (M0-r) for Ξ^0, there exist, for each $i,j\in\mathbf{C}$, a unique cell $\mathfrak{q}_{ij}$ in $\mathbb{K}$ corresponding to a modulation of type 0 on the right side below:

$$\begin{tikzcd}
A^0_i \arrow[r, "A^1_{ij}"] \arrow[d, equal] & A^0_j \arrow[r, "\mathfrak{q}_j"] \arrow[d, phantom, "\mathfrak{q}_{ij}"] & \Xi^0 \arrow[d, equal] \\
A^0_i \arrow[rr, "\mathfrak{q}_i"'] & & \Xi^0
\end{tikzcd} \quad \text{in } \mathbb{K} \quad \Bigg\| \quad
\begin{tikzcd}
A^0_i \arrow[r, "A^1_{ij}"] \arrow[d, equal] & A^0_j \arrow[r, "A^1_{jk}"] \arrow[d, phantom, "A!"] & A^0_k \arrow[d, equal] \\
A^0_i \arrow[rr, "A^1_{ik}"'] & & A^0_k
\end{tikzcd} \quad \text{in } \mathbb{K} \quad (k\in\mathbf{C})$$

Then, $(\mathfrak{q}_i,\mathfrak{q}_{ij})$ uniquely extends to a left F-module $\mathfrak{q}$ by Proposition 3.42 and (M0-r). In particular, $\mathfrak{q}$ is also a left G-module. Thus, by (L-l) for Ξ^0, we obtain a unique loose arrow Ξ^1 in $\mathbb{K}$ and a modulation ${\xi^0}_\dagger\Xi^1$ of type 1 whose components are cartesian:

$$\begin{tikzcd}
A^0_i \arrow[r, "\mathfrak{q}_i"] \arrow[d, "\xi^0_i"'] \arrow[dr, phantom, "({\xi^0}_\dagger\Xi^1)_i\colon \mathsf{cart}"] & \Xi^0 \arrow[d, equal] \\
\Xi^0 \arrow[r, "\Xi^1"'] & \Xi^0
\end{tikzcd} \quad \text{in } \mathbb{K} \quad (i\in\mathbf{C}).$$

In the same way, we can construct a right F-module $\mathfrak{p}=(\mathfrak{p}_i,\mathfrak{p}_{ij})$, a loose arrow $\Xi^{1\prime}$, and a modulation $\Xi^{1\prime}{\xi^0}^\dagger$ of type 1 whose components are cartesian. By replacing $\mathfrak{p}_i$ appropriately, we can assume $\Xi^1=\Xi^{1\prime}$ without loss of generality. We now have cartesian cells as follows:

$$\begin{tikzcd}
A^0_i \arrow[r, "A^1_{ij}"] \arrow[d, "\xi^0_i"'] \arrow[dr, phantom, "\mathsf{cart}"] & A^0_j \arrow[d, "\xi^0_j"] \\
\Xi^0 \arrow[r, "\Xi^1"'] & \Xi^0
\end{tikzcd} =
\begin{tikzcd}
A^0_i \arrow[r, "A^1_{ij}"] \arrow[d, equal] \arrow[dr, phantom, "(\mathfrak{q}_i{\xi^0}^\dagger)_j\colon \mathsf{cart}"] & A^0_j \arrow[d, "\xi^0_j"] \\
A^0_i \arrow[r, "\mathfrak{q}_i"] \arrow[d, "\xi^0_i"'] \arrow[dr, phantom, "({\xi^0}_\dagger\Xi^1)_i\colon \mathsf{cart}"] & \Xi^0 \arrow[d, equal] \\
\Xi^0 \arrow[r, "\Xi^1"'] & \Xi^0
\end{tikzcd} =
\begin{tikzcd}
A^0_i \arrow[r, "A^1_{ij}"] \arrow[d, "\xi^0_i"'] \arrow[dr, phantom, "({\xi^0}_\dagger\mathfrak{p}_j)_i\colon \mathsf{cart}"] & A^0_j \arrow[d, equal] \\
\Xi^0 \arrow[r, "\mathfrak{p}_j"] \arrow[d, equal] \arrow[dr, phantom, "(\Xi^1{\xi^0}^\dagger)_j\colon \mathsf{cart}"] & A^0_j \arrow[d, "\xi^0_j"] \\
\Xi^0 \arrow[r, "\Xi^1"'] & \Xi^0
\end{tikzcd} \quad \text{in } \mathbb{K} \quad (i,j\in\mathbf{C}). \tag{19}$$

By (M2), we have a unique cell Ξ^e below:

$$
\begin{tikzcd}
 & A_i^0 \arrow[d, bend right, "\xi_i^0"'] \arrow[d, bend left, "\xi_i^0"] \arrow[d, phantom, "="] & \\
 & \Xi^0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\Xi^e"] & \\
\Xi^0 \arrow[rr, "\Xi^1"'] & {} & \Xi^0
\end{tikzcd}
=
\begin{tikzcd}
 & A_i^0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "A!"] & \\
A_i^0 \arrow[rr, "A_{ii}^1"'] \arrow[d, "\xi_i^0"'] & \text{cart} & A_i^0 \arrow[d, "\xi_i^0"] \\
\Xi^0 \arrow[rr, "\Xi^1"'] & & \Xi^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad (i \in \mathbf{C}).
$$

By (M0-l), (M0-r), and (M3), we have a unique cell Ξ^m below:

$$
\begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1"] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "\text{cart}"] & A_j^0 \arrow[r, "A_{jk}^1"] \arrow[d, "\xi_j^0"'] \arrow[dr, phantom, "\text{cart}"] & A_k^0 \arrow[d, "\xi_k^0"] \\
\Xi^0 \arrow[r, "\Xi^1"'] \arrow[d, equal] & \Xi^0 \arrow[r, "\Xi^1"'] \arrow[d, phantom, "\Xi^m"] & \Xi^0 \arrow[d, equal] \\
\Xi^0 \arrow[rr, "\Xi^1"'] & & \Xi^0
\end{tikzcd}
=
\begin{tikzcd}
A_i^0 \arrow[r, "A_{ij}^1"] \arrow[d, equal] & A_j^0 \arrow[r, "A_{jk}^1"] \arrow[d, phantom, "A!"] & A_k^0 \arrow[d, equal] \\
A_i^0 \arrow[rr, "A_{ik}^1"'] \arrow[d, "\xi_i^0"'] & \text{cart} & A_k^0 \arrow[d, "\xi_k^0"] \\
\Xi^0 \arrow[rr, "\Xi^1"'] & & \Xi^0
\end{tikzcd}
\quad \text{in } \mathbb{K} \quad (i, j, k \in \mathbf{C}).
$$

Using the functoriality of A and the universal property of versatile colimits, we can verify that $(\Xi^0, \Xi^1, \Xi^e, \Xi^m)$ becomes a monoid Ξ in $\mathbb{K}$.

By the naturality axiom of cells in $\mathbb{M}\mathrm{od}(\mathbb{K})$, the following two composites of cells coincide:

$$
\begin{tikzcd}
 & A_i^0 \arrow[r, "A_i^1"] \arrow[dl, equal] \arrow[d, phantom, "\shortparallel"] & A_i^0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "A!"] & \\
A_i^0 \arrow[r, "A_i^1"'] \arrow[d, equal] & A_i^0 \arrow[r, "A_{ii}^1"'] \arrow[d, phantom, "A_{ii}^l"] & {} & A_i^0 \arrow[d, equal] \\
A_i^0 \arrow[rrr, "A_{ii}^1"'] & & & A_i^0
\end{tikzcd}
=
\begin{tikzcd}
 & A_i^0 \arrow[r, "A_i^1"] \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "A!"] & A_i^0 \arrow[dr, equal] \arrow[d, phantom, "\shortparallel"] & \\
A_i^0 \arrow[r, "A_{ii}^1"'] \arrow[d, equal] & A_i^0 \arrow[r, "A_i^1"'] \arrow[d, phantom, "A_{ii}^r"] & {} & A_i^0 \arrow[d, equal] \\
A_i^0 \arrow[rrr, "A_{ii}^1"'] & & & A_i^0
\end{tikzcd}
\quad \text{in } \mathbb{K}.
$$

Let ξ_i^1 be a cell obtained by the tight composition of the above cell and the cell (19) with $i = j$. Then, we can verify that (ξ_i^0, ξ_i^1) becomes a tight arrow $A_i \xrightarrow{\xi_i} \Xi$ in $\mathbb{M}\mathrm{od}(\mathbb{K})$ for each $i \in \mathbf{C}$.

For objects $i, j \in \mathbf{C}$, the cell (19) yields a cartesian cell ξ_{ij} in $\mathbb{M}\mathrm{od}(\mathbb{K})$ of the following form:

$$
\begin{tikzcd}
A_i \arrow[rr, "A_{ij}"] \arrow[dr, "\xi_i"'] & \xi_{ij} & A_j \arrow[dl, "\xi_j"] \\
 & \Xi &
\end{tikzcd}
\quad : \text{cart} \quad \text{in } \mathbb{M}\mathrm{od}(\mathbb{K}).
$$

Then, the data $(\xi_i, \xi_{ij})_{i,j}$ yield a tight cocone ξ from A with the vertex $\Xi \in \mathbb{M}\mathrm{od}(\mathbb{K})$.

We should show that ξ is a versatile colimit of A. Let us begin with the verification of (T) for ξ. Let $l = (l_i, l_{ij})_{i,j}$ be a tight cocone from A with a vertex $L \in \mathbb{M}\mathrm{od}(\mathbb{K})$. By (T) for the versatile colimit Ξ^0, there is a unique tight arrow $\Xi^0 \xrightarrow{k^0} L^0$ in $\mathbb{K}$ such that, for all i, $\xi_i^0 \fatsemi k^0 = l_i^0$.

By (M1-l) and (M1-r) for the versatile colimit Ξ^0, there is a unique cell k^1 as follows:

$$\begin{tikzcd} A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "\xi_{ij}\colon \mathsf{cart}"] & A_j^0 \arrow[d, "\xi_j^0"] \\ \Xi^0 \arrow[r, "\Xi^1", "\shortmid"{marking}] \arrow[d, "k^0"'] \arrow[dr, phantom, "k^1"] & \Xi^0 \arrow[d, "k^0"] \\ L^0 \arrow[r, "L^1"', "\shortmid"{marking}] & L^0 \end{tikzcd} = \begin{tikzcd} A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, "l_i^0"'] \arrow[dr, phantom, "l_{ij}"] & A_j^0 \arrow[d, "l_j^0"] \\ L^0 \arrow[r, "L^1"', "\shortmid"{marking}] & L^0 \end{tikzcd} \quad \text{in } \mathbb{K} \quad (i, j \in \mathbf{C}).$$

Using (M2)(M1-l)(M1-r)(M3) for Ξ^0, we can verify that (k^0, k^1) becomes a tight arrow $\Xi \xrightarrow{k} L$ in $\mathbb{M}\mathrm{od}(\mathbb{K})$ and that it is a unique one satisfying $\xi ⨾ k = l$.

We next show (L-l) for ξ. Since ξ_i^0 are pulling in $\mathbb{K}$ and since $\mathbb{M}\mathrm{od}(\mathbb{K})$ inherits restrictions from $\mathbb{K}^\flat$ [CS10, 7.4], ξ_i become pulling in $\mathbb{M}\mathrm{od}(\mathbb{K})$. Let $m = (m_i, m_{ij})_{i,j}$ be a left A-module with a vertex $M \in \mathbb{M}\mathrm{od}(\mathbb{K})$. By (L-l) for the versatile colimit Ξ^0, there are loose arrow p^1 and cartesian cells σ_i in $\mathbb{K}$ being a modulation of type 1:

$$\begin{tikzcd} A_i^0 \arrow[r, "m_i^1", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "\sigma_i\colon \mathsf{cart}"] & M^0 \arrow[d, equal] \\ \Xi^0 \arrow[r, "p^1"', "\shortmid"{marking}] & M^0 \end{tikzcd} \quad \text{in } \mathbb{K} \quad (i \in \mathbf{C}).$$

By (M0-l) and (M3) for Ξ^0, there exists a unique cell p^l in $\mathbb{K}$ satisfying the following:

$$\begin{tikzcd} A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] \arrow[dr, phantom, "\xi_{ij}"] & A_j^0 \arrow[r, "m_j^1", "\shortmid"{marking}] \arrow[d, "\xi_j^0"'] \arrow[dr, phantom, "\sigma_j"] & M^0 \arrow[d, equal] \\ \Xi^0 \arrow[r, "\Xi^1", "\shortmid"{marking}] \arrow[d, equal] & \Xi^0 \arrow[r, "p^1", "\shortmid"{marking}] \arrow[d, phantom, "p^l"] & M^0 \arrow[d, equal] \\ \Xi^0 \arrow[rr, "p^1"', "\shortmid"{marking}] & & M^0 \end{tikzcd} = \begin{tikzcd} A_i^0 \arrow[r, "A_{ij}^1", "\shortmid"{marking}] \arrow[d, equal] & A_j^0 \arrow[r, "m_j^1", "\shortmid"{marking}] \arrow[d, phantom, "m_{ij}"] & M^0 \arrow[d, equal] \\ A_i^0 \arrow[rr, "m_i^1", "\shortmid"{marking}] \arrow[d, "\xi_i^0"'] & & M^0 \arrow[d, equal] \\ \Xi^0 \arrow[rr, "p^1"', "\shortmid"{marking}] \arrow[urr, phantom, "\sigma_i"] & & M^0 \end{tikzcd} \quad \text{in } \mathbb{K} \quad (i, j \in \mathbf{C})$$

By (M0-l) for Ξ^0, there exists a unique cell p^r in $\mathbb{K}$ corresponding to a modulation of type 0 on the right below:

$$\begin{tikzcd} \Xi^0 \arrow[r, "p^1", "\shortmid"{marking}] \arrow[d, equal] & M^0 \arrow[r, "M^1", "\shortmid"{marking}] \arrow[d, phantom, "p^r"] & M^0 \arrow[d, equal] \\ \Xi^0 \arrow[rr, "p^1"', "\shortmid"{marking}] & & M^0 \end{tikzcd} \quad \text{in } \mathbb{K} \quad \Bigg\| \quad \begin{tikzcd} A_i^0 \arrow[r, "m_i^1", "\shortmid"{marking}] \arrow[d, equal] & M^0 \arrow[r, "M^1", "\shortmid"{marking}] \arrow[d, phantom, "m_i^r"] & M^0 \arrow[d, equal] \\ A_i^0 \arrow[rr, "m_i^1"', "\shortmid"{marking}] & & M^0 \end{tikzcd} \quad \text{in } \mathbb{K} \quad (i \in \mathbf{C})$$

Then, $p := (p^1, p^l, p^r)$ and the cells σ_i form a loose arrow and cells in $\mathbb{M}\mathrm{od}(\mathbb{K})$. Then, we can verify that the cells σ_i become a modulation (of type 1), which shows (L-l) for ξ. The loosewise dual (L-r) also follows similarly. The rest conditions (M1-l)(M1-r)(M2)(M3) for ξ follow from those for Ξ^0 directly. □

Corollary 4.7. For any AVDC $\mathbb{K}$, $\mathbb{M}\mathrm{od}(\mathbb{K})$ has all versatile collapses.

Proof. Since versatile colimits for the shape $\mathbb{D}1$ are trivial, this follows directly from Theorem 4.6. □

Corollary 4.8. For any AVDC $\mathbb{X}$, $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ has all large versatile collages.

Proof. Combine Lemma 4.5 and Theorem 4.6. □

4.2. Density.

4.2.1. *A general case.*

Definition 4.9. Let $\mathbb{L}$ be an AVDC. An object $A \in \mathbb{L}$ is called ***collage-atomic*** (resp. ***coproduct-atomic***; ***collapse-atomic***) if, for any large versatile collage (resp. coproduct; collapse) $\Xi \in \mathbb{L}$ of $F\colon \mathbb{P}^\flat \mathrm{S} \to \mathbb{L}$ (resp. $\mathbb{D}\mathrm{S} \to \mathbb{L}$; $\mathbb{P}^\flat 1 \to \mathbb{L}$), every tight arrow $A \xrightarrow{f} \Xi$ in $\mathbb{L}$ uniquely factors through a unique coprojection $Fc \xrightarrow{\xi_c} \Xi$:

$$\begin{tikzcd} & A \arrow[dl, "\exists!"'] \arrow[dd, "f"] \\ Fc \arrow[dr, "\xi_c"'] \arrow[r, phantom, "="] & \\ & \Xi \end{tikzcd} \quad \text{in } \mathbb{L} \quad (\exists! c \in \mathrm{S}).$$

♦

Proposition 4.10. Let $\mathbb{X}$ be an AVDC with loose units. An $\mathbb{X}$-enriched large category is collage-atomic in $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ if and only if it is tightwise isomorphic to a semi-object classifier $\mathbf{Z}_c$ for some $c \in \mathbb{X}$.

Proof. Take a versatile collage Ξ of an AVD-functor $A\colon \mathbb{P}^\flat \mathrm{S} \to \mathbb{X}$-$\mathbb{P}\mathrm{rof}$. By the proof of Theorem 4.6, the forgetful AVD-functor $G\colon \mathbb{X}$-$\mathbb{P}\mathrm{rof}^\flat \to \mathbb{X}$-$\mathbb{M}\mathrm{at}$ sends Ξ to a versatile coproduct of $(G\mathbf{A}_i)_{i\in\mathrm{S}}$. Thus, we obtain the following bijections:

$$\mathrm{Hom}_{\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}}\left(\begin{smallmatrix}\mathbf{Z}_c\\ \Xi\end{smallmatrix}\right) \cong \mathrm{Hom}_{\mathbb{X}\text{-}\mathbb{M}\mathrm{at}}\left(\begin{smallmatrix}Y_c\\ G\Xi\end{smallmatrix}\right) \cong \coprod_{i\in\mathrm{S}} \mathrm{Hom}_{\mathbb{X}\text{-}\mathbb{M}\mathrm{at}}\left(\begin{smallmatrix}Y_c\\ G\mathbf{A}_i\end{smallmatrix}\right) \cong \coprod_{i\in\mathrm{S}} \mathrm{Hom}_{\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}}\left(\begin{smallmatrix}Z_c\\ \mathbf{A}_i\end{smallmatrix}\right)$$

This shows that any semi-object classifier $\mathbf{Z}_c$ is collage-atomic in $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$.

To prove the converse direction, take a collage-atomic $\mathbb{X}$-enriched large category $\mathbf{A}$ arbitrarily. By Theorem 4.2, $\mathbf{A}$ can be regarded as a large versatile collage of semi-object classifiers. Since $\mathbf{A}$ is collage-atomic, the identity tight arrow on $\mathbf{A}$ factors through some coprojection $\mathbf{Z}_c \xrightarrow{x} \mathbf{A}$:

$$\begin{tikzcd} & \mathbf{A} \arrow[dl, "\exists! K"'] \arrow[dd, equal] \\ \mathbf{Z}_c \arrow[dr, "x"'] \arrow[r, phantom, "="] & \\ & \mathbf{A} \end{tikzcd} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}.$$

Since $\mathbf{Z}_c$ is also collage-atomic, the tight arrow x must uniquely factor through itself. Thus we have $x ⨟ K = \mathsf{id}$ and $\mathbf{A} \cong \mathbf{Z}_c$. □

A similar proof to Proposition 4.10 works for the following propositions:

Proposition 4.11. Let $\mathbb{K}$ be an AVDC with loose units. Then, $A \in \mathbb{M}\mathrm{od}(\mathbb{K})$ is collapse-atomic if and only if it is tightwise isomorphic to Uc for some $c \in \mathbb{X}$.

Proposition 4.12. Let $\mathbb{X}$ be an AVDC. Then, $A \in \mathbb{X}$-$\mathbb{M}\mathrm{at}$ is coproduct-atomic if and only if it is tightwise isomorphic to Yc for some $c \in \mathbb{X}$.

Definition 4.13. Let $\mathbb{L}$ be an AVDC. A full sub-AVDC $\mathbb{X} \subseteq \mathbb{L}$ is called ***collage-dense*** (resp. ***coproduct-dense***; ***collapse-dense***) if it satisfies following:

- Every object in $\mathbb{X}$ is collage-atomic (resp. coproduct-atomic; collapse-atomic) in $\mathbb{L}$.
- Every object in $\mathbb{L}$ can be written as a large versatile collage (resp. a large versatile coproduct; a versatile collapse) of objects from $\mathbb{X}$. ♦

Remark 4.14. Collage-dense full sub-AVDCs are called ***Cauchy generator*** in the bicategorical setting [Str04]. ◆

Proposition 4.15. Let $\mathbb{X}$ be an AVDC.

(i) If $\mathbb{X}$ has loose units, the full sub-AVDC given by $\mathbb{X} \xhookrightarrow{Z} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ is collage-dense.
(ii) The full sub-AVDC given by $\mathbb{X} \xhookrightarrow{Y} \mathbb{X}\text{-}\mathbb{M}\mathrm{at}$ is coproduct-dense.
(iii) If $\mathbb{X}$ has loose units, the full sub-AVDC given by $\mathbb{X} \xhookrightarrow{U} \mathbb{M}\mathrm{od}(\mathbb{X})$ is collapse-dense.

4.2.2. *The case of virtual equipments.*

Notation 4.16. Let $\mathbb{L}$ be an AVDC, and let $\mathbb{X} \subseteq \mathbb{L}$ be a full sub-AVDC. For an object $L \in \mathbb{L}$, let $\mathbf{T}\mathbb{X}/L$ denote a category defined as follows:

- An object is a pair (X, x) of an object $X \in \mathbb{X}$ and a tight arrow $X \xrightarrow{x} L$ in $\mathbb{L}$.
- A morphism $(X, x) \to (X', x')$ is a tight arrow $X \xrightarrow{f} X'$ in $\mathbb{L}$ such that $f ⨾ x' = x$.

Given $(X, x) \in \mathbf{T}\mathbb{X}/L$, we write Dx for X and identify x with $(Dx, x) \in \mathbf{T}\mathbb{X}/L$. ◆

Definition 4.17. Let $\mathbf{C}$ be a category. An object $m \in \mathbf{C}$ is called ***maximal*** if every parallel morphisms $m \rightrightarrows \cdot$ have a common retraction. Let $\mathbf{Max}(\mathbf{C}) \subseteq \mathbf{C}$ denote the full subcategory of all maximal objects in a category $\mathbf{C}$. ◆

Remark 4.18. The category $\mathbf{Max}(\mathbf{C})$ always becomes a simply connected groupoid. That is, $\mathbf{Max}(\mathbf{C})$ has at most one morphism between any two objects, and such a morphism is an isomorphism. ◆

Definition 4.19. A category $\mathbf{C}$ is called C***-discrete*** if:

- The isomorphism classes of $\mathbf{Max}(\mathbf{C})$ form a large set;
- The inclusion functor $\mathbf{Max}(\mathbf{C}) \hookrightarrow \mathbf{C}$ is final. ◆

Lemma 4.20. The following are equivalent for a category $\mathbf{C}$:

(i) $\mathbf{C}$ is C-discrete.
(ii) There is a final functor $\mathrm{S} \to \mathbf{C}$ from a large discrete category S.
(iii) There is a large set S of objects in $\mathbf{C}$ such that any object in $\mathbf{C}$ has a unique morphism from itself whose codomain lies in S.

Moreover, if these conditions are satisfied, the large set S above becomes isomorphic to a skeleton of $\mathbf{Max}(\mathbf{C})$.

Proof. [(i) $\Longrightarrow$ (ii)] Let S be a skeleton of $\mathbf{Max}(\mathbf{C})$. Since $\mathbf{Max}(\mathbf{C})$ is a simply connected groupoid, the inclusion functor $\mathrm{S} \hookrightarrow \mathbf{Max}(\mathbf{C})$ is final. Since finality is closed under composition, the composite of the inclusions $\mathrm{S} \hookrightarrow \mathbf{Max}(\mathbf{C}) \hookrightarrow \mathbf{C}$ gives a desired final functor.

[(ii) $\Longrightarrow$ (iii)] Let $\Phi \colon \mathrm{S} \to \mathbf{C}$ be a final functor from a large discrete category. By the finality, Φ becomes injective on objects. Then, the image of Φ gives a desired class of objects in $\mathbf{C}$.

[(iii) $\Longrightarrow$ (i)] Let $\mathrm{S} \subseteq \mathrm{Ob}\mathbf{C}$ be the large set in the condition (iii). Let $s \in \mathrm{S}$, and let $f, g \colon s \rightrightarrows c$ be morphisms in $\mathbf{C}$. By the assumption, there is a morphism $h \colon c \to s'$ such that $s' \in \mathrm{S}$. By the uniqueness, we have $f ⨾ h = \mathsf{id} = g ⨾ h$, which shows that s is maximal in $\mathbf{C}$. Thus, the inclusion $\mathrm{S} \hookrightarrow \mathbf{C}$ factors through $\mathbf{Max}(\mathbf{C}) \subseteq \mathbf{C}$, where S is regarded as a large discrete category. Since $\mathrm{S} \hookrightarrow \mathbf{C}$ is final and the inclusion $\mathbf{Max}(\mathbf{C}) \hookrightarrow \mathbf{C}$ is full, the functor $\mathrm{S} \to \mathbf{Max}(\mathbf{C})$ becomes final, hence $\mathbf{Max}(\mathbf{C}) \hookrightarrow \mathbf{C}$ is final. Furthermore, S gives a large skeleton of $\mathbf{Max}(\mathbf{C})$. □

Definition 4.21. Let $\mathbb{E}$ be an AVDC with restrictions. Let $\mathbb{X} \subseteq \mathbb{E}$ be a full sub-AVDC. Fix an object $E \in \mathbb{E}$.

(i) We define an AVD-functor $K_E \colon \mathbb{I}^\flat(\mathbf{T}\mathbb{X}/E) \to \mathbb{X}$ as follows:

- For $x \in \mathbf{T}\mathbb{X}/E$, $K_E(x) := Dx$.
- For $x, y \in \mathbf{T}\mathbb{X}/E$, $K_E(!_{xy}) := E(x, y)$.

$$\begin{tikzcd} Dx \arrow[rr, "K_E(!_{xy})", "\shortmid"{marking}] \arrow[dr, "x"'] \arrow[rr, phantom, shift right=4ex, "\mathsf{cart}"] & & Dy \arrow[dl, "y"] \\ & E & \end{tikzcd} \quad \text{in } \mathbb{E}. \tag{20}$$

- For $x_0, \ldots, x_n \in \mathbf{T}\mathbb{X}/E$ and $x_0 \xrightarrow{f} y, x_n \xrightarrow{g} z$ in $\mathbf{T}\mathbb{X}/E$, the assignment to the unique cell $!$ in $\mathbb{I}^\flat(\mathbf{T}\mathbb{X}/E)$ is defined using the universality of the restrictions:

$$\begin{tikzcd} Dx_0 \arrow[r, "K_E(!_{x_0x_1})", "\shortmid"{marking}] \arrow[d, "f"'] & \cdots \arrow[r, "K_E(!_{x_{n-1}x_n})", "\shortmid"{marking}] \arrow[d, phantom, "K_E(!)"] & Dx_n \arrow[d, "g"] \\ Dy \arrow[rr, "K_E(!_{yz})", "\shortmid"{marking}] \arrow[dr, "y"'] & \mathsf{cart} & Dz \arrow[dl, "z"] \\ & E & \end{tikzcd} = \begin{tikzcd} Dx_0 \arrow[r, "K_E(!)", "\shortmid"{marking}] \arrow[drr, "x_0"'] & Dx_1 \arrow[r, "K_E(!)", "\shortmid"{marking}] \arrow[dr, "x_1"] & \cdots \arrow[r, "K_E(!)", "\shortmid"{marking}] & Dx_{n-1} \arrow[r, "K_E(!)", "\shortmid"{marking}] \arrow[dl, "x_{n-1}"'] & Dx_n \arrow[dll, "x_n"] \\ & \mathsf{cart} & E & \mathsf{cart} & \end{tikzcd} \quad \text{in } \mathbb{E}.$$

(ii) Furthermore, the cartesian cells (20) yield a tight cocone $K_E \Rightarrow E$, which is denoted by κ_E. ♦

Theorem 4.22 (The density theorem)**.** Let $\mathbb{E}$ be an AVDC with restrictions. For a full sub-AVDC $\mathbb{X} \subseteq \mathbb{E}$ whose objects are collage-atomic in $\mathbb{E}$, the following are equivalent:

(i) $\mathbb{X} \subseteq \mathbb{E}$ is collage-dense.
(ii) For every $E \in \mathbb{E}$, the tight cocone κ_E of Definition 4.21 is a versatile colimit and the category $\mathbf{T}\mathbb{X}/E$ is C-discrete.

Proof. [(ii) $\implies$ (i)] Since $\mathbf{T}\mathbb{X}/E$ is C-discrete, there is a final functor $\Phi\colon \mathrm{S} \to \mathbf{T}\mathbb{X}/E$ from a large discrete category S. By Proposition 3.26, Φ induces a final AVD-functor $\mathbb{I}^\flat\Phi\colon \mathbb{I}^\flat\mathrm{S} \to \mathbb{I}^\flat(\mathbf{T}\mathbb{X}/E)$. Then, Theorem 3.28 makes $(\kappa_E)_{\mathbb{I}^\flat\Phi}$ be a versatile collage.

[(i) $\implies$ (ii)] Fix $E \in \mathbb{E}$. Let S be a large set, and let $F\colon \mathbb{I}^\flat\mathrm{S} \to \mathbb{L}$ be an AVD-functor such that $Fi \in \mathbb{X}$ for any $i \in \mathrm{S}$. Let ξ be a tight cocone that exhibits E as a versatile colimit of F. Then, the following assignment yields a functor $\Phi\colon \mathrm{S} \to \mathbf{T}\mathbb{X}/E$:

$$i \in \mathrm{S} \quad \overset{\Phi}{\longmapsto} \quad \begin{tikzcd} Fi \arrow[d, "\xi_i"] \\ E \end{tikzcd} \quad \text{in } \mathbf{T}\mathbb{X}/E.$$

By the definition of collage-atomic objects, the functor Φ becomes final, hence $\mathbf{T}\mathbb{X}/E$ is C-discrete. By virtue of the strongness theorem (Theorem 3.44), we have an invertible AVD-transformation of the following form:

$$\begin{tikzcd} \mathbb{I}^\flat\mathrm{S} \arrow[rr, "F"] \arrow[dr, "\mathbb{I}^\flat\Phi"'] & \Downarrow\cong & \mathbb{E} \\ & \mathbb{I}^\flat(\mathbf{T}\mathbb{X}/E) \arrow[ur, "K_E"'] & \end{tikzcd} \quad \text{in } \mathcal{AVDC}.$$

By Proposition 3.26, the induced AVD functor $\mathbb{I}^\flat\Phi$ is final. Then, Theorem 3.28 implies that the canonical tight cocone κ_L becomes a versatile colimit. □

4.3. Characterization theorems.

Construction 4.23 (Nerve construction)**.** Let $\mathbb{X} \subseteq \mathbb{L}$ be a full sub-AVDC of an AVDC. Suppose that the following conditions hold for every $L \in \mathbb{L}$:

- The category $\mathbf{T}\mathbb{X}/L$ is C-discrete;
- $\mathbf{Max}(\mathbf{T}\mathbb{X}/L)$ has a skeleton whose elements are pulling in $\mathbb{L}$.

Then, we can construct an AVD-functor $N\colon \mathbb{L}^\flat \to \mathbb{X}\text{-}\mathbb{M}\mathrm{at}$ as follows:

(i) Fix $L \in \mathbb{L}$. We choose a skeleton S_L of $\mathbf{Max}(\mathbf{T}\mathbb{X}/L)$ whose elements are pulling in $\mathbb{L}$ and define $NL := \mathrm{S}_L$. For $x \in NL$, its color is defined by $|x| := Dx$.

(ii) For a tight arrow $A \xrightarrow{f} B$ in $\mathbb{L}$, we write Nf for a morphism $NA \to NB$ defined as follows: Let $x \in NA$; since $\mathbf{T}\mathbb{X}/B$ is C-discrete, the tight arrow $x \mathbin{⨟} f$ uniquely factors through a unique $(Nf)^0x \in NB$:

$$\begin{tikzcd} & {|x|} \arrow[dl, "x"'] \arrow[dr, "(Nf)^1x"] & \\ A \arrow[dr, "f"'] & = & {|y|} \arrow[dl, "(Nf)^0x"] \\ & B & \end{tikzcd} \quad \text{in } \mathbb{L},$$

which gives a morphism $x \mapsto (Nf)x$.

(iii) For a loose arrow $A \overset{u}{\nrightarrow} B$ in $\mathbb{L}$, we write Nu for a matrix $NA \nrightarrow NB$ over $\mathbb{X}$ defined as follows: For $x \in NA$ and $y \in NB$, the loose arrow $(Nu)(x,y)$ is defined as a restriction:

$$\begin{tikzcd} {|x|} \arrow[r, "(Nu)(x{,}y)", "\shortmid"{marking}] \arrow[d, "x"'] \arrow[dr, phantom, "\mathsf{cart}"] & {|y|} \arrow[d, "y"] \\ A \arrow[r, "u"', "\shortmid"{marking}] & B \end{tikzcd} \quad \text{in } \mathbb{L}.$$

(iv) For a cell

$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A_n \arrow[d, "g"] \\ B \arrow[r, "v"', "\shortmid"{marking}] & C \end{tikzcd} \quad \text{in } \mathbb{L},$$

we write $N\alpha$ for a cell in $\mathbb{X}\text{-}\mathbb{M}\mathrm{at}$ defined by the following:

$$\begin{tikzcd} {|x_0|} \arrow[r, "Nu_1(x_0{,}x_1)", "\shortmid"{marking}] \arrow[d, equal] & {|x_1|} \arrow[r, "Nu_2(x_1{,}x_2)", "\shortmid"{marking}] \arrow[d, phantom, "(N\alpha)_{x_0x_1\ldots x_n}"] & \cdots \arrow[r, "Nu_n(x_{n-1}{,}x_n)", "\shortmid"{marking}] & {|x_n|} \arrow[d, equal] \\ {|(Nf)^0x_0|} \arrow[rrr, "Nv((Nf)^0x_0{,}(Ng)^0x_n)"', "\shortmid"{marking}] \arrow[d, "(Nf)^0x_0"'] & & & {|(Ng)^0x_n|} \arrow[d, "(Ng)^0x_n"] \\ B \arrow[rrr, "v"', "\shortmid"{marking}] \arrow[rrru, phantom, "\mathsf{cart}"] & & & C \end{tikzcd}$$

$$= \begin{tikzcd} {|x_0|} \arrow[r, "Nu_1(x_0{,}x_1)", "\shortmid"{marking}] \arrow[d, "x_0"'] \arrow[dr, phantom, "\mathsf{cart}"] & {|x_1|} \arrow[r, "Nu_2(x_1{,}x_2)", "\shortmid"{marking}] \arrow[d, "x_1"] \arrow[dr, phantom, "\mathsf{cart}"] & \cdots \arrow[r, "Nu_n(x_{n-1}{,}x_n)", "\shortmid"{marking}] \arrow[dr, phantom, "\mathsf{cart}"] & {|x_n|} \arrow[d, "x_n"] \\ A_0 \arrow[r, "u_1"', "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "u_2"', "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "u_n"', "\shortmid"{marking}] & A_n \arrow[d, "g"] \\ B \arrow[rrr, "v"', "\shortmid"{marking}] & & & C \end{tikzcd} \quad \text{in } \mathbb{L}.$$

♦

Theorem 4.24. The following are equivalent for an AVDC $\mathbb{L}$:

(i) $\mathbb{L}$ is equivalent to $\mathbb{X}$-$\mathbb{P}$rof for some AVDC $\mathbb{X}$ with loose units.
(ii) $\mathbb{L}$ has large versatile collages and a collage-dense full sub-AVDC.

Proof. [(i) $\Longrightarrow$ (ii)] This follows from Corollary 4.8 and Proposition 4.15.

[(ii) $\Longrightarrow$ (i)] In what follows, we write I for the inclusion AVD-functor $\mathbb{X} \hookrightarrow \mathbb{L}$. We first show that the conditions of Construction 4.23 are satisfied for every $L \in \mathbb{L}$. By the collage-density, there are a large set S_L, an AVD-functor $F_L \colon \mathbb{I}^\flat \mathrm{S}_L \to \mathbb{L}$ factoring through $\mathbb{X}$, and a tight cocone ξ^L exhibiting L as a versatile colimit of F_L. Then, by the collage-atomicity, the assignment $s \mapsto \xi^L_s$ yields a final functor $\mathrm{S}_L \to \mathbf{T}\mathbb{X}/L$, which implies C-discreteness. Moreover, the large set $\mathrm{S}_L \cong \{\xi^L_s \mid s \in \mathrm{S}_L\}$ gives a skeleton of $\mathbf{Max}(\mathbf{T}\mathbb{X}/L)$ whose elements are pulling in $\mathbb{L}$. Thus, we obtain the AVD-functor $N \colon \mathbb{L}^\flat \to \mathbb{X}$-$\mathbb{M}$at of Construction 4.23. By Corollary 3.45, $\mathbb{L}$ has all loose units, hence we have the AVD-functor $\mathscr{N} \colon \mathbb{L} \to \mathbb{M}\mathrm{od}(\mathbb{X}\text{-}\mathbb{M}\mathrm{at}) = \mathbb{X}$-$\mathbb{P}$rof corresponding to N.

Let $L \in \mathbb{L}$. By the bijection $\mathrm{S}_L \cong \{\xi^L_s \mid s \in \mathrm{S}_L\}$, the $\mathbb{X}$-enriched large category $\mathbf{N}L := \mathscr{N}(L)$ can be regarded as an AVD-functor of the following form:

$$\mathbb{I}^\flat \mathrm{S}_L \xrightarrow{\mathbf{N}L} \mathbb{X} \xhookrightarrow{I} \mathbb{L}.$$

For $s, t \in \mathrm{S}_L$, $I \circ \mathbf{N}L$ sends the unique loose arrow $!_{st}$ in $\mathbb{I}^\flat \mathrm{S}_L$ to the following restriction:

$$\begin{tikzcd} F_L s \arrow[r, "\mathbf{N}L(\xi^L_s, \xi^L_t)", "\shortmid"{marking}] \arrow[d, "\xi^L_s"'] \arrow[dr, phantom, "\mathsf{cart}"] & F_L t \arrow[d, "\xi^L_t"] \\ L \arrow[r, "\mathsf{U}_L"', "\shortmid"{marking}] & L \end{tikzcd} \quad \text{in } \mathbb{L},$$

where U_L denotes the loose unit on L. Then, by the strongness theorem (Theorem 3.44), $I \circ \mathbf{N}L$ becomes isomorphic to F_L. In what follows, we will regard $F_L = I \circ \mathbf{N}L$.

To show that $\mathscr{N}$ is an equivalence, we will use Theorem 2.15. Let $A, B \in \mathbb{L}$. Since A is a versatile collage of F_A, by (T), the tight arrows $A \longrightarrow B$ in $\mathbb{L}$ bijectively correspond to the tight cocones from F_A with the vertex B. By the collage-atomicity and $F_A = \mathbf{N}A$, those tight cocones correspond to the $\mathbb{X}$-functors $\mathbf{N}A \longrightarrow \mathbf{N}B$.

Take arbitrary data on the left below:

$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] & A_n \arrow[d, "g"] \\ B \arrow[r, dotted, "v"', "\shortmid"{marking}] & C \end{tikzcd} \quad \text{in } \mathbb{L} \qquad \begin{tikzcd} \mathbf{N}A_0 \arrow[r, dashed, "\mathscr{N}\vec{u}", "\shortmid"{marking}] \arrow[d, "\mathscr{N}f"'] & \mathbf{N}A_n \arrow[d, "\mathscr{N}g"] \\ \mathbf{N}B \arrow[r, dotted, "\mathscr{N}v"', "\shortmid"{marking}] & \mathbf{N}C \end{tikzcd} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \tag{21}$$

Using (M1-l)(M1-r)(M2)(M3) for the versatile collages A_i of F_{A_i}, we can straightforwardly show that the cells fitting into the left of (21) correspond to the cells fitting into the right of (21).

Take $\mathbf{A} \in \mathbb{X}$-$\mathbb{P}$rof arbitrarily. Regarding $\mathbf{A}$ as an AVD-functor, we can take a versatile collage ζ with a vertex $Z \in \mathbb{L}$ from the following AVD-functor:

$$\mathbb{I}^\flat \mathrm{Ob}\mathbf{A} \xrightarrow{\mathbf{A}} \mathbb{X} \xhookrightarrow{I} \mathbb{L}.$$

Let $s \in \mathrm{S}_Z$. Since $F_Z s \in \mathbb{L}$ is collage-atomic, the tight arrow ξ^Z_s uniquely factors through $\zeta_{Q^0 s}$ for a unique object $Q^0 s \in \mathbf{A}$:

$$\begin{tikzcd} & F_Z s \arrow[dl, "Q^1 s"'] \arrow[dd, "\xi^Z_s"] \\ {|Q^0 s|}_{\mathbf{A}} \arrow[dr, "\zeta_{Q^0 s}"'] \arrow[r, phantom, "="] & {} \\ & Z \end{tikzcd} \quad \text{in } \mathbb{L}.$$

By the strongness theorem (Theorem 3.44) and the universal property of restrictions, there is a unique cell Q_{st} for $s,t \in \mathrm{S}_Z$ as follows:

$$\begin{tikzcd} F_Z s \arrow[r, "F_Z(!_{st})", "\shortmid"{marking}] \arrow[d, "Q^1 s"'] \arrow[dr, phantom, "Q_{st}"] & F_Z t \arrow[d, "Q^1 t"] \\ {|Q^0 s|} \arrow[r, "\mathbf{A}(Q^0 s, Q^0 t)", "\shortmid"{marking}] \arrow[dr, "\zeta_{Q^0 s}"'] & {|Q^0 t|} \arrow[d, "\zeta_{Q^0 t}"] \arrow[dl, phantom, "\zeta_{Q^0 s Q^0 t}"] \\ & Z \end{tikzcd} = \begin{tikzcd} F_Z s \arrow[rr, "F_Z(!_{st})", "\shortmid"{marking}] \arrow[dr, "\xi^Z_s"'] & \arrow[d, phantom, "\xi^Z_{st}"] & F_Z t \arrow[dl, "\xi^Z_t"] \\ & Z & \end{tikzcd} \quad \text{in } \mathbb{L},$$

which gives an invertible $\mathbb{X}$-functor $Q\colon \mathbf{N}Z \overset{\cong}{\to} \mathbf{A}$.

Let $Q\colon \mathbf{N}Z \overset{\cong}{\to} \mathbf{A}$ and $R\colon \mathbf{N}W \overset{\cong}{\to} \mathbf{B}$ be the invertible $\mathbb{X}$-functors constructed above for $\mathbf{A},\mathbf{B} \in \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$. Let $\mathbf{A} \xrightarrow{P} \mathbf{B}$ be an $\mathbb{X}$-profunctor. Then, by (L-l) for Z and (L-r) for W, we obtain a loose arrow $Z \xrightarrow{p} W$ in $\mathbb{L}$ and a loosewise invertible cell of the following form:

$$\begin{tikzcd} \mathbf{N}Z \arrow[r, "\mathscr{N}p", "\shortmid"{marking}] \arrow[d, "Q"', "\cong"] \arrow[dr, phantom, "\cong"] & \mathbf{N}W \arrow[d, "R", "\cong"'] \\ \mathbf{A} \arrow[r, "P"', "\shortmid"{marking}] & \mathbf{B} \end{tikzcd} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}.$$

Then, we conclude that the AVD-functor $\mathscr{N}\colon \mathbb{L} \to \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ becomes an equivalence. □

We can also prove the following theorems in a similar way to Theorem 4.24:

Theorem 4.25. The following are equivalent for an AVDC $\mathbb{L}$:

(i) $\mathbb{L}$ is equivalent to $\mathbb{X}\text{-}\mathbb{M}\mathrm{at}$ for some AVDC $\mathbb{X}$.
(ii) $\mathbb{L}$ is diminished and has large versatile coproducts and a coproduct-dense full sub-AVDC.

Theorem 4.26. The following are equivalent for an AVDC $\mathbb{L}$:

(i) $\mathbb{L}$ is equivalent to $\mathbb{M}\mathrm{od}(\mathbb{K})$ for some AVDC $\mathbb{K}$ with loose units.
(ii) $\mathbb{L}$ has versatile collapses and a collapse-dense full sub-AVDC.

4.4. **Closedness under slicing.** In this subsection, we prove that the AVDCs of profunctors are closed under "slicing" as a direct consequence of our characterization theorems. We first generalize to AVDCs the notion of slice double categories [Par11], which has been denoted by the double slash "$/\!\!/$."

Definition 4.27. Let $\mathbb{L}$ be an AVDC, and let $L \in \mathbb{L}$. The ***slice*** AVDC, denoted by $\mathbb{L}/L$, is the AVDC defined by the following:

- The tight category is $\mathbf{T}\mathbb{L}/L$;
- A loose arrow $x \xrightarrow{u} y$ in $\mathbb{L}/L$ is a pair (Du, u) of a loose arrow Du and a cell u

$$\begin{tikzcd} Dx \arrow[rr, "Du", "\shortmid"{marking}] \arrow[dr, "x"'] & \arrow[d, phantom, "u"] & Dy \arrow[dl, "y"] \\ & L & \end{tikzcd} \quad \text{in } \mathbb{L};$$

- A cell $\alpha \in \mathrm{Cell}_{\mathbb{L}/L}\left(f \overset{\vec{u}}{\underset{v}{}} g \right)$ is a cell in $\mathbb{L}$ satisfting the following:

$$\begin{tikzcd} Dx_0 \arrow[r, "Du_1"] \arrow[d, "f"'] & \cdots \arrow[r, "Du_n"] & Dx_n \arrow[d, "g"] \\ Dy \arrow[rr, dotted, "Dv"] \arrow[dr, "y"'] \arrow[rr, phantom, "\alpha", shift left=4ex] & & Dz \arrow[dl, "z"] \\ & L \arrow[u, phantom, "v"] & \end{tikzcd} = \begin{tikzcd} Dx_0 \arrow[r, "Du_1"] \arrow[dr, "x_0"'] & \cdots \arrow[r, "Du_n"] \arrow[d, phantom, "u_1 \cdots u_n"] & Dx_n \arrow[dl, "x_n"] \\ & L & \end{tikzcd} \quad \text{in } \mathbb{L}.$$

We write $D_L \colon \mathbb{L}/L \to \mathbb{L}$ for the canonical AVD-functor defined as $x \mapsto Dx$. For a full sub-AVDC $\mathbb{X} \subseteq \mathbb{L}$, we write $\mathbb{X}/L \subseteq \mathbb{L}/L$ for the full sub-AVDC consisting of objects $x \in \mathbb{L}/L$ such that $Dx \in \mathbb{X}$. ♦

Lemma 4.28. Let $F \colon \mathbb{K} \to \mathbb{L}$ be an AVD-functor between AVDCs. Then, a tight cocone from F with a vertex $L \in \mathbb{L}$ is the same thing as an AVD-functor $\mathbb{K} \to \mathbb{L}/L$ where the post-composite with $D_L \colon \mathbb{L}/L \to \mathbb{L}$ is F.

$$\begin{tikzcd} \mathbb{K} \arrow[r] \arrow[dr, "F"'] & \mathbb{L}/L \arrow[d, "D_L"] \\ & \mathbb{L} \end{tikzcd}$$

Lemma 4.29. Let $\mathbb{L}$ be an AVDC, and let $L \in \mathbb{L}$. Let $G \colon \mathbb{K} \to \mathbb{L}/L$ be an AVD-functor from an AVDC. Suppose that we are given a versatile colimit ξ of $D_L G$ with a vertex $\Xi \in \mathbb{L}$. Then, there is a versatile colimit of G, which is sent to ξ by D_L.

Proof. Let l denote the tight cocone from $D_L G$ associated with G, and let $L \in \mathbb{L}$ be its vertex. By (T) for the versatile colimit ξ, we obtain the canonical tight arrow $\Xi \xrightarrow{k} L$ in $\mathbb{L}$. Then, the AVD-functor $H \colon \mathbb{K} \to \mathbb{L}/\Xi$ corresponding to ξ makes the following diagram commute:

$$\begin{tikzcd} \mathbb{K} \arrow[r, "H"] \arrow[drr, "G"'] & \mathbb{L}/\Xi \cong & (\mathbb{L}/L)/k \arrow[d, "D_k"] \\ & & \mathbb{L}/L \end{tikzcd}$$

This gives a tight cocone from G with the vertex k, which becomes a versatile colimit of G straightforwardly. □

Lemma 4.30. Let $\mathbb{X} \subseteq \mathbb{L}$ be a collage-dense (resp. collapse-dense) full sub-AVDC of an AVDC, and let $L \in \mathbb{L}$. Then, $\mathbb{X}/L \subseteq \mathbb{L}/L$ also becomes collage-dense (resp. collapse-dense).

Proof. This follows from Lemma 4.29 directly. □

By the characterization theorems (Theorems 4.24 and 4.26), we now have the following:

Corollary 4.31. Let $\mathbb{X}$ be an AVDC with loose units.

(i) For an $\mathbb{X}$-enriched category $\mathbf{A}$, there is an equivalence $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}/\mathbf{A} \simeq (\mathbb{X}/\mathbf{A})\text{-}\mathbb{P}\mathrm{rof}$ in $\mathcal{AVDC}$.

(ii) For a monoid M in $\mathbb{X}$, there is an equivalence $\mathbb{M}\mathrm{od}(\mathbb{X})/M \simeq \mathbb{M}\mathrm{od}(\mathbb{X}/M)$ in $\mathcal{AVDC}$.

Remark 4.32. Corollary 4.31(i) is a double categorical refinement of the result in [FL24], which treats the (strict) slice 2-category of the 2-category of enriched categories and functors over a bicategory. ♦

References

[AM24] N. Arkor and D. McDermott. *The nerve theorem for relative monads*. coming soon at TAC. 2024. arXiv: 2404.01281 [math.CT]. URL: https://arxiv.org/abs/2404.01281 (cit. on p. 45).

[Bur71] A. Burroni. "T-catégories (catégories dans un triple)". In: *Cahiers Topologie Géom. Différentielle* 12 (1971), pp. 215–321 (cit. on p. 6).

[CKW87] A. Carboni, S. Kasangian, and R. Walters. "An axiomatics for bicategories of modules". In: *J. Pure Appl. Algebra* 45.2 (1987), pp. 127–141.

[CS10] G. S. H. Cruttwell and M. A. Shulman. "A unified framework for generalized multicategories". In: *Theory Appl. Categ.* 24 (2010), No. 21, 580–655 (cit. on pp. 6, 9–11, 13, 48).

[DPP06] R. J. M. Dawson, R. Paré, and D. A. Pronk. "Paths in double categories". In: *Theory Appl. Categ.* 16 (2006), No. 18, 460–521 (cit. on p. 11).

[FL24] S. Fujii and S. Lack. "The oplax limit of an enriched category". In: *Theory Appl. Categ.* 40 (2024), Paper No. 14, 390–412 (cit. on p. 55).

[HK] K. Hoshino and Y. Kawase. *Formal accessibility*. in preparation.

[Kou20] S. R. Koudenburg. "Augmented virtual double categories". In: *Theory Appl. Categ.* 35 (2020), Paper No. 10, 261–325 (cit. on pp. 1, 2, 4–11, 40).

[Kou24] S. R. Koudenburg. "Formal category theory in augmented virtual double categories". In: *Theory Appl. Categ.* 41 (2024), Paper No. 10, 288–413.

[Lei99] T. Leinster. *Generalized Enrichment for Categories and Multicategories*. 1999. arXiv: math/9901139 [math.CT]. URL: https://arxiv.org/abs/math/9901139 (cit. on pp. 6, 11, 17, 18).

[Lei02] T. Leinster. "Generalized enrichment of categories". In: vol. 168. 2-3. Category theory 1999 (Coimbra). 2002, pp. 391–406 (cit. on pp. 6, 17, 18).

[Lei04] T. Leinster. *Higher operads, higher categories*. Vol. 298. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2004, pp. xiv+433 (cit. on pp. 6, 11, 17).

[Par90] R. Paré. "Simply connected limits". In: *Canad. J. Math.* 42.4 (1990), pp. 731–746 (cit. on p. 29).

[Par11] R. Paré. "Yoneda theory for double categories". In: *Theory Appl. Categ.* 25 (2011), No. 17, 436–489 (cit. on pp. 20, 54).

[Sch15] P. Schultz. *Regular and exact (virtual) double categories*. 2015. arXiv: 1505.00712 [math.CT]. URL: https://arxiv.org/abs/1505.00712 (cit. on p. 45).

[Str04] R. Street. "Cauchy characterization of enriched categories [MR0708046]". In: *Repr. Theory Appl. Categ.* 4 (2004), pp. 1–16 (cit. on p. 50).

[Wal82] R. F. C. Walters. "Sheaves on sites as Cauchy-complete categories". In: *J. Pure Appl. Algebra* 24.1 (1982), pp. 95–102 (cit. on p. 18).

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: ykawase@kurims.kyoto-u.ac.jp