

Double categories of profunctors

Yuto Kawase

RIMS, Kyoto University

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← Today's slides

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Abstract

\mathbb{X} : a “double category” (*unital virtual double category*)



The “double category” $\mathbb{X}\text{-Prof}$:

- \mathbb{X} -enriched categories;
- \mathbb{X} -enriched functors;
- \mathbb{X} -enriched profunctors.

Goal

To characterize the “double categories” $\mathbb{X}\text{-Prof}$.

- 1 Enrichment in a monoidal category
- 2 Generalized enrichment
- 3 Colimits in a unital virtual double category
- 4 The characterization theorem

Enrichment in a monoidal category

$(\mathcal{V}, \otimes, I)$: a monoidal category

\mathcal{V} -category \mathbf{A}

- A class $\text{Ob}\mathbf{A}$ of *objects*;
- *Hom-objects* $\mathbf{A}(x, y) \in \mathcal{V}$ ($x, y \in \text{Ob}\mathbf{A}$);
- *Compositions* $\mathbf{A}(x, y) \otimes \mathbf{A}(y, z) \rightarrow \mathbf{A}(x, z)$ in \mathcal{V} ($x, y, z \in \text{Ob}\mathbf{A}$);
- *Identities* $I \rightarrow \mathbf{A}(x, x)$ in \mathcal{V} ($x \in \text{Ob}\mathbf{A}$). (+Axioms)

\mathcal{V} -functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\text{Ob}\mathbf{A} \longrightarrow \text{Ob}\mathbf{B}$;
$$x \longmapsto Fx$$
- $\mathbf{A}(x, y) \rightarrow \mathbf{B}(Fx, Fy)$ in \mathcal{V} ($x, y \in \text{Ob}\mathbf{A}$). (+Axioms)

\mathcal{V} -profunctor $\mathbf{A} \xrightarrow{P} \mathbf{B}$

- *Hom-objects* $P(x, y) \in \mathcal{V}$ ($x \in \mathbf{A}, y \in \mathbf{B}$);
- *Actions* $\mathbf{A}(x', x) \otimes P(x, y) \rightarrow P(x', y)$, $P(x, y) \otimes \mathbf{B}(y, y') \rightarrow P(x, y')$ in \mathcal{V} . (+Axioms)

The virtual double category $\mathcal{V}\text{-}\mathbb{P}\text{rof}$

- (objects) \mathcal{V} -categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$;
- (vertical arrows) \mathcal{V} -functors $\begin{array}{c} \mathbf{A} \\ F \downarrow \\ \mathbf{B} \end{array}, \dots$ and their compositions and identities;

- (horizontal arrows) \mathcal{V} -profunctors $\mathbf{A} \xrightarrow{P} \mathbf{B}, \dots$;

- (cells) “generalized” \mathcal{V} -natural transformations

$$\begin{array}{ccc} \mathbf{A}_0 & \xrightarrow{P_1} \mathbf{A}_1 & \xrightarrow{P_2} \dots \xrightarrow{P_n} \mathbf{A}_n \\ F \downarrow & & \downarrow G \\ \mathbf{B} & \xrightarrow{Q} \mathbf{C} & \end{array} \quad \parallel \quad \begin{array}{c} P_1(x_0, x_1) \otimes P_2(x_1, x_2) \otimes \dots \otimes P_n(x_{n-1}, x_n) \\ \downarrow \\ Q(Fx_0, Gx_n) \end{array}$$

- and their compositions (+ identity cells)

$$\begin{array}{ccc} \mathbf{A}_0 & \xrightarrow{\vec{P}_1} \mathbf{A}_1 & \xrightarrow{\vec{P}_2} \dots \xrightarrow{\vec{P}_n} \mathbf{A}_n \\ F_0 \downarrow & \alpha_1 F_1 \downarrow & \alpha_2 \dots \alpha_n \downarrow F_n \\ \mathbf{B}_0 & \xrightarrow{Q_1} \mathbf{B}_1 & \xrightarrow{Q_2} \dots \xrightarrow{Q_n} \mathbf{B}_n \\ G \downarrow & & \downarrow H \\ \mathbf{C} & \xrightarrow{R} \mathbf{D} & \end{array} \rightsquigarrow \begin{array}{ccc} \mathbf{A}_0 & \xrightarrow{\vec{P}_1} \mathbf{A}_1 & \xrightarrow{\vec{P}_2} \dots \xrightarrow{\vec{P}_n} \mathbf{A}_n \\ F_0 \circ G \downarrow & \alpha \circ \beta & \downarrow F_n \circ H \\ \mathbf{C} & \xrightarrow{R} \mathbf{D} & \end{array}$$

Horizontal composition

\mathbb{X} : a virtual double category (VDC).

Definition

A cell $\begin{array}{ccc} A_0 & \xrightarrow{u_1} A_1 & \xrightarrow{u_2} \dots \xrightarrow{u_n} A_n \\ \parallel & & \parallel \\ A_0 & \xrightarrow{v} & A_n \end{array}$ in \mathbb{X} is **composing**

$$\stackrel{\text{def}}{\Leftrightarrow} \begin{array}{ccc} \cdot & \dashrightarrow & A_0 \dashrightarrow A_n \dashrightarrow \cdot \\ \downarrow & & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \quad \forall \beta \quad = \quad \begin{array}{ccccc} \cdot & \dashrightarrow & A_0 & \dashrightarrow & A_n & \dashrightarrow & \cdot \\ \parallel & & \parallel & & \parallel & & \parallel \\ \cdot & \dashrightarrow & A_0 & \xrightarrow{v} & A_n & \dashrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot & & \cdot & & \cdot \end{array} \quad \exists! \gamma \quad \text{in } \mathbb{X}.$$

Definition

The **unit** on $X \in \mathbb{X}$... $\begin{array}{ccc} & X & \\ \parallel & & \parallel \\ X & \xrightarrow{u_X} & X \end{array}$ in \mathbb{X} . $\left(\text{written } \begin{array}{ccc} & X & \\ \parallel & & \parallel \\ X & \xrightarrow{\text{comp}} & X \end{array} \right)$

Horizontal composition

Examples in $\mathcal{V}\text{-}\mathbb{P}\text{rof}$

$$\begin{array}{ccccc} \textcircled{1} & \mathbf{A} & \xrightarrow{P} & \mathbf{B} & \xrightarrow{Q} & \mathbf{C} \\ & \parallel & & \text{comp} & & \parallel \\ & \mathbf{A} & \xrightarrow{P \odot Q} & & \mathbf{C} \end{array}$$

$$(P \odot Q)(x, z) := \int^{y \in \mathbf{B}} P(x, y) \otimes Q(y, z) \quad \text{in } \mathcal{V}$$

(Suppose that \int is preserved by $X \otimes -, - \otimes Y$.)

$$\begin{array}{ccc} & \mathbf{A} & \\ \swarrow & & \searrow \\ \textcircled{2} & \text{comp} & \\ \swarrow & & \searrow \\ \mathbf{A} & \xrightarrow{U_{\mathbf{A}}} & \mathbf{A} \end{array}$$

$$U_{\mathbf{A}}(x, x') := \mathbf{A}(x, x') \quad \text{in } \mathcal{V}$$

Unital virtual double categories

Definition

A VDC \mathbb{X} is **unital** $\stackrel{\text{def}}{\iff}$ Every object $X \in \mathbb{X}$ has the unit.

Example

$\mathcal{V}\text{-Prof}$ is unital.

In a unital VDC \mathbb{X} ,

$$\begin{array}{c} \cdot \xrightarrow{\vec{u}} \cdot \\ f \searrow \alpha \swarrow g \\ \cdot \end{array} := \begin{array}{c} \cdot \xrightarrow{\vec{u}} \cdot \\ f \downarrow \alpha \downarrow g \\ \cdot \quad \text{---} \end{array} \quad \text{in } \mathbb{X}. \quad (\text{0-coary cell})$$

0-coary cells can be composed horizontally:

$$\begin{array}{c} \cdot \xrightarrow{\vec{u}} \cdot \quad \cdot \xrightarrow{\vec{u}} \cdot \\ \alpha \searrow \downarrow \swarrow \beta \\ \cdot \end{array} \rightsquigarrow \begin{array}{c} \cdot \xrightarrow{\vec{u}} \cdot \quad \cdot \xrightarrow{\vec{u}} \cdot \\ \searrow \quad \alpha \smile \beta \quad \swarrow \\ \cdot \end{array} := \begin{array}{c} \cdot \xrightarrow{\vec{u}} \cdot \quad \cdot \xrightarrow{\vec{u}} \cdot \\ \downarrow \quad \alpha \quad \downarrow \quad \beta \quad \downarrow \\ \cdot \quad \text{---} \quad \cdot \quad \text{---} \quad \cdot \\ \parallel \quad \text{comp} \quad \parallel \\ \cdot \quad \text{---} \quad \cdot \end{array}$$

- 1 Enrichment in a monoidal category
- 2 Generalized enrichment
- 3 Colimits in a unital virtual double category
- 4 The characterization theorem

Monoidal categories vs. bicategories vs. VDC

Monoidal category = single-object bicategory

\mathcal{V} : a mon.cat. $\iff \mathcal{B}(\mathcal{V})$: a single-obj.bicat.

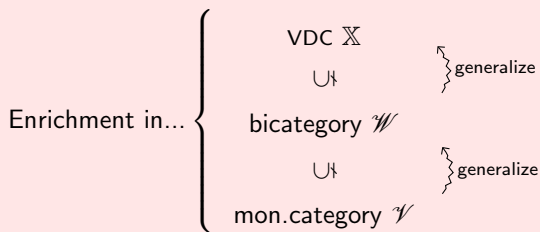
$$\begin{array}{ccc}
 \begin{array}{c} X \otimes Y \\ \alpha \downarrow \\ Z \end{array} & \text{in } \mathcal{V} & \parallel \\
 & & \begin{array}{ccc} X & \xrightarrow{\quad} & * \\ & \searrow \alpha & \swarrow Y \\ * & \xrightarrow{\quad Z \quad} & * \end{array} \text{ in } \mathcal{B}(\mathcal{V})
 \end{array}$$

(Virtual) bicategory = vertically discrete VDC

\mathcal{W} : a (virtual) bicat. $\iff \mathbb{D}(\mathcal{W})$: a vertically discrete VDC

$$\begin{array}{ccc}
 \begin{array}{ccccc} & f & & g & \\ c_0 & \xrightarrow{\quad} & c_1 & \xrightarrow{\quad} & c_2 \\ & \searrow \alpha & & \swarrow & \\ & & c_2 & & \end{array} & \text{in } \mathcal{W} & \parallel \\
 & & & & \begin{array}{ccccc} c_0 & \xrightarrow{f} & c_1 & \xrightarrow{g} & c_2 \\ \parallel & & \alpha & & \parallel \\ c_0 & \xrightarrow{\quad h \quad} & & & c_2 \end{array} \text{ in } \mathbb{D}(\mathcal{W})
 \end{array}$$

Generalization of enriching bases



Remark

We obtain unital VDCs $\mathscr{V}\text{-Prof}$, $\mathscr{W}\text{-Prof}$, and $\mathbb{X}\text{-Prof}$ for any $\mathscr{V}, \mathscr{W}, \mathbb{X}$.

From \mathcal{V} to \mathcal{W}

\mathcal{W} : a bicategory

\mathcal{W} -category \mathbf{A}

- A class \mathbf{ObA} of *objects*;
- *Coloring* $|x| \in \mathcal{W}$ ($x \in \mathbf{ObA}$);
- *Hom-1-cells* $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in \mathcal{W} ($x, y \in \mathbf{ObA}$);

- *Compositions*

A diagram showing the composition of 1-cells. On the left is the object $|x|$. On the top right is $|y|$ and on the bottom right is $|z|$. A curved arrow from $|x|$ to $|y|$ is labeled $\mathbf{A}(x,y)$. A curved arrow from $|y|$ to $|z|$ is labeled $\mathbf{A}(y,z)$. A curved arrow from $|x|$ to $|z|$ is labeled $\mathbf{A}(x,z)$. A pink downward arrow points from the top curved arrow to the bottom curved arrow.

in \mathcal{W} ($x, y, z \in \mathbf{ObA}$);

- *Identities* $|x| \xrightarrow{\mathbf{A}(x,x)} |x|$ in \mathcal{W} ($x \in \mathbf{ObA}$). (+Axioms)

From \mathcal{V} to \mathcal{W}

\mathcal{W} : a bicategory

\mathcal{W} -functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\text{Ob}\mathbf{A} \longrightarrow \text{Ob}\mathbf{B}$ s.t. $|x| = |Fx|$ in \mathcal{W} ;
 $x \longmapsto Fx$

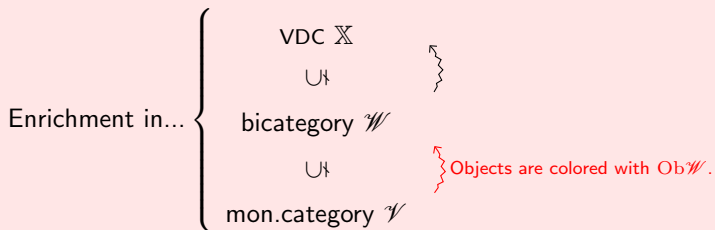
- $|x| \begin{array}{c} \xrightarrow{\mathbf{A}(x,y)} \\ \Downarrow \\ \xrightarrow{\mathbf{B}(Fx,Fy)} \end{array} |y| \quad \text{in } \mathcal{W} \quad (x, y \in \text{Ob}\mathbf{A}).$ (+Axioms)

\mathcal{W} -profunctor $\mathbf{A} \xrightarrow{P}_{+} \mathbf{B}$

- Hom-1-cells $|x| \xrightarrow{P(x,y)} |y|$ in \mathcal{W} ($x \in \mathbf{A}, y \in \mathbf{B}$);

- Actions $\begin{array}{ccccc} \mathbf{A}(x',x) & \rightarrow & |x| & \xrightarrow{P(x,y)} & |y| \\ |x'| & \searrow & \Downarrow & \nearrow & \nearrow \\ & & & & |y'| \end{array} \quad \begin{array}{ccccc} P(x,y) & \rightarrow & |y| & \xrightarrow{\mathbf{B}(y,y')} & |y'| \\ |x| & \searrow & \Downarrow & \nearrow & \nearrow \\ & & & & |y'| \end{array}$ in \mathcal{W} .
 $P(x',y)$ $P(x,y')$ (+Axioms)

From \mathcal{V} to \mathcal{W}

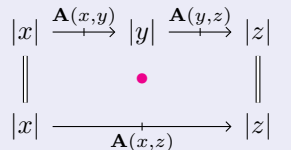


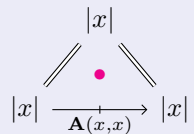
From \mathcal{W} to \mathbb{X}

\mathbb{X} : a VDC

\mathbb{X} -category \mathbf{A}

- A class $\text{Ob}\mathbf{A}$ of *objects*;
- *Coloring* $|x| \in \mathbb{X}$ ($x \in \text{Ob}\mathbf{A}$);
- *Hom-horizontal arrows* $|x| \xrightarrow{\text{A}(x,y)} |y|$ in \mathbb{X} ($x, y \in \text{Ob}\mathbf{A}$);

- *Compositions*
 in \mathbb{X} ($x, y, z \in \text{Ob}\mathbf{A}$);

- *Identities*
 in \mathbb{X} ($x \in \text{Ob}\mathbf{A}$). (+Axioms)

From \mathcal{W} to \mathbb{X}

\mathbb{X} : a VDC

Notation

$\mathcal{H}(\mathbb{X})$... the (virtual) bicat. obtained by forgetting all vertical arrows from \mathbb{X} .

Enrichment in a bicategory		Enrichment in a VDC
$\mathcal{H}(\mathbb{X})$ -categories	=	\mathbb{X} -categories
$\mathcal{H}(\mathbb{X})$ -functors	\subseteq	\mathbb{X} -functors
$\mathcal{H}(\mathbb{X})$ -profunctors	=	\mathbb{X} -profunctors

From \mathcal{W} to \mathbb{X}

\mathbb{X} -functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\text{Ob}\mathbf{A} \longrightarrow \text{Ob}\mathbf{B}$;
 $x \longmapsto F^0 x$

- “*Color-comparing*” vertical arrows $|x| \xrightarrow{F^1 x} |F^0 x|$ in \mathbb{X} ($x \in \text{Ob}\mathbf{A}$);

- $$\begin{array}{ccc}
 |x| & \xrightarrow{\mathbf{A}(x,y)} & |y| \\
 F^1 x \downarrow & \bullet & \downarrow F^1 y \\
 |F^0 x| & \xrightarrow{\mathbf{B}(F^0 x, F^0 y)} & |F^0 y|
 \end{array}
 \quad \text{in } \mathbb{X} \ (x, y \in \text{Ob}\mathbf{A}).$$
(+Axioms)

Enrichment in... $\left\{ \begin{array}{l} \text{VDC } \mathbb{X} \\ \Downarrow \\ \text{bicategory } \mathcal{W} \\ \Downarrow \\ \text{mon.category } \mathcal{V} \end{array} \right.$

} Functors no longer need to preserve the color of objects.

} Objects are colored with $\text{Ob}\mathcal{W}$.

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Collages of a profunctor

\mathcal{V} : a monoidal category with \emptyset (the initial preserved by \otimes)

$\mathbf{A} \xrightarrow{P} \mathbf{B}$: a \mathcal{V} -profunctor

Definition

A **collage** (or *cograph*) of P is the \mathcal{V} -category **Col** P :

$$\text{Ob}(\mathbf{Col} P) := \text{Ob} \mathbf{A} + \text{Ob} \mathbf{B};$$

$$\mathbf{Col} P(x, y) := \begin{cases} \mathbf{A}(x, y) & \text{if } x, y \in \mathbf{A} \\ \mathbf{B}(x, y) & \text{if } x, y \in \mathbf{B} \\ P(x, y) & \text{if } x \in \mathbf{A}, y \in \mathbf{B} \\ \emptyset & \text{if } x \in \mathbf{B}, y \in \mathbf{A} \end{cases}$$

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{P} & \mathbf{B} \\ \text{coproj.} \searrow & \cdot & \swarrow \text{coproj.} \\ & \mathbf{Col} P & \end{array} \quad \text{in } \mathcal{V}\text{-Prof}$$

Universal property (vertical)

$$\begin{array}{ccc}
 \text{Col } P & & \text{A} \xrightarrow{P} \text{B} \\
 \downarrow & \text{in } \mathcal{V}\text{-Prof} & \text{in } \mathcal{V}\text{-Prof} \\
 \text{C} & & \text{C}
 \end{array}$$

(A pink dot is located between the two horizontal arrows.)

Universal property (horizontal)

$$\begin{array}{ccc}
 \text{Col } P \multimap \text{C} & \text{in } \mathcal{V}\text{-Prof} & \begin{array}{ccc} \text{A} \xrightarrow{P} \text{B} \multimap \text{C} \\ \parallel & \bullet & \parallel \\ \text{A} \multimap \text{C} \end{array} & \text{in } \mathcal{V}\text{-Prof} \\
 \\
 \text{C} \multimap \text{Col } P & \text{in } \mathcal{V}\text{-Prof} & \begin{array}{ccc} \text{C} \multimap \text{A} \xrightarrow{P} \text{B} \\ \parallel & \bullet & \parallel \\ \text{C} \multimap \text{B} \end{array} & \text{in } \mathcal{V}\text{-Prof}
 \end{array}$$

Col P has universal properties in the three directions:

$$\begin{array}{c}
 \cdot \multimap \text{Col } P \multimap \cdot \\
 \downarrow \\
 \cdot
 \end{array}$$

Collages in general

S: a set

Notation

\mathbb{IS} ... the VDC described by:

- $\text{Ob}(\mathbb{IS}) := S$;
- For $i, j \in S$, there is a unique horizontal arrow $i \xrightarrow{!ij} j$ in \mathbb{IS} ;

- For $i_0, \dots, i_n \in S$, there is a unique cell

$$\begin{array}{c} i_0 \xrightarrow{!i_0 i_1} i_1 \xrightarrow{!i_1 i_2} \dots \xrightarrow{!i_{n-1} i_n} i_n \\ \parallel \qquad \qquad \qquad !i_0 \dots i_n \qquad \qquad \qquad \parallel \\ i_0 \xrightarrow{\qquad \qquad \qquad !i_0 i_n \qquad \qquad \qquad} i_n \end{array}$$

A virtual double (VD)-functor $\mathbb{IS} \longrightarrow \mathbb{X}\text{-Prof}$ is equivalent to the following data:

- \mathbf{A}_i : \mathbb{X} -categories ($i \in S$)
- $\mathbf{A}_i \xrightarrow{P_{ij}} \mathbf{A}_j$: \mathbb{X} -profunctors ($i, j \in S$)

- $$\begin{array}{ccc} & \mathbf{A}_i & \\ // & & \\ \mathbf{A}_i & \xrightarrow{P_{ii}} & \mathbf{A}_i \\ & \eta_i & \end{array} \quad (i \in S)$$

- $$\begin{array}{ccc} \mathbf{A}_i & \xrightarrow{P_{ij}} & \mathbf{A}_j \xrightarrow{P_{jk}} \mathbf{A}_k \\ \parallel & & \parallel \\ \mathbf{A}_i & \xrightarrow{P_{ik}} & \mathbf{A}_k \end{array} \quad (i, j, k \in S)$$

(+Axioms)

Collages in general

$P := (\mathbf{A}_i, P_{ij}, \eta_i, \mu_{ijk})_{i,j,k \in S} : \text{a VD-functor } \mathbb{S} \xrightarrow{P} \mathbb{X}\text{-Prof}$

Definition

A **collage** of P is the \mathbb{X} -category **Col** P :

$$\text{Ob}(\mathbf{Col} P) := \coprod_{i \in S} \text{Ob} \mathbf{A}_i;$$

$$\mathbf{Col} P(x, y) := P_{ij}(x, y) \quad (\text{where } x \in \mathbf{A}_i, y \in \mathbf{A}_j).$$

$$\begin{array}{ccc} \mathbf{A}_i & \xrightarrow{P_{ij}} & \mathbf{A}_j \\ \searrow \xi_i & \text{\textcolor{violet}{\xi}_{ij}} & \swarrow \xi_j \\ & \text{\textcolor{violet}{Col} } P & \end{array} \quad \text{in } \mathbb{X}\text{-Prof} \quad (\text{coprojections})$$

Again, **Col** P has universal properties in the three directions:

$$\begin{array}{c} \cdot \twoheadrightarrow \mathbf{Col} P \twoheadrightarrow \cdot \\ \downarrow \\ \cdot \end{array}$$

Versatile colimits

\mathbb{K} : a VDC, \mathbb{L} : a unital VDC, $\mathbb{K} \xrightarrow{F} \mathbb{L}$: a VD-functor

Definition (informal)

A **versatile colimit** of F is a “cocone”

$$\left\{ \begin{array}{ccc} FA & \xrightarrow{Fu} & FB \\ & \searrow \xi_A \quad \swarrow \xi_B & \\ & \Xi & \end{array} \right\} \text{ in } \mathbb{L} \quad \left(A \xrightarrow{u} B \text{ in } \mathbb{K} \right)$$

having the universal property in the three directions $\begin{array}{ccc} \cdot & \twoheadrightarrow & \Xi \\ & \downarrow & \\ & \cdot & \end{array} \quad \text{in } \mathbb{L}.$

Example

- 1 **Versatile collages** ($:=$ vers.colim. of shapes $\mathbb{K} = \mathbb{IS}$)
- 2 **Versatile coproducts** ($:=$ vers.colim. of discrete shapes)

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The object classifier in $\mathcal{V}\text{-}\mathbb{P}\text{rof}$

$(\mathcal{V}, \otimes, I)$: a monoidal category

Notation

The *unit \mathcal{V} -category* **I**:

$$\text{Ob}\mathbf{I} := \{*\}, \quad \mathbf{I}(*, *) := I.$$

The unit \mathcal{V} -category classifies objects

For every \mathcal{V} -category **A**,

$$\begin{array}{c} \mathbf{I} \\ \downarrow \\ \mathbf{A} \end{array} \quad \text{in } \mathcal{V}\text{-}\mathbb{P}\text{rof} \quad \Bigg\| \quad \text{an object } x \in \mathbf{A}$$

Object classifiers in \mathcal{W} -Prof

\mathcal{W} : a bicategory

Notation

The unit \mathcal{W} -category \mathbf{I}_c for $c \in \mathcal{W}$:

$$\mathrm{Ob}\mathbf{I}_c := \{*\} \text{ with } |*| := c, \quad \mathbf{I}_c(*, *) := (c \xrightarrow{\mathrm{id}} c \text{ in } \mathcal{W})$$

The unit \mathcal{W} -categories classify objects

For every \mathcal{W} -category \mathbf{A} ,

$$\begin{array}{c} \mathbf{I}_c \\ \downarrow \\ \mathbf{A} \end{array} \quad \text{in } \mathcal{W}\text{-Prof} \quad \Bigg\| \quad \text{an object } x \in \mathbf{A} \text{ s.t. } |x| = c$$

Object classifiers in $\mathbb{X}\text{-Prof}$

\mathbb{X} : a unital VDC

Notation

The *unit* \mathbb{X} -category \mathbf{I}_c for $c \in \mathbb{X}$:

$$\text{Ob}\mathbf{I}_c := \{*\} \text{ with } |*| := c, \quad \mathbf{I}_c(*, *) := (c \neq c \text{ in } \mathbb{X})$$

The unit \mathbb{X} -categories no longer classify objects.

For every \mathbb{X} -category \mathbf{A} ,

$$\begin{array}{c} \mathbf{I}_c \\ \downarrow \\ \mathbf{A} \end{array} \text{ in } \mathbb{X}\text{-Prof} \quad \parallel \quad \begin{array}{c} \text{an object } x \in \mathbf{A}, \text{ and} \\ \text{a vertical arrow } \begin{array}{c} c \\ \downarrow \\ |x| \end{array} \text{ in } \mathbb{X} \end{array} \quad (\textit{semi-object})$$

\rightsquigarrow The unit \mathbb{X} -categories classify *semi-objects*.

The embedding

\mathbb{X} : a unital VDC

Proposition

There is an “embedding” $\mathbb{X} \xrightarrow{\mathbf{I}_\bullet} \mathbb{X}\text{-Prof}$. That is:

$$\begin{array}{ccc} c_0 & \xrightarrow{\quad} c_1 & \xrightarrow{\quad} \cdots \xrightarrow{\quad} c_n \\ \downarrow & & \downarrow \\ d & \xrightarrow{\quad} & e \end{array} \quad \text{in } \mathbb{X} \quad \parallel \quad \begin{array}{ccc} \mathbf{I}_{c_0} & \xrightarrow{\quad} \mathbf{I}_{c_1} & \xrightarrow{\quad} \cdots \xrightarrow{\quad} \mathbf{I}_{c_n} \\ \downarrow & & \downarrow \\ \mathbf{I}_d & \xrightarrow{\quad} & \mathbf{I}_e \end{array} \quad \text{in } \mathbb{X}\text{-Prof}$$

In what follows, we will consider $\mathbb{X} \subseteq \mathbb{X}\text{-Prof}$.

Toward the characterization: density

$\mathbb{X} \subseteq \mathbb{X}\text{-Prof}$ (\mathbb{X} : a unital VDC)

Observation I

Every \mathbb{X} -category is a versatile collage of objects from \mathbb{X} .

\therefore

$$\frac{\mathbf{A}: \text{an } \mathbb{X}\text{-category}}{\mathbb{I}Ob\mathbf{A} \xrightarrow{|\cdot|_{\mathbf{A}}} \mathbb{X} \subseteq \mathbb{X}\text{-Prof}: \text{a VD-functor}} \quad \begin{array}{c} \text{\color{red} } \downarrow \text{\color{red} } \\ \text{\color{red} } \uparrow \end{array} \quad \text{\color{red} Taking the versatile colimit in } \mathbb{X}\text{-Prof}$$

The VD-functor $|\cdot|_{\mathbf{A}}$ is given by:

$$\begin{array}{ccc} \begin{array}{c} x_0 \xrightarrow{!} x_1 \xrightarrow{!} \dots \xrightarrow{!} x_n \\ \parallel \qquad \qquad \parallel \\ x_0 \xrightarrow{\quad ! \quad} x_n \\ \text{in } \mathbb{I}Ob\mathbf{A} \end{array} & \mapsto & \begin{array}{c} |x_0| \xrightarrow{\mathbf{A}(x_0, x_1)} |x_1| \xrightarrow{\mathbf{A}(x_1, x_2)} \dots \xrightarrow{\mathbf{A}(x_{n-1}, x_n)} |x_n| \\ \parallel \qquad \qquad \text{"composition" in } \mathbf{A} \qquad \qquad \parallel \\ |x_0| \xrightarrow{\mathbf{A}(x_0, x_n)} |x_n| \\ \text{in } \mathbb{X} \end{array} \end{array}$$

Toward the characterization: atomicity

Observation II

In $\mathbb{X}\text{-Prof}$, the unit \mathbb{X} -categories can be characterized by “atomicity” w.r.t. versatile collages.

Definition

\mathbb{L} : a unital VDC.

$L \in \mathbb{L}$: **collage-atomic** $\stackrel{\text{def}}{\iff}$ For every (large) versatile collage $\mathbf{Col} P$ in \mathbb{L} , vertical arrows $L \rightarrow \mathbf{Col} P$ uniquely factor through a unique coprojection.

$$\begin{array}{ccc} & L & \\ \swarrow \exists! & \downarrow \vee & \\ Pi & \mathbf{Col} P & \end{array} \quad \text{in } \mathbb{L} \quad (\exists! i)$$

coproj.

Proposition

\mathbb{X} : a unital VDC.

In $\mathbb{X}\text{-Prof}$, collage-atomic \iff vertically isomorphic to some \mathbf{I}_c .

The characterization

Theorem

TFAE for a unital VDC \mathbb{L} :

- ① There is an equivalence $\mathbb{L} \simeq \mathbb{X}\text{-Prof}$ for some unital VDC \mathbb{X} .
(in the 2-category of unital VDCs)
- ②
 - ◇ \mathbb{L} has (large) versatile collages;
 - ◇ Every object in \mathbb{L} is a versatile collage of collage-atomic objects.

Sketch of proof: The only non-trivial part is $2. \implies 1.$

Let $\mathbb{X} := \{\text{collage-atomic objs.}\} \subseteq \mathbb{L}$.

We can construct an adjunction: $\mathbb{X}\text{-Prof} \begin{array}{c} \xrightarrow{R} \\ \perp \\ \xleftarrow{N} \end{array} \mathbb{L}.$

Construction of R : Regarding $\mathbf{A} \in \mathbb{X}\text{-Prof}$ as $\mathbb{I}\text{Ob}\mathbf{A} \longrightarrow \mathbb{X} \subseteq \mathbb{L}$, we take the vers.colim. of it.

Construction of N : Suppose $L \in \mathbb{L}$ is a vers.colim. of $\mathbb{I}\mathbf{S} \xrightarrow{P} \mathbb{X} \subseteq \mathbb{L}$. Define $L \xrightarrow{N} "(S, P)" \in \mathbb{X}\text{-Prof}.$

$R \circ N \cong \text{Id}$ is trivial. $\text{Id} \cong N \circ R$ follows from **uniqueness of (S, P)** . □

Why is $\mathbb{IS} \xrightarrow{P} \mathbb{X}$ unique for each $L \in \mathbb{L}$?

- The uniqueness of the shape S and the object function of P
 \dots From the **collage-atomicity** of \mathbb{X} .
- The uniqueness of the horizontal arrow function & the cell function of P
 \dots From **strongness theorem** on versatile collages:

Strongness theorem

\mathbb{L} : a unital VDC. $\mathbf{Col} P (\in \mathbb{L})$: a versatile collage of $\mathbb{IS} \xrightarrow{P} \mathbb{L}$.
 \implies Coprojections of $\mathbf{Col} P$ form *cartesian cells*:

$$\begin{array}{ccc}
 Pi & \xrightarrow{P!_{ij}} & Pj \\
 \swarrow \text{coproj.} & \text{cart} & \nwarrow \text{coproj.} \\
 & \mathbf{Col} P &
 \end{array}
 \quad \text{in } \mathbb{L} \quad (i, j \in S).$$

Thank you!



Today's slides

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