

# On the decomposition of a strong epimorphism into regular epimorphisms

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Categories in Tokyo 2nd



← slides

- 1 Regular epimorphisms
- 2 Generalized regular epimorphisms
- 3 Locally orthogonal factorizations

## Recall

Every homomorphism  $f: A \rightarrow B$  between groups can be decomposed into a surjective hom. followed by an injective hom.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \nearrow \\ & \text{Im } f & \end{array} \quad \text{in } \mathbf{Grp}$$

**Regular epimorphisms** = morphisms being the coequalizer of some parallel pair of morphisms.

$$\cdot \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \cdot \xrightarrow{\text{regular epi}} \cdot$$

(eg. regular epi = surjective hom. in **Grp**)

**Monomorphisms** = morphisms s.t.  $f \circ -$  is injective.

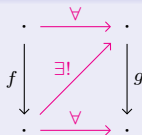
$$\cdot \begin{array}{c} \xrightarrow{g} \\ \xleftarrow{h} \end{array} \cdot \xrightarrow[f]{\text{mono}} \cdot \quad (\text{cofork}) \quad \implies \quad g = h$$

(eg. mono = injective hom. in **Grp**)

## Definition

$f \perp g$   
( $f$  and  $g$  are **orthogonal**)

$\stackrel{\text{def}}{\Leftrightarrow}$



## Example

In every category,  $\{\text{regular epi}\} \perp \{\text{mono}\}$ .

$\mathbf{E} \subseteq \text{Mor } \mathcal{C}$ : a class of morphisms in a category  $\mathcal{C}$ .

## Definition

$$\mathbf{E}: \text{iso-closed} \stackrel{\text{def}}{\iff} \begin{array}{l} \bullet \{ \text{iso} \} \subseteq \mathbf{E}. \\ \bullet \mathbf{E} \ni f \downarrow \begin{array}{c} \cdot \cong \cdot \\ \cdot \cong \cdot \end{array} \downarrow g \implies g \in \mathbf{E}. \end{array}$$

## Definition

$\mathbf{E}$ : iso-closed.

$\mathcal{C}$  admits orthogonal  $\mathbf{E}$ -factorizations  $\stackrel{\text{def}}{\iff}$  Every morphism  $f$  can be decomposed as

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ \mathbf{E} \ni e \searrow & & \nearrow m \in \mathbf{r}(\mathbf{E}) \\ & \cdot & \end{array} \quad \text{where } \mathbf{r}(\mathbf{E}) := \{ m \mid \mathbf{E} \perp m \}.$$

## Example

$\text{Grp}$  admits orthogonal  $\{\text{regular epi}\}$ -factorizations.

## The fundamental homomorphism theorem

$$A / \text{Ker } f \cong \text{Im } f \quad (f: A \rightarrow B)$$

The **kernel pair** of  $f$ :

$$\begin{array}{ccc} \text{Kp } f & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow & \lrcorner & \downarrow f \\ A & \xrightarrow{f} & B \end{array} \quad (\text{pullback})$$

The **coimage** of  $f$ :

$$\text{Kp } f \xrightleftharpoons[\pi_1]{\pi_2} A \twoheadrightarrow \text{Coim } f \quad (\text{coequalizer})$$

## The fundamental homomorphism theorem (categorically)

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \nearrow \text{mono} \\ & \text{Coim } f & \end{array}$$

This holds in any categories of equational algebras.

## Fact

TFAE for a category  $\mathcal{C}$  with finite limits and colimits:

- ①  $\mathcal{C}$  admits orthogonal {regular epi}-factorizations.
- ② “The fundamental homomorphism theorem holds in  $\mathcal{C}$ .” That is, for every  $f: A \rightarrow B$  in  $\mathcal{C}$ , the canonical morphism  $\text{Coim } f \rightarrow B$  is monic.

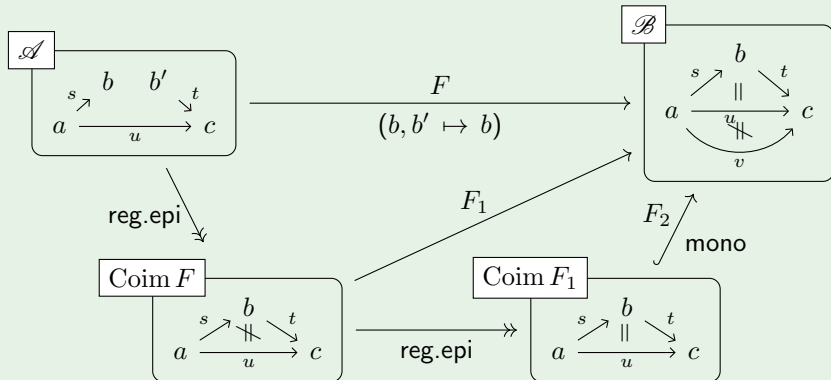
## Question

Does the fundamental hom. thm. hold in any category?

$\rightsquigarrow$  No.

## Counter example

In  $\mathbf{Cat}$ ,





## Definition

A **(regular) decomposition** of  $A \xrightarrow{f} B$  in  $\mathcal{C}$  consists of:

- A functor  $\mathbf{Ord} \xrightarrow{A_\bullet} \mathcal{C}$  with  $A_0 = A$ ;
- A cocone  $f_\bullet = (A_\alpha \xrightarrow{f_\alpha} B)$  over  $A_\bullet$  with  $f_0 = f$

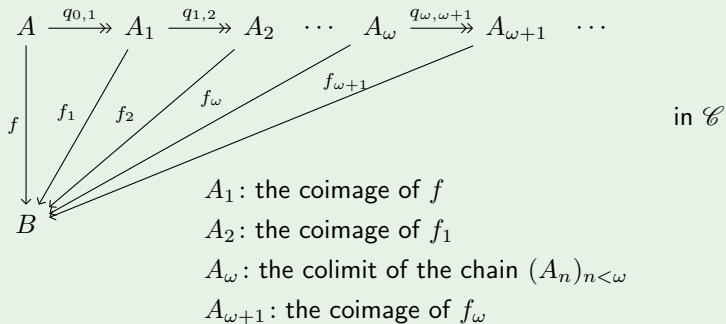
satisfying  $(\forall \alpha, A_{\alpha, \alpha+1}: \text{regular epi})$  and  $(\forall \gamma: \text{limit type}, A_\gamma \cong \text{Colim}_{\alpha < \gamma} A_\alpha)$ .

$$\begin{array}{ccccccc}
 A = A_0 & \xrightarrow{A_{0,1}} & A_1 & \longrightarrow & \cdots & \longrightarrow & A_\alpha \xrightarrow{A_{\alpha, \alpha+1}} A_{\alpha+1} \longrightarrow \cdots \\
 \downarrow f=f_0 & \swarrow f_1 & & \nearrow f_\alpha & & \nearrow f_{\alpha+1} & \\
 & B & & & & & 
 \end{array}
 \quad \text{in } \mathcal{C}$$

## Definition

- 1 A decomp.  $(A_\bullet, f_\bullet)$  **stabilizes at  $\alpha$**   $\stackrel{\text{def}}{\Leftrightarrow} f_\beta: \text{mono } (\alpha \leq \forall \beta) \Leftrightarrow f_\alpha: \text{mono}$
- 2 The smallest  $\alpha$  above is called the **length** of  $(A_\bullet, f_\bullet)$ . (if exists)

$\mathcal{C}$ : a category with kernel pairs and colimits.



$\rightsquigarrow$  This gives a decomposition of  $f$ . (**canonical decomposition**)

## Definition

- ① The **canonical decomposition number**  $\sigma(f) :=$  the length of the canonical decomposition of  $f$ . If it doesn't stabilize,  $\sigma(f)$  is left to be undefined.
- ② If  $\sigma(f)$  is defined for every  $f$  in  $\mathcal{C}$ , the **global canonical decomposition number**  $\sigma(\mathcal{C}) := \min\{\alpha \mid \sigma(f) < \alpha \text{ for every } f \text{ in } \mathcal{C}\}$

## Remark

- ①  $\sigma(\mathcal{C}) = 0 \iff \mathcal{C} = \emptyset$ .
- ②  $\sigma(\mathcal{C}) \leq 1 \iff \sigma(f) = 0 \ (\forall f) \iff f: \text{monic} \ (\forall f)$ .
- ③  $\sigma(\mathcal{C}) \leq 2 \iff \sigma(f) \leq 1 \ (\forall f) \iff \text{"The fund. hom. thm. holds in } \mathcal{C}\text{"}$ .
- ④  $\sigma(\mathcal{C}) \leq \omega \iff \sigma(f): \text{finite} \ (\forall f)$ .

## Example

- ①  $\sigma(\mathbf{Set}) = \sigma(\mathbf{Grp}) = 2$ .
- ②  $\sigma(\mathbf{Cat}) = 3$ .
- ③  $\sigma(n\text{-}\mathbf{Cat}) = n + 2$ .

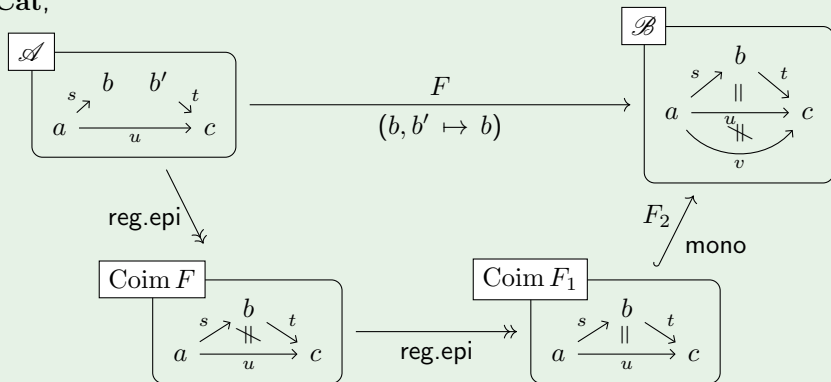
## Theorem [GU71]

$\mathcal{C}$ : locally  $\kappa$ -presentable  $\implies \sigma(f)$  is defined and  $\sigma(f) \leq \kappa$  holds. ( $\forall f$  in  $\mathcal{C}$ )  
(Equivalently,  $\sigma(\mathcal{C}) \leq \kappa + 1$ )

# Minimum decomposition problem

Recall (a canonical decomposition in  $\mathbf{Cat}$ )

In  $\mathbf{Cat}$ ,



## Question

Is there a shorter decomposition of  $F$  than the canonical one in  $\mathbf{Cat}$ ? That is, can  $F$  be decomposed as  $\cdot \xrightarrow{\text{reg.epi}} \cdot \xleftarrow{\text{mono}} \cdot$ ?

# Minimum decomposition problem

## Definition

The **minimum decomposition number**  $\delta(f)$  := the shortest possible length of decompositions of  $f$ . (if at least one decomposition exists)

## Question (restated)

Does  $\sigma(f) = \delta(f)$  hold?

## Theorem (Canonical vs minimum [KN])

In a category with kernel pairs and colimits,

- 1  $\sigma(f)$  is defined  $\iff \delta(f)$  is defined.
- 2 Whenever they are defined,  $\sigma(f) = \delta(f)$  holds.

That is, the canonical (regular) decomposition is minimum.

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## Observation

In a category with kernel pairs,

$q$ : regular epi.  $\iff q$ : the coequalizer of the kernel pair of  $q$ .

- “coequalize the kernel pair of  $q$ ”  $\iff$  coequalize any pair of morphisms that coequalized by  $q$ .  $\cdots$   $(\star)$  (*the kernel condition associated with  $q$* )

$\therefore q$ : regular epi.  $\iff q$ : the most universal among morphisms satisfying  $(\star)$ .

In a category with binary coproducts,

$$X \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} A \text{ is coequalized by } \begin{array}{c} A \\ \downarrow q \\ B \end{array} \iff \begin{array}{ccc} X + X & \xrightarrow{(u,v)} & A \\ \nabla \downarrow & & \downarrow q \\ X & \xrightarrow{\exists} & B \end{array}$$

$$A \xrightarrow{f} C \text{ satisfies } (\star) \iff \forall X, \begin{array}{ccc} X + X & \xrightarrow{\nabla} & A \\ \nabla \downarrow & & \downarrow q \\ X & \xrightarrow{\nabla} & B \end{array} \begin{array}{c} \searrow f \\ \xrightarrow{\exists} C \end{array}$$

## Idea

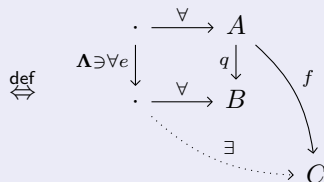
Replace  $\{X + X \xrightarrow{\nabla} X\}_{X: \text{obj}}$  with another class of epimorphisms.

# $\Lambda$ -regular epimorphisms

$\Lambda$ : a class of epimorphisms.    Fix  $A \xrightarrow{q} B$ .

## Definition [KN]

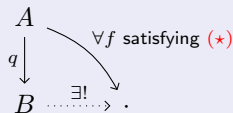
$A \xrightarrow{f} C$  satisfies  $(\star)$   
(the  $\Lambda$ -kernel condition assoc. with  $q$ )



## Definition [KN]

$q$ :  $\Lambda$ -regular epi

$\stackrel{\text{def}}{\Leftrightarrow}$





# Example I

## Example

**Pos**: the category of posets and order-preserving maps.

$$\Lambda := \left\{ \boxed{\bullet \quad \star} \xrightarrow{\iota} \boxed{\bullet < \star} \quad \text{in } \mathbf{Pos} \right\}$$

$\rightsquigarrow$   $\Lambda$ -regular epis = *surjective* order-preserving maps (= epis)

## Example

**Cat**: the category of small categories and functors.

$$\Lambda := \left\{ \boxed{\bullet \rightarrow \star} \xrightarrow{L} \boxed{\bullet \cong \star} \quad \text{in } \mathbf{Cat} \right\}$$

$\rightsquigarrow$   $\Lambda$ -regular epis = *strict localizations*

## Example II

### Example

**Met**: the category of metric spaces and non-expansive maps.

$$\Lambda := \left\{ \boxed{\bullet \xrightarrow{r+1} \star} \xrightarrow{e_r} \boxed{\bullet \xrightarrow{r} \star} \text{ in } \mathbf{Met} \right\}_{r \in \mathbb{Q}_{>0}}$$

$$\rightsquigarrow q: \Lambda\text{-regular epi} \iff d(x, y) - d(q(x), q(y)) \leq 1 \ (\forall x, y). \\ \text{(a shrinking at most 1)}$$

### Example

**Top**: the category of topological spaces and continuous maps.

$$\Lambda := \left\{ K \xrightarrow{!K} 1 \text{ in } \mathbf{Top} \right\}_{K: \text{connected space}}$$

$$\rightsquigarrow q: \Lambda\text{-regular epi} \iff q^{-1}(x): \text{connected} \ (\forall x). \text{ (= monotone map)}$$

category	$\Lambda$	$\mathbf{E} = \{\Lambda\text{-regular epis}\}$	$\mathbf{r}(\mathbf{E})$
any category with bin.coprod.	$\{\text{codiagonals}\}$	$\{\text{regular epis}\}$	$\{\text{monos}\}$
<b>Pos</b>	$\{\iota\}$	$\{\text{surjections}\}$	$\{\text{embeddings}\}$
<b>Cat</b>	$\{L\}$	$\{\text{strict localizations}\}$	$\{\text{conservatives}\}$
<b>Met</b>	$\{e_r\}_r$	$\{\text{shrinkings at most 1}\}$	$\{\text{isometries}\}$
<b>Top</b>	$\{!_K\}_K$	$\{\text{monotone maps}\}$	$\{\text{light maps}\}$

**Pos** admits orthogonal  $\{\text{surj}\}$ -factorizations.

However, neither **Cat**, **Met**, nor **Top** admits orthogonal  $\mathbf{E}$ -factorizations.

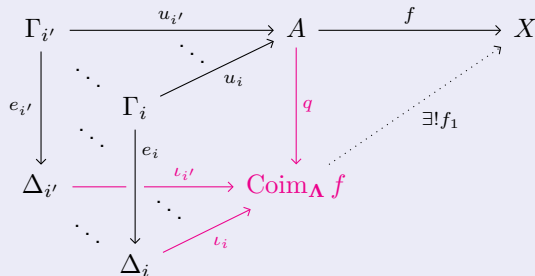
# Coimages in terms of $\Lambda$ -regular epis

## Definition

Let  $A \xrightarrow{f} X$ .

$$\mathcal{S} := \left\{ (e_i, u_i) \mid \Lambda \ni e_i \downarrow \begin{array}{ccc} \Gamma_i & \xrightarrow{u_i} & A \xrightarrow{f} X \\ \Delta_i & \searrow \exists & \end{array} \right\}$$

Then,  $\text{Coim}_{\Lambda} f := \text{Colim } \mathcal{S}$ . (multiple pushout)



$q$ :  $\Lambda$ -regular epi?

## Proposition [KN]

- 1  $\Lambda \subseteq \mathbf{Reg}(\Lambda)$  ( $:= \{\Lambda\text{-regular epis}\}$ ).
- 2 Moreover,  $\mathbf{Reg}(\Lambda)$  is the “multiple pushout closure” of  $\Lambda$ . That is,  
 $q: \Lambda\text{-reg.epi} \iff q: \text{obtained by a mult.pushout of some family of spans}$   
 $(e_i, u_i)$  s.t.  $e_i \in \Lambda$ .

In particular,

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \Lambda\text{-regular epi } \rightsquigarrow q \downarrow & \nearrow f_1 & \\ \text{Coim}_\Lambda f & & \end{array}$$

However,  $f_1 \notin \mathbf{r}(\mathbf{E})$  in general.

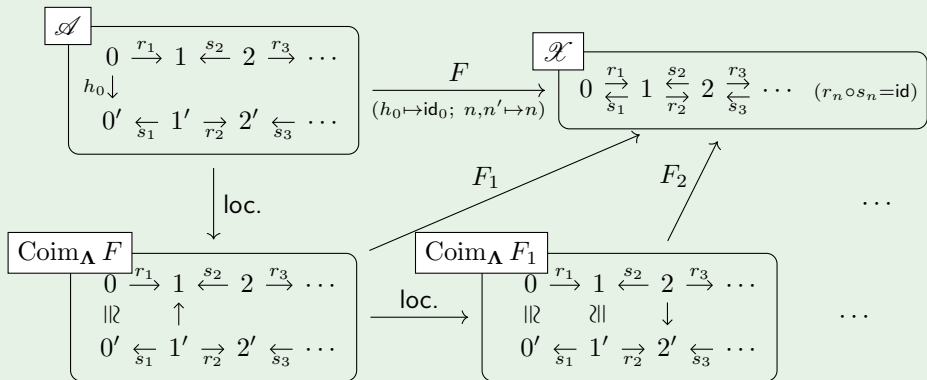
Replacing  $\left( \begin{array}{ccc} \{\text{regular epis}\} & \rightsquigarrow & \mathbf{E} = \mathbf{Reg}(\Lambda) \\ \{\text{monos}\} & \rightsquigarrow & \mathbf{r}(\mathbf{E}) \end{array} \right),$

## Definition

$\sigma_{\mathbf{E}}(f) := \min\{\alpha \mid f_\alpha \in \mathbf{r}(\mathbf{E})\}$ . (the *canonical  $\mathbf{E}$ -decomposition number*)

## Example

In  $\mathbf{Cat}$  with  $\mathbf{E} := \{\text{strict localizations}\}$ ,



$$\rightsquigarrow \sigma_{\mathbf{E}}(F) = \omega.$$

## Theorem (Small object argument)

$\mathcal{C}$ : locally  $\kappa$ -presentable;  $\Lambda$ : small;  $\text{dom } f, \text{cod } f$ :  $\kappa$ -presentable ( $\forall f \in \Lambda$ )  
 $\implies \sigma_{\mathbf{E}}(\mathcal{C}) \leq \kappa + 1$ , where  $\mathbf{E} := \mathbf{Reg}(\Lambda)$ .

## Corollary

$\sigma_{\mathbf{E}}(\mathbf{Cat}) = \omega + 1$ , where  $\mathbf{E} := \{\text{strict localizations}\}$ .

# Minimum decomposition problem

Let  $\mathbf{E} := \mathbf{Reg}(\Lambda)$ .

We can also generalize  $\left( \begin{array}{ccc} \text{regular decomposition} & \rightsquigarrow & \mathbf{E}\text{-decomposition} \\ \text{minimum dec.num. } \delta(f) & \rightsquigarrow & \delta_{\mathbf{E}}(f) \end{array} \right).$

Then,  $\sigma_{\mathbf{E}}$  and  $\delta_{\mathbf{E}}$  still coincide:

## Theorem (Canonical vs minimum [KN])

Let  $\mathcal{C}$ : locally small and cocomplete,  $\mathbf{E} := \mathbf{Reg}(\Lambda)$  with  $\Lambda$ : small.

- ①  $\sigma_{\mathbf{E}}(f)$  is defined  $\iff \delta_{\mathbf{E}}(f)$  is defined.
- ② Whenever they are defined,  $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$  holds.



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In the previous part, we did:

1. Define the class of morphisms  $\mathbf{E} := \mathbf{Reg}(\Lambda)$ ;
2. Define the *coimage-factorizations* in terms of  $\mathbf{E}$ ;
3. Define  $\sigma_{\mathbf{E}}(f)$  as the length of the iterated coimage-factorization of  $f$ ;
4. Compare  $\sigma_{\mathbf{E}}(f)$  with the minimum decomposition number  $\delta_{\mathbf{E}}(f)$ .

### Idea

Abstracting a class  $\mathbf{E}$  with coimage-factorizations, we can follow the same story from 2. to 4.

## Definition [MT82]

$$\begin{array}{c} X \\ f \downarrow \\ Y \end{array} \text{ and } \begin{array}{c} A' \xrightarrow{a} A \\ \downarrow g \\ B \end{array} \text{ are locally orthogonal} \\
 \text{(written } f \perp^a g \text{)} \quad \stackrel{\text{def}}{\iff} \quad \begin{array}{ccccc} X & \xrightarrow{\forall} & A' & \xrightarrow{a} & A \\ f \downarrow & & \exists! \nearrow & & \downarrow g \\ Y & \xrightarrow{\forall} & & & B \end{array}$$

## Recall Definition [MT82]

**E**: iso-closed.

**C** admits locally orthogonal **E**-factorizations  $\stackrel{\text{def}}{\iff} \forall f$  can be decomposed as

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ \mathbf{E} \ni e \searrow & & \nearrow m \\ & \cdot & \end{array} \quad \text{with } \mathbf{E} \perp^e m.$$

## Example

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ e \searrow & & \nearrow m \\ & \cdot & \end{array} \text{ is the coimage-factorization in terms of } \mathbf{E} = \mathbf{Reg}(\Lambda)$$

$\iff$  it is a locally orthogonal **E**-factorization, i.e.,  $e \in \mathbf{E}$  and  $\mathbf{E} \perp^e m$ .

Locally orthogonal factorizations subsume the coimage-factorization!

## Theorem [Tho83; KN]

$\mathcal{C}$ : co-well-powered, having small colimits and products.

TFAE for a class  $\mathbf{E} \subseteq \{\text{epis}\}$ :

- 1  $\mathcal{C}$  admits locally orthogonal  $\mathbf{E}$ -factorizations.
- 2  $\mathbf{E}$  is closed under multiple pushout.
- 3  $\mathbf{E} = \mathbf{Reg}(\Lambda)$  for some  $\Lambda \subseteq \{\text{epis}\}$ .

## Question

Is there a non-epimorphic example?

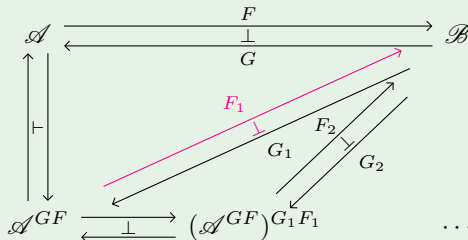
# The Eilengerg–Moore decomposition (monadic towers)

## Example

**AdjCAT**: the category of (large) categories and left adjoints.

$\mathbf{E} := \{\text{monadic adjunctions}\}$ .

$\rightsquigarrow$  **AdjCAT** admits locally orthogonal **E**-factorizations [MT82].



This **stabilizes** at  $\alpha \stackrel{\text{def}}{\Leftrightarrow} \mathbf{E} \perp F_\alpha \Leftrightarrow F_\alpha : \text{fully faithful}$

$\rightsquigarrow \sigma_{\mathbf{E}}(F) = \text{the monadic length of } F \dashv G.$

# Minimum decomposition problem

## Theorem (Canonical vs minimum, the most general [KN])

$\mathcal{C}$ : a category with locally orthogonal  $\mathbf{E}$ -factorizations and colimits of chains.  
 $f$ : a morphism in  $\mathcal{C}$ .

- ①  $\sigma_{\mathbf{E}}(f)$  is defined  $\iff \delta_{\mathbf{E}}(f)$  is defined.
- ②  $\delta_{\mathbf{E}}(f)$ : 0 or successor  $\implies \sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$ .
- ③  $\delta_{\mathbf{E}}(f)$ : limit  $\implies \sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$  or  $\delta_{\mathbf{E}}(f) + 1$ .

## Corollary

The monadic length coincides with the shortest possible length of monadic decompositions if they are finite.

# Thank you!

slides  $\rightsquigarrow$



This talk is based on the first half of [KN].

## References

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