FORMAL ACCESSIBILITY

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ABSTRACT. abstract

CONTENTS

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1. Archery range

An augmented virtual double category (avdc).
an AVDC.
AVDC.
Avdc.
AVDCs.
Avdcs.
augmented virtual double category.
Augmented virtual double category.
augmented virtual double categories.
Augmented virtual double categories.

$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, equal, "\shortmid"{marking}] \arrow[d, "g"] & B \arrow[r, "\shortmid"{marking}] \arrow[d, equal] & C \arrow[dl, "h"] \\
X \arrow[r, dotted, "v"', "\shortmid"{marking}] & Y & B &
\end{tikzcd}$$

An arrow $\relbar\joinrel\mapstochar\joinrel\longrightarrow$ exists.
An arrow $\longrightarrow$ exists.
An arrow $A \longrightarrow B$ exists.
An arrow $A \xrightarrow{u} B$ exists.
An arrow $A \overset{u}{\dashrightarrow} B$ exists.
An arrow $A \overset{u}{\dashrightarrow\dashrightarrow\dashrightarrow\dashrightarrow} B$ exists.

(NN) aaaaabbbbbb
(NN) says nothing.

$u_1 \odot u_2 \odot \ldots \odot u_n$
$f \fatsemi g$

$$\begin{tikzcd}
A \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\mathsf{lan}"] & I \arrow[dl, "\mathrm{Lan}_\varphi k"] \\
\Phi^{\triangledown} A &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha\Nearrow"] & B \arrow[d] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow\alpha"] & B \arrow[d] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nwarrow\alpha"] & B \arrow[d] \\
C \arrow[r] & D & C \arrow[r] & D & C \arrow[r] & D & C \arrow[r] & D
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & B \arrow[d] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha\Swarrow"] & B \arrow[d] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Searrow\alpha"] & B \arrow[d] \\
C \arrow[r] & D & C \arrow[r] & D & C \arrow[r] & D & C \arrow[r] & D
\end{tikzcd}$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\alpha}{\nRightarrow}"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha\not\nearrow"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\Uparrow\alpha"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\nwarrow\alpha"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\alpha}{\nLeftarrow}"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha\not\swarrow"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\Downarrow\alpha"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\searrow\alpha"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
$$

$$\vec{p} \triangleright^f q, \quad q^f \blacktriangleleft \vec{p}.$$

$$\vec{p} \triangleright q, \quad q \blacktriangleleft \vec{p}.$$

$\Phi^{\triangledown} A$ is.
$\Phi^{\triangledown}$ is.
$\Psi^{\blacktriangledown} A$ is.
$\Psi^{\blacktriangledown}$ is.

$$
\begin{tikzcd}
A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "=_f"] \\
B
\end{tikzcd}
\qquad =_f \; = \; =_f
$$

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel_u"] & B \arrow[d, equal] \\
A \arrow[r, "u"'] & B
\end{tikzcd}
\qquad \shortparallel_u = \shortparallel_u
$$

2. Basic notions

By a *graph*, we mean a presheaf for $\mathsf{G}_1 = ([0] \rightrightarrows [1])$, and we write **Gph** for the presheaf category $[\mathsf{G}_1^{\mathrm{op}}, \mathbf{Set}]$. Let us write $\mathtt{fc}$ for the free category monad on **Gph** and write $F_{\mathtt{fc}} \colon \mathbf{Gph} \longrightarrow \mathbf{Cat}$ and $U_{\mathtt{fc}} \colon \mathbf{Cat} \longrightarrow \mathbf{Gph}$ for the free functor and the forgetful functor respectively.

Given a graph X, an arrow in $F_{\mathtt{fc}}(X)$ (or an edge in $\mathtt{fc}(X)$) is a *path* in X, and the composite of arrows $\vec{p}$ and $\vec{q}$ in $F_{\mathtt{fc}}(X)$ is defined by concatenation of paths, which is denoted by $\vec{p} \frown \vec{q}$. We often identify an edge in X with a path of length 1, and such a path is denoted by a solid arrow $A \longrightarrow B$. A general path is often denoted by a dashed arrow $A \dashrightarrow B$. A dotted arrow $A \cdots\rightarrow B$ denote a path of length 0 or 1, and such a path is called a *dotted* paths. A path of length 0 is called a *null* path. A null path on a vertex A is denoted by $()_A$.

We write X^{op} for the graph obtained from a graph X by flipping the directions of its edges. fc は実際には large set の圏上のモナドのはずですで、この辺のサイズマターが若干気になります。

2.1. Augmented virtual double categories.

Although our framework is based on *virtual equipments*, we utilize *augmented virtual double category* [Kou20; Kou24] and define a virtual equipment as a *unital virtual equipment*: an augmented virtual double category with all restrictions. See [Kou20, Section 4] for more detail.

An augmented virtual double category $\mathbb{X}$ [Kou20; Kou24] consists of the following data:

- A class $\mathrm{Obj}(\mathbb{X})$ of *objects*.

- A category $\mathbf{V}(\mathbb{X})$ whose class of objects is $\mathrm{Obj}(\mathbb{X})$, which is called the *underlying category* or *vertical category* of $\mathbb{X}$. A *vertical arrow* in $\mathbb{X}$ is a morphism in $\mathbf{V}(\mathbb{X})$. A vertical arrow is denoted by an arrow, which is often written vertically; we mean by

$$\begin{tikzcd} A \arrow[d, "f"] \\ X \end{tikzcd} \quad \text{or} \quad A \xrightarrow{f} X \qquad \text{in } \mathbb{X}$$

a vertical arrow in $\mathbb{X}$ from A to X. We write id_A for the identity on any object A. The composition of $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{V}(\mathbb{D})$ is denoted by either of $f ⨟ g$ or $g \circ f$.
- A graph $\mathrm{H}(\mathbb{X})$ whose class of vertices is $\mathrm{Obj}(\mathbb{X})$. A *horizontal arrow* in $\mathbb{X}$ is an edge in $\mathrm{H}(\mathbb{X})$. A horizontal arrow is denoted by a slashed arrow, which is often written horizontally; we mean by

$$A \overset{p}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} B \qquad \text{in } \mathbb{X}$$

a horizontal arrow in $\mathbb{X}$ from A to B. As in the notation above for arbitrary graphs, we use slashed dashed arrow for path of horizontal arrows, and we mean by a slashed dotted arrow a path of horizontal arrows whose length is less than 2.
- A class of cells for each square formed by horizontal arrows and vertical arrows in the following way:

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"', "\shortmid"{marking}] & Y \end{tikzcd}$$

where $\vec{p}$ and u are paths of horizontal arrows such that $\mathtt{length}(u) \leq 1$.
- There are two kinds of identity cells denoted as follows:

$$\begin{tikzcd} A \arrow[r, dotted, "u", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel_u"] & B \arrow[d, equal] \\ A \arrow[r, dotted, "u"', "\shortmid"{marking}] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "=_f"] \\ B, \end{tikzcd}$$

The former is called the *vertical identity* or just the identity on u, while the latter is called the *horizontal identity* or the identity on f. If the length of u is 0, those two kinds of identities coincide; i.e., we have $=_{\mathsf{id}_A} = \shortparallel_{()_A}$.
- Composition of cells is defined for each pasting scheme of the following form:

$$\begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{\varphi}_1", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots \arrow[d, phantom, "\cdots"] & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "\psi_1"', "\shortmid"{marking}] \arrow[d, "g"'] & A_1 \arrow[r, dotted, "\psi_2"', "\shortmid"{marking}] & A_2 \arrow[d, phantom, "\beta"] & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"', "\shortmid"{marking}] & A_n \arrow[d, "h"] \\ B_0 \arrow[rrrrr, dotted, "r"', "\shortmid"{marking}] & & & & & B_1 \end{tikzcd} \mapsto \begin{tikzcd} A'_0 \arrow[rr, dashed, "\vec{\varphi}_1 \frown \cdots \frown \vec{\varphi}_n", "\shortmid"{marking}] \arrow[d, "f_0"'] & & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[d, "g"'] & \vec{\alpha} ⨟ \beta & A_n \arrow[d, "h"] \\ B_0 \arrow[rr, dotted, "r"', "\shortmid"{marking}] & & B_1 \end{tikzcd}$$

- By the diagram on the left, we mean the cell obtained by the composite of the diagram on the right when the sum of the lengths of ψ_i $(i = 1 \dots n)$ is lower than or equal to 1.

$$
\begin{tikzcd}
A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\
A_0 \arrow[r, dotted, "\psi_1"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n
\end{tikzcd}
=
\begin{tikzcd}
A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\
A_0 \arrow[r, dotted, "\psi_1"'] \arrow[d, equal] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 \arrow[d, phantom, "="] & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n \arrow[d, equal] \\
A_0 \arrow[rrrrr, dotted, "\psi_1 \frown \dots \frown \psi_n"'] & & & & & A_n
\end{tikzcd}
\tag{1}
$$

Definition 2.1. Let $\mathbb{X}$ be an augmented virtual double category. Define a 2-category $\mathscr{V}\,\mathbb{X}$, the *underlying 2-category* or *vertical 2-category* of $\mathbb{X}$, as follows.

- The underlying category is $\mathbf{V}\,\mathbb{X}$.
- A 2-cell is a cell in $\mathbb{X}$ whose top and bottom sides are null.
- Local composition of 2-cells are the special cases of (1) whose top and bottom sides are null. ♦

2.2. Weak notions for AVDCs.

Definition 2.2. Let $\mathbb{X}$ and $\mathbb{Y}$ be AVDCs. A *strict functor* of AVDCs $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ consists of the following data.

- A functor $F|_{\mathbf{V}}\colon \mathbf{V}\,\mathbb{X} \longrightarrow \mathbf{V}\,\mathbb{Y}$ on the underlying categories.
- A graph homomorphism $F|_{\mathrm{H}}\colon \mathrm{H}\,\mathbb{X} \longrightarrow \mathrm{H}\,\mathbb{X}$ whose action on the vertices coincide with the object part of $F|_{\mathbf{V}}$.
- An assignment on cells of the following form

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "u"'] & Y
\end{tikzcd}
\mapsto
\begin{tikzcd}
FA \arrow[r, dashed, "F\vec{p}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[d, "Fg"] \\
FX \arrow[r, dotted, "Fu"'] & FY
\end{tikzcd}
$$

  that is compatible with compositions and identities. Here, we mean by $F\vec{p}$ and Fu the images of $\vec{p}$ and u respectively, under the graph homomorphism $\mathtt{fc}(F|_{\mathrm{H}})\colon \mathtt{fc}(\mathrm{H}\,\mathbb{Y}) \longrightarrow \mathtt{fc}(\mathrm{H}\,\mathbb{Y})$.

We write **AVDbl** for the category of AVDCs and strict functors. ♦

Let us write $\mathbf{2}$ for the two-element chain: the lattice with exactly two elements.

Definition 2.3 (Cylinder)**.** Let $\mathbb{X}$ be an AVDC. We define the *cylinder* $\mathbf{2}\cdot\mathbb{X}$ of $\mathbb{X}$ as the following AVDC.

- The underlying category is the product $(\mathbf{V}\,\mathbb{X}) \times \mathbf{2}$.
- For each horizontal arrow $X \xrightarrow{p} Y$ in $\mathbb{X}$ and $\varepsilon \in \mathbf{2}$, there is a horizontal arrow $(X, \varepsilon) \xrightarrow{p} (Y, \varepsilon)$, and all horizontal arrows are exhausted by such arrows.
- A cell of the form on the left is defined as a cell in $\mathbb{X}$ of the form on the right:

$$
\begin{tikzcd}
(X_0, \varepsilon) \arrow[r, dashed, "\vec{p}"] \arrow[d, "{(f,l)}"'] \arrow[dr, phantom, "\alpha"] & (X_n, \varepsilon) \arrow[d, "{(g,l)}"] \\
(A_0, \varepsilon') \arrow[r, dotted, "u"'] & (A_1, \varepsilon')
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
X_0 \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X_n \arrow[d, "g"] \\
A_0 \arrow[r, dotted, "u"'] & A_1
\end{tikzcd}
$$

- Identities and compositions of cells are induced from those for $\mathbb{X}$.

For each $\varepsilon \in \mathbf{2}$, the assignment $X \mapsto (X, \varepsilon)$ extends to an embedding $\partial_\varepsilon\colon \mathbb{X} \hookrightarrow \mathbf{2}\cdot\mathbb{X}$. ♦

Definition 2.4. A strict transformation $F \Longrightarrow G \colon \mathbb{X} \longrightarrow \mathbb{Y}$ is a strict functor $\theta\colon\ \mathbf{2}{\cdot}\mathbb{X} \longrightarrow \mathbb{Y}$ satisfying $\theta \circ \partial_0 = F$ and $\theta \circ \partial_1 = G$. In detail, θ consists of the following data.

- For each $X \in \mathbb{X}$, a vertical arrow $\theta_X = \theta(X, \leq)\colon FX \longrightarrow GX$ in $\mathbb{Y}$. They form a natural transformation $F|_{\mathbf{V}} \Longrightarrow G|_{\mathbf{V}}\colon\ \mathbf{V}\,\mathbb{X} \longrightarrow \mathbf{V}\,\mathbb{Y}$.
- For each $u\colon X \longrightarrow\!\!\!\!\!\shortmid\ \ Y$ in $\mathbb{X}$, a cell

$$\begin{tikzcd} FX \arrow[r, "Fu", "\shortmid"{marking}] \arrow[d, "\theta_X"'] \arrow[dr, phantom, "\theta_u"] & FY \arrow[d, "\theta_Y"] \\ GX \arrow[r, "Gu"', "\shortmid"{marking}] & GY \end{tikzcd}$$

  that yields to the followings.
  - Suppose that we are given a cell in $\mathbb{X}$ of the following form.

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[d] \\ X \arrow[r, dotted, "v"', "\shortmid"{marking}] & Y \end{tikzcd}$$

    The following equality holds.

$$\begin{tikzcd} FA \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "F\alpha"] & FB \arrow[d] \\ FX \arrow[r, dotted, "Fv", "\shortmid"{marking}] \arrow[d, "\theta_X"'] \arrow[dr, phantom, "\theta_v"] & FY \arrow[d, "\theta_Y"] \\ GX \arrow[r, dotted, "Gv"', "\shortmid"{marking}] & GY \end{tikzcd} = \begin{tikzcd} FA \arrow[r, dashed, "F\vec{u}", "\shortmid"{marking}] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\theta_{\vec{u}}"] & FB \arrow[d, "\theta_B"] \\ GA \arrow[r, dashed, "G\vec{u}", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "G\alpha"] & GB \arrow[d] \\ GX \arrow[r, dotted, "Gv"', "\shortmid"{marking}] & GY \end{tikzcd}$$

    Here, $\theta_{\vec{u}}$ is a sequence of cells defined inductively as follows.
    * $\theta_{\vec{u}}$ is a null sequence, or the horizontal identitity $=_{\theta_X}$, if $\texttt{length}(\vec{u}) = 0$.
    * $\theta_{u \frown \vec{u}} = \theta_u \frown \theta_{\vec{u}}$. ♦

Proposition 2.5. AVDCs, strict functors, and strict transformations form a 2-category $\mathscr{AVD}bl$.

Proof. ongoing, or citation □

Definition 2.6. Let $\mathbb{X}$ be an AVDC. Define another AVDC $\mathsf{Q}\,\mathbb{X}$ as follows.

- The underlying category $\mathbf{V}(\mathsf{Q}\,\mathbb{X})$ is the free category $F_{\mathtt{fc}}(U_{\mathtt{fc}}(\mathbf{V}\,\mathbb{X}))$.
- The horizontal graph is the same as that of $\mathbb{X}$.
- A cell of the form on the left is defined as a cell in $\mathbb{X}$ of the form on the right.[1]

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, dashed, "\vec{f}"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, dashed, "\vec{g}"] \\ X \arrow[r, dotted, "u"', "\shortmid"{marking}] & Y \end{tikzcd} \text{ in } \mathsf{Q}\,\mathbb{X} \quad \Bigg\| \quad \begin{tikzcd} A \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"', "\shortmid"{marking}] & Y \end{tikzcd} \text{ in } \mathbb{X} \tag{2}$$

  Here, f and g are the composite of $\vec{f}$ and $\vec{g}$ in $\mathbf{V}\,\mathbb{X}$, respectively.
- Compositions and identities are defined from those for $\mathbb{X}$ in an obvious way.

There is a canonical strict functor $\varepsilon_{\mathbb{X}}\colon\ \mathsf{Q}\,\mathbb{X} \longrightarrow \mathbb{X}$ that sends cells of the form on the left of (2) to those of the form on the right. Moreover, the unit for $F_{\mathtt{fc}} \dashv U_{\mathtt{fc}}$ induces a canonical strict functor $\delta_{\mathbb{X}}\colon\ \mathsf{Q}\,\mathbb{X} \longrightarrow \mathsf{Q}\,\mathsf{Q}\,\mathbb{X}$, and they form a comonad $\mathsf{Q} = (\mathsf{Q}, \varepsilon, \delta)$ on **AVDbl**. ♦

[1]The notation shown on the left is more appropriate for Q-coalgebras than for AVDCs; a Q-coalgebra is nothing but an AVDC whose underlying category is free.

Definition 2.7. We write $\mathbf{AVDbl}_{\mathsf{wk}}$ for the coKleisli category of the comonad Q. A morphism in $\mathbf{AVDbl}_{\mathsf{wk}}$ is called a *weak functor* of AVDCs. In detail,ongoing ♦

Notation 2.8. For each AVDC $\mathbb{X}$, the AVDC $\mathsf{Q}\,\mathbb{X}$ is called the *weak functor classifier* of $\mathbb{X}$. For each weak functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$, we write $\bar{F}$ for the classifying strict functor $\mathsf{Q}\,\mathbb{X} \longrightarrow \mathbb{Y}$. ♦

Definition 2.9. Let $F, G\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be weak functors of AVDCs. A *weak transformation* $F \Longrightarrow G$ is a weak functor $\theta\colon \mathbf{2}{\cdot}\mathbb{X} \longrightarrow \mathbb{Y}$ satisfying $\bar{\theta} \circ \mathsf{Q}\,\partial_0 = \bar{F}$ and $\bar{\theta} \circ \mathsf{Q}\,\partial_1 = \bar{G}$.

- For each $X \in \mathbb{X}$, a vertical arrow $\theta_X = \theta(\mathsf{id}_X, \leq)\colon FX \longrightarrow GX$ in $\mathbb{Y}$.
- For each vertical arrow $f\colon X \longrightarrow A$ in $\mathbb{X}$, a vertical arrow $\theta(f, \leq)\colon FX \longrightarrow GA$ in $\mathbb{Y}$.
- Suppose that we are given two paths of paths of vertical arrows $\cdot \overset{\vec{f}_1}{\dashrightarrow} X_1 \overset{k}{\dashrightarrow} X_2 \overset{\vec{f}_2}{\dashrightarrow} \cdot$ and $\cdot \overset{\vec{g}_1}{\dashrightarrow} Y_1 \overset{h}{\dashrightarrow} Y_2 \overset{\vec{g}_2}{\dashrightarrow} \cdot$, and a cell in $\mathbb{X}$ of the form shown on the left below. As a weak functor, θ assigns to such data a cell in $\mathbb{Y}$ of the form on the right below.

$$\begin{tikzcd}[row sep=small] \cdot \arrow[r, dashed, "\vec{u}"] \arrow[d, dashed, "\vec{f}_1"'] & \cdot \arrow[d, dashed, "\vec{g}_1"] \\ \cdot \arrow[d, dotted, "k"'] \arrow[r, phantom, "\alpha"] & \cdot \arrow[d, dotted, "h"] \\ \cdot \arrow[d, dashed, "\vec{f}_2"'] & \cdot \arrow[d, dashed, "\vec{g}_2"] \\ \cdot \arrow[r, dotted, "v"'] & \cdot \end{tikzcd} \quad \mapsto \quad \begin{tikzcd}[row sep=small] \cdot \arrow[r, dashed, "F\vec{u}"] \arrow[d, dashed, "F\vec{f}_1"'] & \cdot \arrow[d, dashed, "F\vec{g}_1"] \\ \cdot \arrow[d, "\theta(k{,}\leq)"'] \arrow[r, phantom, "\theta(\alpha)"] & \cdot \arrow[d, "\theta(h{,}\leq)"] \\ \cdot \arrow[d, dashed, "G\vec{f}_2"'] & \cdot \arrow[d, dashed, "G\vec{g}_2"] \\ \cdot \arrow[r, dotted, "Gv"'] & \cdot \end{tikzcd}$$

Moreover, they are compatible with F and G in an obvious sense. ♦

Proposition 2.10. Let $F, G\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be weak functors. A weak transformation $\theta\colon F \Longrightarrow G$ is determined by the following data:

- A vertical arrow $\theta_X\colon FX \longrightarrow GX$ for each $X \in \mathbb{X}$.
- * A vertical arrow $\theta(f, \leq)\colon FX \longrightarrow GY$ for each $f\colon X \longrightarrow Y$ in $\mathbb{X}$, For each $X \in \mathbb{X}$, we assume $\theta(\mathsf{id}_X, \leq) = \theta_X$.
- Horizontally invertible cells that yield the following equality

$$\begin{tikzcd} & FX \arrow[dl, "\theta_X"'] \arrow[dr, "Ff"] & \\ GX \arrow[dr, "Gf"'] \arrow[rr, phantom, "\theta_f"] & & FY \arrow[dl, "\theta_Y"] \\ & GY & \end{tikzcd} = \begin{tikzcd} & FX \arrow[dl, "\theta_X"'] \arrow[dr, "Ff"] \arrow[dd, "\theta(f{,}\leq)"] & \\ GX \arrow[dr, "Gf"'] \arrow[r, phantom, "\theta_f^1"] & {} \arrow[r, phantom, "\theta_f^2"] & FY \arrow[dl, "\theta_Y"] \\ & GY & \end{tikzcd}$$

for each vertical arrow $f\colon X \longrightarrow Y$ in $\mathbb{X}$.
- A cell

$$\begin{tikzcd} FA \arrow[r, "Fu"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\theta_u"] & FB \arrow[d, "\theta_B"] \\ GA \arrow[r, "Gu"'] & GB \end{tikzcd} \quad \text{in } \mathbb{Y}$$

for each horizontal arrow $A \overset{u}{\longrightarrow} B$ in $\mathbb{X}$.

* Those data subject to the following: For each cell

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}"] \arrow[d, dashed, "\vec{f}"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, dashed, "\vec{g}"] \\ X \arrow[r, dotted, "v"'] & Y \end{tikzcd} \quad \text{in } \mathsf{Q}\,\mathbb{X},$$

the following equality holds.

$$
\begin{tikzcd}[column sep=small]
& FA \arrow[rr, dashed, "F\vec{u}"] \arrow[dl, "\theta_A"'] \arrow[dr, dashed, "F\vec{f}"] \arrow[dr, phantom, bend right=60, "\theta_{\vec{f}}"'] & \arrow[d, phantom, "F\alpha"] & FB \arrow[dr, dashed, "F\vec{g}"] & \\
GA \arrow[dr, dashed, "G\vec{f}"'] & & FX \arrow[rr, dotted, "Fv"'] \arrow[dl, "\theta_X"'] & \arrow[d, phantom, "\theta_v"] & FY \arrow[dl, "\theta_Y"] \\
& GX \arrow[rr, dotted, "Gv"'] & & GY &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& FA \arrow[rr, dashed, "F\vec{u}"] \arrow[dl, "\theta_A"'] & \arrow[d, phantom, "\theta_{\vec{u}}"] & FB \arrow[dl, "\theta_B"] \arrow[dr, dashed, "F\vec{g}"] & \\
GA \arrow[rr, dashed, "F\vec{u}"] \arrow[dr, dashed, "G\vec{f}"'] & \arrow[d, phantom, "G\alpha"] & GB \arrow[dr, dashed, "G\vec{g}"'] \arrow[rr, phantom, "\theta_{\vec{g}}"] & & FY \arrow[dl, "\theta_Y"] \\
& GX \arrow[rr, dotted, "v"'] & & GY &
\end{tikzcd}
\tag{3}
$$

Here $\theta_{\vec{u}}$ and θ_v are defined inductively in the same way as Definition 2.2, while $\theta_{\vec{f}}$ and $\theta_{\vec{g}}$ are defined inductively as follows. Let $\vec{f}\colon A \dashrightarrow X$ be a path of vertical arrows.

- When $\vec{f}$ is null and hence $A = X$, $\theta_{\vec{f}}$ is the horizontal identity cell on θ_A.
- $\theta_{f\frown\vec{f}}$ is defined as the following composite.

$$
\begin{tikzcd}[column sep=small]
& FA \arrow[dl, "\theta_A"'] \arrow[dr, "Ff"] & & \\
GA \arrow[dr, "Gf"'] \arrow[rr, phantom, "\theta_f"] & & F\cdot \arrow[dl] \arrow[dr, dashed, "F\vec{f}"] & \\
& G\cdot \arrow[dr, dashed, "G\vec{f}"'] \arrow[rr, phantom, "\theta_{\vec{f}}"] & & FX \arrow[dl, "\theta_X"] \\
& & GX &
\end{tikzcd}
$$

Remark 2.11. The data $\theta(f,\leq)$ marked by $\bullet^*$ and the cells θ^1_f and θ^2_f can be omitted for merely defining a weak transformation: we can take $\theta(f,\leq) := Ff ⨾ \theta_X$, $\theta^1_f := =_{\theta(f,\leq)}$, and $\theta^2_f := \theta_f$ for each $f\colon X \longrightarrow Y$. These data are necessary, however, for ensuring that the reconstructed weak transformation exactly coincides with the original weak transformation. ♦

Proof of Proposition 2.10. Construct the data described in Definition 2.9 as follows.

- θ_X is already given for each $X \in \mathbb{X}$. For each $f\colon A \longrightarrow X$, we set $\theta(f,\leq) := \theta_A ⨾ Gf$.
- Suppose that we are given two paths of paths of vertical arrows $\cdot \overset{\vec{f}_1}{\dashrightarrow} X_1 \overset{k}{\dashrightarrow} X_2 \overset{\vec{f}_2}{\dashrightarrow} \cdot$ and $\cdot \overset{\vec{g}_1}{\dashrightarrow} Y_1 \overset{h}{\dashrightarrow} Y_2 \overset{\vec{g}_2}{\dashrightarrow} \cdot$, and a cell in $\mathbb{X}$ of the form shown on the left below. We will define another cell $\theta(\alpha)$ of the form on the mid below, as the composite of the diagram on the right below.

$$
\begin{tikzcd}[column sep=small, row sep=small]
\cdot \arrow[rr, dashed, "\vec{u}"] \arrow[d, dashed, "\vec{f}_1"'] & & \cdot \arrow[d, dashed, "\vec{g}_1"] \\
\cdot \arrow[d, dotted, "k"'] & \alpha & \cdot \arrow[d, dotted, "h"] \\
\cdot \arrow[d, dashed, "\vec{f}_2"'] & & \cdot \arrow[d, dashed, "\vec{g}_2"] \\
\cdot \arrow[rr, dotted, "v"'] & & \cdot
\end{tikzcd}
\mapsto
\begin{tikzcd}[column sep=small, row sep=small]
\cdot \arrow[rr, dashed, "F\vec{u}"] \arrow[d, dashed, "F\vec{f}_1"'] & & \cdot \arrow[d, dashed, "F\vec{g}_1"] \\
\cdot \arrow[d, "\theta(k,\leq)"'] & \theta(\alpha) & \cdot \arrow[d, "\theta(h,\leq)"] \\
\cdot \arrow[d, dashed, "G\vec{f}_2"'] & & \cdot \arrow[d, dashed, "G\vec{g}_2"] \\
\cdot \arrow[rr, dotted, "Gv"'] & & \cdot
\end{tikzcd}
=
\begin{tikzcd}[column sep=small, row sep=small]
& \cdot \arrow[rrrrr, dashed, "F\vec{u}"] \arrow[d, dashed, "F\vec{f}_1"'] & & & & & \cdot \arrow[d, dashed, "F\vec{g}_1"] & \\
& \cdot \arrow[dl, bend right=60, "\theta(k,\leq)"'] \arrow[d, dashed, "\theta_{X_1}"'] \arrow[dr, dotted, "Fk"] & & & \arrow[d, phantom, "F\alpha"] & & \cdot \arrow[dl, dotted, "Fh"'] \arrow[d, "\theta_{Y_1}"] \arrow[dr, bend left=60, "\theta(h,\leq)"] & \\
\arrow[r, phantom, "\theta^{1,-1}_k"] & \cdot \arrow[dl, dotted, "Gk"] & \cdot \arrow[dr, dashed, "F\vec{f}_2"] \arrow[d, phantom, "\theta_{k\frown\vec{f}_2}"] & & & \cdot \arrow[dl, dashed, "F\vec{g}_2"'] \arrow[d, phantom, "\theta^{-1}_{h\frown\vec{g}_2}"] & \cdot \arrow[dr, dotted, "Gh"'] & \arrow[l, phantom, "\theta^1_k"] \\
\cdot \arrow[dr, dashed, "G\vec{f}_2"'] & & & \cdot \arrow[r, dotted] \arrow[dll, "\theta."'] & \cdot \arrow[drr, "\theta."] \arrow[d, phantom, "\theta_v"] & & & \cdot \arrow[dl, dashed, "G\vec{g}_2"] \\
& \cdot \arrow[rrrrr, dotted, "v"'] & & & & & \cdot &
\end{tikzcd}
$$

□

Remark 2.12. weak notions do not form a 2-category ♦

2.3. **Cartesian and opcartesian cells.** Let $\mathbb{X}$ be an AVDC.

Definition 2.13 (Cartesian cells)**.** A cell

$$\begin{tikzcd} A \arrow[r, dotted, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"'] & Y \end{tikzcd} \quad \text{in } \mathbb{X}$$

is *cartesian* if for each cell β of the following form, there exists a unique $\bar{\beta}$ satisfying the following equation.

$$\begin{tikzcd} A' \arrow[r, dashed, "\vec{q}"] \arrow[d, "h"'] & B' \arrow[d, "k"] \\ A \arrow[d, "f"'] \arrow[r, phantom, "\beta"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"'] & Y \end{tikzcd} = \begin{tikzcd} A' \arrow[r, dashed, "\vec{q}"] \arrow[d, "h"'] \arrow[dr, phantom, "\bar{\beta}"] & B' \arrow[d, "k"] \\ A \arrow[r, dotted, "p"'] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"'] & Y \end{tikzcd} \quad \text{in } \mathbb{X}$$

If such a cartesian cell exists and the length of p is 1, p is called the *restriction* of u along f and g, and denoted by $p \cong u(f,g)$.

If the length of u is 0 and hence $X = Y$ holds, then we write $X(f,g)$ for the restriction $u(f,g)$. In particular, we write f_* for $X(f,g)$ when g is the identity, and it is called the *companion* of f. Dually, we write g^* for $X(f,g)$ when f is the identity, and it is called the *conjoint* of g. ♦

Definition 2.14 (Units)**.** For each object X, we write U_X for the restriction $X(\mathsf{id}_X, \mathsf{id}_X)$ along the identity vertical arrows, and we say it the *unit* on X. An AVDC is unital if every object has units. ♦

Definition 2.15. A virtual equipment is an AVDC having all restrictions; i.e., for any dotted arrow $u\colon X \dashrightarrow Y$ and vertical arrows $f\colon A \longrightarrow X$ and $g\colon B \longrightarrow Y$, there exists a restriction $u(f,g)\colon A \longrightarrow B$. ♦

Definition 2.16. A vertical arrow $X \xrightarrow{f} Y$ is called *fully faithful* if the identity cell $=_f$ is cartesian.

$$\begin{tikzcd} X \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "=_f"] \\ Y \end{tikzcd} : \mathsf{cart} \quad \text{in } \mathbb{X}$$

♦

Definition 2.17. Suppose that we are given a sequence of cells

$$\begin{tikzcd} A'_0 \arrow[r, dashed, "(\vec{\varphi}_i)_{i=1\ldots n}"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\vec{\alpha}"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dashed, "(\psi_i)_{i=1\ldots n}"'] & A_n \end{tikzcd} := \begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "\psi_1"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n \end{tikzcd} \quad \text{in } \mathbb{X}. \tag{4}$$

- (45) is *weakly opcartesian* if for each cell β of the following form, there exists a unique $\bar{\beta}$ satisfying the following equation.

$$\begin{tikzcd}
A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] & A'_2 & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] & A'_n \arrow[d, "f_n"] \\
A_0 \arrow[d, "g"'] & & \beta & & & A_n \arrow[d, "h"] \\
B_0 \arrow[rrrrr, dotted, "r"'] & & & & & B_1
\end{tikzcd}
=
\begin{tikzcd}
A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\
A_0 \arrow[r, dotted, "\psi_1"'] \arrow[d, "g"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n \arrow[d, "h"] \\
B_0 \arrow[rrrrr, dotted, "r"'] \arrow[rrrrr, phantom, shift left=4, "\bar{\beta}"] & & & & & B_1
\end{tikzcd}$$

  If $\mathbb{X}$ is unital, then we only have to consider the case when r is of length 1 (see Lemma 2.33).
- Suppose that f_0 is the identity in (45). Then the sequence of cells (45) is *left-composing* if for any sequence of horizontal arrows $X \overset{\vec{u}}{\dashrightarrow} A_0$, the following sequence of cells is weakly opcartesian.

$$\begin{tikzcd}
X \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, equal] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[d, equal] \\
X \arrow[r, dashed, "\vec{u}"'] & A_0 \arrow[r, dotted, "\psi_1"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n
\end{tikzcd}$$

  Right-composing sequences of cells are also defined as the horizontal dual of left-composing sequences.
- Suppose that both f_0 and f_n are the identities in (45). Then the sequence of cells (45) is *composing* if it is left-composing and right-composing.
- The sequence (45) is *strongly opcartesian* if it is weakly opcartesian and any sequence of cells of the following form is composing:

$$\begin{tikzcd}
X \arrow[r, dotted, "p"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[r, dotted, "q"] \arrow[d, "f_n"] \arrow[dr, phantom, "\mathsf{cart}"] & Y \arrow[d, equal] \\
X \arrow[r, dotted, "u"] & A_0 \arrow[r, dotted, "\psi_1"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n \arrow[r, dotted, "v"] & Y
\end{tikzcd}$$

- (45) is nanntoka *cartesian* if any sequence of cells obtained by the following concatenation of sequences

$$\begin{tikzcd}
B'_0 \arrow[r, dashed] \arrow[d, "g_0"'] \arrow[dr, phantom, "\vec{\beta}"] & B'_m = A'_0 \arrow[r, dashed, "(\vec{\varphi}_i)_{i=1\dots n}"] \arrow[d, "g_n = f_0"] \arrow[dr, phantom, "\vec{\alpha}"] & A'_n = C'_0 \arrow[r, dashed] \arrow[d, "f_n = h_0"'] \arrow[dr, phantom, "\vec{\gamma}"] & C'_k \arrow[d, "h_k"] \\
B_0 \arrow[r, dashed] & B_n = A_0 \arrow[r, dashed, "(\vec{\psi}_i)_{i=1\dots n}"'] & A_n = C_0 \arrow[r, dashed] & C_k
\end{tikzcd}$$

  is strongly opcartesian whenever $\vec{\beta}$ and $\vec{\gamma}$ are as well.

We say a cell is weakly opcartesian/strongly opcartesian/nanntoka cartesian if the sequence of length 1 is weakly opcartesian/strongly opcartesian/nanntoka cartesian in the above sense. ♦

Definition 2.18. Let $\mathbb{A}$ and $\mathbb{B}$ be AVDC, and $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be a weak functor. We say F *preserves compositions* if for each cell in $\mathsf{Q}\,\mathbb{A}$ of the following form

$$\begin{tikzcd}
\cdot \arrow[r, dashed, bend left] \arrow[r, bend right] \arrow[r, phantom, "c"] & \cdot
\end{tikzcd}$$

whose left and right sides are null and the corresponding cell in $\mathbb{X}$ is composing, the cell Fc in $\mathbb{B}$ is composing. ♦

Remark 2.19. If both f_0 and f_n are the identities in (45), then the sequence of cells (45) is composing if and only if it is strongly opcartesian. ♦

Remark 2.20. Suppose we are given a sequence of cells (45) whose total length of the bottom sides is less than 2. Then the composite is weakly opcartesian (resp. strongly opcartesian) as a cell if and only if the sequence is weakly opcartesian (resp. strongly opcartesian). why ♦

Definition 2.21 (Vertically invertible cells/sequences)**.**

- A sequence of cells

$$\begin{tikzcd} A \arrow[r, dashed, "(\vec{p}_i)_i"] \arrow[d, "f"'] \arrow[dr, phantom, "(\alpha_i)_i"] & B \arrow[d, "g"] \\ X \arrow[r, dashed, "(q_i)_i"'] & Y \end{tikzcd}$$

  is called *vertically invertible* if there exists a sequence $\vec{\beta}$ of cells satisfying the following:

$$\begin{tikzcd} A \arrow[r, dashed, "(\vec{p}_i)_i"] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\alpha}"] & B \arrow[d, "g"] \\ X \arrow[r, dashed, "\vec{q}"'] \arrow[d, "f'"'] \arrow[dr, phantom, "\vec{\beta}"] & Y \arrow[d, "g'"] \\ A \arrow[r, dashed, "(\vec{p}_i)_i"'] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "(\vec{p}_i)_i"] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B \arrow[d, equal] \\ A \arrow[r, dashed, "(\vec{p}_i)_i"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dashed, "\vec{q}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\vec{\beta}"] & Y \arrow[d, "g'"] \\ A \arrow[r, dashed, "(\vec{p}_i)_i"'] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\alpha}"] & B \arrow[d, "g"] \\ X \arrow[r, dashed, "\vec{q}"'] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & Y \arrow[d, equal] \\ X \arrow[r, dashed, "\vec{q}"'] & Y \end{tikzcd}$$

- A cell is called *vertically invertible* if it is so as a sequence of length 1.

♦

Lemma 2.22. A cell

$$\begin{tikzcd} A \arrow[r, dotted, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "q"'] & Y \end{tikzcd}$$

is vertically invertible if and only if there exists a cell β of the following form such that $\alpha \fatsemi \beta = \Vert_p$ and $\beta \fatsemi \alpha = \Vert_q$:

$$\begin{tikzcd} A \arrow[r, dotted, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "q"'] \arrow[d, "f'"'] \arrow[dr, phantom, "\beta"] & Y \arrow[d, "g'"] \\ A \arrow[r, dotted, "p"'] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, dotted, "p"] \arrow[d, equal] \arrow[dr, phantom, "\Vert_p"] & B \arrow[d, equal] \\ A \arrow[r, dotted, "p"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dotted, "q"] \arrow[d, "f'"'] \arrow[dr, phantom, "\beta"] & Y \arrow[d, "g'"] \\ A \arrow[r, dotted, "p"'] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "q"'] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[r, dotted, "q"] \arrow[d, equal] \arrow[dr, phantom, "\Vert_q"] & Y \arrow[d, equal] \\ X \arrow[r, dotted, "q"'] & Y \end{tikzcd}$$

The inverse β is called the *vertical inverse* of α.

Proof. ongoing □

Remark 2.23. Note that a vertical arrow f is an isomorphism in $\mathbf{V}\mathbb{X}$ if and only if the cell $=_f$ is vertically invertible. ♦

Definition 2.24 (Horizontally invertible cell)**.** A cell

$$\begin{tikzcd} A \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\alpha"] \\ B \end{tikzcd} \tag{5}$$

is called *horizontally invertible* if it is an isomorphism in the underlying 2-category $\mathscr{V}(\mathbb{X})$. The inverse of α is called the *horizontal inverse* of α. ◆

Lemma 2.25. A horizontally invertible cell (5) is vertically invertible if and only if the vertical arrows f, g are isomorphisms.

Proof. The only if part is trivial. Suppose that f and g are isomorphisms, and let α' for the horizotnal inverse of α. The vertical inverse is given by the following composite.

$$\begin{tikzcd}
B \arrow[d, "f^{-1}"'] \arrow[dd, bend left=60, equal] \\
A \arrow[d, bend right, "g"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "\alpha'"] \arrow[dd, bend right=60, equal] \\
B \arrow[d, "g^{-1}"] \\
A
\end{tikzcd}$$

□

Lemma 2.26. The cartesian cell defining a unit U_X is vertically invertible.

Proof. Let us write η for the cartesian cell defining U_X. By the universality of the cartesian cell, the identity cell on X factors as follows.

$$\begin{tikzcd}
X \arrow[d, bend right, equal] \arrow[d, bend left, equal] \arrow[d, phantom, "="] \\
X
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\bar{\eta}"] & \\
X \arrow[rr, "\mathsf{U}_X"] \arrow[dr, equal] & {} \arrow[d, phantom, "\eta"] & X \\
& X \arrow[ur, equal] &
\end{tikzcd}$$

The other identity follows from the uniqueness part of the universality of η by considering $\eta ⨾ \bar{\eta} ⨾ \eta = \eta$. □

Definition 2.27 (Restricting cells/sequences)**.**

- A cell
$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "v"'] & Y
\end{tikzcd}
\quad \text{in } \mathbb{X}$$
is called *restricting* if there exist a composing cell β and a cartesian cell γ of the following form such that their composite becomes α:
$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "v"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "\beta\colon \mathsf{comp}"] & B \arrow[d, equal] \\
A \arrow[r, dotted, "w"] \arrow[d, "f"'] \arrow[dr, phantom, "\gamma\colon \mathsf{cart}"] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "v"'] & Y
\end{tikzcd}
\quad \text{in } \mathbb{X}.$$

- In general, a sequence of cells

$$\begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{\varphi}_1", "|"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2", "|"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n", "|"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "\psi_1"', "|"{marking}] & A_1 \arrow[r, dotted, "\psi_2"', "|"{marking}] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"', "|"{marking}] & A_n \end{tikzcd} \quad \text{in } \mathbb{X}.$$

is called *restricting* if there exists a composing cell β of the following form

$$\begin{tikzcd} A_0 \arrow[r, "(\psi_i)_i", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\beta\colon \mathsf{comp}"] & A_n \arrow[d, equal] \\ A_0 \arrow[r, dotted, "\chi"', "|"{marking}] & A_n \end{tikzcd} \quad \text{in } \mathbb{X}$$

such that the composite of the following cells becomes a restricting cell:

$$\begin{tikzcd} A'_0 \arrow[r, dashed, "(\vec{\varphi}_i)_i", "|"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "(\alpha_i)_i"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dashed, "(\psi_i)_i"', "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\beta\colon \mathsf{comp}"] & A_n \arrow[d, equal] \\ A_0 \arrow[r, dotted, "\chi"', "|"{marking}] & A_n \end{tikzcd}$$

♦

Lemma 2.28. Suppose that we are given a sequence of cells $\vec{\alpha}$ and a composing cell β of the following forms:

$$\begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{\varphi}_1", "|"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2", "|"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n", "|"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "\psi_1"', "|"{marking}] \arrow[d, equal] & A_1 \arrow[r, dotted, "\psi_2"', "|"{marking}] & A_2 \arrow[d, phantom, "\beta\colon \mathsf{comp}"] & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"', "|"{marking}] & A_n \arrow[d, equal] \\ A_0 \arrow[rrrrr, dotted, "\chi"', "|"{marking}] & & & & & A_n \end{tikzcd} \tag{6}$$

If the composite of (6) is a restricting cell, then the sequence $\vec{\alpha}$ becomes restricting.

Proof. ongoing □

Corollary 2.29. restricting cell = restricting seq of length 1

Remark 2.30. Every composing sequence is restricting whenever ongoing ♦

Lemma 2.31. Every vertically invertible cell/sequence is restricting and weakly opcartesian.

Proof. ongoing □

Lemma 2.32. A vertically invertible cell/sequence is strongly opcartesian if its left side has a conjoint and the right side has a companion.

Proof. ongoing □

Lemma 2.33. Suppose that $\mathbb{X}$ is unital and we are given a sequence of the following form.

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{p}", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\alpha}"] & B \arrow[d, "g"] \\ X \arrow[r, dashed, "\vec{q}"', "|"{marking}] & Y \end{tikzcd}$$

The following are equivalent.

(i) $\vec{\alpha}$ is weakly opcartesian.
(ii) For each cell β of the following form, there exists a unique $\bar{\beta}$ satisfying the following equation.

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X \arrow[r, phantom, "\beta"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
X' \arrow[r, "v"'] & Y'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\alpha}"] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "\vec{q}"] \arrow[d, "x"'] \arrow[dr, phantom, "\bar{\beta}"] & Y \arrow[d, "y"] \\
X' \arrow[r, "v"'] & Y'
\end{tikzcd}
$$

Proof. (i)$\Rightarrow$(ii) is trivial. Suppose $\vec{\alpha}$ satisfies the universality (ii) and we are given a cell β as above. If the length of v is 1, then the desired $\bar{\beta}$ exists uniquely by the universality of $\vec{\alpha}$. For the case when $\texttt{length}(v) = 0$ and $X' = Y'$, consider the unit $\mathsf{U}_{X'}$ and the vertical composite $\beta ⨾ \bar{\eta}$, where $\bar{\eta}$ is the vertical inverese of the cartesian cell η defining U_X (see Lemma 2.26). Note that Lemma 2.31 shows that $\bar{\eta}$ is cartesian. The universality of the weakly opcartesian cell $\vec{\alpha}$ and the cartesian cell $\bar{\eta}$ give the desired factorisation $\beta = \vec{\alpha} ⨾ \bar{\beta}$. ☐

2.3.1. *Vertical cancellation.*

Lemma 2.34 (Vertical cancellation for cartesian cells)**.** Suppose that we are given a cartesian cell β of the following form. Then there is a bijective correspondence between cells of the form α and cells of the form γ under the following equation.

$$
\begin{tikzcd}
A \arrow[r, dotted, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
A' \arrow[r, dotted, "p'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\beta\colon \mathsf{cart}"] & B' \arrow[d, "g'"] \\
A'' \arrow[r, dotted, "p''"'] & B''
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dotted, "p"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
A' \arrow[r, phantom, "\gamma"] \arrow[d, "f'"'] & B' \arrow[d, "g'"] \\
A'' \arrow[r, dotted, "p''"'] & B''
\end{tikzcd}
$$

For such a pair (α, γ), α is cartesian if and only if γ is cartesian.

Proof. Given arbitrary h, k, and $\vec{u}$, consider cells of the following forms.

$$
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{u}"] \arrow[d, "h"'] \arrow[dr, phantom, "\delta_1"] & \cdot \arrow[d, "k"] \\
A \arrow[r, dotted, "p"'] & B
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{u}"] \arrow[d, "h"'] & \cdot \arrow[d, "k"] \\
A \arrow[r, phantom, "\delta_2"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
A' \arrow[r, dotted, "p'"'] & B'
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{u}"] \arrow[d, "h"'] & \cdot \arrow[d, "k"] \\
A \arrow[d, "f"'] \arrow[r, phantom, "\delta_3"] & B \arrow[d, "g"] \\
A' \arrow[d, "f'"'] & B' \arrow[d, "g'"] \\
A'' \arrow[r, dotted, "p''"'] & B''
\end{tikzcd}
$$

There are functions $\delta_1 \mapsto \delta_1 ⨾ \alpha$ and $\delta_2 \mapsto \delta_2 ⨾ \beta$ from left to right, and the composite of those functions assigns $\delta_1 ⨾ \gamma$ to δ_1. Since β is cartesian, $\delta_2 \mapsto \delta_3$ is bijective, and hence $\delta_1 \mapsto \delta_2$ is bijective if and only if so is $\delta_1 \mapsto \delta_3$. This shows that α and γ are cartesian simultaneously. ☐

Notation 2.35. Suppose that we are given sequences of cells of the following forms.

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\vec{\alpha}"] & A_l \arrow[d] \\
B_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\vec{\beta}"] & B_m \arrow[d] \\
C_0 \arrow[r, dashed, "\shortmid" marking, "\vec{p}"'] & C_n
\end{tikzcd}$$

In general, there might be multiple ways to composite cells in $\vec{\alpha}$ and $\vec{\beta}$ and obtain another sequence of cells.augmented ならではの現象なので，その点を言及したい. We write

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\alpha}"] & A_l \arrow[d, "f'"] \\
B_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\vec{\beta}"] & B_m \arrow[d, "g'"] \\
C_0 \arrow[r, dashed, "\shortmid" marking, "\vec{p}"'] & C_n
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "f"'] & A_l \arrow[d, "f'"] \\
B_0 \arrow[r, phantom, "\vec{\gamma}"] \arrow[d, "g"'] & B_m \arrow[d, "g'"] \\
C_0 \arrow[r, dashed, "\shortmid" marking, "\vec{p}"'] & C_n
\end{tikzcd} \tag{7}$$

if one of the compositions coincides with $\vec{\gamma}$.

Observe that given any cell δ of the following form, the composite of δ to those sequences as shown below is uniquely determined and we have an equality $\vec{\alpha}\fatsemi(\vec{\beta}\fatsemi\delta) = \vec{\gamma}\fatsemi\delta$ as a cell.

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\alpha}"] & A_l \arrow[d, "f'"] \\
B_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\vec{\beta}"] & B_m \arrow[d, "g'"] \\
C_0 \arrow[r, dashed, "\shortmid" marking, "\vec{p}"'] \arrow[d] \arrow[dr, phantom, "\delta"] & C_n \arrow[d] \\
\cdot \arrow[r, "\shortmid" marking] & \cdot
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "f"'] & A_l \arrow[d, "f'"] \\
B_0 \arrow[r, phantom, "\vec{\gamma}"] \arrow[d, "g"'] & B_m \arrow[d, "g'"] \\
C_0 \arrow[r, dashed, "\shortmid" marking, "\vec{p}"'] \arrow[d] \arrow[dr, phantom, "\delta"] & C_n \arrow[d] \\
\cdot \arrow[r, "\shortmid" marking] & \cdot
\end{tikzcd}$$

♦

Lemma 2.36 (Vertical cancellation for opcartesian cells/sequences I)**.** Suppose we are given sequences of cells satisfying (46).

(i) If the sequence $\vec{\alpha}$ is weakly opcartesian, then $\vec{\beta}$ is weakly opcartesian if and only if $\vec{\gamma}$ is as well.
(ii) Suppose that f and g are the identities. If the sequence $\vec{\alpha}$ is left-composing, then $\vec{\beta}$ is left-composing if and only if $\vec{\gamma}$ is as well.
(iii) Suppose that $\mathbb{X}$ is a virtual equipment. If the sequence $\vec{\alpha}$ is strongly opcartesian, then $\vec{\beta}$ is strongly opcartesian if and only if $\vec{\gamma}$ is as well.

Proof. (i) is shown in a way similar to the proof for Lemma 11.9. One can also easily check that (ii) follows from (i).

For (iii), suppose $\mathbb{X}$ is a virtual equipment. Then, utilizing Lemma 11.9, we conclude that any cartesian cell of the form of ζ_1 below extends to a cartesian cell of the form ζ_2, while any cartesian cell of the form of ζ_2 factors as composite of some cartesian cells ξ and ζ_1, through taking restrictions.

$$
\begin{tikzcd}
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & \cdot \arrow[d, "f'"] \\
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\zeta_1"] & \cdot \arrow[d, "g'"] \\
\cdot \arrow[r, dotted, "\shortmid" marking] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[d, "f"'] & \cdot \arrow[d, "f'"] \\
\cdot \arrow[d, "g"'] \arrow[r, phantom, "\zeta_2"] & \cdot \arrow[d, "g'"] \\
\cdot \arrow[r, dotted, "\shortmid" marking] & \cdot
\end{tikzcd}
$$

Now consider the following sequences of cells, where cart denote some cartesian cells.

$$
\begin{tikzcd}
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & B_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\vec{\beta}"] & B_m \arrow[r, dotted, "\shortmid" marking] \arrow[d, "g'"] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, dotted, "\shortmid" marking] & C_0 \arrow[r, dashed, "\shortmid" marking] & C_n \arrow[r, dotted, "\shortmid" marking] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[dd, equal] & A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "f"'] & A_l \arrow[r, dotted, "\shortmid" marking] \arrow[d, "f'"] & \cdot \arrow[dd, equal] \\
\mathsf{cart} & B_0 \arrow[d, "g"'] \arrow[r, phantom, "\vec{\gamma}"] & B_m \arrow[d, "g'"] & \mathsf{cart} \\
\cdot \arrow[r, dotted, "\shortmid" marking] & C_0 \arrow[r, dashed, "\shortmid" marking] & C_n \arrow[r, dotted, "\shortmid" marking] & \cdot
\end{tikzcd}
$$

For a sequence of the form on the left, there exists a sequence of the form on the right by extending cartesian cells in the above way. Since $\vec{\alpha}$ is strongly opcartesian, (ii) shows that the left one is strongly opcartesian if and only if the right one is as well. In the same way, given a sequence of the form on the right, through factorising cartesian cells, we have a sequence of the form on the left such that they are strongly opcartesian simultaneously. □

Lemma 2.37 (Vertical cancellation for opcartesian cells/sequences II)**.** Suppose that we are given sequences of cells satisfying (46) and that $\mathbb{X}$ is a virtual equipment. If the sequence $\vec{\beta}$ is composing, then $\vec{\alpha}$ is strongly opcartesian if and only if $\vec{\gamma}$ is as well.

Lemma 2.38 (Vertical cancellation for restricting cells/sequences)**.** Suppose that we are given sequences of cells satisfying (46).

(i) If the sequence $\vec{\beta}$ is restricting or composing, then $\vec{\alpha}$ is restricting if and only if $\vec{\gamma}$ is as well.
(ii) If the sequence $\vec{\alpha}$ is composing, then $\vec{\beta}$ is restricting if and only if $\vec{\gamma}$ is as well.

2.3.2. *Companions and conjoints.*

Definition 2.39. Let $\mathbb{X}$ be an AVDC. Let $A \xrightarrow{f} B$ be a vertical arrow in $\mathbb{X}$.

(i) A *companion* of f consists of the following data:
- A horizontal arrow $A \xrightarrow{f_*} B$;
- Cells of the following forms:

$$
\begin{tikzcd}
 & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\eta"] & \\
A \arrow[rr, "f_*"', "\shortmid" marking] & {} & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f_*", "\shortmid" marking] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\varepsilon"] & B \arrow[dl, equal] \\
 & B &
\end{tikzcd}
$$

These are required to satisfy the following equations:

$$
\begin{tikzcd}
 & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\eta"] & \\
A \arrow[rr, "f_*", "\shortmid" marking] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\varepsilon"] & B \arrow[dl, equal] \\
 & B &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[rr, "f_*", "\shortmid" marking] \arrow[dl, equal] \arrow[d, "f"] \arrow[dl, phantom, "\eta"] & \arrow[d, phantom, "\varepsilon"] & B \arrow[dl, equal] \\
A \arrow[r, "f_*"', "\shortmid" marking] & B & &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f_*", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & B \arrow[d, equal] \\
A \arrow[r, "f_*"', "\shortmid" marking] & B
\end{tikzcd}
$$

Given a companion (f_*, η, ε) of f, we call the cell η *unit* and call ε *counit*. We also call f_* a *companion* of f.

(ii) A *conjoint* of f consists of the following data:

- A horizontal arrow $B \xrightarrow{f^*} A$;
- Cells of the following forms:

$$
\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\
B \arrow[rr, "f^*"'] & {} & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, "f^*"] \arrow[dr, equal] & {} \arrow[d, phantom, "\varepsilon"] & A \arrow[dl, "f"] \\
 & B &
\end{tikzcd}
$$

These are required to satisfy the following equations:

$$
\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\
B \arrow[rr, "f^*"] \arrow[dr, equal] & {} \arrow[d, phantom, "\varepsilon"] & A \arrow[dl, "f"] \\
 & B &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, "f^*"] \arrow[dr, equal] & {} \arrow[d, phantom, "\varepsilon"] & A \arrow[dl, "f"] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\
 & B \arrow[rr, "f^*"'] & {} & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "f^*"] \arrow[d, equal] \arrow[dr, phantom, "="] & A \arrow[d, equal] \\
B \arrow[r, "f^*"'] & A
\end{tikzcd}
$$

We also call f^* a *conjoint* of f.

♦

Proposition 2.40. Let (f_*, η, ε) be a companion of f.

(i) The counit ε becomes cartesian.

(ii) The unit η becomes strongly opcartesian.

$$
\begin{tikzcd}
 & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\mathsf{s.opc}"] & \\
A \arrow[rr, "f_*"'] & {} & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f_*"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\mathsf{cart}"] & B \arrow[dl, equal] \\
 & B &
\end{tikzcd}
$$

In particular, we have $f_* \cong B(f, \mathrm{id})$, hence a companion f_* of f is unique up to isomorphism. The similar statement holds for conjoints.

Proof. ongoing □

2.3.3. *Sandwich lemmas.*

Lemma 2.41 (Sandwich lemma for restricting sequences)**.** Let $\mathbb{E}$ be a virtual equipment. Then, the concatenation of the sequences of cells in (8) becomes restricting whenever there exists a composing cell β of the form (9):

$$
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_{1\le i\le l}"] \arrow[d, "a"'] \arrow[dr, phantom, "(\alpha_{0,i})_i\colon \mathsf{res}"] & B \arrow[r, dashed, "(\vec{u}_{1,j})_{1\le j\le m}"] \arrow[d, "b"'] \arrow[dr, phantom, "(\alpha_{1,j})_j\colon \mathsf{s.opc}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_{1\le k\le n}"] \arrow[d, "c"] \arrow[dr, phantom, "(\alpha_{2,k})_k\colon \mathsf{res}"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "(v_{0,i})_i"'] & X \arrow[r, dashed, "(v_{1,j})_j"'] & Y \arrow[r, dashed, "(v_{2,k})_k"'] & Z
\end{tikzcd}
\quad \text{in } \mathbb{E}. \tag{8}
$$

$$
\begin{tikzcd}
W \arrow[r, dashed, "(v_{0,i})_i"] \arrow[d, equal] & X \arrow[r, dashed, "(v_{1,j})_j"] \arrow[d, phantom, "\beta\colon \mathsf{comp}"] & Y \arrow[r, dashed, "(v_{2,k})_k"] & Z \arrow[d, equal] \\
W \arrow[rrr, dotted, "w"'] & {} & {} & Z
\end{tikzcd}
\tag{9}
$$

Proof.

`Claim 1.` The composite of the following cells becomes restricting:

$$
\begin{tikzcd}
A \arrow[r, "f_*"] \arrow[dr, "f"'] & X \arrow[r, dotted, "p"] \arrow[d, equal] \arrow[dr, phantom, "="] \arrow[dl, phantom, "\mathsf{cart}"] & Y \arrow[r, "g^*"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[dl, "g"] \\
& X \arrow[r, dotted, "p"'] & Y &
\end{tikzcd}
\quad \text{in } \mathbb{E}. \tag{10}
$$

$\because$ We have to show that the following cell α is composing:

$$
\begin{tikzcd}
A \arrow[r, "f_*"] \arrow[d, equal] & X \arrow[r, dotted, "p"] & Y \arrow[r, "g^*"] & B \arrow[d, equal] \\
A \arrow[rrr, "p(f,g)"] \arrow[d, "f"'] \arrow[rrru, phantom, "\alpha"] & & & B \arrow[d, "g"] \\
X \arrow[rrr, dotted, "p"'] \arrow[rrru, phantom, "\mathsf{cart}"] & & & Y
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f_*"] \arrow[dr, "f"'] & X \arrow[r, dotted, "p"] \arrow[d, equal] \arrow[dr, phantom, "="] \arrow[dl, phantom, "\mathsf{cart}"] & Y \arrow[r, "g^*"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[dl, "g"] \\
& X \arrow[r, dotted, "p"'] & Y &
\end{tikzcd}
\quad \text{in } \mathbb{E}. \tag{11}
$$

By the universality of $p(f,g)$, we have the following equation:

$$
\begin{tikzcd}
& A \arrow[rr, "p(f,g)"] \arrow[dl, equal] \arrow[d, "f"] \arrow[dl, phantom, "\mathsf{s.opc}"] \arrow[drr, phantom, "\mathsf{cart}"] & & B \arrow[d, "g"'] \arrow[dr, equal] \arrow[dr, phantom, "\mathsf{s.opc}"] & \\
A \arrow[r, "f_*"'] \arrow[d, equal] & X \arrow[rr, dotted, "p"'] & & Y \arrow[r, "g^*"'] & B \arrow[d, equal] \\
A \arrow[rrrr, "p(f,g)"'] \arrow[rrrru, phantom, "\alpha"] & & & & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "p(f,g)"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[r, "p(f,g)"'] & B
\end{tikzcd}
$$

This implies that α is vertically invertible. Indeed, the remaining equation for vertical inverse follows from (11) and the equational definition of companions and conjoints. Then, Lemma 2.32 implies that α is strongly opcartesian. $\lozenge$

`Claim 2.` The sequence of the following cells (12) becomes restricting whenever there exists a composing cell of the form (13):

$$
\begin{tikzcd}
A \arrow[r, "f_*"] \arrow[dr, "f"'] & X \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, equal] \arrow[dr, phantom, "(\beta_i)_i\colon \mathsf{comp}"] \arrow[dl, phantom, "\mathsf{cart}"] & Y \arrow[r, "g^*"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[dl, "g"] \\
& X \arrow[r, dashed, "(v_i)_i"'] & Y &
\end{tikzcd}
\tag{12}
$$

$$
\begin{tikzcd}
X \arrow[r, dashed, "(v_i)_i"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{comp}"] & Y \arrow[d, equal] \\
X \arrow[r, dotted, "w"'] & Y
\end{tikzcd}
\tag{13}
$$

$\because$

$$
\begin{tikzcd}
A \arrow[r, "f_*"] \arrow[dr, "f"'] & X \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, equal] \arrow[dr, phantom, "(\beta_i)_i\colon \mathsf{comp}"] \arrow[dl, phantom, "\mathsf{cart}"] & Y \arrow[r, "g^*"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[dl, "g"] \\
& X \arrow[r, dashed, "(v_i)_i"'] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{comp}"] & Y \arrow[d, equal] & \\
& X \arrow[r, dotted, "w"'] & Y &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f_*"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, equal] \arrow[dr, phantom, "(\beta_i)_i\colon \mathsf{comp}"] & Y \arrow[r, "g^*"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[r, "f_*"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[r, dashed, "(v_i)_i"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{comp}"] & Y \arrow[r, "g^*"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[r, "f_*"] \arrow[dr, "f"'] & X \arrow[r, dotted, "w"] \arrow[d, equal] \arrow[dr, phantom, "="] \arrow[dl, phantom, "\mathsf{cart}"] & Y \arrow[r, "g^*"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[dl, "g"] \\
& X \arrow[r, dotted, "w"'] & Y &
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
A \arrow[r, "f_*"] \arrow[d, equal] & X \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[dr, phantom, "\mathsf{comp}"] & Y \arrow[r, "g^*"] & B \arrow[d, equal] \\
A \arrow[r, "f_*"] \arrow[dr, "f"'] \arrow[r, phantom, "\mathsf{cart}", shift right=4ex] & X \arrow[r, dotted, "w"] \arrow[d, equal] \arrow[dr, phantom, "="] & Y \arrow[r, "g^*"] \arrow[d, equal] \arrow[r, phantom, "\mathsf{cart}", shift right=4ex] & B \arrow[dl, "g"] \\
& X \arrow[r, dotted, "w"'] & Y &
\end{tikzcd}
\tag{14}
$$

By Claim 1, the bottom sequence in (14) is restricting, hence the composite of cells (14) also becomes restricting. ◊

`Claim 3.` Consider a strongly opcartesian sequence of cells $(\beta_i)_i$ and cartesian cells α and γ of the following form:

$$
\begin{tikzcd}
A \arrow[r, dotted, "p"] \arrow[d, "a"'] \arrow[dr, phantom, "\alpha\colon \mathsf{cart}"] & B \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, "b"'] \arrow[dr, phantom, "(\beta_i)_i\colon \mathsf{s.opc}"] & C \arrow[r, dotted, "r"] \arrow[d, "c"] \arrow[dr, phantom, "\gamma\colon \mathsf{cart}"] & D \arrow[d, "d"] \\
W \arrow[r, dotted, "q"'] & X \arrow[r, dashed, "(v_i)_i"'] & Y \arrow[r, dotted, "s"'] & Z
\end{tikzcd}
$$

Then, the concatenation of the above sequences becomes restricting whenever there exists a composing cell of the following form:

$$
\begin{tikzcd}
W \arrow[r, dotted, "q"] \arrow[d, equal] & X \arrow[r, dashed, "(v_i)_i"] \arrow[d, phantom, "\mathsf{comp}"] & Y \arrow[r, dotted, "s"] & Z \arrow[d, equal] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
$$

∵ By Claim 1, we have composing cells α' and γ' satisfying the following equations:

$$
\begin{tikzcd}
A \arrow[r, "a_*"] \arrow[d, equal] & W \arrow[r, dotted, "q"] \arrow[d, phantom, "\alpha'\colon \mathsf{comp}"] & X \arrow[r, "b^*"] & B \arrow[d, equal] \\
A \arrow[rrr, dotted, "p"'] \arrow[d, "a"'] & {} \arrow[d, phantom, "\alpha\colon \mathsf{cart}"] & & B \arrow[d, "b"] \\
W \arrow[rrr, dotted, "q"'] & {} & & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "a_*"] \arrow[dr, "a"'] \arrow[r, phantom, "\mathsf{cart}", shift right=4ex] & W \arrow[r, dotted, "q"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[r, "b^*"] \arrow[d, equal] \arrow[r, phantom, "\mathsf{cart}", shift right=4ex] & B \arrow[dl, "b"] \\
& W \arrow[r, dotted, "q"'] & X &
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "c_*"] \arrow[d, equal] & Y \arrow[r, dotted, "s"] \arrow[d, phantom, "\gamma'\colon \mathsf{comp}"] & Z \arrow[r, "d^*"] & D \arrow[d, equal] \\
C \arrow[rrr, dotted, "r"'] \arrow[d, "c"'] & {} \arrow[d, phantom, "\gamma\colon \mathsf{cart}"] & & D \arrow[d, "d"] \\
Y \arrow[rrr, dotted, "s"'] & {} & & Z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, "c_*"] \arrow[dr, "c"'] \arrow[r, phantom, "\mathsf{cart}", shift right=4ex] & Y \arrow[r, dotted, "s"] \arrow[d, equal] \arrow[dr, phantom, "="] & Z \arrow[r, "d^*"] \arrow[d, equal] \arrow[r, phantom, "\mathsf{cart}", shift right=4ex] & D \arrow[dl, "d"] \\
& Y \arrow[r, dotted, "s"'] & Z &
\end{tikzcd}
$$

Then, the following equations show that a sequence of cells (16) can be presented as a vertical composite of two sequences in (15):

$$
\begin{tikzcd}
A \arrow[r, "a_*"] \arrow[d, equal] & W \arrow[r, dotted, "q"] \arrow[d, phantom, "\alpha'\colon \mathsf{comp}"] & X \arrow[r, "b^*"] & B \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "c_*"] \arrow[d, equal] & Y \arrow[r, dotted, "s"] \arrow[d, phantom, "\gamma'\colon \mathsf{comp}"] & Z \arrow[r, "d^*"] & D \arrow[d, equal] \\
A \arrow[rrr, dotted, "p"] \arrow[d, "a"'] & {} \arrow[d, phantom, "\alpha\colon \mathsf{cart}"] & & B \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, "b"'] \arrow[dr, phantom, "(\beta_i)_i\colon \mathsf{s.opc}"] & C \arrow[rrr, dotted, "r"] \arrow[d, "c"] & {} \arrow[d, phantom, "\gamma\colon \mathsf{cart}"] & & D \arrow[d, "d"] \\
W \arrow[rrr, dotted, "q"'] & {} & & X \arrow[r, dashed, "(v_i)_i"'] & Y \arrow[rrr, dotted, "s"'] & {} & & Z
\end{tikzcd}
\tag{15}
$$

$$
= \quad
\begin{tikzcd}
A \arrow[r, "a_*"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathsf{cart}", near start] & W \arrow[r, dotted, "q"] \arrow[dl, equal] & X \arrow[r, "b^*"] \arrow[dr, equal] \arrow[d, phantom, "\shortparallel"] & B \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, "b"] \arrow[dl, phantom, "\mathsf{cart}"] \arrow[dr, phantom, "(\beta_i)_i\colon \mathsf{s.opc}"] & C \arrow[r, "c_*"] \arrow[d, "c"'] \arrow[dr, phantom, "\mathsf{cart}"] & Y \arrow[r, dotted, "s"] \arrow[dl, equal] \arrow[dr, phantom, "\shortparallel"] & Z \arrow[r, "d^*"] \arrow[dr, equal] & D \arrow[d, "d"] \arrow[dl, phantom, "\mathsf{cart}"] \\
W \arrow[rrr, dotted, "q"'] & & & X \arrow[r, dashed, "(v_i)_i"'] & Y \arrow[rrr, dotted, "s"'] & & & Z
\end{tikzcd}
\tag{16}
$$

By Claim 2, the sequence of cells (16) is restricting. Since the top sequence in (15) is composing, the vertical cancellation implies that the bottom sequence in (15) is restricting. $\Diamond$

Since the sequences $\vec{\alpha}_0$ and $\vec{\alpha}_2$ are restricting, we have composing cells of the following form:

$$
\begin{tikzcd}
W \arrow[r, dashed, "\vec{v}_0"] \arrow[d, equal] \arrow[dr, phantom, "\beta_0\colon \mathsf{comp}"] & X \arrow[d, equal] \\
W \arrow[r, dotted, "w_0"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, dashed, "\vec{v}_2"] \arrow[d, equal] \arrow[dr, phantom, "\beta_2\colon \mathsf{comp}"] & Z \arrow[d, equal] \\
Y \arrow[r, dotted, "w_2"'] & Z
\end{tikzcd}
$$

Since the top sequence on the left side of the following is composing, we obtain a unique cell γ satisfying the following equation:

$$
\begin{tikzcd}
W \arrow[r, dashed, "\vec{v}_0"] \arrow[d, equal] \arrow[dr, phantom, "\beta_0\colon \mathsf{comp}"] & X \arrow[r, dashed, "\vec{v}_1"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & Y \arrow[r, dashed, "\vec{v}_2"] \arrow[d, equal] \arrow[dr, phantom, "\beta_2\colon \mathsf{comp}"] & Z \arrow[d, equal] \\
W \arrow[r, dotted, "w_0"'] \arrow[d, equal] & X \arrow[r, dashed, "\vec{v}_1"'] \arrow[d, phantom, "\gamma"] & Y \arrow[r, dotted, "w_2"'] & Z \arrow[d, equal] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
=
\begin{tikzcd}
W \arrow[r, dashed, "\vec{v}_0"] \arrow[d, equal] & X \arrow[r, dashed, "\vec{v}_1"] \arrow[d, phantom, "\beta\colon \mathsf{comp}"] & Y \arrow[r, dashed, "\vec{v}_2"] & Z \arrow[d, equal] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
$$

By the cancellation for composing sequences, γ is composing. By the definition of restricting sequences, we have $A \overset{p_0}{\dashrightarrow} B$ and $A \overset{p_2}{\dashrightarrow} B$, and the cells $\vec{\alpha}_0 ⨾ \beta_0$ and $\vec{\alpha}_2 ⨾ \beta_2$ are decomposed into a composing cell and a cartesian cell. Then, we have the following:

$$
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, "a"'] \arrow[dr, phantom, "\vec{\alpha}_0\colon \mathsf{res}"] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, "b"'] \arrow[dr, phantom, "\vec{\alpha}_1\colon \mathsf{s.opc}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, "c"] \arrow[dr, phantom, "\vec{\alpha}_2\colon \mathsf{res}"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "\vec{v}_0"'] \arrow[d, equal] & X \arrow[r, dashed, "\vec{v}_1"'] \arrow[d, phantom, "\beta\colon \mathsf{comp}"] & Y \arrow[r, dashed, "\vec{v}_2"'] & Z \arrow[d, equal] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, "a"'] \arrow[dr, phantom, "\vec{\alpha}_0\colon \mathsf{res}"] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, "b"'] \arrow[dr, phantom, "\vec{\alpha}_1\colon \mathsf{s.opc}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, "c"] \arrow[dr, phantom, "\vec{\alpha}_2\colon \mathsf{res}"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "\vec{v}_0"] \arrow[d, equal] \arrow[dr, phantom, "\beta_0\colon \mathsf{comp}"] & X \arrow[r, dashed, "\vec{v}_1"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & Y \arrow[r, dashed, "\vec{v}_2"] \arrow[d, equal] \arrow[dr, phantom, "\beta_2\colon \mathsf{comp}"] & Z \arrow[d, equal] \\
W \arrow[r, dotted, "w_0"'] \arrow[d, equal] & X \arrow[r, dashed, "\vec{v}_1"'] \arrow[d, phantom, "\gamma\colon \mathsf{comp}"] & Y \arrow[r, dotted, "w_2"'] & Z \arrow[d, equal] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{comp}"] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{comp}"] & D \arrow[d, equal] \\
A \arrow[r, dotted, "p_0"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, "b"'] \arrow[dr, phantom, "\vec{\alpha}_1\colon \mathsf{s.opc}"] & C \arrow[r, dotted, "p_2"] \arrow[d, "c"] \arrow[dr, phantom, "\mathsf{cart}"] & D \arrow[d, "d"] \\
W \arrow[r, dotted, "w_0"'] \arrow[d, equal] & X \arrow[r, dashed, "\vec{v}_1"'] \arrow[d, phantom, "\gamma\colon \mathsf{comp}"] & Y \arrow[r, dotted, "w_2"'] & Z \arrow[d, equal] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, equal] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, phantom, "\mathsf{comp}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] & D \arrow[d, equal] \\
A \arrow[r, dotted, "p_0"] \arrow[d, "a"'] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, phantom, "\mathsf{res}"] & C \arrow[r, dotted, "p_2"] & D \arrow[d, "d"] \\
W \arrow[r, dotted, "w_0"'] \arrow[d, equal] & X \arrow[r, dashed, "\vec{v}_1"'] \arrow[d, phantom, "\gamma\colon \mathsf{comp}"] & Y \arrow[r, dotted, "w_2"'] & Z \arrow[d, equal] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
$$

$$
= \begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, equal] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, phantom, "\mathsf{comp}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] & D \arrow[d, equal] \\
A \arrow[r, dotted, "p_0"] \arrow[d, equal] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, phantom, "\mathsf{comp}"] & C \arrow[r, dotted, "p_2"] & D \arrow[d, equal] \\
A \arrow[rrr, dotted] \arrow[d, "a"'] & \arrow[d, phantom, "\mathsf{cart}"] & & D \arrow[d, "d"] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
= \begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, equal] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, phantom, "\mathsf{comp}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] & D \arrow[d, equal] \\
A \arrow[rrr, dotted] \arrow[d, "a"'] & \arrow[d, phantom, "\mathsf{cart}"] & & D \arrow[d, "d"] \\
W \arrow[rrr, dotted, "w"'] & & & Z
\end{tikzcd}
$$

Here, the third equation follows from Claim 3. This finishes the proof. □

Lemma 2.42 (Sandwich lemma for strongly opcartesian sequences)**.** Let $\mathbb{E}$ be a virtual equipment. Then, the concatenation of the following sequences of cells becomes strongly opcartesian:

$$
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_{1\le i\le l}"] \arrow[d, "a"'] \arrow[dr, phantom, "(\alpha_{0,i})_i\colon \mathsf{s.opc}"] & B \arrow[r, dashed, "(\vec{u}_{1,j})_{1\le j\le m}"] \arrow[d, "b"'] \arrow[dr, phantom, "(\alpha_{1,j})_j\colon \mathsf{res}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_{1\le k\le n}"] \arrow[d, "c"] \arrow[dr, phantom, "(\alpha_{2,k})_k\colon \mathsf{s.opc}"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "(v_{0,i})_i"'] & X \arrow[r, dashed, "(v_{1,j})_j"'] & Y \arrow[r, dashed, "(v_{2,k})_k"'] & Z
\end{tikzcd}
\quad \text{in } \mathbb{E}. \tag{17}
$$

Proof. We begin with the following special case:

`Claim 1.` In general, the sequence of the following form becomes strongly opcartesian:

$$
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, "f"'] \arrow[dr, phantom, "(\alpha_i)_i\colon \mathsf{s.opc}"] & B \arrow[r, "g_*"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathsf{cart}"] & Y \arrow[r, dashed, "\vec{w}"] \arrow[dl, equal] \arrow[d, phantom, "="] & Z \arrow[d, equal] \\
X \arrow[r, dashed, "(v_i)_i"'] & Y \arrow[rr, dashed, "\vec{w}"'] & & Z
\end{tikzcd}
\tag{18}
$$

∵ It suffices to show that the sequence (18) is weakly opcartesian since the remaining part is trivial. Take $X \xrightarrow{x} X'$, $Z \xrightarrow{z} Z'$, and $X' \overset{r}{\dashrightarrow} Z'$ arbitrarily. Then, there exist bijective correspondences among the cells of the following forms:

$$
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, "f"'] & B \arrow[r, "g_*"] & Y \arrow[r, dashed, "\vec{w}"] & Z \arrow[dd, "z"] \\
X \arrow[d, "x"'] & \cdot & & \\
X' \arrow[rrr, dotted, "r"'] & & & Z'
\end{tikzcd}
\;\Bigg\|\;
\begin{tikzcd}
X \arrow[r, "f^*"] \arrow[d, "x"'] & A \arrow[r, dashed, "(\vec{u}_i)_i"] & B \arrow[r, "g_*"] \arrow[d, phantom, "\cdot"] & Y \arrow[r, dashed, "\vec{w}"] & Z \arrow[d, "z"] \\
X' \arrow[rrrr, dotted, "r"'] & & & & Z'
\end{tikzcd}
\;\Bigg\|\;
\begin{tikzcd}
X \arrow[r, dashed, "\vec{v}"] \arrow[d, "x"'] & Y \arrow[r, dashed, "\vec{w}"] \arrow[d, phantom, "\cdot"] & Z \arrow[d, "z"] \\
X' \arrow[rr, dotted, "r"'] & & Z'
\end{tikzcd}
$$

Here, the correspondence between the left and the middle cells follows from the equational definition of the companion f_*, and the correspondence between the middle and the right cells follows from the strong opcartesianess of $\vec{\alpha}$. This proves that (18) is weakly opcartesian. ◊

Since $\vec{\alpha}_1 := (\alpha_{1,j})_j$ is restricting, we obtain $X \overset{p}{\dashrightarrow} Y$, $B \overset{q}{\dashrightarrow} C$, composing cells β and γ, and a cartesian cell δ satisfying the following equation:

$$
\begin{tikzcd}
B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, "b"'] \arrow[dr, phantom, "\vec{\alpha}_1"] & C \arrow[d, "c"] \\
X \arrow[r, dashed, "\vec{v}_1"] \arrow[d, equal] \arrow[dr, phantom, "\beta\colon \mathsf{comp}"] & Y \arrow[d, equal] \\
X \arrow[r, dotted, "p"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, equal] \arrow[dr, phantom, "\gamma\colon \mathsf{comp}"] & C \arrow[d, equal] \\
B \arrow[r, dotted, "q"] \arrow[d, "b"'] \arrow[dr, phantom, "\delta\colon \mathsf{cart}"] & C \arrow[d, "c"] \\
X \arrow[r, dotted, "p"'] & Y
\end{tikzcd}
\quad \text{in } \mathbb{E}.
$$

The first Sandwich lemma (Lemma 2.41) implies that the right side of the following is restricting, hence a unique cell ε becomes composing:

$$
\begin{tikzcd}
B \arrow[r, "b_*"] \arrow[d, equal] \arrow[drrr, phantom, "\varepsilon"] & X \arrow[r, dotted, "p"] & Y \arrow[r, "c^*"] & C \arrow[d, equal] \\
B \arrow[rrr, dotted, "q"'] \arrow[d, "b"'] \arrow[drrr, phantom, "\delta\colon \mathsf{cart}"] & & & C \arrow[d, "c"] \\
X \arrow[rrr, dotted, "p"'] & & & Y
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "b_*"] \arrow[d, "b"'] \arrow[dr, phantom, "\mathsf{cart}"] & X \arrow[r, dotted, "p"] \arrow[dl, equal] \arrow[dr, phantom, "="] & Y \arrow[r, "c^*"] \arrow[dr, equal] & C \arrow[d, "c"] \arrow[dl, phantom, "\mathsf{cart}"] \\
X \arrow[rrr, dotted, "p"'] & & & Y
\end{tikzcd}
\quad \text{in } \mathbb{E}.
$$

Then, Claim 1 and its horizontal dual imply that a vertical composite of the following sequences is strongly opcartesian:

$$
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "b_*"] \arrow[d, equal] \arrow[drrr, phantom, "\varepsilon\colon \mathsf{comp}"] & X \arrow[r, dotted, "p"] & Y \arrow[r, "c^*"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] \\
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, "a"'] \arrow[dr, phantom, "\vec{\alpha}_0\colon \mathsf{s.opc}"] & B \arrow[rrr, dotted, "q"] \arrow[d, "b"] \arrow[drrr, phantom, "\delta\colon \mathsf{cart}"] & & & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, "c"'] \arrow[dr, phantom, "\vec{\alpha}_2\colon \mathsf{s.opc}"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "\vec{v}_0"'] & X \arrow[rrr, dotted, "p"'] & & & Y \arrow[r, dashed, "\vec{v}_2"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, "a"'] \arrow[dr, phantom, "\vec{\alpha}_0\colon \mathsf{s.opc}"] & B \arrow[r, "b_*"] \arrow[d, "b"] & X \arrow[r, dotted, "p"] \arrow[dl, equal] \arrow[dr, phantom, "="] & Y \arrow[r, "c^*"] \arrow[dr, equal] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, "c"'] \arrow[dr, phantom, "\vec{\alpha}_2\colon \mathsf{s.opc}"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "\vec{v}_0"'] & X \arrow[rrr, dotted, "p"'] & & & Y \arrow[r, dashed, "\vec{v}_2"'] & Z
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, "a"'] \arrow[dr, phantom, "\vec{\alpha}_0\colon \mathsf{s.opc}"] & B \arrow[r, "b_*"] \arrow[d, "b"] & X \arrow[r, dotted, "p"] \arrow[dl, equal] \arrow[dr, phantom, "="] & Y \arrow[r, "c^*"] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] \\
W \arrow[r, dashed, "\vec{v}_0"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[rr, dotted, "p"] \arrow[d, equal] \arrow[drr, phantom, "="] & & Y \arrow[r, "c^*"] \arrow[dr, equal] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, "c"] \arrow[dr, phantom, "\vec{\alpha}_2\colon \mathsf{s.opc}"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "\vec{v}_0"'] & X \arrow[rrr, dotted, "p"'] & & & Y \arrow[r, dashed, "\vec{v}_2"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, "a"'] \arrow[drrrrr, phantom, "\mathsf{s.opc}"] & B \arrow[r, "b_*"] & X \arrow[r, dotted, "p"] & Y \arrow[r, "c^*"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] & D \arrow[d, equal] \\
W \arrow[r, dashed, "\vec{v}_0"] \arrow[d, equal] \arrow[drrrrr, phantom, "\mathsf{s.opc}"] & X \arrow[rr, dotted, "p"] & & Y \arrow[r, "c^*"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "\vec{v}_0"'] & X \arrow[rr, dotted, "p"'] & & Y \arrow[rr, dashed, "\vec{v}_2"'] & & Z
\end{tikzcd}
$$

Thus, by vertical cancellation, the sequence $(\vec{\alpha}_0, \delta, \vec{\alpha}_2)$ becomes strongly opcartesian. This implies that a vertical composite of the following sequences is strongly opcartesian:

$$
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, "a"'] \arrow[dr, phantom, "\vec{\alpha}_0"] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, "b"'] \arrow[dr, phantom, "\vec{\alpha}_1"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, "c"] \arrow[dr, phantom, "\vec{\alpha}_2"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "\vec{v}_0"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[r, dashed, "\vec{v}_1"] \arrow[d, equal] \arrow[dr, phantom, "\beta\colon \mathsf{comp}"] & Y \arrow[r, dashed, "\vec{v}_2"] \arrow[d, equal] \arrow[dr, phantom, "="] & Z \arrow[d, equal] \\
W \arrow[r, dashed, "\vec{v}_0"'] & X \arrow[r, dotted, "p"'] & Y \arrow[r, dashed, "\vec{v}_2"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, dashed, "(\vec{u}_{1,j})_j"] \arrow[d, equal] \arrow[dr, phantom, "\gamma\colon \mathsf{comp}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] \\
A \arrow[r, dashed, "(\vec{u}_{0,i})_i"] \arrow[d, "a"'] \arrow[dr, phantom, "\vec{\alpha}_0"] & B \arrow[r, dotted, "q"] \arrow[d, "b"'] \arrow[dr, phantom, "\delta\colon \mathsf{cart}"] & C \arrow[r, dashed, "(\vec{u}_{2,k})_k"] \arrow[d, "c"] \arrow[dr, phantom, "\vec{\alpha}_2"] & D \arrow[d, "d"] \\
W \arrow[r, dashed, "\vec{v}_0"'] & X \arrow[r, dotted, "p"'] & Y \arrow[r, dashed, "\vec{v}_2"'] & Z
\end{tikzcd}
\ \colon \mathsf{s.opc}
$$

Then, Lemma 2.37 implies that the sequence $(\vec{\alpha}_0, \vec{\alpha}_1, \vec{\alpha}_2)$ is strongly opcartesian, which is the desired conclusion. □

Corollary 2.43. Let $\mathbb{E}$ be a virtual equipment.

(i) Suppose that we are given data satisfying the following equation:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\
X \arrow[r, dotted, "v"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, equal] & & B \arrow[d, equal] \arrow[dl, phantom, "="] \\
X \arrow[r, "f^*"'] \arrow[d, equal] \arrow[drr, phantom, "\beta"] & A \arrow[r, dashed, "\vec{u}"'] & B \arrow[d, equal] \\
X \arrow[rr, dotted, "v"'] & & B
\end{tikzcd}
$$

Then, the cell α is strongly opcartesian if and only if β is composing.

(ii) Suppose that we are given data satisfying the following equation:

$$
\begin{tikzcd}
A \arrow[rr, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha"] & & B \arrow[d, "g"] \\
X \arrow[rr, dotted, "v"'] & & Y
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rrrr, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, equal] \arrow[drrrr, phantom, "="] & & & & B \arrow[d, "g"] \arrow[dl, equal] \\
X \arrow[r, "f^*"'] \arrow[d, equal] \arrow[drrrr, phantom, "\beta"] & A \arrow[rr, dashed, "\vec{u}"'] & & B \arrow[r, "g_*"'] & Y \arrow[d, equal] \\
X \arrow[rrrr, dotted, "v"'] & & & & Y
\end{tikzcd}
\tag{19}
$$

Then, the cell α is strongly opcartesian if and only if β is composing.

Proof. The top sequence on the right side of (19) is strongly opcartesian by Lemma 2.42. Then, (ii) follows from the vertical cancellation Lemma 2.37. The proof of (i) is similar. □

2.4. **Extensions and lifts.**

Definition 2.44 (Extending cells)**.** Let $\mathbb{X}$ be an AVDC. A cell

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha"] & B \arrow[r, "p"] & C \arrow[d, equal] \\
X \arrow[rr, dotted, "v"'] & & C
\end{tikzcd}
\quad \text{in } \mathbb{X}
\tag{20}
$$

is called *extending* if, for any Y, $\vec{q}$, g, and a cell β of the following form, there exists a unique cell $\bar{\beta}$ satisfying the following equation:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\beta"] & B \arrow[r, dashed, "\vec{q}"] & Y \arrow[d, "g"] \\
X \arrow[rr, dotted, "v"'] & & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, dashed, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "\bar{\beta}"] & Y \arrow[d, "g"] \\
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha"] & B \arrow[r, "p"] & C \arrow[d, equal] \\
X \arrow[rr, dotted, "v"'] & & C
\end{tikzcd}
\quad \text{in } \mathbb{X}.
$$

By the universal property, a horizontal arrow p in the extending cell (20) is unique up to isomorphism, hence we write p for $\vec{u} \triangleright^f v$ and call it an *extension of* (f, v) *along* $\vec{u}$. When f is the identity, we also use a notation $\vec{u} \triangleright v$. When $\mathtt{length}(v) = 0$ (hence $X = C$), we also use a notation $\vec{u} \triangleright^f X$. An extending cell is denoted by the following:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\mathsf{ext}"] & B \arrow[r, "\vec{u} \triangleright^f v"] & C \arrow[d, equal] \\
X \arrow[rr, dotted, "v"'] & & C
\end{tikzcd}
$$

♦

Definition 2.45 (Lifting cells)**.** *Lifting* cells are defined to be the horizontal dual of extending cells. It is denoted by the following:

$$
\begin{tikzcd}
C \arrow[r, "v^f \blacktriangleleft \vec{u}"] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{lift}"] & B \arrow[r, dashed, "\vec{u}"] & A \arrow[d, "f"] \\
C \arrow[rr, dotted, "v"'] & & X
\end{tikzcd}
$$

The horizontal arrow $v^f \blacktriangleleft \vec{u}$ is called a *lift of* (v, f) *along* $\vec{u}$. ♦

Proposition 2.46. Let $\mathbb{X}$ be an AVDC. Suppose that we are given data satisfying the following equation:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha"] & B \arrow[r, "p"] & C \arrow[d, equal] \\
X \arrow[rr, dotted, "v"'] & & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[drr, phantom, "\beta"] & B \arrow[r, "p"] & C \arrow[d, equal] \\
A \arrow[rr, dotted, "w"'] \arrow[d, "f"'] \arrow[drr, phantom, "\mathsf{cart}"] & & C \arrow[d, equal] \\
X \arrow[rr, dotted, "v"'] & & C
\end{tikzcd}
$$

Then, the cell α is extending if and only if so is β.

Proof. This immediately follows from the universal properties of cartesian cells and extending cells. □

Proposition 2.47. Let $\mathbb{E}$ be a virtual equipment. Suppose that there exists a lifting cell and an extending cell of the following forms. Then cells of the form α and those of the form β bijectively corresponds each other, through the following equation:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha"] & A' \arrow[r, "u"] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
C \arrow[rr, "q^g\blacktriangleleft\vec{b}"'] \arrow[d, equal] \arrow[drrr, phantom, "\mathsf{lift}"] & & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
C \arrow[rrr, dotted, "q"'] & & & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[drr, phantom, "\beta"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drrr, phantom, "\mathsf{ext}"] & A' \arrow[rr, "\vec{a}\triangleright^f q"'] & & D \arrow[d, equal] \\
C \arrow[rrr, dotted, "q"'] & & & D
\end{tikzcd}
$$

Moreover, α is extending if and only if β is a lifting. This can be presented as follows:

$$\vec{a} \triangleright^f (q^g \blacktriangleleft \vec{b}) \cong (\vec{a} \triangleright^f q)^g \blacktriangleleft \vec{b}.$$

Proof. ongoingSuppose that α is extending. To show that β is lifting, take X, $X \xrightarrow{x} A'$, and $X \overset{\vec{v}}{\dashrightarrow} B'$ arbitrarily. Then, there are bijective correspondences among cells of the following forms:

$$
\begin{tikzcd}
X \arrow[r, dashed, "\vec{v}"] \arrow[d, "x"'] \arrow[drr, phantom, "\cdot"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
A' \arrow[rr, "\vec{a}\triangleright^f q"'] & & D
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
A' \arrow[r, "x^*"] \arrow[d, equal] \arrow[drrr, phantom, "\cdot"] & X \arrow[r, dashed, "\vec{v}"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
A' \arrow[rrr, "\vec{a}\triangleright^f q"'] & & & D
\end{tikzcd}
\quad\Bigg\|\quad a
$$

□

Definition 2.48. Let $\mathbb{X}$ be an AVDC. Suppose that we are given an extending cell α, a lifting cell β, and x, y of the following forms:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha\colon \mathsf{ext}"] & A' \arrow[r, "\vec{a}\triangleright^f p"] & D \arrow[d, equal] & \\
C \arrow[rr, dotted, "p"'] & & D \arrow[r, dashed, "x"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
& D \arrow[r, "q^g\blacktriangleleft\vec{b}"] \arrow[d, equal] \arrow[drr, phantom, "\beta\colon \mathsf{lift}"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
Y \arrow[r, dashed, "y"] & D \arrow[rr, dotted, "q"'] & & E
\end{tikzcd}
$$

We say that x *preserves the extension* $\vec{a} \rhd^f p$ *(or the extending cell α)* if there exist two composing cells and an extending cell satisfying the following equality.

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[rr, phantom, shift right=4ex, "\alpha\colon \mathsf{ext}"] & A' \arrow[r, "\vec{a}\rhd^f p"] & D \arrow[r, dashed, "x"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[d, equal] \\
C \arrow[rr, dotted, "p"'] \arrow[d, equal] \arrow[rrr, phantom, shift right=4ex, "\mathsf{comp}"] & & D \arrow[r, dashed, "x"'] & X \arrow[d, equal] \\
C \arrow[rrr, dotted] & & & X
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "\vec{a}\rhd^f p"] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "\mathsf{comp}"] & D \arrow[r, dashed, "x"] & X \arrow[d, equal] \\
A \arrow[r, dashed, "\vec{a}"'] \arrow[d, "f"'] \arrow[rrr, phantom, shift right=4ex, "\mathsf{ext}"] & A' \arrow[rr] & & X \arrow[d, equal] \\
C \arrow[rrr, dotted] & & & E
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

Dually, y *preserves the lift* $q^g\blacktriangleleft \vec{b}$ *(or, the lifting cell β)* if:

$$
\begin{tikzcd}
Y \arrow[r, dashed, "y"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "q^g\blacktriangleleft\vec{b}"] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "\beta\colon \mathsf{lift}"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
Y \arrow[r, dashed, "y"'] \arrow[d, equal] \arrow[rrr, phantom, shift right=4ex, "\mathsf{comp}"] & D \arrow[rr, dotted, "q"'] & & E \arrow[d, equal] \\
Y \arrow[rrr, dotted] & & & E
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, dashed, "y"] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "\mathsf{comp}"] & D \arrow[r, "q^g\blacktriangleleft\vec{b}"] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
Y \arrow[rr] \arrow[d, equal] \arrow[rrr, phantom, shift right=4ex, "\mathsf{lift}"] & & B' \arrow[r, dashed, "\vec{b}"'] & E \arrow[d, "g"] \\
Y \arrow[rrr, dotted] & & & E
\end{tikzcd}
\quad \text{in } \mathbb{E}.
$$

♦

Proposition 2.49 (Ext–lift lemma)**.** Let $\mathbb{E}$ be a virtual equipment. Suppose that we are given an extending cell α and a lifting cell β of the following forms:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[rr, phantom, shift right=4ex, "\alpha\colon \mathsf{ext}"] & A' \arrow[r, "u"] & D \arrow[d, equal] \\
C \arrow[rr, dotted, "p"'] & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, "v"] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "\beta\colon \mathsf{lift}"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
D \arrow[rr, dotted, "q"'] & & E
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

Suppose that p preserves the lifting cell β and that q preserves the extending cell α. Then, the following are equivalent:

(i) u preserves the lifting cell β.
(ii) v preserves the extending cell α.

Proof. We show (i) $\Longrightarrow$ (ii), and the converse is shown in a similar way. By the assumption, we have a composing cell of the following form:

$$
\begin{tikzcd}
C \arrow[r, dotted, "p"] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "\mathsf{comp}"] & D \arrow[r, dotted, "q"] & E \arrow[d, equal] \\
C \arrow[rr, dotted, "c_{p,q}"'] & & E
\end{tikzcd}
$$

Therefore, we can consider the following composites of cells:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[rr, phantom, shift right=4ex, "\alpha\colon \mathsf{ext}"] & A' \arrow[r, "u"] & D \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "="] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[ddr, phantom, "="] & B \arrow[dd, equal] \\
C \arrow[rr, dotted, "p"'] \arrow[d, equal] \arrow[rrr, phantom, shift right=4ex, "\mathsf{comp}"] & & D \arrow[r, "v"] & B' \arrow[d, equal] & \\
C \arrow[rrr] \arrow[d, equal] \arrow[rrrr, phantom, shift right=4ex, "\mathsf{lift}"] & & & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
C \arrow[rrrr, dotted, "c_{p,q}"'] & & & & E
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[rr, phantom, shift right=4ex, "\alpha\colon \mathsf{ext}"] & A' \arrow[r, "u"] & D \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "="] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
C \arrow[rr, dotted, "p"'] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "="] & & D \arrow[r, "v"] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "\beta\colon \mathsf{lift}"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
C \arrow[rr, dotted, "p"] \arrow[d, equal] \arrow[rrrr, phantom, shift right=4ex, "\mathsf{comp}"] & & D \arrow[rr, dotted, "q"] & & E \arrow[d, equal] \\
C \arrow[rrrr, dotted, "c_{p,q}"'] & & & & E
\end{tikzcd}
$$

$$
= \begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "v"] \arrow[d, equal] \arrow[drr, phantom, "\beta\colon \mathsf{lift}"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[drrr, phantom, "\mathsf{comp}"] & D \arrow[rr, "q"] & & E \arrow[d, equal] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drrrr, phantom, "\mathsf{ext}"] & A' \arrow[rrr] & & & E \arrow[d, equal] \\
C \arrow[rrrr, dotted, "c_{p,q}"'] & & & & E
\end{tikzcd}
= \begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{comp}"] & D \arrow[r, "v"] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[rr] \arrow[d, equal] \arrow[drrr, phantom, "\mathsf{lift}"] & & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drrrr, phantom, "\mathsf{ext}"] & A' \arrow[rrr] & & & E \arrow[d, equal] \\
C \arrow[rrrr, dotted, "c_{p,q}"'] & & & & E
\end{tikzcd}
$$

$$
= \begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{comp}"] & D \arrow[r, "v"] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[ddr, phantom, "="] & B \arrow[dd, equal] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drrr, phantom, "\mathsf{ext}"] & A' \arrow[rr] & & B' \arrow[d, equal] & \\
C \arrow[rrr] \arrow[d, equal] \arrow[drrrr, phantom, "\mathsf{lift}"] & & & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
C \arrow[rrrr, dotted, "c_{p,q}"'] & & & & E
\end{tikzcd}
$$

Here, the first equality follows from that p preserves β, the second one follows from that q preserves α, the third one follows from that u preserves β, and the last one follows from Proposition 2.47. Then, the universality of lifting cells proves that v preserves the extending cell α. □

2.5. Kan extensions.

Definition 2.50. Let $\mathbb{X}$ be an AVDC. A cell

$$
\begin{tikzcd}
A \arrow[rr, dashed, "\vec{u}"] \arrow[dr, "f"'] & \alpha & B \arrow[dl, "g"] \\
& X &
\end{tikzcd}
\quad \text{in } \mathbb{X} \tag{21}
$$

is called a *lan-cell* if, for any C, $\vec{v}$, h, and a cell β of the following form, there exists a unique cell $\bar{\beta}$ satisfying the following equation:

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[dr, "f"'] & B \arrow[r, dashed, "\vec{v}"] \arrow[d, phantom, "\beta"] & C \arrow[dl, "h"] \\
& X &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[dr, "f"'] \arrow[dr, phantom, "\alpha", shift left=3] & B \arrow[r, dashed, "\vec{v}"] \arrow[d, "g"] \arrow[dr, phantom, "\bar{\beta}", shift right=3] & C \arrow[dl, "h"] \\
& X &
\end{tikzcd}
\quad \text{in } \mathbb{X}.
$$

If the cell α in (21) is a lan-cell, g is uniquely determined up to horizontally invertible cells by the universal property. Then, g is called a left Kan extension of f along $\vec{u}$ and denoted by $\mathrm{Lan}_{\vec{u}} f$. A lan-cell is sometimes denoted by the following:

$$
\begin{tikzcd}
A \arrow[rr, dashed, "\vec{u}"] \arrow[dr, "f"'] & \mathsf{lan} & B \arrow[dl, "\mathrm{Lan}_{\vec{u}} f"] \\
& X &
\end{tikzcd}
$$

♦

Definition 2.51. *Ran-cells* are defined as the horizontal dual of lan-cells. A ran-cell is sometimes denoted by the following:

$$\begin{tikzcd}
B \arrow[rr, dashed, "\vec{u}"] \arrow[dr, "\mathrm{Ran}_{\vec{u}} f"'] & \arrow[d, phantom, "\mathsf{ran}"] & A \arrow[dl, "f"] \\
& X &
\end{tikzcd}$$

The vertical arrow $\mathrm{Ran}_{\vec{u}} f$ is called a *right Kan extension of f along $\vec{u}$.* ♦

Proposition 2.52. Let $\mathbb{X}$ be an AVDC. Suppose that we are given cells α, β, and a lan-cell of the following forms:

$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[dr, "f"'] \arrow[r, phantom, shift right=3ex, "\mathsf{lan}"] & B \arrow[r, dashed, "\vec{v}"] \arrow[d, "g"] \arrow[r, phantom, shift right=3ex, "\alpha"] & C \arrow[dl, "h"] \\
& X &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[dr, "f"'] & B \arrow[r, dashed, "\vec{v}"] \arrow[d, phantom, "\beta"] & C \arrow[dl, "h"] \\
& X &
\end{tikzcd}
\quad \text{in } \mathbb{X}.$$

Then, α is a lan-cell if and only if β is as well.

Proof. This immediately follows from the definition of lan-cells. □

Proposition 2.53. Let $\mathbb{X}$ be an AVDC. Suppose that we are given a companion g_* and cells of the following forms:

$$\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[dr, "f"'] \arrow[r, phantom, shift right=3ex, "\alpha"] & B \arrow[r, "g_*"] \arrow[d, "g"] \arrow[r, phantom, shift right=3ex, "\mathsf{cart}"] & C \arrow[dl, equal] \\
& C &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{u}"] \arrow[dr, "f"'] & B \arrow[r, "g_*"] \arrow[d, phantom, "\beta"] & C \arrow[dl, equal] \\
& C &
\end{tikzcd}
\quad \text{in } \mathbb{X}.$$

Then, α is a lan-cell if and only if β is extending.

Proof. ongoing □

Proposition 2.54. Let $\mathbb{X}$ be an AVDC. Suppose that we are given a right-composing sequence $(\alpha_i)_i$ and a cell β of the following forms:

$$\begin{tikzcd}
A \arrow[r, dashed, "(\vec{u}_i)_i"] \arrow[d, "d"'] \arrow[dr, phantom, "(\alpha_i)_i : \mathsf{r.cmp}"] & C \arrow[d, equal] \\
B \arrow[r, dashed, "(v_i)_i"] \arrow[d, "f"'] \arrow[dr, phantom, "\beta"] & C \arrow[dl, "g"] \\
X &
\end{tikzcd}$$

Then, β is a lan-cell if and only if the composite $\vec{\alpha} ⨾ \beta$ is as well.

Proof. By the universality of the right-composing sequence $\vec{\alpha}$, there is a bijective correspondence between cells of the following forms:

□

Lemma 2.55.

$$\begin{tikzcd}
C \arrow[r, "p_*"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{lan}"] & D \arrow[dl, "l"] \\
A &
\end{tikzcd}$$

If p is fully faithful, then $f \cong p ⨾ l$.

Proof. ongoing

$$\begin{tikzcd} C \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & C \arrow[d, "p"] \\ C \arrow[r, "p_*", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{lan}" near start] & D \arrow[dl, "l"] \\ A & \end{tikzcd}$$

□

aaaaa

3. Class of weights

3.1. Weights.

Definition 3.1 (Weights)**.** Let $\mathbb{E}$ be a virtual equipment. Let us consider the following conditions for a horizontal arrow $X \xrightarrow{u} Y$ in $\mathbb{E}$:

(LW1) For any horizontal arrow $\cdot \xrightarrow{v} X$ in $\mathbb{E}$, the composite $v \odot u$ exists.

(LW2) For any $X \xrightarrow{f} Z$ and $Z \xrightarrow{v} W$ in $\mathbb{E}$, the following has its extension:

$$\begin{tikzcd} X \arrow[r, "u", "\shortmid" marking] \arrow[d, "f"'] & Y \\ Z \arrow[r, "v"', "\shortmid" marking] & W \end{tikzcd} \quad \text{in } \mathbb{E}$$

(RW1) For any horizontal arrow $Y \xrightarrow{v} \cdot$ in $\mathbb{E}$, the composite $u \odot v$ exists.

(RW2) For any $Y \xrightarrow{f} W$ and $Z \xrightarrow{v} W$ in $\mathbb{E}$, the following has its lifting:

$$\begin{tikzcd} X \arrow[r, "u", "\shortmid" marking] & Y \arrow[d, "f"] \\ Z \arrow[r, "v"', "\shortmid" marking] & W \end{tikzcd} \quad \text{in } \mathbb{E}$$

A horizontal arrow $u \xrightarrow{u} Y$ is called a *left weight* if it satisfies (LW1). It is called a *strong left weight* if it satisfies (LW1) and (LW2). (Strong) right weights are dually defined by (RW1) and (RW2). ♦

Definition 3.2 (Φ-cocompleteness)**.** Let $\mathbb{E}$ be a virtual equipment.

(i) Let Φ be a (not necessarily small) class of left weights. An object $X \in \mathbb{E}$ is Φ*-cocomplete* if for any $A \xrightarrow{\varphi} I$ in Φ and $A \xrightarrow{f} X$ in $\mathbb{E}$, the left Kan extension $\mathrm{Lan}_\varphi f$ exists.

(ii) Let Ψ be a (not necessarily small) class of right weights. An object $X \in \mathbb{E}$ is Ψ*-complete* if for any $J \xrightarrow{\psi} A$ in Ψ and $A \xrightarrow{f} X$ in $\mathbb{E}$, the right Kan extension $\mathrm{Ran}_\psi f$ exists.

$$\begin{tikzcd} J \arrow[r, "\psi", "\shortmid" marking] \arrow[dr, "\mathrm{Ran}_\psi f"'] & A \arrow[r, "\varphi", "\shortmid" marking] \arrow[d, "f"] \arrow[dl, phantom, "\mathsf{ran}" near start] \arrow[dr, phantom, "\mathsf{lan}" near start] & I \arrow[dl, "\mathrm{Lan}_\varphi f"] \\ & X & \end{tikzcd}$$

♦

Definition 3.3 (Φ-cocontinuity)**.** Let $\mathbb{E}$ be a virtual equipment.

(i) Let Φ be a (not necessarily small) class of left weights. A vertical arrow $X \xrightarrow{f} Y$ in $\mathbb{E}$ is *Φ-cocontinuous* if for any $A \xrightarrow{\varphi} I$ in Φ and $A \xrightarrow{g} X$, the following becomes a lan cell whenever the left Kan extension of g along φ exists:

$$
\begin{tikzcd}
A \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\mathsf{lan}" near start] & I \arrow[dl, "\mathrm{Lan}_\varphi g"] \\
X \arrow[d, "f"'] & \\
Y &
\end{tikzcd}
$$

(ii) Let Ψ be a (not necessarily small) class of right weights. A vertical arrow $X \xrightarrow{f} Y$ in $\mathbb{E}$ is *Ψ-continuous* if for any $J \xrightarrow{\psi} A$ in Ψ and $A \xrightarrow{g} X$, the following becomes a ran cell whenever the right Kan extension of g along ψ exists:

$$
\begin{tikzcd}
J \arrow[r, "\psi", "\shortmid"{marking}] \arrow[dr, "\mathrm{Ran}_\psi g"'] & A \arrow[d, "g"] \arrow[dl, phantom, "\mathsf{ran}" near start] \\
 & X \arrow[d, "f"] \\
 & Y
\end{tikzcd}
$$

♦

3.2. **Modestness.**

Definition 3.4 (Modest objects)**.** Let $\mathbb{E}$ be a virtual equipment. An object $C \in \mathbb{E}$ is *modest* if the following three conditions hold:

(MCmp) Every pair of two horizontal arrows $(\cdot \xrightarrow{\shortmid} C, C \xrightarrow{\shortmid} \cdot)$ has its composite.

(MExt) Given $C \xrightarrow{u} X$, $C \xrightarrow{f} Y$, and $Y \xrightarrow{v} Z$, the extension of them always exists.

$$
\begin{tikzcd}
C \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\mathsf{ext}"] & X \arrow[r, "\exists", "\shortmid"{marking}] & Z \arrow[d, equal] \\
Y \arrow[rr, "v"', "\shortmid"{marking}] & & Z
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

(MLift) Given $X \xrightarrow{u} C$, $C \xrightarrow{f} Y$, and $Z \xrightarrow{v} Y$, the lift of them always exists.

$$
\begin{tikzcd}
Z \arrow[r, "\exists", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{lift}"] & X \arrow[r, "u", "\shortmid"{marking}] & C \arrow[d, "f"] \\
Z \arrow[rr, "v"', "\shortmid"{marking}] & & Y
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

♦

Definition 3.5. Let $\mathbb{E}$ be a virtual equipment.

(i) A horizontal arrow $X \xrightarrow{u} Y$ is called *left-modest* if it has the following expression with some modest object C:

$$\begin{tikzcd} C \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\mathsf{s.opc}"] & Y \arrow[d, equal] \\ X \arrow[r, "u"', "|" marking] & Y \end{tikzcd} \quad \text{in } \mathbb{E}.$$

(ii) A horizontal arrow $Y \xrightarrow{u} X$ is called *right-modest* if it has the following expression with some modest object C:

$$\begin{tikzcd} Y \arrow[r, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{s.opc}"] & C \arrow[d] \\ Y \arrow[r, "u"', "|" marking] & X \end{tikzcd} \quad \text{in } \mathbb{E}.$$

♦

Notation 3.6. Let $\mathbb{E}$ be a virtual equipment. We denote by $\mathsf{LM}_{\mathbb{E}}$ the class of all left-modest horizontal arrows in $\mathbb{E}$, and denote by $\mathsf{RM}_{\mathbb{E}}$ the class of all right-modest horizontal arrows in $\mathbb{E}$. ♦

Proposition 3.7. Let $\mathbb{E}$ be a virtual equipment. Then, every left-modest horizontal arrow is a left weight. Dually, every right-modest horizontal arrow is a right weight.

Proof. ongoing □

Lemma 3.8. Let $\mathbb{E}$ be a virtual equipment.

(i) The class $\mathsf{LM}_{\mathbb{E}}$ are closed under horizontal positive composites; that is, if $n \geq 1$ and all u_i $(1 \leq i \leq n)$ are left-modest, then their composite $u_1 \odot u_2 \odot \ldots \odot u_n$ exists and becomes left-modest.

(ii) Suppose that $X \xrightarrow{u} Y$ and a fully faithful $Y \xrightarrow{f} Z$ are given. Then, u is left-modest whenever the composite $X \xrightarrow{u} Y \xrightarrow{f_*} Z$ exists and is left-modest.

3.3. **Scales.**

Definition 3.9. Let $\mathbb{E}$ be a virtual equipment. Let us consider the following conditions for a class Φ of left (or right) weights in $\mathbb{E}$:

(VInv) If we are given a vertically invertible cell of the following form:

$$\begin{tikzcd} \cdot \arrow[r, "p", "|" marking] \arrow[d] \arrow[dr, phantom, "\cong"] & \cdot \arrow[d] \\ \cdot \arrow[r, "q"', "|" marking] & \cdot \end{tikzcd} \quad \text{in } \mathbb{E},$$

then $p \in \Phi$ if and only if $q \in \Phi$.

(Cmpo) $u, v \in \Phi$ implies $u \odot v \in \Phi$ for any $X \xrightarrow{u} Y$ and $Y \xrightarrow{v} Z$,

(Conj) $f^* \in \Phi$ for any vertical arrow f.

(Cmpa) $f_* \in \Phi$ for any vertical arrow f.

(LPt) If we are given a cartesian cell of the following form:

$$\begin{tikzcd} \cdot \arrow[r, "v", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "u"', "|" marking] & \cdot \end{tikzcd} \quad \text{in } \mathbb{E},$$

then $u \in \Phi$ implies $v \in \Phi$.

(RPt) If we are given a cartesian cell of the following form:

$$\begin{tikzcd} \cdot \arrow[r, "v", "|"'] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "u"', "|"] & \cdot \end{tikzcd} \quad \text{in } \mathbb{E},$$

then $u \in \Phi$ implies $v \in \Phi$.

(Fin) If we are given a strongly opcartesian cell of the following form:

$$\begin{tikzcd} \cdot \arrow[r, "u", "|"'] \arrow[d] \arrow[dr, phantom, "\mathsf{s.opc}"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "v"', "|"] & \cdot \end{tikzcd} \quad \text{in } \mathbb{E},$$

then, $u \in \Phi$ implies $v \in \Phi$.

(Ini) If we are given a strongly opcartesian cell of the following form:

$$\begin{tikzcd} \cdot \arrow[r, "u", "|"'] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{s.opc}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "v"', "|"] & \cdot \end{tikzcd} \quad \text{in } \mathbb{E},$$

then, $u \in \Phi$ implies $v \in \Phi$.

A class Φ of left weights is called a *left scale* if it satisfies the conditions (VInv), (Cmpo), and (Conj). A class Φ of left weights is called a *left pre-scale* if it satisfies the conditions (VInv), (Cmpo), (LPt), and (Fin). ♦

Example 3.10. The class $\mathsf{LM}_{\mathbb{E}}$ becomes a left pre-scale on $\mathbb{E}$. Dually, the class $\mathsf{RM}_{\mathbb{E}}$ becomes a right pre-scale on $\mathbb{E}$. ♦

3.4. **Soundness.**

Definition 3.11 (Commutativity)**.** Let $\mathbb{E}$ be a virtual equipment.

(i) A pair $(A \overset{\varphi_0}{\longrightarrow} I, B \overset{\varphi_1}{\longrightarrow} J)$ of left weights *commutes* (as left weights) if: for any X, $A \xrightarrow{f} X$, and $X \xrightarrow{u} B$, the following gives the extension of $(f, u \odot \varphi_1)$ along φ_0:

$$\begin{tikzcd} A \arrow[r, "\varphi_0", "|"'] \arrow[d, "f"'] & I \arrow[r, "|"] \arrow[d, phantom, "\mathsf{ext}"] & B \arrow[r, "\varphi_1", "|"'] \arrow[d, equal] \arrow[dr, phantom, "="] & J \arrow[d, equal] \\ X \arrow[rr, "u"', "|"] & & B \arrow[r, "\varphi_1"', "|"] & J \end{tikzcd} \quad \text{in } \mathbb{E}.$$

(ii) A pair $(I \overset{\psi_0}{\longrightarrow} A, J \overset{\psi_1}{\longrightarrow} B)$ of right weights *commutes* (as right weights) if: for any X, $B \xrightarrow{f} X$, and $A \xrightarrow{u} X$, the following gives the lift of $(\psi_0 \odot u, f)$ along ψ_1:

$$\begin{tikzcd} I \arrow[r, "\psi_0", "|"'] \arrow[d, equal] \arrow[dr, phantom, "="] & A \arrow[r, "|"] \arrow[d, equal] & J \arrow[r, "\psi_1", "|"'] \arrow[d, phantom, "\mathsf{lift}"] & B \arrow[d, "f"] \\ I \arrow[r, "\psi_0"', "|"] & A \arrow[rr, "u"', "|"] & & X \end{tikzcd} \quad \text{in } \mathbb{E}.$$

♦

Notation 3.12. We will write $\varphi_0 \mathbin{/\!/} \varphi_1$ when a pair (φ_0, φ_1) of left weights commutes as left weights. Similarly, we will write $\psi_0 \mathbin{\backslash\!\backslash} \psi_1$ when a pair (ψ_0, ψ_1) of right weights commutes as right weights. ♦

Definition 3.13 (Weak commutativity)**.** Let $\mathbb{E}$ be a virtual equipment.

(i) A pair $(A \xrightarrow{\varphi_0} I, B \xrightarrow{\varphi_1} J)$ of left weights *weakly commutes* (as left weights) if: for any $A \xrightarrow{f} B$, the following gives the extension of $(f, \mathsf{Id}_B \odot \varphi_1)$ along φ_0:

$$\begin{tikzcd} A \arrow[r, "\varphi_0"] \arrow[d, "f"'] \arrow[drr, phantom, "\mathsf{ext}"] & I \arrow[r] & B \arrow[r, "\varphi_1"] \arrow[d, equal] \arrow[dr, phantom, "="] & J \arrow[d, equal] \\ B \arrow[rr, equal] & & B \arrow[r, "\varphi_1"'] & J \end{tikzcd} \quad \text{in } \mathbb{E}.$$

(ii) A pair $(I \xrightarrow{\psi_0} A, J \xrightarrow{\psi_1} B)$ of right weights *weakly commutes* (as right weights) if: for any $B \xrightarrow{f} A$, the following gives the lift of $(\psi_0 \odot \mathsf{Id}_A, f)$ along ψ_1:

$$\begin{tikzcd} I \arrow[r, "\psi_0"] \arrow[d, equal] \arrow[dr, phantom, "="] & A \arrow[r] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{lift}"] & J \arrow[r, "\psi_1"] & B \arrow[d, "f"] \\ I \arrow[r, "\psi_0"'] & A \arrow[rr, equal] & & A \end{tikzcd} \quad \text{in } \mathbb{E}.$$

♦

Notation 3.14. We will write $\varphi_0 \mathbin{/} \varphi_1$ when a pair (φ_0, φ_1) of left weights weakly commutes as left weights. Similarly, we will write $\psi_0 \mathbin{\backslash} \psi_1$ when a pair (ψ_0, ψ_1) of right weights weakly commutes as right weights. ♦

Notation 3.15. Given a class Φ of left weights, $\Phi_{/\!/}$ and $\Phi_{/}$ denote the classes of left weights defined by the following:

$$\begin{aligned} \Phi_{/\!/} \ni \varphi' &\quad \text{if and only if} \quad \varphi \mathbin{/\!/} \varphi' \text{ for all } \varphi \in \Phi; \\ \varphi' \in {}^{/\!/}\Phi &\quad \text{if and only if} \quad \varphi' \mathbin{/\!/} \varphi \text{ for all } \varphi \in \Phi; \\ \Phi_{/} \ni \varphi' &\quad \text{if and only if} \quad \varphi \mathbin{/} \varphi' \text{ for all } \varphi \in \Phi; \\ \varphi' \in {}^{/}\Phi &\quad \text{if and only if} \quad \varphi' \mathbin{/} \varphi \text{ for all } \varphi \in \Phi. \end{aligned}$$

Similarly, given a class Ψ of right weights, ${}_{\backslash\backslash}\Psi$ and ${}_{\backslash}\Psi$ denote the classes of right weights defined by the following:

$$\begin{aligned} \psi' \in {}_{\backslash\backslash}\Psi &\quad \text{if and only if} \quad \psi' \mathbin{\backslash\backslash} \psi \text{ for all } \psi \in \Psi; \\ \Psi^{\backslash\backslash} \ni \psi' &\quad \text{if and only if} \quad \psi \mathbin{\backslash\backslash} \psi' \text{ for all } \psi \in \Psi; \\ \psi' \in {}_{\backslash}\Psi &\quad \text{if and only if} \quad \psi' \mathbin{\backslash} \psi \text{ for all } \psi \in \Psi; \\ \Psi^{\backslash} \ni \psi' &\quad \text{if and only if} \quad \psi \mathbin{\backslash} \psi' \text{ for all } \psi \in \Psi. \end{aligned}$$

♦

Proposition 3.16. $\Phi_{/\!/}$ and ${}^{/\!/}\Phi$ are left dogmas. ${}_{\backslash\backslash}\Psi$ and $\Psi^{\backslash\backslash}$ are right dogmas.

Definition 3.17. A class Φ of left weights is *sound* if $\Phi_{/} \subseteq \Phi_{/\!/}$, equivalently, if $\Phi_{/} = \Phi_{/\!/}$. Dually, a class Ψ of right weights is *sound* if ${}_{\backslash}\Psi \subseteq {}_{\backslash\backslash}\Psi$. ♦

Proposition 3.18. If a class Φ of left weights is sound, then so is ${}^{/\!/}(\Phi_{/\!/})$.

experimental file for AVDC version for RCBadjoint

4. Ind-completions

4.1. Characterizations.

Proposition 4.1. Let $\mathbb{E}$ be a virtual equipment and let Φ be a class of horizontal arrows in $\mathbb{E}$. For each vertical arrow $A \xrightarrow{k} X$ in $\mathbb{E}$, the following conditions (i) and (ii) are equivalent:

(i) (a) For any $A \xrightarrow{\varphi} \cdot$ in Φ, the left Kan extension of k along φ exists, and it also yields a cartesian cell.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "k"'] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[dl, "\mathrm{Lan}_\varphi k"] \\ X & \end{tikzcd}$$

(b) For any vertical arrow $\cdot \xrightarrow{f} X$ in $\mathbb{E}$, the restriction $X(k,f)$ belongs to Φ, and it also yields a lan cell.

$$\begin{tikzcd} A \arrow[r, "X(k{,}f)"] \arrow[d, "k"'] \arrow[dr, phantom, "\mathsf{lan}"] & \cdot \arrow[dl, "f"] \\ X & \end{tikzcd}$$

(ii) k yields the following adjoint equivalence of categories for every $I \in \mathbb{E}$:

$$\mathbf{Hom}_\Phi(A, I) \underset{X(k,-)}{\overset{\mathrm{Lan}_- k}{\underset{\simeq}{\rightleftarrows}}} \mathbf{Hom}_\mathbb{E}\left(\begin{smallmatrix} I \\ X \end{smallmatrix} \right).$$

Here, $\mathbf{Hom}_\Phi(A, I)$ denotes the category of horizontal arrows from A to I belonging to Φ, and $\mathbf{Hom}_\mathbb{E}\left(\begin{smallmatrix} I \\ X \end{smallmatrix} \right)$ denotes the category of vertical arrows from I to X.

If Φ satisfies (LPt), the above conditions are also equivalent to the following condition (iii):

(iii) (a) The companion $A \xrightarrow{k_*} X$ belongs to Φ.

(b) The cartesian cell

$$\begin{tikzcd} A \arrow[r, "k_*"] \arrow[d, "k"'] \arrow[dr, phantom, "\mathsf{cart}"] & X \arrow[dl, equal] \\ X & \end{tikzcd}$$

is also a lan cell.

(c) Every $A \xrightarrow{\varphi} \cdot$ in Φ is a restriction of k and some $\cdot \xrightarrow{f} X$ in $\mathbb{E}$.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "k"'] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[dl, "f"] \\ X & \end{tikzcd}$$

Proof. ongoing □

Definition 4.2. Let $\mathbb{E}$ be a virtual equipment and let Φ be a class of horizontal arrows in $\mathbb{E}$. A vertical arrow $A \xrightarrow{k} X$ in $\mathbb{E}$ is called a Φ-*ind-morphism* if it satisfies the condition (i) or (ii) of Proposition 4.1. ♦

Definition 4.3. Let Ψ be a right dogma on a virtual equipment $\mathbb{E}$. A vertical arrow in $\mathbb{E}$ is a Ψ-*pro-morphism* if it is a Ψ-ind-morphism in the horizontal opposite of $\mathbb{E}$. ♦

Proposition 4.4. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let $A \xrightarrow{k_i} X_i$ $(i = 0, 1)$ be Φ-ind-morphisms in $\mathbb{E}$. Then, X_0 and X_1 are equivalent in the vertical 2-category of $\mathbb{E}$.

Notation 4.5. We will denote by $A \longrightarrow \Phi^{\triangledown}A$ a Φ-ind-morphism, and we will denote by $A \longrightarrow \Psi^{\blacktriangledown}A$ a Ψ-pro-morphism. ♦

Definition 4.6. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. A vertical arrow $X \xrightarrow{f} Y$ in $\mathbb{E}$ is called Φ-*atomic* if for any $A \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} I$ in Φ and $A \xrightarrow{g} Y$, the composite of the following cells

$$\begin{tikzcd} X \arrow[r, "Y(f,g)", "\shortmid"{marking}] \arrow[dr, "f"'] & A \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "g"] \arrow[dl, phantom, "\mathsf{cart}"] & I \arrow[dl] \arrow[l, phantom, ""] \\ & Y \arrow[ur, phantom, "\mathsf{lan}"] & \end{tikzcd}$$

is restricting whenever the left Kan extension of g along φ exists. ♦

Theorem 4.7. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. For each vertical arrow $A \xrightarrow{k} X$ in $\mathbb{E}$, the following are equivalent:

(i) $A \xrightarrow{k} X$ is a Φ-ind-morphism.
(ii) (a) X is Φ-cocomplete.
  (b) $A \xrightarrow{k} X$ is Φ-atomic and fully faithful.
  (c) For any vertical arrow $I \xrightarrow{f} X$ in $\mathbb{E}$, there exist an object $B \in \mathbb{E}$, $B \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} I$ in Φ, $B \xrightarrow{g} A$, and the following lan cell:

$$\begin{tikzcd} B \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\mathsf{lan}"] & I \arrow[ddl, "f"] \\ A \arrow[d, "k"'] & \\ X & \end{tikzcd}$$

4.2. As a cocompletion.

Lemma 4.8. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let $A \xrightarrow{a} \Phi^{\triangledown}A$ be a Φ-ind-completion in $\mathbb{E}$. Then, the following are equivalent for a vertical arrow $\Phi^{\triangledown}A \xrightarrow{f} B$ in $\mathbb{E}$:

(i) f is Φ-cocontinuous.
(ii) f preserves the left Kan extension

$$\begin{tikzcd} A \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\mathsf{lan}"] & \cdot \arrow[dl] \\ \Phi^{\triangledown}A & \end{tikzcd} \qquad \text{in } \mathbb{E}$$

for all $\varphi \in \Phi$.

(iii) f is a left Kan extension along a_*, i.e., there exists a vertical arrow $A \xrightarrow{g} B$ in $\mathbb{E}$ such that

$$\begin{tikzcd} A \arrow[r, "a_*"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathsf{lan}"] & \Phi^{\triangledown}A \arrow[dl, "f"] \\ B & \end{tikzcd} \quad \text{in } \mathbb{E}.$$

4.3. **Relative company-biadjunctions.** In this subsection, we fix the following data:

- AVDCs $\mathbb{A}'$, $\mathbb{A}$, and $\mathbb{B}$.
- Weak functors $\mathbb{A}' \xrightarrow{I} \mathbb{A}$, $\mathbb{A}' \xrightarrow{F} \mathbb{B}$, and $\mathbb{B} \xrightarrow{G} \mathbb{A}$.
- A weak transformation $I \xRightarrow{\eta} GF$ satisfying the following conditions.
  - Each component of η has a companion in $\mathbb{B}$.
  - For any horizontal arrow $A_0 \xrightarrow{v} A_1$ in $\mathbb{A}'$, there exists a strongly opcartesian cell of the following form:

$$\begin{tikzcd} IA_0 \arrow[r, "Iv"] \arrow[d, equal] \arrow[dr, phantom, "c_u\colon \mathsf{s.opc}"] & IA_1 \arrow[d, "\eta_{A_1}"] \\ IA_0 \arrow[r, "Iv\odot\eta_{A_1,*}"'] & GFA_1 \end{tikzcd} \tag{22}$$

  - For any $A \in \mathbb{A}'$, $B \in \mathbb{B}$, and $u\colon FA \longrightarrow B$ in $\mathbb{B}$, there exists a restriction $\widehat{u} = Gu(\eta_A, \mathsf{id}_B)$;

$$\begin{tikzcd} IA \arrow[r, "\widehat{u}"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\kappa_u\colon \mathsf{cart}"] & GB \arrow[d, equal] \\ GFA \arrow[r, "Gu"'] & GB \end{tikzcd}$$

We choose $\widehat{u}$ and the cartesian cell κ_u for any $A \in \mathbb{A}'$, $B \in \mathbb{B}$, and $u\colon FA \longrightarrow B$.

Given such data, we employ the following notations.

- For any $A \in \mathbb{A}'$ and $B \in \mathbb{B}$, we write

$$\mathbf{Hom}_{\mathbb{B}}\left(\begin{smallmatrix} FA \\ B \end{smallmatrix}\right) \xrightarrow{G} \mathbf{Hom}_{\mathbb{A}}\left(\begin{smallmatrix} GFA \\ GB \end{smallmatrix}\right) \xrightarrow{\eta_A \fatsemi -} \mathbf{Hom}_{\mathbb{A}}\left(\begin{smallmatrix} IA \\ GB \end{smallmatrix}\right)\colon f \mapsto \widehat{f} \tag{23}$$

- For any $A \in \mathbb{A}'$ and $B \in \mathbb{B}$, we write

$$\mathbf{Hom}_{\mathbb{B}}(FA, B) \longrightarrow \mathbf{Hom}_{\mathbb{A}}(IA, GB)\colon u \mapsto \widehat{u} := Gu(\eta_A, \mathsf{id}_B) \tag{24}$$

  for the restriction along η_A.

- For any $A \in \mathbb{A}'$ and a path $B_0 \overset{\vec{p}}{\dashrightarrow} B_n$ in $\mathbb{B}$ with $B_0 = FA$, define another path $IA \overset{A^*\vec{p}}{\dashrightarrow} GB_n$ and a horizontal sequence of cells $\kappa_{\vec{p}}$ in $\mathbb{A}$ as follows.
  - If $n = 0$ and $\vec{p}$ is null, then $A^*\vec{p} := \eta_{A,*}\colon IA \longrightarrow GFA$. $\kappa_{\vec{p}}$ is the canonical cartesian cell of the following form:

$$\begin{tikzcd} IA \arrow[rr, "\eta_{A,*}"] \arrow[dr, "\eta_A"'] & \arrow[d, phantom, "\kappa_{\vec{p}}"] & GFA \arrow[dl, equal] \\ & GFA & \end{tikzcd}$$

– If $\vec{p} = p \frown \vec{q}$, we define $A^*\vec{p}$ by $\widehat{p} \frown G\vec{q}$. $\kappa_{\vec{p}}$ is the following sequence of cells:

$$
\begin{tikzcd}
IA \arrow[r, "\widehat{p}"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\kappa_p"] & GB_1 \arrow[r, dashed, "G\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & GB_n \arrow[d, equal] \\
GFA \arrow[r, "Gp"'] & GB_1 \arrow[r, dashed, "G\vec{q}"'] & GB_n
\end{tikzcd}
$$

Let us consider several properties as follows for these data $(\mathbb{A}', \mathbb{A}, \mathbb{B}, I, F, G, \eta)$:

$\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ For any $A \in \mathbb{A}'$ and $B \in \mathbb{B}$, the functor (23) is essentially surjective.

$\left(\begin{smallmatrix} F & G \end{smallmatrix}\right)$ For any $A \in \mathbb{A}'$ and $B \in \mathbb{B}$, the functor (24) is essentially surjective.

(HTrans) For every horizontal arrow $A_0 \xrightarrow{u} A_1$ in $\mathbb{A}'$, the following cell is restricting.

$$
\begin{tikzcd}
IA_0 \arrow[r, "Iu"] \arrow[d, "\eta_{A_0}"'] \arrow[dr, phantom, "\eta_u"] & IA_1 \arrow[r, "\eta_{A_1,*}"] \arrow[d, "\eta_{A_1}"'] & GFA_1 \\
GFA_0 \arrow[r, "GFu"] & GFA_1 \arrow[ur, equal] &
\end{tikzcd}
\tag{25}
$$

$\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$ For any $A \in \mathbb{A}'$, paths $B_0 \overset{\vec{p}}{\dashrightarrow} B_n$ and $B'_0 \overset{v}{\cdots\to} B'_1$ in $\mathbb{B}$ with $B_0 = FA$, and vertical arrows f, g:

$$
\begin{tikzcd}
FA \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] & B_n \arrow[d, "g"] \\
B'_0 \arrow[r, dotted, "v"'] & B'_1
\end{tikzcd}
\quad \text{in } \mathbb{B},
$$

the assignment described as follows yields a bijection $\mathsf{Cell}_{\mathbb{B}}\left(f \substack{\vec{p} \\ \\ v} g\right) \cong \mathsf{Cell}_{\mathbb{A}}\left(\widehat{f} \substack{A^*\vec{p} \\ \\ Gv} Gg\right)$:

$$
\begin{tikzcd}
FA \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B_n \arrow[d, "g"] \\
B'_0 \arrow[r, dotted, "v"'] & B'_1
\end{tikzcd}
\quad \text{in } \mathbb{B}, \quad \mapsto \quad
\begin{tikzcd}
IA \arrow[r, dashed, "A^*\vec{p}"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\kappa_{\vec{p}}"] & GB_n \arrow[d, equal] \\
GFA \arrow[r, dashed, "G\vec{p}"'] \arrow[d, "Gf"'] \arrow[dr, phantom, "G\alpha"] & GB_n \arrow[d, "Gg"] \\
GB'_0 \arrow[r, dotted, "Gv"'] & GB'_1
\end{tikzcd}
$$

$\left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)$ For any vertical arrows $A_0 \xrightarrow{f} A_1$ in $\mathbb{A}'$, $B_0 \xrightarrow{g} B_1$ in $\mathbb{B}$, and horizontal paths u, v in $\mathbb{B}$:

$$
\begin{tikzcd}
FA_0 \arrow[r, dotted, "u"] \arrow[d, "Ff"'] & B_0 \arrow[d, "g"] \\
FA_1 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}
\quad \text{in } \mathbb{B},
$$

the assignment described as follows yields a bijection $\alpha \mapsto \widehat{\alpha} \colon \mathsf{Cell}_{\mathbb{B}}\big(Ff \ {}^{u}_{v}\ g\big) \cong \mathsf{Cell}_{\mathbb{A}}\big(If \ {}^{A_0^*u}_{A_1^*v}\ Gg\big)$: for each $\alpha \in \mathsf{Cell}_{\mathbb{B}}\big(Ff \ {}^{u}_{v}\ g\big)$, $\widehat{\alpha}$ is the unique cell satisfying the following equality.

$$
\begin{tikzcd}
IA_0 \arrow[r, "A_0^*u"] \arrow[d, "If"'] \arrow[dr, phantom, "\widehat{\alpha}"] & GB_0 \arrow[d, dashed, "Gg"] \arrow[dr, equal] & \\
IA_1 \arrow[r, "A_1^*v"] \arrow[d, "\eta_{A_1}"'] \arrow[dr, phantom, "\kappa_v"] & GFA_0 \arrow[d, equal] \arrow[r, phantom, "="] & GB_0 \arrow[dl, "Gg"] \\
GFA_1 \arrow[r, dotted, "Gv"'] & GFA_1 &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
IA_0 \arrow[r, "A_0^*u"] \arrow[d, "If"'] \arrow[dr, "\eta_{A_0}"] & GB_0 \arrow[dr, equal] \arrow[d, phantom, "\kappa_u"] & \\
IA_1 \arrow[r, phantom, "\cong"] \arrow[d, "\eta_{A_1}"'] & GFA_0 \arrow[r, dotted, "Gu"'] \arrow[dl, "GFf"'] \arrow[d, phantom, "G\alpha"] & GB_0 \arrow[dl, "Gg"] \\
GFA_1 \arrow[r, dotted, "Gv"'] & GFA_1 &
\end{tikzcd}
$$

Here, the horizontally invertible cell on the right is the image under η of the canonical horizontally invertible cell $(f,0) \frown (A_1, \leq) \cong (A_0, \leq) \frown (f,1)$ in $\mathsf{Q}(\mathbf{2}\cdot\mathbb{A}')$.

$\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right)$ For any sequence horizontal arrows $A_0 \overset{\vec{u}}{\dashrightarrow} A_1$ in $\mathbb{A}'$, a dotted sequence $B_0 \overset{v}{\dashrightarrow} B_1$ in $\mathbb{B}$, and vertical arrows f, g in $\mathbb{B}$:

$$
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "f"'] & FA_1 \arrow[d, "g"] \\
B_0 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}
\quad \text{in } \mathbb{B},
$$

the assignment described as follows yields a bijection $\mathsf{Cell}_{\mathbb{B}}\big(f \ {}^{F\vec{u}}_{v}\ g\big) \cong \mathsf{Cell}_{\mathbb{A}}\big(\widehat{f} \ {}^{I\vec{u}}_{Gv}\ \widehat{g}\big)$:

$$
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & FA_1 \arrow[d, "g"] \\
B_0 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{u}"] \arrow[d, "\eta_{A_0}"'] \arrow[dr, phantom, "\eta_u"] & IA_1 \arrow[d, "\eta_{A_1}"] \\
GFA_0 \arrow[r, dashed, "GF\vec{u}"] \arrow[d, "Gf"'] \arrow[dr, phantom, "G\alpha"] & GFA_1 \arrow[d, "Gg"] \\
GB_0 \arrow[r, dotted, "Gv"'] & GB_1
\end{tikzcd}
$$

$\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$ For any $A_0, A_1, A_2 \in \mathbb{A}'$, $B \in \mathbb{B}$, vertical arrows f, g, and horizontal paths $\vec{p}, v$:

$$
\begin{tikzcd}
A_0 \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] & A_1 \\
A_2 &
\end{tikzcd}
\quad \text{in } \mathbb{A} \qquad\qquad
\begin{tikzcd}
& FA_1 \arrow[d, "g"] \\
FA_2 \arrow[r, dotted, "v"'] & B
\end{tikzcd}
\quad \text{in } \mathbb{B},
$$

the assignment described as follows yields a bijection $\alpha \mapsto \tilde{\alpha} \colon \mathsf{Cell}_{\mathbb{B}}\big(Ff \ {}^{F\vec{p}}_{v}\ g\big) \cong \mathsf{Cell}_{\mathbb{A}}\big(If \ {}^{I\vec{p}}_{A_2^*v}\ \widehat{g}\big)$: for each $\alpha \in \mathsf{Cell}_{\mathbb{B}}\big(Ff \ {}^{F\vec{p}}_{v}\ g\big)$, the cell $\tilde{\alpha}$ is the unique cell satisfying the following equality.

$$
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, "If"'] \arrow[dr, "\eta_{A_0}"] & IA_1 \arrow[dr, "\eta_{A_1}"] \arrow[d, phantom, "\eta_{\vec{p}}"] & \\
IA_2 \arrow[r, phantom, "\cong"] \arrow[d, "\eta_{A_2}"'] & GFA_0 \arrow[r, dashed, "GF\vec{p}"'] \arrow[dl, "GFf"'] \arrow[d, phantom, "G\alpha"] & GFA_1 \arrow[dl, "Gg"] \\
GFA_2 \arrow[r, dotted, "Gv"'] & GB &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, "If"'] \arrow[dr, phantom, "\tilde{\alpha}"] & IA_1 \arrow[d, "\widehat{g}"] \\
IA_2 \arrow[r, "A_2^*v"] \arrow[d, "\eta_{A_2}"'] \arrow[dr, phantom, "\kappa_v"] & GB \arrow[d, equal] \\
GFA_2 \arrow[r, dotted, "Gv"'] & GB
\end{tikzcd}
$$

Lemma 4.9. Under $\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} F & G \end{smallmatrix}\right)$, the following implications hold:

(i) $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right) \Longrightarrow \left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)$
(ii) $\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right) \Longrightarrow \left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$

Proof. almost done [$\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right) \Longrightarrow \left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)$] By $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$, the following cells correspond to each other:

$$
\begin{tikzcd}
FA_0 \arrow[r, dotted, "u"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\alpha"] & B_0 \arrow[d, "g"] \\
FA_1 \arrow[r, dotted, "v"'] & B_1
\end{tikzcd}
\text{ in } \mathbb{B}
\qquad \Bigg\| \qquad
\begin{tikzcd}
IA_0 \arrow[r, "A_0^* u"] \arrow[d, "\widehat{Ff}"'] \arrow[dr, phantom, "\cdot"] & GB_0 \arrow[d, "Gg"] \\
GFA_1 \arrow[r, dotted, "Gv"'] & GB_1
\end{tikzcd}
\text{ in } \mathbb{A}
$$

Since $\widehat{Ff} = \eta_{A_0} ⨾ GFf \cong If ⨾ \eta_{A_1}$ and $\widehat{v} = (Gv)(\eta_{A_1}, \mathsf{id})$, the above cells also correspond to the following cells:

$$
\begin{tikzcd}
IA_0 \arrow[r, "A_0^* u"] \arrow[d, "If"'] & GB_0 \arrow[dd, "Gg"] \\
IA_1 \arrow[d, "\eta_{A_1}"'] \arrow[r, phantom, "\cdot"] & \\
GFA_1 \arrow[r, dotted, "Gv"'] & GB_1
\end{tikzcd}
\text{ in } \mathbb{A}
\qquad \Bigg\| \qquad
\begin{tikzcd}
IA_0 \arrow[r, "A_0^* u"] \arrow[d, "If"'] \arrow[dr, phantom, "\cdot"] & GB_0 \arrow[d, "Gg"] \\
IA_1 \arrow[r, "A_1^* v"'] & GB_1
\end{tikzcd}
\text{ in } \mathbb{A}
$$

This proves the condition $\left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)$. maybe there must be some English.
[$\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right) \Longrightarrow \left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$] By $\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right)$, the following cells correspond to each other:

$$
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{p}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\alpha"] & FA_1 \arrow[d, "g"] \\
FA_2 \arrow[r, dotted, "v"'] & B
\end{tikzcd}
\text{ in } \mathbb{B}
\qquad \Bigg\| \qquad
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, "\widehat{Ff}"'] \arrow[dr, phantom, "\cdot"] & IA_1 \arrow[d, "\widehat{g}"] \\
GFA_2 \arrow[r, dotted, "Gv"'] & GB
\end{tikzcd}
\text{ in } \mathbb{A}
$$

The invertible cell $\widehat{Ff} = \eta_{A_0} ⨾ GFf \cong If ⨾ \eta_{A_2}$ and the universality of the cell defining $A_2^* v$ implies that the above cells also correspond to the following cells:

$$
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, "If"'] & IA_1 \arrow[dd, "\widehat{g}"] \\
IA_2 \arrow[d, "\eta_{A_2}"'] \arrow[r, phantom, "\cdot"] & \\
GFA_2 \arrow[r, dotted, "Gv"'] & GB
\end{tikzcd}
\text{ in } \mathbb{A}
\qquad \Bigg\| \qquad
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, "If"'] \arrow[dr, phantom, "\cdot"] & IA_1 \arrow[d, "\widehat{g}"] \\
IA_2 \arrow[r, "A_2^* v"'] & GB
\end{tikzcd}
\text{ in } \mathbb{A}
$$

This proves the condition $\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$. maybe there must be some English. □

Lemma 4.10. Let $\mathbb{X}$ be an AVDC and consider a weakly opcartesian cell c and a cell c' of the following forms in $\mathbb{X}$.

$$
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] \arrow[dr, phantom, "c : \mathsf{w.opc}"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] \arrow[dr, phantom, "c'"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "u"'] & \cdot
\end{tikzcd}
$$

Then c' is weakly opcartesian if and only if the following condition holds.

For each horizontal arrow v that is parallel to u, there is a bijective correspondence between the following cells

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} \cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, "f"'] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \tag{26}$$

that relates α to the vertical composite $c' ⨟ \alpha$.

In this situation, c' is strongly opcartesian if and only if so is c.

Proof. ongoing □

Lemma 4.11. Under the conditions $\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} F & G \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$, the following are equivalent:

$$\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right) \Leftrightarrow \left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right) \Leftrightarrow \text{(HTrans)}$$

Proof. [$\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right) \implies \left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$] This follows from Lemma 4.9.

[$\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right) \implies$ (HTrans)] almost done Take an arbitrary horizontal arrow $A_0 \xrightarrow{u} A_1$ in $\mathbb{A}'$. In order to show (HTrans), it suffices to check the cell

$$\begin{tikzcd} IA_0 \arrow[r, "Iu"] \arrow[d, equal] \arrow[dr, phantom, "\cdot"] & IA_1 \arrow[d, "\eta_{A_1}"] \\ IA_0 \arrow[r, "\widehat{Fu}"'] & GFA_1 \end{tikzcd} \tag{27}$$

induced from the cell η_u and κ_u defining $\widehat{Fu}$, is strongly opcartesian. However, since we have assumed that there exists a strongly opcartesian cell of the form (22), it suffices to show the condition (26). For any horizontal arrow $FA_0 \xrightarrow{v} FA_1$ in $\mathbb{A}$, there are bijective correspondences among cells of the following forms:

$$\begin{tikzcd} IA_0 \arrow[r, "\widehat{Fu}"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & GFA_1 \arrow[d, equal] \\ IA_0 \arrow[r, "A_0^*v"'] & GFA_1 \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} FA_0 \arrow[r, "Fu"] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & FA_1 \arrow[d, equal] \\ FA_0 \arrow[r, "v"'] & FA_1 \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} IA_0 \arrow[r, "Iu"] \arrow[d, equal] \arrow[dr, phantom, "\gamma"] & IA_1 \arrow[d, "\eta_{A_1}"] \\ IA_0 \arrow[r, "A_0^*v"'] & GFA_1 \end{tikzcd}$$

In the first correspondence, we have used $\left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)$. Note that $\left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)$ follows from $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$ (Lemma 4.9). In the second correspondence, we have used $\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$. We can check that the resulting bijective correspondence between α and γ coincides with the pre-composition of the cell (27). Finally, since $\left(\begin{smallmatrix} F & G \end{smallmatrix}\right)$ shows that any horizontal morphism $IA_0 \longrightarrow GFA_1$ is vertically isomoprhic to one of the form A_0^*v, the above correspondence $\alpha \mapsto \gamma$ shows the condition (26).

[(HTrans) $\implies \left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right)$] By $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$, the following cells bijectively correspond to each other:

$$\begin{tikzcd} FA_0 \arrow[r, dashed, "F\vec{p}"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & FA_1 \arrow[d, "g"] \\ B_0 \arrow[r, dotted, "v"'] & B_1 \end{tikzcd} \text{ in } \mathbb{B} \quad \Bigg\| \quad \begin{tikzcd} IA_0 \arrow[r, dashed, "A_0^*(F\vec{p})"] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\beta"] & GFA_1 \arrow[d, "Gg"] \\ GB_0 \arrow[r, dotted, "Gv"'] & GB_1 \end{tikzcd} \text{ in } \mathbb{A}$$

It follows from (HTrans) that there is a strongly opcartesian sequence of cells of the following form.

$$\begin{tikzcd} IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "\text{s.opc}"] & IA_1 \arrow[d, "\eta_{A_1}"] \\ IA_0 \arrow[r, dashed, "A_0^*(F\vec{p})"] & GFA_1 \end{tikzcd}$$

Therefore, there is also a bijective correspondence between cells of the following forms.

$$
\begin{tikzcd}
IA_0 \arrow[r, dashed, "A_0^*(F\vec{p})"] \arrow[d, "\widehat{f}"'] \arrow[dr, phantom, "\beta"] & GFA_1 \arrow[d, "Gg"] \\
GB_0 \arrow[r, dotted, "Gv"'] & GB_1
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, "\widehat{f}"'] \arrow[dr, phantom, "\gamma"] & IA_1 \arrow[d, "\widehat{g}"] \\
GB_0 \arrow[r, dotted, "Gv"'] & GB_1
\end{tikzcd}
\quad \text{in } \mathbb{A}
$$

We now get the bijective correspondence between α and γ, which proves $\left(\begin{smallmatrix}F&F\\G&G\end{smallmatrix}\right)$. □

Definition 4.12. We say that $(\mathbb{A}', \mathbb{A}, \mathbb{B}, I, F, G, \eta)$ exhibits a *left I-relative companied biadjoint* of G if the following equivalent conditions hold:

(i) $\left(\begin{smallmatrix}F\\G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&G\end{smallmatrix}\right)$, (HTrans), and $\left(\begin{smallmatrix}F&G\\G&G\end{smallmatrix}\right)$.
(ii) $\left(\begin{smallmatrix}F\\G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&G\\G&G\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}F&F\\G&G\end{smallmatrix}\right)$.
(iii) $\left(\begin{smallmatrix}F\\G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&G\\G&G\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}F&F\\F&G\end{smallmatrix}\right)$.
(iv) $\left(\begin{smallmatrix}F\\G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&G\end{smallmatrix}\right)$, (HTrans), $\left(\begin{smallmatrix}F&G\\G&G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&G\\F&G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&F\\G&G\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}F&F\\F&G\end{smallmatrix}\right)$.

The equivalence among (i) to (iv) follows from Lemmas 4.9 and 4.11. ♦

Theorem 4.13 (Reconstruction theorem of the left relative companied biadjoint)**.** Consider the following data:

- AvDCs $\mathbb{A}'$, $\mathbb{A}$, and $\mathbb{B}$.
- Weak functors $\mathbb{A}' \xrightarrow{I} \mathbb{A}$ and $\mathbb{B} \xrightarrow{G} \mathbb{A}$.
- For each object $A \in \mathbb{A}'$, an object $FA \in \mathbb{B}$ and a vertical arrow $IA \xrightarrow{\eta_A} GFA$ in $\mathbb{A}$ satisfying the following.
  - η_A has a companion in $\mathbb{A}$.
  - For any horizontal arrow $A_0 \xrightarrow{v} A_1$ in $\mathbb{A}'$, there exists a strongly opcartesian cell of the following form.

$$
\begin{tikzcd}
IA_0 \arrow[r, "Iv"] \arrow[d, equal] \arrow[dr, phantom, "c_v\colon \mathsf{s.opc}"] & IA_1 \arrow[d, "\eta_{A_1}"] \\
IA_0 \arrow[r, "Iv \odot \eta_{A_1,*}"'] & GFA_1
\end{tikzcd}
\tag{28}
$$

  - For any $A \in \mathbb{A}'$, $B \in \mathbb{B}$, and $FA \xrightarrow{u} B$ in $\mathbb{B}$, there exists a restriction $\widehat{u} = Gu(\eta_A, \mathsf{id}_{GB})$;

$$
\begin{tikzcd}
IA \arrow[r, "\widehat{u}"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\kappa_u\colon \mathsf{cart}"] & GB \arrow[d, equal] \\
GFA \arrow[r, "Gu"'] & GB
\end{tikzcd}
$$

  We choose $\widehat{u}$ and the cartesian cell κ_u for any $A \in \mathbb{A}'$, $B \in \mathbb{B}$, and $FA \xrightarrow{u} B$.

Suppose that the same conditions hold as $\left(\begin{smallmatrix}F\\G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F&G\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}F&G\\G&G\end{smallmatrix}\right)$ (note here that $\left(\begin{smallmatrix}F&G\\G&G\end{smallmatrix}\right)$ does not require the weak naturality of η). Then, these data extend to an I-relative companied biadjunction.

Proof. Firstly, we construct a weak functor $\mathbb{A}' \xrightarrow{F} \mathbb{B}$ and a weak transformation $I \xRightarrow{\eta} G \circ F$ simulataneously. Here, we give data described in Proposition 2.10 for the definition of η.

- For each object $A \in \mathbb{A}'$, $FA \in \mathbb{B}$ and $\eta_A\colon IA \longrightarrow GFA$ are already given.

- Suppose we are given a vertical arrow $A^0 \xrightarrow{f} A^1$ in $\mathbb{A}'$. Applying $\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ to $If ⨾ \eta_{A^1}$, we take $Ff \colon FA^0 \longrightarrow FA^1$ with a horizontally invertible cell

$$\begin{tikzcd} & IA^0 \arrow[dl, "\eta_{A^0}"'] \arrow[dr, "If"] & \\ GFA^0 \arrow[dr, "GFf"'] & \eta_f & IA^1 \arrow[dl, "\eta_{A^1}"] \\ & GFA^1 & \end{tikzcd}$$

in $\mathbb{A}$.

- For each horizontal arrow $A_0 \xrightarrow{v} A_1$ in $\mathbb{A}'$, by (28) and $(\,F\ G\,)$, we take $FA_0 \xrightarrow{Fv} FA_1$ with the following strongly opcartesian cell and a cartesian cell. We write η_v for the composite of those cells.

$$\begin{tikzcd} IA_0 \arrow[r, "Iv"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{s.opc}"] & IA_1 \arrow[d, "\eta_{A_1}"] \\ IA_0 \arrow[r, "\widehat{Fv}"] \arrow[d, "\eta_{A_0}"'] \arrow[dr, phantom, "\mathsf{cart}"] & GFA_1 \arrow[d, equal] \\ GFA_0 \arrow[r, "GFv"'] & GFA_1 \end{tikzcd} \quad =: \quad \begin{tikzcd} IA_0 \arrow[r, "Iv"] \arrow[d, "\eta_{A_0}"'] \arrow[dr, phantom, "\eta_v"] & IA_1 \arrow[d, "\eta_{A_1}"] \\ GFA_0 \arrow[r, "GFv"'] & GFA_1 \end{tikzcd} \tag{29}$$

* Observe that for each horizontal arrow $\vec{u} \colon A_0 \dashrightarrow A_n$ in $\mathbb{A}'$, there exists a strongly opcartesian sequence in $\mathbb{A}$ of the following form.

$$\begin{tikzcd} IA_0 \arrow[r, dashed, "I\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{s.opc}"] & IA_n \arrow[d, "\eta_{A_n}"] \\ IA_0 \arrow[r, dashed, "A_0^* F\vec{u}"'] & GFA_n \end{tikzcd} \tag{30}$$

This is obtained inductively as follows:
  - If $\vec{u}$ is null, its images $I\vec{u}$ and $F\vec{u}$ are also null, and we take the canonical strongly opcartesian cell with respect to the companion $\eta_{A_n,*} =: A_0^* F\vec{u}$.
  - If the length of $\vec{u}$ is exactly 1, take the opcartesian cell obtained in (29).
  - Suppose we have two horizontal arrows v_1 and v_2 with $\vec{u} = v_1 \frown v_2 \frown \vec{v}$. Recall that $A_1^*(Fv_2 \frown F\vec{v})$ is defined as $\widehat{Fv_2} \frown GF\vec{v}$. The following composite gives the desired opcartesian sequence by doko.

$$\begin{tikzcd} IA_0 \arrow[r, "Iv_1"] \arrow[d, equal] \arrow[dr, phantom, "="] & IA_1 \arrow[rr, dashed, "Iv_2 \frown I\vec{v}"] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{s.opc}"] & & IA_n \arrow[d, "\eta_{A_n}"] \\ IA_0 \arrow[r, "Iv_1"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{s.opc}"] & IA_1 \arrow[r, "\widehat{Fv_2}"] \arrow[d, "\eta_{A_1}"] \arrow[dr, phantom, "\mathsf{cart}"] & GFA_2 \arrow[r, dashed, "GF\vec{v}"] \arrow[d, equal] \arrow[dr, phantom, "="] & GFA_n \arrow[d, equal] \\ IA_0 \arrow[r, "\widehat{Fv_1}"'] & GFA_1 \arrow[r, "GFv_2"'] & GFA_2 \arrow[r, dashed, "GF\vec{v}"'] & GFA_n \end{tikzcd}$$

Here the top-right opcartesian sequence is obtained by the inductive hypothesis.

* Let us write $\fatsemi(\vec{f})$ for the composite of a sequence $\vec{f}$ of vertical arrows. Observe the following correspondences of cells in $\mathbb{A}$ and $\mathbb{B}$:

$$
\begin{tikzcd}
\cdot \arrow[r, dashed, "I\vec{u}"] \arrow[d, dashed, "I\vec{f}"'] & \cdot \arrow[d, dashed, "I\vec{g}"] \\
\cdot \arrow[d, "\eta"'] \arrow[r, phantom, "\alpha_1"] & \cdot \arrow[d, "\eta"] \\
\cdot \arrow[r, dotted, "GFv"'] & \cdot
\end{tikzcd}
\Bigg\Vert
\begin{tikzcd}
\cdot \arrow[r, dashed, "I\vec{u}"] \arrow[d, "\eta"'] & \cdot \arrow[d, "\eta"] \\
\cdot \arrow[d, dashed, "GF\vec{f}"'] \arrow[r, phantom, "\alpha_2"] & \cdot \arrow[d, dashed, "GF\vec{g}"] \\
\cdot \arrow[r, dotted, "GFv"'] & \cdot
\end{tikzcd}
\Bigg\Vert
\begin{tikzcd}
\cdot \arrow[r, dashed, "I\vec{u}"] \arrow[dd, "\widehat{\fatsemi(F(\vec{f}))}"'] & \cdot \arrow[d, "\eta"] \\
\arrow[r, phantom, "\alpha_3"] & \cdot \arrow[d, "G(\fatsemi(F\vec{g}))"] \\
\cdot \arrow[r, dotted, "GFv"'] & \cdot
\end{tikzcd}
$$

$$
\Bigg\Vert
\begin{tikzcd}
\cdot \arrow[r, dashed, "A_0^*(F\vec{u})"] \arrow[d, "\widehat{\fatsemi(F(\vec{f}))}"'] \arrow[dr, phantom, "\alpha_4"] & \cdot \arrow[d, "G(\fatsemi(F\vec{g}))"] \\
\cdot \arrow[r, dotted, "GFv"'] & \cdot
\end{tikzcd}
\Bigg\Vert
\begin{tikzcd}
\cdot \arrow[r, dashed, "F\vec{u}"] \arrow[d, dashed, "F(\vec{f})"'] \arrow[dr, phantom, "\alpha_5"] & \cdot \arrow[d, dashed, "F\vec{g}"] \\
\cdot \arrow[r, dotted, "Fv"'] & \cdot
\end{tikzcd}
\tag{31}
$$

The first is obtained by composing the canonical cells defining components of the path $F\vec{f}$. The second is obtained by composing the *coherence cells* for the weak functor G: the images of the canonical horizontally invertible cells $F\vec{f} \cong \fatsemi(F\vec{f})$ and $F\vec{g} \cong \fatsemi(F\vec{g})$ in $\mathsf{Q}\,\mathbb{B}$. The third is obtained by factoring through the strongly opcartesian cell (30). The fourth is $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$.

An easy calculation shows that given α_1, the cell α_5 is the unique cell satisfying the following equality.

$$
\begin{tikzcd}
& IA \arrow[rr, dashed, "I\vec{u}"] \arrow[dl, "\eta_A"'] \arrow[dr, dashed, "I\vec{f}"] & & IB \arrow[dr, dashed, "I\vec{g}"] & \\
GFA \arrow[dr, dashed, "GF\vec{f}"'] \arrow[rr, phantom, "\eta_{\vec{f}}"] & & IX \arrow[dl, "\eta_X"] \arrow[rr, phantom, "\alpha_1"] & & IY \arrow[dl, "\eta_Y"] \\
& GFX \arrow[rr, dotted, "GFv"'] & & GFY &
\end{tikzcd}
=
\begin{tikzcd}
& IA \arrow[rr, dashed, "I\vec{u}"] \arrow[dl, "\eta_A"'] \arrow[r, phantom, "\eta_{\vec{u}}"] & {} & IB \arrow[dl, "\eta_B"'] \arrow[dr, dashed, "I\vec{g}"] & \\
GFA \arrow[rr, dashed, "GF\vec{u}"] \arrow[dr, dashed, "GF\vec{f}"'] & & GFB \arrow[dr, dashed, "GF\vec{g}"] \arrow[rr, phantom, "\eta_{\vec{g}}"] \arrow[d, phantom, "G\alpha_5"] & & IY \arrow[dl, "\eta_Y"] \\
& GFX \arrow[rr, dotted, "GFv"'] & {} & GFY &
\end{tikzcd}
$$

* Suppose we are given a cell in $\mathsf{Q}\,\mathbb{A}'$ of the following form.

$$
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{u}"] \arrow[d, dashed, "\vec{f}"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, dashed, "\vec{g}"] \\
\cdot \arrow[r, dotted, "v"'] & \cdot
\end{tikzcd}
$$

Using the above correspondence, we define $F\alpha$ as the unique cell satisfying the following equality.

$$
\begin{tikzcd}
& IA \arrow[rr, dashed, "I\vec{u}"] \arrow[dl, "\eta_A"'] \arrow[dr, dashed, "I\vec{f}"] & & IB \arrow[dr, dashed, "I\vec{g}"] \arrow[d, phantom, "I\alpha"] & \\
GFA \arrow[dr, dashed, "GF\vec{f}"'] \arrow[rr, phantom, "\eta_{\vec{f}}"] & & IX \arrow[dl, "\eta_X"] \arrow[rr, dotted, "Iv"] & {} \arrow[d, phantom, "\eta_v"] & IY \arrow[dl, "\eta_Y"] \\
& GFX \arrow[rr, dotted, "GFv"'] & & GFY &
\end{tikzcd}
=
\begin{tikzcd}
& IA \arrow[rr, dashed, "I\vec{u}"] \arrow[dl, "\eta_A"'] \arrow[r, phantom, "\eta_{\vec{u}}"] & {} & IB \arrow[dl, "\eta_B"'] \arrow[dr, dashed, "I\vec{g}"] & \\
GFA \arrow[rr, dashed, "GF\vec{u}"] \arrow[dr, dashed, "GF\vec{f}"'] & & GFB \arrow[dr, dashed, "GF\vec{g}"] \arrow[rr, phantom, "\eta_{\vec{g}}"] \arrow[d, phantom, "GF\alpha"] & & IY \arrow[dl, "\eta_Y"] \\
& GFX \arrow[rr, dotted, "GFv"'] & {} & GFY &
\end{tikzcd}
\tag{32}
$$

Observe that these data obviously satisfies (HTrans) by (29). Moreover, (32) shows the condition (3) in Proposition 2.10, for which η is a weak transformation.

It only remains to show that F is a weak functor, which easily follows from the functoriality of I and G, and the uniqueness property with respect to (32). $\square$

4.4. **Basic properties.** In this subsection, suppose that the following diagram exhibits an I-relative companied biadjunction:

$$\begin{tikzcd}
\mathbb{A}' \arrow[r, "I"] \arrow[dr, "F"'] & \mathbb{A} \arrow[d, phantom, "\Downarrow\eta"', pos=0.3] \\
& \mathbb{B} \arrow[u, "G"']
\end{tikzcd}$$

Lemma 4.14. Suppose that the following cells corresponds to each other through $\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$.

$$\begin{tikzcd}
FA_0 \arrow[r, dashed, "\shortmid" marking, "F\vec{p}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\alpha"] & FA_2 \arrow[d, "g"] \\
FA_1 \arrow[r, dotted, "\shortmid" marking, "v"'] & B
\end{tikzcd}
\text{ in } \mathbb{B}
\quad\Bigg\|\quad
\begin{tikzcd}
IA_0 \arrow[r, dashed, "\shortmid" marking, "I\vec{p}"] \arrow[d, "If"'] \arrow[dr, phantom, "\beta"] & IA_2 \arrow[d, "\widehat{g}"] \\
IA_1 \arrow[r, "\shortmid" marking, "A_1^* v"'] & GB
\end{tikzcd}
\text{ in } \mathbb{A}$$

Then α is weakly opcartesian if β is as well.

Proof. almost done $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$ gives the following bijective correspondence.

$$\begin{tikzcd}
FA_1 \arrow[r, dotted, "\shortmid" marking, "v"] \arrow[d, "x"'] \arrow[dr, phantom, "\gamma_1"] & B \arrow[d, "y"] \\
X \arrow[r, dotted, "\shortmid" marking, "r"'] & Y
\end{tikzcd}
\text{ in } \mathbb{B}
\quad\Bigg\|\quad
\begin{tikzcd}
IA_1 \arrow[r, "\shortmid" marking, "A_1^* v"] \arrow[d, "\widehat{x}"'] \arrow[dr, phantom, "\delta_1"] & GB \arrow[d, "Gy"] \\
GX \arrow[r, "\shortmid" marking, "Gr"'] & GB
\end{tikzcd}
\text{ in } \mathbb{A}$$

On the other hand, $\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right)$ gives the following bijective correspondence.

$$\begin{tikzcd}
FA_0 \arrow[r, dashed, "\shortmid" marking, "F\vec{p}"] \arrow[d, "Ff"'] & FA_2 \arrow[d, "g"] \\
FA_1 \arrow[r, phantom, "\gamma_2"] \arrow[d, "x"'] & B \arrow[d, "y"] \\
X \arrow[r, dotted, "\shortmid" marking, "r"'] & Y
\end{tikzcd}
\text{ in } \mathbb{B}
\quad\Bigg\|\quad
\begin{tikzcd}
IA_0 \arrow[r, dashed, "\shortmid" marking, "I\vec{p}"] \arrow[d, "If"'] & IA_2 \arrow[d, "\widehat{g}"] \\
IA_1 \arrow[r, phantom, "\delta_2"] \arrow[d, "\widehat{x}"] & GB \arrow[d, "Gy"] \\
GX \arrow[r, "\shortmid" marking, "Gr"'] & GY
\end{tikzcd}
\text{ in } \mathbb{A}$$

One can easily check that the precomposition $\gamma_1 \mapsto \alpha \fatsemi \gamma_1 := \gamma_2$ of α is compatible with the precomposition $\delta_1 \mapsto \beta \fatsemi \delta_1 := \delta_2$ of β through these bijective correspondences. This shows that α is weakly opcartesian if β is as well. $\square$

Observe here that through the same discussion as that in the proof of Theorem 4.13, the image under F of a cell is determined by the naturality of η: for each cell

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid" marking, "\vec{p}"] \arrow[d, dashed, "\vec{f}"'] \arrow[dr, phantom, "\alpha"] & A_2 \arrow[d, dashed, "\vec{g}"] \\
A_1 \arrow[r, dotted, "\shortmid" marking, "u"'] & A_3
\end{tikzcd} \tag{33}$$

in $\mathsf{Q}\,\mathbb{A}'$, $F\alpha$ is the unique cell satisfying the equality (32).

Proposition 4.15. Suppose that we are given a cell α in **??** in $\mathsf{Q}\,\mathbb{A}'$ such that the image $I\alpha$ in $\mathbb{A}$ is weakly opcartesian. Then, $F\alpha$ is also weakly opcartesian.

Proof. Suppose that the following cells corresponds to each other by $\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$. Here, the horizontally invertible cell on the left is the image under F of the canonical cell $\fatsemi\left(\vec{f}\right) \cong \vec{f}$ in

$\mathsf{Q}\,\mathbb{A}'$.

$$
\begin{tikzcd}
FA_0 \arrow[r, dashed, "F\vec{p}"] \arrow[d, dashed, "F\vec{f}"'] \arrow[d, bend right=60, "F⨾(\vec{f})"'] \arrow[d, phantom, shift right=4ex, "\cong"] \arrow[dr, phantom, "F\alpha"] & FA_2 \arrow[d, "⨾(F\vec{g})"] \\
FA_1 \arrow[r, dotted, "Fu"'] & FA_3
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, "I⨾(\vec{f})"'] \arrow[dr, phantom, "\beta"] & IA_2 \arrow[d, "\widehat{⨾(F\vec{g})}"] \\
IA_1 \arrow[r, "A_1^* u"'] & GFA_3
\end{tikzcd}
$$

Repharase the naturality for η as follows:

$$
\begin{tikzcd}
& IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[dl, dashed, "I\vec{f}"'] \arrow[d, "\eta_{A_0}"] \arrow[dr, phantom, "\eta_{\vec{p}}"] & IA_1 \arrow[dr, dashed, "I\vec{g}"] \arrow[d, "\eta_{A_0}"] & \\
IA_2 \arrow[dr, "\eta_{A_2}"'] \arrow[r, phantom, "\eta_{\vec{f}}^{-1}"] & GFA_0 \arrow[r, dashed, "GF\vec{p}"] \arrow[d, dashed, "GF\vec{f}"] \arrow[dr, phantom, "GF\alpha"] & GFA_1 \arrow[d, dashed, "GF\vec{g}"] \arrow[r, phantom, "\eta_{\vec{g}}"] & IA_3 \arrow[dl, "\eta_{A_3}"] \\
& GFA_2 \arrow[r, dotted, "GFu"'] & GFA_3 &
\end{tikzcd}
=
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, dashed, "I\vec{f}"'] \arrow[dr, phantom, "I\alpha"] & IA_1 \arrow[d, dashed, "I\vec{g}"] \\
IA_2 \arrow[r, dotted, "Iu"] \arrow[d, "\eta_{A_2}"'] \arrow[dr, phantom, "\eta_u"] & IA_3 \arrow[d, "\eta_{A_3}"] \\
GFA_2 \arrow[r, dotted, "GFu"'] & GFA_3
\end{tikzcd}
$$

The universality of the cartesian cell κ_u, (HTrans), and the definition of β induce the following equality.

$$
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, dashed, "I\vec{f}"'] \arrow[dr, phantom, "I\alpha"] & IA_1 \arrow[d, dashed, "I\vec{g}"] \\
IA_2 \arrow[r, dotted, "Iu"] \arrow[d, equal] \arrow[dr, phantom, "c_u\colon \mathsf{s.opc}"] & IA_3 \arrow[d, "\eta_{A_3}"] \\
IA_2 \arrow[r, "A_2^* u"'] & GFA_3
\end{tikzcd}
=
\begin{tikzcd}
IA_0 \arrow[r, dashed, "I\vec{p}"] \arrow[d, "I⨾(\vec{f})"] \arrow[d, dashed, bend right=60, "I\vec{f}"'] \arrow[d, phantom, shift right=4ex, "\cong"] \arrow[dr, phantom, "\beta"] & IA_1 \arrow[d, "\widehat{⨾(F\vec{g})}"'] \arrow[r, dashed, "I\vec{g}"] \arrow[dr, phantom, "\cong"] & IA_3 \arrow[dl, "\eta_{A_3}"] \\
IA_2 \arrow[r, "A_2^* u"'] & GFA_3 &
\end{tikzcd}
$$

The horizontally invertible cells are those induced from the image under I of the canonical cell $\vec{f} \cong ⨾(\vec{f})$ in $\mathsf{Q}\,\mathbb{A}'$, the image under G of the canonical cell $⨾(F\vec{g}) \cong \vec{Fg}$ in $\mathsf{Q}\,\mathbb{B}$, and $\eta_{\vec{g}}$. Since $I\alpha$ is weakly opcartesian, the composite of the right hand side is weakly opcartesian, and so is β. Therefore, by Lemma 4.14, we conclude $F\alpha$ is weakly opcartesian. $\square$

Given paths of horizontal arrows $\vec{p}\colon FA \dashrightarrow X_1$ and $\vec{q}\colon X_1 \dashrightarrow X_2$, observe that there is a canonical strongly opcartesian sequence satisfying the following equation. This follows from (i).

$$
\begin{tikzcd}
IA \arrow[r, dashed, "A^*\vec{p}"] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{s.opc}"] & GX_1 \arrow[r, dashed, "G\vec{q}"] & GX_2 \arrow[d, equal] \\
IA \arrow[rr, dashed, "A^*(\vec{p}\frown\vec{q})"'] \arrow[d, "\eta_A"'] \arrow[drr, phantom, "\kappa_{\vec{p}\frown\vec{q}}"] & & GX_2 \arrow[d, equal] \\
GFA \arrow[rr, dashed, "G(\vec{p}\frown\vec{q})"'] & & GX_2
\end{tikzcd}
=
\begin{tikzcd}
IA \arrow[r, dashed, "A^*\vec{p}"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\kappa_{\vec{p}}"] & GX_1 \arrow[r, dashed, "G\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & GX_2 \arrow[d, equal] \\
GFA \arrow[r, dashed, "G(\vec{p})"'] & GX_1 \arrow[r, dashed, "G(\vec{q})"'] & GX_2
\end{tikzcd}
\tag{34}
$$

Lemma 4.16. nannka eigo

- Consider the following cells, and suppose that γ_1, γ_2, and γ_3 corresponds to δ_1, δ_2, and δ_3 by $\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right)$, respectively.

$$\begin{tikzcd} FA_1 \arrow[r, dashed, "F\vec{p}", "|" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "\gamma_1"] & FA_3 \arrow[d, "x'"] \\ FA_2 \arrow[r, dotted, "v", "|" marking] \arrow[d, "x"'] \arrow[dr, phantom, "\gamma_2"] & X_2 \arrow[d, "g"] \\ X_1 \arrow[r, dotted, "r"', "|" marking] & X_3 \end{tikzcd} , \quad \begin{tikzcd} FA_1 \arrow[r, dashed, "F\vec{p}", "|" marking] \arrow[d, "Ff"'] & FA_3 \arrow[d, "x'"] \\ FA_2 \arrow[d, "x"'] \arrow[r, phantom, "\gamma_3"] & X_2 \arrow[d, "g"] \\ X_1 \arrow[r, dotted, "r"', "|" marking] & X_3 \end{tikzcd} \quad \text{in } \mathbb{B} \qquad \begin{tikzcd} IA_1 \arrow[r, dashed, "I\vec{p}", "|" marking] \arrow[d, "If"'] \arrow[dr, phantom, "\delta_1"] & IA_3 \arrow[d, "\widehat{x'}"] \\ IA_2 \arrow[r, "A_2^* v", "|" marking] \arrow[d, "\widehat{x}"'] \arrow[dr, phantom, "\delta_2"] & GX_2 \arrow[d, "Gg"] \\ GX_1 \arrow[r, dotted, "Gr"', "|" marking] & GX_3 \end{tikzcd} , \quad \begin{tikzcd} IA_1 \arrow[r, dashed, "I\vec{p}", "|" marking] \arrow[d, "If"'] & IA_3 \arrow[d, "\widehat{x'}"] \\ IA_2 \arrow[d, "\widehat{x}"'] \arrow[r, phantom, "\delta_3"] & GX_2 \arrow[d, "Gg"] \\ GX_1 \arrow[r, dotted, "Gr"', "|" marking] & GX_3 \end{tikzcd} \quad \text{in } \mathbb{A}$$

  Then $\gamma_1 ⨾ \gamma_2 = \gamma_3$ holds if and only if $\delta_1 ⨾ \delta_2 = \delta_3$ holds.
- Consider the following cells, and suppose that γ_1, γ_2, and γ_3 corresponds to δ_1, δ_2, and δ_3 by $\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right)$, respectively.

$$\begin{matrix} \begin{tikzcd} FA_1 \arrow[r, dashed, "F\vec{p}", "|" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "\gamma_1"] & FA_3 \arrow[r, dashed, "F\vec{q}", "|" marking] \arrow[d, "x"] \arrow[dr, phantom, "\gamma_2"] & X_2 \arrow[d, "g"] \\ FA_2 \arrow[r, dotted, "u"', "|" marking] & X_1 \arrow[r, dotted, "r"', "|" marking] & X_3 \end{tikzcd} \\ , \\ \begin{tikzcd} FA_1 \arrow[r, dashed, "F\vec{p}", "|" marking] \arrow[d, "Ff"'] & FA_3 \arrow[r, dashed, "F\vec{q}", "|" marking] \arrow[d, phantom, "\gamma_3"] & X_2 \arrow[d, "g"] \\ FA_2 \arrow[r, dotted, "u"', "|" marking] & X_1 \arrow[r, dotted, "r"', "|" marking] & X_3 \end{tikzcd} \end{matrix} \quad \text{in } \mathbb{B}, \qquad \begin{matrix} \begin{tikzcd} IA_1 \arrow[r, dashed, "I\vec{p}", "|" marking] \arrow[d, "If"'] \arrow[dr, phantom, "\delta_1"] & IA_3 \arrow[r, dashed, "I\vec{q}", "|" marking] \arrow[d, "\widehat{x}"] \arrow[dr, phantom, "\delta_2"] & GX_2 \arrow[d, "Gg"] \\ IA_2 \arrow[r, "A_2^* u"', "|" marking] & GX_1 \arrow[r, dotted, "Gr"', "|" marking] & GX_3 \end{tikzcd} \\ , \\ \begin{tikzcd} IA_1 \arrow[r, dashed, "I\vec{p}", "|" marking] \arrow[d, "If"'] & IA_3 \arrow[r, dashed, "I\vec{q}", "|" marking] \arrow[d, phantom, "\delta_3"] & GX_2 \arrow[d, "Gg"] \\ IA_2 \arrow[rr, "A_2^*(u \frown r)"', "|" marking] & & GX_3 \end{tikzcd} \end{matrix} \quad \text{in } \mathbb{A}$$

  Then $\gamma_1 \frown \gamma_2 = \gamma_3$ holds if and only if $(\delta_1 \frown \delta_2) ⨾ c = \delta_3$ holds, where c is the strongly opcartesian cell obtained in (34).

Proof. ongoing, but trivial □

Theorem 4.17 (Companion theorem)**.** nannka eigo

(i) A vertical arrow $FA \xrightarrow{f} B$ in $\mathbb{B}$ has a companion if and only if so does $IA \xrightarrow{\widehat{f}} GB$ in $\mathbb{A}$.
(ii) A horizontal arrow $FA \xrightarrow{u} B$ in $\mathbb{B}$ is a companion of some vertical arrow if and only if so is $IA \xrightarrow{\widehat{u}} GB$ in $\mathbb{A}$.

Proof. Suppose that we are given a horizontal arrow $FA \xrightarrow{u} B$ and a vertical arrow $FA \xrightarrow{f} B$, along with four cells which are related through $\left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$ in the following way.

$$\begin{tikzcd} & FA \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\alpha_1"] & \\ FA \arrow[rr, "u"', "|" marking] & {} & B \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} & IA \arrow[dl, equal] \arrow[dr, "\widehat{f}"] \arrow[d, phantom, "\beta_1"] & \\ IA \arrow[rr, "\widehat{u}"', "|" marking] & {} & GB \end{tikzcd} \qquad \left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right)$$

$$\begin{tikzcd} FA \arrow[rr, "u", "|" marking] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\alpha_2"] & B \arrow[dl, equal] \\ & B & \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} IA \arrow[rr, "\widehat{u}", "|" marking] \arrow[dr, "\widehat{f}"'] & {} \arrow[d, phantom, "\beta_2"] & GB \arrow[dl, equal] \\ & GB & \end{tikzcd} \qquad \left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$$

We show that (u, α_1, α_2) forms a companion of f if and only if $(\widehat{u}, \beta_1, \beta_2)$ forms a companion of $\widehat{f}$. This shows (i) and (ii) by considering $\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} F & G \end{smallmatrix}\right)$. Now observe that $\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right)$ and

$\left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)$ relate horizontal and vertical identity cells in the following way.

$$
\begin{tikzcd} FA \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ B \end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd} IA \arrow[d, bend right, "\widehat{f}"'] \arrow[d, bend left, "\widehat{f}"] \arrow[d, phantom, "="] \\ GB \end{tikzcd}
\qquad \left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right)
$$

$$
\begin{tikzcd} FA \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\ FA \arrow[r, "u"'] & B \end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd} IA \arrow[r, "\widehat{u}"] \arrow[d, equal] \arrow[dr, phantom, "="] & GB \arrow[d, equal] \\ IA \arrow[r, "\widehat{u}"'] & GB \end{tikzcd}
\qquad \left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)
$$

Therefore, considering the equations for cells defining a companion, the tuples (u, α_1, α_2) and $(\widehat{u}, \beta_1, \beta_2)$ simultaneously form companions by Lemma 4.16. ☐

Theorem 4.18. Suppose the following for the relative companied adjunction:

- $\mathbb{A}'$ is a full sub-pseudo-double category of $\mathbb{A}$ and I is the inclusion.
- Every component of η has a conjunction in $\mathbb{A}$.
- Every component of η is fully faithful in $\mathbb{A}$.

Then, the following are equivalent for a horizontal arrow $FA_0 \xrightarrow{u} FA_1$ in $\mathbb{B}$:

(i) u lies in the essential image of F, i.e., $u \cong Fv$ for some $A_0 \xrightarrow{v} A_1$ in $\mathbb{A}$.

(ii) The following forms a cartesian cell:

$$
\begin{tikzcd} A_0 \arrow[r] \arrow[d, "\eta_{A_0}"'] \arrow[dr, phantom, "\mathsf{cart}"] & A_1 \arrow[r, "\eta_{A_1 *}"] \arrow[d, "\eta_{A_1}"'] & GFA_1 \\ GFA_0 \arrow[r, "Gu"'] & GFA_1 \arrow[ur, equal, "\mathsf{cart}"] & \end{tikzcd}
\qquad \text{in } \mathbb{A}.
$$

Proof. [(i) $\Longrightarrow$ (ii)] Consider the following equations of cells:

$$
\begin{tikzcd} A_0 \arrow[r, "u"] \arrow[d, "\eta_{A_0}"'] \arrow[dr, phantom, "\eta_u"] & A_1 \arrow[d, "\eta_{A_1}"] \\ GFA_0 \arrow[r, "GFu"'] & GFA_1 \end{tikzcd}
=
\begin{tikzcd} A_0 \arrow[rr, "u"] \arrow[d, equal] \arrow[dr, phantom, "="] & & A_1 \arrow[d, "\eta_{A_1}"] \\ A_0 \arrow[r, "u"] \arrow[d, "\eta_{A_0}"'] \arrow[dr, phantom, "\eta_u"] & A_1 \arrow[ur, equal, "\mathsf{s.opc}"] \arrow[r, "\eta_{A_1 *}"] \arrow[dr, "\eta_{A_1}"', "\mathsf{cart}"] & GFA_1 \arrow[d, equal] \\ GFA_0 \arrow[rr, "GFu"'] & & GFA_1 \end{tikzcd}
=
\begin{tikzcd} A_0 \arrow[rr, "u"] \arrow[d, equal] \arrow[drr, phantom, "\alpha\colon \mathsf{cart}"] & & A_1 \arrow[d, "\eta_{A_1}"] \\ A_0 \arrow[r, "u"] \arrow[d, "\eta_{A_0}"'] \arrow[drr, phantom, "\beta\colon \mathsf{cart}"] & A_1 \arrow[r, "\eta_{A_1 *}"] & GFA_1 \arrow[d, equal] \\ GFA_0 \arrow[rr, "GFu"'] & & GFA_1 \end{tikzcd}
$$

Since η_{A_1} is fully faithful, the cell α is cartesian. By (HTrans), the cell β is also cartesian. Thus, η_u is cartesian. Then, the cartesianness of β implies (ii).

[(ii) $\Longrightarrow$ (i)] Let us define $V := Gv(\eta_{A_0}, \eta_{A_1})$. By (HTrans), the cell

$$
\begin{tikzcd} A_0 \arrow[r, "\eta_{A_0 *}"] \arrow[d, equal] & GFA_0 \arrow[r, "GFv"] \arrow[d, phantom, "\cong"] & GFA_1 \arrow[d, equal] \\ A_0 \arrow[r, "v"] \arrow[d, "\eta_{A_0}"'] \arrow[dr, phantom, "\mathsf{cart}"] & A_1 \arrow[r, "\eta_{A_1 *}"] \arrow[d, "\eta_{A_1}"'] & GFA_1 \\ GFA_0 \arrow[r, "Gu"'] & GFA_1 \arrow[ur, equal, "\mathsf{cart}"] & \end{tikzcd}
\qquad \text{in } \mathbb{A}.
$$

becomes cartesian. In other words, we get an iso-cell

$$\begin{tikzcd} A_0 \arrow[r, "\widehat{Fv}"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & GFA_1 \arrow[d, equal] \\ A_0 \arrow[r, "\widehat{u}"'] & GFA_1 \end{tikzcd} \quad \text{in } \mathbb{A}.$$

By $\left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right)$, we obtain a desired iso-morphism $Fv \cong u$. □

4.5. Ind-completions as a double functor.

Definition 4.19. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let us denote by $\mathbb{E}_\Phi$ the pseudo-double category defined as follows:

- $\mathbb{E}_\Phi$ contains all the objects of $\mathbb{E}$;
- $\mathbb{E}_\Phi$ contains all the vertical arrows in $\mathbb{E}$;
- A horizontal arrow in $\mathbb{E}_\Phi$ is that in Φ;
- $\mathbb{E}_\Phi$ contains all suitable cells in $\mathbb{E}$. ♦

Notation 4.20. Let us denote by $\mathbb{E}_\Phi{}'$ the full sub-pseudo-double category of $\mathbb{E}_\Phi$ consisting of all objects whose Φ-ind-completion exists in $\mathbb{E}$. ♦

Definition 4.21. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let us denote by $\mathbb{Q}\Phi^\triangledown$ the (strict) double category defined as follows:

- An object in $\mathbb{Q}\Phi^\triangledown$ is a Φ-cocomplete object in $\mathbb{E}$;
- A vertical arrow in $\mathbb{Q}\Phi^\triangledown$ is a Φ-cocontinuous vertical arrow in $\mathbb{E}$;
- A horizontal arrow $X \longrightarrow Y$ in $\mathbb{Q}\Phi^\triangledown$ is a vertical arrow $Y \longrightarrow X$ in $\mathbb{E}$.
- A cell

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g"] \\ Z \arrow[r, "v"'] & W \end{tikzcd} \quad \text{in } \mathbb{Q}\Phi^\triangledown$$

is a cell

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, "u"'] & Y \arrow[d, "g"] \\ X \arrow[d, "f"'] \arrow[r, phantom, "\alpha"] & W \arrow[d, "v"] \\ Z \arrow[r, equal] & Z \end{tikzcd} \quad \text{in } \mathbb{E}.$$

♦

Definition 4.22 (This does not work in pre-dogmas)**.** Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. The following assignments yields a pseudo-double functor $\mathbb{Q}\Phi^\triangledown \xrightarrow{U} \mathbb{E}_\Phi$:

- For each object $X \in \mathbb{Q}\Phi^\triangledown$, $UX := X$.
- For each vertical arrow $X \xrightarrow{f} Y$ in $\mathbb{Q}\Phi^\triangledown$, $Uf := f$.
- For each horizontal arrow $X \xrightarrow{u} Y$ in $\mathbb{Q}\Phi^\triangledown$, the conjoint $Uu := u^*$ in $\mathbb{E}$.
- For each cell

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g"] \\ Z \arrow[r, "v"'] & W \end{tikzcd} \quad \text{in } \mathbb{Q}\Phi^\triangledown,$$

the cell

$$
\begin{tikzcd}
UX \arrow[r, "Uu"] \arrow[d, "Uf"'] \arrow[dr, phantom, "U\alpha"] & UY \arrow[d, "Ug"] \\
UZ \arrow[r, "Uv"'] & UW
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
X \arrow[r, "u^*"] \arrow[d, equal] & Y \arrow[d, "g"] \arrow[dl, "u"'] \\
X \arrow[d, "f"'] \arrow[r, phantom, "\alpha"] & W \arrow[d, equal] \arrow[dl, "v"'] \\
Z \arrow[r, "v^*"'] & W
\end{tikzcd}
\quad \text{in } \mathbb{E}.
$$

♦

Theorem 4.23. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Then, the Φ-ind-completions $A \xrightarrow{\eta_A} U\Phi^\triangledown A$ for $A \in \mathbb{E}_\Phi{}'$ yield a left I-relative company-biadjoint $\Phi^\triangledown$ of U:

$$
\begin{tikzcd}
\mathbb{E}_\Phi{}' \arrow[r, phantom, "\overset{I}{\subseteq}"] \arrow[dr, "\Phi^\triangledown"'] & \mathbb{E}_\Phi \\
& \mathbb{Q}\Phi^\triangledown \arrow[u, "U"'] \arrow[ul, phantom, "\Downarrow\eta"]
\end{tikzcd}
$$

Here, I denotes the inclusion.

Proof. By the definition of Φ-ind-completions, each η_A has a companion. Thus, by Theorem 4.13, it suffices to prove the conditions $\left(\begin{smallmatrix}F\\G\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}F & G\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}F & G\\G & G\end{smallmatrix}\right)$. ongoing □

Definition 4.24 (Explicit definition of $\Phi^\triangledown$)**.** Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. The following assignments yield a pseudo-double functor $\mathbb{E}_\Phi{}' \xrightarrow{\Phi^\triangledown} \mathbb{Q}\Phi^\triangledown$:

- For each object $A \in \mathbb{E}_\Phi{}'$, its Φ-ind-completion $\Phi^\triangledown A$.
- For each vertical arrow $A \xrightarrow{f} B$ in $\mathbb{E}$, the left Kan extension $\Phi^\triangledown f$ of $f ⨟ b$ along a_*.

$$
\begin{tikzcd}
A \arrow[r, "a_*"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{lan}"] & \Phi^\triangledown A \arrow[ddl, "\Phi^\triangledown f"] \\
B \arrow[d, "b"'] & \\
\Phi^\triangledown B &
\end{tikzcd}
$$

  Here, a and b denote Φ-ind-morphisms.
- For each horizontal arrow $A \xrightarrow{u} B$ in Φ, the left Kan extension $\Phi^\triangledown u$ of a along the path $A \xrightarrow{u} B \xrightarrow{b_*} \Phi^\triangledown B$.

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathsf{lan}"] & B \arrow[r, "b_*"] & \Phi^\triangledown B \arrow[dll, "\Phi^\triangledown u"] \\
\Phi^\triangledown A & &
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

  Here, a and b denote Φ-ind-morphisms.
- For each cell

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
C \arrow[r, "v"'] & D
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

with $u, v \in \Phi$, a unique cell $\Phi^\triangledown\alpha$ such that

$$
\begin{tikzcd}
A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, "a"] & B \arrow[r, "b_*", "\shortmid"{marking}] \arrow[d, phantom, "\mathsf{lan}"] & \Phi^\triangledown B \arrow[dl, "\Phi^\triangledown u"'] \arrow[d, "\Phi^\triangledown g"] \\
C \arrow[r, phantom, "\cong"] \arrow[dr, "c"'] & \Phi^\triangledown A \arrow[d, "\Phi^\triangledown f"] \arrow[r, phantom, "\Phi^\triangledown\alpha"] & \Phi^\triangledown D \arrow[dl, "\Phi^\triangledown v"] \\
& \Phi^\triangledown C &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "b_*", "\shortmid"{marking}] \arrow[d, "g"] \arrow[dr, phantom, "\mathsf{lan}"] & \Phi^\triangledown B \arrow[dd, "\Phi^\triangledown g"] \\
C \arrow[r, "v"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] \arrow[dr, "d"] & \\
C \arrow[r, "v"', "\shortmid"{marking}] \arrow[dr, "c"'] & D \arrow[r, "d_*"', "\shortmid"{marking}] \arrow[d, phantom, "\mathsf{lan}"] & \Phi^\triangledown D \arrow[dl, "\Phi^\triangledown v"] \\
& \Phi^\triangledown C &
\end{tikzcd}
\quad \text{in } \mathbb{E}.
$$

Note that by Lemma 4.8, $\Phi^\triangledown u ⨾ \Phi^\triangledown f$ is a left Kan extension of $f⨾c \cong a⨾\Phi^\triangledown f$ along the path $A \xrightarrow{u} B \xrightarrow{b_*} \Phi^\triangledown B$. ♦

4.6. **Three corollaries: nerves and realizations.** The companion theorem (Theorem 4.17) provides us several useful corollaries.

Corollary 4.25. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let $A \xrightarrow{a} \Phi^\triangledown A$ be a Φ-ind-completion in $\mathbb{E}$. Fix a left Kan extension of some f along a_*

$$
\begin{tikzcd}
A \arrow[r, "a_*", "\shortmid"{marking}] \arrow[d, "f"'] & \Phi^\triangledown A \arrow[dl, "l"] \arrow[dl, phantom, "\mathsf{lan}"', near start] \\
E &
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

where E is Φ-cocomplete. Then, the following are equivalent:

(i) f_* belongs to Φ.
(ii) l has a right adjoint in the vertical 2-category of $\mathbb{E}$.

Proof. Regarding l as a vertical arrow $\Phi^\triangledown A \xrightarrow{l} E$ in $\mathbb{Q}\Phi^\triangledown$, (ii) is equivalent to saying that l has a companion in $\mathbb{Q}\Phi^\triangledown$. Thus, this follows from Theorem 4.17(i). $\square$

Corollary 4.26. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let $A \xrightarrow{a} \Phi^\triangledown A$ be a Φ-ind-completion in $\mathbb{E}$. Fix a left Kan extension of a along some $\varphi \in \Phi$

$$
\begin{tikzcd}
A \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "a"'] & E \arrow[dl, "r", "\shortmid"{marking}] \arrow[dl, phantom, "\mathsf{lan}"', near start] \\
\Phi^\triangledown A &
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

where E is Φ-cocomplete. Then, the following are equivalent:

(i) φ is a companion, i.e., $\varphi \cong f_*$ for some f.
(ii) r has a left adjoint in the vertical 2-category of $\mathbb{E}$.

Proof. Regarding r as a horizontal arrow $\Phi^\triangledown A \xrightarrow{r} E$ in $\mathbb{Q}\Phi^\triangledown$, (ii) is equivalent to saying that r is a companion of some vertical arrow in $\mathbb{Q}\Phi^\triangledown$. Thus, this follows from Theorem 4.17(ii). $\square$

Lemma 4.27. Let $\mathbb{D}$ be a pseudo-double category such that a conjunction always exists. Let $B \xrightarrow{g} C$ be a fully faithful vertical arrow in $\mathbb{D}$ that has a companion. Then, a vertical arrow $A \xrightarrow{f} B$ in $\mathbb{D}$ has a companion if and only if so does $f⨾g$.

Proof. Suppose that $f \fatsemi g$ has a companion. Since g is fully faithful, we can take a canonical cell α as follows:

$$
\begin{tikzcd}
A \arrow[r, "(f\fatsemi g)_*(\mathsf{id},g)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[d, "g"] \\
A \arrow[r, "(f\fatsemi g)_*", "\shortmid"{marking}] \arrow[d, "f"'] & C \arrow[dd, equal] \\
B \arrow[d, "g"'] \arrow[r, phantom, "\mathsf{cart}"] & \\
C \arrow[r, equal, "\shortmid"{marking}] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "(f\fatsemi g)_*(\mathsf{id},g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\
B \arrow[r, equal, "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[d, "g"] \\
C \arrow[r, equal, "\shortmid"{marking}] & C
\end{tikzcd}
\quad \text{in } \mathbb{D}.
$$

By the cancellation property of cartesian cells, α is cartesian. Thus, $(f \fatsemi g)_*(\mathsf{id}, g)$ becomes a companion of f.

We now show the opposite direction. Suppose that f has a companion. Then, the restriction $g_*(f, \mathsf{id})$ exists and the composite of the following cells becomes cartesian:

$$
\begin{tikzcd}
A \arrow[r, "g_*(f,\mathsf{id})", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{cart}"] & C \arrow[d, equal] \\
B \arrow[r, "g_*", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\mathsf{cart}"] & C \arrow[d, equal] \\
C \arrow[r, equal, "\shortmid"{marking}] & C
\end{tikzcd}
\quad \text{in } \mathbb{D}.
$$

Thus, $f \fatsemi g$ has a companion. □

Corollary 4.28 ([DL07, Prop 3.3])**.** Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let $A \xrightarrow{a} \Phi^\triangledown A$ and $B \xrightarrow{b} \Phi^\triangledown B$ be Φ-ind-completions in $\mathbb{E}$. Then, the following are equivalent for a vertical arrow $A \xrightarrow{f} B$ in $\mathbb{E}$:

(i) $\Phi^\triangledown A \xrightarrow{\Phi^\triangledown f} \Phi^\triangledown B$ has a right adjoint in the vertical 2-category of $\mathbb{E}$.

(ii) $A \xrightarrow{f_*} B$ belongs to Φ.

Proof. By Corollary 4.25, (i) is equivalent to $(f \fatsemi b)_* \in \Phi$, in other words, $f \fatsemi b$ has a companion in $\mathbb{E}_\Phi$. Moreover, by Lemma 4.27, this is equivalent to that f has a companion in $\mathbb{E}_\Phi$, i.e., the companion f_* in $\mathbb{E}$ belongs to Φ. predogmaで成り立つように証明を修正しても良さそう. □

5. Cauchy-completeness

5.1. **Adjunction of weights.**

Definition 5.1. Let $\mathbb{E}$ be a virtual equipment. A *left adjoint* of a left weight $X \xrightarrow{u} Y$ in $\mathbb{E}$ consists of:

- A horizontal arrow $Y \xrightarrow{v} X$ in $\mathbb{E}$;
- Cells

$$
\begin{tikzcd}
& Y \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\
Y \arrow[rr, "v\odot u"', "\shortmid"{marking}] & {} & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "u", "\shortmid"{marking}] \arrow[dr, equal] & Y \arrow[r, "v", "\shortmid"{marking}] \arrow[d, phantom, "\varepsilon"] & X \arrow[dl, equal] \\
& X &
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

such that the following equations hold.

$$\begin{tikzcd}
& Y \arrow[rr, "v"] \arrow[dl, equals] \arrow[dr, equals] \arrow[d, phantom, "\eta"] & & X \arrow[dr, equals] \arrow[dl, phantom, "="] & \\
Y \arrow[rr, "v\odot u"'] \arrow[d, equals] \arrow[drrrr, phantom, "v\odot\varepsilon"] & & Y \arrow[rr, "v"'] & & X \arrow[d, equals] \\
Y \arrow[rrrr, "v"'] & & & & X
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, equals] \arrow[dr, phantom, "="] & X \arrow[d, equals] \\
Y \arrow[r, "v"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
& X \arrow[rr, "u"] \arrow[dl, equals] \arrow[dr, phantom, "="] & & Y \arrow[dl, equals] \arrow[dr, equals] \arrow[d, phantom, "\eta"] & \\
X \arrow[rr, "u"'] \arrow[d, equals] \arrow[drrrr, phantom, "\varepsilon\odot u"] & & Y \arrow[rr, "v\odot u"'] & & Y \arrow[d, equals] \\
X \arrow[rrrr, "u"'] & & & & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equals] \arrow[dr, phantom, "="] & Y \arrow[d, equals] \\
X \arrow[r, "u"'] & Y
\end{tikzcd}$$

Here, $v\odot\varepsilon$ and $\varepsilon\odot u$ are unique cells determined as follows:

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, equals] \arrow[drr, phantom, "\mathsf{comp}"] & X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, equals] \arrow[dr, phantom, "="] & X \arrow[d, equals] \\
Y \arrow[rr, "v\odot u"'] \arrow[d, equals] \arrow[drrr, phantom, "v\odot\varepsilon"] & & Y \arrow[r, "v"'] & X \arrow[d, equals] \\
Y \arrow[rrr, "v"'] & & & X
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, equals] \arrow[dr, phantom, "="] & X \arrow[r, "u"] \arrow[drr, equals] & Y \arrow[r, "v"] \arrow[d, phantom, "\varepsilon"] & X \arrow[d, equals] \\
Y \arrow[rrr, "v"'] & & & X
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equals] \arrow[dr, phantom, "="] & Y \arrow[r, "v"] \arrow[d, equals] \arrow[drr, phantom, "\mathsf{comp}"] & X \arrow[r, "u"] & Y \arrow[d, equals] \\
X \arrow[r, "u"'] \arrow[d, equals] \arrow[drrr, phantom, "\varepsilon\odot u"] & Y \arrow[rr, "v\odot u"'] & & Y \arrow[d, equals] \\
X \arrow[rrr, "u"'] & & & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equals] \arrow[dr, phantom, "\varepsilon"] & Y \arrow[r, "v"] \arrow[dl, equals] & X \arrow[r, "u"] \arrow[d, phantom, "="] & Y \arrow[d, equals] \\
X \arrow[rrr, "u"'] & & & Y
\end{tikzcd}$$

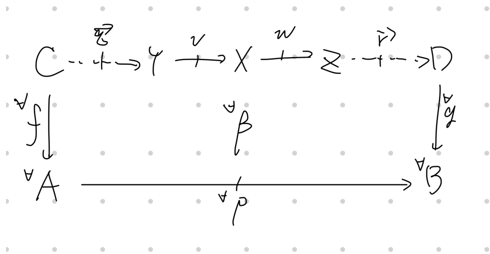
♦

Theorem 5.2. Let $X \xrightarrow{u} Y$ be a left weight in a virtual equipment $\mathbb{E}$. Then, the left adjoint $Y \xrightarrow{v} X$ of u becomes a right weight in $\mathbb{E}$.

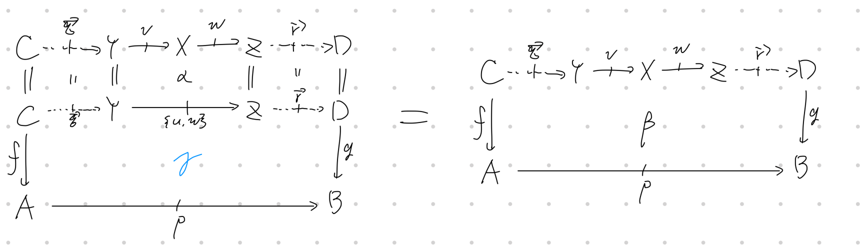
Proof of (RW1). Take an arbitrary horizontal arrow $X \xrightarrow{w} Z$ in $\mathbb{E}$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[drr, equals] & Y \arrow[r, "v"] \arrow[d, phantom, "\mathrm{ext}"] & X \arrow[r, "w"] \arrow[dl, equals] \arrow[dr, phantom, "="] & Z \arrow[d, equals] \\
& X \arrow[rr, "w"'] & & Z
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equals] \arrow[dr, phantom, "="] & Y \arrow[r, "v"] \arrow[d, equals] \arrow[drr, phantom, "\alpha"] & X \arrow[r, "w"] & Z \arrow[d, equals] \\
X \arrow[r, "u"'] \arrow[d, equals] \arrow[drrr, phantom, "\mathrm{ext}"] & Y \arrow[rr, "\{u,w\}"'] & & Z \arrow[d, equals] \\
X \arrow[rrr, "w"'] & & & Z
\end{tikzcd}$$

We will show that α is opcartesian. Take $C \overset{\vec{q}}{\dashrightarrow} Y$ and $Z \overset{\vec{r}}{\dashrightarrow} D$. Take



We have to prove that the following γ exists uniquely.



The existence of γ follows from here.
The uniqueness of γ follows from here. □

Proof of (RW2). Take $X \xrightarrow{f} W$ and $Z \xrightarrow{w} W$ arbitrarily. The proof is here. □

Lemma 5.3. Let $\mathbb{E}$ be a virtual equipment. Suppose that a left weight $X \xrightarrow{u} Y$ and a right weight $Y \xrightarrow{v} X$ form an adjunction in $\mathbb{E}$. Let us consider the following cells:

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "f"'] & \arrow[d, phantom, "\alpha"] & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "v"] \arrow[dr, "g"'] & \arrow[d, phantom, "\beta"] & X \arrow[dl, "f"] \\ & Z & \end{tikzcd} \quad \text{in } \mathbb{E} \tag{35}$$

such that:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "f"'] \arrow[dr, phantom, bend left, "\alpha"] & Y \arrow[d, "g"] \arrow[r, "v"] & X \arrow[dl, "f"] \arrow[dl, phantom, bend right, "\beta"'] \\ & Z & \end{tikzcd} = \begin{tikzcd} X \arrow[r, "u"] \arrow[dr, equal] & Y \arrow[r, "v"] \arrow[d, phantom, "\varepsilon"] & X \arrow[dl, equal] \\ & X \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] & \\ & Z & \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\ Y \arrow[rr, "v\odot u"] \arrow[dr, "g"'] & \arrow[d, phantom, "\beta\odot\alpha"] & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} = \begin{tikzcd} Y \arrow[d, bend right, "g"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "="] \\ Z \end{tikzcd} \tag{36}$$

Here, the cell $\beta\odot\alpha$ is a unique one satisfying

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, phantom, "\mathsf{comp}"] & Y \arrow[d, equal] \\ Y \arrow[rr, "v\odot u"] \arrow[dr, "g"'] & \arrow[d, phantom, "\beta\odot\alpha"] & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} = \begin{tikzcd} Y \arrow[r, "v"] \arrow[dr, "g"'] \arrow[dr, phantom, bend left, "\beta"] & X \arrow[d, "f"] \arrow[r, "u"] & Y \arrow[dl, "g"] \arrow[dl, phantom, bend right, "\alpha"'] \\ & Z & \end{tikzcd}$$

Then, α is a lan-cell and β is a ran-cell.

Corollary 5.4 ([Gar14, 1.2 Theorem])**.** Let $\mathbb{E}$ be a virtual equipment. Suppose that a left weight $X \xrightarrow{u} Y$ and a right weight $Y \xrightarrow{v} X$ form an adjunction in $\mathbb{E}$. Let $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ be vertical arrows in $\mathbb{E}$. Then, there is a bijective correspondence between data of the following forms:

(i) A lan-cell

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "f"'] & \alpha & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

(ii) A ran-cell

$$\begin{tikzcd} Y \arrow[rr, "v"] \arrow[dr, "g"'] & \beta & X \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

(iii) Cells α and β in (35) satisfying the equations (36).

Theorem 5.5. Let $\mathbb{E}$ be a virtual equipment. For a left weight $X \xrightarrow{u} Y$ in $\mathbb{E}$, the following are equivalent:

(i) u has a left adjoint.

(ii) $X \xrightarrow{u} Y$ preserves the extension

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, equal] & Y \arrow[r, "v"] \arrow[d, phantom, "\mathsf{ext}"] & X \arrow[dl, equal] \\ & X & \end{tikzcd} \quad \text{in } \mathbb{E}.$$

(iii) Any horizontal arrow $X \longrightarrow \cdot$ in $\mathbb{E}$ preserves the extension [2]

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, equal] & Y \arrow[r, "v"] \arrow[d, phantom, "\mathsf{ext}"] & X \arrow[dl, equal] \\ & X & \end{tikzcd} \quad \text{in } \mathbb{E}.$$

Notation 5.6. Let $\mathbb{E}$ be a virtual equipment. We denote by $\mathsf{RA}_{\mathbb{E}}$ the class of all left weights having a left adjoint in $\mathbb{E}$. Dually, we denote by $\mathsf{LA}_{\mathbb{E}}$ the class of all right weights having a right adjoint in $\mathbb{E}$. ♦

Proposition 5.7. Let $\mathbb{E}$ be a virtual equipment. The class $\mathsf{RA}_{\mathbb{E}}$ becomes a left dogma on $\mathbb{E}$.

Theorem 5.8 (Cauchy completeness)**.** Let $\mathbb{E}$ be a virtual equipment. The following are equivalent for an object $X \in \mathbb{E}$:

(i) X is $\mathsf{RA}_{\mathbb{E}}$-cocomplete.

(ii) X is $\mathsf{LA}_{\mathbb{E}}$-complete.

(iii) Every $X \xrightarrow{u} \cdot$ in $\mathsf{RA}_{\mathbb{E}}$ is a companion for some vertical arrow.

(iv) Every $\cdot \xrightarrow{v} X$ in $\mathsf{LA}_{\mathbb{E}}$ is a conjoint for some vertical arrow.

Proof. [(i) $\Longrightarrow$ (iii)]

[(ii) $\Longrightarrow$ (i)] □

Definition 5.9. An object $X \in \mathbb{E}$ is called *Cauchy complete* if it satisfies the equivalent conditions of Theorem 5.8. ♦

[2] This makes a sense since v becomes a right weight.

5.2. **Absolute limits.**

Definition 5.10 (Absolute Kan extensions)**.** A lan-cell in a virtual equipment is called *absolute* if it is preserved by post-composing with arbitrary vertical arrows. ♦

Definition 5.11 (Absolute weights)**.** Let $\mathbb{E}$ be a virtual equipment. A left weight $X \xrightarrow{u} Y$ in $\mathbb{E}$ is called *left-absolute* if every left Kan extension along u is, whenever it exists, absolute in $\mathbb{E}$. ♦

Definition 5.12. We say that a virtual equipment $\mathbb{E}$ has *anti-restrictions* if for every horizontal arrow $X \xrightarrow{u} Y$ in $\mathbb{E}$, there exist an object $Z \in \mathbb{E}$, vertical arrows $X \xrightarrow{f} Z \xleftarrow{g} Y$, and a cartesian cell

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=4, "\mathsf{cart}"] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \quad \text{in } \mathbb{E}.$$

♦

Definition 5.13. A cell

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\ C \arrow[r] & B \end{tikzcd}$$

is called *right composing* if for every path $B \overset{\vec{p}}{\dashrightarrow} D$, the path of cells

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \arrow[r, dashed, "\vec{p}"] \arrow[dr, phantom, "="] & D \arrow[d, equal] \\ C \arrow[r] & B \arrow[r, dashed, "\vec{p}"'] & D \end{tikzcd}$$

is weakly opcartesian. ♦

Lemma 5.14. Let $\mathbb{E}$ be a virtual equipment having anti-restrictions. Then, for a cell

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[r, phantom, shift right=4, "\alpha"] & Y \arrow[dl, "l"] \\ A & \end{tikzcd} \quad \text{in } \mathbb{E},$$

the following are equivalent:

(i) The cell α is an absolute lan-cell.
(ii) The cell

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[r, phantom, shift right=4, "\alpha"] & Y \arrow[dl, "l"] \arrow[d, equal] \\ A \arrow[r, "l^*"'] \arrow[ur, phantom, shift right=4, "\mathsf{s.opc}"] & Y \end{tikzcd} \quad \text{in } \mathbb{E} \tag{37}$$

is right composing.

Proof. Proof □

Lemma 5.15 (Ext-cart Lemma)**.** Let $\mathbb{E}$ be a virtual equipment and let Φ be a left dogma on $\mathbb{E}$. Let $A \xrightarrow{a} \Phi^{\triangledown}A$ be a Φ-ind-morphism in $\mathbb{E}$. Let $B \xrightarrow{u} C$ be a horizontal arrow and let

$B \xrightarrow{f} \Phi^{\triangledown}A \xleftarrow{g} C$ be vertical arrows in $\mathbb{E}$. Then, there is a bijective correspondence between cells of the following forms:

$$
\begin{tikzcd}
A \arrow[r, "{\Phi^{\triangledown}A(a,f)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\alpha"] & B \arrow[r, "u", "\shortmid"{marking}] & C \arrow[d, equal] \\
A \arrow[rr, "{\Phi^{\triangledown}A(a,g)}"', "\shortmid"{marking}] & & C
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
B \arrow[rr, "u", "\shortmid"{marking}] \arrow[dr, "f"'] & \beta & C \arrow[dl, "g"] \\
& \Phi^{\triangledown}A &
\end{tikzcd}
\quad \text{in } \mathbb{E}.
$$

Moreover, under this correspondence, α is extending if and only if β is cartesian.

Proof. The desired correspondence follows from that $\Phi^{\triangledown}A(a,g)$ is a restriction and that f is a left Kan extension of a along $\Phi^{\triangledown}A(a,f)$.

Suppose that α is extending. To show that β is cartesian, take vertical arrows $B' \xrightarrow{s} B, C' \xrightarrow{t} C$, a horizontal path $B' \overset{\vec{p}}{\dashrightarrow} C'$. Consider the following opcartesian path of cells:

$$
\begin{tikzcd}
B' \arrow[rr, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "s"'] \arrow[dr, equal] & \arrow[d, phantom, "="] & C' \arrow[d, equal] \\
B \arrow[r, "s^*"', "\shortmid"{marking}] & B' \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] & C'
\end{tikzcd}
\quad \text{in } \mathbb{E}. \tag{38}
$$

Then, there is a bijective correspondence between cells of the following forms:

$$
\begin{tikzcd}
B' \arrow[rr, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "s"'] & & C' \arrow[d, "t"] \\
B \arrow[dr, "f"'] & \gamma_0 & C \arrow[dl, "g"] \\
& \Phi^{\triangledown}A &
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
B \arrow[r, "s^*", "\shortmid"{marking}] \arrow[ddr, "f"'] & B' \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & C' \arrow[d, "t"] \\
& \gamma_1 & C \arrow[dl, "g"] \\
& \Phi^{\triangledown}A &
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
A \arrow[r, "{\Phi^{\triangledown}A(a,f)}", "\shortmid"{marking}] \arrow[ddrr, "a"'] & B \arrow[r, "s^*", "\shortmid"{marking}] & B' \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & C' \arrow[d, "t"] \\
& & \gamma_2 & C \arrow[dl, "g"] \\
& & \Phi^{\triangledown}A &
\end{tikzcd}
$$

$$
\Bigg\|\quad
\begin{tikzcd}
A \arrow[r, "{\Phi^{\triangledown}A(a,f)}", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "s^*", "\shortmid"{marking}] \arrow[d, phantom, "\gamma_3"] & B' \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & C' \arrow[d, "t"] \\
A \arrow[rrr, "{\Phi^{\triangledown}A(a,g)}"', "\shortmid"{marking}] & & & C
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
B \arrow[r, "s^*", "\shortmid"{marking}] \arrow[d, equal] & B' \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, phantom, "\gamma_4"] & C' \arrow[d, "t"] \\
B \arrow[rr, "u"', "\shortmid"{marking}] & & C
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
B' \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "s"'] \arrow[dr, phantom, "\gamma_5"] & C' \arrow[d, "t"] \\
B \arrow[r, "u"', "\shortmid"{marking}] & C
\end{tikzcd}
$$

Indeed, since the path (38) of cells is opcartesian, γ_0 corresponds to γ_1, and γ_4 corresponds to γ_5. Since f is a left Kan extension, γ_1 corresponds to γ_2. Since $\Phi^{\triangledown}A(a,g)$ is a restriction, γ_2 corresponds to γ_3. Since u is an extension, γ_3 corresponds to γ_4. Consequently, we get a bijective correspondence between γ_0 and γ_5, which concludes that β is cartesian.

We next show that α is extending when β is cartesian. Take a horizontal path $B \overset{\vec{p}}{\dashrightarrow} C'$ and a vertical arrow $C' \xrightarrow{t} C$. Then, there is a bijective correspondence between cells of the following forms:

$$
\begin{tikzcd}
A \arrow[r, "{\Phi^{\triangledown}A(a,f)}", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, phantom, "\delta_0"] & C' \arrow[d, "t"] \\
A \arrow[rr, "{\Phi^{\triangledown}A(a,g)}"', "\shortmid"{marking}] & & C
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
A \arrow[r, "{\Phi^{\triangledown}A(a,f)}", "\shortmid"{marking}] \arrow[ddr, "a"'] & B \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] & C' \arrow[d, "t"] \\
& \delta_1 & C \arrow[dl, "g"] \\
& \Phi^{\triangledown}A &
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
B \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[dd, "f"'] & C' \arrow[d, "t"] \\
\delta_2 & C \arrow[dl, "g"] \\
\Phi^{\triangledown}A &
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
B \arrow[r, "\vec{p}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\delta_3"] & C' \arrow[d, "t"] \\
B \arrow[r, "u"', "\shortmid"{marking}] & C
\end{tikzcd}
$$

Indeed, since $\Phi^{\triangledown}A(a,g)$ is a restriction, δ_0 corresponds to δ_1. Since f is a left Kan extension, δ_1 corresponds to δ_2. Since β is cartesian, δ_2 corresponds to δ_3. Thus, α is extending. $\square$

Theorem 5.16. Let $\mathbb{E}$ be a virtual equipment having anti-restrictions, and let $X \xrightarrow{u} Y$ be a left weight in $\mathbb{E}$. Suppose that there exists a left dogma Φ on $\mathbb{E}$ such that:

- $u \in \Phi$;
- $\mathbb{E}$ has a Φ-ind-cocompletion of X, i.e., there exists a Φ-ind-morphism $X \xrightarrow{k} \Phi^{\triangledown}X$.

Then, the following are equivalent for u:

(i) u is left-absolute.
(ii) u has a left adjoint.

Proof. [(ii) $\Longrightarrow$ (i)] This follows from Corollary 5.4. detail

[(i) $\Longrightarrow$ (ii)] Let $\hat{u}$ be the left Kan extension of k along u and let $v := \Phi^{\triangledown}X(\hat{u}, k)$. Since k is fully faithful, we have a unique cell ε' satisfying the following equation:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, "k"'] & Y \arrow[r, "v"] \arrow[d, "\hat{u}"] \arrow[dl, phantom, "\alpha\colon \mathsf{lan}"] \arrow[dr, phantom, "\beta\colon \mathsf{cart}"] & X \arrow[dl, "k"] \\
& \Phi^{\triangledown}X &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, phantom, "\varepsilon'"] & X \arrow[d, equal] \\
X \arrow[rr, equal] \arrow[dr, "k"'] & \mathsf{cart} & X \arrow[dl, "k"] \\
& \Phi^{\triangledown}X &
\end{tikzcd}
$$

By the Ext-cart lemma (Lemma 5.15), the cell ε' is extending, hence so is the following composite ε:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, equal] & Y \arrow[r, "v"] \arrow[d, phantom, "\varepsilon"] & X \arrow[dl, equal] \\
& X &
\end{tikzcd}
:=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, phantom, "\varepsilon'\colon \mathsf{ext}"] & X \arrow[d, equal] \\
X \arrow[rr, equal] \arrow[dr, equal] & \mathsf{cart} & X \arrow[dl, equal] \\
& X &
\end{tikzcd}
$$

We have to show that the extending ε is preserved by the horizontal arrow u. That is, we have to show that the following cell γ becomes extending:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, equal] & Y \arrow[r, "v"] \arrow[d, phantom, "\varepsilon\colon \mathsf{ext}"] & X \arrow[r, "u"] \arrow[dl, equal] \arrow[d, phantom, "="] & Y \arrow[d, equal] \\
& X \arrow[rr, "u"'] & & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "="] & Y \arrow[r, "v"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, phantom, "\mathsf{comp}"] & Y \arrow[d, equal] \\
X \arrow[r, "u"'] \arrow[d, equal] & Y \arrow[rr, "v\odot u"'] & \gamma & Y \arrow[d, equal] \\
X \arrow[rrr, "u"'] & & & Y
\end{tikzcd}
$$

By Lemma 5.14, the following cell (39) is a right composing:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "k"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, equal] \arrow[dl, "\hat{u}"'] \arrow[dl, phantom, "\mathsf{s.opc}"] \\
\Phi^{\triangledown}X \arrow[r, "\hat{u}^*"'] & Y
\end{tikzcd}
\tag{39}
$$

Since u is a left weight, the horizontal composite $k^* \odot u$ exists and it is vertically isomorphic to $\hat{u}^*$, hence the cell (39) becomes strongly opcartesian. Then, using the Sandwich lemma, we can show that the following cell (40) is restricting:

$$
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[dr, "\hat{u}"'] & X \arrow[r, "u"] \arrow[d, "k"] \arrow[dl, phantom, "\beta" near start] \arrow[dr, phantom, "\alpha" near start] & Y \arrow[dl, "\hat{u}"] \\
& \Phi^{\triangledown}X &
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[dr, "\hat{u}"'] & X \arrow[r, "u"] \arrow[d, "k"] \arrow[dl, phantom, "\beta\colon \mathsf{cart}" near start] \arrow[dr, phantom, "\alpha" near start] & Y \arrow[dl, "\hat{u}"] \arrow[d, equal] \\
& \Phi^{\triangledown}X \arrow[r, "\hat{u}^{*}"'] \arrow[dr, equal] \arrow[r, phantom, shift left=4, "\cdot"] & Y \arrow[d, "\hat{u}"] \\
& & \Phi^{\triangledown}X \arrow[ul, phantom, "\mathsf{cart}" near start]
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[dr, "\hat{u}"'] & X \arrow[r, "u"] \arrow[d, phantom, "\mathsf{res}"] & Y \arrow[d, equal] \\
& \Phi^{\triangledown}X \arrow[r, "\hat{u}^{*}"'] \arrow[dr, equal] & Y \arrow[d, "\hat{u}"] \\
& & \Phi^{\triangledown}X \arrow[ul, phantom, "\mathsf{cart}" near start]
\end{tikzcd}
: \mathsf{res} \tag{40}
$$

By the definition of restricting cells, we obtain a cartesian cell δ satisfying the following equation:

$$
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[dr, "\hat{u}"'] & X \arrow[r, "u"] \arrow[d, "k"] \arrow[dl, phantom, "\beta" near start] \arrow[dr, phantom, "\alpha" near start] & Y \arrow[dl, "\hat{u}"] \\
& \Phi^{\triangledown}X &
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, phantom, "\mathsf{comp}"] & Y \arrow[d, equal] \\
Y \arrow[rr, "v\odot u"] \arrow[dr, "\hat{u}"'] & \arrow[d, phantom, "\delta\colon \mathsf{cart}" near start] & Y \arrow[dl, "\hat{u}"] \\
& \Phi^{\triangledown}X &
\end{tikzcd}
$$

Then, the following calculation shows that the cells δ and γ corresponds to each other in the way of the Ext-cart lemma (Lemma 5.15), which concludes that γ is extending:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] \arrow[dl, phantom, "="] & X \arrow[r, "u"] \arrow[d, phantom, "\mathsf{comp}"] & Y \arrow[d, equal] \\
X \arrow[r, "u"] \arrow[dr, "k"'] & Y \arrow[rr, "v\odot u"] \arrow[d, "\hat{u}"] \arrow[l, phantom, shift right=4, "\alpha"] & \arrow[d, phantom, "\delta\colon \mathsf{cart}" near start] & Y \arrow[dll, "\hat{u}"] \\
& \Phi^{\triangledown}X & &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[drr, "k"'] & Y \arrow[r, "v"] \arrow[dr, "\hat{u}"] \arrow[d, phantom, "\alpha"] & X \arrow[r, "u"] \arrow[d, "k"] \arrow[d, phantom, shift right=4, "\beta"] \arrow[dr, phantom, "\alpha" near start] & Y \arrow[dl, "\hat{u}"] \\
& \; & \Phi^{\triangledown}X &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, phantom, "\varepsilon'"] & X \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "="] & Y \arrow[d, equal] \\
X \arrow[rr, equal] \arrow[dr, "k"'] & \arrow[d, phantom, "\mathsf{cart}" near start] & X \arrow[r, "u"] \arrow[dl, "k"] \arrow[dr, phantom, "\alpha" near start] & Y \arrow[dll, "\hat{u}"] \\
& \Phi^{\triangledown}X & &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[dl, phantom, "\varepsilon"] & X \arrow[r, "u"] \arrow[dll, equal] \arrow[d, phantom, "="] & Y \arrow[d, equal] \\
X \arrow[rrr, "u"] \arrow[d, "k"'] & \arrow[d, phantom, "\alpha" near start] & & Y \arrow[dlll, "\hat{u}"] \\
\Phi^{\triangledown}X & & &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] \arrow[dl, phantom, "="] & X \arrow[r, "u"] \arrow[d, phantom, "\mathsf{comp}"] & Y \arrow[d, equal] \\
X \arrow[r, "u"'] \arrow[d, equal] & Y \arrow[rr, "v\odot u"'] \arrow[d, phantom, "\gamma"] & & Y \arrow[d, equal] \\
X \arrow[rrr, "u"] \arrow[d, "k"'] & \arrow[d, phantom, "\alpha" near start] & & Y \arrow[dlll, "\hat{u}"] \\
\Phi^{\triangledown}X & & &
\end{tikzcd}
$$

$\square$

6. Local presentability

6.1. Relative completeness.

Definition 6.1. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let Ψ be a class of right weights in $\mathbb{E}$. An object $A \in \mathbb{E}$ is called $\Phi^{\triangledown}$-*relatively* Ψ-*complete* if, for any $A \xrightarrow{\varphi} B$ in Φ and $C \xrightarrow{\psi} B$ in Ψ, the lift $A \xrightarrow{\varphi \blacktriangleleft \psi} C$ belongs to Φ.

$$\begin{tikzcd} A \arrow[r, "\varphi\blacktriangleleft\psi", "|" marking] \arrow[d, equal] & C \arrow[r, "\psi", "|" marking] \arrow[d, phantom, "\text{lift}"] & B \arrow[d, equal] \\ A \arrow[rr, "\varphi"', "|" marking] & & B \end{tikzcd} \quad \text{in } \mathbb{E}$$

When $\Phi = \mathsf{LW}_{\mathbb{E}}$, the class of all left weights, we use '*virtually*' instead of '$\mathsf{LW}_{\mathbb{E}}{}^{\triangledown}$-relatively.' When $\Psi = \mathsf{RW}_{\mathbb{E}}$, the class of all right weights, we use '*complete*' instead of '$\mathsf{RW}_{\mathbb{E}}$-complete.' By the similar way, we define $\Psi'^{\blacktriangledown}$-*relatively* Φ'-*cocompleteness* for a right dogma Ψ' and a class Φ' of left weights. ♦

Theorem 6.2. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let Ψ be a class of right weights in $\mathbb{E}$. For an object $A \in \mathbb{E}$ such that $\Phi^{\triangledown}A$ exists, the following are equivalent:

(i) A is $\Phi^{\triangledown}$-relatively Ψ-complete.
(ii) $\Phi^{\triangledown}A$ is Ψ-complete.

Theorem 6.3 (Completeness of "locally presentable" objects)**.** Let Φ be a sound class of left weights in a virtual equipment $\mathbb{E}$. Then, Φ-cocomplete and virtually complete objects are also $\Phi_{/\!/}^{\triangledown}$-relatively complete.

7. Duality

Definition 7.1. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. A vertical arrow $A \xrightarrow{f} B$ in $\mathbb{E}$ is called $\Phi^{\triangledown}$-*weighty* if $f_* \in \Phi$. ♦

Definition 7.2. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. An object $X \in \mathbb{E}$ is called $\Phi^{\triangledown}$-*accessible* if it is vertically equivalent to some Cauchy complete object in $\mathbb{E}$. ♦

Notation 7.3. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let $\mathbb{C}\mathrm{au}\Phi^{\triangledown}$ denote the AVDC defined as follows:

- An object in $\mathbb{C}\mathrm{au}\Phi^{\triangledown}$ is a Cauchy complete object $A \in \mathbb{E}$ such that its Φ-ind-completion $\Phi^{\triangledown}A$ exists;
- A vertical arrow in $\mathbb{C}\mathrm{au}\Phi^{\triangledown}$ is a $\Phi^{\triangledown}$-weighty vertical arrow in $\mathbb{E}$;
- A horizontal arrow in $\mathbb{C}\mathrm{au}\Phi^{\triangledown}$ is that in Φ;
- $\mathbb{C}\mathrm{au}\Phi^{\triangledown}$ contains all suitable cells in $\mathbb{E}$.

Note that $\mathbb{C}\mathrm{au}\Phi^{\triangledown}$ is a sub-AVDC of $\mathbb{E}_{\Phi}{}'$ and that $\mathbb{C}\mathrm{au}\Phi^{\triangledown}$ becomes a pseudo-double category.

$$\mathbb{C}\mathrm{au}\Phi^{\triangledown} \subseteq \mathbb{E}_{\Phi}{}' \subseteq \mathbb{E}_{\Phi}.$$ ♦

Notation 7.4. Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Let $\mathbb{A}\mathrm{cc}\Phi^{\triangledown}$ denote the (strict) double category defined as follows:

- An object in $\mathbb{A}\mathrm{cc}\Phi^{\triangledown}$ is a $\Phi^{\triangledown}$-accessible object in $\mathbb{E}$;
- A vertical arrow in $\mathbb{A}\mathrm{cc}\Phi^{\triangledown}$ is a Φ-cocontinuous vertical arrow in $\mathbb{E}$ that is a right adjoint in the vertical 2-category $\mathscr{V}\,\mathbb{E}$;
- A horizontal arrow $X \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; Y$ in $\mathbb{A}\mathrm{cc}\Phi^{\triangledown}$ is a Φ-cocontinuous vertical arrow $Y \longrightarrow X$ in $\mathbb{E}$;

- A cell
$$\begin{tikzcd} X \arrow[r, "u", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g"] \\ Z \arrow[r, "v"', "|"{marking}] & W \end{tikzcd} \quad \text{in } \mathbb{A}\mathrm{cc}\Phi^{\triangledown}$$
is a $(0,0)$-cell
$$\begin{tikzcd} & Y \arrow[dl, "u"'] \arrow[dr, "g"] & \\ X \arrow[dr, "f"'] \arrow[rr, phantom, "\alpha"] & & W \arrow[dl, "v"] \\ & Z & \end{tikzcd} \quad \text{in } \mathbb{E}.$$

Note that $\mathbb{A}\mathrm{cc}\Phi^{\triangledown}$ is a sub-double category of $\mathbb{Q}\Phi^{\triangledown}$. ♦

Definition 7.5. A week equivalence between AVDCs is a weak functor that is surj on obj up to vert.equiv. + local equivalence + fully faithful. ♦

Theorem 7.6 (The duality theorem)**.** Let Φ be a left dogma on a virtual equipment $\mathbb{E}$. Then, there is a weak equivalence $\mathbb{C}\mathrm{au}\Phi^{\triangledown} \xrightarrow{\simeq} \mathbb{A}\mathrm{cc}\Phi^{\triangledown}$. Note that this weak functor is induced by $\mathbb{E}_{\Phi}{}' \xrightarrow{\Phi^{\triangledown}} \mathbb{Q}\Phi^{\triangledown}$, hence the following commutes:
$$\begin{tikzcd} \mathbb{C}\mathrm{au}\Phi^{\triangledown} \arrow[r, "\simeq"] \arrow[d, hook] & \mathbb{A}\mathrm{cc}\Phi^{\triangledown} \arrow[d, hook] \\ \mathbb{E}_{\Phi}{}' \arrow[r, "\Phi^{\triangledown}"'] & \mathbb{Q}\Phi^{\triangledown} \end{tikzcd}$$

8. Examples

8.1. **Sliced double categories.**

Notation 8.1. Let $\mathbb{D}$ be a virtual double category and let $X \in \mathbb{D}$. Let $\mathbb{D}/X$ denotes ongoing ♦

Theorem 8.2. Let $\mathbb{E}$ be a virtual equipment and let $X \in \mathbb{E}$. restriction

Theorem 8.3 (Nasu-construction)**.** Let $\mathbb{E}$ be a virtual equipment and let $X \in \mathbb{E}$. Fix $(A,a) \xrightarrow{(u,\alpha)} (B,b)$ and $(A,a) \xrightarrow{(v,\beta)} (C,c)$ in $\mathbb{E}/X$. If the extension $B \xrightarrow{u \triangleright v} C$ exists in $\mathbb{E}$ and the category $\mathbf{Hom}_{\mathbb{E}}(B,C)$ has pullbacks, then the extension $(u,\alpha)\triangleright(v,\beta)$ exists in $\mathbb{E}/X$.
$$\begin{tikzcd} (A,a) \arrow[r, "(u,\alpha)", "|"{marking}] \arrow[d, equal] & (B,b) \arrow[r, "(u,\alpha)\triangleright(v,\beta)", "|"{marking}] \arrow[d, phantom, "\mathsf{ext}"] & (C,c) \arrow[d, equal] \\ (A,a) \arrow[rr, "(v,\beta)"', "|"{marking}] & & (C,c) \end{tikzcd} \quad \text{in } \mathbb{E}/X$$

Proof. proof □

8.2. **Bicategory-enriched categories.**

Notation 8.4. Let $\mathscr{W}$ be a bicategory. We denote by $\mathscr{W}$-$\mathbb{P}\mathrm{rof}$ the virtual equipment of $\mathscr{W}$-enriched categories, $\mathscr{W}$-functors, and $\mathscr{W}$-profunctors. ♦

Lemma 8.5. Let $\mathscr{W}$ be a bicategory. Consider the following data:

$$\begin{tikzcd} \mathbf{A}_0 \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "F"'] & \mathbf{A}_1 & \mathbf{C} \arrow[d, equal] \\ \mathbf{B} \arrow[rr, "Q"', "\shortmid"{marking}] & & \mathbf{C} \end{tikzcd} \tag{41}$$

Then, the following are sufficient to construct an extension $\mathbf{A}_1 \xrightarrow{R} \mathbf{C}$ of (41).

- For any $a_0 \in \mathbf{A}_0$, $a_1 \in \mathbf{A}_1$, and $c \in \mathbf{C}$, the right Kan extension of

$$\begin{tikzcd} & |a_1| \\ |a_0| \arrow[ur, "P(a_0,a_1)"] \arrow[rr, "Q(Fa_0,c)"'] & & |c| \end{tikzcd} \quad \text{in } \mathscr{W},$$

  denoted by $|a_1| \xrightarrow{P(a_0,a_1) \triangleright Q(Fa_0,c)} |c|$, exists.
- For any $a_0, a_0' \in \mathbf{A}_0$, $a_1 \in \mathbf{A}_1$, and $c \in \mathbf{C}$, the right Kan extension of

$$\begin{tikzcd} & & |a_1| \\ & |a_0'| \arrow[ur, "P(a_0',a_1)"] \\ |a_0| \arrow[ur, "\mathbf{A}_0(a_0,a_0')"] \arrow[rrr, "Q(Fa_0,c)"'] & & & |c| \end{tikzcd} \quad \text{in } \mathscr{W},$$

  denoted by $|a_1| \xrightarrow{(\mathbf{A}_0(a_0,a_0') \odot P(a_0',a_1)) \triangleright Q(Fa_0,c)} |c|$, exists.
- For any $a_1 \in \mathbf{A}_1$ and $c \in \mathbf{C}$, the end $\int_{a_0} P(a_0,a_1) \triangleright Q(Fa_0,c)$ exists in the category $\mathbf{Hom}_{\mathscr{W}}(|a_1|,|c|)$.

Notation 8.6. Let $\mathscr{W}$ be a bicategory and let $c \in \mathscr{W}$. Let $\mathbf{1}_c$ denote the single-object $\mathscr{W}$-enriched category whose unique object is indexed by c and whose unique hom-1-cell $c \xrightarrow{\mathbf{1}_c(*,*)} c$ in $\mathscr{W}$ is the identity. ♦

Definition 8.7 ([FL22])**.** Let $\mathscr{W}$ be a bicategory and let $\mathbf{X}$ be a $\mathscr{W}$-enriched category. Let $\mathscr{W}/\!\!/\mathbf{X}$ denote the bicategory defined by the following:

- An object in $\mathscr{W}/\!\!/\mathbf{X}$ is simply an object $x \in \mathbf{X}$;
- A 1-cell $x \longrightarrow x'$ in $\mathscr{W}/\!\!/\mathbf{X}$ is a pair (S,s) of the following form:

$$\begin{tikzcd} \mathbf{1}_{|x|} \arrow[rr, "S", "\shortmid"{marking}] \arrow[dr, "\ulcorner x \urcorner"'] \arrow[rr, phantom, shift right=4ex, "s"] & & \mathbf{1}_{|x'|} \arrow[dl, "\ulcorner x' \urcorner"] \\ & \mathbf{X} \end{tikzcd} \quad \text{in } \mathscr{W}\text{-}\mathbb{P}\mathrm{rof};$$

- A 2-cell $\begin{tikzcd} x \arrow[r, bend left, "{(S,s)}"] \arrow[r, bend right, "{(T,t)}"'] \arrow[r, phantom, "\Downarrow"] & x' \end{tikzcd}$ in $\mathscr{W}/\!\!/\mathbf{X}$ is a cell α as follows:

$$\begin{tikzcd} \mathbf{1}_{|x|} \arrow[r, "S", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & \mathbf{1}_{|x'|} \arrow[d, equal] \\ \mathbf{1}_{|x|} \arrow[r, "T", "\shortmid"{marking}] \arrow[dr, "\ulcorner x \urcorner"'] \arrow[r, phantom, shift right=3ex, "t"] & \mathbf{1}_{|x'|} \arrow[d, "\ulcorner x' \urcorner"] \\ & \mathbf{X} \end{tikzcd} = \begin{tikzcd} \mathbf{1}_{|x|} \arrow[rr, "S", "\shortmid"{marking}] \arrow[dr, "\ulcorner x \urcorner"'] \arrow[rr, phantom, shift right=3ex, "s"] & & \mathbf{1}_{|x'|} \arrow[dl, "\ulcorner x' \urcorner"] \\ & \mathbf{X} \end{tikzcd} \quad \text{in } \mathscr{W}\text{-}\mathbb{P}\mathrm{rof}$$

identity, composition ♦

Remark 8.8. A $\mathscr{W}/\!\!/\mathbf{X}$-enriched category is simply a $\mathscr{W}$-functor to $\mathbf{X}$, denoted by $\mathbf{A} \xrightarrow{p_{\mathbf{A}}} \mathbf{X}$, and a $\mathscr{W}/\!\!/\mathbf{X}$-functor $\mathbf{A} \longrightarrow \mathbf{B}$ is simply a $\mathscr{W}$-functor that commutes with $p_{\mathbf{A}}$ and $p_{\mathbf{B}}$ [FL22]. Furthermore, a $\mathscr{W}/\!\!/\mathbf{X}$-profunctor $\mathbf{A} \nrightarrow \mathbf{B}$ is simply a pair (P, p_P) of the following form:

$$\begin{tikzcd} \mathbf{A} \arrow[rr, "P", "\shortmid"{marking}] \arrow[dr, "p_{\mathbf{A}}"'] \arrow[rr, phantom, shift right=3ex, "p_P"] & & \mathbf{B} \arrow[dl, "p_{\mathbf{B}}"] \\ & \mathbf{X} \end{tikzcd} \quad \text{in } \mathscr{W}\text{-}\mathbb{P}\mathrm{rof}.$$

cell... Consequently, we get an isomorphism $(\mathscr{W}/\!\!/\mathbf{X})\text{-}\mathbb{P}\mathrm{rof} \cong (\mathscr{W}\text{-}\mathbb{P}\mathrm{rof})/\mathbf{X}$. ♦

8.2.1. $\mathbb{P}\mathrm{rof}/\mathbf{X}$. We fix a locally small category $\mathbf{X}$ and consider the virtual equipment $\mathbb{P}\mathrm{rof}/\mathbf{X}$.

Example 8.9 (Extensions)**.** Let $(\mathbf{A}, p_{\mathbf{A}}) \xrightarrow{(P,p_P)} (\mathbf{B}, p_{\mathbf{B}})$ and $(\mathbf{A}, p_{\mathbf{A}}) \xrightarrow{(Q,p_Q)} (\mathbf{C}, p_{\mathbf{C}})$ be horizontal arrows in $\mathbb{P}\mathrm{rof}/\mathbf{X}$. The extension

$$\begin{tikzcd} (\mathbf{A}, p_{\mathbf{A}}) \arrow[r, "{(P,p_P)}", "\shortmid"{marking}] \arrow[d, equal] & (\mathbf{B}, p_{\mathbf{B}}) \arrow[r, "{(P \triangleright Q, p_{P \triangleright Q})}", "\shortmid"{marking}] \arrow[d, phantom, "\varepsilon\colon \mathsf{ext}"] & (\mathbf{C}, p_{\mathbf{C}}) \arrow[d, equal] \\ (\mathbf{A}, p_{\mathbf{A}}) \arrow[rr, "{(Q,p_Q)}"', "\shortmid"{marking}] & & (\mathbf{C}, p_{\mathbf{C}}) \end{tikzcd} \quad \text{in } \mathbb{P}\mathrm{rof}/\mathbf{X}$$

is described as follows:

$$(P \triangleright Q)(b,c) := \left\{ (f, \theta) \;\middle|\; \begin{array}{c} p(b) \xrightarrow{f} p(c) \quad \text{in } \mathbf{X}, \quad P(-,b) \xRightarrow{\theta} Q(-,c) \quad \text{in } \widehat{\mathbf{A}} \\ \text{s.t., for all } a \in \mathbf{A}, \quad \begin{tikzcd} P(a,b) \arrow[r, "\theta_a"] \arrow[d, "p"'] & Q(a,c) \arrow[d, "p"] \\ \mathrm{Hom}_{\mathbf{X}}(p(a), p(b)) \arrow[r, "-\,\mathbin{;}\,f"'] & \mathrm{Hom}_{\mathbf{X}}(p(a), p(c)) \end{tikzcd} \quad \text{commutes.} \end{array} \right\};$$

$$\begin{array}{rclcl} (P \triangleright Q)(b,c) & \xrightarrow{p_{P \triangleright Q}} & \mathrm{Hom}_{\mathbf{X}}(p(b), p(c))\colon & (f, \theta) & \mapsto & f; \\ P(a,b) \times (P \triangleright Q)(b,c) & \xrightarrow{\varepsilon} & Q(a,c)\colon & (\alpha, (f,\theta)) & \mapsto & \theta_a(\alpha). \end{array}$$

♦

Example 8.10 (Weighted cocones)**.** Fix the following data:

$$\begin{tikzcd}
(\mathbf{A},p_{\mathbf{A}}) \arrow[r, "{(P,p_P)}", "\shortmid"{marking}] \arrow[d, "F"'] & (\mathbf{1},x) \\
(\mathbf{C},p_{\mathbf{C}})
\end{tikzcd} \quad \text{in } (\mathbb{P}\mathrm{rof})/\mathbf{X}, \tag{42}$$

where x represents an object $x \in \mathbf{X}$. A *P-weighted cocone* from F to $c \in \mathbf{C}$ consists of:

- an arrow $x \xrightarrow{f} p(c)$ in $\mathbf{X}$;
- for each $a \in \mathbf{A}$ and $i \in P(a)$, an arrow $Fa \xrightarrow{\theta_{a,i}} c$ in $\mathbf{C}$

such that:

- for every $a' \xrightarrow{t} a$ in $\mathbf{A}$, the following commutes:

$$\begin{tikzcd}
Fa' \arrow[rr, "Ft"] \arrow[dr, "\theta_{a',t^*i}"'] & & Fa \arrow[dl, "\theta_{a,i}"] \\
& c &
\end{tikzcd} \quad \text{in } \mathbf{C};$$

- for every $a \in \mathbf{A}$ and $i \in P(a)$, the following commutes:

$$\begin{tikzcd}
p(Fa)=p(a) \arrow[r, "p_P(i)"] \arrow[dr, "p(\theta_{a,i})"'] & x \arrow[d, "f"] \\
& p(c)
\end{tikzcd} \quad \text{in } \mathbf{X}.$$

A P-weighted cocone from F to c bijectively corresponds to an element of the set $P \rhd F_*(c)$, equivalently, the following data:

$$\begin{tikzcd}
(\mathbf{A},p_{\mathbf{A}}) \arrow[r, "{(P,p)}", "\shortmid"{marking}] \arrow[dr, "F"'] & (\mathbf{1},x) \arrow[r, "{(\Delta 1,f)}", "\shortmid"{marking}] \arrow[d, phantom, "\theta"] & (\mathbf{1},p(c)) \arrow[dl, "c"] \\
& (\mathbf{C},p_{\mathbf{C}}) &
\end{tikzcd} \quad \text{in } (\mathbb{P}\mathrm{rof})/\mathbf{X}.$$

♦

Example 8.11 (Weighted colimits)**.** Fix the same data as (42). A *P-weighted colimit* of F consists of:

- an object $l \in \mathbf{C}$ such that $p(l) = x$;
- for each $a \in \mathbf{A}$ and $i \in P(a)$, an arrow $Fa \xrightarrow{\lambda_{a,i}} l$ in $\mathbf{C}$

such that:

- the identity $x \xrightarrow{\mathsf{id}} p(l)$ and arrows $\lambda_{a,i}$ forms a P-weighted cocone;
- for every $c \in \mathbf{C}$ and every P-weighted cocone $(f, \theta_{a,i})$ from F to c, there exists a unique arrow $l \xrightarrow{k} c$ in $\mathbf{C}$ such that $p(k) = f$ and $\lambda_{a,i} \mathbin{⨾} k = \theta_{a,i}$.

A P-weighted colimit of F corresponds to the lan-cell:

$$\begin{tikzcd}
(\mathbf{A},p_{\mathbf{A}}) \arrow[r, "{(P,p_P)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\lambda\colon \mathsf{lan}"] & (\mathbf{1},x) \arrow[dl, "l"] \\
(\mathbf{C},p_{\mathbf{C}}) &
\end{tikzcd} \quad \text{in } (\mathbb{P}\mathrm{rof})/\mathbf{X}.$$

♦

8.3. **Fibred categories.**

Notation 8.12. Let $\mathbf{B}$ be a (locally small) category.

- For each functor $p_{\mathbf{E}}\colon \mathbf{E} \to \mathbf{B}$ over a category $\mathbf{B}$, we write $\mathbf{B} \downarrow p_{\mathbf{E}}$ for the oplax limit of $p_{\mathbf{E}}$. Or equivalently, the tabulator of the profunctor $(p_{\mathbf{E}})^*\colon \mathbf{B} \longrightarrow \mathbf{E}$. An object of $\mathbf{B} \downarrow p_{\mathbf{E}}$ is a triple (b, u, e) such that $b \in \mathbf{B}$, $e \in \mathbf{E}$, and $u\colon b \longrightarrow p(e)$. $\mathbf{B} \downarrow p_{\mathbf{E}}$ is equipped with the canonical functor $(b, u, e) \mapsto b\colon \mathbf{B} \downarrow p_{\mathbf{E}}$, and seen as a object of the slice 2-category $\mathscr{CAT}/\mathbf{B}$.
- A *fibration* over $\mathbf{B}$ is a functor $p_{\mathbf{E}}\colon \mathbf{E} \longrightarrow \mathbf{B}$ over $\mathbf{B}$ equipped with the right adjoint $(b, u, e) \mapsto u^*(e)\colon \mathbf{B} \downarrow p_{\mathbf{E}} \longrightarrow \mathbf{E}$ in $\mathscr{CAT}/\mathbf{B}$ of the canonical functor $e \mapsto (e, \mathsf{id}_{p_{\mathbf{E}}(e)}, p(e))\colon \mathbf{E} \longrightarrow \mathbf{B} \downarrow p_{\mathbf{E}}$.
- A functor over $\mathbf{B}$ between fibrations are called *fibred* if it preserves the cleavage up to isomorphism.iwjdwoj Fibrations, fibred functors, and 2-cells in $\mathscr{CAT}/\mathbf{B}$ form $\mathscr{Fib}\,\mathbf{B}$.
- The Grothendieck construction gives an 2-equivalence $\int\colon \mathscr{Ps}\,[\mathbf{B}^{\mathrm{op}}, \mathscr{CAT}] \xrightarrow{\sim} \mathscr{Fib}\,\mathbf{B}$, where the domain is the 2-category of pseudo-functors, pseudo-natural transformations, and modifications. Objects in this 2-category are called *indexed categories* over $\mathbf{B}$.
- A fibration is called *split* if it is sent to a 2-functor by the pseudo inverse of the Grothendieck construction. Split fibrations and fibred functors preserving the cleavage up-to identities, form a sub-2-category $s\mathscr{Fib}\,\mathbf{B}$ of $\mathscr{Fib}\,\mathbf{B}$. The Grothendieck construction restricts to another 2-equivalence $\int\colon [\mathbf{B}^{\mathrm{op}}, \mathscr{CAT}] \xrightarrow{\sim} s\mathscr{Fib}\,\mathbf{B}$, where the domain is the 2-category of 2-functors, 2-natural transformations, and modifications. ♦

Definition 8.13. Suppose we are given a horizontal arrow $u\colon \mathbf{C} \longrightarrow \mathbf{D}$ in $\mathbf{Set}$-$\mathbb{P}$rof. We define a locally small category $\mathbf{Gl}(u)$ as follows.

- The set of objects is the disjoint union $\operatorname{Obj}\mathbf{C} \sqcup \operatorname{Obj}\mathbf{D}$.
- The homsets are defined as follows.

$$\operatorname{Hom}_{\mathbf{Gl}(u)}(e, e') = \begin{cases} \operatorname{Hom}_{\mathbf{C}}(e, e') & e \in \mathbf{C}, e' \in \mathbf{C} \\ u(e, e') & e \in \mathbf{C}, e' \in \mathbf{D} \\ \operatorname{Hom}_{\mathbf{D}}(e, e') & e \in \mathbf{D}, e' \in \mathbf{D} \\ \emptyset & \text{otherwise} \end{cases}$$

- Compositions and identities are those for $\mathbf{C}$ and $\mathbf{D}$, and left and right actions for u of $\mathbf{C}$ and $\mathbf{D}$.

There are two fully faithful functors $\kappa\colon \mathbf{C} \hookrightarrow \mathbf{Gl}(u)$ and $\kappa'\colon \mathbf{D} \hookrightarrow \mathbf{Gl}(u)$ and the restriction $\mathbf{Gl}(u)(\kappa, \kappa')\colon \mathbf{C} \longrightarrow \mathbf{D}$ coincides with p. ♦

Proposition 8.14. Suppose we are given a horizontal arrow $p\colon \mathbf{C} \longrightarrow \mathbf{D}$ in $\mathbf{Set}$-$\mathbb{P}$rof. Then the canonical cartesian cell

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "u"] \arrow[dr, "\kappa"'] & \arrow[d, phantom, "\mathsf{cart}"] & \mathbf{D} \arrow[dl, "\kappa'"] \\ & \mathbf{Gl}(u) & \end{tikzcd}$$

defines a (strong) cotabulator of u.

Definition 8.15. Suppose that we are given a category $\mathbf{B}$ and two fibrations $\mathbf{E}$ and $\mathbf{D}$ over $\mathbf{B}$. A *fibred profunctor* $(u, \alpha)\colon \mathbf{E} \longrightarrow \mathbf{D}$ is a profunctor $u\colon \mathbf{E} \to \mathbf{D}$ on the total category equipped with a cell

$$\begin{tikzcd} \mathbf{E} \arrow[rr, "u"] \arrow[dr, "p_{\mathbf{E}}"'] & \arrow[d, phantom, "\alpha"] & \mathbf{D} \arrow[dl, "p_{\mathbf{D}}"] \\ & \mathbf{B} & \end{tikzcd}$$

such that the induced functor $\bar{\alpha}\colon \mathbf{Gl}(p) \longrightarrow \mathbf{Gl}(\mathsf{Id}_{\mathbf{B}})$ through the universality of the cotabulator, is a fibration over $\mathbf{Gl}(\mathsf{Id}_{\mathbf{B}})$ and the coprojections give rise to fibred functors from $\mathbf{E}$ and $\mathbf{D}$. ♦

Definition 8.16. Let $\mathbf{C}$ be a category. Define a virtual double category $\mathbf{Set}\text{-}\mathbb{M}\mathrm{at}^{\mathbf{C}^{\mathrm{op}}}$ as follows.

- The vertical category is the category of presheaves $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$.
- A horizontal arrow $p\colon F \longrightarrow\!\!\!\!\!\!\mapsto G$ is a section of the Grothendieck construction of $C \mapsto \mathbf{Hom}_{\mathbf{Set}\text{-}\mathbb{M}\mathrm{at}}(F(C), G(C)) : \mathbf{C} \longrightarrow \mathscr{C}AT$.
  In other words, p consists of the following data:
  - A family of $\mathbf{Set}$-matrices $\left(p_C\colon FC \longrightarrow\!\!\!\!\!\!\mapsto GC\right)_{C\in\mathbf{C}}$ indexed by objects of $\mathbf{C}$.
  - A family of cells $(p_f)_{f\colon C'\to C}$ in $\mathbf{Set}\text{-}\mathbb{M}\mathrm{at}$ indexed by arrows in $\mathbf{C}$ satisfying the followings.
    * For each $f\colon C' \longrightarrow C$ in $\mathbf{C}$, p_f is of the following form.

$$\begin{tikzcd} FC \arrow[r, "p_C", "|"] \arrow[d, "Ff"'] \arrow[dr, phantom, "p_f"] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "p_{C'}"', "|"] & GC' \end{tikzcd}$$

    * p_{id_C} is the identity cell on p_C, and for each composable pair (f, g) of arrows in $\mathbf{C}$, we have the following equality of cells.

$$\begin{tikzcd} \cdot \arrow[r, "p_\cdot", "|"] \arrow[d, "Ff"'] \arrow[dr, phantom, "p_f"] & \cdot \arrow[d, "Gf"] \\ \cdot \arrow[r, "p_\cdot", "|"] \arrow[d, "Fg"'] \arrow[dr, phantom, "p_g"] & \cdot \arrow[d, "Gg"] \\ \cdot \arrow[r, "p_\cdot"', "|"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "p_\cdot", "|"] \arrow[d, "Ff"'] & \cdot \arrow[d, "Gf"] \\ \cdot \arrow[d, "Fg"'] \arrow[r, phantom, "p_{f\fatsemi g}"] & \cdot \arrow[d, "Gg"] \\ \cdot \arrow[r, "p_\cdot"', "|"] & \cdot \end{tikzcd}$$

-

♦

9. Hoshino's writings

Proposition 9.1. Let $\mathbb{E}$ be a virtual equipment. Suppose that there exists a lifting cell and an extension cell of the following forms. Then cells of the form α and those of the form β bijectively corresponds each other, through the following equation.

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{a}", "|"] \arrow[d, "f"'] \arrow[drr, phantom, "\alpha"] & A' \arrow[r, "u", "|"] & B' \arrow[r, dashed, "\vec{b}", "|"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\ C \arrow[rr, "|"] \arrow[d, equal] \arrow[drrr, phantom, "\mathsf{lift}"] & & B' \arrow[r, dashed, "\vec{b}", "|"] & B \arrow[d, "g"] \\ C \arrow[rrr, dotted, "\vec{q}"', "|"] & & & E \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{a}", "|"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u", "|"] \arrow[d, equal] \arrow[drr, phantom, "\beta"] & B' \arrow[r, dashed, "\vec{b}", "|"] & B \arrow[d, "g"] \\ A \arrow[r, dashed, "\vec{a}", "|"] \arrow[d, "f"'] \arrow[drrr, phantom, "\mathsf{ext}"] & A' \arrow[rr, "|"] & & E \arrow[d, equal] \\ C \arrow[rrr, dotted, "\vec{q}"', "|"] & & & E \end{tikzcd}$$

Moreover, α is an extension if and only if β is a lifting.

Proof. ongoing □

Definition 9.2. Let $\mathbb{E}$ be a virtual equipment. Suppose that we are given an extension cell α, a lifting cell β, and horizontal arrows x, y of the following forms.

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] & A' \arrow[r, "u"] \arrow[d, phantom, "\alpha\colon \mathsf{ext}"] & D \arrow[d, equal] \\
C \arrow[rr, dotted, "\vec{p}"'] & & D \arrow[r, "x"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D \arrow[r, "v"] \arrow[d, equal] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, phantom, "\beta\colon \mathsf{lift}"] & B \arrow[d, "g"] \\
Y \arrow[r, "y"] & D \arrow[rr, dotted, "\vec{q}"'] & & E
\end{tikzcd}
$$

We say x *preserves the extension* α if there exists two opcartesian cells and an extension cell satisfying the following equality.

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] & A' \arrow[r, "u"] \arrow[d, phantom, "\alpha\colon \mathsf{ext}"] & D \arrow[r, "x"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[d, equal] \\
C \arrow[rr, dotted, "\vec{p}"'] \arrow[d, equal] & \arrow[d, phantom, "\mathsf{s.opc}"] & D \arrow[r, "x"] & X \arrow[d, equal] \\
C \arrow[rrr] & & & X
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] & D \arrow[r, "x"] \arrow[d, phantom, "\mathsf{s.opc}"] & X \arrow[d, equal] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] & A' \arrow[rr] & \arrow[d, phantom, "\mathsf{ext}"] & X \arrow[d, equal] \\
C \arrow[rrr] & & & E
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

Dually, y *preserves the lifting* β if:

$$
\begin{tikzcd}
Y \arrow[r, "y"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "v"] \arrow[d, equal] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, phantom, "\beta\colon \mathsf{lift}"] & B \arrow[d, "g"] \\
Y \arrow[r, "y"'] \arrow[d, equal] & D \arrow[rr, dotted, "\vec{q}"'] \arrow[d, phantom, "\mathsf{s.opc}"] & & E \arrow[d, equal] \\
Y \arrow[rrr] & & & E
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "y"] \arrow[d, equal] & D \arrow[r, "v"] \arrow[d, phantom, "\mathsf{s.opc}"] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
Y \arrow[rr] \arrow[d, equal] & \arrow[d, phantom, "\mathsf{lift}"] & B' \arrow[r, dashed, "\vec{b}"] & E \arrow[d, "g"] \\
Y \arrow[rrr] & & & E
\end{tikzcd}
\quad \text{in } \mathbb{E}.
$$

♦

Proposition 9.3 (Ext–lift lemma)**.** Let $\mathbb{E}$ be a virtual equipment. Given an extension α and a lifting β of the following forms, suppose that p preserves the lifing β and q preserves the extension α.

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] & A' \arrow[r, "u"] \arrow[d, phantom, "\alpha\colon \mathsf{ext}"] & D \arrow[d, equal] \\
C \arrow[rr, "p"'] & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, "v"] \arrow[d, equal] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, phantom, "\beta\colon \mathsf{lift}"] & B \arrow[d, "g"] \\
D \arrow[rr, "q"'] & & E
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

Then, the following are equivalent:

(i) u preserves the lifting β.
(ii) v preserves the extension α.

Proof. ongoing We show $i) \to ii)$, and the converse is shown in a similar way. Since p preserves β, in particular there exists the composite $p \odot v$. Therefore we can consider the following composites of cells.

$$
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] & A' \arrow[r, "u"] \arrow[d, phantom, "\alpha\colon \mathsf{ext}"] & D \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "="] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[d, phantom, shift left=4ex, "="] & B \arrow[dd, equal] \\
C \arrow[rr, "p"'] \arrow[d, equal] & \arrow[d, phantom, "\mathsf{s.opc}"] & D \arrow[r, "v"] & B' \arrow[d, equal] & \\
C \arrow[rrr] \arrow[d, equal] & & \arrow[d, phantom, "\mathsf{lift}"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
C \arrow[rrrr] & & & & E
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] & A' \arrow[r, "u"] \arrow[d, phantom, "\alpha\colon \mathsf{ext}"] & D \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "="] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
C \arrow[rr, "p"'] \arrow[d, equal] & \arrow[d, phantom, "="] & D \arrow[r, "v"] \arrow[d, equal] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, phantom, "\beta\colon \mathsf{lift}"] & B \arrow[d, "g"] \\
C \arrow[rr, "p"] \arrow[d, equal] & & D \arrow[rr, "q"] \arrow[d, phantom, "\mathsf{s.opc}"] & & E \arrow[d, equal] \\
C \arrow[rrrr] & & & & E
\end{tikzcd}
=
$$

$$
= \begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "v"] \arrow[d, equal] \arrow[drr, phantom, "\beta\colon \mathsf{lift}"] & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[drrr, phantom, "\mathsf{s.opc}"] & D \arrow[rr, "q"] & & E \arrow[d, equal] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drrrr, phantom, "\mathsf{ext}"] & A' \arrow[rrr] & & & E \arrow[d, equal] \\
C \arrow[rrrr] & & & & E
\end{tikzcd}
= \begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{s.opc}"] & D \arrow[r, "v"] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[rr] \arrow[d, equal] \arrow[drrr, phantom, "\mathsf{lift}"] & & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drrrr, phantom, "\mathsf{ext}"] & A' \arrow[rrr] & & & E \arrow[d, equal] \\
C \arrow[rrrr] & & & & E
\end{tikzcd}
$$

$$
= \begin{tikzcd}
A \arrow[r, dashed, "\vec{a}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A' \arrow[r, "u"] \arrow[d, equal] \arrow[drr, phantom, "\mathsf{s.opc}"] & D \arrow[r, "v"] & B' \arrow[r, dashed, "\vec{b}"] \arrow[d, equal] \arrow[ddr, phantom, "="] & B \arrow[dd, equal] \\
A \arrow[r, dashed, "\vec{a}"] \arrow[d, "f"'] \arrow[drrr, phantom, "\mathsf{ext}"] & A' \arrow[rr] & & B' \arrow[d, equal] & \\
C \arrow[rrr] \arrow[d, equal] \arrow[drrrr, phantom, "\mathsf{lift}"] & & & B' \arrow[r, dashed, "\vec{b}"] & B \arrow[d, "g"] \\
C \arrow[rrrr] & & & & E
\end{tikzcd}
$$

□

Lemma 9.4. Let $\mathbb{E}$ be a virtual double category and $\mathbb{D}$ be a sub-virtual double category of $\mathbb{E}$. Then the following are equivalent.

(i) For any $\vec{u}\colon X \dashrightarrow Y$ and $f\colon X \longrightarrow Z$ in $\mathbb{D}$, there exists an opcartesian cell

$$
\begin{tikzcd}
X \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{s.opc}"] & Y \arrow[d, equal] \\
Z \arrow[r, "v"'] & Y
\end{tikzcd}
\quad \text{in } \mathbb{E} \tag{43}
$$

that is also belongs to $\mathbb{D}$.

(ii) $\mathbb{D}$ is a double category with conjoints and the inclusion $\mathbb{D} \hookrightarrow \mathbb{E}$ preserves and creates composition: i.e., any opcartesian cell (43) exists in $\mathbb{E}$ and belongs to $\mathbb{D}$ whenever $\vec{u}$ belongs to $\mathbb{D}$ and $f = \mathsf{id}$.

Definition 9.5. Let $\mathbb{E}$ be a virtual equipment. An *left admissible sub-double category* of $\mathbb{E}$ is a sub-virtual double category $\mathbb{D}$ of $\mathbb{E}$ satisfying the equivalent conditions in Lemma 9.4. ♦

Proposition 9.6. Let $\mathbb{E}$ be a virtual equipment. Left weights and all vertical arrows in $\mathbb{E}$ form an left admissible sub-double category $\mathbb{LW}(\mathbb{E}) \subseteq \mathbb{E}$. Dually, right weights form an left admissible sub-double category of $\mathbb{E}^{\mathrm{hop}}$ and we write $\mathbb{RW}(\mathbb{E})$ for its horizontal opposite.

Proof. ongoing By Lemma 9.4, it suffices to show that given any opcartesian cell

$$
\begin{tikzcd}
X \arrow[r, dashed, "\vec{u}"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{s.opc}"] & Y \arrow[d, equal] \\
Z \arrow[r, "v"'] & Y
\end{tikzcd}
\quad \text{in } \mathbb{E}
$$

such that each component of $\vec{u}$ is a left weight, v is also a left weight. □

Definition 9.7. Let $\mathbb{E}$ be a virtual equipment. A *left dogma* is an left admissible sub-double category of $\mathbb{LW}(\mathbb{E})$. Dually, a *right dogma* is the horizontal opposite of an left admissible sub-double category of $\mathbb{RW}(\mathbb{E})^{\mathrm{hop}}$. ♦

Lemma 9.8. Let $\mathbb{E}$ be a virtual equipment. Left dogmas are closed under meets with respect to the inclusion of virtual double categories. In particular, for any class of left weights Φ and class of vertical arrows A, there exists the smallest left dogma $\mathbb{S}_{\Phi,\mathrm{A}}$ including Φ and A.

Proposition 9.9. Let $\mathbb{E}$ be a virtual equipment. Suppose we are given a class of left weights Φ and a class of vertical arrows A. Then for any object $X \in \mathbb{E}$, the following are equivalent.

(i) X is $\mathbb{S}_{\Phi,\mathrm{A}}$-cocomplete.
(ii) For any $\varphi \in \Phi$ and $f \in \mathrm{A}$ with $\mathrm{cod}(f) = X$, there exists a left Kan extension of f along φ.

Proof. ongoing □

10. Hoshino's writings 2

10.1. Cartesian monads.

Definition 10.1. Define a 2-category $\mathscr{C}ART$ as follows.

- Objects are finitely complete large categories and 1-cells are functors preserving pullbacks.
- 2-cells are cartesian natural transformations: A natural transformation $\alpha\colon F \Rightarrow G\colon \mathbf{E} \longrightarrow \mathbf{G}$ is *cartesian* if for each arrow $f\colon E \longrightarrow E'$ in $\mathbf{E}$, the naturality square

$$\begin{tikzcd} F(E) \arrow[r, "\alpha_E"] \arrow[d, "Ff"'] & G(E) \arrow[d, "Gf"] \\ F(E') \arrow[r, "\alpha_{E'}"'] & G(E') \end{tikzcd}$$

is a pullback square in $\mathbf{D}$.

A *cartesian monad* is a monad in $\mathscr{C}ART$. ♦

Proposition 10.2. Suppose we are given $\mathbf{E} \in \mathscr{C}ART$ and $E \in \mathbf{E}$. Then, the canonical adjoint $\begin{tikzcd} \mathbf{E}/E \arrow[r, shift left, "\mathrm{dom}"] \arrow[r, phantom, "\perp"] & \mathbf{E} \arrow[l, shift left, "E\times -"] \end{tikzcd}$ is in $\mathscr{C}ART$.

Proposition 10.3. Let $\mathscr{K}$ be a 2-category and $\begin{tikzcd} Y \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & X \arrow[l, shift left, "G"] \end{tikzcd}$ be an adjoint in $\mathscr{K}$. Then the functor

$$\mathbf{Hom}_{\mathscr{K}}(X, X) \xrightarrow{F \fatsemi - \fatsemi G} \mathbf{Hom}_{\mathscr{K}}(Y, Y)$$

defines a lax monoidal functor. In particular, a monad on X is sent to a monad on Y.

Definition 10.4. For each cartesian monad T on $\mathbf{E}$ and objects $E, E' \in \mathbf{E}$, we write $T_{/E,E'}$ for the functor $\mathrm{dom} \fatsemi T \fatsemi (E' \times -)\colon \mathbf{E}/E \longrightarrow \mathbf{E}/E'$. Observe that for each object $E \in \mathbf{E}$, the endo-functor $T_{/E,E}$ on $\mathbf{E}/E$ is equipped with a structure of monad induced from the adjoint in *Proposition* 10.2 through *Proposition* 10.3. We denote the monad by $T_{/E}$. ♦

In general, for any monoidal category $\mathbf{C}$ and a monoid $M \in \mathbf{C}$, the slice category $\mathbf{C}/M$ has a monoidal structure. This construction is generalised to bicategories:

Construction 10.5. Let $\mathscr{B}$ be a bicategory and $T\colon \mathscr{I}(\mathrm{Obj}\,\mathscr{B}) \longrightarrow \mathscr{B}$ be a lax functor that is identity on object, whose domain is the set $\mathrm{Obj}\,\mathscr{B}$ seen as an indiscrete 2-category. Then we construct another bicategory $\mathscr{B}/T$ in the following way.

- Objects are the same as those of $\mathscr{B}$: i.e., we have $\mathrm{Obj}\,\mathscr{B} = \mathrm{Obj}(\mathscr{B}/T)$.

- Hom categories are defined as the slice category: we have
$$\mathbf{Hom}_{\mathscr{B}/T}(A,B)=\mathbf{Hom}_{\mathscr{B}}(A,B)/T(A,B)$$
for any objects $A,B\in\operatorname{Obj}\mathscr{B}$. Here, we write $T(A,B)$ for the image of the unique morphism $A\longrightarrow B$ in $\mathscr{I}(\operatorname{Obj}\mathscr{B})$ under the lax functor T.
- The composition is defined as the composite of the following functors.

$$\begin{aligned}\mathbf{Hom}_{\mathscr{B}/T}(A,B)\times\mathbf{Hom}_{\mathscr{B}/T}(B,C) &\cong (\mathbf{Hom}_{\mathscr{B}}(A,B)\times\mathbf{Hom}_{\mathscr{B}}(B,C))\,/\,(T(A,B),T(B,C))\\ &\longrightarrow\mathbf{Hom}_{\mathscr{B}}(A,C)/\,(T(A,B)\,⨾\,T(B,C))\\ &\longrightarrow\mathbf{Hom}_{\mathscr{B}}(A,C)/T(A,C)=\mathbf{Hom}_{\mathscr{B}/T}(A,C)\end{aligned}$$

The first one is a canonical isomorphism, the second one is obtained by the composite of $\mathscr{B}$, and the last one is obtained by post-composing the multiplication of T.

For more detailed description and proof for that this construction indeed gives a bicategory, see fujiilack. ♦

Proposition 10.6. Let F be an morphism $\mathbf{E}'\longrightarrow\mathbf{E}$ in $\mathscr{C}ART$. The following functor defined through evaluation at the terminal object

$$\begin{array}{ccc}\mathbf{Hom}_{\mathscr{C}ART}(\mathbf{E}',\mathbf{E})/F & \xrightarrow{\quad\mathrm{ev}_1\quad} & \mathbf{E}/F1\\ \cup & & \cup\\ (p\colon G\to F) & \longmapsto & (p_1\colon G1\to F1)\end{array}$$

is an equivalence of categories. If $\mathbf{E}'=\mathbf{E}$ and $F=T$ is a monad, this induces a monoidal structure on $\mathbf{E}/T1$.

Corollary 10.7. Suppose we are given an monad T on $\mathbf{E}$ in $\mathscr{C}ART$. For any objects $E',E\in\mathbf{E}$, the composite

$$\mathbf{Hom}_{\mathscr{C}ART}(\mathbf{E}/E',\mathbf{E}/E)/T_{/E',E}\xrightarrow{\mathrm{ev}_{1_{E'}}}(\mathbf{E}/E)\,/T_{/E',E}1_{E'}\xrightarrow{\mathrm{dom}}(\mathbf{E}/TE'\times E)$$

is an equivalence.

Definition 10.8. Suppose that we are given a finitely complete category $\mathbf{E}$ and a cartesian monad T on $\mathbf{E}$.

- Define a 2-category $\mathscr{B}_{\mathbf{E}}\subseteq\mathscr{C}ART$ as follows.
  - Objects are those of $\mathbf{E}$.
  - The hom category is $\mathbf{Hom}_{\mathscr{B}_{\mathbf{E}}}(A,B)=\mathscr{C}ART(\mathbf{E}/A,\mathbf{E}/B)$ for any objects $A,B\in\operatorname{Obj}\mathbf{E}$.
  - Composition and identities are those for $\mathscr{C}ART$.
- Define an identity-on-object lax functor $\tilde{T}\colon\mathscr{I}(\operatorname{Obj}\mathbf{E})\longrightarrow\mathscr{B}_{\mathbf{E}}$ as follows.
  - The unique map $A\longrightarrow B$ in $\mathscr{I}(\operatorname{Obj}\mathbf{E})$ is sent to $T_{/A,B}\colon\mathbf{E}/A\longrightarrow\mathbf{E}/B$ defined in Definition 10.4.
  - The unit $\mathsf{id}_{\mathbf{E}/A}\Longrightarrow T_{/A,A}$ is that for the monad $T_{/A}$, for each $A\in\operatorname{Obj}\mathbf{E}$.
  - The multiplication $T_{/A,B}\,⨾\,T_{/B,C}\Longrightarrow T_{/A,C}$ is defined as the composite of the following 2-cells in $\mathscr{C}ART$.

$$\begin{aligned}T_{/A,B}\,⨾\,T_{/B,C}&\xRightarrow{\mathrm{dom}\,⨾\,T\,⨾\,\varepsilon\,⨾\,T\,⨾\,-\times A}\mathrm{dom}\,⨾\,T\,⨾\,T\,⨾\,-\times A\\ &\xRightarrow{\mathrm{dom}\,⨾\,\mu^T\,⨾\,-\times A}\mathrm{dom}\,⨾\,T\,⨾\,-\times A\end{aligned}$$

    for any $A,B\in\operatorname{Obj}\mathbf{E}$.
    For the case $A=B=C$, this multiplication coincide with that of the monad $T_{/A}$ observed in Definition 10.4.

- Now we have the slice bicategory
$$\mathscr{B}_{\mathbf{E}}/\tilde{T}$$
using Construction 10.5, whose hom categories are defined as
$$\mathbf{Hom}_{\mathscr{B}_{\mathbf{E}}/\tilde{T}}(A,B) = \mathbf{Hom}_{\mathscr{C}ART}(\mathbf{E}/A, \mathbf{E}/B)/T_{/A,B}$$
for any $A, B \in \mathbf{E}$.
Through the equivalence Corollary 10.7, we obtain a bicategory $\mathscr{S}\mathrm{pan}_T\,\mathbf{E}$ whose hom category is $\mathbf{E}/TA \times B$ for any objects $A, B \in \mathbf{E}$, that is equivalent to the slice $\mathscr{B}_{\mathbf{E}}/\tilde{T}$. In particular, we have a monoidal structure on $\mathbf{E}/TE \times E$. ♦

Lemma 10.9. Let $f\colon E \longrightarrow E'$ be a morphism in a finitely complete category $\mathbf{E}$ and T be a cartesian monad on $\mathbf{E}$. The reindexing functor $\mathbf{E}/TE' \times E' \longrightarrow \mathbf{E}/TE \times E$ along $Tf \times f$ defines a lax monoidal functor, where the domain and the codomain are equipped with the monoidal structures obtained in the Definition 10.8.

Moreover, this gives rise to a pseudo functor $\mathbf{E}^{\mathrm{op}} \longrightarrow \mathscr{M}on\mathscr{C}AT_{lax}$ that sends an object $E \in \mathbf{E}$ to $\mathbf{E}/TE \times E$, where $\mathscr{M}on\mathscr{C}AT_{lax}$ is the 2-category of monoidal categories, lax monoidal functors, and monoidal natural transformations.

Definition 10.10. For each cartesian monad T on $\mathbf{E}$, a *T-category* is a pair (C_0, C) of an object $C_0 \in \mathbf{E}$ and a monoid C in the monoidal category $\mathbf{E}/TC_0 \times C_0$ obtained in Definition 10.8. For any two T-categories C and C', a *T-functor* $F\colon C \longrightarrow C'$ is a pair (F_0, F_1) of a morphism $F_0\colon C_0 \longrightarrow C_0'$ and a monoid morphism $F_1\colon C \longrightarrow (TF_0 \times F_0)^*(C')$ in $\mathbf{E}/TC_0 \times C_0$.

We write T-**Cat** for the category of T-categories and T-functors. In other words, T-**Cat** is the Grothendieck construction of the composite of the pseudo functors
$$\mathbf{E}^{\mathrm{op}} \longrightarrow \mathscr{M}on\mathscr{C}AT_{lax} \xrightarrow{\mathbf{Mon}} \mathscr{C}AT,$$
where the first one is what is obtained in Lemma 10.9. ♦

Example 10.11. The identity monad $\mathsf{id}_{\mathbf{E}}$ on any finitely complte category $\mathbf{E}$ is cartesian and an $\mathsf{id}_{\mathbf{E}}$-category is precisely the same as a category internal to $\mathbf{E}$. ♦

For any monad T on $\mathbf{E}$, we write
$$\begin{tikzcd} \mathbf{E} \arrow[r, shift left, "F_T"] \arrow[r, phantom, "\perp"] & T\text{-}\mathbf{Alg} \arrow[l, shift left, "U_T"] \end{tikzcd}$$
for the free-forgetful adjunction for T-algebras.

Definition 10.12. Define the following categories.

- G_1 is a category freely generated by the following graph.
$$\begin{tikzcd} 0 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & 1 \end{tikzcd}$$
- G_1^r is presented as follows.
  - Objects are the same as G_1.
  - Morphisms generated by
    * two *face maps*
$$\begin{tikzcd} 0 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & 1 \end{tikzcd}$$
    and
    * a *degeneracy map*
$$0 \xleftarrow{\mathtt{i}} 1$$

subject to

$$\sigma \mathbin{⨟} \mathtt{i} = \mathsf{id}_0 = \tau \mathbin{⨟} \mathtt{i}.$$

- A *graph* is a presheaf of G_1. The category of graphs, **Gph**, is the presheaf category $[\mathsf{G}_1^{\mathrm{op}}, \mathbf{Set}]$.
  Given a graph X, the value at 0, X_0, is the set of *vertices*, while the value at 1, X_1, is called the set of *edges*. For each edge $e \in X_1$, $X_\sigma(e)$ and $X_\tau(e)$ are the *source* and the *target* of e, and e is denoted as $e\colon x \to y$ if $X_\sigma(e) = x$ and $X_\tau(e) = y$.
- A *reflexive graph* is a presheaf of G_1^r. The category of reflexive graphs, **rGph**, is the presheaf category $[\mathsf{G}_1^{r\,\mathrm{op}}, \mathbf{Set}]$.
  The reindexing along the inclusion $\mathsf{G}_1 \rightarrowtail \mathsf{G}_1^r$ defines a conservative and faithful functor $U_{\mathtt{fr}}\colon \mathbf{rGph} \longrightarrow \mathbf{Gph}$. We often assume **rGph** as a subcategory of **Gph** through this functor. ♦

Proposition 10.13. Let **Cat** be the category of categories.

- **Cat** is monadic over **Gph** and **rGph**.
- **rGph** is monadic over **Gph**.

The monads induced by those monadic functors are denoted by $\mathtt{fc}$, $\mathtt{fc}'$, and $\mathtt{fr}$, respectively. Moreover, they satisfy $U_{\mathtt{fc}} = U_{\mathtt{fr}} \circ U_{\mathtt{fc}'}$.

Definition 10.14. A *virtual double category* is an $\mathtt{fc}$-category and a *virtual double functor* is an $\mathtt{fc}$-functor. We write **VDbl** for $\mathtt{fc}$-**Cat**. ♦

Definition 10.15. An *augmented virtual double category* $\mathbb{X} = (\mathrm{H}\,\mathbb{X}, \mathbb{X})$ is an $\mathtt{fc}'$-category $\mathbb{X}$ equipped with a graph $\mathrm{H}\,\mathbb{X}$ such that the underlying reflexive graph of $\mathbb{X}$ is the free reflexive graph $F_{\mathtt{fr}}(\mathrm{H}\,\mathbb{X})$. An *augmented virtual double functor* $\mathbb{X} \longrightarrow \mathbb{Y}$ is an $\mathtt{fc}'$-functor between AVDCs such that its underlying morphism $F_{\mathtt{fr}}\,\mathrm{H}\,\mathbb{X} \longrightarrow F_{\mathtt{fr}}\,\mathrm{H}\,\mathbb{Y}$ in **rGph** preserves the generators. AVDCs and augmented virtual double functors form a subcategory of $\mathtt{fc}'$-**Cat**, which is denoted by **AVDbl**. ♦

Proposition 10.16. Let T and S be cartesian monads on $\mathbf{E}$ and $\mathbf{D}$ and let $F\colon T \longrightarrow S$ be a lax monad morphism in $\mathscr{CART}$. For each object $E \in \mathbf{E}$, F induces a lax monoidal functor $\mathbf{E}/TE \times E \longrightarrow \mathbf{D}/SFE \times FE$ and a functor T-**Cat** $\longrightarrow S$-**Cat**.

Lemma 10.17. Let us write $\mathrm{V}\colon \mathbf{rGph} \longrightarrow \mathbf{Set}$ for the functor sending a reflexive graph to its set of vertices, obtained by reindexing along $\ulcorner 0 \urcorner\colon 1 \longrightarrow \mathsf{G}_1^r$. Suppose that we are given a graph X. Then $\mathrm{V}\colon \mathbf{rGph}/\mathtt{fc}'(F_{\mathtt{fr}}(X)) \times F_{\mathtt{fr}}(X) \longrightarrow \mathbf{Set}/\mathrm{V}(X) \times \mathrm{V}(X)$ extends to a strong monoidal functor.

Notation 10.18. Suppose we are given an augmented virtual double category $\mathbb{X}$.

- Vertices of $\mathrm{H}\,\mathbb{X}$ are called *objects* of $\mathbb{X}$, while edges of $\mathrm{H}\,\mathbb{X}$ are called *horizontal arrows* of $\mathbb{X}$. We write $\mathrm{Obj}(\mathbb{X})$ for the set of objects of $\mathbb{X}$.
- For any objects A, B, we write

$$A \xrightarrow{p} B, \quad A \overset{\vec{p}}{\dashrightarrow} B, \text{ and } \quad A \overset{u}{\cdots\rightarrow} B$$

  for edges in $\mathrm{H}\,\mathbb{X}$, $\mathtt{fc}(\mathrm{H}\,\mathbb{X})$, and $\mathtt{fr}(\mathrm{H}\,\mathbb{X})$, respectively.
- The strong monoidal functor obtained in Lemma 10.17 gives a monoid $\mathbf{V}(\mathbb{X})$ in $\mathbf{Set}/\mathrm{Obj}(\mathbb{X}) \times \mathrm{Obj}(\mathbb{X})$ (= a category whose set of objects is $\mathrm{Obj}(\mathbb{X})$), which is called the *vertical category* of $\mathbb{X}$.
- Arrows in $\mathbf{V}(\mathbb{X})$ are called *vertical arrows* in $\mathbb{X}$. A vertical arrow is denoted as $f\colon A \longrightarrow B$.
- When $\mathbb{X}$ is seen as an arrow in **rGph** towards $\mathtt{fc}'(F_{\mathtt{fr}}(\mathrm{H}\,\mathbb{X})) \times F_{\mathtt{fr}}(\mathrm{H}\,\mathbb{X})$, vertical arrows are precisely vertices of the domain of $\mathbb{X}$. On the other hand, edges of the domain of $\mathbb{X}$ are called *double cells* or just *cells* of $\mathbb{X}$.

- Assume again that $\mathbb{X}$ is an arrow in **rGph**. We write

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"', "\shortmid"{marking}] & Y \end{tikzcd} \tag{44}$$

when the following hold.
  - f and g are vertices of the domain of $\mathbb{X}$, while $\alpha\colon f \to g$ is an edge in the reflexive graph.
  - $\mathbb{X}$, as a morphism in **rGph**, sends the edge $\alpha\colon f \to g$ to the edge $(\vec{p}, u)\colon (A, X) \to (B, Y)$ in $\mathtt{fc}'(F_{\mathfrak{fr}}(\mathrm{H}\,\mathbb{X})) \times F_{\mathfrak{fr}}(\mathrm{H}\,\mathbb{X})$.

  When the lengths of $\vec{p}$ and $\vec{u}$ is n and ε, we say α is an (n, ε)-*cell* in $\mathbb{X}$.
- We write the cell (44) by

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ X \arrow[rr, dotted, "u"', "\shortmid"{marking}] & {} & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[rr, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\alpha"] & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

  for the case when the length of either $\vec{p}$ or u is 0.
- $\mathbb{X}$, as a morphism in **rGph**, sends a reflection to a reflection in $\mathtt{fc}'(F_{\mathfrak{fr}}(\mathrm{H}\,\mathbb{X})) \times F_{\mathfrak{fr}}(\mathrm{H}\,\mathbb{X})$. Therefore for each object f in the domain of $\mathbb{X}$, the reflection is described as

$$\begin{tikzcd} A \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ B, \end{tikzcd}$$

  and it is called the *unit* on f.
- When $\mathbb{X}$ is considered to be an object in $\mathbf{rGph}/\mathtt{fc}'(F_{\mathfrak{fr}}(\mathrm{H}\,\mathbb{X})) \times (F_{\mathfrak{fr}}(\mathrm{H}\,\mathbb{X}))$, an edge of the tensor product $\mathbb{X} \otimes \mathbb{X}$ in this monoidal category is expressed in the either of the following forms.
  -

$$\begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{\varphi}_1", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "\psi_1"', "\shortmid"{marking}] \arrow[d, "g"'] & A_1 \arrow[r, dotted, "\psi_2"', "\shortmid"{marking}] & A_2 \arrow[d, phantom, "\beta"] & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"', "\shortmid"{marking}] & A_n \arrow[d, "h"] \\ B_0 \arrow[rrrrr, dotted, "r"', "\shortmid"{marking}] & & & & & B_1 \end{tikzcd}$$

    where α_i is not an identity on vertical arrow for each $i = 1 \ldots n$.
  -

$$\begin{tikzcd} & A' \arrow[d, bend right, "f_0"'] \arrow[d, bend left, "f_0"] \arrow[d, phantom, "="] & \\ & A \arrow[dl, "g"'] \arrow[dr, "h"] \arrow[d, phantom, "\beta"] & \\ B_0 \arrow[rr, dotted, "u"', "\shortmid"{marking}] & {} & B_1 \end{tikzcd}$$

The composition $\mathbb{X} \otimes \mathbb{X} \longrightarrow \mathbb{X}$ sends them to cells in $\mathbb{X}$, whose left and right legs result in the composites $f_0 \mathbin{⨾} g$ and $f_n \mathbin{⨾} h$ in $\mathbf{V}(\mathbb{X})$.

- $F_{\mathtt{fr}}(\mathrm{H}\,\mathbb{X}) \xrightarrow{\langle \eta, \mathsf{id}\rangle} \mathtt{fc}'(F_{\mathtt{fr}}(\mathrm{H}\,\mathbb{X})) \times F_{\mathtt{fr}}(\mathrm{H}\,\mathbb{X})$ is the monoidal unit of $\mathbf{rGph}/\mathtt{fc}'(F_{\mathtt{fr}}(\mathrm{H}\,\mathbb{X})) \times F_{\mathtt{fr}}(\mathrm{H}\,\mathbb{X})$. Therefore the unit for the monoid $\mathbb{X}$ gives cells of the following form:

$$\begin{tikzcd} A \arrow[r, "u", dotted, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & B \arrow[d, equal] \\ A \arrow[r, "u"', dotted, "\shortmid" marking] & B \end{tikzcd}$$

for each $u\colon A \dashrightarrow B$, where the left and right legs are the identities in $\mathbf{V}(\mathbb{X})$. Since the unit for $\mathbb{X}$ is a morphism in $\mathbf{rGph}$, it preserves reflections. Therefore, we have

$$\begin{tikzcd} A \arrow[d, equal, bend right] \arrow[d, equal, bend left] \arrow[d, phantom, "="] \\ A \end{tikzcd} = \begin{tikzcd} A \arrow[d, equal, bend right] \arrow[d, equal, bend left] \arrow[d, phantom, "\shortparallel"] \\ A. \end{tikzcd}$$

- By the diagram on the left, we mean the cell obtained by the composite of the diagram on the right when the sum of the lengths of ψ_i $(i = 1 \dots n)$ is lower than or equal to 1.

$$\begin{tikzcd} A'_0 \arrow[r, "\vec{\varphi}_1", dashed] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, "\vec{\varphi}_2", dashed] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 & \cdots & A'_{n-1} \arrow[r, "\vec{\varphi}_n", dashed] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, "\psi_1"', dotted] & A_1 \arrow[r, "\psi_2"', dotted] & A_2 & \cdots & A_{n-1} \arrow[r, "\psi_n"', dotted] & A_n \end{tikzcd} = \begin{tikzcd} A'_0 \arrow[r, "\vec{\varphi}_1", dashed] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, "\vec{\varphi}_2", dashed] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, "\vec{\varphi}_n", dashed] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, "\psi_1"', dotted] \arrow[d, equal] & A_1 \arrow[r, "\psi_2"', dotted] & A_2 \arrow[d, phantom, "\shortparallel"] & \cdots & A_{n-1} \arrow[r, "\psi_n"', dotted] & A_n \arrow[d, equal] \\ A_0 \arrow[rrrrr, "\psi_1 \dots \psi_n"', dotted] & & & & & A_n \end{tikzcd}$$

- Objects, vertical arrows, and $(0,0)$-ary cells form a 2-category $\mathscr{V}(\mathbb{X})$. Moreover, a 2-category $\mathscr{K}$ is precisely the same as an augmented virtual double category $\mathbb{V}(\mathscr{K})$ such that $\mathrm{H}\,\mathbb{V}(\mathscr{K})$ is discrete, and augmented virtual double functors between 2-categories are 2-functors. $2\mathbf{Cat}$ is a coreflexive full subcategory of $\mathbf{AVDbl}$, and $\mathscr{V}$ is the reflection $\mathbb{V}\colon 2\mathbf{Cat} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{AVDbl} : \mathscr{V}$. ♦

Remark 10.19. A virtual double category is precisely the same as an augmented virtual double category such that for any cell

$$\begin{tikzcd} A \arrow[rr, "\vec{p}", dashed] \arrow[dr, "f"'] & \alpha & B \arrow[dl, "g"] \\ & X, & \end{tikzcd}$$

the length of $\vec{p}$ is 0, $f = g$, and α is the unit on f. Through this identification, $\mathbf{VDbl}$ is a full subcategory of $\mathbf{AVDbl}$. ♦

Lemma 10.20. $\mathbf{VDbl}$ is a coreflexive full subcategory of $\mathbf{AVDbl}$.

11. Hoshino's writings 3

Let $\mathbb{X}$ be an augmented virtual double category.

Definition 11.1. A cell

$$\begin{tikzcd} A \arrow[r, "p", dotted] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "u"', dotted] & Y \end{tikzcd}$$

is *cartesian* if for each cell β of the following form, there exists a unique $\bar{\beta}$ satisfying the following equation.

$$\begin{tikzcd} A' \arrow[r, dashed, "\vec{q}"] \arrow[d, "h"'] & B' \arrow[d, "k"] \\ A \arrow[r, phantom, "\beta"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"'] & Y \end{tikzcd} = \begin{tikzcd} A' \arrow[r, dashed, "\vec{q}"] \arrow[d, "h"'] \arrow[dr, phantom, "\bar{\beta}"] & B' \arrow[d, "k"] \\ A \arrow[r, dotted, "p"'] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"'] & Y \end{tikzcd}$$

If such a cartesian cell exists and the length of p is 1, p is called the *restriction* of u along f and g, and denoted by $p \cong u(f,g)$.

If the length of u is 0 and hence $X = Y$ holds, then we write $X(f,g)$ for the restriction $u(f,g)$. In particular, we write f_* for $X(f,g)$ when g is an identity, and it is called the *companion* of f. Dually, we write g^* for $X(f,g)$ when f is an identity, and it is called the *conjoint* of g. ♦

Definition 11.2. For each object X, we write U_X for the restriction $X(\mathsf{id}_X, \mathsf{id}_X)$ along the identity vertical arrows, and we say it the *unit* on X. ♦

Definition 11.3. A *unital virtual double category* is an augmented virtual double category $\mathbb{X}$ with all units. We write **UVDbl** for the full subcategory of **AVDbl** consisting of unital virtual double categories. ♦

Definition 11.4. A *virtual equipment* is an augmented virtual double category with all restrictions. In particular, an equipment is a unital virtual double category. ♦

Definition 11.5. A *virtual company* is an augmented virtual double category with all restrictions of the following form.

$$\begin{tikzcd} A \arrow[r, "u(f,\mathsf{id})"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{cart}"] & B \arrow[d, equal] \\ X \arrow[r, dotted, "u"'] & B \end{tikzcd}$$

♦

Definition 11.6. Suppose that we are given a sequence of cells

$$\begin{tikzcd} A'_0 \arrow[r, dashed, "(\vec{\varphi}_i)_{i=1\ldots n}"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\vec{\alpha}"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dashed, "(\vec{\psi}_i)_{i=1\ldots n}"'] & A_n \end{tikzcd} := \begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "\psi_1"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n. \end{tikzcd} \tag{45}$$

- (45) is *weakly opcartesian* if for each cell β of the following form, there exists a unique $\bar{\beta}$ satisfying the following equation.

$$\begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] & A'_2 & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[d, "g"'] & & & \beta & & A_n \arrow[d, "h"] \\ B_0 \arrow[rrrrr, dotted, "r"'] & & & & & B_1 \end{tikzcd} = \begin{tikzcd} A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\ A_0 \arrow[r, dotted, "\psi_1"'] \arrow[d, "g"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n \arrow[d, "h"] \\ B_0 \arrow[rrrrr, dotted, "r"'] & & & \bar{\beta} & & B_1 \end{tikzcd}$$

- Suppose f_0 and f_n are identities in (45). Then it is *strongly opcartesian* if for any sequences of horizontal arrows

$$X \overset{\vec{u}}{\dashrightarrow} A_0 \text{ and } A_n \overset{\vec{v}}{\dashrightarrow} Y,$$

the following sequence of cells is weakly opcartesian.

$$\begin{tikzcd}
X \arrow[r, dashed, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, equal] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A_n \arrow[r, dashed, "\vec{v}"] \arrow[d, equal] \arrow[dr, phantom, "="] & Y \arrow[d, equal] \\
X \arrow[r, dashed, "\vec{u}"'] & A_0 \arrow[r, dotted, "\psi_1"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n \arrow[r, dashed, "\vec{v}"'] & Y
\end{tikzcd}$$

- In general, the sequence (45) is *strongly opcartesian* if it is weakly opcartesian and any sequence of cells of the following form is strongly opcartesian:

$$\begin{tikzcd}
X \arrow[r, dotted, "p"] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & A'_0 \arrow[r, dashed, "\vec{\varphi}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, dashed, "\vec{\varphi}_2"] \arrow[d] \arrow[dr, phantom, "\alpha_2"] & A'_2 \arrow[d] & \cdots & A'_{n-1} \arrow[r, dashed, "\vec{\varphi}_n"] \arrow[d] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[r, dotted, "q"] \arrow[d, "f_n"] \arrow[dr, phantom, "\mathsf{cart}"] & Y \arrow[d, equal] \\
X \arrow[r, dotted, "u"'] & A_0 \arrow[r, dotted, "\psi_1"'] & A_1 \arrow[r, dotted, "\psi_2"'] & A_2 & \cdots & A_{n-1} \arrow[r, dotted, "\psi_n"'] & A_n \arrow[r, dotted, "v"'] & Y
\end{tikzcd}$$

- (45) is *nanntoka cartesian* if any sequence of cells obtained by the following concatenation of sequences

$$\begin{tikzcd}
B'_0 \arrow[r, dashed] \arrow[d, "g_0"'] \arrow[dr, phantom, "\vec{\beta}"] & B'_m = A'_0 \arrow[r, dashed, "(\vec{\varphi}_i)_{i=1\dots n}"] \arrow[d, "g_n = f_0"] \arrow[dr, phantom, "\vec{\alpha}"] & A'_n = C'_0 \arrow[r, dashed] \arrow[d, "f_n = h_0"'] \arrow[dr, phantom, "\vec{\gamma}"] & C'_k \arrow[d, "h_k"] \\
B_0 \arrow[r, dashed] & B_n = A_0 \arrow[r, dashed, "(\vec{\psi}_i)_{i=1\dots n}"'] & A_n = C_0 \arrow[r, dashed] & C_k
\end{tikzcd}$$

is strongly opcartesian whenever $\vec{\beta}$ and $\vec{\gamma}$ are as well.

We say a cell is weakly opcartesian/strongly opcartesian/nanntoka cartesian if the sequence of length 1 is weakly opcartesian/strongly opcartesian/nanntoka cartesian in the above sense. ♦

Remark 11.7. Suppose we are given a sequence of cells (45) whose coarity (i.e., the total length of $(\psi_i)_{i=1\dots n}$) is less than 2. Then the composite is (weakly/strongly) opcartesian as a cell if and only if the sequence is (weakly/strongly) opcartesian. why ♦

Theorem 11.8. Let us write I for the composite of functors

$$\mathbf{UVDbl} \hookrightarrow \mathbf{AVDbl} \longrightarrow \mathbf{VDbl},$$

where the last one is the reflection observed in Lemma 10.20. I satisfies the following conditions.

(i) I has a right adjoint $\mathbb{M}\mathrm{od}$.
(ii) I is conservative.
(iii) The essential image of I consisting of virtual double categories such that for any object X, there exists an opcartesian cell of the following form.

$$\begin{tikzcd}
& X \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\mathsf{s.opc}"] & \\
X \arrow[rr] & {} & X
\end{tikzcd}$$

Lemma 11.9 (Vertical cancellation)**.** Suppose that we are given a cartesian cell β of the following form. Then there is a bijective correspondence between cells of the form α and cells of the form γ under the following equation.

$$
\begin{tikzcd}
A \arrow[r, dotted, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
A' \arrow[r, dotted, "p'", "\shortmid"{marking}] \arrow[d, "f'"'] \arrow[dr, phantom, "\beta"] & B' \arrow[d, "g'"] \\
A'' \arrow[r, dotted, "p''"', "\shortmid"{marking}] & B''
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, dotted, "p", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\
A' \arrow[r, phantom, "\gamma"] \arrow[d, "f'"'] & B' \arrow[d, "g'"] \\
A'' \arrow[r, dotted, "p''"', "\shortmid"{marking}] & B''
\end{tikzcd}
$$

For such a pair (α, γ), α is cartesian if and only if γ is cartesian.

Proof. Given arbitrary h, k, and $\vec{u}$, consider cells of the following forms.

$$
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\delta_1"] & \cdot \arrow[d, "k"] \\
A \arrow[r, dotted, "p"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad \Bigg| \quad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "h"'] & \cdot \arrow[d, "k"] \\
A \arrow[r, phantom, "\delta_2"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
A' \arrow[r, dotted, "p'"', "\shortmid"{marking}] & B'
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
\cdot \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, "h"'] & \cdot \arrow[d, "k"] \\
A \arrow[d, "f"'] & B \arrow[d, "g"] \\
A' \arrow[r, phantom, "\delta_3"] \arrow[d, "f'"'] & B' \arrow[d, "g'"] \\
A'' \arrow[r, dotted, "p''"', "\shortmid"{marking}] & B''
\end{tikzcd}
$$

There are functions $\delta_1 \mapsto \delta_1 ⨟ \alpha$ and $\delta_2 \mapsto \delta_2 ⨟ \beta$ from left to right, and the composite of those functions assigns $\delta_1 ⨟ \gamma$ to δ_1. Since β is cartesian, $\delta_2 \mapsto \delta_3$ is bijective, and hence $\delta_1 \mapsto \delta_2$ is bijective if and only if so is $\delta_1 \mapsto \delta_3$. This shows that α and γ are cartesian simultaneously. $\square$

Notation 11.10. Suppose that we are given sequences of cells of the following forms.

$$
\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\vec{\alpha}"] & A_l \arrow[d] \\
B_0 \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\vec{\beta}"] & B_m \arrow[d] \\
C_0 \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] & C_n
\end{tikzcd}
$$

In general, there might be multiple ways to composite cells in $\vec{\alpha}$ and $\vec{\beta}$ and obtain another sequence of cells. We write

$$
\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\alpha}"] & A_l \arrow[d, "f'"] \\
B_0 \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\vec{\beta}"] & B_m \arrow[d, "g'"] \\
C_0 \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] & C_n
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "f"'] & A_l \arrow[d, "f'"] \\
B_0 \arrow[r, phantom, "\vec{\gamma}"] \arrow[d, "g"'] & B_m \arrow[d, "g'"] \\
C_0 \arrow[r, dashed, "\vec{p}"', "\shortmid"{marking}] & C_n
\end{tikzcd}
\tag{46}
$$

if one of the compositions coincides with $\vec{\gamma}$.

Observe that given any cell δ of the following form, the composite of δ to those sequences as shown below is uniquely determined and we have an equality $\vec{\gamma}\,⨟\,\delta = \vec{\alpha}\,⨟\,(\vec{\beta}\,⨟\,\delta)$ as a cell.

$$
\begin{tikzcd}
A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\vec{\alpha}"] & A_l \arrow[d, "f'"] & & A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "f"'] & A_l \arrow[d, "f'"] \\
B_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\vec{\beta}"] & B_m \arrow[d, "g'"] & = & B_0 \arrow[r, phantom, "\vec{\gamma}"] \arrow[d, "g"'] & B_m \arrow[d, "g'"] \\
C_0 \arrow[r, dashed, "\shortmid" marking, "\vec{p}"'] \arrow[d] \arrow[dr, phantom, "\delta"] & C_n \arrow[d] & & C_0 \arrow[r, dashed, "\shortmid" marking, "\vec{p}"'] \arrow[d] \arrow[dr, phantom, "\delta"] & C_n \arrow[d] \\
\cdot \arrow[r, "\shortmid" marking] & \cdot & & \cdot \arrow[r, "\shortmid" marking] & \cdot
\end{tikzcd}
$$

This follows from the associativity of the multiplication of the monoid $\mathbb{X}$ in $\mathbf{rGph}/\mathtt{fc}'(F_{\mathfrak{fr}}\,\mathrm{H}\,\mathbb{X})\times F_{\mathfrak{fr}}\,\mathrm{H}\,\mathbb{X}$. ♦

Lemma 11.11. Suppose we are given sequences of cells satisfying (46).

(i) If the sequence $\vec{\alpha}$ is weakly opcartesian, then $\vec{\beta}$ is weakly opcartesian if and only if $\vec{\gamma}$ is as well.
(ii) Suppose that f, f', g, and g' are identities. If the sequence $\vec{\alpha}$ is weakly opcartesian, then $\vec{\beta}$ is weakly opcartesian if and only if $\vec{\gamma}$ is as well.
(iii) Suppose that $\mathbb{X}$ is a virtual equipment. If the sequence $\vec{\alpha}$ is strongly opcartesian, then $\vec{\beta}$ is strongly opcartesian if and only if $\vec{\gamma}$ is as well.

Proof. (i) is shown in a way similar to the proof for Lemma 11.9. One can also easily check that (ii) follows from (i).

For (iii), suppose $\mathbb{X}$ is a virtual equipment. Then, utilizing Lemma 11.9, we conclude that any cartesian cell of the form of ζ_1 below extends to a cartesian cell of the form ζ_2, while any cartesian cell of the form of ζ_2 factors as composite of some cartesian cells ξ and ζ_1, through taking restrictions.

$$
\begin{tikzcd}
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & \cdot \arrow[d, "f'"] & \cdot \arrow[r, dotted, "\shortmid" marking] \arrow[d, "f"'] & \cdot \arrow[d, "f'"] \\
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\zeta_1"] & \cdot \arrow[d, "g'"] & \cdot \arrow[r, phantom, "\zeta_2"] \arrow[d, "g"'] & \cdot \arrow[d, "g'"] \\
\cdot \arrow[r, dotted, "\shortmid" marking] & \cdot & \cdot \arrow[r, dotted, "\shortmid" marking] & \cdot
\end{tikzcd}
$$

Now consider the following sequences of cells, where cart denote some cartesian cells.

$$
\begin{tikzcd}
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathsf{cart}"] & B_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\vec{\beta}"] & B_m \arrow[r, dotted, "\shortmid" marking] \arrow[d, "g'"] \arrow[dr, phantom, "\mathsf{cart}"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, dotted, "\shortmid" marking] & C_0 \arrow[r, dashed, "\shortmid" marking] & C_n \arrow[r, dotted, "\shortmid" marking] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, dotted, "\shortmid" marking] \arrow[dd, equal] \arrow[ddr, phantom, "\mathsf{cart}"] & A_0 \arrow[r, dashed, "\shortmid" marking] \arrow[d, "f"'] & A_l \arrow[r, dotted, "\shortmid" marking] \arrow[d, "f'"] \arrow[ddr, phantom, "\mathsf{cart}"] & \cdot \arrow[dd, equal] \\
& B_0 \arrow[r, phantom, "\vec{\gamma}"] \arrow[d, "g"'] & B_m \arrow[d, "g'"] & \\
\cdot \arrow[r, dotted, "\shortmid" marking] & C_0 \arrow[r, dashed, "\shortmid" marking] & C_n \arrow[r, dotted, "\shortmid" marking] & \cdot
\end{tikzcd}
$$

For a sequence of the form on the left, there exists a sequence of the form on the right by extending cartesian cells in the above way. Since $\vec{\alpha}$ is strongly opcartesian, (ii) shows that the left one is strongly opcartesian if and only if the right one is as well. In the same way, given a sequence of the form on the right, through factorising cartesian cells, we have a sequence of the form on the left such that they are strongly opcartesian simultaneously. □

Lemma 11.12. Let $\mathbb{X}$ be an augmented virtual double category. Consider cells of the forms on the left, and suppose we are given a cartesian cell α' of the form on the right. Let β be the unique cell satisfying the equation.

$$
\begin{tikzcd}
X \arrow[r, dotted, "u"] \arrow[dr, "f"'] \arrow[dr, phantom, "\alpha_0", near start, shift left=2] & \cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, dashed, "l"] \arrow[dr, phantom, "\beta"] & \cdot \arrow[r, dotted, "v"] \arrow[d, dashed, "r"'] & Y \arrow[dl, "g"] \arrow[dl, phantom, "\alpha_1", near start, shift right=2] \\
 & \cdot \arrow[r, "q"'] & \cdot &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dotted, "u"] \arrow[dr, equal] & \cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, phantom, "\beta'"] & \cdot \arrow[r, dotted, "v"] & Y \arrow[dl, equal] \\
 & X \arrow[r, "q(f,g)"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha'"] & Y \arrow[d, "g"] & \\
 & \cdot \arrow[r, "q"'] & \cdot &
\end{tikzcd}
$$

If α_0 and α_1 are cartesian and β is strongly opcartesian, then β' is strongly opcartesian.

Proof. ongoing To show that β' is strongly opcartesian, suppose we are given sequences of horizontal arrows $\vec{\varphi}\colon X' \dashrightarrow X$ and $\vec{\psi}\colon Y \dashrightarrow Y'$ and a cell γ of the following form.

$$
\begin{tikzcd}
X' \arrow[r, "\vec{\varphi}"] \arrow[dr] & X \arrow[r, dotted, "u"] & \cdot \arrow[r, dashed, "\vec{p}"] \arrow[d, phantom, "\gamma"] & \cdot \arrow[r, dotted, "v"] & Y \arrow[r, "\vec{\psi}"] & Y' \arrow[dl] \\
 & X \arrow[rrr, dotted] & & & Y &
\end{tikzcd}
$$

We show γ factors through β' in a unique way.

Case1. For the case when f and g are identities, this follows from the definition of strongly opcartesian cell.

Case2. Suppose l and r are identities. Observe that there are cells β_0 and β_1 satisfying the following equalities since α_0 and α_1 are cartesian.

$$
\begin{tikzcd}
 & X \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\beta_0"] & \\
\cdot \arrow[rr, dotted] \arrow[dr, "f"'] & \arrow[d, phantom, "\alpha_0"] & X \arrow[dl, equal] \\
 & \cdot &
\end{tikzcd}
=
\begin{tikzcd}
 & X \arrow[dl, equal] \arrow[dr, "f"] & \\
\cdot \arrow[rr, phantom, "="] \arrow[dr, "f"'] & & X \arrow[dl, equal] \\
 & \cdot &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dl, "g"'] \arrow[dr, equal] \arrow[d, phantom, "\beta_1"] & \\
\cdot \arrow[rr, dotted] \arrow[dr, equal] & \arrow[d, phantom, "\alpha_1"] & X \arrow[dl, "g"] \\
 & \cdot &
\end{tikzcd}
=
\begin{tikzcd}
 & X \arrow[dl, "g"'] \arrow[dr, equal] & \\
\cdot \arrow[rr, phantom, "="] \arrow[dr, equal] & & X \arrow[dl, "g"] \\
 & \cdot &
\end{tikzcd}
$$

Case3. □

□

Lemma 11.13. Suppose we are given a cell

$$
\begin{tikzcd}
A \arrow[r, dotted, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, dotted, "u"'] & Y
\end{tikzcd}
$$

in an augmented virtual double category $\mathbb{X}$. The following hold.

(i) α is nanntoka cartesian if g or f is identity and α is cartesian.
(ii) α is nanntoka cartesian if it is cartesian and the restriction $u(f, \mathsf{id}_Y)$ exists.
(iii) If $\mathbb{X}$ is a virtual equipment, α is cartesian if and only if it is nanntoka cartesian.

The last argument shows that, in a virtual equipment, cartesian cells are precisely nanntoka cartesian cells whose arities and coarities are less than 2.

Proof. ongoing (i) follows directly from the definition of strongly opcartesian sequence.

For (ii), suppose we are given strongly opcartesian sequences $\vec{\beta}, \vec{\gamma}$ of the following forms.

$$\begin{tikzcd} S'_0 \arrow[r, dashed, "\vec{\beta}" phantom] \arrow[d, "f_0"'] & S'_m = A \arrow[r, dotted, "p"] \arrow[d, "f_n = f"] & B = T'_0 \arrow[r, dashed] \arrow[d, "g = g_0"'] & T'_k \arrow[d, "g_k"] \\ S_0 \arrow[r, dashed] \arrow[ur, phantom, "\vec{\beta}"] & S_n = X \arrow[r, dotted, "u"'] \arrow[ur, phantom, "\alpha"] & Y = T_0 \arrow[r, dashed] \arrow[ur, phantom, "\vec{\gamma}"] & T_k \end{tikzcd}$$

If α is cartesian, it factors through the restriction $u(f, \mathsf{id}_Y)$ as follows, and $\bar{\alpha}$ is also cartesian by Lemma 11.9.

$$\begin{tikzcd} A \arrow[r, dotted, "p"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"'] \arrow[ur, phantom, "\alpha"] & Y \end{tikzcd} = \begin{tikzcd} A \arrow[r, dotted, "p"] \arrow[d, equal] & B \arrow[d, "g"] \\ X \arrow[r, "u(f,\mathsf{id}_Y)"'] \arrow[d, "f"'] \arrow[ur, phantom, "\bar{\alpha}"] & Y \arrow[d, equal] \\ X \arrow[r, dotted, "u"'] \arrow[ur, phantom, "\beta"] & Y \end{tikzcd}$$

β and $\bar{\alpha}$ are nanntoka cartesian by (i), and Lemma 11.11 shows the above sequence is strongly opcartesian, which concludes α is nanntoka cartesian.

The only if part of (iii) follows from (ii). It remains to show that α is cartesian provided that $\mathbb{X}$ is a virtual equipment and α is nanntoka cartesian. □

Definition 11.14. A virtual double category $\mathbb{X}$ is *representable* if for any sequence of horizontal arrows $\vec{p} \colon A \dashrightarrow B$, there exists a horizontal arrow $p \colon A \longrightarrow B$ and an strongly opcartesian cell of the following form.

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{p}"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "p"'] \arrow[ur, phantom, "\mathsf{s.opc}"] & B \end{tikzcd}$$

♦

Fact 11.15. A *pseudo double category* is essentially the same as a representable virtual double category: There is a way to see a pseudo double category as a virtual double category, and a virtual double category is isomorphic to one obtained from a pseudo double category if and only if it is representable.

Using this fact, we use the term pseudo double category to mean representable virtual double category. ♦

Definition 11.16. Suppose we are given a cartesian monad T on a finitely complete category $\mathbf{E}$. We define $\mathbb{S}\mathrm{pan}_T\,\mathbf{E}$ as follows.

- The vertical category $\mathbf{V}(\mathbb{S}\mathrm{pan}_T\,\mathbf{E})$ is $\mathbf{E}$.
- For each objects $A, B \in \mathbf{E}$, a horizontal arrow $A \longrightarrow B$ in $\mathbb{S}\mathrm{pan}_T\,\mathbf{E}$ is an object of $\mathbf{E}/TA \times B$.
- A cell

$$A \longrightarrow B$$

♦

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Definition 11.17. Suppose we are given a cell

$$\begin{tikzcd} A \arrow[rr, dashed, "\vec{p}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\alpha"] & B \arrow[dl, "l"] \\ & X. & \end{tikzcd}$$

We say α exhibits l as a *left Kan extension* of f along $\vec{p}$ if for any cell β of the following form, there exists a unique $\bar{\beta}$ satisfying the following equation

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{p}"] \arrow[dr, "f"'] & B \arrow[r, dashed, "\vec{q}"] \arrow[d, dashed, "l"] \arrow[dl, phantom, "\alpha"] & B \arrow[dl, "g"] \arrow[dll, phantom, "\bar{\beta}"] \\ & A & \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{p}"] \arrow[dr, "f"'] & B \arrow[r, dashed, "\vec{q}"] \arrow[d, phantom, "\beta"] & B \arrow[dl] \\ & A & \end{tikzcd}$$

The same cell α exhibits f as a *right Kan extension* of l along $\vec{p}$ if it exhibits f as a left Kan extension of l along $\vec{p}$ in $\mathbb{X}^{\mathrm{hop}}$. We write

$$\begin{tikzcd} A \arrow[rr, dashed, "\vec{p}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\mathsf{lan}"] & B \arrow[dl, "l"] \\ & X & \end{tikzcd} \left(\text{resp.} \begin{tikzcd} A \arrow[rr, dashed, "\vec{p}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\mathsf{ran}"] & B \arrow[dl, "l"] \\ & X & \end{tikzcd} \right)$$

if such a cell α exists and it is a cell exhibiting left (resp. right) Kan extension. ♦

12. Virtuality in terms of strict double category

Notation 12.1. By a *graph*, we mean a presheaf for $\mathsf{G}_1 = ([0] \rightrightarrows [1])$, and we write **Gph** for the presheaf category $[\mathsf{G}_1^{\mathrm{op}}, \mathbf{Set}]$. Let us write $\mathtt{fc}$ for the free category monad on **Gph** and write $F_{\mathtt{fc}} \colon \mathbf{Gph} \longrightarrow \mathbf{Cat}$ for the free functor.

Given a graph X, an arrow in $F_{\mathtt{fc}}(X)$ (or an edge in $\mathtt{fc}(X)$) is a *path* in X, and the composite of arrows $\vec{p}$ and $\vec{q}$ in $F_{\mathtt{fc}}(X)$ is defined by concatenation of paths, which is denoted by $\vec{p} \frown \vec{q}$. We often identify an arrow in X with a path of length 1, and such a path is denoted by a solid arrow $A \longrightarrow B$. A general path is often denoted by a dashed arrow $A \dashrightarrow B$. A dotted arrow $A \cdots\!\!\to B$ denote a path of length 0 or 1, and such a path is called a *dotted* paths. A path of length 0 is called a *null* path. A null path on a vertex X is denoted by $()_X$.

We write X^{op} for the graph obtained from a graph X by flipping the directions of its edges. ♦

Definition 12.2. A *double graph* is a presheaf of $\mathsf{G}_1 \times \mathsf{G}_1$. We write **DGph** for $[\mathsf{G}_1^{\mathrm{op}} \times \mathsf{G}_1^{\mathrm{op}}, \mathbf{Set}]$. A double graph X consists of the following sets and functions

$$\begin{tikzcd} X_{11} \arrow[r, shift left, "\mathtt{l}"] \arrow[r, shift right, "\mathtt{r}"'] \arrow[d, shift right, "\mathtt{t}"'] \arrow[d, shift left, "\mathtt{b}"] & X_{01} \arrow[d, shift right, "\mathtt{t}"'] \arrow[d, shift left, "\mathtt{b}"] \\ X_{10} \arrow[r, shift left, "\mathtt{l}"] \arrow[r, shift right, "\mathtt{r}"'] & X_{00} \end{tikzcd}$$

satisfying $\mathtt{bside} \circ \mathtt{rside} = \mathtt{rside} \circ \mathtt{bside}$, $\mathtt{bside} \circ \mathtt{lside} = \mathtt{lside} \circ \mathtt{bside}$, $\mathtt{tside} \circ \mathtt{lside} = \mathtt{lside} \circ \mathtt{tside}$, and $\mathtt{tside} \circ \mathtt{rside} = \mathtt{rside} \circ \mathtt{tside}$.

- Elements of X_{11}, X_{01}, X_{10}, and X_{00} are called *cells*, *vertical edges*, *horizontal edges*, and *vertices*, respectively.
- Given a cell α in X, the images of α under $\mathtt{lside}$, $\mathtt{rside}$, $\mathtt{tside}$, and $\mathtt{bside}$ are called the *left*, *right*, *top*, and *bottom side* of α respevetively. ♦

Definition 12.3. A *strict double category* $\mathbb{D}$ is a category internal to **Cat**. In detail, $\mathbb{D}$ consists of the following data.

$$\begin{tikzcd} \mathbf{V}_1\,\mathbb{D}_{\mathtt{rside}} \times_{\mathtt{lside}} \mathbf{V}_1\,\mathbb{D} \arrow[r, shift left=2] \arrow[r, "\odot" description] \arrow[r, shift right=2] & \mathbf{V}_1\,\mathbb{D} \arrow[r, shift left=2, "\mathtt{lside}"] \arrow[r, shift right=2, "\mathtt{rside}"'] & \mathbf{V}\,\mathbb{D} \arrow[l, "=" description] \end{tikzcd} \quad \text{in } \mathbf{Cat}$$

where $\mathbf{V}_1\,\mathbb{D}_{\mathtt{rside}} \times_{\mathtt{lside}} \mathbf{V}_1\,\mathbb{D}$ is the fibred product of $\mathtt{rside}$ and $\mathtt{lside}$.

- An *object* of $\mathbb{D}$ is an object of the category $\mathbf{V}\,\mathbb{D}$. We write $\mathrm{Obj}\,\mathbb{D}$ for the set of objects.
- A *vertical arrow* of $\mathbb{D}$ is an arrow in $\mathbf{V}\,\mathbb{D}$.

- A *horizontal arrow* of $\mathbb{D}$ is an object in $\mathbf{V}_1\,\mathbb{D}$. We write $p\colon A \longrightarrow B$ if $A, B \in \mathrm{Obj}\,\mathbb{D}$, $p \in \mathbf{V}_1\,\mathbb{D}$, $\mathtt{lside}(p) = A$, and $\mathtt{rside}(p) = B$.
- A *cell* of $\mathbb{D}$ is an arrow in $\mathbf{V}_1\,\mathbb{D}$. A cell α is described as

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "q"'] & Y \end{tikzcd}$$

if $\alpha\colon p \longrightarrow q$ in $\mathbf{V}_1\,\mathbb{D}$, $f\colon A \longrightarrow X$ and $g\colon B \longrightarrow Y$ in $\mathbf{V}\,\mathbb{D}$, $\mathtt{lside}(\alpha) = f$, and $\mathtt{rside}(\alpha) = g$.
- Objects and horizontal arrows form a category $\mathbf{H}\,\mathbb{D}$, whose composition and identity are induced from the object parts of $-\odot-\colon \mathbf{V}_1\,\mathbb{D}_{\mathtt{rside}}\times_{\mathtt{lside}}\mathbf{V}_1\,\mathbb{D} \longrightarrow \mathbf{V}_1\,\mathbb{D}$ and $=_{-}\colon \mathbf{V}\,\mathbb{D} \longrightarrow \mathbf{V}_1\,\mathbb{D}$. In other words, $\mathbf{H}\,\mathbb{D}$ is the image of $\mathbb{D}$ under the cartesian functor $\mathrm{Obj} : \mathbf{Cat} \to \mathbf{Set}$.
- Vertical arrows and cells form a category $\mathbf{H}_1\,\mathbb{D}$ in the same way.
- The functions dom and cod for the categories $\mathbf{V}\,\mathbb{D}$ and $\mathbf{V}_1\,\mathbb{D}$ defines functors $\mathtt{tside}\colon \mathbf{H}_1\,\mathbb{D} \longrightarrow \mathbf{H}\,\mathbb{D}$ and $\mathtt{bside}\colon \mathbf{H}_1\,\mathbb{D} \longrightarrow \mathbf{H}\,\mathbb{D}$. In the same way composition and identity for those two categories form functors $\shortparallel_{-}\colon \mathbf{H}\,\mathbb{D} \longrightarrow \mathbf{H}_1\,\mathbb{D}$ and $-\fatsemi-\colon \mathbf{H}_1\,\mathbb{D}_{\mathtt{bside}}\times_{\mathtt{tside}}\mathbf{H}_1\,\mathbb{D} \longrightarrow \mathbf{H}_1\,\mathbb{D}$.

  Consequently, we have another strict double category

$$\begin{tikzcd} \mathbf{H}_1\,\mathbb{D}_{\mathtt{bside}}\times_{\mathtt{tside}}\mathbf{H}_1\,\mathbb{D} \arrow[r, shift left=2] \arrow[r, "\fatsemi" description] \arrow[r, shift right=2] & \mathbf{H}_1\,\mathbb{D} \arrow[r, shift left=2, "\mathtt{tside}"] \arrow[r, shift right=2, "\mathtt{bside}"'] & \mathbf{H}\,\mathbb{D} \arrow[l, "\shortparallel" description] \end{tikzcd} \quad \text{in } \mathbf{Cat},$$

which we write $\mathbb{D}^t$.
- Flipping $\mathtt{lside}$ and $\mathtt{rside}$, we have yet another strict double category $\mathbb{D}^{\mathrm{vop}}$. In other words, $\mathbb{D}^{\mathrm{vop}}$ is the image of $\mathbb{D}$ under the cartesian functor $(-)^{\mathrm{op}}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}$. We write $\mathbb{D}^{\mathrm{hop}}$ for $((\mathbb{D}^t)^{\mathrm{vop}})^t$. ♦

Definition 12.4. Let $\mathbb{D}$ be a strict double category. Consider a cell α of the following form.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "q"'] & Y \end{tikzcd}$$

- α is *cartesian* if it is cartesian arrow in $\mathbf{V}_1\,\mathbb{D}$ with respect to the functor

$$\mathbf{V}_1\,\mathbb{D} \xrightarrow{\langle \mathtt{lside}, \mathtt{rside}\rangle} \mathbf{V}\,\mathbb{D}\times\mathbf{V}\,\mathbb{D}.$$

- Dually, it is *opcartesian* if it is cartesian in $\mathbb{D}^{\mathrm{vop}}$.
- α is *(horizontally) globular* if $f = \mathtt{lside}(\alpha)$ and $g = \mathtt{rside}(\alpha)$ are identities in $\mathbf{V}\,\mathbb{D}$. ♦

Definition 12.5. Define a category $\mathbf{BVDbl}$ by the following pullback square.

$$\begin{tikzcd} \mathbf{BVDbl} \arrow[r, "|-|"] \arrow[d, "\mathrm{H}"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbf{SDbl} \arrow[d, "\mathbf{H}"] \\ \mathbf{Gph} \arrow[r, "F_{\mathrm{fc}}"'] & \mathbf{Cat} \end{tikzcd}$$

A *bivirtual double category* is an object of $\mathbf{BVDbl}$. In detail, a bivirtual double category $\mathbb{X} = (|\mathbb{X}|, \mathrm{H}(\mathbb{X}))$ consists of the followings.
- A strict double category $|\mathbb{X}|$.
- A graph $\mathrm{H}(\mathbb{X})$ satisfying $F_{\mathrm{fc}}(\mathrm{H}(\mathbb{X})) = \mathbf{H}(|\mathbb{X}|)$. ♦

Notation 12.6. Suppose that we are given a bivirtual double category $\mathbb{D}$.

- *Objects*, *vertical arrows*, and *cells* of $\mathbb{D}$ are those of $|\mathbb{D}|$.
- A *horizontal arrow* in $\mathbb{D}$ is an edge in the graph $\mathrm{H}(\mathbb{D})$.
- By a solid arrow $p\colon A \xrightarrow{\shortmid} B$, we mean a horizontal arrow in $\mathbb{D}$. It is identified with an arrow in $\mathbf{H}|\mathbb{D}|$ of length 1, and sometimes called a *solid* path.
- A horizontal arrow in $|\mathbb{D}|$, which is a path of horizontal arrows in $\mathbb{D}$, is denoted by a slashed dashed arrow $\vec{p}\colon A \dashrightarrow B$.
- We denote by a slashed dotted arrow $p\colon A \cdots\shortmid\cdots\!\!> B$ for an arrow in $|\mathbb{D}|$ whose length is shorter than 2.
- An (m,n)-*cell* is a cell whose top side and bottom side are of length m and n, respectively.
- We write $\mathbb{D}^{\mathrm{hop}}$ for the bivirtual double category $(|\mathbb{D}|^{\mathrm{hop}}, \mathrm{H}(\mathbb{D})^{\mathrm{op}})$ ♦

Definition 12.7. Suppose that we are given a bivirtual double category $\mathbb{D}$ and a cell α in $\mathbb{D}$. A *decomposition* of α is a path of arrows in $\mathbf{H}_1|\mathbb{D}|$ whose composite is α. By a *decomposition*, we mean a decomposition of some cell in $\mathbb{D}$.

- A decomposition is *null* if its length is 0. In particular, a null decomposition is a decomposition of an identity cell on a vertical arrow.
- A decomposition is *solid* if its components have solid bottom sides.
- A decomposition is *dotted* if its components have dotted bottom sides. ♦

There exists a double graph whose

- vertices, vertical edges, and horizontal edges are those for $|\mathbb{D}|$ and
- cells are decompositions in $\mathbb{D}$,

so that we can define `lside`, `rside`, `tside`, and `bside` for decompositions.

Definition 12.8. Suppose that we are given a bivirtual double category $\mathbb{D}$.

- Let $\vec{\beta}$ be a dotted decomposition and γ be an $(-,0)$-cell whose right side coincides with the left side of $\vec{\beta}$. Then we define another dotted decomposition $\gamma \wedge \vec{\beta}$ inductively as follows.
  - When the length of $\vec{\beta}$ is 0, $\gamma \wedge \vec{\beta}$ is just γ seen as a dotted decomposition of γ.
  - $\gamma \wedge (\beta_0 \frown \vec{\beta})$ is defined by $(\gamma \odot \beta_0) \frown \vec{\beta}$.

  Dually, we define $\vec{\alpha} \wedge \gamma$ for the case when the left side of γ coincides with the right side of $\vec{\alpha}$.
- Define an equivalence relation $\sim$ on the set of dotted decompositions of cells in $\mathbb{D}$ as that generated from

$$\left(\vec{\alpha} \frown (\gamma \wedge \vec{\beta})\right) \sim \left(\vec{\alpha} \frown \gamma \frown \vec{\beta}\right) \sim \left((\vec{\alpha} \wedge \gamma) \frown \vec{\beta}\right) \text{ and } \tag{47}$$

$$\left(\vec{\zeta} \frown ()_f \frown \vec{\xi}\right) \sim \left(\vec{\zeta} \frown =_f \frown \vec{\xi}\right). \tag{48}$$

  Here $\vec{\alpha}$ and $\vec{\beta}$ are decompositions, f is a vertical arrow satisfying $f = \mathtt{rside}(\vec{\zeta}) = \mathtt{lside}(\vec{\xi})$, and γ is a $(-,0)$-cell satisfying $\mathtt{lside}(\gamma) = \mathtt{rside}(\vec{\alpha})$ and $\mathtt{rside}(\gamma) = \mathtt{lside}(\vec{\beta})$.

  Note that this equivalence relation is closed under concatenation. ♦

Definition 12.9.

- A *virtual double category* is a bivirtual double category such that every cell has a unique solid decomposition. We write **VDbl** for the full subcategory of **BVDbl** consisting of virtual double categories.

- An *augmented virtual double category* is a bivirtual double category such that every cell has a dotted decomposition that is unique up to the equivalence relation $\sim$ defined in Definition 12.8. We write **AVDbl** for the full subcategory of **BVDbl** consisting of augmented virtual double categories. ◆

Remark 12.10. A virtual double category is an augmented virtual double category $\mathbb{D}$ without non-trivial $(-,0)$-cells; i.e., any $(-,0)$-cell is the identity on its left (or right) side in $|\mathbb{D}|$, and hence $n = 0$. Therefore, **VDbl** is a full subcategory of **AVDbl**. ◆

Notation 12.11. For an augmented virtual double category $\mathbb{E}$, we mean by a *cell* in $\mathbb{E}$ a cell whose bottom is dotted. In order to distinguish cells in $\mathbb{E}$ in this sence and those in the bivirtual double category, we write Path for both of the inclusions $\mathbf{VDbl} \hookrightarrow \mathbf{BVDbl}$ and $\mathbf{AVDbl} \hookrightarrow \mathbf{BVDbl}$: A cell in $\mathbb{E}$ is a cell in $\mathsf{Path}\,\mathbb{E}$ whose bottom is dotted, while a cell in $\mathsf{Path}\,\mathbb{E}$ is an equivalence class of sequences of cells in $\mathbb{E}$. ◆

Proposition 12.12. The two inclusions $\mathbf{VDbl} \hookrightarrow \mathbf{AVDbl}$ and $\mathbf{AVDbl} \hookrightarrow \mathbf{BVDbl}$ have right adjoints $\mathtt{sd}$ and $\mathtt{dd}$ respectively.

Proof. For the right adjoint of the first inclusion, consider a bivirtual double category $\mathbb{D}$ and define a virtual double category $\mathtt{sd}(\mathbb{D})$ as follows.

- Objects and horizontal and vertical arrows are defined as the same as those of $\mathbb{D}$.
- (n,m)-cells of $\mathtt{sd}(\mathbb{D})$ are solid decompositions of (n,m)-cells in $\mathbb{D}$.
- The horizontal composition is defined by the concatenation of cells.
- The vertical composition is defined by piecewisely composing One can readily check that this is well defined.

Through a straightforward induction on lengths of decompositions, it is shown that this indeed define a virtual double category $\mathtt{sd}(\mathbb{D})$. Now suppose that we are given a virtual double category $\mathbb{X}$ and a bivirtual double category $\mathbb{D}$. Any functor $F\colon \mathbb{X} \to \mathbb{D}$ uniquely factors through the composition $\mathtt{sd}(\mathbb{D}) \twoheadrightarrow \mathbb{D}$, since any morphism in **BVDbl** preserves solid decompositions. This shows that the inclusion $\mathbf{VDbl} \hookrightarrow \mathbf{BVDbl}$ has a right adjoint, hence we also obtain a right adjoint of the inclusion $\mathbf{VDbl} \hookrightarrow \mathbf{AVDbl}$.

Given a bivirtual double category $\mathbb{D}$, we define another bivirtual double category $\mathtt{dd}(\mathbb{D})$ as follows.

- Objects and horizontal and vertical arrows are defined as the same as those of $\mathbb{D}$.
- (n,m)-cells of $\mathtt{sd}(\mathbb{D})$ are equivalence classes of dotted decompositions of (n,m)-cells in $\mathbb{D}$ with repects to the equivalence relation $\sim$ defined in Definition 12.8.
- The horizontal composition is defined by the concatenation of cells. This is well-defined since $\sim$ is defined to be closed under concatenation.

Suppose we are given two dotted decompositions $\vec{\alpha}, \vec{\beta}$ satisfying $\mathtt{bside}(\vec{\alpha}) = \mathtt{tside}(\vec{\beta})$. Define another dotted decomposition $\vec{\alpha} \mathbin{⨟_l} \vec{\beta}$ inductively as follows. For each dotted decomposition $\vec{\gamma}$ and a subsequence $\vec{p}$ of $\mathtt{bside}(\vec{\gamma})$, we write $\vec{\gamma}_{\vec{p}}$ for the largest subsequence of $\vec{\gamma}$ satisfying $\mathtt{bside}(\vec{\gamma}_{\vec{p}}) = \vec{p}$.

$$\vec{\alpha} \mathbin{⨟_l} \vec{\beta} = \begin{cases} ()_{g ⨟ f} & \text{if } \vec{\alpha} = ()_g \text{ and } \vec{\beta} = ()_f \\ \vec{\alpha} ⨟ f & \text{if } \mathtt{length}(\vec{\alpha}) \geq 1 \text{ and } \vec{\beta} = ()_f \\ \vec{\alpha}_{\mathtt{tside}(\beta)} ⨟ \beta \frown \vec{\alpha}' ⨟ \vec{\beta}' & \text{if } \vec{\beta} = \beta \frown \vec{\beta}' \end{cases}$$

In this definition,

- $\vec{\alpha}'$ is the unique decomposition satisfying $\vec{\alpha} = \vec{\alpha}_{\mathtt{tside}(\beta)} \frown \vec{\alpha}'$, and
- $\vec{\alpha} ⨟ f$ and $\vec{\alpha}_{\mathtt{tside}(\beta)} ⨟ \beta$ are cells defined by compositon in $\mathbb{D}$, which are seen as dotted decompositions of themselves.

We will define the vertical composision of the equivalence classes of $\vec{\alpha}$ and $\vec{\beta}$ to be the equivalence class of $\vec{\alpha} \mathbin{⨟_l} \vec{\beta}$.

`Claim 1.` $(\vec{\alpha}\fatsemi_l\vec{\beta}) \sim (\vec{\alpha}'\fatsemi_l\vec{\beta})$ for each $\vec{\alpha} \sim \vec{\alpha}'$.
$\because$ It suffices to show for the case when $\vec{\alpha}$ and $\vec{\alpha}'$ are related in the way described in (47) and (48).

- Consider a dotted decomposition $\vec{\alpha} = \vec{\alpha}_1 \frown \gamma \frown \vec{\alpha}_2$ with $\mathtt{length}(\mathtt{bside}(\gamma)) = 0$ and $\mathtt{length}(\gamma) = 1$. We write $\vec{\alpha}^r = \vec{\alpha}_1 \frown \gamma \wedge \vec{\alpha}_2$ and $\vec{\alpha}^l = \vec{\alpha}_1 \wedge \gamma \frown \vec{\alpha}_2$ and show $\vec{\alpha}\fatsemi_l\vec{\beta} \sim \vec{\alpha}^r\fatsemi_l\vec{\beta} \sim \vec{\alpha}^l\fatsemi_l\vec{\beta}$ by induction on the length of $\vec{\beta}$.
  - If $\vec{\beta} = ()_f$ or $\mathtt{length}(\vec{\beta}) = 1$, the associativity of the composition in $\mathbb{D}$ shows that $\vec{\alpha}\fatsemi_l\vec{\beta} = \vec{\alpha}^r\fatsemi_l\vec{\beta} = \vec{\alpha}^l\fatsemi_l\vec{\beta}$.
  - Suppose we have $\vec{\beta} = \beta \frown \vec{\beta}'$ for some cell β and a dotted decomposition $\vec{\beta}'$.
    (i) If $\mathtt{length}(\mathtt{bside}(\vec{\alpha}_1)) < \mathtt{length}(\mathtt{tside}(\beta))$, there exists a subsequence $\vec{\alpha}_2'$ of $\vec{\alpha}_2$ satisfying $\vec{\alpha}_{\mathtt{tside}(\beta)} = \vec{\alpha}_1 \frown \gamma \frown \vec{\alpha}_2'$ and $\mathtt{length}(\mathtt{bside}(\vec{\alpha}_2)) > 0$. Therefore $\vec{\alpha}\fatsemi_l\vec{\beta}$, $\vec{\alpha}^r\fatsemi_l\vec{\beta}$, and $\vec{\alpha}^l\fatsemi_l\vec{\beta}$ coincide for the associativity of the composition of $\mathbb{D}$.
    (ii) If $\mathtt{length}(\mathtt{bside}(\vec{\alpha}_1)) > \mathtt{length}(\mathtt{tside}(\beta))$, there exists a dotted decomposition $\vec{\alpha}_1'$ satisfying $\vec{\alpha}_1 = \vec{\alpha}_{1,\mathtt{tside}(\beta)} \frown \vec{\alpha}_1'$ and $\mathtt{length}(\vec{\alpha}_1') > 0$. Let us write $\vec{\alpha}'$, $\vec{\alpha}'^l$, and $\vec{\alpha}'^r$ for $\vec{\alpha}_1' \frown \gamma \frown \vec{\alpha}_2$, $(\vec{\alpha}_1' \wedge \gamma) \frown \vec{\alpha}_2$, and $\vec{\alpha}_1' \frown (\gamma \wedge \vec{\alpha}_2)$ respectively. By the inductive hypothesis, we have
    $$\vec{\alpha}'\fatsemi_l\vec{\beta}' \sim \vec{\alpha}'^l\fatsemi_l\vec{\beta}' \sim \vec{\alpha}'^r\fatsemi_l\vec{\beta}' \tag{49}$$
    . Since $\sim$ is closed under concatenation, we obtaine the desired sequence of relations by concatenating $\vec{\alpha}_{1,\mathtt{tside}(\beta)}$ to the left of (49).
    (iii) If $\mathtt{length}(\mathtt{bside}(\vec{\alpha}_1)) = \mathtt{length}(\mathtt{tside}(\beta))$, $\vec{\alpha}_1 \frown \gamma$ and $\vec{\alpha}_1 \wedge \gamma$ are included in $\vec{\alpha}_{\mathtt{tside}(\beta)}$ and $\vec{\alpha}^l_{\mathtt{tside}(\beta)}$ respectively, and hence we have $\vec{\alpha}\fatsemi_l\vec{\beta} = \vec{\alpha}^l\fatsemi_l\vec{\beta}$ by the associativity of the composition in $\mathbb{D}$. It remains to show $\vec{\alpha}\fatsemi_l\vec{\beta} = \vec{\alpha}^r\fatsemi_l\vec{\beta}$. $\vec{\alpha} = \vec{\alpha}^r$ if $\mathtt{length}(\alpha_2) = 0$. Therefore, we assume there exists $\vec{\alpha}_2 = \alpha_2 \frown \vec{\alpha}_2'$.
      * If $\mathtt{length}(\mathtt{bside}(\alpha_2)) = 0$, then we have $\vec{\alpha}\fatsemi_l\vec{\beta} = \vec{\alpha}^r\fatsemi_l\vec{\beta}$ for the same reason as above; $\gamma \frown \alpha_2$ and $\gamma \wedge \alpha_2$ are included in $\vec{\alpha}_{\mathtt{bside}(\beta)}$ and $\vec{\alpha}^r_{\mathtt{bside}(\beta)}$ respectively.
      * If $\mathtt{length}(\mathtt{bside}(\alpha_2)) = 1$, then we have $\vec{\alpha}_{\mathtt{tside}(\beta)} = \vec{\alpha}_1 \frown \gamma$ and $\vec{\alpha}^r_{\mathtt{tside}(\beta)} = \vec{\alpha}_1$. If we write f for $\mathtt{rside}(\beta) = \mathtt{lside}(\vec{\beta}')$, we have the following sequence.
      $$\begin{aligned}\vec{\alpha}\fatsemi_l\vec{\beta} &= (\vec{\alpha}_1 \frown \gamma)\fatsemi\beta \frown \vec{\alpha}_2\fatsemi_l\vec{\beta}' \\ &= (\vec{\alpha}_1\fatsemi\beta \wedge \gamma\fatsemi f) \frown \vec{\alpha}_2\fatsemi_l\vec{\beta}' \\ &\sim \vec{\alpha}_1\fatsemi\beta \frown \gamma\fatsemi f \frown \vec{\alpha}_2\fatsemi_l\vec{\beta}' \\ &\sim \vec{\alpha}_1\fatsemi\beta \frown (\gamma\fatsemi f \wedge \vec{\alpha}_2\fatsemi_l\vec{\beta}') \\ &= \vec{\alpha}_1\fatsemi\beta \frown (\gamma \wedge \vec{\alpha}_2)\fatsemi_l\vec{\beta}' \\ &= \vec{\alpha}^r\fatsemi_l\vec{\beta}\end{aligned}$$
      Here the second from last equation follows from
      $$\gamma\fatsemi f \wedge \vec{\alpha}_2'\fatsemi_l\vec{\beta}' = (\gamma \wedge \vec{\alpha}_2')\fatsemi_l\vec{\beta}', \tag{50}$$
      which is shown by induction on the length of $\vec{\beta}'$: If $\vec{\beta}' = ()_f$, then the equation (50) trivially follows from the definition of $\wedge$ and the associativity of the composition in $\mathbb{D}$. If $\vec{\beta}' = \beta' \frown \vec{\beta}''$, then, again by the

associativity of the composition in $\mathbb{D}$, we have

$$\gamma \fatsemi f \wedge \vec{\alpha}'_{2,\mathtt{tside}(\beta')} \fatsemi \beta' = (\gamma \wedge \vec{\alpha}'_{2,\mathtt{tside}(\beta')}) \fatsemi \beta'.$$

Since $\mathtt{length}(\mathtt{bside}(\gamma)) = 0$, we have $(\gamma \wedge \vec{\alpha}'_2)_{\mathtt{tside}(\beta')} = \gamma \wedge \vec{\alpha}'_{2,\mathtt{tside}(\beta')}$, and hence we conclude the equation (50).

- Consider a dotted decomposition $\vec{\alpha} = \vec{\alpha}_1 \frown \vec{\alpha}_2 = \vec{\alpha}_1 \frown ()_f \frown \vec{\alpha}_2$. We write $\vec{\alpha}'$ for $\vec{\alpha}_1 \frown =_f \frown \vec{\alpha}_2$.
  - If $\vec{\alpha} = ()_f$ and $\vec{\beta} = ()_g$ for some vertical arrow g, then $\vec{\alpha} \fatsemi_l \vec{\beta} = ()_{f \fatsemi g} \sim =_f \fatsemi g = \vec{\alpha}' \fatsemi_l \vec{\beta}$.
  - If $\mathtt{length}(\vec{\alpha}) > 0$ and $\vec{\beta} = ()_g$ for some vertical arrow g, we have $\vec{\alpha} \fatsemi_l \vec{\beta} = \vec{\alpha} \fatsemi g = \vec{\alpha}' \fatsemi g = \vec{\alpha}' \fatsemi_l \vec{\beta}$ by the unitality and the associativity for $\mathbb{D}$.
  - Suppose we have $\vec{\beta} = \beta \frown \vec{\beta}'$ for some cell β and a dotted decomposition $\vec{\beta}'$.
    * For the case when

      $$\mathtt{length}(\mathtt{bside}(\vec{\alpha}_1)) < \mathtt{length}(\mathtt{tside}(\beta)) \text{ or}$$
      $$\mathtt{length}(\mathtt{bside}(\vec{\alpha}_1)) > \mathtt{length}(\mathtt{tside}(\beta)),$$

      see the proof (i) and (ii) above.
    * Suppose $\mathtt{length}(\mathtt{bside}(\vec{\alpha}_1)) = \mathtt{length}(\mathtt{tside}(\beta))$. Then we have $\vec{\alpha}_{\mathtt{tside}(\beta)} = \vec{\alpha}_1$ and $\vec{\alpha}'_{\mathtt{tside}(\beta)} = \vec{\alpha}_1 \frown =_f$. Therefore, $\vec{\alpha} \fatsemi_l \vec{\beta} = \vec{\alpha}_1 \fatsemi \beta \frown \vec{\alpha}_2 \fatsemi_l \vec{\beta}' = (\vec{\alpha}_1 \frown =_f) \fatsemi \beta \frown \vec{\alpha}_2 \fatsemi_l \vec{\beta}' = \vec{\alpha}' \fatsemi_l \vec{\beta}$, where the third equation follows from the unitality for $\mathbb{D}$. $\Diamond$

`Claim 2.` $(\vec{\alpha} \fatsemi_l \vec{\beta}) \sim (\vec{\alpha} \fatsemi_l \vec{\beta}')$ for each $\vec{\beta} \sim \vec{\beta}'$.

$\because$ It suffices to show for the case when $\vec{\beta}$ and $\vec{\beta}'$ are related in the way described in (47) and (48).

- Suppose we are given dotted decompositions $\vec{\beta} = \vec{\beta}_1 \frown \gamma \frown \vec{\beta}_2$ satisfying $\mathtt{length}(\gamma) = 1$, $\mathtt{length}(\mathtt{bside}(\gamma)) = 0$, $\mathtt{rside}(\vec{\beta}_1) = \mathtt{lside}(\gamma)$, and $\mathtt{rside}(\gamma) = \mathtt{lside}(\vec{\beta}_2)$. Let us write $\vec{\beta}^r$ and $\vec{\beta}^l$ for $\vec{\beta}_1 \frown \gamma \wedge \vec{\beta}_2$ and $\vec{\beta}_1 \wedge \gamma \frown \vec{\beta}_2$ respectively. We show $\vec{\alpha} \fatsemi_l \vec{\beta} \sim \vec{\alpha} \fatsemi_l \vec{\beta}^r \sim \vec{\alpha} \fatsemi_l \vec{\beta}^l$ by induction on the length of $\vec{\beta}_1$. ongoing
- ongoing

ongoing $\Diamond$

Now that we have defined the vertical composition for $\mathtt{dd}(\mathbb{D})$, it remains to show that those indeed define a strict double category $|\mathtt{dd}(\mathbb{D})|$.

`Claim 3.` $(\vec{\alpha}_1 \fatsemi_l \vec{\alpha}_2) \fatsemi_l \vec{\alpha}_3 \sim \vec{\alpha}_1 \fatsemi_l (\vec{\alpha}_2 \fatsemi_l \vec{\alpha}_3)$

$\because$ ongoing $\Diamond$

`Claim 4.` $(\vec{\alpha}_1 \frown \vec{\alpha}_2) \fatsemi_l (\vec{\beta}_1 \frown \vec{\beta}_2) \sim (\vec{\alpha}_1 \fatsemi_l \vec{\alpha}_2) \frown (\vec{\beta}_1 \fatsemi_l \vec{\beta}_2)$

$\because$ ongoing $\Diamond$

$\square$

As in the proof of the above proposition, we denote by $\mathtt{sd}$ for both of the right adjoints of $\mathbf{VDbl} \hookrightarrow \mathbf{AVDbl}$ and $\mathbf{VDbl} \hookrightarrow \mathbf{BVDbl}$.

Remark 12.13. In the definition of $\mathtt{sd}(\mathbb{D})$ in Proposition 12.12, only cells in $\mathbb{D}$ with solid bottoms are used. Also, the composites used are only for *pasting schemes* where each component has a solid bottom and the bottom of the result will also be solid. Therefore, a virtual double category is determined by specifying such cells and compositions. This recovers the ordinary definition of a virtual double category. Moreover, the definition of $\mathtt{sd}(\mathbb{D})$ is essentially the same as the *path construction* in [DPP06].

In the same way, in the definition of $\mathtt{dd}(\mathbb{D})$, only cells in $\mathbb{D}$ with dotted bottoms are used and the composites used are only for pasting schemes where each component has a dotted bottom and the bottom of the result will also be dotted. Therefore an augmented virtual double category is determined by secifying such data, which is the same as what is discussed in [Kou20]. ♦

Definition 12.14. Suppose we are given a bivirtual double category $\mathbb{D}$ and a cell α of the following form.

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dashed, "\vec{q}"', "\shortmid"{marking}] & Y \end{tikzcd}$$

- α is (op)cartesian if it is (op)cartesian in the strict double category $|\mathbb{D}|$.
- If α is cartesian and $\vec{p}$ is a solid arrow $p\colon A \longrightarrow B$, then p is called the *restriction* of $\vec{q}$ along the pair (f, g).
- α is *weakly opcartesian* if given a cell β whose bottom side is solid, there exists a unique cell $\bar{\beta}$ of the following form satisfying the equality.

$$\begin{tikzcd} A \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, dashed, "\vec{q}", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\bar{\beta}"] & Y \arrow[d, "k"] \\ X' \arrow[r, "r"', "\shortmid"{marking}] & Y' \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[d, "h"'] & \beta \quad Y \arrow[d, "k"] \\ X' \arrow[r, "r"', "\shortmid"{marking}] & Y' \end{tikzcd}$$

- α is *left composing* if it is weakly opcartesian and any composite

$$\begin{tikzcd} \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[r, dashed, "\vec{p}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ \cdot \arrow[r, dashed, "\shortmid"{marking}] & X \arrow[r, dashed, "\vec{q}"', "\shortmid"{marking}] & Y \end{tikzcd}$$

is weakly opcartesian whenever β is cartesian.
- α is *right composing* if it is left composing in $\mathbb{D}^{\mathrm{hop}}$.
- α is *composing* if it is left composing and right composing. ♦

Definition 12.15. Define a category **CSDbl** by the following pullback square.

$$\begin{tikzcd} \mathbf{CSDbl} \arrow[r, "|-|"] \arrow[d, "\langle \mathrm{V},\mathrm{H} \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{SDbl} \arrow[d, "\langle \mathbf{V},\mathbf{H} \rangle"] \\ \mathbf{Gph} \times \mathbf{Gph} \arrow[r, "F_{\mathsf{fc}} \times F_{\mathsf{fc}}"'] & \mathbf{Cat} \times \mathbf{Cat} \end{tikzcd}$$

A *cellular strict double category (CSDC)* is an object of **CSDbl**. In detail, a cellular strict double category $\mathbb{X} = (|\mathbb{X}|, \mathrm{V}(\mathbb{X}), \mathrm{H}(\mathbb{X}))$ consists of the followings.

- A strict double category $|\mathbb{X}|$.
- Two graphs $\mathrm{V}(\mathbb{X})$ and $\mathrm{H}(\mathbb{X})$ satisfying $F_{\mathsf{fc}}(\mathrm{V}(\mathbb{X})) = \mathbf{V}(|\mathbb{X}|)$ and $F_{\mathsf{fc}}(\mathrm{H}(\mathbb{X})) = \mathbf{H}(|\mathbb{X}|)$. ♦

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