

# Double categories of profunctors

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← Today's slides

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- 1 Enrichment in a monoidal category
- 2 Generalized enrichment
- 3 Colimits in a unital virtual double category
- 4 The characterization theorem

# Enrichment in a monoidal category

$(\mathcal{V}, \otimes, I)$ : a monoidal category

## $\mathcal{V}$ -category $\mathbf{A}$

- A class  $\mathbf{ObA}$  of *objects*;
- *Hom-objects*  $\mathbf{A}(x, y) \in \mathcal{V}$  ( $x, y \in \mathbf{ObA}$ );
- *Compositions*  $\mathbf{A}(x, y) \otimes \mathbf{A}(y, z) \rightarrow \mathbf{A}(x, z)$  in  $\mathcal{V}$  ( $x, y, z \in \mathbf{ObA}$ );
- *Identities*  $I \rightarrow \mathbf{A}(x, x)$  in  $\mathcal{V}$  ( $x \in \mathbf{ObA}$ ). (+Axioms)

## $\mathcal{V}$ -functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map  $\mathbf{ObA} \longrightarrow \mathbf{ObB}$ ;  
$$x \longmapsto Fx$$
- $\mathbf{A}(x, y) \rightarrow \mathbf{B}(Fx, Fy)$  in  $\mathcal{V}$  ( $x, y \in \mathbf{ObA}$ ). (+Axioms)

## $\mathcal{V}$ -profunctor $\mathbf{A} \xrightarrow{P} \mathbf{B}$

- *Hom-objects*  $P(x, y) \in \mathcal{V}$  ( $x \in \mathbf{A}, y \in \mathbf{B}$ );
- *Actions*  $\mathbf{A}(x', x) \otimes P(x, y) \rightarrow P(x', y)$ ,  $P(x, y) \otimes \mathbf{B}(y, y') \rightarrow P(x, y')$  in  $\mathcal{V}$ . (+Axioms)

# The virtual double category $\mathcal{V}\text{-}\mathbb{P}\text{rof}$

- (objects)  $\mathcal{V}$ -categories  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ ;
- (vertical arrows)  $\mathcal{V}$ -functors  $\begin{array}{c} \mathbf{A} \\ F \downarrow \\ \mathbf{B} \end{array}, \dots$  and their compositions and identities;

- (horizontal arrows)  $\mathcal{V}$ -profunctors  $\mathbf{A} \xrightarrow{P} \mathbf{B}, \dots$ ;

- (cells) “generalized”  $\mathcal{V}$ -natural transformations

$$\begin{array}{ccc} \mathbf{A}_0 & \xrightarrow{P_1} \mathbf{A}_1 & \xrightarrow{P_2} \dots \xrightarrow{P_n} \mathbf{A}_n \\ F \downarrow & & \downarrow G \\ \mathbf{B} & \xrightarrow{Q} & \mathbf{C} \end{array} \quad \parallel \quad \begin{array}{c} P_1(x_0, x_1) \otimes P_2(x_1, x_2) \otimes \dots \otimes P_n(x_{n-1}, x_n) \\ \downarrow \\ Q(Fx_0, Gx_n) \end{array}$$

- and their compositions (+ identity cells)

$$\begin{array}{ccc} \mathbf{A}_0 & \xrightarrow{\vec{P}_1} \mathbf{A}_1 & \xrightarrow{\vec{P}_2} \dots \xrightarrow{\vec{P}_n} \mathbf{A}_n \\ F_0 \downarrow & \alpha_1 F_1 \downarrow & \alpha_2 \dots \alpha_n \downarrow F_n \\ \mathbf{B}_0 & \xrightarrow{Q_1} \mathbf{B}_1 & \xrightarrow{Q_2} \dots \xrightarrow{Q_n} \mathbf{B}_n \\ G \downarrow & & \downarrow H \\ \mathbf{C} & \xrightarrow{R} & \mathbf{D} \end{array} \rightsquigarrow \begin{array}{ccc} \mathbf{A}_0 & \xrightarrow{\vec{P}_1} \mathbf{A}_1 & \xrightarrow{\vec{P}_2} \dots \xrightarrow{\vec{P}_n} \mathbf{A}_n \\ F_0 \circ G \downarrow & \alpha \circ \beta & \downarrow F_n \circ H \\ \mathbf{C} & \xrightarrow{R} & \mathbf{D} \end{array}$$

# Horizontal composition

$\mathbb{X}$ : a virtual double category (VDC).

## Definition

A cell  $\begin{array}{ccc} A_0 & \xrightarrow{u_1} A_1 & \xrightarrow{u_2} \dots \xrightarrow{u_n} A_n \\ \parallel & & \parallel \\ A_0 & \xrightarrow[v]{} & A_n \end{array}$  in  $\mathbb{X}$  is **composing**

$$\stackrel{\text{def}}{\Leftrightarrow} \begin{array}{ccc} \cdot & \dashrightarrow & A_0 \dashrightarrow A_n \dashrightarrow \cdot \\ \downarrow & & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \quad \forall \beta \quad = \quad \begin{array}{ccccc} \cdot & \dashrightarrow & A_0 & \dashrightarrow & A_n & \dashrightarrow & \cdot \\ \parallel & & \parallel & & \parallel & & \parallel \\ \cdot & \dashrightarrow & A_0 & \xrightarrow[v]{} & A_n & \dashrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot & & \cdot & & \cdot \end{array} \quad \exists! \gamma \quad \text{in } \mathbb{X}.$$

## Definition

The **unit** on  $X \in \mathbb{X}$  ...  $\begin{array}{ccc} & X & \\ \parallel & & \parallel \\ X & \xrightarrow[u_X]{} & X \end{array}$  in  $\mathbb{X}$ .  $\left( \text{written } \begin{array}{ccc} & X & \\ \parallel & & \parallel \\ X & \xrightarrow{\quad} & X \end{array} \right)$

# Horizontal composition

## Examples in $\mathcal{V}\text{-}\mathbb{P}\text{rof}$

$$\begin{array}{ccccc} \textcircled{1} & \mathbf{A} & \xrightarrow{P} & \mathbf{B} & \xrightarrow{Q} & \mathbf{C} \\ & \parallel & & \text{comp} & & \parallel \\ & \mathbf{A} & \xrightarrow{P \odot Q} & \mathbf{C} & & \end{array}$$

$$(P \odot Q)(x, z) := \int^{y \in \mathbf{B}} P(x, y) \otimes Q(y, z) \quad \text{in } \mathcal{V}$$

(Suppose that  $\int$  is preserved by  $X \otimes -, - \otimes Y$ .)

$$\begin{array}{ccc} & \mathbf{A} & \\ \swarrow & & \searrow \\ \textcircled{2} & \text{comp} & \\ \swarrow & & \searrow \\ \mathbf{A} & \xrightarrow{U_{\mathbf{A}}} & \mathbf{A} \end{array}$$

$$U_{\mathbf{A}}(x, x') := \mathbf{A}(x, x') \quad \text{in } \mathcal{V}$$

# Unital virtual double categories

## Definition

A VDC  $\mathbb{X}$  is **unital**  $\stackrel{\text{def}}{\iff}$  Every object  $X \in \mathbb{X}$  has the unit.

## Example

$\mathcal{V}\text{-Prof}$  is unital.

In a unital VDC  $\mathbb{X}$ ,

$$\begin{array}{c} \cdot \overset{\vec{u}}{\dashrightarrow} \cdot \\ f \searrow \alpha \swarrow g \\ \cdot \end{array} := \begin{array}{c} \cdot \overset{\vec{u}}{\dashrightarrow} \cdot \\ f \downarrow \alpha \downarrow g \\ \cdot \text{---} \end{array} \text{ in } \mathbb{X}. \quad (\text{0-coary cell})$$

0-coary cells can be composed horizontally:

$$\begin{array}{c} \cdot \overset{\vec{u}}{\dashrightarrow} \cdot \overset{\vec{v}}{\dashrightarrow} \cdot \\ \alpha \searrow \downarrow \swarrow \beta \\ \cdot \end{array} \rightsquigarrow \begin{array}{c} \cdot \overset{\vec{u}}{\dashrightarrow} \cdot \overset{\vec{v}}{\dashrightarrow} \cdot \\ \searrow \textcolor{violet}{\alpha \smile \beta} \swarrow \\ \cdot \end{array} := \begin{array}{c} \cdot \overset{\vec{u}}{\dashrightarrow} \cdot \overset{\vec{v}}{\dashrightarrow} \cdot \\ \downarrow \alpha \downarrow \beta \downarrow \\ \cdot \text{---} \cdot \text{---} \cdot \\ \parallel \text{comp} \parallel \\ \cdot \text{---} \cdot \end{array}$$

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# Monoidal categories vs. bicategories vs. VDC

Monoidal category = single-object bicategory

$\mathcal{V}$ : a mon.cat.  $\iff \mathcal{B}(\mathcal{V})$ : a single-obj.bicat.

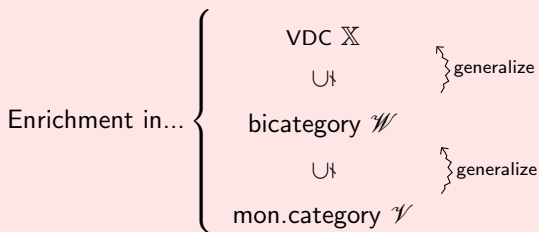
$$\begin{array}{ccc}
 \begin{array}{c} X \otimes Y \\ \alpha \downarrow \\ Z \end{array} & \text{in } \mathcal{V} & \parallel \\
 & & \begin{array}{ccc} X & \xrightarrow{\quad} & * \\ & \searrow \alpha & \swarrow Y \\ * & \xrightarrow{\quad Z \quad} & * \end{array} \text{ in } \mathcal{B}(\mathcal{V})
 \end{array}$$

(Virtual) bicategory = vertically discrete VDC

$\mathcal{W}$ : a (virtual) bicat.  $\iff \mathbb{D}(\mathcal{W})$ : a vertically discrete VDC

$$\begin{array}{ccc}
 \begin{array}{ccccc} & f & & g & \\ c_0 & \xrightarrow{\quad} & c_1 & \xrightarrow{\quad} & c_2 \\ & \searrow \alpha & & \swarrow & \\ & & c_2 & & \end{array} & \text{in } \mathcal{W} & \parallel \\
 & & & & \begin{array}{ccccc} c_0 & \xrightarrow{f} & c_1 & \xrightarrow{g} & c_2 \\ \parallel & & \alpha & & \parallel \\ c_0 & \xrightarrow{\quad h \quad} & & & c_2 \end{array} \text{ in } \mathbb{D}(\mathcal{W})
 \end{array}$$

# Generalization of enriching bases



## Remark

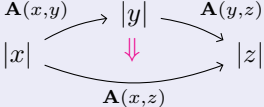
We obtain unital VDCs  $\mathscr{V}\text{-Prof}$ ,  $\mathscr{W}\text{-Prof}$ , and  $\mathbb{X}\text{-Prof}$  for any  $\mathscr{V}, \mathscr{W}, \mathbb{X}$ .

# From $\mathcal{V}$ to $\mathcal{W}$

$\mathcal{W}$ : a bicategory

## $\mathcal{W}$ -category $\mathbf{A}$

- A class  $\mathbf{ObA}$  of *objects*;
- *Coloring*  $|x| \in \mathcal{W}$  ( $x \in \mathbf{ObA}$ );
- *Hom-1-cells*  $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$  in  $\mathcal{W}$  ( $x, y \in \mathbf{ObA}$ );

- *Compositions*  

$$\begin{array}{ccccc} & \mathbf{A}(x,y) & & \mathbf{A}(y,z) & \\ |x| & \xrightarrow{\quad} & |y| & \xrightarrow{\quad} & |z| \\ & \text{pink } \Downarrow & & & \\ & \mathbf{A}(x,z) & & & \end{array} \quad \text{in } \mathcal{W} \quad (x, y, z \in \mathbf{ObA});$$

- *Identities*  $|x| \xrightarrow{\mathbf{A}(x,x)} |x|$  in  $\mathcal{W}$  ( $x \in \mathbf{ObA}$ ). (+Axioms)

# From $\mathcal{V}$ to $\mathcal{W}$

$\mathcal{W}$ : a bicategory

$\mathcal{W}$ -functor  $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map  $\text{Ob}\mathbf{A} \longrightarrow \text{Ob}\mathbf{B}$  s.t.  $|x| = |Fx|$  in  $\mathcal{W}$ ;  
 $x \longmapsto Fx$

- $|x| \begin{array}{c} \xrightarrow{\mathbf{A}(x,y)} \\ \Downarrow \\ \xrightarrow{\mathbf{B}(Fx,Fy)} \end{array} |y| \quad \text{in } \mathcal{W} \quad (x, y \in \text{Ob}\mathbf{A}).$  (+Axioms)

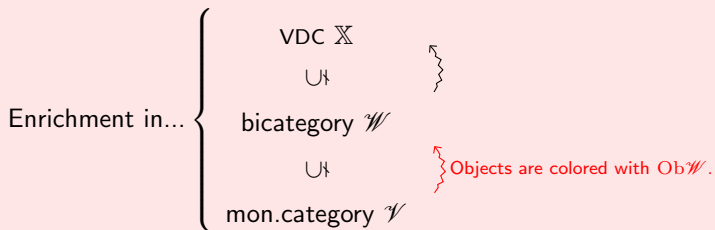
$\mathcal{W}$ -profunctor  $\mathbf{A} \xrightarrow{P}_+ \mathbf{B}$

- Hom-1-cells  $|x| \xrightarrow{P(x,y)} |y|$  in  $\mathcal{W}$  ( $x \in \mathbf{A}, y \in \mathbf{B}$ );

- Actions  $\begin{array}{ccccc} \mathbf{A}(x',x) & \rightarrow & |x| & \xrightarrow{P(x,y)} & |y| \\ & & \Downarrow & & \\ |x'| & \xrightarrow{\quad} & & \xrightarrow{P(x,y)} & |y| \\ & & \xrightarrow{P(x',y)} & & \end{array} \quad \begin{array}{ccccc} & & P(x,y) & \rightarrow & |y| \\ & & \Downarrow & & \xrightarrow{\mathbf{B}(y,y')} \\ & & & & |y'| \\ & & \xrightarrow{P(x,y')} & & \end{array} \quad \text{in } \mathcal{W}.$

(+Axioms)

# From $\mathcal{V}$ to $\mathcal{W}$

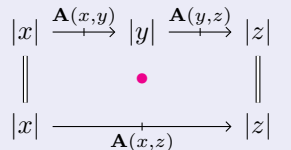


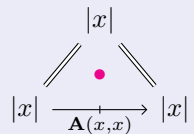
# From $\mathcal{W}$ to $\mathbb{X}$

$\mathbb{X}$ : a VDC

## $\mathbb{X}$ -category $\mathbf{A}$

- A class  $\text{Ob}\mathbf{A}$  of *objects*;
- *Coloring*  $|x| \in \mathbb{X}$  ( $x \in \text{Ob}\mathbf{A}$ );
- *Hom-horizontal arrows*  $|x| \xrightarrow{\text{A}(x,y)} |y|$  in  $\mathbb{X}$  ( $x, y \in \text{Ob}\mathbf{A}$ );

- *Compositions*  
 in  $\mathbb{X}$  ( $x, y, z \in \text{Ob}\mathbf{A}$ );

- *Identities*  
 in  $\mathbb{X}$  ( $x \in \text{Ob}\mathbf{A}$ ). (+Axioms)

# From $\mathcal{W}$ to $\mathbb{X}$

$\mathbb{X}$ : a VDC

## Notation

$\mathcal{H}(\mathbb{X})$  ... the (virtual) bicat. obtained by forgetting all vertical arrows from  $\mathbb{X}$ .

Enrichment in a bicategory		Enrichment in a VDC
$\mathcal{H}(\mathbb{X})$ -categories	=	$\mathbb{X}$ -categories
$\mathcal{H}(\mathbb{X})$ -functors	$\subseteq$	$\mathbb{X}$ -functors
$\mathcal{H}(\mathbb{X})$ -profunctors	=	$\mathbb{X}$ -profunctors

# From $\mathcal{W}$ to $\mathbb{X}$

$\mathbb{X}$ -functor  $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map  $\text{Ob}\mathbf{A} \longrightarrow \text{Ob}\mathbf{B}$ ;  
 $x \longmapsto F^0 x$

- “*Color-comparing*” vertical arrows  $|x| \xrightarrow{F^1 x} |F^0 x|$  in  $\mathbb{X}$  ( $x \in \text{Ob}\mathbf{A}$ );

- $$\begin{array}{ccc}
 |x| & \xrightarrow{\mathbf{A}(x,y)} & |y| \\
 F^1 x \downarrow & \bullet & \downarrow F^1 y \\
 |F^0 x| & \xrightarrow{\mathbf{B}(F^0 x, F^0 y)} & |F^0 y|
 \end{array}
 \quad \text{in } \mathbb{X} \ (x, y \in \text{Ob}\mathbf{A}).$$
(+Axioms)

Enrichment in...  $\left\{ \begin{array}{l} \text{VDC } \mathbb{X} \\ \Downarrow \\ \text{bicategory } \mathcal{W} \\ \Downarrow \\ \text{mon.category } \mathcal{V} \end{array} \right.$

} Functors no longer need to preserve the color of objects.

} Objects are colored with  $\text{Ob}\mathcal{W}$ .



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# Collages of a profunctor

$\mathcal{V}$ : a monoidal category with  $\emptyset$  (the initial preserved by  $\otimes$ )

$\mathbf{A} \xrightarrow{P} \mathbf{B}$ : a  $\mathcal{V}$ -profunctor

## Definition

A **collage** (or *cograph*) of  $P$  is the  $\mathcal{V}$ -category **Col**  $P$ :

$$\text{Ob}(\mathbf{Col} P) := \text{Ob} \mathbf{A} + \text{Ob} \mathbf{B};$$

$$\mathbf{Col} P(x, y) := \begin{cases} \mathbf{A}(x, y) & \text{if } x, y \in \mathbf{A} \\ \mathbf{B}(x, y) & \text{if } x, y \in \mathbf{B} \\ P(x, y) & \text{if } x \in \mathbf{A}, y \in \mathbf{B} \\ \emptyset & \text{if } x \in \mathbf{B}, y \in \mathbf{A} \end{cases}$$

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{P} & \mathbf{B} \\ \text{coproj.} \searrow & \cdot & \swarrow \text{coproj.} \\ & \mathbf{Col} P & \end{array} \quad \text{in } \mathcal{V}\text{-Prof}$$

## Universal property (vertical)

$$\begin{array}{ccc}
 \text{Col } P & & \\
 \downarrow & & \\
 \mathbf{C} & & 
 \end{array}
 \quad \text{in } \mathcal{V}\text{-Prof}
 \quad \parallel \quad
 \begin{array}{ccc}
 \mathbf{A} & \xrightarrow{P} & \mathbf{B} \\
 & \searrow \quad \swarrow & \\
 & \bullet & \\
 & \searrow \quad \swarrow & \\
 & \mathbf{C} & 
 \end{array}
 \quad \text{in } \mathcal{V}\text{-Prof}$$

## Universal property (horizontal)

$$\begin{array}{ccc}
 \text{Col } P \multimap \mathbf{C} & \text{in } \mathcal{V}\text{-Prof} & \\
 \parallel & & \parallel \\
 \mathbf{A} \xrightarrow{P} \mathbf{B} \multimap \mathbf{C} & \text{in } \mathcal{V}\text{-Prof} & 
 \end{array}
 \quad \parallel \quad
 \begin{array}{ccc}
 \mathbf{A} \xrightarrow{P} \mathbf{B} \multimap \mathbf{C} & & \\
 \parallel & \bullet & \parallel \\
 \mathbf{A} \multimap \mathbf{C} & & 
 \end{array}$$
  

$$\begin{array}{ccc}
 \mathbf{C} \multimap \text{Col } P & \text{in } \mathcal{V}\text{-Prof} & \\
 \parallel & & \parallel \\
 \mathbf{C} \multimap \mathbf{A} \xrightarrow{P} \mathbf{B} & \text{in } \mathcal{V}\text{-Prof} & 
 \end{array}
 \quad \parallel \quad
 \begin{array}{ccc}
 \mathbf{C} \multimap \mathbf{A} \xrightarrow{P} \mathbf{B} & & \\
 \parallel & \bullet & \parallel \\
 \mathbf{C} \multimap \mathbf{B} & & 
 \end{array}$$

$\text{Col } P$  has also universal properties in the three directions:

$$\begin{array}{c}
 \cdot \multimap \text{Col } P \multimap \cdot \\
 \downarrow \\
 \cdot
 \end{array}$$

# Collages in general

S: a set

## Notation

$\mathbb{IS}$  ... the VDC described by:

- $\text{Ob}(\mathbb{IS}) := S$ ;
- For  $i, j \in S$ , there is a unique horizontal arrow  $i \xrightarrow{!ij} j$  in  $\mathbb{IS}$ ;

- For  $i_0, \dots, i_n \in S$ , there is a unique cell

$$\begin{array}{c} i_0 \xrightarrow{!i_0 i_1} i_1 \xrightarrow{!i_1 i_2} \dots \xrightarrow{!i_{n-1} i_n} i_n \\ \parallel \qquad \qquad \qquad !i_0 \dots i_n \qquad \qquad \parallel \\ i_0 \xrightarrow{\qquad \qquad \qquad !i_0 i_n \qquad \qquad \qquad} i_n \end{array}$$

A virtual double (VD)-functor  $\mathbb{IS} \longrightarrow \mathbb{X}\text{-Prof}$  is equivalent to the following data:

- $\mathbf{A}_i$ :  $\mathbb{X}$ -categories ( $i \in S$ )
- $\mathbf{A}_i \xrightarrow{P_{ij}} \mathbf{A}_j$ :  $\mathbb{X}$ -profunctors ( $i, j \in S$ )

- $$\begin{array}{ccc} & \mathbf{A}_i & \\ // & & \\ \mathbf{A}_i & \xrightarrow{P_{ii}} & \mathbf{A}_i \\ & \eta_i & \end{array} \quad (i \in S)$$

- $$\begin{array}{ccc} \mathbf{A}_i & \xrightarrow{P_{ij}} & \mathbf{A}_j \xrightarrow{P_{jk}} \mathbf{A}_k \\ \parallel & & \parallel \\ \mathbf{A}_i & \xrightarrow{P_{ik}} & \mathbf{A}_k \end{array} \quad (i, j, k \in S)$$

(+Axioms)

# Collages in general

$P := (\mathbf{A}_i, P_{ij}, \eta_i, \mu_{ijk})_{i,j,k \in S}$  : a VD-functor  $\mathbb{S} \xrightarrow{P} \mathbb{X}\text{-Prof}$

## Definition

A **collage** of  $P$  is the  $\mathbb{X}$ -category **Col**  $P$ :

$$\text{Ob}(\mathbf{Col} P) := \coprod_{i \in S} \text{Ob} \mathbf{A}_i;$$

$$\mathbf{Col} P(x, y) := P_{ij}(x, y) \quad (\text{where } x \in \mathbf{A}_i, y \in \mathbf{A}_j).$$

$$\begin{array}{ccc} \mathbf{A}_i & \xrightarrow{P_{ij}} & \mathbf{A}_j \\ \searrow \xi_i & \downarrow \xi_{ij} & \swarrow \xi_j \\ & \mathbf{Col} P & \end{array} \quad \text{in } \mathbb{X}\text{-Prof} \quad (\text{coprojections})$$

Again, **Col**  $P$  has universal properties in the three directions:

$$\begin{array}{ccc} \cdot & \twoheadrightarrow & \mathbf{Col} P \twoheadrightarrow \cdot \\ & & \downarrow \\ & & \cdot \end{array}$$

# Versatile colimits

$\mathbb{K}$ : a VDC,  $\mathbb{L}$ : a unital VDC,  $\mathbb{K} \xrightarrow{F} \mathbb{L}$ : a VD-functor

## Definition (informal)

A **versatile colimit** of  $F$  is a “cocone”

$$\left\{ \begin{array}{c} FA \xrightarrow{Fu} FB \\ \searrow \quad \swarrow \\ \xi_A \quad \xi_B \\ \quad \Xi \end{array} \right\} \text{ in } \mathbb{L} \quad \left( A \xrightarrow{u} B \text{ in } \mathbb{K} \right)$$

having the universal property in the three directions  $\begin{array}{ccc} \cdot & \twoheadrightarrow & \Xi \\ & & \downarrow \\ & & \cdot \end{array} \text{ in } \mathbb{L}.$

## Example

- ① **Versatile collages** ( $:=$  vers.colim. of shapes  $\mathbb{K} = \mathbb{IS}$ )
- ② **Versatile coproducts** ( $:=$  vers.colim. of discrete shapes)

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# The object classifier in $\mathcal{V}\text{-}\mathbb{P}\text{rof}$

$(\mathcal{V}, \otimes, I)$ : a monoidal category

## Notation

The *unit  $\mathcal{V}$ -category* **I**:

$$\text{Ob}\mathbf{I} := \{*\}, \quad \mathbf{I}(*, *) := I.$$

## The unit $\mathcal{V}$ -category classifies objects

For every  $\mathcal{V}$ -category **A**,

$$\begin{array}{c} \mathbf{I} \\ \downarrow \\ \mathbf{A} \end{array} \quad \text{in } \mathcal{V}\text{-}\mathbb{P}\text{rof} \quad \Bigg\| \quad \text{an object } x \in \mathbf{A}$$



# Object classifiers in $\mathcal{W}\text{-Prof}$

$\mathcal{W}$ : a bicategory

## Notation

The unit  $\mathcal{W}$ -category  $\mathbf{I}_c$  for  $c \in \mathcal{W}$ :

$$\text{Ob}\mathbf{I}_c := \{*\} \text{ with } |*| := c, \quad \mathbf{I}_c(*, *) := (c \xrightarrow{\text{id}} c \text{ in } \mathcal{W})$$

## The unit $\mathcal{W}$ -categories classify objects

For every  $\mathcal{W}$ -category  $\mathbf{A}$ ,

$$\begin{array}{c} \mathbf{I}_c \\ \downarrow \\ \mathbf{A} \end{array} \text{ in } \mathcal{W}\text{-Prof} \quad \Bigg\| \quad \text{an object } x \in \mathbf{A} \text{ s.t. } |x| = c$$

# Object classifiers in $\mathbb{X}\text{-Prof}$

$\mathbb{X}$ : a unital VDC

## Notation

The *unit*  $\mathbb{X}$ -category  $\mathbf{I}_c$  for  $c \in \mathbb{X}$ :

$$\text{Ob}\mathbf{I}_c := \{*\} \text{ with } |*| := c, \quad \mathbf{I}_c(*, *) := (c \neq c \text{ in } \mathbb{X})$$

The unit  $\mathbb{X}$ -categories do not classify objects.

For every  $\mathbb{X}$ -category  $\mathbf{A}$ ,

$$\begin{array}{c} \mathbf{I}_c \\ \downarrow \\ \mathbf{A} \end{array} \text{ in } \mathbb{X}\text{-Prof} \quad \parallel \quad \begin{array}{c} \text{an object } x \in \mathbf{A}, \text{ and} \\ \text{a vertical arrow } \begin{array}{c} c \\ \downarrow \\ |x| \end{array} \text{ in } \mathbb{X} \end{array} \quad (\textit{semi-object})$$

$\rightsquigarrow$  The unit  $\mathbb{X}$ -categories classify *semi-objects*.

# The embedding

$\mathbb{X}$ : a unital VDC

## Proposition

There is an “embedding”  $\mathbb{X} \xhookrightarrow{\mathbf{I}_\bullet} \mathbb{X}\text{-Prof}$ . That is:

$$\begin{array}{ccc} c_0 & \xrightarrow{\quad} c_1 & \xrightarrow{\quad} \cdots \xrightarrow{\quad} c_n \\ \downarrow & & \downarrow \\ d & \xrightarrow{\quad} & e \end{array} \quad \text{in } \mathbb{X} \quad \parallel \quad \begin{array}{ccc} \mathbf{I}_{c_0} & \xrightarrow{\quad} \mathbf{I}_{c_1} & \xrightarrow{\quad} \cdots \xrightarrow{\quad} \mathbf{I}_{c_n} \\ \downarrow & & \downarrow \\ \mathbf{I}_d & \xrightarrow{\quad} & \mathbf{I}_e \end{array} \quad \text{in } \mathbb{X}\text{-Prof}$$

In what follows, we will consider  $\mathbb{X} \subseteq \mathbb{X}\text{-Prof}$ .

# Toward the characterization: density

$\mathbb{X} \subseteq \mathbb{X}\text{-Prof}$  ( $\mathbb{X}$ : a unital VDC)

## Observation I

Every  $\mathbb{X}$ -category is a versatile collage of objects from  $\mathbb{X}$ .

$\therefore$

$$\frac{\mathbf{A}: \text{an } \mathbb{X}\text{-category}}{\mathbb{I}Ob\mathbf{A} \xrightarrow{|\cdot|_{\mathbf{A}}} \mathbb{X} \subseteq \mathbb{X}\text{-Prof}: \text{a VD-functor}} \quad \begin{array}{c} \text{\color{red} \Downarrow} \\ \text{\color{red} \Uparrow} \end{array} \quad \text{\color{red} Taking the versatile colimit in } \mathbb{X}\text{-Prof}$$

The VD-functor  $|\cdot|_{\mathbf{A}}$  is given by:

$$\begin{array}{ccc} \begin{array}{c} x_0 \xrightarrow{!} x_1 \xrightarrow{!} \dots \xrightarrow{!} x_n \\ \parallel \qquad \qquad \parallel \\ x_0 \xrightarrow{\quad ! \quad} x_n \\ \text{in } \mathbb{I}Ob\mathbf{A} \end{array} & \mapsto & \begin{array}{c} |x_0| \xrightarrow{\mathbf{A}(x_0, x_1)} |x_1| \xrightarrow{\mathbf{A}(x_1, x_2)} \dots \xrightarrow{\mathbf{A}(x_{n-1}, x_n)} |x_n| \\ \parallel \qquad \qquad \qquad \text{"composition" in } \mathbf{A} \qquad \qquad \parallel \\ |x_0| \xrightarrow{\mathbf{A}(x_0, x_n)} |x_n| \\ \text{in } \mathbb{X} \end{array} \end{array}$$

# Toward the characterization: atomicity

## Observation II

In  $\mathbb{X}\text{-Prof}$ , the unit  $\mathbb{X}$ -categories can be characterized by “atomicity” w.r.t. versatile collages.

## Definition

$\mathbb{L}$ : a unital VDC.

$L \in \mathbb{L}$ : **collage-atomic**  $\stackrel{\text{def}}{\iff}$  For every (large) versatile collage  $\mathbf{Col} P$  in  $\mathbb{L}$ , vertical arrows  $L \rightarrow \mathbf{Col} P$  uniquely factor through a unique coprojection.

$$\begin{array}{ccc} & L & \\ \swarrow \exists! & \downarrow \vee & \\ P_i & \downarrow \text{coproj.} & \mathbf{Col} P \end{array} \quad \text{in } \mathbb{L} \quad (\exists! i)$$

## Proposition

$\mathbb{X}$ : a unital VDC.

In  $\mathbb{X}\text{-Prof}$ , **collage-atomic**  $\iff$  vertically isomorphic to some  $\mathbf{I}_c$ .

# The characterization

## Theorem

TEAE for a unital VDC  $\mathbb{L}$ :

- ① There is an equivalence  $\mathbb{L} \simeq \mathbb{X}\text{-Prof}$  for some unital VDC  $\mathbb{X}$ .  
(in the 2-category of unital VDCs)
- ②
  - ◇  $\mathbb{L}$  has (large) versatile collages;
  - ◇ Every object in  $\mathbb{L}$  is a versatile collage of collage-atomic objects.

**Sketch of proof:** The only non-trivial part is  $2. \implies 1.$

Let  $\mathbb{X} := \{\text{collage-atomic objs.}\} \subseteq \mathbb{L}$ .

We can construct an adjunction:  $\mathbb{X}\text{-Prof} \begin{array}{c} \xrightarrow{R} \\ \perp \\ \xleftarrow{N} \end{array} \mathbb{L}.$

Construction of  $R$ : Regarding  $\mathbf{A} \in \mathbb{X}\text{-Prof}$  as  $\mathbb{I}\text{Ob}\mathbf{A} \longrightarrow \mathbb{X} \subseteq \mathbb{L}$ , we take the vers.colim. of it.

Construction of  $N$ : Suppose  $L \in \mathbb{L}$  is a vers.colim. of  $\mathbb{I}\mathbf{S} \xrightarrow{P} \mathbb{X} \subseteq \mathbb{L}$ . Define  $L \xrightarrow{N} "(S, P)" \in \mathbb{X}\text{-Prof}.$

$R \circ N \cong \text{Id}$  is trivial.  $\text{Id} \cong N \circ R$  follows from **uniqueness of  $(S, P)$** . □

Why is  $\mathbb{IS} \xrightarrow{P} \mathbb{X}$  unique for each  $L \in \mathbb{L}$ ?

- The uniqueness of both  $S$  and the object part of  $P$   
 ... From the **collage-atomicity** of  $\mathbb{X}$ .
- The uniqueness of the horizontal arrow & cell part of  $P$   
 ... From **strongness theorem** on versatile collages:

### Strongness theorem

$\mathbb{L}$ : a unital VDC.  $\mathbf{Col} P ( \in \mathbb{L} )$ : a versatile collage of  $\mathbb{IS} \xrightarrow{P} \mathbb{L}$ .  
 $\implies$  Coprojections of  $\mathbf{Col} P$  form *cartesian cells*:

$$\begin{array}{ccc}
 Pi & \xrightarrow{P!_{ij}} & Pj \\
 \swarrow \text{coproj.} & \text{cart} & \nwarrow \text{coproj.} \\
 & \mathbf{Col} P &
 \end{array}
 \quad \text{in } \mathbb{L} \quad (i, j \in S).$$

Thank you!



Today's slides



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