On the decomposition of a strong epimorphism into regular epimorphisms

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⤳ slides

Recall

Every homomorphism $f\colon A \to B$ between groups can be decomposed into a surjective hom. followed by an injective hom.

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
 & \searrow \quad \nearrow & \\
 & \operatorname{Im} f &
\end{array}
\qquad \text{in } \mathbf{Grp}
$$

Regular epimorphisms = morphisms being the coequalizer of some parallel pair of morphisms.

$$
\cdot \overset{\longrightarrow}{\underset{\smile}{}} \cdot \xrightarrow{\text{regular epi}} \cdot
$$

(eg. regular epi = surjective hom. in $\mathbf{Grp}$)
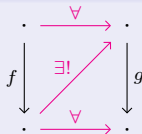

Monomorphisms = morphisms s.t. $f \circ -$ is injective.

$$
\cdot \mathrel{\substack{g \\ \longrightarrow \\ \longrightarrow \\ h}} \cdot \xhookrightarrow[f]{\text{mono}} \cdot \quad (\text{cofork}) \qquad \Longrightarrow \qquad g = h
$$

(eg. mono = injective hom. in $\mathbf{Grp}$)

Definition

$$f \perp g$$
(f and g are orthogonal)

$\overset{\text{def}}{\Longleftrightarrow}$

Example

In every category, {regular epi} $\perp$ {mono}.

$\mathbf{E} \subseteq \mathrm{Mor}\mathscr{C}$: a class of morphisms in a category $\mathscr{C}$.

Definition

$\mathbf{E}$: iso-closed $\quad\overset{\mathsf{def}}{\Leftrightarrow}$

- $\{\mathsf{iso}\} \subseteq \mathbf{E}$.
- $\begin{array}{ccc} \cdot & \cong & \cdot \\ {\scriptstyle\mathbf{E}\ni f}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \cong & \cdot \end{array} \quad\Longrightarrow\quad g \in \mathbf{E}.$

Definition

$\mathbf{E}$: iso-closed.

$\mathscr{C}$ admits orthogonal $\mathbf{E}$-factorizations $\overset{\mathsf{def}}{\Leftrightarrow}$ Every morphism f can be decomposed as

$$\begin{array}{ccc} \cdot & \overset{f}{\longrightarrow} & \cdot \\ {\scriptstyle\mathbf{E}\ni e}\searrow & & \nearrow{\scriptstyle m\in\mathbf{r}(\mathbf{E})} \\ & \cdot & \end{array} \quad \text{where } \mathbf{r}(\mathbf{E}) := \{m \mid \mathbf{E} \perp m\}.$$

Example

$\mathbf{Grp}$ admits orthogonal $\{\text{regular epi}\}$-factorizations.

The fundamental homomorphism theorem

$$A/\operatorname{Ker} f \cong \operatorname{Im} f \qquad (f\colon A \to B)$$

The kernel pair of f:

$$
\begin{array}{ccc}
\operatorname{Kp} f & \xrightarrow{\ \pi_2\ } & A \\
{\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
A & \xrightarrow{\ f\ } & B
\end{array}
\qquad \text{(pullback)}
$$

The coimage of f:

$$\operatorname{Kp} f \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} A \twoheadrightarrow \operatorname{Coim} f \qquad \text{(coequalizer)}$$

The fundamental homomorphism theorem (categorically)

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
 \searrow & & \nearrow {\scriptstyle \text{mono}} \\
 & \operatorname{Coim} f &
\end{array}
$$

This holds in any categories of equational algebras.

Fact

TFAE for a category $\mathscr{C}$ with finite limits and colimits:

1. $\mathscr{C}$ admits orthogonal {regular epi}-factorizations.
2. "The fundamental homomorphism theorem holds in $\mathscr{C}$." That is, for every $f\colon A \to B$ in $\mathscr{C}$, the canonical morphism $\operatorname{Coim} f \to B$ is monic.
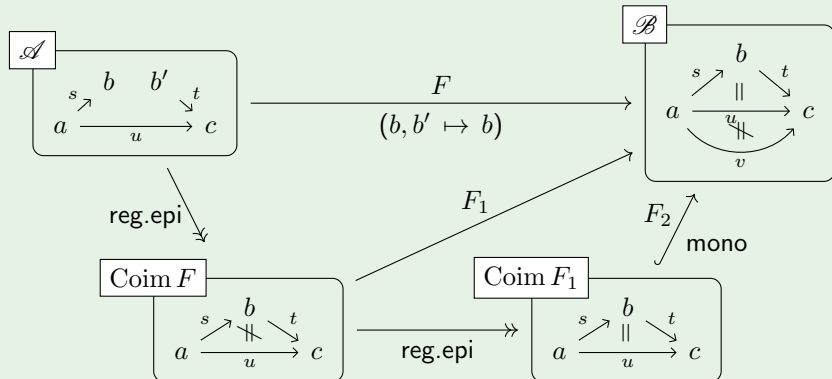
Question

Does the fundamental hom. thm. hold in any category?
$\rightsquigarrow$ No.

Counter example

In **Cat**,

Definition

A (regular) decomposition of $A \xrightarrow{f} B$ in $\mathscr{C}$ consists of:

- A functor $\mathbf{Ord} \xrightarrow{A_\bullet} \mathscr{C}$ with $A_0 = A$;
- A cocone $f_\bullet = (A_\alpha \xrightarrow{f_\alpha} B)$ over $A_\bullet$ with $f_0 = f$
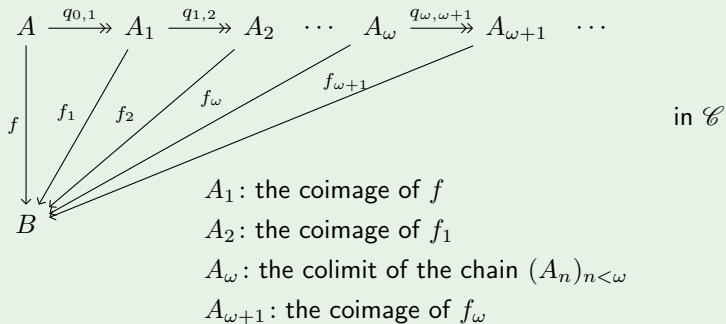
satisfying $(\forall \alpha, \ A_{\alpha,\alpha+1}:$ regular epi$)$ and $(\forall \gamma:$ limit type, $A_\gamma \cong \mathrm{Colim}_{\alpha < \gamma} A_\alpha)$.

$$A = A_0 \xtwoheadrightarrow{A_{0,1}} A_1 \longrightarrow \cdots \longrightarrow A_\alpha \xtwoheadrightarrow{A_{\alpha,\alpha+1}} A_{\alpha+1} \longrightarrow \cdots$$

in $\mathscr{C}$

with $f = f_0$, f_1, f_α, $f_{\alpha+1}$ mapping to B.

Definition

1. A decomp. $(A_\bullet, f_\bullet)$ stabilizes at $\alpha \overset{\mathsf{def}}{\Leftrightarrow} f_\beta:$ mono $(\alpha \leq \forall \beta)$ $(\Leftrightarrow f_\alpha:$ mono$)$
2. The smallest α above is called the length of $(A_\bullet, f_\bullet)$. (if exists)

$\mathscr{C}$: a category with kernel pairs and colimits.

$A \xrightarrow{q_{0,1}} A_1 \xrightarrow{q_{1,2}} A_2 \cdots A_\omega \xrightarrow{q_{\omega,\omega+1}} A_{\omega+1} \cdots$ in $\mathscr{C}$

A_1: the coimage of f

A_2: the coimage of f_1

A_ω: the colimit of the chain $(A_n)_{n<\omega}$

$A_{\omega+1}$: the coimage of f_ω

$\rightsquigarrow$ This gives a decomposition of f. (canonical decomposition)

Definition

① The canonical decomposition number $\sigma(f) \coloneqq$ the length of the canonical decomposition of f. If it doesn't stabilizes, $\sigma(f)$ is left to be undefined.

② If $\sigma(f)$ is defined for every f in $\mathscr{C}$, the global canonical decomposition number $\sigma(\mathscr{C}) \coloneqq \min\{\alpha \mid \sigma(f) < \alpha \text{ for every } f \text{ in } \mathscr{C}\}$

Remark

1. $\sigma(\mathscr{C}) = 0 \iff \mathscr{C} = \varnothing$.
2. $\sigma(\mathscr{C}) \leq 1 \iff \sigma(f) = 0 \ (\forall f) \iff f$: monic $(\forall f)$.
3. $\sigma(\mathscr{C}) \leq 2 \iff \sigma(f) \leq 1 \ (\forall f) \iff$ "The fund. hom. thm. holds in $\mathscr{C}$."

Example

1. $\sigma(\mathbf{Set}) = \sigma(\mathbf{Grp}) = 2$.
2. $\sigma(\mathbf{Cat}) = 3$.
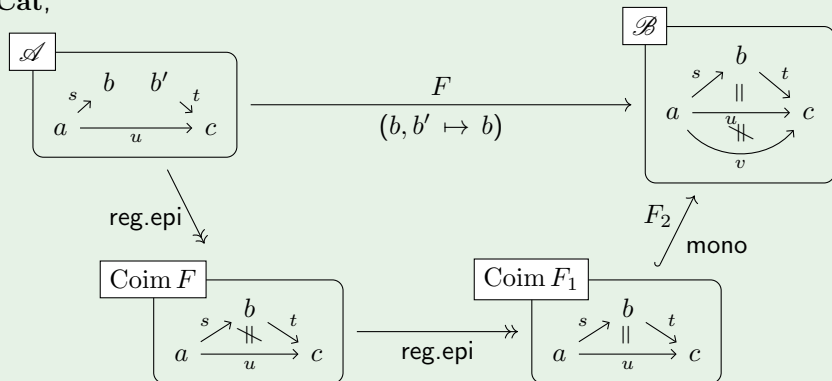3. $\sigma(n\text{-}\mathbf{Cat}) = n + 2$.

Theorem [GU71]

$\mathscr{C}$: locally κ-presentable $\implies \sigma(f)$ is defined and $\sigma(f) \leq \kappa$ holds. $(\forall f$ in $\mathscr{C})$
(Equivalently, $\sigma(\mathscr{C}) \leq \kappa + 1$)

Minimum decomposition problem

Recall (a canonical decomposition in **Cat**)

In **Cat**,

Question

Is there a shorter decomposition of F than the canonical one in **Cat**? That is, can F be decomposed as $\cdot \xrightarrow{\text{reg.epi}} \cdot \xhookrightarrow{\text{mono}} \cdot$?

Minimum decomposition problem

Definition

The minimum decomposition number $\delta(f) \coloneqq$ the shortest possible length of decompositions of f. (if at least one decomposition exists)

Question (restated)

Does $\sigma(f) = \delta(f)$ hold?

Theorem (Canonical vs minimum [KN])

In a category with kernel pairs and colimits,

1. $\sigma(f)$ is defined $\iff$ $\delta(f)$ is defined.
2. Whenever they are defined, $\sigma(f) = \delta(f)$ holds.

That is, the canonical (regular) decomposition is minimum.

Observation

In a category with kernel pairs,

q: regular epi. $\iff$ q: the coequalizer of the kernel pair of q.

- "coequalize the kernel pair of q" $\iff$ coequalize any pair of morphisms that coequalized by q. $\cdots$ $(\star)$ (*the kernel condition associated with* q)

$\therefore$ q: regular epi. $\iff$ q: the most universal among morphisms satisfying $(\star)$.

In a category with binary coproducts,

$$X \underset{v}{\overset{u}{\rightrightarrows}} A \text{ is coequalized by } \begin{array}{c} A \\ \downarrow q \\ B \end{array} \iff \begin{array}{ccc} X + X & \xrightarrow{(u,v)} & A \\ \nabla \downarrow & & \downarrow q \\ X & \underset{\exists}{\dashrightarrow} & B \end{array}$$

$$A \xrightarrow{f} C \text{ satisfies } (\star) \iff \forall X, \quad \begin{array}{ccc} X + X & \xrightarrow{\forall} & A \\ \nabla \downarrow & & \downarrow q \\ X & \xrightarrow{\forall} & B \\ & \searrow_{\exists} & \downarrow f \\ & & C \end{array}$$
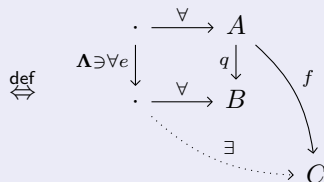
Idea

Replace $\{X + X \xrightarrow{\nabla} X\}_{X \,:\, \mathrm{obj}}$ with another class of epimorphisms.

Λ-regular epimorphisms

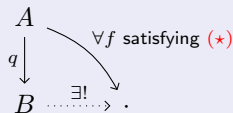**Λ**: a class of epimorphisms.   Fix $A \xrightarrow{q} B$.

Definition [KN]

$A \xrightarrow{f} C$ satisfies $(\star)$
(the **Λ**-*kernel condition assoc. with* q)

$\overset{\text{def}}{\Leftrightarrow}$

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\ \forall\ } & A \\
{\scriptstyle \mathbf{\Lambda}\ni\forall e}\downarrow & & \downarrow{\scriptstyle q} \\
\cdot & \xrightarrow{\ \forall\ } & B
\end{array}
$$

with f and $\exists$ dotted arrow to C

Definition [KN]

q: **Λ**-regular epi $\overset{\text{def}}{\Leftrightarrow}$

$$
\begin{array}{ccc}
& A & \\
{\scriptstyle q}\downarrow & & {\scriptstyle \forall f\ \text{satisfying}\ (\star)} \\
B & \dashrightarrow[\exists!]{} & \cdot
\end{array}
$$

Example I

Example

Pos: the category of posets and order-preserving maps.

$$\Lambda := \left\{ \boxed{\bullet \quad \star} \xrightarrow{\iota} \boxed{\bullet < \star} \quad \text{in } \mathbf{Pos} \right\}$$

$\rightsquigarrow$ Λ-regular epis $=$ *surjective* order-preserving maps ($=$ epis)

Example

Cat: the category of small categories and functors.

$$\Lambda := \left\{ \boxed{\bullet \to \star} \xrightarrow{L} \boxed{\bullet \cong \star} \quad \text{in } \mathbf{Cat} \right\}$$

$\rightsquigarrow$ Λ-regular epis $=$ *strict localizations*

Example II

Example

Met: the category of metric spaces and non-expansive maps.

$$\Lambda := \left\{ \boxed{\bullet \overset{r+1}{-} \bigstar} \overset{e_r}{\longrightarrow} \boxed{\bullet \overset{r}{-} \bigstar} \quad \text{in } \mathbf{Met} \right\}_{r \in \mathbb{Q}_{>0}}$$

$\rightsquigarrow q$: Λ-regular epi $\iff d(x,y) - d(q(x), q(y)) \leq 1 \ (\forall x, y).$
(*a shrinking at most 1*)

Example

Top: the category of topological spaces and continuous maps.

$$\Lambda := \left\{ K \overset{!_K}{\to} 1 \quad \text{in } \mathbf{Top} \right\}_{K:\text{connected space}}$$

$\rightsquigarrow q$: Λ-regular epi $\iff q^{-1}(x)$: connected $(\forall x).$ (= *monotone map*)

Orthogonal {Λ-regular epi}-factorizations

category	Λ	$\mathbf{E} = \{\Lambda$-regular epis$\}$	$r(\mathbf{E})$
any category with bin.coprod.	{codiagonals}	{regular epis}	{monos}
Pos	$\{\iota\}$	{surjections}	{embeddings}
Cat	$\{L\}$	{strict localizations}	{conservatives}
Met	$\{e_r\}_r$	{shrinkings at most 1}	{isometries}
Top	$\{!_K\}_K$	{monotone maps}	{*light maps*}

Pos admits orthogonal {surj}-factorizations.
However, neither **Cat**, **Met**, nor **Top** admits orthogonal **E**-factorizations.

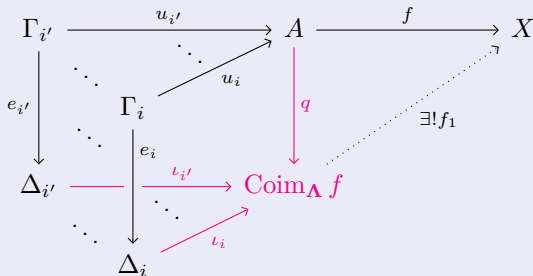# Coimages in terms of $\mathbf{\Lambda}$-regular epis

Definition

Let $A \xrightarrow{f} X$.

$$\mathcal{S} := \left\{ (e_i, u_i) \;\middle|\; \begin{array}{c} \Gamma_i \xrightarrow{u_i} A \xrightarrow{f} X \\ \mathbf{\Lambda} \ni e_i \downarrow \quad\quad\;\; \exists \\ \Delta_i \end{array} \right\}$$

Then, $\mathrm{Coim}_{\mathbf{\Lambda}} f := \mathrm{Colim}\, \mathcal{S}$. (multiple pushout)

Proposition [KN]

1. $\mathbf{\Lambda} \subseteq \mathbf{Reg}(\mathbf{\Lambda})$ ($:= \{\mathbf{\Lambda}\text{-regular epis}\}$).

2. Moreover, $\mathbf{Reg}(\mathbf{\Lambda})$ is the "multiple pushout closure" of $\mathbf{\Lambda}$. That is,
   q: $\mathbf{\Lambda}$-reg.epi $\iff$ q: obtained by a mult.pushout of some family of spans
   (e_i, u_i) s.t. $e_i \in \mathbf{\Lambda}$.

In particular,

$$
\begin{array}{ccc}
 & A \xrightarrow{\ f\ } X & \\
\mathbf{\Lambda}\text{-regular epi} \rightsquigarrow q \downarrow & \nearrow{\scriptstyle f_1} & \\
 & \mathrm{Coim}_{\mathbf{\Lambda}}\, f &
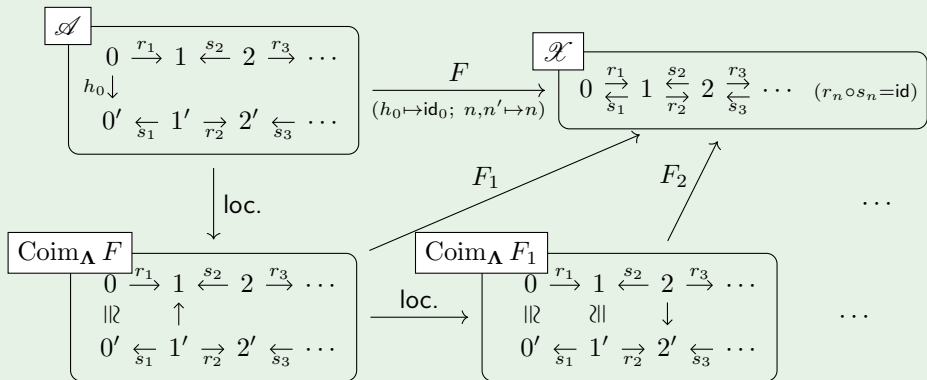\end{array}
$$

Replacing $\left(\begin{array}{ccc} \{\text{regular epis}\} & \rightsquigarrow & \mathbf{E} = \mathbf{Reg}(\mathbf{\Lambda}) \\ \{\text{monos}\} & \rightsquigarrow & \mathbf{r}(\mathbf{E}) \end{array} \right)$,

Definition

$\sigma_{\mathbf{E}}(f) := \min\{\alpha \mid f_\alpha \in \mathbf{r}(\mathbf{E})\}$. (the *canonical $\mathbf{E}$-decomposition number*)

Example

In **Cat** with $\mathbf{E} := \{\text{strict localizations}\}$,

$\leadsto \sigma_{\mathbf{E}}(F) = \omega.$

Theorem (Small object argument)

$\mathscr{C}$: locally κ-presentable; $\boldsymbol{\Lambda}$: small; $\operatorname{dom} f, \operatorname{cod} f$: κ-presentable ($\forall f \in \boldsymbol{\Lambda}$)
$\implies \sigma_{\mathbf{E}}(\mathscr{C}) \leq \kappa + 1$, where $\mathbf{E} \coloneqq \mathbf{Reg}(\boldsymbol{\Lambda})$.

Corollary

$\sigma_{\mathbf{E}}(\mathbf{Cat}) = \omega + 1$, where $\mathbf{E} \coloneqq \{\text{strict localizations}\}$.

Minimum decomposition problem

Let $\mathbf{E} \coloneqq \mathbf{Reg}(\mathbf{\Lambda})$.

We can also generalize $\left(\begin{array}{ccc} \text{regular decomposition} & \rightsquigarrow & \mathbf{E}\text{-decomposition} \\ \text{minimum dec.num. } \delta(f) & \rightsquigarrow & \delta_{\mathbf{E}}(f) \end{array} \right)$.

Then, $\sigma_{\mathbf{E}}$ and $\delta_{\mathbf{E}}$ still coincide:

Theorem (Canonical vs minimum [KN])

Let $\mathscr{C}$: locally small and cocomplete, $\mathbf{E} \coloneqq \mathbf{Reg}(\mathbf{\Lambda})$ with $\mathbf{\Lambda}$: small.

1. $\sigma_{\mathbf{E}}(f)$ is defined $\iff \delta_{\mathbf{E}}(f)$ is defined.
2. Whenever they are defined, $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$ holds.

In the previous part, we did:

1. Define the class of morphisms $\mathbf{E} := \mathbf{Reg}(\mathbf{\Lambda})$;
2. Define the *coimage-factorizations* in terms of $\mathbf{E}$;
3. Define $\sigma_{\mathbf{E}}(f)$ as the length of the iterated coimage-factorization of f;
4. Compare $\sigma_{\mathbf{E}}(f)$ with the minimum decomposition number $\delta_{\mathbf{E}}(f)$.

Idea

Abstracting a class $\mathbf{E}$ with coimage-factorizations, we can follow the same story from 2. to 4.

Definition [MT82]

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array} \quad \text{and} \quad \begin{array}{ccc} A' & \xrightarrow{a} & A \\ & & \downarrow g \\ & & B \end{array} \quad \text{are \color{red}locally orthogonal} \quad \overset{\mathsf{def}}{\Leftrightarrow}$$ (written $f \perp^a g$)

$$\begin{array}{ccccc} X & \xrightarrow{\forall} & A' & \xrightarrow{a} & A \\ f\downarrow & & \downarrow {\exists!} & & \downarrow g \\ Y & \xrightarrow[\forall]{} & & & B \end{array}$$

RecallDefinition [MT82]

$\mathbf{E}$: iso-closed.

$\mathscr{C}$ \color{red}admits locally orthogonal $\mathbf{E}$-factorizations$\color{black} \overset{\mathsf{def}}{\Leftrightarrow} \forall f$ can be decomposed as

$$\begin{array}{ccc} & \xrightarrow{f} & \cdot \\ {\mathbf{E}\ni e}\searrow & & \nearrow{m} \\ & \cdot & \end{array} \quad \text{with } \mathbf{E} \perp^e m.$$

Example

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ {e}\searrow & & \nearrow{m} \\ & \cdot & \end{array}$$ is the coimage-factorization in terms of $\mathbf{E} = \mathbf{Reg}(\Lambda)$

$\iff$ it is a locally orthogonal $\mathbf{E}$-factorization, i.e., $e \in \mathbf{E}$ and $\mathbf{E} \perp^e m$.

Locally orthogonal factorizations subsume the coimage-factorization!

Theorem [Tho83; KN]

$\mathscr{C}$: co-well-powered, having small colimits and products.
TFAE for a class $\mathbf{E} \subseteq \{\text{epis}\}$:

1. $\mathscr{C}$ admits locally orthogonal $\mathbf{E}$-factorizations.

2. $\mathbf{E}$ is closed under multiple pushout.

3. $\mathbf{E} = \mathbf{Reg}(\mathbf{\Lambda})$ for some $\mathbf{\Lambda} \subseteq \{\text{epis}\}$.

Question

Is there a non-epimorphic example?
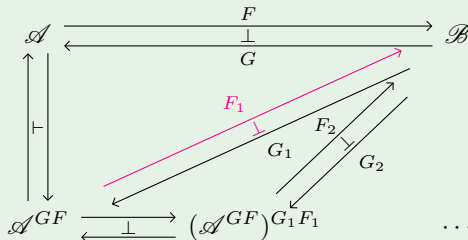
The Eilengerg–Moore decomposition (monadic towers)

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Example

AdjCAT: the category of (large) categories and left adjoints.
$\mathbf{E} := \{$monadic adjunctions$\}$.
  $\rightsquigarrow$ **AdjCAT** admits locally orthogonal $\mathbf{E}$-factorizations [MT82].

This stabilizes at $\alpha \overset{\text{def}}{\Leftrightarrow} \mathbf{E} \perp F_\alpha \Leftrightarrow F_\alpha$: fully faithful
  $\rightsquigarrow \sigma_{\mathbf{E}}(F) =$ the *monadic length of $F \dashv G$*.

</div>

Minimum decomposition problem

Theorem (Canonical vs minimum, the most general [KN])

$\mathscr{C}$: a category with locally orthogonal $\mathbf{E}$-factorizations and colimits of chains.
f: a morphism in $\mathscr{C}$.

1. $\sigma_{\mathbf{E}}(f)$ is defined $\iff$ $\delta_{\mathbf{E}}(f)$ is defined.
2. $\delta_{\mathbf{E}}(f)$: 0 or successor $\implies$ $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$.
3. $\delta_{\mathbf{E}}(f)$: limit $\implies$ $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$ or $\delta_{\mathbf{E}}(f) + 1$.

Corollary

The monadic length coincides with the shortest possible length of monadic decompositions if they are finite.

Thank you!

slides ⤳

This talk is based on the first half of [KN].

References

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