Double categories of profunctors

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← Today's slides

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Abstract

$\mathbb{X}$: a "double category" (*unital virtual double category*)

$\rightsquigarrow$

The "double category" $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$:

- $\mathbb{X}$-enriched categories;
- $\mathbb{X}$-enriched functors;
- $\mathbb{X}$-enriched profunctors.

Goal

To characterize the "double categories" $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$.

Enrichment in a monoidal category

$(\mathscr{V}, \otimes, I)$: a monoidal category

$\mathscr{V}$-category $\mathbf{A}$

- A class $\mathrm{Ob}\mathbf{A}$ of *objects*;
- *Hom-objects* $\mathbf{A}(x, y) \in \mathscr{V}$ $(x, y \in \mathrm{Ob}\mathbf{A})$;
- *Compositions* $\mathbf{A}(x, y) \otimes \mathbf{A}(y, z) \to \mathbf{A}(x, z)$ in $\mathscr{V}$ $(x, y, z \in \mathrm{Ob}\mathbf{A})$;
- *Identities* $I \to \mathbf{A}(x, x)$ in $\mathscr{V}$ $(x \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

$\mathscr{V}$-functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\begin{array}{ccc} \mathrm{Ob}\mathbf{A} & \longrightarrow & \mathrm{Ob}\mathbf{B}; \\ x & \longmapsto & Fx \end{array}$
- $\mathbf{A}(x, y) \to \mathbf{B}(Fx, Fy)$ in $\mathscr{V}$ $(x, y \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

$\mathscr{V}$-profunctor $\mathbf{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\mapstochar\;\;\;\;} \mathbf{B}$

- *Hom-objects* $P(x, y) \in \mathscr{V}$ $(x \in \mathbf{A}, y \in \mathbf{B})$;
- *Actions* $\mathbf{A}(x', x) \otimes P(x, y) \to P(x', y)$, $P(x, y) \otimes \mathbf{B}(y, y') \to P(x, y')$ in $\mathscr{V}$. (+Axioms)

The virtual double category $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

- (objects) $\mathscr{V}$-categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$;
- (vertical arrows) $\mathscr{V}$-functors $\begin{array}{c}\mathbf{A}\\ {\scriptstyle F}\downarrow\\ \mathbf{B}\end{array}$, ... and their compositions and identities;
- (horizontal arrows) $\mathscr{V}$-profunctors $\mathbf{A} \xrightarrow{P} \mathbf{B}$, ...;
- (cells) "generalized" $\mathscr{V}$-natural transformations

$$
\begin{tikzcd}
\mathbf{A}_0 \arrow[r, "P_1", "\shortmid"{marking}] \arrow[d, "F"'] & \mathbf{A}_1 \arrow[r, "P_2", "\shortmid"{marking}] & \cdots \arrow[r, "P_n", "\shortmid"{marking}] & \mathbf{A}_n \arrow[d, "G"] \\
\mathbf{B} \arrow[rrr, "Q"', "\shortmid"{marking}] \arrow[rrr, phantom, bend left=20, "\alpha"] & & & \mathbf{C}
\end{tikzcd}
\quad \Bigg\| \quad
\begin{array}{c}
P_1(x_0, x_1) \otimes P_2(x_1, x_2) \otimes \cdots \otimes P_n(x_{n-1}, x_n) \\
\downarrow \\
Q(Fx_0, Gx_n)
\end{array}
$$

- and their compositions (+ identity cells)

$$
\begin{tikzcd}
\mathbf{A}_0 \arrow[r, dashed, "\vec{P}_1", "\shortmid"{marking}] \arrow[d, "F_0"'] \arrow[dr, phantom, "\alpha_1"] & \mathbf{A}_1 \arrow[r, dashed, "\vec{P}_2", "\shortmid"{marking}] \arrow[d, "F_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{P}_n", "\shortmid"{marking}] \arrow[dr, phantom, "\alpha_n"] & \mathbf{A}_n \arrow[d, "F_n"] \\
\mathbf{B}_0 \arrow[r, "Q_1"', "\shortmid"{marking}] \arrow[d, "G"'] & \mathbf{B}_1 \arrow[r, "Q_2"', "\shortmid"{marking}] & \cdots \arrow[r, "Q_n"', "\shortmid"{marking}] & \mathbf{B}_n \arrow[d, "H"] \\
\mathbf{C} \arrow[rrr, "R"', "\shortmid"{marking}] \arrow[rrr, phantom, bend left=20, "\beta"] & & & \mathbf{D}
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathbf{A}_0 \arrow[r, dashed, "\vec{P}_1", "\shortmid"{marking}] \arrow[d, "F_0 \fatsemi G"'] & \mathbf{A}_1 \arrow[r, dashed, "\vec{P}_2", "\shortmid"{marking}] & \cdots \arrow[r, dashed, "\vec{P}_n", "\shortmid"{marking}] & \mathbf{A}_n \arrow[d, "F_n \fatsemi H"] \\
\mathbf{C} \arrow[rrr, "R"', "\shortmid"{marking}] \arrow[rrr, phantom, bend left=20, "\vec{\alpha} \fatsemi \beta"] & & & \mathbf{D}
\end{tikzcd}
$$

Horizontal composition

$\mathbb{X}$: a virtual double category (VDC).

Definition

A cell
$$\begin{tikzcd} A_0 \arrow[r, "u_1", "\shortmid"{marking}] \arrow[d, equal] & A_1 \arrow[r, "u_2", "\shortmid"{marking}] & \cdots \arrow[r, "u_n", "\shortmid"{marking}] & A_n \arrow[d, equal] \\ A_0 \arrow[rrr, "v"', "\shortmid"{marking}] \arrow[urrr, phantom, "\alpha"] & & & A_n \end{tikzcd}$$
in $\mathbb{X}$ is composing

$\overset{\text{def}}{\Leftrightarrow}$
$$\begin{tikzcd} \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] & A_0 \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] & A_n \arrow[r, dashed, "\shortmid"{marking}] & \cdot \arrow[d] \\ \cdot \arrow[rrr, "\shortmid"{marking}] \arrow[urrr, phantom, "\forall\beta"] & & & \cdot \end{tikzcd}
=
\begin{tikzcd} \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, equal] & A_0 \arrow[r, dashed, "\vec{u}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dl, phantom, "="] & A_n \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, equal] \arrow[dl, phantom, "\alpha"] & \cdot \arrow[d, equal] \arrow[dl, phantom, "="] \\ \cdot \arrow[r, dashed, "\shortmid"{marking}] \arrow[d] & A_0 \arrow[r, "v"', "\shortmid"{marking}] & A_n \arrow[r, dashed, "\shortmid"{marking}] & \cdot \arrow[d] \\ \cdot \arrow[rrr, "\shortmid"{marking}] \arrow[urrr, phantom, "\exists!\gamma"] & & & \cdot \end{tikzcd}$$
in $\mathbb{X}$.

Definition

The unit on $X \in \mathbb{X}$ $\cdots$
$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, equal] & \\ X \arrow[rr, "\mathsf{U}_X"', "\shortmid"{marking}] \arrow[ur, phantom, ""] & \text{comp} & X \end{tikzcd}$$
in $\mathbb{X}$. (written
$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, equal] & \\ X \arrow[rr, equal, "\shortmid"{marking}] & \text{comp} & X \end{tikzcd}$$
)

Horizontal composition

Examples in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

1. $$\begin{tikzcd} \mathbf{A} \arrow[r, "P", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\text{comp}"] & \mathbf{B} \arrow[r, "Q", "\shortmid" marking] & \mathbf{C} \arrow[d, equal] \\ \mathbf{A} \arrow[rr, "P \odot Q"', "\shortmid" marking] & & \mathbf{C} \end{tikzcd}$$ $$(P \odot Q)(x, z) := \int^{y \in \mathbf{B}} P(x, y) \otimes Q(y, z) \quad \text{in } \mathscr{V}$$

   (Suppose that $\int$ is preserved by $X \otimes -, - \otimes Y$.)

2. $$\begin{tikzcd} & \mathbf{A} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\text{comp}"] & \\ \mathbf{A} \arrow[rr, equal, "\mathsf{U}_{\mathbf{A}}"', "\shortmid" marking] & & \mathbf{A} \end{tikzcd}$$ $$\mathsf{U}_{\mathbf{A}}(x, x') := \mathbf{A}(x, x') \quad \text{in } \mathscr{V}$$

Unital virtual double categories

Definition

A VDC $\mathbb{X}$ is unital $\overset{\text{def}}{\Leftrightarrow}$ Every object $X \in \mathbb{X}$ has the unit.

Example

$\mathscr{V}$-$\mathbb{P}\mathrm{rof}$ is unital.

In a unital VDC $\mathbb{X}$,

$$\begin{tikzcd} \cdot \arrow[rr, dashed, "\vec{u}", "|"] \arrow[dr, "f"'] & \arrow[d, phantom, "\alpha"] & \cdot \arrow[dl, "g"] \\ & \cdot & \end{tikzcd} := \begin{tikzcd} \cdot \arrow[r, dashed, "\vec{u}", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal, "|"] & \cdot \end{tikzcd} \quad \text{in } \mathbb{X}. \qquad (\text{0-coary cell})$$

0-coary cells can be composed horizontally:

$$\begin{tikzcd} \cdot \arrow[r, dashed, "|"] \arrow[dr, phantom, "\alpha"] \arrow[dr] & \cdot \arrow[r, dashed, "|"] \arrow[d] & \cdot \arrow[dl] \arrow[dl, phantom, "\beta"] \\ & \cdot & \end{tikzcd} \rightsquigarrow \begin{tikzcd} \cdot \arrow[r, dashed, "|"] \arrow[dr] & \cdot \arrow[r, dashed, "|"] \arrow[d, phantom, "\alpha\frown\beta"] & \cdot \arrow[dl] \\ & \cdot & \end{tikzcd} := \begin{tikzcd} \cdot \arrow[r, dashed, "|"] \arrow[d] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[r, dashed, "|"] \arrow[d] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d] \\ \cdot \arrow[r, equal, "|"] \arrow[d, equal] & \cdot \arrow[r, equal, "|"] \arrow[d, phantom, "\text{comp}"] & \cdot \arrow[d, equal] \\ \cdot \arrow[rr, equal, "|"] & & \cdot \end{tikzcd}$$

Monoidal categories vs. bicategories vs. VDC

Monoidal category = single-object bicategory

$\mathscr{V}$: a mon.cat. $\quad\leftrightsquigarrow\quad$ $\mathcal{B}(\mathscr{V})$: a single-obj.bicat.

$$\begin{tikzcd} X \otimes Y \arrow[d, "\alpha"'] \\ Z \end{tikzcd} \text{ in } \mathscr{V} \qquad \Bigg\| \qquad \begin{tikzcd} & * \arrow[dr, bend left, "Y"] & \\ * \arrow[ur, bend left, "X"] \arrow[rr, bend right, "Z"'] & \arrow[u, phantom, "\Downarrow\alpha"] & * \end{tikzcd} \text{ in } \mathcal{B}(\mathscr{V})$$

(Virtual) bicategory = vertically discrete VDC

$\mathscr{W}$: a (virtual) bicat. $\quad\leftrightsquigarrow\quad$ $\mathbb{D}(\mathscr{W})$: a vertically discrete VDC

$$\begin{tikzcd} & c_1 \arrow[dr, bend left, "g"] & \\ c_0 \arrow[ur, bend left, "f"] \arrow[rr, bend right, "h"'] & \arrow[u, phantom, "\Downarrow\alpha"] & c_2 \end{tikzcd} \text{ in } \mathscr{W} \qquad \Bigg\| \qquad \begin{tikzcd} c_0 \arrow[r, "f", "|"' marking] \arrow[d, equal] & c_1 \arrow[r, "g", "|"' marking] \arrow[d, phantom, "\alpha"] & c_2 \arrow[d, equal] \\ c_0 \arrow[rr, "h"', "|" marking] & & c_2 \end{tikzcd} \text{ in } \mathbb{D}(\mathscr{W})$$

Generalization of enriching bases

$$\text{Enrichment in...} \begin{cases} \text{VDC } \mathbb{X} \\ \cup\!\!\!\shortmid \\ \text{bicategory } \mathscr{W} \\ \cup\!\!\!\shortmid \\ \text{mon.category } \mathscr{V} \end{cases}$$

generalize

generalize

Remark

We obtain unital VDCs $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$, $\mathscr{W}$-$\mathbb{P}\mathrm{rof}$, and $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ for any $\mathscr{V}, \mathscr{W}, \mathbb{X}$.

From $\mathscr{V}$ to $\mathscr{W}$

$\mathscr{W}$: a bicategory

$\mathscr{W}$-category $\mathbf{A}$

- A class $\mathrm{Ob}\mathbf{A}$ of *objects*;
- *Coloring* $|x| \in \mathscr{W}$ $(x \in \mathrm{Ob}\mathbf{A})$;
- *Hom-1-cells* $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathscr{W}$ $(x, y \in \mathrm{Ob}\mathbf{A})$;
- *Compositions* $\begin{tikzcd} & {|y|} \arrow[dr, bend left, "\mathbf{A}(y,z)"] \arrow[d, phantom, "\Downarrow"] & \\ {|x|} \arrow[ur, bend left, "\mathbf{A}(x,y)"] \arrow[rr, bend right, "\mathbf{A}(x,z)"'] & {} & {|z|} \end{tikzcd}$ in $\mathscr{W}$ $(x, y, z \in \mathrm{Ob}\mathbf{A})$;
- *Identities* $\begin{tikzcd} {|x|} \arrow[r, bend left, equal] \arrow[r, bend right, "\mathbf{A}(x,x)"'] \arrow[r, phantom, "\Downarrow"] & {|x|} \end{tikzcd}$ in $\mathscr{W}$ $(x \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

From $\mathscr{V}$ to $\mathscr{W}$

$\mathscr{W}$: a bicategory

$\mathscr{W}$-functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\begin{array}{ccc} \mathrm{Ob}\mathbf{A} & \longrightarrow & \mathrm{Ob}\mathbf{B} \\ x & \longmapsto & Fx \end{array}$ s.t. $|x| = |Fx|$ in $\mathscr{W}$;
- $\begin{tikzcd} {|x|} \arrow[r, bend left, "\mathbf{A}(x{,}y)"] \arrow[r, bend right, "\mathbf{B}(Fx{,}Fy)"'] \arrow[r, phantom, "\Downarrow"] & {|y|} \end{tikzcd}$ in $\mathscr{W}$ $(x, y \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

$\mathscr{W}$-profunctor $\mathbf{A} \overset{P}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbf{B}$

- *Hom-1-cells* $|x| \xrightarrow{P(x,y)} |y|$ in $\mathscr{W}$ $(x \in \mathbf{A}, y \in \mathbf{B})$;
- *Actions* $\begin{tikzcd} & {|x|} \arrow[dr, "P(x{,}y)"] \arrow[d, phantom, "\Downarrow"] & \\ {|x'|} \arrow[ur, "\mathbf{A}(x'{,}x)"] \arrow[rr, bend right, "P(x'{,}y)"'] & {} & {|y|} \end{tikzcd}$ $\begin{tikzcd} & {|y|} \arrow[dr, "\mathbf{B}(y{,}y')"] \arrow[d, phantom, "\Downarrow"] & \\ {|x|} \arrow[ur, "P(x{,}y)"] \arrow[rr, bend right, "P(x{,}y')"'] & {} & {|y'|} \end{tikzcd}$ in $\mathscr{W}$.

(+Axioms)

From $\mathscr{V}$ to $\mathscr{W}$

$$
\text{Enrichment in...}\left\{
\begin{array}{c}
\text{VDC } \mathbb{X} \\
\rotatebox{90}{$\subsetneq$} \\
\text{bicategory } \mathscr{W} \\
\rotatebox{90}{$\subsetneq$} \\
\text{mon.category } \mathscr{V}
\end{array}
\right.
$$

Objects are colored with $\mathrm{Ob}\mathscr{W}$.

From $\mathscr{W}$ to $\mathbb{X}$

$\mathbb{X}$: a VDC

$\mathbb{X}$-category $\mathbf{A}$

- A class $\mathrm{Ob}\mathbf{A}$ of *objects*;
- *Coloring* $|x| \in \mathbb{X}$ $(x \in \mathrm{Ob}\mathbf{A})$;
- *Hom-horizontal arrows* $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathbb{X}$ $(x, y \in \mathrm{Ob}\mathbf{A})$;
- *Compositions*
$$\begin{tikzcd} {|x|} \arrow[r, "{\mathbf{A}(x,y)}"] \arrow[d, equal] & {|y|} \arrow[r, "{\mathbf{A}(y,z)}"] \arrow[d, phantom, "\bullet"] & {|z|} \arrow[d, equal] \\ {|x|} \arrow[rr, "{\mathbf{A}(x,z)}"'] & & {|z|} \end{tikzcd}$$
in $\mathbb{X}$ $(x, y, z \in \mathrm{Ob}\mathbf{A})$;
- *Identities*
$$\begin{tikzcd} & {|x|} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\bullet"] & \\ {|x|} \arrow[rr, "{\mathbf{A}(x,x)}"'] & {} & {|x|} \end{tikzcd}$$
in $\mathbb{X}$ $(x \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

From $\mathscr{W}$ to $\mathbb{X}$

$\mathbb{X}$: a VDC

Notation

$\mathcal{H}(\mathbb{X})$ $\cdots$ the (virtual) bicat. obtained by forgetting all vertical arrows from $\mathbb{X}$.

Enrichment in a bicategory		Enrichment in a VDC
$\mathcal{H}(\mathbb{X})$-categories	$=$	$\mathbb{X}$-categories
$\mathcal{H}(\mathbb{X})$-functors	$\subseteq$	$\mathbb{X}$-functors
$\mathcal{H}(\mathbb{X})$-profunctors	$=$	$\mathbb{X}$-profunctors

From $\mathscr{W}$ to $\mathbb{X}$

$\mathbb{X}$-functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\begin{array}{ccc} \mathrm{Ob}\mathbf{A} & \longrightarrow & \mathrm{Ob}\mathbf{B}; \\ x & \longmapsto & F^0 x \end{array}$
- "*Color-comparing*" vertical arrows $|x| \xrightarrow{F^1 x} |F^0 x|$ in $\mathbb{X}$ $(x \in \mathrm{Ob}\mathbf{A})$;
- $\begin{tikzcd} {|x|} \arrow[r, "\mathbf{A}(x{,}y)", "\shortmid" marking] \arrow[d, "F^1 x"'] \arrow[dr, phantom, "\bullet"] & {|y|} \arrow[d, "F^1 y"] \\ {|F^0 x|} \arrow[r, "\mathbf{B}(F^0 x{,}F^0 y)"', "\shortmid" marking] & {|F^0 y|} \end{tikzcd}$ in $\mathbb{X}$ $(x, y \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

Enrichment in... $\left\{ \begin{array}{c} \text{VDC } \mathbb{X} \\ \cup\!\!\!\nmid \\ \text{bicategory } \mathscr{W} \\ \cup\!\!\!\nmid \\ \text{mon.category } \mathscr{V} \end{array} \right.$

Functors no longer need to preserve the color of objects.

Objects are colored with $\mathrm{Ob}\mathscr{W}$.

Collages of a profunctor

$\mathscr{V}$: a monoidal category with $\varnothing$ (the initial preserved by $\otimes$)

$\mathbf{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbf{B}$: a $\mathscr{V}$-profunctor

Definition

A collage (or *cograph*) of P is the $\mathscr{V}$-category $\mathbf{Col}\,P$:

$$\mathrm{Ob}(\mathbf{Col}\,P) := \mathrm{Ob}\mathbf{A} + \mathrm{Ob}\mathbf{B};$$

$$\mathbf{Col}\,P(x,y) := \begin{cases} \mathbf{A}(x,y) & \text{if } x, y \in \mathbf{A} \\ \mathbf{B}(x,y) & \text{if } x, y \in \mathbf{B} \\ P(x,y) & \text{if } x \in \mathbf{A}, y \in \mathbf{B} \\ \varnothing & \text{if } x \in \mathbf{B}, y \in \mathbf{A} \end{cases}$$

$$\begin{tikzcd} \mathbf{A} \arrow[rr, "P", "\shortmid"{marking}] \arrow[dr, "\text{coproj.}"'] & \cdot & \mathbf{B} \arrow[dl, "\text{coproj.}"] \\ & \mathbf{Col}\,P & \end{tikzcd}$$

in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

Universal property (vertical)

$$\begin{tikzcd} \mathbf{Col}\,P \arrow[d] \\ \mathbf{C} \end{tikzcd} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \qquad \Bigg\| \qquad \begin{tikzcd} \mathbf{A} \arrow[rr, "P", "\shortmid"{marking}] \arrow[dr] \arrow[rr, phantom, shift right=4, "\bullet"] & & \mathbf{B} \arrow[dl] \\ & \mathbf{C} & \end{tikzcd} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$$

Universal property (horizontal)

$$\begin{tikzcd} \mathbf{Col}\,P \arrow[r, "\shortmid"{marking}] & \mathbf{C} \end{tikzcd} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \qquad \Bigg\| \qquad \begin{tikzcd} \mathbf{A} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, equal] & \mathbf{B} \arrow[r, "\shortmid"{marking}] \arrow[d, phantom, "\bullet"] & \mathbf{C} \arrow[d, equal] \\ \mathbf{A} \arrow[rr, "\shortmid"{marking}] & & \mathbf{C} \end{tikzcd} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$$

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\shortmid"{marking}] & \mathbf{Col}\,P \end{tikzcd} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \qquad \Bigg\| \qquad \begin{tikzcd} \mathbf{C} \arrow[r, "\shortmid"{marking}] \arrow[d, equal] & \mathbf{A} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, phantom, "\bullet"] & \mathbf{B} \arrow[d, equal] \\ \mathbf{C} \arrow[rr, "\shortmid"{marking}] & & \mathbf{B} \end{tikzcd} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$$

$\mathbf{Col}\,P$ has universal properties in the three directions:

$$\begin{tikzcd} \cdot \arrow[r, "\shortmid"{marking}] & \mathbf{Col}\,P \arrow[r, "\shortmid"{marking}] \arrow[d] & \cdot \\ & \cdot & \end{tikzcd}$$

Collages in general

S: a set

Notation

$\mathbb{I}\mathrm{S}$ $\cdots$ the VDC described by:

- $\mathrm{Ob}(\mathbb{I}\mathrm{S}) := \mathrm{S}$;
- For $i, j \in \mathrm{S}$, there is a unique horizontal arrow $i \xrightarrow{!_{ij}} j$ in $\mathbb{I}\mathrm{S}$;
- For $i_0, \ldots, i_n \in \mathrm{S}$, there is a unique cell

$$\begin{tikzcd} i_0 \arrow[r, "!_{i_0 i_1}"] \arrow[d, equal] & i_1 \arrow[r, "!_{i_1 i_2}"] & \cdots \arrow[r, "!_{i_{n-1} i_n}"] & i_n \arrow[d, equal] \\ i_0 \arrow[rrr, "!_{i_0 i_n}"'] \arrow[urrr, phantom, "!_{i_0 \ldots i_n}"] & & & i_n \end{tikzcd}$$

A virtual double (VD)-functor $\mathbb{I}\mathrm{S} \longrightarrow \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ is equivalent to the following data:

- $\mathbf{A}_i$: $\mathbb{X}$-categories $\quad (i \in \mathrm{S})$
- $\mathbf{A}_i \xrightarrow{P_{ij}} \mathbf{A}_j$: $\mathbb{X}$-profunctors $\quad (i, j \in \mathrm{S})$
- $$\begin{tikzcd} & \mathbf{A}_i \arrow[dl, equal] \arrow[dr, equal] & \\ \mathbf{A}_i \arrow[rr, "P_{ii}"'] \arrow[rr, phantom, bend left, "\eta_i"] & & \mathbf{A}_i \end{tikzcd} \quad (i \in \mathrm{S})$$
- $$\begin{tikzcd} \mathbf{A}_i \arrow[r, "P_{ij}"] \arrow[d, equal] & \mathbf{A}_j \arrow[r, "P_{jk}"] & \mathbf{A}_k \arrow[d, equal] \\ \mathbf{A}_i \arrow[rr, "P_{ik}"'] \arrow[urr, phantom, "\mu_{ijk}"] & & \mathbf{A}_k \end{tikzcd} \quad (i, j, k \in \mathrm{S})$$

(+Axioms)

Collages in general

$P := (\mathbf{A}_i, P_{ij}, \eta_i, \mu_{ijk})_{i,j,k\in \mathrm{S}}$: a VD-functor $\mathbb{IS} \xrightarrow{P} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$

Definition

A collage of P is the $\mathbb{X}$-category $\mathbf{Col}\,P$:

$$\mathrm{Ob}(\mathbf{Col}\,P) := \coprod_{i\in \mathrm{S}} \mathrm{Ob}\mathbf{A}_i;$$

$$\mathbf{Col}\,P(x,y) := P_{ij}(x,y) \quad (\text{where } x\in \mathbf{A}_i, y \in \mathbf{A}_j).$$

$$\begin{tikzcd} \mathbf{A}_i \arrow[rr, "P_{ij}", "|" marking] \arrow[dr, "\xi_i"'] & \arrow[d, phantom, "\xi_{ij}"] & \mathbf{A}_j \arrow[dl, "\xi_j"] \\ & \mathbf{Col}\,P & \end{tikzcd}$$

in $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ (*coprojections*)

Again, $\mathbf{Col}\,P$ has universal properties in the three directions:

$$\begin{tikzcd} \cdot \arrow[r, "|" marking] & \mathbf{Col}\,P \arrow[r, "|" marking] \arrow[d] & \cdot \\ & \cdot & \end{tikzcd}$$

Versatile colimits

$\mathbb{K}$: a VDC, $\mathbb{L}$: a unital VDC, $\mathbb{K} \xrightarrow{F} \mathbb{L}$: a VD-functor

Definition (informal)

A versatile colimit of F is a "cocone"

$$\left\{ \begin{tikzcd} FA \arrow[rr, "Fu", "|" marking] \arrow[dr, "\xi_A"'] & \arrow[d, phantom, "\xi_u"] & FB \arrow[dl, "\xi_B"] \\ & \Xi & \end{tikzcd} \quad \text{in } \mathbb{L} \right\}_{\left(A \xrightarrow{u} B \quad \text{in } \mathbb{K}\right)}$$

having the universal property in the three directions

$$\begin{tikzcd} \cdot \arrow[r, "|" marking] & \Xi \arrow[r, "|" marking] \arrow[d] & \cdot \\ & \cdot & \end{tikzcd} \qquad \text{in } \mathbb{L}.$$

Example

1. Versatile collages ($:=$ vers.colim. of shapes $\mathbb{K} = \mathbb{I}\mathbb{S}$)
2. Versatile coproducts ($:=$ vers.colim. of discrete shapes)

The object classifier in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

$(\mathscr{V}, \otimes, I)$: a monoidal category

Notation

The *unit $\mathscr{V}$-category* $\mathbf{I}$:

$$\mathrm{Ob}\mathbf{I} := \{*\}, \quad \mathbf{I}(*,*) := I.$$

The unit $\mathscr{V}$-category classifies objects

For every $\mathscr{V}$-category $\mathbf{A}$,

$$\begin{array}{c|c} \begin{array}{c} \mathbf{I} \\ \downarrow \\ \mathbf{A} \end{array} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof} & \text{an object } x \in \mathbf{A} \end{array}$$

Object classifiers in $\mathscr{W}$-$\mathbb{P}\mathrm{rof}$

$\mathscr{W}$: a bicategory

Notation

The *unit $\mathscr{W}$-category* $\mathbf{I}_c$ for $c \in \mathscr{W}$:

$$\mathrm{Ob}\mathbf{I}_c := \{*\} \text{ with } |*| := c, \quad \mathbf{I}_c(*,*) := (c \xrightarrow{\text{id}} c \quad \text{in } \mathscr{W})$$

The unit $\mathscr{W}$-categories classify objects

For every $\mathscr{W}$-category $\mathbf{A}$,

$$\frac{\mathbf{I}_c \to \mathbf{A} \quad \text{in } \mathscr{W}\text{-}\mathbb{P}\mathrm{rof}}{\text{an object } x \in \mathbf{A} \text{ s.t. } |x| = c}$$

Object classifiers in $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$

$\mathbb{X}$: a <u>unital</u> VDC

Notation

The *unit $\mathbb{X}$-category* $\mathbf{I}_c$ for $c \in \mathbb{X}$:

$$\mathrm{Ob}\mathbf{I}_c := \{*\} \text{ with } |\,*\,| := c, \quad \mathbf{I}_c(*,*) := (c \relbar\joinrel\mapstochar\joinrel\rightarrow c \quad \text{in } \mathbb{X})$$

The unit $\mathbb{X}$-categories do not classify objects.

For every $\mathbb{X}$-category $\mathbf{A}$,

$$\begin{array}{c}\mathbf{I}_c\\ \Big\downarrow\\ \mathbf{A}\end{array} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \quad \Bigg\| \quad \begin{array}{l}\text{an object } x \in \mathbf{A}\text{, and}\\ \text{a vertical arrow } \begin{array}{c} c\\ \Big\downarrow\\ |x|\end{array} \text{ in } \mathbb{X}\end{array} \qquad (\textit{semi-object})$$

$\rightsquigarrow$ The unit $\mathbb{X}$-categories classify *semi-objects*.

The embedding

$\mathbb{X}$: a unital VDC

Proposition

There is an "embedding" $\mathbb{X} \xhookrightarrow{\mathbf{I}_\bullet} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$. That is:

$$
\begin{tikzcd}
c_0 \arrow[r, "\shortmid" marking] \arrow[d] \arrow[drrr, phantom, "\bullet"] & c_1 \arrow[r, "\shortmid" marking] & \cdots \arrow[r, "\shortmid" marking] & c_n \arrow[d] \\
d \arrow[rrr, "\shortmid" marking] & & & e
\end{tikzcd}
\quad \text{in } \mathbb{X}
\quad \Bigg\| \quad
\begin{tikzcd}
\mathbf{I}_{c_0} \arrow[r, "\shortmid" marking] \arrow[d] \arrow[drrr, phantom, "\bullet"] & \mathbf{I}_{c_1} \arrow[r, "\shortmid" marking] & \cdots \arrow[r, "\shortmid" marking] & \mathbf{I}_{c_n} \arrow[d] \\
\mathbf{I}_d \arrow[rrr, "\shortmid" marking] & & & \mathbf{I}_e
\end{tikzcd}
\quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}
$$

In what follows, we will consider $\mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$.

Toward the characterization: density

$\mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ ($\mathbb{X}$: a unital VDC)

Observation I

Every $\mathbb{X}$-category is a versatile collage of objects from $\mathbb{X}$.

$\because$

$$\begin{array}{c} \mathbf{A}: \text{an } \mathbb{X}\text{-category} \\ \hline\hline \mathbb{I}\mathrm{Ob}\mathbf{A} \xrightarrow{|\cdot|_{\mathbf{A}}} \mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}: \text{a VD-functor} \end{array}$$

Taking the versatile colimit in $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$

The VD-functor $|\cdot|_{\mathbf{A}}$ is given by:

$$\begin{tikzcd} x_0 \arrow[r, "!", "\shortmid"{marking}] \arrow[d, equal] & x_1 \arrow[r, "!", "\shortmid"{marking}] & \cdots \arrow[r, "!", "\shortmid"{marking}] & x_n \arrow[d, equal] \\ x_0 \arrow[rrr, "!"', "\shortmid"{marking}] \arrow[rrru, phantom, "!"] & & & x_n \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} {|x_0|} \arrow[r, "{\mathbf{A}(x_0,x_1)}", "\shortmid"{marking}] \arrow[d, equal] & {|x_1|} \arrow[r, "{\mathbf{A}(x_1,x_2)}", "\shortmid"{marking}] & \cdots \arrow[r, "{\mathbf{A}(x_{n-1},x_n)}", "\shortmid"{marking}] & {|x_n|} \arrow[d, equal] \\ {|x_0|} \arrow[rrr, "{\mathbf{A}(x_0,x_n)}"', "\shortmid"{marking}] \arrow[rrru, phantom, "\text{“composition” in } \mathbf{A}"] & & & {|x_n|} \end{tikzcd}$$

in $\mathbb{I}\mathrm{Ob}\mathbf{A}$ in $\mathbb{X}$

Toward the characterization: atomicity

Observation II

In $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$, the unit $\mathbb{X}$-categories can be characterized by "atomicity" w.r.t. versatile collages.

Definition

$\mathbb{L}$: a unital VDC.
$L \in \mathbb{L}$: collage-atomic $\overset{\text{def}}{\Leftrightarrow}$ For every (large) versatile collage $\mathbf{Col}\,P$ in $\mathbb{L}$, vertical arrows $L \to \mathbf{Col}\,P$ uniquely factor through a unique coprojection.

$$\begin{tikzcd} & L \arrow[dl, dotted, "\exists!"'] \arrow[d, "\forall"] \\ Pi \arrow[r, "\text{coproj.}"'] & \mathbf{Col}\,P \end{tikzcd} \quad \text{in } \mathbb{L} \quad (\exists! i)$$

Proposition

$\mathbb{X}$: a unital VDC.
In $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$, collage-atomic $\iff$ vertically isomorphic to some $\mathbf{I}_c$.

The characterization

Theorem

TFAE for a unital VDC $\mathbb{L}$:

1. There is an equivalence $\mathbb{L} \simeq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ for some unital VDC $\mathbb{X}$.
   (in the 2-category of unital VDCs)
2.
   - $\mathbb{L}$ has (large) versatile collages;
   - Every object in $\mathbb{L}$ is a versatile collage of collage-atomic objects.

Sketch of proof: The only non-trivial part is 2. $\Longrightarrow$ 1.
Let $\mathbb{X} := \{\text{collage-atomic objs.}\} \subseteq \mathbb{L}$.

We can construct an adjunction: $\begin{tikzcd} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \arrow[r, shift left=2, "R"] \arrow[r, phantom, "\perp"] & \mathbb{L} \arrow[l, shift left=2, "N"] \end{tikzcd}$.

Construction of R: Regarding $\mathbf{A} \in \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ as $\mathbb{I}\mathrm{Ob}\mathbf{A} \longrightarrow \mathbb{X} \subseteq \mathbb{L}$, we take the vers.colim. of it.

Construction of N: Suppose $L \in \mathbb{L}$ is a vers.colim. of $\mathbb{I}\mathrm{S} \xrightarrow{P} \mathbb{X} \subseteq \mathbb{L}$. Define $L \xmapsto{N}$ "(S, P)" $\in \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$.

$R \circ N \cong \mathrm{Id}$ is trivial. $\mathrm{Id} \cong N \circ R$ follows from uniqueness of (S, P). □

Why is $\mathbb{IS} \xrightarrow{P} \mathbb{X}$ unique for each $L \in \mathbb{L}$?

- The uniqueness of both S and the object part of P
  $\cdots$ From the collage-atomicity of $\mathbb{X}$.
- The uniqueness of the horizontal arrow & cell part of P
  $\cdots$ From strongness theorem on versatile collages:

Strongness theorem

$\mathbb{L}$: a unital VDC. $\mathbf{Col}\,P(\in \mathbb{L})$: a versatile collage of $\mathbb{IS} \xrightarrow{P} \mathbb{L}$.
$\Longrightarrow$ Coprojections of $\mathbf{Col}\,P$ form *cartesian cells*:

$$\begin{tikzcd} Pi \arrow[rr, "P!_{ij}", "|" marking] \arrow[dr, "\text{coproj.}"'] \arrow[rr, phantom, shift right=4ex, "\text{cart}"] & & Pj \arrow[dl, "\text{coproj.}"] \\ & \mathbf{Col}\,P & \end{tikzcd} \quad \text{in } \mathbb{L} \quad (i, j \in \mathrm{S}).$$

Thank you!

Today's slides

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