

# On the decomposition of a strong epimorphism into regular epimorphisms

Yuto Kawase (joint with Hayato Nasu)

RIMS, Kyoto University

February 8, 2026. Categories in Tokyo 2nd



~~~~~ slides

- 1 Regular epimorphisms
- 2 Generalized regular epimorphisms
- 3 Locally orthogonal factorizations

## Recall

Every homomorphism  $f: A \rightarrow B$  between groups can be decomposed into a surjective hom. followed by an injective hom.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \swarrow \\ & \text{Im } f & \end{array} \quad \text{in } \mathbf{Grp}$$

**Regular epimorphisms** = morphisms being the coequalizer of some parallel pair of morphisms.

$$\cdot \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \cdot \xrightarrow{\text{regular epi}} \cdot$$

(eg. regular epi = surjective hom. in  $\mathbf{Grp}$ )

**Monomorphisms** = morphisms s.t.  $f \circ -$  is injective.

$$\cdot \begin{array}{c} \overset{g}{\curvearrowright} \\ \underset{h}{\curvearrowright} \end{array} \cdot \xleftarrow[\text{f}]{\text{mono}} \cdot \text{(cofork)} \quad \Rightarrow \quad g = h$$

(eg. mono = injective hom. in  $\mathbf{Grp}$ )

## Definition

$f \perp g$   
( $f$  and  $g$  are orthogonal)



## Example

In every category, {regular epi}  $\perp$  {mono}.

$\mathbf{E} \subseteq \text{Mor}\mathcal{C}$ : a class of morphisms in a category  $\mathcal{C}$ .

## Definition

- $\{\text{iso}\} \subseteq \mathbf{E}$ .
- $\mathbf{E} \text{ iso-closed} \stackrel{\text{def}}{\iff} \begin{array}{c} \cdot \xrightarrow{\cong} \cdot \\ \mathbf{E} \ni f \downarrow \quad \downarrow g \\ \cdot \xrightarrow{\cong} \cdot \end{array} \implies g \in \mathbf{E}$ .

## Definition

$\mathbf{E}$ : iso-closed.

$\mathcal{C}$  admits orthogonal  $\mathbf{E}$ -factorizations  $\stackrel{\text{def}}{\iff}$  Every morphism  $f$  can be decomposed as

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ \mathbf{E} \ni e \searrow & \nearrow_{m \in \mathbf{r}(\mathbf{E})} & \end{array} \quad \text{where } \mathbf{r}(\mathbf{E}) := \{m \mid \mathbf{E} \perp m\}.$$

## Example

$\text{Grp}$  admits orthogonal  $\{\text{regular epi}\}$ -factorizations.

## The fundamental homomorphism theorem

$$A / \text{Ker } f \cong \text{Im } f \quad (f: A \rightarrow B)$$

The **kernel pair** of  $f$ :

$$\begin{array}{ccc} \text{Kp } f & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow & \lrcorner & \downarrow f \\ A & \xrightarrow{f} & B \end{array} \quad (\text{pullback})$$

The **coimage** of  $f$ :

$$\text{Kp } f \rightrightarrows_{\pi_1}^{\pi_2} A \longrightarrow \text{Coim } f \quad (\text{coequalizer})$$

The fundamental homomorphism theorem (categorically)

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \nearrow \text{mono} \\ & & \text{Coim } f \end{array}$$

This holds in any categories of equational algebras.

## Fact

TFAE for a category  $\mathcal{C}$  with finite limits and colimits:

- ①  $\mathcal{C}$  admits orthogonal  $\{\text{regular epi}\}$ -factorizations.
- ② “The fundamental homomorphism theorem holds in  $\mathcal{C}$ .” That is, for every  $f: A \rightarrow B$  in  $\mathcal{C}$ , the canonical morphism  $\text{Coim } f \rightarrow B$  is monic.

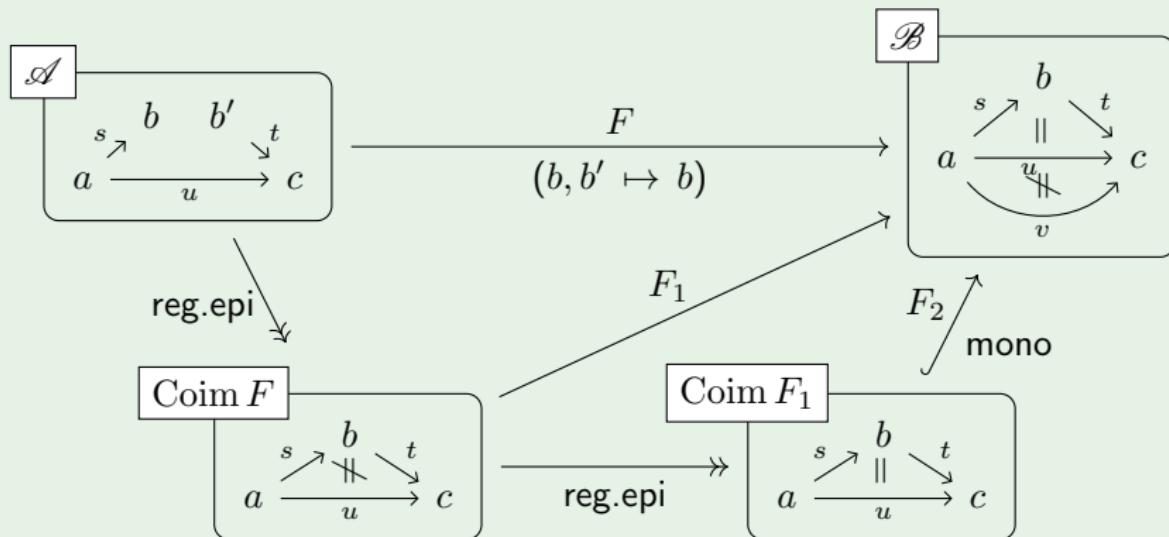
## Question

Does the fundamental hom. thm. hold in any category?

~~~ No.

## Counter example

In  $\text{Cat}$ ,



## Definition

A (**regular**) **decomposition** of  $A \xrightarrow{f} B$  in  $\mathcal{C}$  consists of:

- A functor  $\mathbf{Ord} \xrightarrow{A_\bullet} \mathcal{C}$  with  $A_0 = A$ ;
- A cocone  $f_\bullet = (A_\alpha \xrightarrow{f_\alpha} B)$  over  $A_\bullet$  with  $f_0 = f$

satisfying  $(\forall \alpha, A_{\alpha,\alpha+1} \text{ regular epi})$  and  $(\forall \gamma \text{ limit type}, A_\gamma \cong \text{Colim}_{\alpha < \gamma} A_\alpha)$ .

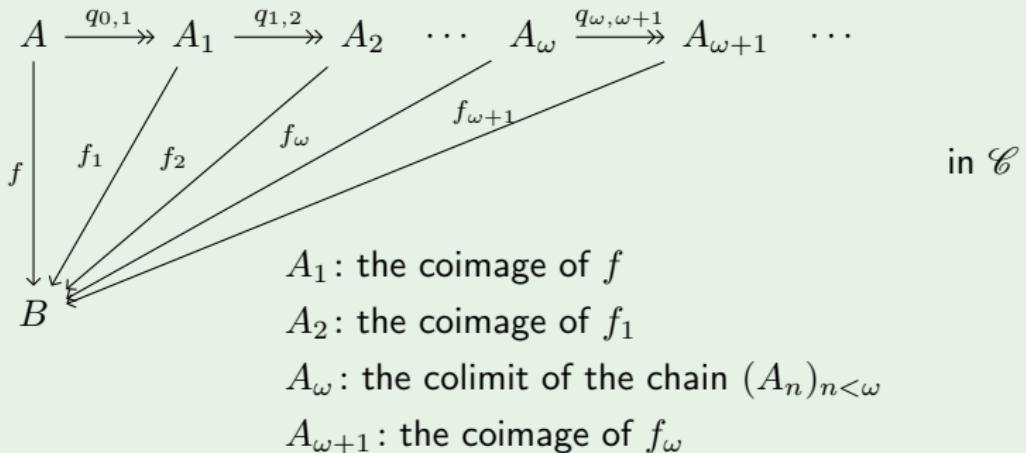
$$A = A_0 \xrightarrow{A_{0,1}} A_1 \longrightarrow \cdots \longrightarrow A_\alpha \xrightarrow{A_{\alpha,\alpha+1}} A_{\alpha+1} \longrightarrow \cdots$$

in  $\mathcal{C}$

## Definition

- ① A decomp.  $(A_\bullet, f_\bullet)$  **stabilizes at  $\alpha$**   $\stackrel{\text{def}}{\Leftrightarrow} f_\beta \text{ mono } (\alpha \leq \forall \beta) \quad (\Leftrightarrow f_\alpha \text{ mono})$
- ② The smallest  $\alpha$  above is called the **length** of  $(A_\bullet, f_\bullet)$ . (if exists)

$\mathcal{C}$ : a category with kernel pairs and colimits.



~~~ This gives a decomposition of  $f$ . (**canonical decomposition**)

## Definition

- ① The **canonical decomposition number**  $\sigma(f) :=$  the length of the canonical decomposition of  $f$ . If it doesn't stabilize,  $\sigma(f)$  is left to be undefined.
- ② If  $\sigma(f)$  is defined for every  $f$  in  $\mathcal{C}$ , the **global canonical decomposition number**  $\sigma(\mathcal{C}) := \min\{\alpha \mid \sigma(f) < \alpha \text{ for every } f \text{ in } \mathcal{C}\}$

## Remark

- ①  $\sigma(\mathcal{C}) = 0 \iff \mathcal{C} = \emptyset.$
- ②  $\sigma(\mathcal{C}) \leq 1 \iff \sigma(f) = 0 \ (\forall f) \iff f: \text{monic } (\forall f).$
- ③  $\sigma(\mathcal{C}) \leq 2 \iff \sigma(f) \leq 1 \ (\forall f) \iff \text{"The fund. hom. thm. holds in } \mathcal{C} \text{".}$
- ④  $\sigma(\mathcal{C}) \leq \omega \iff \sigma(f): \text{finite } (\forall f).$

## Example

- ①  $\sigma(\mathbf{Set}) = \sigma(\mathbf{Grp}) = 2.$
- ②  $\sigma(\mathbf{Cat}) = 3.$
- ③  $\sigma(n\text{-}\mathbf{Cat}) = n + 2.$

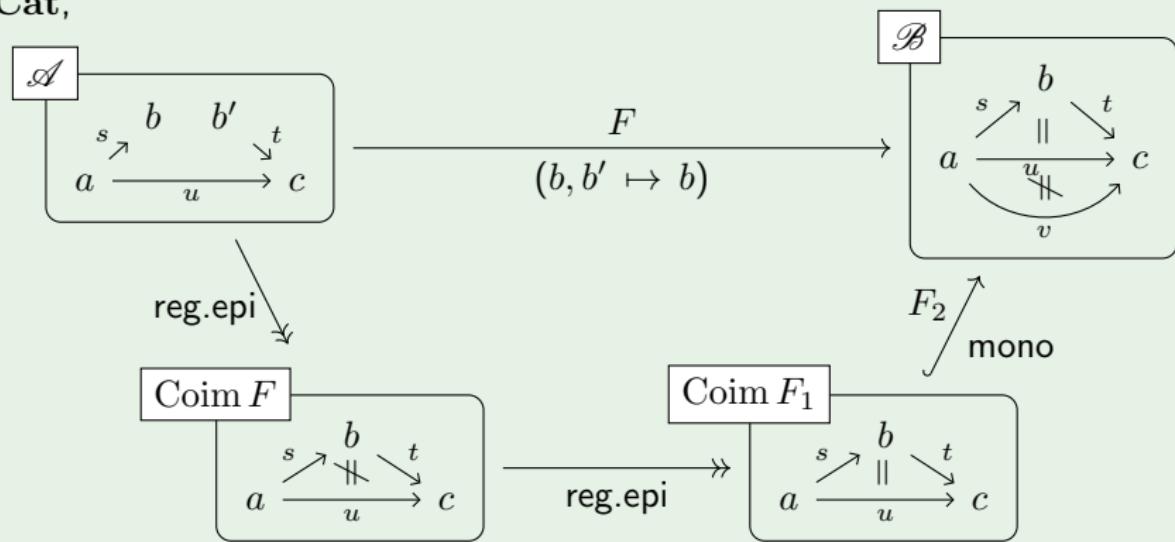
## Theorem [GU71]

$\mathcal{C}$ : locally  $\kappa$ -presentable  $\implies \sigma(f)$  is defined and  $\sigma(f) \leq \kappa$  holds.  $(\forall f \text{ in } \mathcal{C})$   
(Equivalently,  $\sigma(\mathcal{C}) \leq \kappa + 1$ )

# Minimum decomposition problem

Recall (a canonical decomposition in **Cat**)

In **Cat**,



## Question

Is there a shorter decomposition of  $F$  than the canonical one in **Cat**? That is,  
can  $F$  be decomposed as  $\xrightarrow{\text{reg.epi}} \xrightarrow{\text{mono}} \dots ?$

# Minimum decomposition problem

## Definition

The **minimum decomposition number**  $\delta(f)$  := the shortest possible length of decompositions of  $f$ . (if at least one decomposition exists)

## Question (restated)

Does  $\sigma(f) = \delta(f)$  hold?

## Theorem (Canonical vs minimum [KN])

In a category with kernel pairs and colimits,

- ①  $\sigma(f)$  is defined  $\iff \delta(f)$  is defined.
- ② Whenever they are defined,  $\sigma(f) = \delta(f)$  holds.

That is, the canonical (regular) decomposition is minimum.

- 1 Regular epimorphisms
- 2 Generalized regular epimorphisms
- 3 Locally orthogonal factorizations

## Observation

In a category with kernel pairs,

$q$ : regular epi.  $\iff q$ : the coequalizer of the kernel pair of  $q$ .

- “coequalize the kernel pair of  $q$ ”  $\iff$  coequalize any pair of morphisms that coequalized by  $q$ .  $\cdots$   $(\star)$  (*the kernel condition associated with  $q$* )

$\therefore q$ : regular epi.  $\iff q$ : the most universal among morphisms satisfying  $(\star)$ .

In a category with binary coproducts,

$X \xrightarrow{\begin{smallmatrix} u \\ v \end{smallmatrix}} A$  is coequalized by

$$\begin{array}{ccc} A & X + X & \xrightarrow{(u,v)} A \\ \downarrow q & \nabla \downarrow & \downarrow q \\ B & X & \xrightarrow{\exists} B \end{array}$$

$A \xrightarrow{f} C$  satisfies  $(\star)$   $\iff \forall X,$

$$\begin{array}{ccc} X + X & \xrightarrow{\forall} & A \\ \nabla \downarrow & & q \downarrow \\ X & \xrightarrow{\forall} & B \\ & \swarrow \exists & \searrow f \\ & & C \end{array}$$

## Idea

Replace  $\{X + X \xrightarrow{\nabla} X\}_{X: \text{obj}}$  with another class of epimorphisms.

# $\Lambda$ -regular epimorphisms

$\Lambda$ : a class of epimorphisms. Fix  $A \xrightarrow{q} B$ .

## Definition [KN]

$A \xrightarrow{f} C$  satisfies  $(\star)$   
(the  $\Lambda$ -kernel condition assoc. with  $q$ )

$$\Leftrightarrow \begin{array}{ccc} \cdot & \xrightarrow{\forall} & A \\ \Lambda \exists \forall e \downarrow & & q \downarrow \\ \cdot & \xrightarrow{\forall} & B \\ & \swarrow \exists! & f \\ & & C \end{array}$$

## Definition [KN]

$q$ :  **$\Lambda$ -regular epi**

$\stackrel{\text{def}}{\Leftrightarrow}$

$$\begin{array}{c} A \\ q \downarrow \\ B \\ \exists! \rightarrow \end{array} \quad \begin{array}{l} \forall f \text{ satisfying } (\star) \\ \curvearrowright \end{array}$$

## Example I

### Example

**Pos:** the category of posets and order-preserving maps.

$$\Lambda := \left\{ \begin{array}{c} \boxed{\bullet \quad \star} \\ \xrightarrow{\iota} \end{array} \boxed{\bullet < \star} \quad \text{in } \mathbf{Pos} \right\}$$

$\rightsquigarrow$   $\Lambda$ -regular epis = *surjective* order-preserving maps (= epis)

### Example

**Cat:** the category of small categories and functors.

$$\Lambda := \left\{ \begin{array}{c} \boxed{\bullet \rightarrow \star} \\ \xrightarrow{L} \end{array} \boxed{\bullet \cong \star} \quad \text{in } \mathbf{Cat} \right\}$$

$\rightsquigarrow$   $\Lambda$ -regular epis = *strict localizations*

## Example II

### Example

**Met:** the category of metric spaces and non-expansive maps.

$$\Lambda := \left\{ \boxed{\bullet \xrightarrow{r+1} \star} \xrightarrow{e_r} \boxed{\bullet \xrightarrow{r} \star} \quad \text{in Met} \right\}_{r \in \mathbb{Q}_{>0}}$$

$\rightsquigarrow q: \Lambda\text{-regular epi} \iff d(x, y) - d(q(x), q(y)) \leq 1 \ (\forall x, y).$   
*(a shrinking at most 1)*

### Example

**Top:** the category of topological spaces and continuous maps.

$$\Lambda := \left\{ K \xrightarrow{!_K} 1 \quad \text{in Top} \right\}_{K: \text{connected space}}$$

$\rightsquigarrow q: \Lambda\text{-regular epi} \iff q^{-1}(x): \text{connected } (\forall x). \ (= \text{monotone map})$

| category                         | $\Lambda$                      | $\mathbf{E} = \{\Lambda\text{-regular epis}\}$ | $\mathbf{r}(\mathbf{E})$ |
|----------------------------------|--------------------------------|------------------------------------------------|--------------------------|
| any category<br>with bin.coprod. | {codiagonals}                  | {regular epis}                                 | {monos}                  |
| <b>Pos</b>                       | { $\iota$ }                    | {surjections}                                  | {embeddings}             |
| <b>Cat</b>                       | { $L$ }                        | {strict localizations}                         | {conservatives}          |
| <b>Met</b>                       | { $e_r$ } <sub>r</sub>         | {shrinkings at most 1}                         | {isometries}             |
| <b>Top</b>                       | {! <sub>K</sub> } <sub>K</sub> | {monotone maps}                                | {light maps}             |

**Pos** admits orthogonal {surj}-factorizations.

However, neither **Cat**, **Met**, nor **Top** admits orthogonal  $\mathbf{E}$ -factorizations.

# Coimages in terms of $\Lambda$ -regular epis

## Definition

Let  $A \xrightarrow{f} X$ .

$$\mathcal{S} := \left\{ \begin{array}{c|c} (\mathbf{e}_i, \mathbf{u}_i) & \Lambda \ni e_i \downarrow \\ \hline \Gamma_i \xrightarrow{\mathbf{u}_i} A \xrightarrow{f} X & \Delta_i \end{array} \right\}$$

Then,  $\text{Coim}_{\Lambda} f := \text{Colim } \mathcal{S}$ . (multiple pushout)

$$\begin{array}{ccccc}
 \Gamma_{i'} & \xrightarrow{u_{i'}} & A & \xrightarrow{f} & X \\
 \downarrow e_{i'} & \nearrow u_i & \downarrow q & \nearrow \exists! f_1 & \\
 \Gamma_i & & \text{Coim}_{\Lambda} f & & \\
 \downarrow e_i & \xrightarrow{\iota_{i'}} & & & \\
 \Delta_{i'} & \xrightarrow{\iota_i} & & & \\
 \downarrow & \nearrow \iota_i & & &
 \end{array}$$

$q$ :  $\Lambda$ -regular epi?

## Proposition [KN]

- ①  $\Lambda \subseteq \mathbf{Reg}(\Lambda)$  ( $\coloneqq \{\Lambda\text{-regular epis}\}$ ).
- ② Moreover,  $\mathbf{Reg}(\Lambda)$  is the “multiple pushout closure” of  $\Lambda$ . That is,  
 $q: \Lambda\text{-reg.epi} \iff q: \text{obtained by a mult.pushout of some family of spans}$   
 $(e_i, u_i) \text{ s.t. } e_i \in \Lambda.$

In particular,

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \Lambda\text{-regular epi} \rightsquigarrow q \downarrow & & f_1 \\ & \nearrow & \\ & \text{Coim}_{\Lambda} f & \end{array}$$

However,  $f_1 \notin \mathbf{r}(\mathbf{E})$  in general.

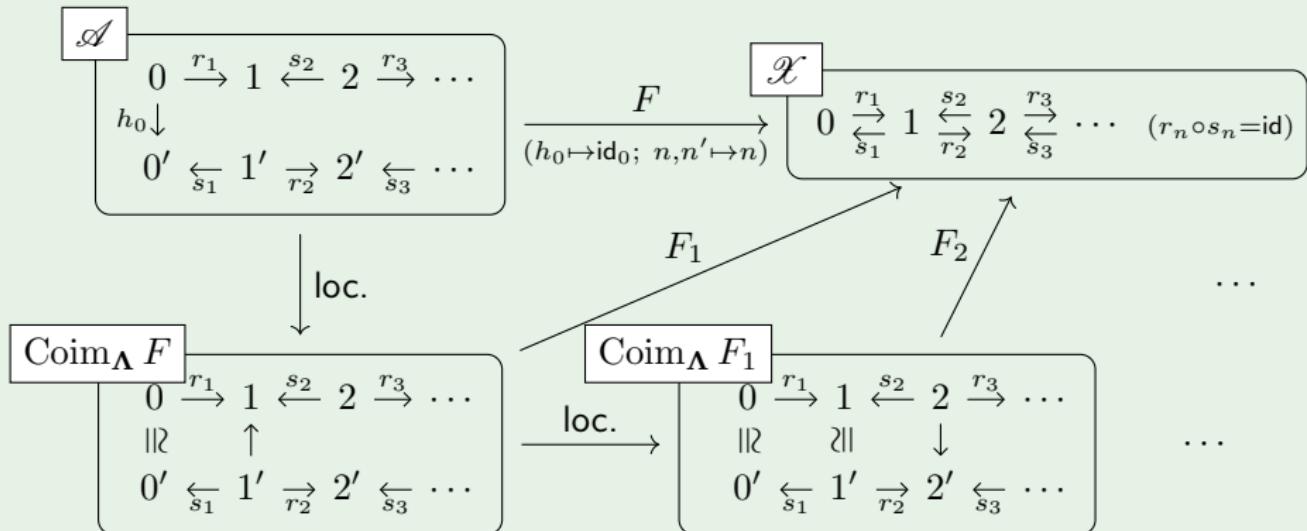
Replacing  $\begin{pmatrix} \{\text{regular epis}\} & \rightsquigarrow & \mathbf{E} = \mathbf{Reg}(\Lambda) \\ \{\text{monos}\} & \rightsquigarrow & \mathbf{r}(\mathbf{E}) \end{pmatrix},$

## Definition

$\sigma_{\mathbf{E}}(f) := \min\{\alpha \mid f_\alpha \in \mathbf{r}(\mathbf{E})\}$ . (the *canonical E-decomposition number*)

## Example

In  $\mathbf{Cat}$  with  $\mathbf{E} := \{\text{strict localizations}\}$ ,



$$\rightsquigarrow \sigma_{\mathbf{E}}(F) = \omega.$$

## Theorem (Small object argument)

$\mathcal{C}$ : locally  $\kappa$ -presentable;  $\Lambda$ : small;  $\text{dom } f, \text{cod } f$ :  $\kappa$ -presentable ( $\forall f \in \Lambda$ )  
 $\implies \sigma_{\mathbf{E}}(\mathcal{C}) \leq \kappa + 1$ , where  $\mathbf{E} := \mathbf{Reg}(\Lambda)$ .

## Corollary

$\sigma_{\mathbf{E}}(\mathbf{Cat}) = \omega + 1$ , where  $\mathbf{E} := \{\text{strict localizations}\}$ .

# Minimum decomposition problem

Let  $\mathbf{E} := \mathbf{Reg}(\Lambda)$ .

We can also generalize  $\begin{pmatrix} \text{regular decomposition} & \rightsquigarrow & \mathbf{E}\text{-decomposition} \\ \text{minimum dec.num. } \delta(f) & \rightsquigarrow & \delta_{\mathbf{E}}(f) \end{pmatrix}$ .

Then,  $\sigma_{\mathbf{E}}$  and  $\delta_{\mathbf{E}}$  still coincide:

## Theorem (Canonical vs minimum [KN])

Let  $\mathcal{C}$ : locally small and cocomplete,  $\mathbf{E} := \mathbf{Reg}(\Lambda)$  with  $\Lambda$ : small.

- ①  $\sigma_{\mathbf{E}}(f)$  is defined  $\iff \delta_{\mathbf{E}}(f)$  is defined.
- ② Whenever they are defined,  $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$  holds.

- 1 Regular epimorphisms
- 2 Generalized regular epimorphisms
- 3 Locally orthogonal factorizations

In the previous part, we did:

1. Define the class of morphisms  $\mathbf{E} := \text{Reg}(\Lambda)$ ;
2. Define the *coimage-factorizations* in terms of  $\mathbf{E}$ ;
3. Define  $\sigma_{\mathbf{E}}(f)$  as the length of the iterated coimage-factorization of  $f$ ;
4. Compare  $\sigma_{\mathbf{E}}(f)$  with the minimum decomposition number  $\delta_{\mathbf{E}}(f)$ .

### Idea

Abstracting a class  $\mathbf{E}$  with coimage-factorizations, we can follow the same story from 2. to 4.



## Definition [MT82]

$X$  and  $A' \xrightarrow{a} A$  are locally orthogonal  
 $f \downarrow$  and  $\downarrow g$  are locally orthogonal  
 $Y$  and  $B$  (written  $f \perp^a g$ )

$$\begin{array}{c} X \xrightarrow{\forall} A' \xrightarrow{a} A \\ f \downarrow \qquad \qquad \downarrow g \\ Y \qquad \qquad B \end{array} \stackrel{\text{def}}{\iff} \begin{array}{c} X \xrightarrow{\forall} A' \xrightarrow{a} A \\ f \downarrow \qquad \qquad \downarrow g \\ Y \xrightarrow{\exists!} B \\ \swarrow \qquad \searrow \\ \forall \end{array}$$

## Recall Definition [MT82]

$E$ : iso-closed.

$\mathcal{C}$  admits locally orthogonal  $E$ -factorizations  $\stackrel{\text{def}}{\iff} \forall f$  can be decomposed as

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ E \ni e \searrow & \nearrow m & \end{array} \quad \text{with } E \perp^e m.$$

## Example

$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ e \searrow & \nearrow m & \end{array}$  is the coimage-factorization in terms of  $E = \text{Reg}(\Lambda)$

$\iff$  it is a locally orthogonal  $E$ -factorization, i.e.,  $e \in E$  and  $E \perp^e m$ .

Locally orthogonal factorizations subsume the coimage-factorization!

## Theorem [Tho83; KN]

$\mathcal{C}$ : co-well-powered, having small colimits and products.  
TFAE for a class  $\mathbf{E} \subseteq \{\text{epis}\}$ :

- ①  $\mathcal{C}$  admits locally orthogonal  $\mathbf{E}$ -factorizations.
- ②  $\mathbf{E}$  is closed under multiple pushout.
- ③  $\mathbf{E} = \mathbf{Reg}(\Lambda)$  for some  $\Lambda \subseteq \{\text{epis}\}$ .

## Question

Is there a non-epimorphic example?

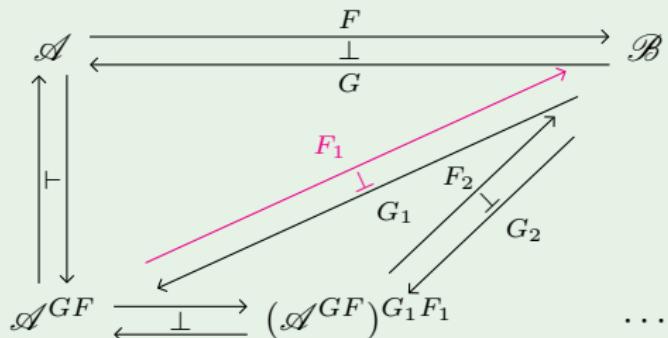
# The Eilenberg–Moore decomposition (monadic towers)

## Example

**AdjCAT**: the category of (large) categories and left adjoints.

$\mathbf{E} := \{\text{monadic adjunctions}\}$ .

~~~ **AdjCAT** admits locally orthogonal  $\mathbf{E}$ -factorizations [MT82].



This **stabilizes** at  $\alpha \stackrel{\text{def}}{\Leftrightarrow} \mathbf{E} \perp F_\alpha \Leftrightarrow F_\alpha$  : fully faithful  
~~~  $\sigma_{\mathbf{E}}(F) =$  the *monadic length* of  $F \dashv G$ .

# Minimum decomposition problem

## Theorem (Canonical vs minimum, the most general [KN])

$\mathcal{C}$ : a category with locally orthogonal  $\mathbf{E}$ -factorizations and colimits of chains.

$f$ : a morphism in  $\mathcal{C}$ .

- ①  $\sigma_{\mathbf{E}}(f)$  is defined  $\iff \delta_{\mathbf{E}}(f)$  is defined.
- ②  $\delta_{\mathbf{E}}(f)$ : 0 or successor  $\implies \sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$ .
- ③  $\delta_{\mathbf{E}}(f)$ : limit  $\implies \sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$  or  $\delta_{\mathbf{E}}(f) + 1$ .

## Corollary

The monadic length coincides with the shortest possible length of monadic decompositions if they are finite.

# Thank you!

slides ↵



This talk is based on the first half of [KN].

## References

- [BBP99] M. A. Bednarczyk, A. M. Borzyszkowski, and W. Pawłowski. "Generalized congruences—epimorphisms in *Cat*". In: *Theory Appl. Categ.* 5 (1999), No. 11, 266–280.
- [Bör91] R. Börger. "Making factorizations compositive". In: *Comment. Math. Univ. Carolin.* 32.4 (1991), pp. 749–759.
- [GU71] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Vol. Vol. 221. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1971.
- [KN] Y. Kawase and H. Nasu. *On the decomposition of a strong epimorphism into regular epimorphisms*. in preparation.
- [MT82] J. MacDonald and W. Tholen. "Decomposition of morphisms into infinitely many factors". In: *Category Theory (Gummersbach, 1981)*. Vol. 962. Lecture Notes in Math. Springer, Berlin-New York, 1982, pp. 175–189.
- [MS82] J. L. MacDonald and A. Stone. "The tower and regular decomposition". In: *Cahiers Topologie Géom. Différentielle* 23.2 (1982), pp. 197–213.
- [Tho83] W. Tholen. "Factorizations, localizations, and the orthogonal subcategory problem". In: *Math. Nachr.* 114 (1983), pp. 63–85.