Double categories of profunctors

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 \leftarrow Today's slides

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Abstract

 \mathbb{X} : a "double category" (unital virtual double category)

----×;

The "double category" X-Prof:

- X-enriched categories;
- X-enriched functors;
- X-enriched profunctors.

Goal

To characterize the "double categories" X-Prof.

Enrichment in a monoidal category

@ Generalized enrichment

3 Colimits in a unital virtual double category

4 The characterization theorem

Enrichment in a monoidal category $(\mathscr{V}, \otimes, I)$: a monoidal category

\mathscr{V} -category ${f A}$

- A class ObA of objects;
- Hom-objects $\mathbf{A}(x,y) \in \mathscr{V} \ (x,y \in \mathrm{Ob}\mathbf{A});$
- Compositions A(x,y) ⊗ A(y,z) → A(x,z) in V (x,y,z ∈ ObA);
 Identities I → A(x,x) in V (x ∈ ObA).

\mathscr{V} -functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\mathrm{Ob}\mathbf{A} \longrightarrow \mathrm{Ob}\mathbf{B}$;
- $x \longmapsto Fx$
- \mathscr{V} -profunctor $\mathbf{A} \stackrel{P}{\longrightarrow} \mathbf{B}$

• $\mathbf{A}(x,y) \to \mathbf{B}(Fx,Fy)$ in $\mathscr{V}(x,y \in \mathrm{Ob}\mathbf{A})$.

- a Ham chiests $P(m, u) \in \mathcal{U} (m \in A, u \in \mathbf{R})$:
- Hom-objects $P(x,y) \in \mathcal{V} \ (x \in \mathbf{A}, y \in \mathbf{B});$ • Actions $\mathbf{A}(x',x) \otimes P(x,y) \to P(x',y), \ P(x,y) \otimes \mathbf{B}(y,y') \to P(x,y') \text{ in } \mathcal{V}.$

(+Axioms)

(+Axioms)

The virtual double category $\mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$

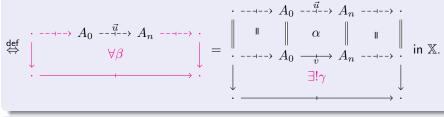
- (objects) \mathscr{V} -categories A, B, C, ...;
- ullet (vertical arrows) $\mathscr V$ -functors $\displaystyle\frac{\mathbf A}{F\downarrow}$,... and their compositions and identities; $\mathbf B$
- (horizontal arrows) \mathscr{V} -profunctors $\mathbf{A} \stackrel{P}{\longrightarrow} \mathbf{B} , \dots;$
- (cells) "generalized" \mathcal{V} -natural transformations

and their compositions (+ identity cells)

Horizontal composition

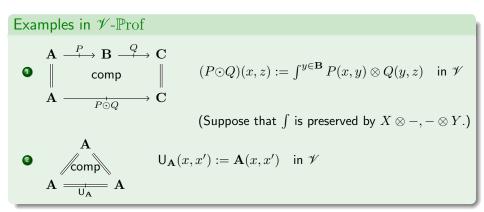
X: a virtual double category (VDC).

Definition



Definition

Horizontal composition



Unital virtual double categories

Definition

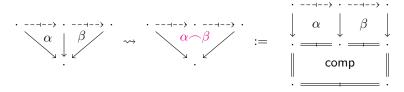
A VDC \mathbb{X} is unital $\overset{\text{def}}{\Leftrightarrow}$ Every object $X \in \mathbb{X}$ has the unit.

Example

 \mathscr{V} - \mathbb{P} rof is unital.

In a unital VDC X,

0-coary cells can be composed horizontally:



Enrichment in a monoidal category

2 Generalized enrichment

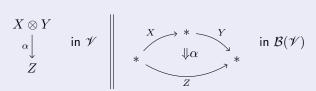
3 Colimits in a unital virtual double category

4 The characterization theorem

Monoidal categories vs. bicategories vs. VDC

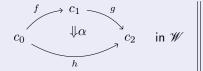
Monoidal category = single-object bicategory

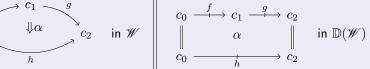
 \mathscr{V} : a mon.cat. \iff $\mathcal{B}(\mathscr{V})$: a single-obj.bicat.



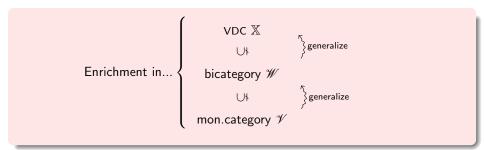
(Virtual) bicategory = vertically discrete VDC

 \mathscr{W} : a (virtual) bicat. $\iff \mathbb{D}(\mathscr{W})$: a vertically discrete VDC





Generalization of enriching bases



Remark

We obtain unital VDCs \mathcal{V} -Prof, \mathcal{W} -Prof, and \mathbb{X} -Prof for any \mathcal{V} , \mathcal{W} , \mathbb{X} .

From \mathscr{V} to \mathscr{W}

₩: a bicategory

\mathscr{W} -category \mathbf{A}

- A class ObA of objects;
- Coloring $|x| \in \mathcal{W} \ (x \in \mathrm{Ob}\mathbf{A});$
- Hom-1-cells $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathscr{W}(x,y \in \mathrm{Ob}\mathbf{A})$;
- Compositions $A(x,y) \rightarrow |y| \rightarrow A(y,z)$ in $\mathscr{W}(x,y,z \in \mathrm{Ob}\mathbf{A});$ $|x| \rightarrow |z|$
- Identities |x| |x| in \mathscr{W} $(x \in \mathrm{Ob}\mathbf{A})$. $(+\mathsf{Axioms})$

From \mathscr{V} to \mathscr{W} ₩: a bicategory

$$\mathscr{W}$$
-functor $\mathbf{A} \stackrel{F}{\longrightarrow} \mathbf{B}$

 $\mathbf{A}(x,y)$

 $\mathbf{B}(Fx,Fy)$

- A map $ObA \longrightarrow ObB$ s.t. |x| = |Fx| in \mathcal{W} ;
 - $x \longmapsto Fx$
- |x| |y| in $\mathcal{W}(x, y \in \mathrm{Ob}\mathbf{A})$.
- \mathscr{W} -profunctor $\mathbf{A} \stackrel{P}{\longrightarrow} \mathbf{B}$
 - Hom-1-cells $|x| \xrightarrow{\mathbf{P}(x,y)} |y|$ in $\mathscr{W} (x \in \mathbf{A}, y \in \mathbf{B})$;

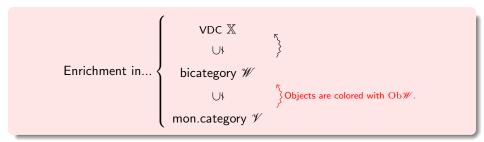
 - $\bullet \ \, \textit{Actions} \ \, \begin{matrix} \mathbf{A}(x',x) \\ |x'| \end{matrix} \qquad \begin{matrix} |x| \\ \downarrow \end{matrix} \qquad \begin{matrix} P(x,y) \\ |y| \end{matrix} \qquad \begin{matrix} P(x,y) \\ |x| \end{matrix} \qquad \begin{matrix} |y| \\ \downarrow \end{matrix} \qquad \begin{matrix} \mathbf{B}(y,y') \\ |y'| \end{matrix}$ P(x',y)P(x,y')

(+Axioms)

(+Axioms)

in W.

From \mathscr{V} to \mathscr{W}



From \mathscr{W} to \mathbb{X}

 \mathbb{X} : a VDC

\mathbb{X} -category \mathbf{A}

- A class ObA of objects;
- Coloring $|x| \in \mathbb{X} \ (x \in \mathrm{Ob}\mathbf{A});$
- Hom-horizontal arrows $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathbb{X}(x,y \in \mathrm{Ob}\mathbf{A})$;
- $|x| \xrightarrow{\mathbf{A}(x,y)} |y| \xrightarrow{\mathbf{A}(y,z)} |z|$ Compositions $\| \qquad \qquad \| \qquad \text{in } \mathbb{X} \ (x,y,z \in \mathrm{Ob}\mathbf{A});$ $|x| \xrightarrow{\mathbf{A}(x,z)} |z|$
- Identities



in \mathbb{X} $(x \in \mathrm{Ob}\mathbf{A})$.

(+Axioms)

From \mathscr{W} to \mathbb{X}

 \mathbb{X} : a VDC

Notation

 $\mathcal{H}(\mathbb{X})$ $\ \cdots$ the (virtual) bicat. obtained by forgetting all vertical arrows from $\mathbb{X}.$

Enrichment		Enrichment	
in a bicategory		in a VDC	
$\mathcal{H}(\mathbb{X})$ -categories	=	X-categories	
$\mathcal{H}(\mathbb{X})$ -functors	\subseteq	\mathbb{X} -functors	
$\mathcal{H}(\mathbb{X})$ -profunctors	=	\mathbb{X} -profunctors	

From \mathscr{W} to \mathbb{X}

\mathbb{X} -functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $ObA \longrightarrow ObB$; $x \longmapsto F^0 x$
- - "Color-comparing" vertical arrows $|x| \xrightarrow{F^1 x} |F^0 x|$ in \mathbb{X} $(x \in \mathrm{Ob}\mathbf{A})$;

 - $$\begin{split} |x| & \xrightarrow{\mathbf{A}(x,y)} |y| \\ \bullet & F^1x \downarrow \qquad \bullet \qquad \downarrow F^1y \quad \text{ in } \mathbb{X} \ (x,y \in \mathrm{Ob}\mathbf{A}). \\ |F^0x|_{\overrightarrow{\mathbf{B}(F^0x,F^0y)}}|F^0y| \end{split}$$
- Functors no longer need to preserve the color of objects.
- bicategory W Enrichment in... <

(+Axioms)

1 Enrichment in a monoidal category

Generalized enrichment

3 Colimits in a unital virtual double category

4 The characterization theorem

Collages of a profunctor

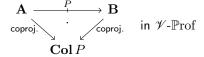
 \mathscr{V} : a monoidal category with \varnothing (the initial preserved by \otimes)

 $\mathbf{A} \stackrel{P}{\longrightarrow} \mathbf{B}$: a \mathscr{V} -profunctor

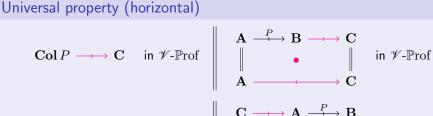
Definition

A collage (or *cograph*) of P is the \mathcal{V} -category $\mathbf{Col} P$:

$$\mathrm{Col}\,P(x,y) := \begin{cases} \mathbf{A}(x,y) & \text{if } x,y \in \mathbf{A} \\ \mathbf{B}(x,y) & \text{if } x,y \in \mathbf{B} \\ P(x,y) & \text{if } x \in \mathbf{A}, y \in \mathbf{B} \\ \varnothing & \text{if } x \in \mathbf{B}, y \in \mathbf{A} \end{cases}$$



Universal property (vertical) $\mathbf{A} \xrightarrow{P} \mathbf{B}$ in \mathscr{V} -Prof $\operatorname{Col} P$ in $\mathscr{V} ext{-}\mathbb{P} ext{rof}$



 $\mathbf{C} \xrightarrow{P} \mathbf{A} \xrightarrow{P} \mathbf{B}$ lacksquare in $\mathscr{V} ext{-}\mathbb{P}\mathrm{rof}$ $\mathbf{C} \longrightarrow \mathbf{Col} P$ in \mathscr{V} -Prof

 $\cdot \longrightarrow \mathbf{Col} P \longrightarrow \cdot$ $\operatorname{Col} P$ has universal properties in the three directions:

Collages in general

S: a set

Notation

IS · · · the VDC described by:

- $Ob(\mathbb{I}S) := S;$
 - For $i, j \in S$, there is a unique horizontal arrow $i \xrightarrow{|i_j|} j$ in $\mathbb{I}S$;

A virtual double (VD)-functor $\mathbb{I}S \longrightarrow \mathbb{X}\text{-}\mathbb{P}rof$ is equivalent to the following data:

- $\mathbf{A}_{i} \xrightarrow{P_{ii}} \mathbf{A}_{i} \qquad (i \in \mathbf{S}) \qquad \mathbf{A}_{i} \xrightarrow{J} \mathbf{A}_{j} \xrightarrow{J} \mathbf{A}_{k} \qquad (i, j, k \in \mathbf{S}) \qquad \mathbf{A}_{i} \xrightarrow{P_{ik}} \mathbf{A}_{k} \qquad (+\text{Axioms})$

Collages in general

 $P := (\mathbf{A}_i, P_{ij}, \eta_i, \mu_{ijk})_{i,j,k \in \mathbf{S}} : \text{a VD-functor } \mathbb{IS} \xrightarrow{P} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$

Definition

A collage of P is the \mathbb{X} -category $\operatorname{Col} P$:

$$Ob(\mathbf{Col}\,P) := \coprod_{i \in S} Ob\mathbf{A}_i;$$

 $\operatorname{Col} P(x,y) := P_{ij}(x,y) \text{ (where } x \in \mathbf{A}_i, y \in \mathbf{A}_j).$

$$\mathbf{A}_{i} \xrightarrow{P_{ij}} \mathbf{A}_{j}$$

$$\xi_{i} \xrightarrow{\xi_{ij}} \xi_{j} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \quad (coprojections)$$

$$\mathbf{Col} P$$

Again, $\operatorname{Col} P$ has universal properties in the three directions: $\overset{\cdot \longrightarrow}{} \operatorname{Col} P \xrightarrow{\cdot} \overset{\cdot}{}$

Versatile colimits

 \mathbb{K} : a VDC, \mathbb{L} : a unital VDC, $\mathbb{K} \xrightarrow{F} \mathbb{L}$: a VD-functor

Definition (informal)

A versatile colimit of F is a "cocone"

$$\left\{\begin{array}{ccc} FA & \xrightarrow{Fu} & FB \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & \\ & & & \\ & & \\ & & & \\ &$$

having the universal property in the three directions $\stackrel{\cdot \longrightarrow \Xi}{\downarrow} \stackrel{\cdots}{\longrightarrow} \stackrel{\cdot}{\sqsubseteq} \dots$ in $\mathbb{L}.$

Example

- $\textbf{ Versatile collages} \; (:= \text{vers.colim. of shapes} \; \mathbb{K} = \mathbb{I} S)$
- Versatile coproducts (:= vers.colim. of discrete shapes)

Enrichment in a monoidal category

@ Generalized enrichment

3 Colimits in a unital virtual double category

The characterization theorem

The object classifier in \mathscr{V} -Prof

 $(\mathscr{V}, \otimes, I)$: a monoidal category

Notation

The unit \mathcal{V} -category \mathbf{I} :

$$ObI := \{*\}, \quad I(*,*) := I.$$

The unit \mathscr{V} -category classifies objects

For every \mathscr{V} -category \mathbf{A} ,



Object classifiers in \mathcal{W} -Prof

 \mathcal{W} : a bicategory

Notation

The unit \mathcal{W} -category \mathbf{I}_c for $c \in \mathcal{W}$:

$$\mathrm{Ob}\mathbf{I}_c := \{*\} \text{ with } |*| := c, \quad \mathbf{I}_c(*,*) := (c \xrightarrow{\mathsf{id}} c \quad \mathsf{in} \ \mathscr{W})$$

The unit \mathcal{W} -categories classify objects

For every \mathcal{W} -category \mathbf{A} ,



Object classifiers in $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$

 \mathbb{X} : a unital VDC

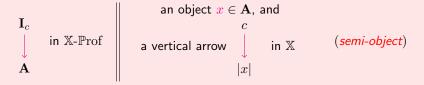
Notation

The unit X-category I_c for $c \in X$:

$$\mathrm{Ob}\mathbf{I}_c := \{*\} \text{ with } |*| := c, \quad \mathbf{I}_c(*,*) := (c =\!\!\!\!- c \quad \text{in } \mathbb{X})$$

The unit X-categories do not classify objects.

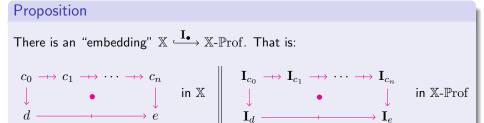
For every X-category A,



The unit X-categories classify *semi-objects*.

The embedding

 \mathbb{X} : a unital VDC



In what follows, we will consider $\mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$.

Toward the characterization: density

 $\mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ (\mathbb{X} : a unital VDC)

Observation I

Every X-category is a versatile collage of objects from X.

 $\mathbb{I}\mathbf{ObA} \xrightarrow{|\cdot|_{\mathbf{A}}} \mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}: \mathsf{a}\;\mathsf{VD}\text{-}\mathsf{functor}$

Taking the versatile colimit in X-Prof

The VD-functor $|\cdot|_{\mathbf{A}}$ is given by:

Toward the characterization: atomicity

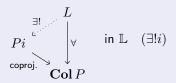
Observation II

In $\mathbb{X}\text{-}\!\operatorname{Prof},$ the unit $\mathbb{X}\text{-}\!\operatorname{categories}$ can be characterized by "atomicity" w.r.t. versatile collages.

Definition

 \mathbb{L} : a unital VDC.

 $L \in \mathbb{L}$: collage-atomic $\stackrel{\text{def}}{\Leftrightarrow}$ For every (large) versatile collage $\operatorname{Col} P$ in \mathbb{L} , vertical arrows $L \to \operatorname{Col} P$ uniquely factor through a unique coprojection.



Proposition

 \mathbb{X} : a unital VDC.

In \mathbb{X} -Prof, collage-atomic \iff vertically isomorphic to some \mathbf{I}_c .

The characterization

Theorem

TFAF for a unital VDC I.:

 $\bullet \ \ \, \text{There is an equivalence } \mathbb{L} \simeq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \,\, \text{for some unital VDC } \mathbb{X}.$

(in the 2-category of unital VDCs)

- ② ⋄ L has (large) versatile collages;
- \diamond Every object in $\mathbb L$ is a versatile collage of collage-atomic objects.

Sketch of proof: The only non-trivial part is $2. \Longrightarrow 1$. Let $\mathbb{X} := \{ \text{collage-atomic objs.} \} \subset \mathbb{L}$.

We can construct an adjunction: $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \xleftarrow{\mathbb{R}} \mathbb{L}$.

<u>Construction of R:</u> Regarding $A \in X$ -Prof as $IObA \longrightarrow X \subseteq L$, we take the vers.colim. of it.

<u>Construction of N:</u> Suppose $L \in \mathbb{L}$ is a vers.colim. of $\mathbb{I}S \xrightarrow{P} \mathbb{X} \subseteq \mathbb{L}$. Define $L \xrightarrow{N}$ "(S, P)" $\in \mathbb{X}$ -Prof.

 $\Sigma \longmapsto (S, P) \in \mathbb{A}\text{-Proj}$

 $R \circ N \cong \operatorname{Id}$ is trivial. $\operatorname{Id} \cong N \circ R$ follows from uniqueness of (S, P).

Why is $\mathbb{I}S \xrightarrow{P} \mathbb{X}$ unique for each $L \in \mathbb{L}$?

- The uniqueness of both S and the object part of P \cdots From the collage-atomicity of X.
- The uniqueness of the horizontal arrow & cell part of P \cdots From strongness theorem on versatile collages:

Strongness theorem

 \mathbb{L} : a unital VDC. $\operatorname{\mathbf{Col}} P(\in \mathbb{L})$: a versatile collage of $\mathbb{I}S \xrightarrow{P} \mathbb{L}$. \Longrightarrow Coprojections of $\operatorname{\mathbf{Col}} P$ form *cartesian cells*:

$$\begin{array}{ccc} Pi & \xrightarrow{P!_{ij}} & Pj \\ & & \text{coproj.} & & \text{in } \mathbb{L} & (i,j \in \mathbf{S}). \\ & & & \mathbf{Col} \, P \end{array}$$

Thank you!



Today's slides

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