Yusuf Kerim
Ciger

Data Analysis
Technique

Assignment - 1

1)

1. Starting from the general expressions for the mean μ and variance σ^2 of a pdf $f(x)$, namely

$$\mu = \int x f(x)\, dx,$$
$$\sigma^2 = \int (x-\bar{x})^2 f(x)\, dx,$$

show that, for the uniform distribution $U(0,1)$, these quantities are:

$$\mu = 0.5,$$
$$\sigma = \sqrt{\frac{1}{12}}.$$

[4]

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

→ To show for the uniform distribution with its mean and standard deviation, we should perform the integration with its lower limit $-\infty$, and upper limit $+\infty$.

$$\mu = \int_{-\infty}^{+\infty} x f(x)\, dx \Rightarrow \int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-0.5)^2 f(x)\, dx \Rightarrow \int_0^1 (x-0.5)^2 dx = \int_0^1 (x^2 - x + 0.25)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + 0.25x\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12} \Leftrightarrow \sigma = \sqrt{\frac{1}{12}}$$

2. A defence system is 99.5% efficient in intercepting ballistic missiles. (a) What is the probability that it will intercept all of 100 missiles launched against it? (b) How many missiles must an aggressor launch to have a better than evens (i.e. > 50%) chance of one or more penetrating the defences? (c) How many missiles would be needed to ensure a better than evens chance of more than two missiles evading the defences? (Hint: Do you think the number would be large enough that you could consider an approximation?) [10]

a) Let X be the number of missiles intercepted, that follows a binomial distribution

$X \sim \text{Binomial}(100, 0.995)$

$P(X=100) = p^{100} = (0.995)^{100}$

$= \approx 0.6058 = 60.58\%$

b)

$P(\text{at least one penetrates}) = 1 - p^n$

And we need,

$1 - p^n > 0.5$

so, $p^n < 0.5$

$n \ln(p) < \ln(0.5)$

$n \ln(0.995) < \ln(0.5)$

$n < \frac{\ln(0.5)}{\ln(0.995)} \Rightarrow n > \frac{-0.69}{-0.005} \approx 138.27$

$n = \underline{139}.$

c)

Let Y be the number of missiles that evade interception.

$Y \to$ Binomial Distribution. We want $P(Y>2) > 0.5 \Rightarrow 1 - P(Y \leq 2) > 0.5$

$\Rightarrow P(Y \leq 2) < 0.5 \Rightarrow [P(X=0) + P(X=1) + P(X=2)] < 0.5$

$P(X=0) = \binom{n}{0} p^0 (1-p)^n = (0.995)^n$

$P(X=1) = \binom{n}{1} p^1 (1-p)^{n-1} = (0.005).(0.995)^{n-1}$

$P(X=2) = \binom{n}{2} p^2 (1-p)^{n-2} = \frac{n(n-1)}{2}.(0.005)^2.(0.995)^{n-2}$

$\Rightarrow \left[(0.995)^n + \left[(0.005)^n.(0.995)^{n-1}\right] + \left[\frac{n(n-1)}{2}.(0.005)^2.(0.995)^{n-2}\right]\right] < 0.5$

To find the value of n, I am going to provide a Python script

```python
from scipy.stats import binom

p = 0.005
threshold_prob = 0.5
n = 1

while True:
    prob_X_leq_2 = binom.cdf(2, n, p)

    if prob_X_leq_2 < threshold_prob:
        break
    n += 1

print("The smallest number of missiles needed is:", n)
```

0.5s

```
The smallest number of missiles needed is: 535
```

As we can see from the output we need at least $n = 535$ missiles.

3) For this question, below, I add the screenshot of my Python notebook

```python
data = pd.read_csv('ccd_image.txt', header = None, delimiter=',')

data.head()
```

[3] 0.0s

	0	1	2	3	4	5	6	7	8	9	...	245	246	247	248	249	250	251	252	253	254
0	1279	1244	1285	1283	1329	1303	1296	1297	1292	1232	...	1146	1190	1116	1180	1208	1137	1169	1132	1141	1137
1	1301	1334	1220	1313	1329	1285	1295	1262	1258	1308	...	1207	1123	1158	1150	1193	1168	1131	1229	1126	1156
2	1352	1350	1299	1255	1276	1267	1265	1366	1297	1327	...	1177	1195	1192	1145	1194	1142	1181	1175	1147	1127
3	1267	1303	1323	1368	1243	1298	1332	1277	1316	1314	...	1192	1214	1139	1161	1143	1154	1136	1153	1128	1153
4	1267	1274	1324	1293	1294	1328	1258	1258	1320	1303	...	1171	1206	1192	1134	1194	1195	1169	1181	1166	1195

5 rows × 255 columns

A) Finding the statistics of data

```python
values = data.values.flatten()


mean = np.mean(values)
std = np.std(values)
median = np.median(values)
mode = np.bincount(values).argmax()

print( f'Mean: {mean: .2f}\nStandard Deviation: {std: .2f}\nMedian: {median}\nMode: {mode}')
```

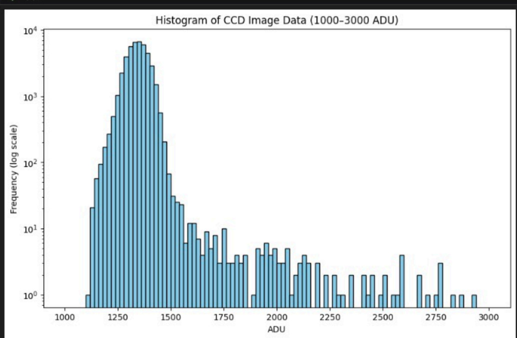
[12] 0.0s

```
Mean:  1350.91
Standard Deviation:  249.41
Median: 1343.0
Mode: 1352
```

B) Plotting the data

```python
plt.figure(figsize=(10, 6))
plt.hist(values, bins=100, range=(1000, 3000), log=True, color='skyblue', edgecolor='black')
plt.xlabel('ADU')
plt.ylabel('Frequency (log scale)')
plt.title('Histogram of CCD Image Data (1000-3000 ADU)')
plt.show()
```

[11] 0.5s



C)

- If we just checked the histogramic data, we observed that 'Median' is actually our best candidate.
- It is because of sensivity to extreme values. We can also observe that 'Median' provides the central tendency of data that is not influenced by the star brightness levels

4)

An experiment carried out twice produces the following results:

$$12 \pm 2$$
$$9 \pm 3.$$

Calculate the weighted mean result and its uncertainty. What are the 90% confidence limits on the weighted mean? (By x% confidence limits we mean the (usually symmetric) boundaries within which x% of the area falls. Hint: $\frac{1}{\sqrt{2\pi}}\int_{-1.645}^{1.645} \exp\left(-z^2/2\right)\,dz = 0.90$.)

We know that $\mu_1 = 12$, $\sigma_1 = 2$ and $\mu_2 = 9$, $\sigma_2 = 3$

↳ To find the weigthed mean, I am going to use this formula: $\bar{x}_{weighted} = \dfrac{\sum\left(\frac{x_i}{\sigma_i^2}\right)}{\sum\left(\frac{1}{\sigma_i^2}\right)}$

$$\bar{x}_{weighted} = \frac{\left(\frac{12}{4} + \frac{9}{9}\right)}{\left(\frac{1}{4} + \frac{1}{9}\right)} = \frac{4}{\frac{13}{36}} = \frac{4 \cdot 36}{13} \cong 11.08$$

↳ To find the uncertainty in weighted mean: $\sigma_{\bar{x}_{weighted}} = \dfrac{1}{\sqrt{\sum\left(\frac{1}{\sigma_i^2}\right)}}$

$$= \frac{1}{\sqrt{\frac{13}{36}}} \cong 1.66$$

↳ So, determining the 90% Confidence Interval by: $\bar{x}_{weigthed} \pm 1.645 \times \sigma_{\bar{x}_{weigthed}}$.

$$\Rightarrow \left(11.08 - 1.645 \times 1.66 \,,\, 11.08 + 1.645 \times 1.66\right)$$

$$\Rightarrow \left(8.34 \,,\, 13.81\right)$$

5)

When a muon decays, an electron is emitted at an angle θ relative to the spin, such that the probability density function of the angular distribution is

$$dP = k(1 + \alpha x)\, dx,$$

where $x = \cos\theta$ and α is a constant that must be determined experimentally. Find the value of the constant k, and the mean and variance of the x distribution (in terms of α). [5]

↳ To normalize the PDF, we need the total probability over the range of $\cos\theta\,(x)$ which is -1 to 1 to be equal to 1.

$$\Rightarrow \int_{-1}^{1} k\,dx + \int_{-1}^{1} k\alpha x\,dx \Rightarrow k[x]_{-1}^{1} + k\alpha\left[\frac{x^2}{2}\right]_{-1}^{1} = 1$$

$$\Rightarrow 2k + k\alpha 0 = 1 \iff 2k = 1 \qquad k = \frac{1}{2},$$

so $dP = \frac{1}{2}(1+\alpha x)\,dx$

↳ To find the mean:

$$\mu = \int_{-1}^{1} x \cdot \frac{1}{2}(1+\alpha x)\,dx = \underbrace{\int_{-1}^{1} \frac{1}{2} x\,dx}_{0} + \frac{\alpha}{2}\int_{-1}^{1} x^2\,dx$$

$$= \frac{\alpha}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{\alpha}{2}\cdot\frac{2}{3} = \boxed{\frac{\alpha}{3} = \mu}$$

↳ To calculate the $\mathrm{var}(x) = \langle x^2\rangle - \langle x\rangle^2$

$$\langle x^2\rangle = \int_{-1}^{1} x^2 \frac{1}{2}(1+\alpha x)\,dx = \frac{1}{2}\int_{-1}^{1} x^2\,dx + \frac{\alpha}{2}\int_{-1}^{1} x^3\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} + \frac{\alpha}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{3} + \frac{\alpha}{2}\cdot 0 = 1/3$$

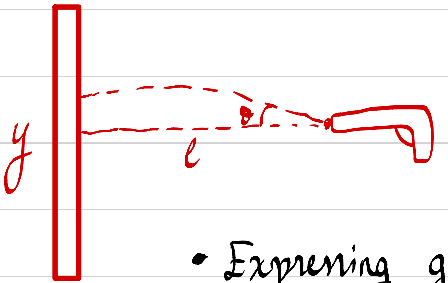
$\langle x\rangle^2 = \frac{\alpha^2}{9}$, so $\mathrm{var}(x) = \frac{1}{3} - \frac{\alpha^2}{9} = \boxed{\frac{3-\alpha^2}{9} = \mathrm{Var}(x)}$

6)

6. This question is a simple exercise in changing variables. A gun fires at random in the angular range $-\pi/2 < \theta < \pi/2$ towards a wall a distance l away. If y is the coordinate along the wall, show that

$$g(y)dy = \frac{1}{\pi}\frac{1}{1+(y/l)^2}\frac{dy}{l}.$$

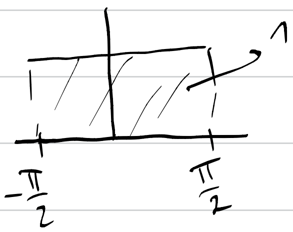
This, as we saw in the lectures, is the Cauchy distribution. Assuming that the mean should be zero from symmetry considerations, try to find the standard deviation; what problem do you have? Truncate the distribution at a distance $|y| = L$ (not the same as l) either side of the peak. Calculate the new normalisation constant, and find the standard deviation. [12]



so, $y = l\tan\theta$, so the range of y is $(-\frac{\pi}{2}, \frac{\pi}{2})$

$\frac{dy}{d\theta} = l\sec^2\theta \Rightarrow dy = l\sec^2\theta\, d\theta$, $d\theta = \frac{dy}{l\sec^2\theta}$

• Expressing $g(y)dy$ in terms of θ: we already know that pdf $f(\theta)$ for θ has intervals of $(-\frac{\pi}{2}, \frac{\pi}{2})$. Means that $f(\theta)d\theta = \frac{1}{\pi}d\theta$



$$f(\theta)d\theta = \frac{1}{\pi}\cdot\frac{1}{l\sec^2\theta}dy$$

↳ we know that $\sec^2\theta \Rightarrow 1+\tan^2\theta$, and $\tan\theta = \frac{y}{l}$

↳ $f(\theta)d\theta = \frac{1}{\pi}\cdot\frac{1}{l}\cdot\frac{1}{1+(\frac{y}{l})^2}dy$

↳ $g(y)dy = \frac{1}{\pi}\cdot\frac{1}{l}\cdot\frac{1}{1+(\frac{y}{l})^2}dy$

• Finding standard deviation assuming $\mu = 0$.

$$\sigma^2 = \int_{-\infty}^{+\infty}(y-\mu)^2 g(y)\,dy \Rightarrow \int_{-\infty}^{\infty} y^2\cdot\frac{1}{\pi l}\cdot\frac{1}{1+(y/l)^2}dy$$

$$\Rightarrow \frac{1}{\pi l}\cdot\int_{-\infty}^{+\infty}\frac{y^2}{1+(y/l)^2}dy .$$

→ After this point, since the Cauchy Distribution are too heavy, the integral for the variance diverges.

↳ Truncate distribution

$$\int_{-L}^{L} g(y)\,dy = 1 \quad , \text{ substitute } \quad g(y) = \frac{c}{\pi \cdot \ell} \cdot \frac{1}{1+(y/\ell)^2} \quad , \quad c \Rightarrow \text{normalization constant.}$$

$$\hookrightarrow \int_{-L}^{L} \frac{c}{\pi\ell} \cdot \frac{1}{1+(y/\ell)^2}\,dy = 1 \iff \frac{c}{\pi\ell} \int_{-L}^{L} \frac{1}{1+(y/\ell)^2}\,dy = 1$$

$$\boxed{c = \pi\ell \int_{-L}^{L} \left(\frac{1}{1+(y/\ell)^2}\right)^{-1} dy}$$

7)

7. A simple example of changing variables.

(a) Let x_1, x_2 be independent random variables. Let

$$\left.\begin{aligned} u_1 &= x_1 - x_2, \\ u_2 &= x_1 + x_2. \end{aligned}\right\} \quad u = A \cdot x \Rightarrow u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad := A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Assume we are given

$$\underline{\underline{V}} = \begin{pmatrix} \sigma_{x1}^2 & 0 \\ 0 & \sigma_{x2}^2 \end{pmatrix}. \qquad A^T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Find $\underline{\underline{V}}'$, the error matrix for u_1 and u_2. Substitute in $\underline{\sigma_{x_1} = 5}$, $\underline{\sigma_{x_2} = 3}$, and evaluate $\underline{\underline{V}}'$, and hence σ_{u_1} and σ_{u_2}.

(b) Undoing the error propagation:

$$x_1 = \frac{u_1 + u_2}{2},$$
$$x_2 = \frac{u_2 - u_1}{2}.$$

What do you get for $\sigma_{x_1}^2$ and $\sigma_{x_2}^2$ (using the numerical values for σ_{u_1} and σ_{u_2} found above) if you *ignore* correlations?

Now use the correct matrix method to recover the expected

$$\underline{\underline{V}} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}.$$

a) $V = \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix}$ $\qquad V' = A \cdot V \cdot A^T$

$$A \cdot V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix} = \begin{pmatrix} \sigma_{x_1}^2 & -\sigma_{x_2}^2 \\ \sigma_{x_1}^2 & \sigma_{x_2}^2 \end{pmatrix}$$

$$A \cdot V \cdot A^T = \begin{pmatrix} \sigma_{x_1}^2 & -\sigma_{x_2}^2 \\ \sigma_{x_1}^2 & \sigma_{x_2}^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_{x_1}^2 + \sigma_{x_2}^2 & \sigma_{x_1}^2 - \sigma_{x_2}^2 \\ \sigma_{x_1}^2 - \sigma_{x_2}^2 & \sigma_{x_1}^2 + \sigma_{x_2}^2 \end{pmatrix}$$

↳ Now just replace $\sigma_{x_1} = 5$ and $\sigma_{x_2} = 3$

$$V' = \begin{pmatrix} 34 & 16 \\ 16 & 34 \end{pmatrix}$$ → The variances for u_1 and u_2

↳ $Var(u_1) = Var(u_2) = 34$

↳ $\sigma_{u_1} = \sigma_{u_2} = \sqrt{34}$ ✗

b) • Ignoring Correlations.

↳ We already know that $\sigma_{u_1}^2 = \sigma_{u_2}^2 = 34$, and by ignoring correlations we assume $Cov(u_1, u_2) = 0$.

$$\sigma_{x_1}^2 = \left(\frac{1}{2}\right)^2 \sigma_{u_1}^2 + \left(\frac{1}{2}\right)^2 \sigma_{u_2}^2 \Rightarrow \underbrace{\frac{1}{4} \cdot 34}_{17} + \underbrace{\frac{1}{4} \cdot 34}_{17} = 17$$

$\sigma^2_{x_2}$ = By using same formula, $var(x_2) = var(x_1) = 17$.

$$\begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix} \neq \underbrace{\begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}}_{V}$$

- Correct Matrix Method.

$$V' = \begin{pmatrix} \sigma_{u_1}^2 & Cov(u_1, u_2) \\ Cov(u_1, u_2) & \sigma^2_{u_2} \end{pmatrix} = \begin{pmatrix} 34 & 16 \\ 16 & 34 \end{pmatrix}$$

$$, \ x_2 = \frac{u_2 - u_1}{2} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\frac{1}{2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{A^{-1}}$$

$$(A^{-1})^T = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$V = A^{-1} V' (A^{-1})^T$$

$$A^{-1} V' = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 34 & 16 \\ 16 & 34 \end{pmatrix} = \begin{pmatrix} 25 & 25 \\ -9 & 9 \end{pmatrix}$$

$$V = \begin{pmatrix} 25 & 25 \\ -9 & 9 \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}$$

$\Rightarrow$ By correct matrix method, we find the equivalent.