Homework assignment 4 – Symbolic Systems I – UvA, June 2020

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Question 1

TBox:

$$Republic \sqcup ConstitutionalMonarchy \sqsubseteq Country \tag{1}$$
$$(Person \sqcap Country) \sqcup (Person \sqcap City) \sqcup (Country \sqcap City) \sqsubseteq \bot \tag{2}$$
$$Republic \sqcap ConstitutionalMonarchy \sqsubseteq \bot \tag{3}$$
$$County \sqsubseteq \exists hasCapital.City \tag{4}$$
$$County \sqsubseteq \forall hasCapital.City \tag{5}$$
$$County \sqcup City \sqsubseteq \exists hasHead.Person \tag{6}$$
$$President \sqcup Monarch \sqcup Mayor \sqsubseteq Person \tag{7}$$
$$ConstitutionalMonarchy \sqsubseteq \forall hasHead.Monarch \tag{8}$$
$$City \sqsubseteq \forall hasHead.Mayor \tag{9}$$

ABox:

$$\text{'}NL\text{'} : Country \sqcap ConstitutionalMonarchy \tag{10}$$
$$(\text{'}NL\text{'}, \text{'}WillemAlexander\text{'}) : hasHead \tag{11}$$
$$(\text{'}NL\text{'}, \text{'}Amsterdam\text{'}) : hasCapital \tag{12}$$
$$(\text{'}Amsterdam\text{'}, \text{'}Femke\text{'}) : hasHead \tag{13}$$

Statements expressed as general concept inclusion or assertion in ALC:

$$Country \sqcap Republic \sqsubseteq \neg\exists hasHead.Mayor \tag{14}$$
$$\text{'}Femke\text{'} : Mayor \tag{15}$$
$$\text{'}NL\text{'} : \neg Republic \tag{16}$$
$$\text{'}WillemAlaxander\text{'} : \neg Mayor \tag{17}$$

In the following we will show whether (14) to (17) follow from the knowledge base.

(14): $Country \sqcap Republic \sqsubseteq \neg\exists hasHead.Mayor$
We know from (9) that *City* is only connected by *hasHead* to objects in *Mayor*. So first we must verify that *Country* or *Republic* does not overlap with *City* (verification of one of the two is enough, since we take their intersection). From (2) we have that *Country* and *City* are disjunct, so $Country \sqcap Republic$ does not overlap with *City*. However, we cannot conclude (14) yet, since both *Country* and *Republic* can still be connected by *hasHead* to objects in *Mayor* from themselves. To take away this possibility, we must say that either *Country* or *Republic* is only connected by *hasHead* to objects in a concept other than (and disjunct from) *Mayor*. The following two lines will therefore allow us to conclude (14) from the knowledge base:

$$President \sqcap Mayor \sqsubseteq \bot \tag{18}$$
$$Republic \sqsubseteq \forall hasHead.President \tag{19}$$

(15): $\text{'}Femke\text{'} : Mayor$
From (10), (12) and (13) we have that *'NL'* is in concept *Country*, connected by *hasCapital* to *'Amsterdam'*, which in turn is connected by *hasHead* to *'Femke'*. Now, since (5) states that objects in *Country* are only related by *hasCapital* to objects in *City*, we can conclude that *'Amsterdam'* is in *City*. And since (9) states that objects in *City* are only related by *hasHead* to objects in *Mayor*, we can conclude that *'Femke'* is in *Mayor*, without additional set inclusions or assertions.

(16): *'NL'* : $\neg$*Republic*
According to (10), *'NL'* is in *ConstitutionalMonarchy*. Given (3), we know that *ConstitutionalMonarchy* and *Republic* are disjunct, so we can conclude (16) from the knowledge base.

(17): *'WillemAlaxander'* : $\neg$*Mayor*
According to (10) and (11), *'NL'* is in *ConstitutionalMonarchy*, and is connected by *hasHead* to *'WillemAlexander'*. Since (8) states that objects in *ConstitutionalMonarchy* are only connected by *hasHead* to objects in *Monarch*, we can conclude that *'WillemAlexander'* is in *Monarch*. However we cannot conclude (17) yet, because it might be that *Monarch* and *Mayor* are overlapping. We can conclude (17) after ensuring that *Monarch* and *Mayor* are disjoint, i.e. after adding:

$$\mathit{Monarch} \sqcap \mathit{Mayor} \sqsubseteq \bot \tag{20}$$

Question 2

In order to encode 3SAT into ALC satisfiability, we take the following steps: First, define for each propositional variable $(x_1, \ldots, x_n)$ a concept $(A_1, \ldots, A_n)$ and assert $x_1 : A_1, \ldots, x_n : A_n$. Then for each clause define a concept $(C_1, \ldots, C_k)$, which is encoded such that each disjunction in the original clause in the 3CNF formula is replaced by $\sqcup$, and the variables $(x_1, \ldots, x_n)$ by their corresponding concepts $(A_1, \ldots, A_n)$. Negation $\neg$ will also be encoded as $\neg$ in ALC. For example, a clause (say the third clause in the 3CNF formula) $\neg x_1 \vee x_2 \vee x_4$ is encoded as $C_3 \sqsubseteq \neg A_1 \sqcup A_2 \sqcup A_4$ in ALC. Then finally, we combine all clauses similar to the original 3CNF formula, but with each $\wedge$ replaced by $\sqcap$, i.e. $C_\varphi \sqsubseteq C_1 \sqcap C_2 \sqcap \cdots \sqcap C_k$. C_φ is satisfiable if and only if φ is satisfiable, because whenever some inconsistency occurs, the final combination $(C_1 \sqcap C_2 \sqcap \cdots \sqcap C_k)$ will become the empty set, resulting in C_φ to be empty, which is not the case if φ is satisfiable.

Question 3

To enforce that every object in concept A_0 is related by some r to an object in $B_{1,a} \sqcap A_1$ and to an object in $B_{1,b} \sqcap A_1$, we use the following set inclusions:

$$A_0 \sqsubseteq \exists r.(B_{1,a} \sqcap A_1)$$
$$A_0 \sqsubseteq \exists r.(B_{1,b} \sqcap A_1)$$

Then to make sure that on object in $B_{1,a}$ is not also an object in $B_{1,b}$ (i.e. $B_{1,a}$ and $B_{1,b}$ are disjoint) we have:

$$B_{1,a} \sqcap B_{1,b} \sqsubseteq \bot$$

For KB_2 we add the same set inclusions, but now for one index higher, i.e.

$$A_1 \sqsubseteq \exists r.(B_{2,a} \sqcap A_2)$$
$$A_1 \sqsubseteq \exists r.(B_{2,b} \sqcap A_2)$$
$$B_{2,a} \sqcap B_{2,b} \sqsubseteq \bot$$

However, to enforce that we really only have a tree structure (and no possibility for cheating), we must set some additional constraints. This is more easily seen if the structures are drawn on a piece of paper, but the following is to ensure that (i) we have nodes/points/objects in each A_i that do not lie in A_{i+1}; and (ii) we have nodes/points/objects in each $B_{i,a}$ and $B_{i,b}$ that do not lie in $B_{i+1,a}$ or $B_{i+1,b}$. This last part is implicitly done by making sure that if an object is in concept $B_{1,a}$, all objects that it is related to (by role r) must also be in $B_{1,a}$. We get:

$$A_1 \equiv B_{1,a} \sqcup B_{1,b}$$
$$A_2 \equiv B_{2,a} \sqcup B_{2,b}$$
$$B_{1,a} \sqsubseteq \exists r.B_{1,a}$$
$$B_{1,b} \sqsubseteq \exists r.B_{1,b}$$
$$B_{2,a} \sqsubseteq \exists r.B_{2,a}$$
$$B_{2,b} \sqsubseteq \exists r.B_{2,b}$$

The first two lines, together with the fact that $B_{1,a}$ and $B_{1,b}$ resp. $B_{2,a}$ and $B_{2,b}$ are disjoint, state that A_1 is partitioned into $B_{1,a}$ and $B_{1,b}$, and A_2 is partitioned into $B_{2,a}$ and $B_{2,b}$. To make this general, for KB_n we have the

TBox:

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\begin{aligned}
A_0 &\sqsubseteq \exists r.(B_{1,a} \sqcap A_1) \\
A_0 &\sqsubseteq \exists r.(B_{1,b} \sqcap A_1) \\
A_1 &\equiv B_{1,a} \sqcup B_{1,b} \\
B_{1,a} &\sqsubseteq \exists r.B_{1,a} \\
B_{1,b} &\sqsubseteq \exists r.B_{1,b} \\
B_{1,a} \sqcap B_{1,b} &\sqsubseteq \bot \\
&\vdots \\
A_{n-1} &\sqsubseteq \exists r.(B_{n,a} \sqcap A_n) \\
A_{n-1} &\sqsubseteq \exists r.(B_{n,b} \sqcap A_n) \\
A_n &\equiv B_{n,a} \sqcup B_{n,b} \\
B_{n,a} &\sqsubseteq \exists r.B_{n,a} \\
B_{n,b} &\sqsubseteq \exists r.B_{n,b} \\
B_{n,a} \sqcap B_{n,b} &\sqsubseteq \bot,
\end{aligned}
$$

with the assertion $x : A_0$ in the ABox, which together can only be satisfied by a tree and not by a single object, or a path of objects, since we eliminated that possibility by the given constraints.

Question 4

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