

Large deviations of active work in systems of active Brownian particles

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- 1 Active matter
 - Non-equilibrium systems
 - Active Brownian particles
- 2 Large deviation theory
 - Concepts and applications
 - Cloning algorithm
- 3 Large deviations of active work
 - PSA and CM transitions
 - Brownian rotors
- 4 Conclusion

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Non-equilibrium dynamics breaks time-reversal symmetry and thus detailed balance.

Ludovic Berthier and Jorge Kurchan. "Lectures on non-equilibrium active systems". In: *arXiv preprint arXiv:1906.04039* (2019).

Michael E Cates and Julien Tailleur. "Motility-induced phase separation". In: *Annu. Rev. Condens. Matter Phys.* 6.1 (2015), pp. 219–244.

Non-equilibrium dynamics breaks time-reversal symmetry and thus detailed balance. We can identify 3 general classes:

- Systems relaxing towards equilibrium.

Example

Thermal system adapting to its thermostat, glasses.

Ludovic Berthier and Jorge Kurchan. "Lectures on non-equilibrium active systems". In: *arXiv preprint arXiv:1906.04039* (2019).

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Non-equilibrium dynamics breaks time-reversal symmetry and thus detailed balance. We can identify 3 general classes:

- Systems relaxing towards equilibrium.
- Systems with boundary conditions imposing steady currents.

Example

Sheared liquid, metal rod between two thermostats.

Ludovic Berthier and Jorge Kurchan. "Lectures on non-equilibrium active systems". In: *arXiv preprint arXiv:1906.04039* (2019).

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Non-equilibrium dynamics breaks time-reversal symmetry and thus detailed balance. We can identify 3 general classes:

- Systems relaxing towards equilibrium.
- Systems with boundary conditions imposing steady currents.
- Active matter.

Definition

System composed of self-driven units, *active particles*, each capable of converting stored or ambient free energy into systematic movement.

M Cristina Marchetti et al. "Hydrodynamics of soft active matter". In: *Reviews of Modern Physics* 85.3 (2013), p. 1143.

Example

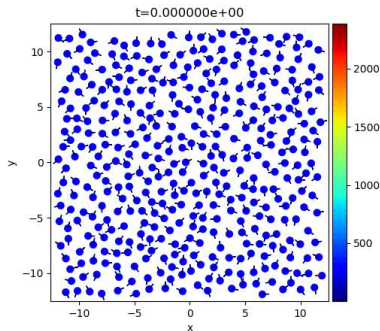
Cell tissues, swarms of bacteria, schools of fish, flocks of birds.

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→ Aligning self-propelled disks with repulsive interactions (Vicsek model).

$$\underline{\dot{r}}_i = v_0 \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix} - \mu \sum_{j=1}^N \nabla U_{ij}, \quad \dot{\theta}_i = \frac{1}{\tau} (\varphi_i - \theta_i) + \xi_i, \quad \varphi_i = \arg(\underline{\dot{r}}_i)$$



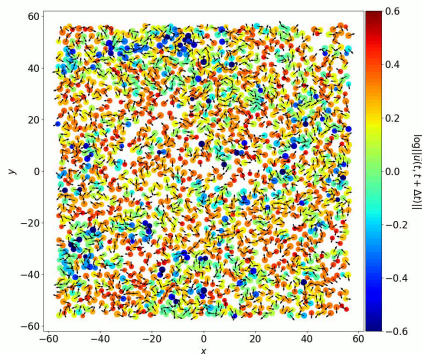
Tamás Vicsek et al. "Novel type of phase transition in a system of self-driven particles". In: *Physical review letters* 75.6 (1995), p. 1226.

Balint Szabo et al. "Phase transition in the collective migration of tissue cells: experiment and model". In: *Physical Review E* 74.6 (2006), p. 061908.

→ Active Brownian particles with repulsive soft interactions.

$$\dot{\underline{r}}_i = v_0 \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix} - \mu \sum_{j=1}^N \nabla U_{ij}, \quad \dot{\theta}_i = \sqrt{2\tau^{-1}} \xi_i$$

$$N = 2.00e+03, \phi = 0.50, \tilde{v} = 1.00e-02, \tilde{v}_r = 5.00e-06, L = 1.128e+02 \\ t = 0.00000e+00, \Delta t = 5.00000e+02$$



Yann-Edwin Keta and Jörg Rottler. "Cooperative motion and shear strain correlations in dense 2D systems of self-propelled soft disks". In: *EPL (Europhysics Letters)* 125.5 (2019), p. 58004.

→ For $\{\underline{r}_i, \underline{u}_i\}_0^\tau$ a translational and orientational trajectory...

$$\dot{S}_N[\{\underline{r}_i, \underline{u}_i\}_0^\tau] = \frac{1}{\tau} \log \frac{\mathcal{P}_N[\{\underline{r}_i, \underline{u}_i\}_0^\tau]}{\mathcal{P}_N^R[\{\underline{r}_i, \underline{u}_i\}_0^\tau]},$$

$$\dot{S}_N = \lim_{\tau \rightarrow \infty} \left\langle \dot{S}_N[\{\underline{r}_i, \underline{u}_i\}_0^\tau] \right\rangle,$$

→ ... defines a distance to equilibrium which cancels at equilibrium.

$$\dot{S}_N[\{\underline{r}_i, \underline{u}_i\}_0^\tau] \propto \frac{1}{\tau} \Delta F \Rightarrow \dot{S}_N = 0.$$

Étienne Fodor et al. "How far from equilibrium is active matter?" In: *Physical review letters* 117.3 (2016), p. 038103.

Cesare Nardini et al. "Entropy production in field theories without time-reversal symmetry: quantifying the non-equilibrium character of active matter". In: *Physical Review X* 7.2 (2017), p. 021007.

→ N ABPs with evolution

$$\begin{aligned}\dot{\underline{r}}_i &= -\mu \sum_{j=1}^N \nabla U_{ij} + v_0 \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix} + \sqrt{2D} \underline{\eta}_i, \\ \dot{\theta}_i &= \sqrt{2D_r} \xi_i,\end{aligned}$$

→ $U_{ij} \equiv$ WCA potential, $\underline{\eta}_i$, $\xi_i \equiv$ Gaussian white noises with unit variance and zero mean, $\sigma \equiv$ diameter, $\phi \equiv$ packing fraction.

Takahiyo Nemoto et al. "Optimizing active work: Dynamical phase transitions, collective motion, and jamming".
In: *Physical Review E* 99.2 (2019), p. 022605.

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7 control parameters: N , ϕ , σ , μ , v_0 , D , D_r .

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5 control parameters: N , ϕ , μ , D , D_r .

→ units of space and time: $\sigma = 1$, $\sigma/v_0 = 1$

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5 control parameters: N , ϕ , μ , D , $\frac{l_p}{\sigma} = D_r^{-1}$.

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$$\dot{\underline{r}}_i = -\mu \sum_{j=1}^N \nabla U_{ij} + \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix} + \sqrt{\frac{2}{3} \frac{\sigma}{l_p}} \underline{\eta}_i,$$

$$\dot{\theta}_i = \sqrt{2 \frac{\sigma}{l_p}} \xi_i,$$

→ $U_{ij} \equiv$ WCA potential, $\underline{\eta}_i$, $\xi_i \equiv$ Gaussian white noises with unit variance and zero mean, $\phi \equiv$ packing fraction.

4 control parameters: N , ϕ , μ , $\frac{l_p}{\sigma} = D_r^{-1}$.

→ units of space and time: $\sigma = 1$, $\sigma/v_0 = 1$

→ Stokes-Einstein-Debye relation: $D = \frac{1}{3} D_r$

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3 control parameters: N , ϕ , $\frac{l_p}{\sigma} = D_r^{-1}$.

→ units of space and time: $\sigma = 1$, $\sigma/v_0 = 1$

→ Stokes-Einstein-Debye relation: $D = \frac{1}{3} D_r$

→ $\mu = D$

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$$\dot{S}_N = \lim_{\tau \rightarrow \infty} \left\langle \frac{1}{\tau} \log \frac{\mathcal{P}_N[\{\underline{r}_i, \underline{u}_i\}_0^\tau]}{\mathcal{P}_N^R[\{\underline{r}_i, \underline{u}_i\}_0^\tau]} \right\rangle = 3 \frac{l_p}{\sigma} N \lim_{\tau \rightarrow \infty} \langle w_\tau \rangle$$

$$\begin{aligned} w_\tau &= \frac{1}{N\tau} \int_0^\tau \sum_{i=1}^N \dot{\underline{r}}_i(t) \cdot \underline{u}(\theta_i(t)) \, dt \\ &= \frac{1}{N\tau} \int_0^\tau \sum_{i=1}^N \left(1 - \frac{1}{3} \frac{\sigma}{l_p} \sum_{j=1}^N \underline{u}(\theta_i) \cdot \nabla U_{ij} + \sqrt{\frac{2}{3}} \frac{\sigma}{l_p} \underline{u}(\theta_i) \cdot \underline{\eta}_i \right) \, dt \end{aligned}$$

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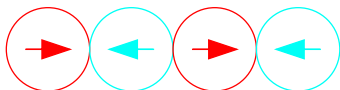
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Flocking



$$\nabla U_{ij} = 0 \Rightarrow w_\tau \approx 1$$

Jamming



$$\dot{\underline{r}}_i \approx 0 \Rightarrow w_\tau \approx 0$$

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→ X_1, \dots, X_n a sequence of random numbers and its sample average

$$S_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

$$S_n \text{ satisfies a LDP} \Leftrightarrow \lim_{n \rightarrow \infty} -\frac{1}{n} \log P(S_n = s) = I(s)$$

$$I \equiv \text{rate function of } S_n \Leftrightarrow P(S_n = s) \asymp \exp(-nI(s))$$

→ X_1, \dots, X_n random numbers from a Gaussian distribution (μ, σ) .

$$S_n = \frac{1}{n} \sum_{i=1}^N X_i,$$

$$P(S_n = s) = \sqrt{\frac{n}{2\pi\sigma^2}} e^{-n(s-\mu)^2/(2\sigma^2)}.$$

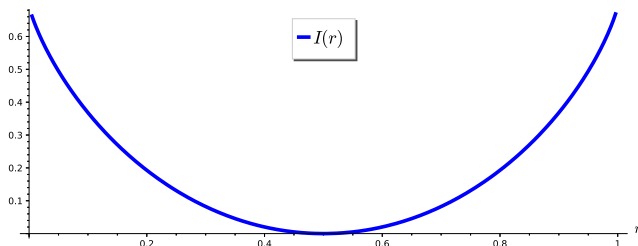
$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log P(S_n = s) = (s - \mu)^2/(2\sigma^2) = I(s).$$

⇒ S_n satisfies a large deviation principle.

→ B_1, \dots, B_n random bits, taking value 0 or 1 with equal probability.

$$R_n = \frac{1}{n} \sum_{i=1}^n B_i$$

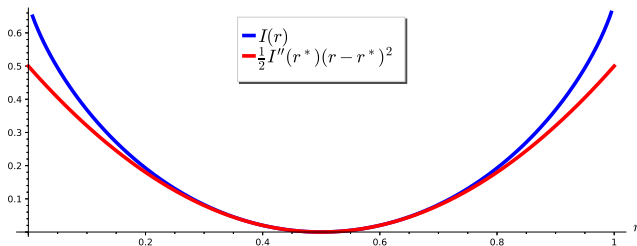
$$P(R_n = r) \asymp \exp(-nI(r)), \quad I(r) = \log 2 + r \log r + (1 - r) \log(1 - r).$$



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→ Deviations from the Gaussian fluctuations predicted by the Central Limit Theorem \Rightarrow *large* deviations.

Scaled cumulant generating function (SCGF)

$$\lambda(k) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int e^{nka} P(A_n = a) da = \lim_{n \rightarrow \infty} \frac{1}{n} \log \langle e^{nkA_n} \rangle.$$

$$\lambda \text{ is differentiable} \Rightarrow I(a) = \sup_k \{ka - \lambda(k)\} = k(a)a - \lambda(k(a))$$

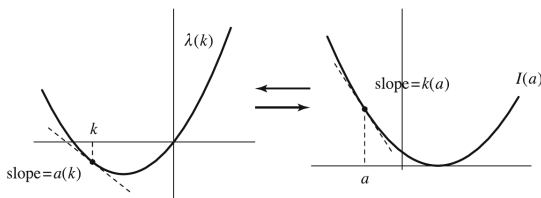


Figure: [from: Hugo Touchette. “The large deviation approach to statistical mechanics”. In: *Physics Reports* 478.1-3 (2009), pp. 1–69]

$$E_n \equiv \text{energy of } n \text{ particles} \Rightarrow P_\beta(\omega) = \frac{e^{-\beta n E_n(\omega)}}{Z_n(\beta)} \equiv \text{Boltzmann distribution}$$

$$\begin{aligned} \psi(\Delta\beta) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \int e^{\Delta\beta n E_n(\omega)} P_\beta(\omega) d\omega = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{Z_n(\beta - \Delta\beta)}{Z_n(\beta)} \\ &= \beta F(\beta) - (\beta - \Delta\beta) F(\beta - \Delta\beta) \end{aligned}$$

$$F(\beta) = -\frac{1}{\beta} \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\beta) \equiv \text{free energy density}$$

$$\psi \text{ is differentiable} \xrightarrow[\text{Gärtner-Ellis}]{} P_\beta(E_n) \asymp \exp(-nI_\beta(E_n))$$

$$P_\beta(E_n) \asymp \exp(n(\underbrace{s(E_n)}_{\substack{\text{entropy} \\ \equiv \text{number of states}}} - \underbrace{\beta E_n}_{\substack{\text{Boltzmann} \\ \text{function}}} + \underbrace{\beta F(\beta)}_{\substack{\text{partition} \\ \text{function}}}))$$

$$I_\beta(E_n) = -s(E_n) - \beta E_n + \beta F(\beta)$$

$$I_\beta(E_n) = 0 \Leftrightarrow F(\beta) = E_n - \frac{1}{\beta}s(E_n)$$

→ d -dimensional system of size N , quantities $a_i(t)$ over trajectories.

$$A_{N\tau} = \frac{1}{N\tau} \int_0^\tau \sum_{i=1}^N a_i(t) dt$$

$$I_N(a) = \lim_{\tau \rightarrow \infty} -\frac{1}{\tau} \log P(A_\tau = a), \quad \psi_N(s) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log \left\langle e^{sN\tau A_\tau} \right\rangle$$

Quantity	Equilibrium counterpart
a_i	Microstates of $(d + 1)$ -dimensional system
$A_{N\tau}$	Mean energy
s	Inverse temperature (conjugate to the energy)
ψ_N	Free energy difference
I_N	$-s(E_n) - \beta E_n + \beta F(\beta)$

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⇒ Singularities in I_N/N and ψ_N/N in the limit $\tau \rightarrow \infty$ and $N \rightarrow \infty \Rightarrow$ *dynamical phase transitions*.

$$Z_\tau(s) = \left\langle e^{sN_\tau A_{N_\tau}} \right\rangle \equiv \begin{array}{l} \text{dynamical partition function} \\ \text{of a Boltzmann-like measure} \end{array}$$

→ $s \neq 0 \Rightarrow$ average dominated by trajectories with rare events.

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→ *Cloning algorithm* to generate the biased measure.

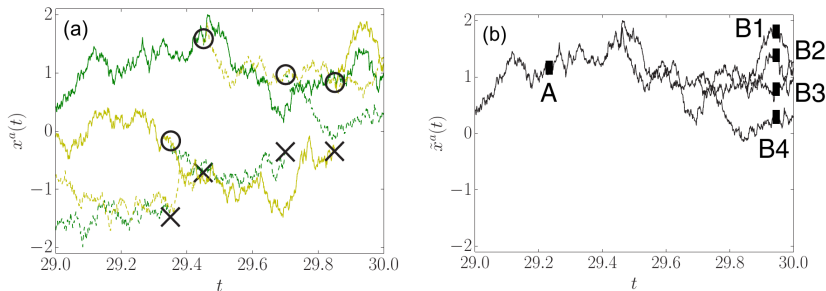


Figure: $Z_\tau(s) = \left\langle \exp \left(s \int_0^\tau x(t)(1 + x(t)) dt \right) \right\rangle$. [from: Takahiro Nemoto et al. "Population-dynamics method with a multicanonical feedback control". In: *Physical Review E* 93.6 (2016), p. 062123]

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⇒ Cloning algorithm → generate trajectories of systems of ABPs where large deviations of the active work are typical.

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→ Compute SCGF...

$$\psi_N(s, \tau) = \frac{1}{\tau} \log \langle e^{-sN\tau w_\tau} \rangle,$$

$s > 0 \Leftrightarrow$ large **negative** fluctuations of $w(s)$

... biased average of the active work...

$$w(s) = \langle w \rangle_s = -\psi'_N(s)/N,$$

... and rate function.

$$I_N(w) = \sup_s \{-sNw - \psi_N(s)\} = -s(w)Nw - \psi_N(s(w)).$$

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→ Look for singularities in I_N/N and $\psi_N/N \Rightarrow$ fundamental changes in the mechanisms to produce the associated fluctuations of the active work.

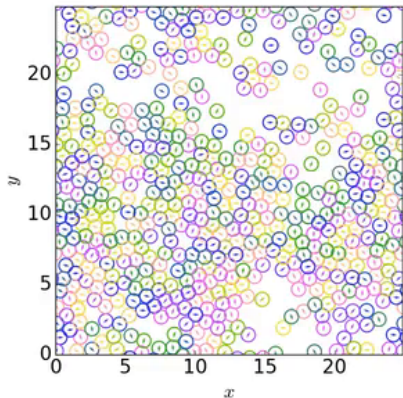


Figure: (Movie) Unbiased trajectory for $\phi = 0.65$, $l_p/\sigma = 40$. [from: Takahiro Nemoto et al. "Optimizing active work: Dynamical phase transitions, collective motion, and jamming". In: *Physical Review E* 99.2 (2019), p. 022605]

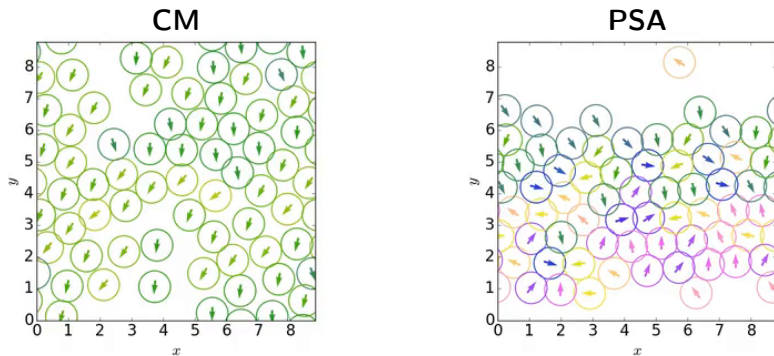


Figure: (Movie) Biased trajectories for $N = 64$, $\phi = 0.65$, $l_p/\sigma = 40$. **(left)** $s = -3.2$. **(right)** $s = 0.8$. [from: Takahiro Nemoto et al. “Optimizing active work: Dynamical phase transitions, collective motion, and jamming”. In: *Physical Review E* 99.2 (2019), p. 022605]

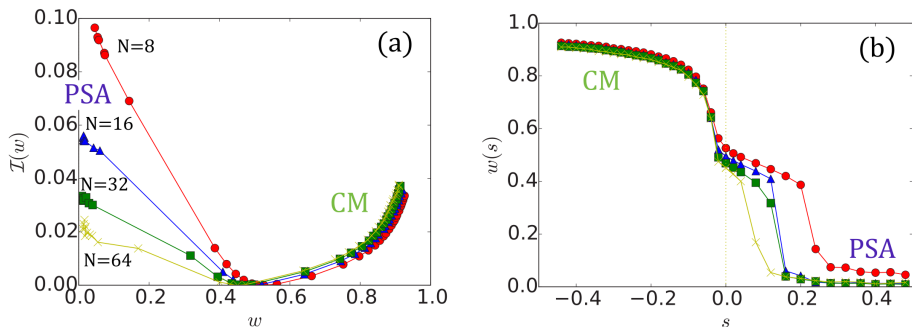


Figure: (a) Rescaled rate function $\mathcal{I} = I_N/N$. (b) Biased average of active work $w(s) = \langle w \rangle_s$. $\phi = 0.65$, $l_p/\sigma = 40$. [from: Takahiro Nemoto et al. "Optimizing active work: Dynamical phase transitions, collective motion, and jamming". In: *Physical Review E* 99.2 (2019), p. 022605]

$$\hat{\nu} = \left| \frac{1}{N} \sum_{i=1}^N \underline{u}_i \right| \equiv \text{global order parameter}, \quad \nu_\tau = \frac{1}{\tau} \int_0^\tau \hat{\nu}(t) dt$$

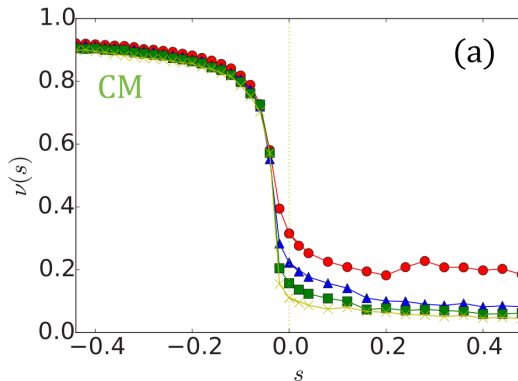


Figure: Biased average of global order parameter $\nu(s) = \langle \nu \rangle_s$. $\phi = 0.65$, $l_p/\sigma = 40$. [from: Takahiro Nemoto et al. "Optimizing active work: Dynamical phase transitions, collective motion, and jamming". In: *Physical Review E* 99.2 (2019), p. 022605]

Two limits to this study of the CM transition:

- (1) Only two high persistence lengths have been considered.
- (2) No claim on the locus of the CM transition.

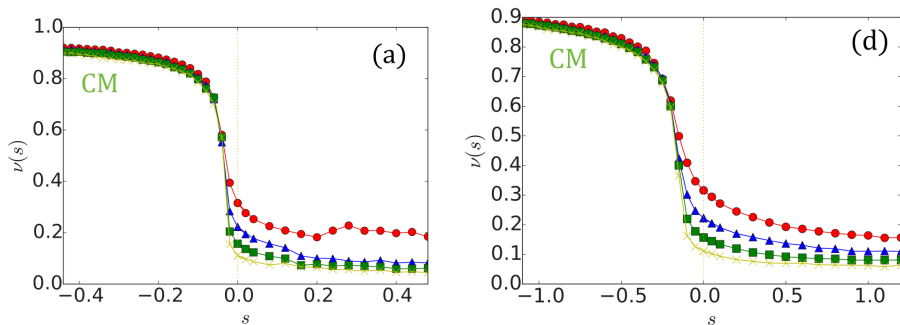
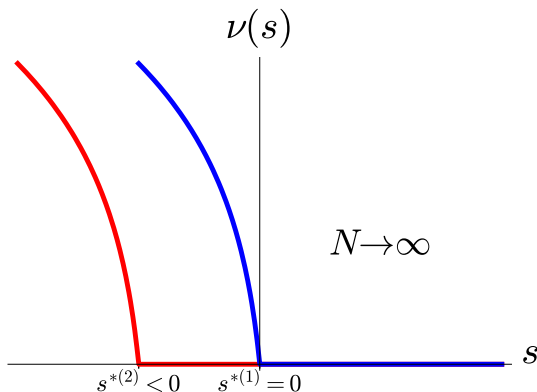


Figure: Biased average of global order parameter $\nu(s) = \langle \nu \rangle_s$. $\phi = 0.65$. **(a)** $l_p/\sigma = 40$ **(d)** $l_p/\sigma = 6.7$. [from: Takahiro Nemoto et al. "Optimizing active work: Dynamical phase transitions, collective motion, and jamming". In: *Physical Review E* 99.2 (2019), p. 022605]

→ Yet there is evidence that this locus may be affected by l_p/σ !



⇒ Important to disentangle active work and the coupling of orientation, and to understand the relation between them.

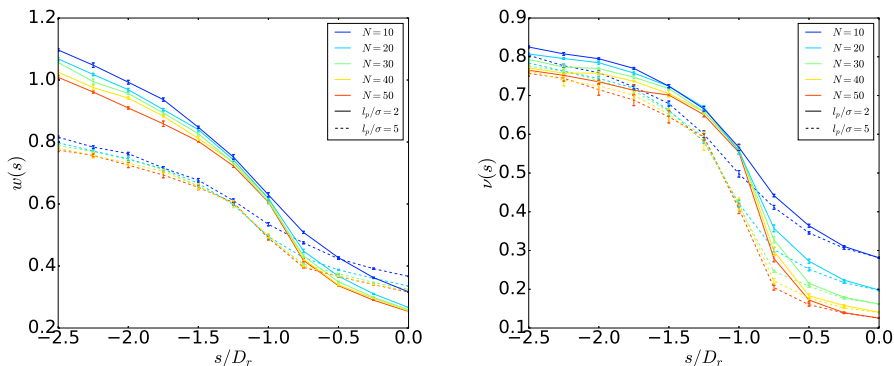


Figure: $\phi = 0.65$, $n_c = 10^2$, $t_{\text{obs}} = 10^2$. $D_r^{-1} = l_p/\sigma$. **(left)** Biased average of active work. **(right)** Biased average of the global order parameter.

$$s^* \approx -(l_p/\sigma)^{-1}$$

→ N independent Brownian rotors

$$\dot{\theta}_i = \sqrt{2D_r}\xi_i,$$

which trajectories we bias with respect to

$$\epsilon_\tau[f] = \frac{1}{\tau} \int_0^\tau f(\nu(t)) dt.$$

$\epsilon_\tau[f] \equiv$ mean energy of microstates (trajectories)

$$\begin{aligned}\epsilon_{\tau}^{(1)} &= \frac{1}{\tau} \int_0^{\tau} \nu(t) dt = \frac{1}{N\tau} \int_0^{\tau} \sum_{i=1}^N \cos(\theta_i(t) - \varphi(t)) dt \\ &\approx \frac{1}{N\tau} \int_0^{\tau} \sum_{i=1}^N \cos(\theta_i(t)) dt\end{aligned}$$

⇒ Coupling to an external field.

θ_i relax significantly faster than φ when $N \rightarrow \infty \Rightarrow \varphi$ treated as fixed external parameter.

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⇒ Coupling to an external field.

$$\epsilon_{\tau}^{(2)} = \frac{1}{\tau} \int_0^{\tau} \nu^2(t) dt = \frac{1}{N^2\tau} \int_0^{\tau} \sum_{i,j=1}^N \cos(\theta_i(t) - \theta_j(t)) dt$$

⇒ Coupling between each couples of rotors.

θ_i relax significantly faster than φ when $N \rightarrow \infty \Rightarrow \varphi$ treated as fixed external parameter.

$$\frac{\partial}{\partial t} P[\{\theta_i\}] = D_r \sum_{i=1}^N \frac{\partial^2}{\partial \theta_i^2} P[\{\theta_i\}] = \mathcal{L} P[\{\theta_i\}] \equiv \text{Fokker-Planck equation}$$

$$N\psi_{N,f}(s) = \lim_{t \rightarrow \infty} \frac{1}{\tau} \log \left\langle e^{-sN\tau\epsilon_\tau[f]} \right\rangle$$

$$\mathcal{W}_{s,f} = \mathcal{L} - sNf(\nu) \equiv \text{tilted generator}, \quad \mathcal{W}_{s,f} P[\{\theta_i\}] = N\psi_{N,f}(s) P[\{\theta_i\}]$$

$$\psi_{N,f}(s) = \frac{1}{N} \sup_P \frac{\int d^N \{\theta_i\} P[\{\theta_i\}] \mathcal{W}_{s,f} P[\{\theta_i\}]}{\int d^N \{\theta_i\} P[\{\theta_i\}] P[\{\theta_i\}]}$$

Hugo Touchette. "Introduction to dynamical large deviations of Markov processes". In: *Physica A: Statistical Mechanics and its Applications* 504 (2018), pp. 5–19.

Robert L Jack. "Ergodicity and large deviations in physical systems with stochastic dynamics". In: *arXiv preprint arXiv:1910.09883* (2019).

$$P[\{\theta_i\}] \propto \exp \left(h(s) \sum_i \cos \theta_i \right) \equiv \text{distribution ansatz}$$

$$\psi_{N,f}(s) \geq \frac{1}{N} \sup_{h(s) \in \mathbb{R}} \frac{\int d^N \{\theta_i\} P[\{\theta_i\}] \mathcal{W}_{s,f} P[\{\theta_i\}]}{\int d^N \{\theta_i\} P[\{\theta_i\}] P[\{\theta_i\}]} = \sup_{h(s) \in \mathbb{R}} B_{s,f}(h(s))$$

$$\langle f(\nu) \rangle_s \approx -\frac{\partial}{\partial s} \sup_{h(s) \in \mathbb{R}} B_{s,f}(h(s)) \equiv \text{approximate order parameter}$$

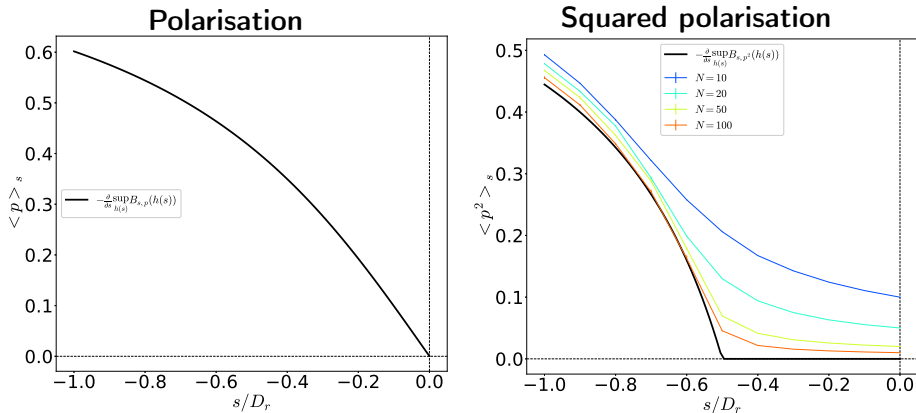


Figure: (left) Biasing with respect to polarisation. (right) Biasing with respect to squared polarisation. Numerical results from cloning with $n_c = 10^3$, $t_{\text{obs}} = 10^2$.

- 1 Active matter
 - Non-equilibrium systems
 - Active Brownian particles
- 2 Large deviation theory
 - Concepts and applications
 - Cloning algorithm
- 3 Large deviations of active work
 - PSA and CM transitions
 - Brownian rotors
- 4 Conclusion

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- An explicit link between active work and global polar order remains to be found.

Thank you!

Consider a sequence of random variables A_n satisfying a LDP with rate function I_A and an other sequence $B_n = h(A_n)$ ¹. We then have

$$P(B_n = b) = \int_{a:h(a)=b} P(A_n = a) da,$$

therefore with Laplace's approximation we can write

$$P(B_n = b) \asymp \exp \left(-n \inf_{a:h(a)=b} I_A(a) \right),$$

which is equivalent to saying that B_n satisfies a LDP with a rate function

$$I_B(b) = \inf_{a:h(a)=b} I_A(a).$$

- Since probabilities are measured on the exponential scale, the probability of any large fluctuation should be approximated by the probability of the least improbable event leading to this fluctuation.

¹ h is continuous and called a contraction of A_n .

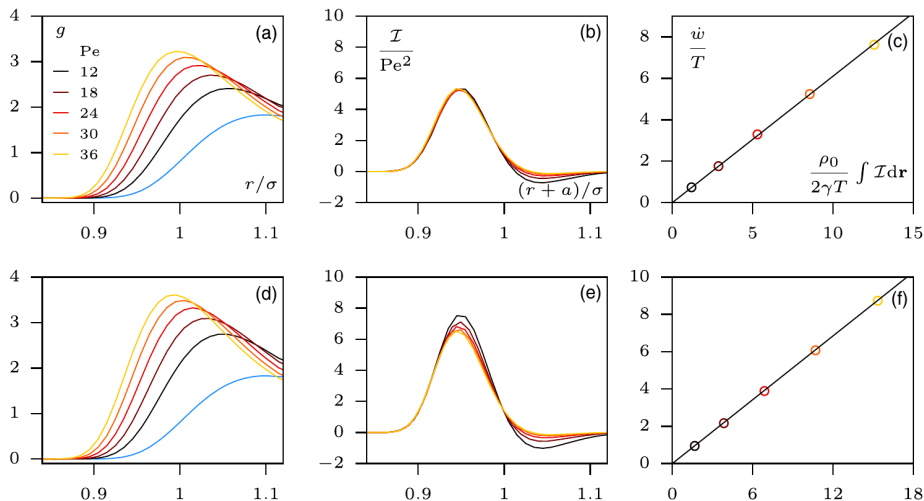


Figure: $\dot{w} \sim -w_{f,\tau}$. $\mathcal{I} = [(\nabla v)^2 - T \nabla^2 v](g - g_{eq})$. [from: Laura Tociu et al. “How Dissipation Constrains Fluctuations in Nonequilibrium Liquids: Diffusion, Structure, and Biased Interactions”. In: *Physical Review X* 9.4 (2019), p. 041026]

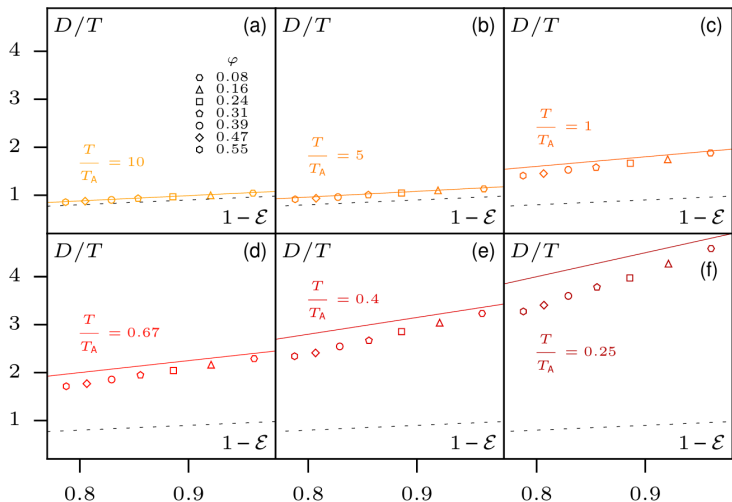


Figure: $1 - \varepsilon \sim w$. [from: Étienne Fodor, Takahiro Nemoto, and Suriyanarayanan Vaikuntanathan. “Dissipation controls transport and phase transitions in active fluids: Mobility, diffusion and biased ensembles”. In: *New Journal of Physics* (2019)]

Consider n_c copies of a system, $A_{N\tau,i}^\beta$ the value of observable $A_{N\tau}$ for copy i on interval $[(\beta - 1)\tau, \beta\tau]$, and

$$\gamma_i^\beta = e^{sN\tau A_{N\tau,i}^\beta}, \quad \gamma^\beta = \frac{1}{n_c} \sum_{i=1}^{n_c} \gamma_i^\beta, \quad \omega_i^\beta = \frac{\gamma_i^\beta}{\gamma^\beta},$$

the associated weight factors. At each cloning time step τ , we clone each copies ω_i^β times, so that we get the probability of observing a given trajectory with this algorithm²

$$P_{\text{clo}}(\{A_{N\tau,i}^\beta\}_{\beta=1}^\gamma) = P_0(\{A_{N\tau,i}^\beta\}_{\beta=1}^\gamma) \frac{\prod_{\beta=1}^\gamma \gamma_i^\beta}{\prod_{\beta=1}^\gamma \gamma^\beta} = \frac{P_s(\{A_{N\tau,i}^\beta\}_{\beta=1}^\gamma)}{\prod_{\beta=1}^\gamma \gamma^\beta},$$

then for $n_c \gg 1$

$$\prod_{\beta=1}^\gamma \gamma^\beta \approx \int P_s(A_{N\gamma\tau}) dA_{N\gamma\tau} \Rightarrow \psi_N(s, \gamma\tau) \approx \frac{1}{\gamma\tau} \sum_{\beta=1}^\gamma \log \left(\frac{1}{n_c} \sum_{i=1}^{n_c} \gamma_i^\beta \right).$$

²Tobias Brewer et al. "Efficient characterisation of large deviations using population dynamics". In: *Journal of Statistical Mechanics: Theory and Experiment* 2018.5 (2018), p. 053204, Thibault Lestang. "Numerical simulation and rare events algorithms for the study of extreme fluctuations of the drag force acting on an obstacle immersed in a turbulent flow". *PhD thesis*. 2018.

$$J(\bar{\nu}) = \lim_{\tau \rightarrow \infty} -\frac{1}{\tau} \log P \left(\int_0^\tau \nu(t) dt = \bar{\nu} \right),$$

$$I(w) = \inf_{\nu} I_2(w, \nu) = I_2(w, \nu(s(w))) \geq \inf_{w'} I_2(w', \nu(s(w))) = J(\nu(s(w)))$$

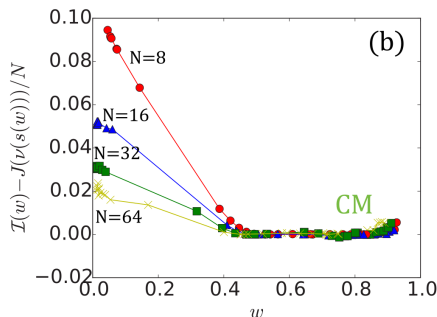


Figure: Difference of rate functions of active work and corresponding global order parameter. $\phi = 0.65$, $l_p/\sigma = 40$. We recall $\langle w \rangle_0 \approx 0.4 - 0.45$. [from: Takahiro Nemoto et al. “Optimizing active work: Dynamical phase transitions, collective motion, and jamming”. In: *Physical Review E* 99.2 (2019), p. 022605]