2 RTPs on a ring

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I. MODEL

A. Fokker-Planck equations

We consider 2 run-and-tumble particles (RTPs), with swim speed v_0 and tumble rate τ^{-1} on a linear ring of length L. With $r_1, r_2 \in [0, L]$ the positions and $\alpha_1, \alpha_2 = \pm$ the states of each particles, and $V(r_1, r_2) = V(|r_1 - r_2|)$ the potential of interactions between these particles, we have the equations of motion

$$\dot{r}_i = \alpha_i v_0 - \partial_{r_i} V(|r_1 - r_2|) \tag{1}$$

and the Fokker-Planck equations for the joint distribution of positions

$$\begin{aligned}
\dot{P}_{\alpha_1,\alpha_2}(r_1, r_2) = & -v_0(\alpha_1 \partial_{r_1} + \alpha_2 \partial_{r_2}) P_{\alpha_1\alpha_2}(r_1, r_2) && \to \text{Propulsion} \\
& + \partial_{r_1}(P_{\alpha_1\alpha_2}(r_1, r_2) \partial_{r_1} V(|r_1 - r_2|)) && \to \text{Interaction} \\
& + \partial_{r_2}(P_{\alpha_1\alpha_2}(r_1, r_2) \partial_{r_2} V(|r_1 - r_2|)) && \\
& + \tau^{-1}(P_{\overline{\alpha_1}\alpha_2}(r_1, r_2) + P_{\alpha_1\overline{\alpha_2}}(r_1, r_2) - 2P_{\alpha_1\alpha_2}(r_1, r_2)) && \to \text{Tumble} \\
= & \mathcal{L}^\dagger_{\alpha_1\alpha_2} P_{\alpha_1\alpha_2}(r_1, r_2)
\end{aligned} \tag{2}$$

with $\mathcal{L}^\dagger_{\alpha_1\alpha_2}$ the Fokker-Planck operator.

We introduce $r \equiv r_2 - r_1 > 0$, such that $\partial_{r_1} \equiv -\partial_r$ and $\partial_{r_2} = \partial_r$, with which we can define the symmetries

$$P_{++}(r) = P_{--}(r), \tag{3}$$

$$P_{+-}(r) = P_{-+}(L - r), \tag{4}$$

and the interaction potential

$$V(r) = \begin{cases} 0 & \text{if } r \in]0, L[\\ \infty & \text{otherwise} \end{cases}$$

such that, with Eq. (1), we get for particles with opposite directions

$$-\partial_r V(0) = \partial_r V(L) = v_0. \tag{5}$$

because of mutual hindrance.

We can solve the steady state problem with the normalisation condition

$$\int_0^L \mathrm{d}r \sum_{\alpha_1\alpha_2} P_{\alpha_1\alpha_2}(r) = 1 \tag{6}$$

the bulk equations

$$\dot{P}_{\alpha_1\alpha_2}(r \in]0, L[) = 0 \tag{7}$$

the left boundary equations

$$\int_{0^-}^{0^+} \mathrm{d}r \, \dot{P}_{\alpha_1\alpha_2}(r) = 0 \tag{8}$$

and the right boundary equations

$$\int_{L^-}^{L^+} \mathrm{d}r \, \dot{P}_{\alpha_1\alpha_2}(r) = 0. \tag{9}$$

B. Biased ensemble

We define our biasing quantity

$$\dot{Z} = \sum_{i=1}^{2} -\partial_r V(r) \alpha_i v_0 = v_0(\alpha_1 - \alpha_2)\partial_r V(r) \tag{10}$$

which is non-zero only at contact, and the tilted generator

$$\mathcal{W}^\dagger_{\alpha_1\alpha_2} = \mathcal{L}^\dagger_{\alpha_1\alpha_2} - s\dot{Z} \tag{11}$$

with s the biasing parameter, which eigenvalue problem

$$\psi P_{\alpha_1\alpha_2}(r) = \mathcal{W}^\dagger_{\alpha_1\alpha_2} P_{\alpha_1\alpha_2}(r) \tag{12}$$

we have to solve for $\psi(s)$ its largest eigenvalue, which defines the dynamical free energy.

While the symmetries (Eqs. (3, 4)) and the normalisation condition (Eq. (6)) remains unchanged, we have that the bulk equations (Eq. (7)), the left boundary equations (Eq. 8), and the right boundary equations (Eq. 9) become

$$\psi P_{\alpha_1\alpha_2}(r \in]0, L[) = \mathcal{W}^\dagger_{\alpha_1\alpha_2} P_{\alpha_1\alpha_2}(r \in]0, L[) \tag{13}$$

$$\int_{0^-}^{0^+} \mathrm{d}r\, \psi P_{\alpha_1\alpha_2}(r) = \int_{0^-}^{0^+} \mathrm{d}r\, \mathcal{W}^\dagger_{\alpha_1\alpha_2} P_{\alpha_1\alpha_2}(r) \tag{14}$$

$$\int_{L^-}^{L^+} \mathrm{d}r\, \psi P_{\alpha_1\alpha_2}(r) = \int_{L^-}^{L^+} \mathrm{d}r\, \mathcal{W}^\dagger_{\alpha_1\alpha_2} P_{\alpha_1\alpha_2}(r) \tag{15}$$

respectively.

II. UNBIASED STEADY STATE DISTRIBUTION

We use the following ansatz for the unbiased ($s = 0$) steady state distribution

$$P_{\alpha_1\alpha_2}(r) = a_{\alpha_1\alpha_2} + b_{\alpha_1\alpha_2}\delta(r) + c_{\alpha_1\alpha_2}\delta(L - r) \tag{16}$$

which satisfies the normalisation condition (Eq. (6))

$$\sum_{\alpha_1\alpha_2} (L a_{\alpha_1\alpha_2} + b_{\alpha_1\alpha_2} + c_{\alpha_1\alpha_2}) = 1 \tag{17}$$

the bulk equations (Eq. (7))

$$\tau^{-1}(a_{+-} + a_{-+} - 2a_{++}) = 0 \tag{18}$$
$$\tau^{-1}(a_{+-} + a_{-+} - 2a_{--}) = 0 \tag{19}$$
$$\tau^{-1}(a_{++} + a_{--} - 2a_{+-}) = 0 \tag{20}$$
$$\tau^{-1}(a_{++} + a_{--} - 2a_{-+}) = 0 \tag{21}$$

the left boundary equations (Eq. (8))

$$\tau^{-1}(b_{+-} + b_{-+} - 2b_{++}) = 0 \tag{22}$$
$$\tau^{-1}(b_{+-} + b_{-+} - 2b_{--}) = 0 \tag{23}$$
$$2v_0 a + \tau^{-1}(b_{++} + b_{--} - 2b_{+-}) = 0 \tag{24}$$
$$-2v_0 a + \tau^{-1}(b_{++} + b_{--} - 2b_{-+}) = 0 \tag{25}$$

the right boundary equations (Eq. (9))

$$\tau^{-1}(c_{+-} + c_{-+} - 2c_{++}) = 0 \tag{26}$$
$$\tau^{-1}(c_{+-} + c_{-+} - 2c_{--}) = 0 \tag{27}$$
$$-2v_0 a + \tau^{-1}(c_{++} + c_{--} - 2c_{+-}) = 0 \tag{28}$$
$$2v_0 a + \tau^{-1}(c_{++} + c_{--} - 2c_{-+}) = 0 \tag{29}$$

and the symmetries (Eqs. (3, 4)).

We get

$$P_{++}(r) = a + la\delta(r) + la\delta(L-r) \tag{30}$$
$$P_{--}(r) = a + la\delta(r) + la\delta(L-r) \tag{31}$$
$$P_{+-}(r) = a + 2la\delta(r) \tag{32}$$
$$P_{-+}(r) = a + 2la\delta(L-r) \tag{33}$$

with

$$a = \frac{1}{4(L+2l)} \tag{34}$$

and the persistence length $l = v_0\tau$.

III. BIASED STEADY STATE DISTRIBUTION

A. Exact solution

We use the following ansätze for the steady state distribution

$$P_{++}(r) = P_{--}(r) = \beta(r) + \gamma_-\delta(r) + \gamma_+\delta(L-r) \tag{35}$$
$$P_{+-}(r) = \varepsilon(r) + \zeta\delta(r) \tag{36}$$
$$P_{-+}(r) = \theta(r) + \zeta\delta(L-r) \tag{37}$$
$$\tag{38}$$

with $\varepsilon(r) = \theta(L-r)$ according to Eq. (4).

We write the eigenvalue equations (Eq. (12))

$$\psi P_{\alpha\alpha}(r) = 2\partial_r(P_{\alpha\alpha}(r)\partial_r V(r)) + \tau^{-1}(P_{+-}(r) + P_{-+}(r) - 2P_{\alpha\alpha}(r)) \tag{39}$$
$$\psi P_{+-}(r) = 2v_0\partial_r P_{+-}(r) + 2\partial_r(P_{+-}(r)\partial_r V(r)) + \tau^{-1}(2P_{\alpha\alpha}(r) - 2P_{+-}(r)) - 2sv_0\partial_r V(r)P_{+-}(r) \tag{40}$$
$$\psi P_{-+}(r) = -2v_0\partial_r P_{-+}(r) + 2\partial_r(P_{-+}(r)\partial_r V(r)) + \tau^{-1}(2P_{\alpha\alpha}(r) - 2P_{-+}(r)) + 2sv_0\partial_r V(r)P_{-+}(r) \tag{41}$$

and integrate Eq. (41) between 0^- and L^+ to get

$$\psi = 2sv_0\partial_r V(L)\zeta = 2sv_0^2\zeta \tag{42}$$

linking the dynamical free energy ψ and the sticking term ζ.

We have the bulk equations (Eq. (13))

$$\psi\beta(r) = \tau^{-1}(\varepsilon(r) + \theta(r) - 2\beta(r)) \tag{43}$$
$$\psi\epsilon(r) = 2v_0\varepsilon'(r) + \tau^{-1}(2\beta(r) - 2\varepsilon(r)) \tag{44}$$
$$\psi\theta(r) = -2v_0\theta'(r) + \tau^{-1}(2\beta(r) - 2\theta(r)) \tag{45}$$

such that Eq. (43) gives

$$\beta(r) = (2+\tau\psi)^{-1}(\epsilon(r) + \theta(r)) \tag{46}$$

and the difference and sum of Eqs. (44, 45) give

$$2l(\varepsilon(r) + \theta(r))' = (2\tau + \psi)(\varepsilon(r) - \theta(r)) \tag{47}$$
$$\begin{aligned} 2l(\varepsilon(r) - \theta(r))' &= -4\beta(r) + (\tau\psi + 2)(\varepsilon(r) + \theta(r)) \\ &= \left((\tau\psi + 2) - \frac{4}{\tau\psi + 2}\right)(\varepsilon(r) + \theta(r)) \end{aligned} \tag{48}$$

where we set $A(r) = \varepsilon(r) + \theta(r)$ and $B(r) = \varepsilon(r) - \theta(r)$, which on the one hand verify

$$A''(r) - k^2 A(r) = 0 \tag{49}$$

where

$$k^2 l^2 = \frac{\tau\psi}{2}\left(\frac{\tau\psi}{2} + 2\right) \tag{50}$$

and which general solution is

$$A(r) = A_+ e^{-kr} + A_- e^{-k(L-r)} \tag{51}$$

and on the other hand

$$B(r) = l(1 + \tau\psi/2)^{-1} A'(r) = kl(1 + \tau\psi/2)^{-1}(A_- e^{-k(L-r)} - A_+ e^{-kr}) \tag{52}$$

from which we infer

$$\begin{aligned}\varepsilon(r) &= \frac{1}{2}(A(r) + B(r)) \\ &= \frac{1}{2}\left(1 - kl(1 + \psi\tau/2)^{-1}\right) A_+ e^{-kr} + \frac{1}{2}\left(1 + kl(1 + \psi\tau/2)^{-1}\right) A_- e^{-k(L-r)}\end{aligned} \tag{53}$$

$$\begin{aligned}\theta(r) &= \frac{1}{2}(A(r) - B(r)) \\ &= \frac{1}{2}\left(1 + kl(1 + \psi\tau/2)^{-1}\right) A_+ e^{-kr} + \frac{1}{2}\left(1 - kl(1 + \psi\tau/2)^{-1}\right) A_- e^{-k(L-r)}\end{aligned} \tag{54}$$

where we note that the symmetry condition of Eq. (4) implies that $A_- = A_+$, and

$$\begin{aligned}\beta(r) &= (\tau\psi + 2)^{-1}(\varepsilon(r) + \theta(r)) = (\tau\psi + 2)^{-1} A(r) \\ &= \frac{1}{2}(1 + \tau\psi/2)^{-1}(A_+ e^{-kr} + A_- e^{-k(L-r)})\end{aligned} \tag{55}$$

where we need to determine $A_+ = A_-$.

We have the left and right boundary equations (Eqs. (14, 15)) for $P_{\alpha\alpha}$

$$\psi\gamma_- = \tau^{-1}(\zeta - 2\gamma_-) \tag{56}$$

$$\psi\gamma_+ = \tau^{-1}(\zeta - 2\gamma_+) \tag{57}$$

such that

$$\gamma = \gamma_- = \gamma_+ = (\tau\psi + 2)^{-1}\zeta \tag{58}$$

which with Eq. (42) gives

$$\psi = 2sv_0^2(\tau\psi + 2)\gamma \tag{59}$$

where we need to determine γ.

We have the left boundary equations (Eq. (14)) for $P_{\alpha\overline{\alpha}}$

$$\psi\zeta = 2v_0\varepsilon(0^+) + \tau^{-1}(2\gamma - 2\zeta) - 2sv_0\partial_r V(0)\zeta \tag{60}$$

$$0 = -2v_0\varepsilon(L^-) + \tau^{-1}2\gamma \tag{61}$$

from which we get

$$[(\tau\psi + 2)(\tau\psi + 2 - 2slv_0) - 2]\,\gamma - l\left(1 - kl(1 + \tau\psi/2)^{-1}\right) A_+ - l\left(1 + kl(1 + \tau\psi/2)^{-1}\right) e^{-kL} A_- = 0 \tag{62}$$

$$2\gamma - l\left(1 - kl(1 + \tau\psi/2)^{-1}\right) e^{-kL} A_+ - l\left(1 + kl(1 + \tau\psi/2)^{-1}\right) A_- = 0 \tag{63}$$

and the normalisation condition (Eq. (6))

$$(2\tau\psi + 8)\gamma + \frac{1}{k}(1 - e^{-kL})(1 + (1 + \tau\psi/2)^{-1})(A_+ + A_-) = 1 \tag{64}$$

so we can solve the system of Eqs. (62, 63, 64) with respect to γ, A_+, A_-, or equivalently with $A_+ = A_-$

$$[(\tau\psi + 2)(\tau\psi + 2 - 2slv_0) - 2]\,\gamma - l\left[\left(1 - kl(1 + \tau\psi/2)^{-1}\right) + \left(1 + kl(1 + \tau\psi/2)^{-1}\right)e^{-kL}\right]A_- = 0 \tag{65}$$

$$(2\tau\psi + 8)\gamma + \frac{2}{k}(1 - e^{-kL})(1 + (1 + \tau\psi/2)^{-1})A_+ = 1 \tag{66}$$

with respect to A_+ and γ.

We use SageMath to solve Eqs. (65, 66)

```
# variables
psi = var('psi')
tau = var('tau')
k = var('k')
L = var('L')
l = var('l')
v0 = var('v0')
s = var('s')

# system [gamma, A]
system = Matrix([

    [(tau*psi + 2)*(tau*psi + 2 - 2*s*l*v0) - 2,
    -l*((1 - k*l/(1 + (tau*psi)/2)) + (1 + k*l/(1 + (tau*psi)/2))*exp(-k*L))],

    [2*tau*psi + 8,
    (2/k)*(1 - exp(-k*L))*(1 + 1/(tau*psi + 2))]

])

# solution
[[gamma], [A]] = system \ Matrix([[0], [1]])
```

and get

$$\gamma = -\frac{1}{2\left(\frac{(\psi\tau+4)\left(\left(\frac{2\,kl}{\psi\tau+2}+1\right)e^{(-Lk)}-\frac{2\,kl}{\psi\tau+2}+1\right)l}{(2\,lsv_0-\psi\tau-2)(\psi\tau+2)+2}+\frac{\left(\frac{1}{\psi\tau+2}+1\right)\left(e^{(-Lk)}-1\right)}{k}\right)} \tag{67}$$

$$A_+ = A_- = \frac{\left(\left(\frac{2\,kl}{\psi\tau+2}+1\right)e^{(-Lk)}-\frac{2\,kl}{\psi\tau+2}+1\right)l}{2\left((2\,lsv_0-\psi\tau-2)(\psi\tau+2)+2\right)\left(\frac{(\psi\tau+4)\left(\left(\frac{2\,kl}{\psi\tau+2}+1\right)e^{(-Lk)}-\frac{2\,kl}{\psi\tau+2}+1\right)l}{(2\,lsv_0-\psi\tau-2)(\psi\tau+2)+2}+\frac{\left(\frac{1}{\psi\tau+2}+1\right)\left(e^{(-Lk)}-1\right)}{k}\right)}. \tag{68}$$

B. Scaling regime

We assume there exists a scaling form

$$\psi(s) = L^{-2}\Psi(sL) \tag{69}$$

so that for $s = \mathcal{O}(L^{-1})$ we have $\psi(s) = \mathcal{O}(L^{-2})$ and $k = \mathcal{O}(L^{-1})$ from Eq. (50).

We give an expression of Eqs. (63, 64) at the lowest order of L^{-1} and using $A_+ = A_-$

$$2\gamma - l(1 + e^{-kL})A_+ = 0 \tag{70}$$

$$8\gamma + \frac{4}{k}(1 - e^{-kL})A_+ = 1 \tag{71}$$

from which we infer

$$\gamma = \frac{1}{8}kl\frac{1+e^{-kL}}{1-e^{-kL}} \tag{72}$$

and an expression of Eq. (59) at the lowest order of L^{-1}

$$\gamma = \frac{\psi}{4sv_0^2} \tag{73}$$

which with Eq.(50) linking k and ψ, also at the lowest order of L^{-1},

$$k^2l^2 = \tau\psi \Rightarrow k = \begin{cases} \frac{1}{l}\sqrt{\tau\psi} & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ \mathrm{i}\,\frac{1}{l}\sqrt{-\tau\psi} & \text{if } s < 0, \end{cases} \tag{74}$$

give

$$\frac{4}{sLv_0} = \frac{\coth\left(\sqrt{\frac{\psi L^2}{4lv_0}}\right)}{\sqrt{\frac{\psi L^2}{4lv_0}}} \Leftrightarrow \frac{1}{\Lambda} = \frac{\coth\left(\sqrt{\Phi}\right)}{\sqrt{\Phi}} \qquad s > 0, \psi > 0 \tag{75}$$

$$-\frac{4}{sLv_0} = \frac{\cot\left(\sqrt{\frac{|\psi|L^2}{4lv_0}}\right)}{\sqrt{\frac{|\psi|L^2}{4lv_0}}} \Leftrightarrow -\frac{1}{\Lambda} = \frac{\cot\left(\sqrt{|\Phi|}\right)}{\sqrt{|\Phi|}} \qquad s < 0, \psi < 0 \tag{76}$$

where we have set

$$\Lambda = \frac{sLv_0}{4} \tag{77}$$

$$\Phi = \frac{\psi L^2}{4lv_0} = \frac{\Psi}{4lv_0} \tag{78}$$

for convenience, and introduce

$$\Gamma = \frac{\Phi}{\Lambda} = \frac{\psi}{4sv_0^2}\frac{4L}{l} = \gamma\frac{4L}{l} \tag{79}$$

which we plot in Fig. 1.

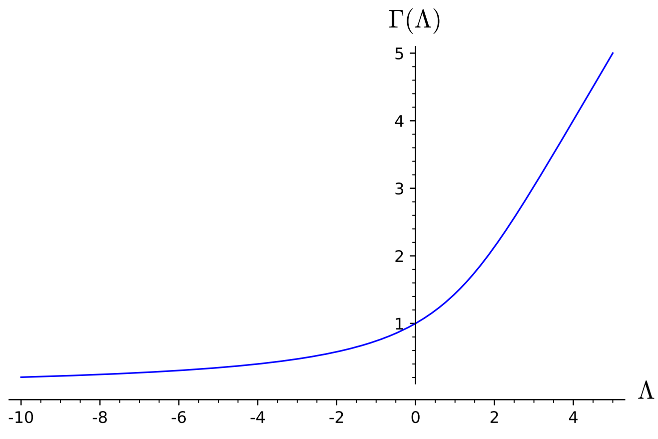

FIG. 1: Rescaled sticking term $\Gamma = (4L/l)\,\gamma$ as a function of rescaled biasing parameter $\Lambda = (Lv_0/4)\,s$.

At the lowest order in L^{-1} we have from Eqs. (53, 54, 55)

$$\varepsilon(r) = \theta(r) = \beta(r) = \frac{1}{2}A_+(e^{-kr} + e^{-k(L-r)}) \tag{80}$$

and from Eqs. (70, 71, 72)

$$A_+ = \frac{2\gamma}{l}\frac{1}{1+e^{-kL}} = \frac{k}{4(1-e^{-kL})} \tag{81}$$

so we can write with Eqs. (74, 77, 78, 79)

$$L\varepsilon(r) = \frac{1}{4}\Gamma \begin{cases} \frac{\cosh\left(\sqrt{\Phi}(1-2r/L)\right)}{\cosh\left(\sqrt{\Phi}\right)} & \text{if } s>0, \\ 1 & \text{if } s=0, \\ \frac{\cos\left(\sqrt{|\Phi|}(1-2r/L)\right)}{\cos\left(\sqrt{|\Phi|}\right)} & \text{if } s<0, \end{cases} \tag{82}$$

which we plot in Fig. 2.

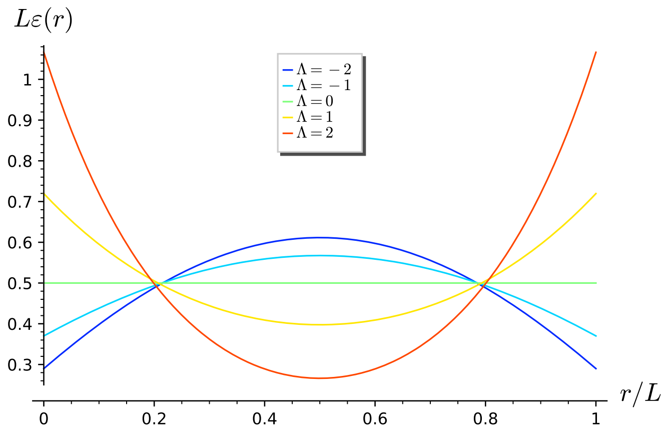

FIG. 2: Rescaled regular part $L\varepsilon(r)$ as a function of rescaled distance r/L for different values of the rescaled biasing parameter $\Lambda = (Lv_0/4)\,s$.

[SEB16] AB Slowman, MR Evans, and RA Blythe. Jamming and attraction of interacting run-and-tumble random walkers. *Physical review letters*, 116(21):218101, 2016.