Forces in vertex models

Yann-Edwin Keta
(Dated: 10/01/2024, 14:27)

I. CELL POTENTIAL ENERGY

We introduce for each cell i a reference perimeter P_i^0 and a reference area A_i^0, and for each of its cell corner $\mu \in \Omega_i$ a force whose effect is to bring the cell's perimeter P_i and area A_i to their reference quantities. A possible force derives from the following potential energy [1–4]

$$E_{\mathrm{VM}} = \sum_{\text{cells } i} \left[\frac{1}{2} K_i (A_i - A_i^0)^2 + \frac{1}{2} \Gamma_i (P_i - P_i^0)^2 \right], \tag{1}$$

where K_i and Γ_i are respectively area and perimeter elastic constants. We denote $\boldsymbol{r}_\mu$, $\boldsymbol{r}_i$ the position of vertices μ, i (see Fig. 1). The cell's perimeter can be written

$$P_i = \sum_{\mu \in \Omega_i} |\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|, \tag{2}$$

and the cell's area can be computed with the shoelace formula

$$A_i = \sum_{\mu \in \Omega_i} \frac{1}{2} [(\boldsymbol{r}_\mu - \boldsymbol{r}_i) \times (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i)] \cdot \hat{\boldsymbol{e}}_z. \tag{3}$$

Note that, by convention of ordering of cell corners, each term in this sum *must be* positive. With these notations, we thus write the force acting on vertex μ

$$\boldsymbol{F}_{\mathrm{VM},\mu} = -\nabla_\mu E_{\mathrm{VM}}, \tag{4}$$

where $\nabla_\mu \equiv \partial / \partial \boldsymbol{r}_\mu$. We compute

$$\nabla_\mu \left(|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}| \right) = \frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|}, \tag{5}$$

as well as

$$\nabla_\mu \left([(\boldsymbol{r}_\mu - \boldsymbol{r}_i) \times (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i)] \cdot \hat{\boldsymbol{e}}_z \right) = (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z \tag{6}$$

to write (4) in its full form

$$\begin{aligned} \boldsymbol{F}_{\mathrm{VM},\mu} = &- \sum_{\text{cells i, } \mu \in \Omega_i} \Bigg[\frac{1}{2} K_i (A_i - A_i^0) \left[(\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z - (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z \right] \\ &\qquad + \Gamma_i (P_i - P_i^0) \left[\frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|} + \frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|} \right] \Bigg] \\ = &\sum_{\text{cells i, } \mu \in \Omega_i} \Bigg[\frac{1}{2} K_i (A_i - A_i^0) \underbrace{(\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_{\mu+1}) \times \hat{\boldsymbol{e}}_z}_{\text{towards cell interior}} + \Gamma_i (P_i - P_i^0) \Bigg[\underbrace{\frac{\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_\mu}{|\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_\mu|}}_{\text{towards neighbour}} + \underbrace{\frac{\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_\mu}{|\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_\mu|}}_{\text{towards neighbour}} \Bigg] \Bigg], \end{aligned} \tag{7}$$

where underbraced vectors are represented in Fig. 1. It is noteworthy that, with potential (1), the force acting on each cell centre is zero

$$\boldsymbol{F}_{\mathrm{VM},i} = -\nabla_i E_{\mathrm{VM}} = -\frac{1}{2} K_i (A_i - A_i^0) \sum_{\mu \in \Omega_i} (\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}) \times \hat{\boldsymbol{e}}_z = 0. \tag{8}$$

[1] R. Farhadifar, J.-C. Röper, B. Aigouy, S. Eaton, and F. Jülicher, Current Biology **17**, 2095 (2007).
[2] A. G. Fletcher, M. Osterfield, R. E. Baker, and S. Y. Shvartsman, Biophysical Journal **106**, 2291 (2014).
[3] D. Bi, X. Yang, M. C. Marchetti, and M. L. Manning, Physical Review X **6**, 021011 (2016).
[4] R. Sknepnek, I. Djafer-Cherif, M. Chuai, C. Weijer, and S. Henkes, eLife **12**, e79862 (2023).

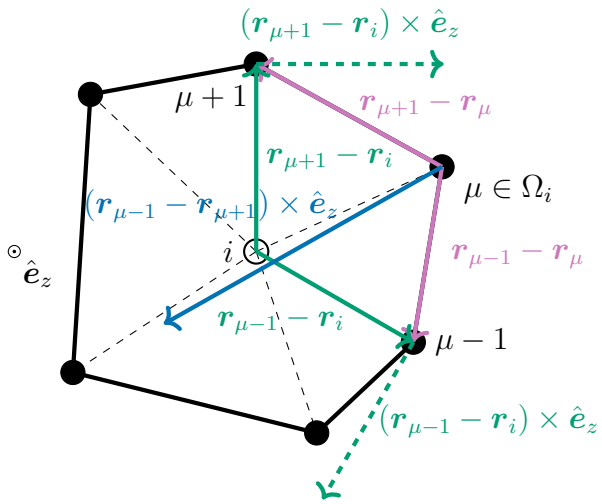

FIG. 1. Vertex representation of a single cell. By convention, cell centres are denoted by latin indices, and cell corners are denotes by greek indices. We denote Ω_i the ensemble of cell corners μ of the cell whose centre is i. By convention, $\mu + 1$ (respectively $\mu - 1$) denotes the next cell corner of i after μ in anticlockwise (respectively clockwise) order.