(Active) vertex model

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(Dated: 20/11/2023, 20:11)

I. MESH

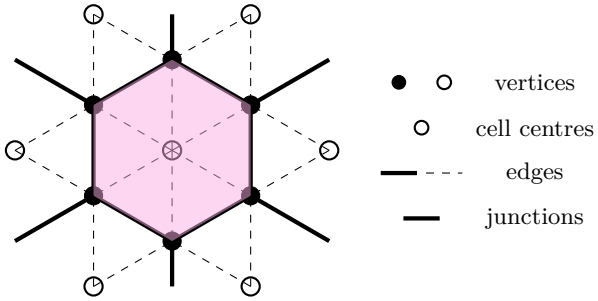

FIG. 1. Schematic of a cell (highlighted in pink) in the vertex model. A *cell centre* is enclosed by *cell corners* (or *vertices*). These are linked between themselves by *junctions*.

We will assume (i) that cells are always convex so that the mesh remains planar, and (ii) that no edge joins two cell centres. The ensemble of the vertices and the edges that link them constitutes the *geometric mesh*. The specification of the cell centres and the junctions between non-cell-centres defines the *physical mesh*.

II. CELL POTENTIAL ENERGY

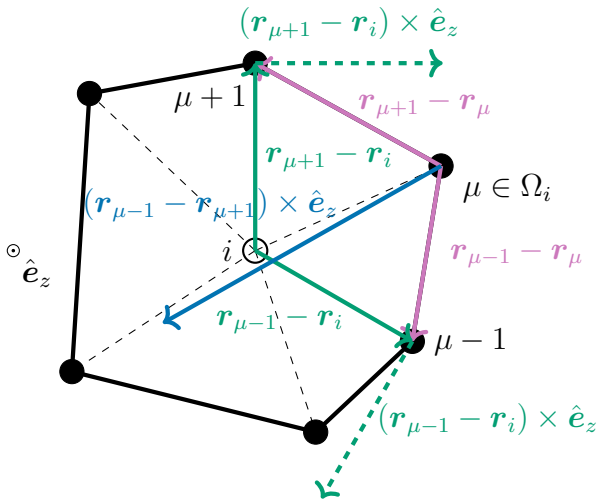

FIG. 2. Vertex representation of a single cell. By convention, cell centres are denoted by latin indices, and cell corners are denotes by greek indices. We denote Ω_i the ensemble of cell corners μ of the cell whose centre is i. By convention, $\mu + 1$ (respectively $\mu - 1$) denotes the next cell corner of i after μ in anticlockwise (respectively clockwise) order.

We introduce for each cell i a reference perimeter P_i^0 and a reference area A_i^0, and for each of its cell corner $\mu \in \Omega_i$ a force whose effect is to bring the cell's perimeter P_i and area A_i to their reference quantities. A possible force derives from the following potential energy [1–3]

$$E_{\mathrm{VM}} = \sum_{\text{cells } i} \left[\frac{1}{2} K_i (A_i - A_i^0)^2 + \frac{1}{2} \Gamma_i (P_i - P_i^0)^2 \right], \tag{1}$$

where K_i and Γ_i are respectively area and perimeter elastic constants. We denote $\boldsymbol{r}_\mu$, $\boldsymbol{r}_i$ the position of vertices μ, i. The cell's perimeter can be written

$$P_i = \sum_{\mu \in \Omega_i} |\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|, \tag{2}$$

and the cell's area can be computed with the shoelace formula

$$A_i = \sum_{\mu \in \Omega_i} \frac{1}{2}[(\boldsymbol{r}_\mu - \boldsymbol{r}_i) \times (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i)] \cdot \hat{\boldsymbol{e}}_z. \tag{3}$$

Note that, by convention of ordering of cell corners, each term in this sum *must be* positive. With these notations, we thus write the force acting on vertex μ

$$\boldsymbol{F}_{\mathrm{VM},\mu} = -\nabla_\mu E_{\mathrm{VM}}, \tag{4}$$

where $\nabla_\mu \equiv \partial/\partial \boldsymbol{r}_\mu$. We compute

$$\nabla_\mu \left(|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|\right) = \frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|}, \tag{5}$$

as well as

$$\nabla_\mu \left([(\boldsymbol{r}_\mu - \boldsymbol{r}_i) \times (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i)] \cdot \hat{\boldsymbol{e}}_z\right) = (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z \tag{6}$$

to write (4) in its full form

$$\begin{aligned} \boldsymbol{F}_{\mathrm{VM},\mu} = -\sum_{\text{cells i, } \mu \in \Omega_i} \Bigg[& \frac{1}{2} K_i (A_i - A_i^0) \left[(\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z - (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z \right] \\ & + \Gamma_i (P_i - P_i^0) \left[\frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|} + \frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|} \right] \Bigg] \\ = \sum_{\text{cells i, } \mu \in \Omega_i} \Bigg[& \frac{1}{2} K_i (A_i - A_i^0) \underbrace{(\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_{\mu+1}) \times \hat{\boldsymbol{e}}_z}_{\text{towards cell interior}} + \Gamma_i (P_i - P_i^0) \Bigg[\underbrace{\frac{\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_\mu}{|\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_\mu|}}_{\text{towards neighbour}} + \underbrace{\frac{\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_\mu}{|\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_\mu|}}_{\text{towards neighbour}} \Bigg] \Bigg], \end{aligned} \tag{7}$$

where underbraced vectors are represented in Fig. 2. It is noteworthy that, with potential (1), the force acting on each cell centre is zero

$$\boldsymbol{F}_{\mathrm{VM},i} = -\nabla_i E_{\mathrm{VM}} = -\frac{1}{2} K_i (A_i - A_i^0) \sum_{\mu \in \Omega_i} (\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}) \times \hat{\boldsymbol{e}}_z = 0. \tag{8}$$

III. HALF-EDGE PROCEDURE

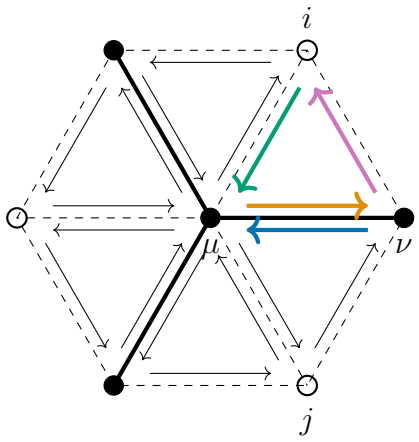

FIG. 3.

IV. T1 TRANSITION

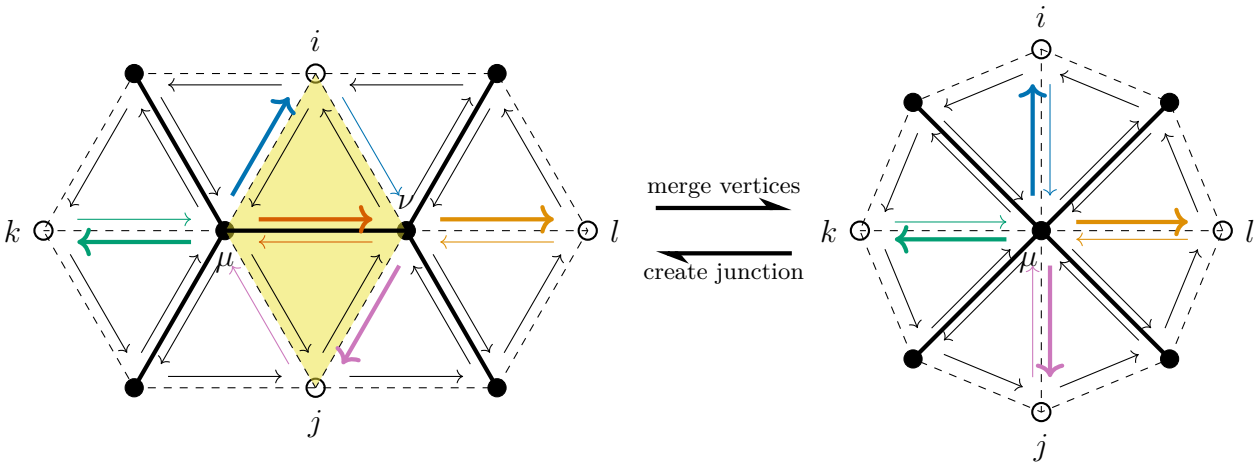

FIG. 4.

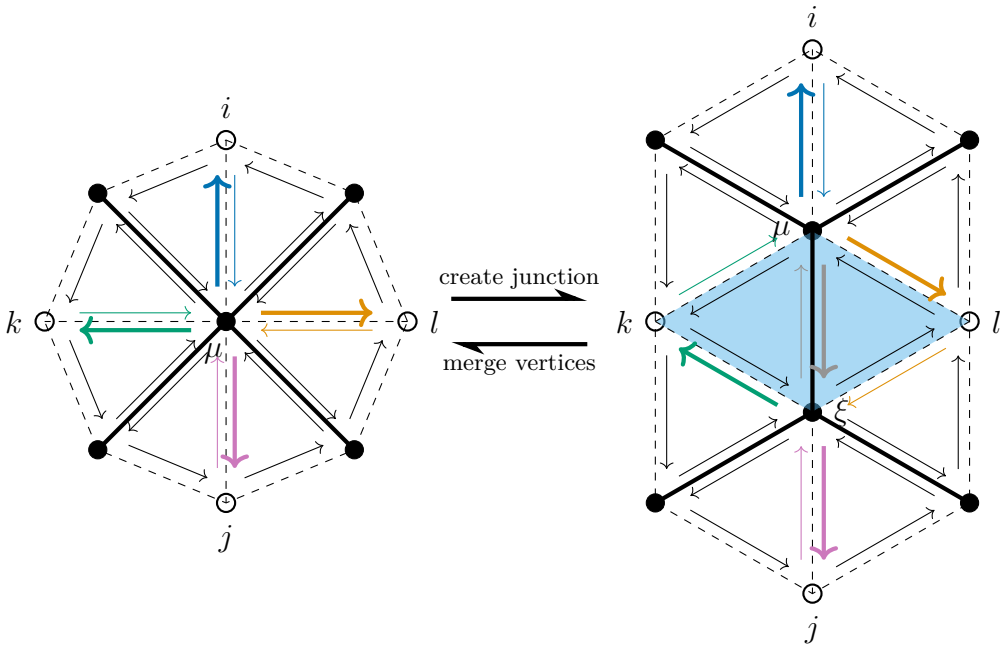

FIG. 5.

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[3] R. Sknepnek, I. Djafer-Cherif, M. Chuai, C. Weijer, and S. Henkes, eLife **12**, e79862 (2023).