(Active) vertex model

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I. MESH

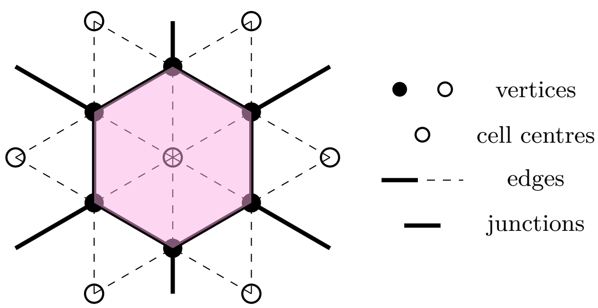

FIG. 1. Schematic of a cell (highlighted in pink) in the vertex model.

A *cell centre* is enclosed by *cell corners* (or *vertices*). These are linked between themselves by *junctions*. We will assume (i) that cells are always convex so that the mesh remains planar, and (ii) that no edge joins two cell centres. The ensemble of the vertices and the edges that link them constitutes the *geometric mesh*. The specification of the cell centres and the junctions between non-cell-centres defines the *physical mesh*.

II. CELL POTENTIAL ENERGY

We introduce for each cell i a reference perimeter P_i^0 and a reference area A_i^0, and for each of its cell corner $\mu \in \Omega_i$ a force whose effect is to bring the cell's perimeter P_i and area A_i to their reference quantities. A possible force derives from the following potential energy [1–4]

$$E_{\mathrm{VM}} = \sum_{\text{cells } i} \left[\frac{1}{2} K_i (A_i - A_i^0)^2 + \frac{1}{2} \Gamma_i (P_i - P_i^0)^2 \right], \tag{1}$$

where K_i and Γ_i are respectively area and perimeter elastic constants. We denote $\boldsymbol{r}_\mu$, $\boldsymbol{r}_i$ the position of vertices μ, i. The cell's perimeter can be written

$$P_i = \sum_{\mu \in \Omega_i} |\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|, \tag{2}$$

and the cell's area can be computed with the shoelace formula

$$A_i = \sum_{\mu \in \Omega_i} \frac{1}{2} [(\boldsymbol{r}_\mu - \boldsymbol{r}_i) \times (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i)] \cdot \hat{\boldsymbol{e}}_z. \tag{3}$$

Note that, by convention of ordering of cell corners, each term in this sum *must be* positive. With these notations, we thus write the force acting on vertex μ

$$\boldsymbol{F}_{\mathrm{VM},\mu} = -\nabla_\mu E_{\mathrm{VM}}, \tag{4}$$

where $\nabla_\mu \equiv \partial / \partial \boldsymbol{r}_\mu$. We compute

$$\nabla_\mu \left(|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}| \right) = \frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|}, \tag{5}$$

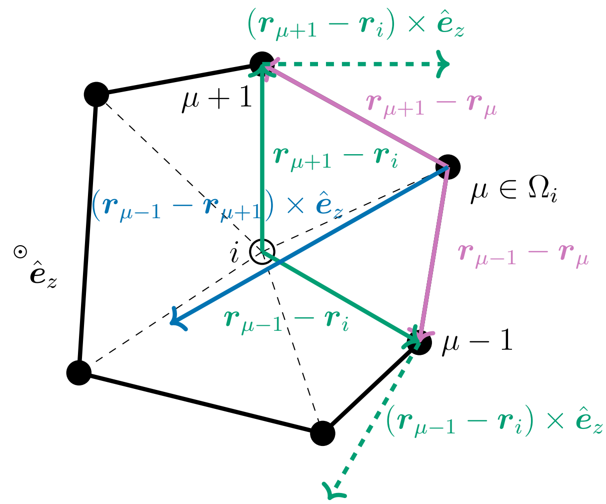

FIG. 2. Vertex representation of a single cell. By convention, cell centres are denoted by latin indices, and cell corners are denotes by greek indices. We denote Ω_i the ensemble of cell corners μ of the cell whose centre is i. By convention, $\mu+1$ (respectively $\mu-1$) denotes the next cell corner of i after μ in anticlockwise (respectively clockwise) order.

as well as

$$\nabla_\mu \left([(\boldsymbol{r}_\mu - \boldsymbol{r}_i) \times (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i)] \cdot \hat{\boldsymbol{e}}_z \right) = (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z \tag{6}$$

to write (4) in its full form

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{VM},\mu} = &- \sum_{\text{cells i, } \mu \in \Omega_i} \left[\frac{1}{2} K_i (A_i - A_i^0) \left[(\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z - (\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_i) \times \hat{\boldsymbol{e}}_z \right] \right. \\
&\qquad \left. + \Gamma_i (P_i - P_i^0) \left[\frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|} + \frac{\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}}{|\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}|} \right] \right] \\
= &\sum_{\text{cells i, } \mu \in \Omega_i} \left[\frac{1}{2} K_i (A_i - A_i^0) \underbrace{(\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_{\mu+1}) \times \hat{\boldsymbol{e}}_z}_{\text{towards cell interior}} + \Gamma_i (P_i - P_i^0) \left[\underbrace{\frac{\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_\mu}{|\boldsymbol{r}_{\mu-1} - \boldsymbol{r}_\mu|}}_{\text{towards neighbour}} + \underbrace{\frac{\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_\mu}{|\boldsymbol{r}_{\mu+1} - \boldsymbol{r}_\mu|}}_{\text{towards neighbour}} \right] \right],
\end{aligned} \tag{7}$$

where underbraced vectors are represented in Fig. 2. It is noteworthy that, with potential (1), the force acting on each cell centre is zero

$$\boldsymbol{F}_{\mathrm{VM},i} = -\nabla_i E_{\mathrm{VM}} = -\frac{1}{2} K_i (A_i - A_i^0) \sum_{\mu \in \Omega_i} (\boldsymbol{r}_\mu - \boldsymbol{r}_{\mu-1}) \times \hat{\boldsymbol{e}}_z = 0. \tag{8}$$

III. HALF-EDGE CONSTRUCTION

The edges of the mesh (Fig. 1) divide the entire system in adjacent and non-overlapping *triangles* (or *faces*). We endow all of these triangles with three arrows. These are oriented such that, for an arbitrary vertex (*e.g.*, μ in Fig. 3) and an arbitrary triangle (*e.g.*, (μ, ν, i)), the cross product of the half-edges pointing toward this vertex in this triangle (*i.e.*, $i \to \mu$) on the one hand and the half-edge pointing out of this vertex in this triangle (*i.e.*, $\mu \to \nu$) on the other hand, has a positive scalar product with $\hat{\boldsymbol{e}}_z$. We introduce three relations between half-edges, which we exemplify in Fig. 3 using an arbitrary reference half-egde $\mu \to \nu$.

1. The *next* half-edge departs from the vertex pointed at by the reference half-edge (*i.e.*, ν here) inside the same triangle. "Next" is an order-3 bijection, meaning that the next half-edge of the next half-edge of any half-edge is this same half-edge.

2. The *previous* half-edge (or *before*) points towards the vertex from which departs the reference half-edge (*i.e.*, μ here) inside the same triangle. "Previous" is also an order-3 bijection.

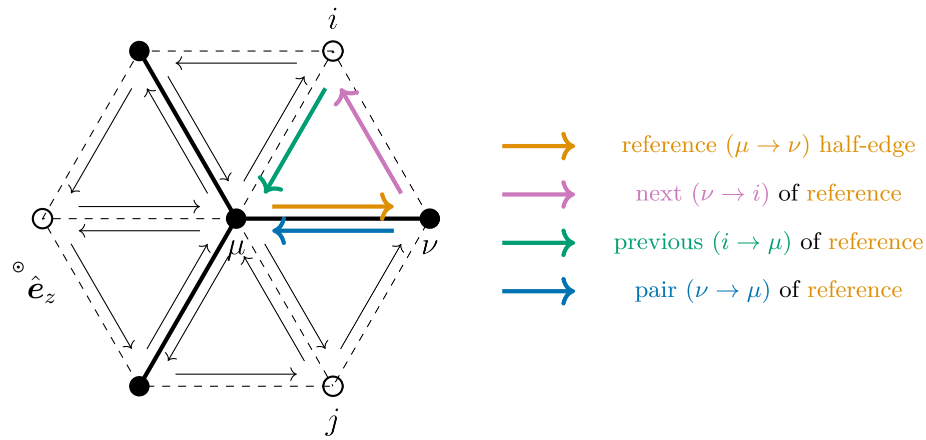

FIG. 3. Half-edge construction in 6 adjacent triangles, highlighting an arbitrary reference half-edge and its associated half-edges.

3. The *pair* half-edge has inverse depart and arrival vertices with respect to the reference half-edge (*i.e.*, ν and μ here) in the adjacent triangle sharing the same edge the reference half-edge belongs to. “Pair” is an order-2 bijecton, meaning that the pair half-edge of any pair half-edge is this same half-edge.

We refer to the ensemble of half-edges and vertices as the *half-edge construction*. We propose in Alg. 2 a way to check its consistency. While this construction contains the same information as the geometric mesh (thus indicating that there is redundancy in this contruction) it enables for faster and in some cases easier computations. We show with Alg. 3 how to identify all neighbours of any vertex given that each vertex contains the information of (at least) one half-edge leaving the vertex.

IV. T1 TRANSITION

T1 transitions correspond to changes of neighbours of 4 cells resulting in a modification of the mesh topology, see Fig. 4. First (1) two vertices are merged (*i.e.*, a junction is deleted) creating a *four-vertex*. In principle, if one of the vertices of the deleted junction had initially more than 3 neighbouring cells, there would be more than 4 cells at the merged vertex. While we expect this not to be frequent, the algorithm designed to perform this merge has to take this into account. Then (2) a junction is created (*i.e.*, a vertex is created).

It is noteworthy that states (A) and (B) (Fig. 4) are described within the half-edge construction with the same number of half-edges and vertices. Therefore it is possible to implement a direct change from (A) to (B) through a clever relabelling of objects. We propose here to implement the two steps (1) and (2) separately in order to enable also the stabilisation of higher order vertices.

Deleting a junction amounts, in the half-edge construction, to deleting 2 triangles (*i.e.*, sets of 3 half-edges) and one vertex. We represent this operation in Fig. 5, where the junction to delete is $[\mu, \nu]$ (μ is merged into ν) and the triangles to delete are shaded. Relations between half-edges, as well as the knowledge for each vertex of a half-edge

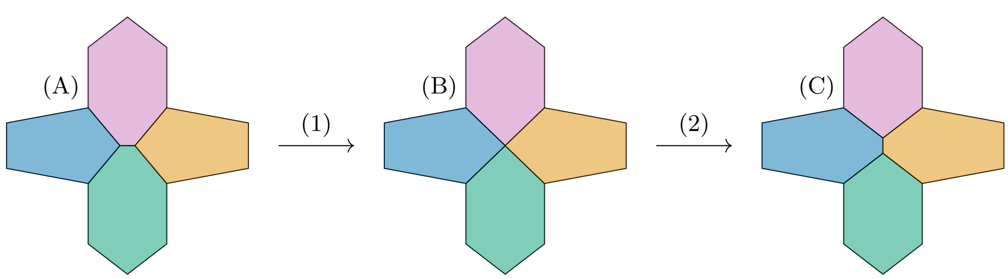

FIG. 4. Four cells undergoing a T1 transition. (A) Cells at the top and at the bottom are in contact while cells on the left and on the right are disjointed. (B) All four cells are touching, at the centre is a four-vertex. (C) Cells at the top and at the bottom are disjointed while cells on the left and on the right are touching.

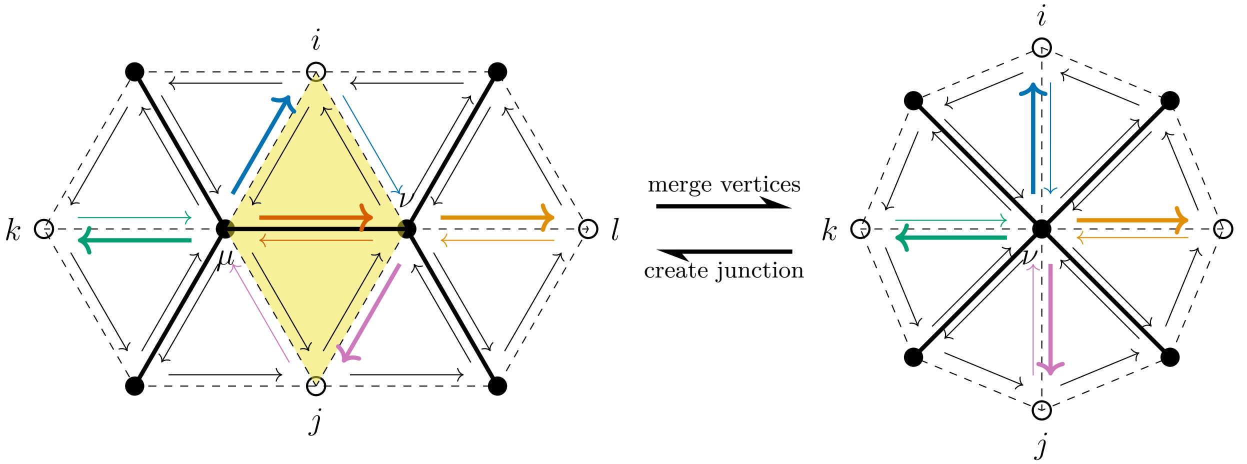

FIG. 5. Representation of vertex merging and junction creation in the half-edge construction. (See Fig. 6 for representation in the orthogonal direction.)

going out of itself, imply that we need to relabel and reaffect half-edges and vertices before deleting anything in order for the construction to remain consistent at the end of the operation (see Alg. 1).

ALG. 1. Merge vertices.

Input: Half-edge $\boldsymbol{h}$ belonging to the junction to delete. ▷ *Thick red half-edge in Fig. 5.*
1: $\mu, \nu =$ origin and destination vertices of $\boldsymbol{h}$.
// *Relabel half-edge origins and destinations for all half-edges connected to* μ.
2: Set $\boldsymbol{b} =$ previous half-edge of $\boldsymbol{h}$.
3: Set $\boldsymbol{b}^0 =$ pair half-edge of $\boldsymbol{h}$.
4: **while** TRUE **do** ▷ *See break condition on line 9.*
5: Set destination vertex of $\boldsymbol{b} = \nu$ (instead of μ).
6: Set $\boldsymbol{p} =$ pair half-edge of $\boldsymbol{b}$.
7: Set origin vertex of $\boldsymbol{p} = \nu$ (instead of μ).
8: **if** $\boldsymbol{b} = \boldsymbol{b}^0$ **then** ▷ *All half-edges going out of* μ *were found, we should also then have* $\boldsymbol{p} = \boldsymbol{h}$.
9: **break**
10: **end if**
11: Set $\boldsymbol{b} =$ previous half-edge of $\boldsymbol{p}$.
12: **end while**
// *Reassign half-edges associated to vertices belonging to the deleted triangles.*
13: Set $\boldsymbol{p} =$ pair half-edge of previous half-edge of $\boldsymbol{h}$. ▷ *Thick blue half-edge in Fig. 5.*
14: Associate $\boldsymbol{p}$ to ν as half-edge going out of it.
15: Set $\boldsymbol{p} =$ pair half-edge of next half-edge of $\boldsymbol{h}$. ▷ *Thin blue half-edge in Fig. 5.*
16: Set $i =$ origin vertex of $\boldsymbol{p}$.
17: Associate $\boldsymbol{p}$ to i as half-edge goigng out of it.
18: Set $\boldsymbol{p} =$ pair half-edge of $\boldsymbol{h}$. ▷ *Thin red half-edge in Fig. 5.*
19: Set $\boldsymbol{p} =$ pair half-edge of next half-edge of $\boldsymbol{p}$. ▷ *Thin purple half-edge in Fig. 5.*
20: Set $j =$ origin vertex of $\boldsymbol{p}$.
21: Associate $\boldsymbol{p}$ to j as half-edge goigng out of it.
// *Relabel half-edge pairs.*
22: Set $\boldsymbol{f} =$ as pair half-edge of previous half-edge of $\boldsymbol{h}$. ▷ *Thick blue half-edge in Fig. 5.*
23: Set $\boldsymbol{t} =$ as pair half-edge of next half-edge of $\boldsymbol{h}$. ▷ *Thin blue half-edge in Fig. 5.*
24: Associate $\boldsymbol{f}$ as pair half-edge of $\boldsymbol{t}$ and vice-versa.
25: Set $\boldsymbol{p} =$ pair half-edge of $\boldsymbol{h}$. ▷ *Thin red half-edge in Fig. 5.*
26: Set $\boldsymbol{f} =$ as pair half-edge of previous half-edge of $\boldsymbol{p}$. ▷ *Thick purple half-edge in Fig. 5.*
27: Set $\boldsymbol{t} =$ as pair half-edge of next half-edge of $\boldsymbol{p}$. ▷ *Thin purple half-edge in Fig. 5.*
28: Associate $\boldsymbol{f}$ as pair half-edge of $\boldsymbol{t}$ and vice-versa.
// *At this point the vertex* μ *and the half-edges in the triangles adjacent at* $\boldsymbol{h}$ *are separated from the rest of the mesh.*
29: Delete junction and triangles.

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[3] D. Bi, X. Yang, M. C. Marchetti, and M. L. Manning, Physical Review X **6**, 021011 (2016).
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Appendix A: Half-edge construction algorithms

ALG. 2. Check half-edge construction.

Input: Ensemble $\mathcal{V}$ of all vertices, ensemble $\mathcal{H}$ of all half-edges.

Make copies $\mathcal{V}'$ and $\mathcal{H}'$ of $\mathcal{V}$ and $\mathcal{H}$.
for $\boldsymbol{h} \in \mathcal{H}$ **do**
  if $\boldsymbol{h} \notin \mathcal{H}'$ **then** ▷ *Half-edge* $\boldsymbol{h}$ *has already been checked.*
    continue
  end if
  Set $\mathcal{T} = \{\boldsymbol{h},\ \text{next of } \boldsymbol{h},\ \text{previous of } \boldsymbol{h}\}$. ▷ *Triangle to which* $\boldsymbol{h}$ *belongs.*
  for $\boldsymbol{h}' \in \mathcal{T}$ **do** ▷ *Loop over the half-edges in the triangle.*
    Set o and d as the origin and destination vertices of $\boldsymbol{h}'$.
    if $o \in \mathcal{V}'$ **then** ▷ *Origin vertex* o *has not been checked.*
      assert o knows one half-edge which *does* leave from o
      Remove o from $\mathcal{V}'$.
    end if
    assert $[\boldsymbol{h}' \times (\boldsymbol{h}'+1)] \cdot \hat{\boldsymbol{e}}_z > 0$ ▷ *Check orientation,* $\boldsymbol{h}'+1$ *is meant as next element in order in* $\mathcal{T}$.
    assert pair half-edge of $\boldsymbol{h}'$ has opposite origin and destination vertices ▷ *Check half-edge.*
    assert $\boldsymbol{h}'$ is the pair of half-edge of its pair half-edge
    assert $\boldsymbol{h}'$ is previous half-edge of $\boldsymbol{h}'+1$. ▷ *Check next half-edge.*
    assert destination vertex of $\boldsymbol{h}'$ is the origin vertex of $\boldsymbol{h}'+1$
    assert $\boldsymbol{h}'$ is next half-edge of $\boldsymbol{h}'-1$. ▷ *Check previous half-edge.*
    assert origin vertex of $\boldsymbol{h}'$ is the destination vertex of $\boldsymbol{h}'-1$
    Remove $\boldsymbol{h}'$ from $\mathcal{H}'$.
  end for
end for
assert $\mathcal{V}' = \emptyset$ and $\mathcal{H}' = \emptyset$

ALG. 3. Find all neighbours of an arbitrary vertex and all half-edges from this vertex to its neighbours. All vertices (cell centre or not) are denoted with latin indices.

Input: Arbitrary half-edge $\boldsymbol{h}$ departing from a given vertex v.
Output: Ensemble $\mathcal{D}$ of neighbours of vertex v, ensemble $\mathcal{H}$ of half-edges from vertex v to its neighbours.

1: Set d^0 = destination vertex of $\boldsymbol{h}$. ▷ *Save first neighbour.*
2: **while** TRUE **do** ▷ *See break condition on line 7.*
3:   Set $\boldsymbol{h}$ = pair half-edge of previous half-edge of $\boldsymbol{h}$.
4:   Set d = destination vectex of $\boldsymbol{h}$.
5:   Add d to $\mathcal{D}$, add $\boldsymbol{h}$ to $\mathcal{H}$. ▷ *By convention of the half-edge construction, these are added in anticlockwise order.*
6:   **if** $d = d^0$ **then** ▷ *All neighbours have been found.*
7:     **break**
8:   **end if**
9: **end while**
10: **return** $\mathcal{D},\ \mathcal{H}$

Appendix B: T1 transition with half-edge construction

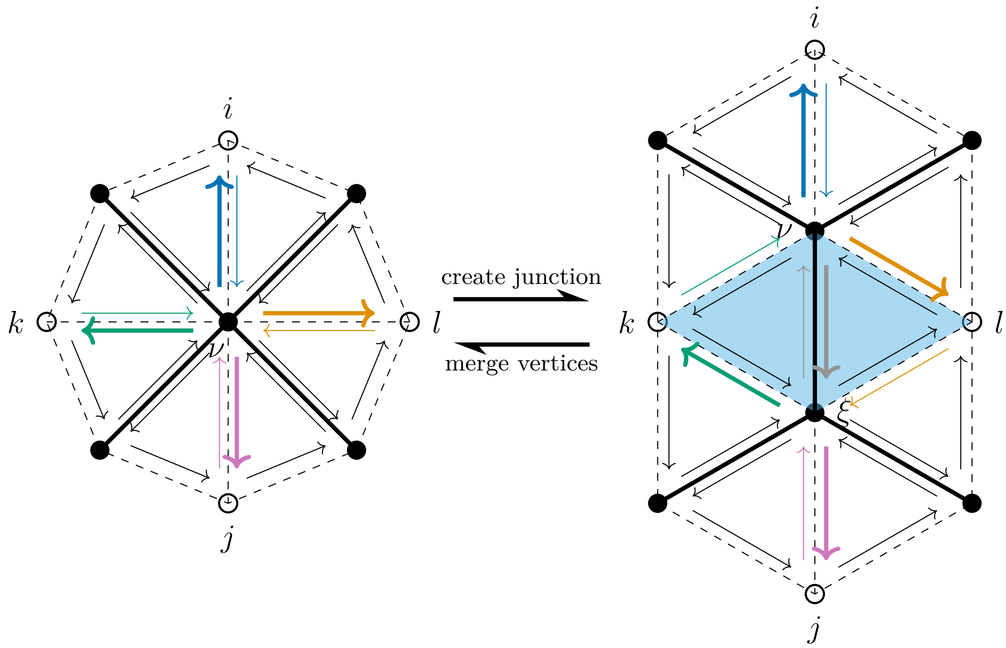

FIG. 6. Representation of vertex merging and junction creation in the half-edge construction. (See Fig. 5 for representation in the orthogonal direction.)