Survey in Intersection Theory

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Contents

1 Localization, local rings

We review necessary commutative algebra tools in this section. Assume ring R is always commutative and unital.

1.1 Noetherian rings

We want to construct the general ring structure for polynomial rings.

Definition 1 (noetherian ring)**.** *A ring R is* ***noetherian*** *if it satisfies any of the following three equivalent conditions:*

1. *ACC condition: every ascending chain of ideals of R terminates.*
2. *Every ideal is finitely generated.*
3. *Given any collection of ideals, there exists a maximal element.*

Example 1 (PID)**.** *For* ***principal ideal domains (PID)*** *R, every ideal is principal, hence finitely generated. Thus PIDs are always noetherian. Moreover, every* ***euclidean domain (ED)*** *is a PID tells they are also noetherian.*

To clarify the claim that euclidean domains are PIDs, let $f \in I \triangleleft R$ where f is of minimal degree in I, then:

$$\forall g \in I \qquad g = qf + r, \qquad \deg r < \deg f.$$

This implies $r = 0$, so $g \in (f)$, hence $(f) = I$.

An important example of PID would be $k[x]$, where k a field. $k[x]$ is a PID since $k[x]$ is moreover a euclidean domain, given the euclidean algorithm:

$$\forall f, 0 \neq g \in k[x] \qquad f = qg + r \qquad r = 0 \text{ or } \deg r < \deg g.$$

From the above we see that for any field k, $k[x]$ is a PID. In fact, for arbitrary ring R,

$$R[x] \text{ is a PID} \qquad \Longleftrightarrow \qquad R \text{ is a field.}$$

We prove the other direction as follows. Assume $R[x]$ is a PID, then the consider the ideal $I = (x, a)$, where a arbitrarily chosen from R. We wish to prove $a \in R^\times$. PID gives existence of some $f \in R[x]$ such that $(f) = (x, a)$. Thus $x \in (f)$, which gives $\deg f \leq 1$.

- *If $\deg f = 0$, then $x = rf$ for some $r \in R[x]$ gives f a unit, hence $1 \in (x, a)$, and a is a unit.*
- *If $\deg f = 1$, then $f(x) = ux + b$ with $u, b \in R$. Since $a \in (f)$, we have $a = h(x)(ux + b) = h(0)b$, thus we must have $a \in (b)$. Since $x \in (f)$, we have $x = g(x)(ux + b)$ for some $g(x) \in R[x]$. This forces $\deg g = 0$, hence $x = (gu)x + (gb)$. This tells $gb = 0$ and g, u both units, which implies $b = 0$. Thus $(x, a) = (x)$, which implies $a \in (x)$, so $a = 0$.*

Example 2 (polynomial rings with finite variables)**.** *Given from the previous example that every $R[x]$ is noetherian, we deduce by induction that for any $n \in \mathbb{Z}$, the ring $R[x_1, \cdots x_n] := (R[x_1, \cdots, x_{n-1}])[x_n]$ is noetherian.*

Proposition 1. *Quotients of noetherian rings are noetherian.*

1.2 Prime ideals

We want the algebraic sets to become algebraic varieties that are irreducible. Such irreducible property needs structures on ideals, which corresponds to the property of being prime.

Prime ideals would work the best in domains.

Definition 2 (domain)**.** *An* ***integral domain*** *or a* ***domain*** *R is a commutative unital ring without zero-divisors, i.e., $\forall a, b \in R$, $ab = 0$ implies $a = 0$ or $b = 0$.*

Proposition 2. *Within a domain R,*

1. *$a = bx$, i.e. a is a **multiple** of b or b is a **divisor** of a, implies $(a) \subset (b)$.*

2. *$d|a$ and $d|b$ means d is a **common divisor** of a and b, and it gives $d|ab$*

3. *$gcd(a, b) := d' \in \{d' : d'|d, d|a, d|b\}$.*

4. *If $a = bu$ with $u \in R^\times$, then $(a) = (b)$. We call such a and b **associate**. Being associate elements is a equivalence relation.*

5. *If $(a) = R$, then $a \in R^\times$.*

Example 3. *We give two counter-examples to show that domains and noetherian rings have no direct relations.*

1. *$\mathbb{Z}/6\mathbb{Z}$ is noetherian, since it is a quotient of noetherian rings. It is not a domain, since $\overline{2} \cdot \overline{3} = \overline{0}$, giving zero-divisors.*

2. *The polynomial ring with infinite variables $k[x_1, x_2, \cdots]$ is a domain, but not noetherian.*

For a short conclusion, we now have such inclusions:

$$\begin{tikzcd}
 & & & \textit{noetherian ring} \arrow[dr, hook] & \\
\textit{field} \arrow[r, hook] & \textit{ED} \arrow[r, hook] & \textit{PID} \arrow[ur, hook] \arrow[dr, hook] & & \textit{commutative unital ring} \\
 & & & \textit{domain} \arrow[ur, hook] &
\end{tikzcd}$$

We now introduce prime ideals as well as related structures.

Definition 3 (prime ideals)**.** *An ideal $\mathfrak{p}$ is **prime** if $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.*
Equivalently, if $a, b \notin \mathfrak{p}$, then $ab \notin \mathfrak{p}$.
*An ideal $\mathfrak{m}$ is **maximal** if there are no nontrivial larger ideals containing $\mathfrak{m}$.*

Proposition 3. *Given commutative unital ring R,*

1. *$R/\mathfrak{p}$ is a domain.*

2. *$R/\mathfrak{m}$ is a field.*

This tells that maximal ideals are necessarily prime.

Proof. We prove one by one.

1. We want to show $R/\mathfrak{p}$ has no zero-divisors. Let $\overline{a}, \overline{b} \in R/\mathfrak{p}$ with $\overline{a} \cdot \overline{b} = \overline{0}$. This refers to $ab \in \mathfrak{p}$. Since $\mathfrak{p}$ prime, $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, so $\overline{a} = 0$ or $\overline{b} = 0$.

2. We want to show any nonzero element in $R/\mathfrak{m}$ is a unit. Let $\overline{0} \neq \overline{a} \in R/\mathfrak{m}$, this means $a \notin \mathfrak{m}$. Consider the ideal $(\mathfrak{m}, a)$, it must equal R. Thus $\exists a \in \mathfrak{m}$ and $\exists r \in R$ such that $m + ra = 1$. This means $\overline{r} \cdot \overline{a} = 1$. Thus we have found a inverse for $\overline{a}$.

$\square$

Example 4. *We review the example $k[x]$ as a PID. If k algebraically closed, then every prime ideal of $k[x]$ is of the form (f) with $\deg f = 1$. This is since any polynomial of degree larger than one could then be decomposed into polynomials of smaller degrees.*

Due to the same reason, for PIDs, prime ideals are maximal, so being prime and maximal are the same.

Definition 4. *An element x is called **prime** if $x|ab$ implies $x|a$ or $x|b$.*
An element x is called irreducible if $a|x$ implies a a unit or an associate of x.

Proposition 4. *Given commutative unital ring,* *1. x is equivalent to (x) is prime.*

2. x is prime implies x is irreducible.

Proof. We prove one by one.

1. $x|a$ is equivalent to $a \in (x)$. Thus the statement $x|ab \implies x|a$ *or* $x|b$ is equivalent to $ab \in (x) \implies a \in (x)$ *or* $b \in (x)$.
2. We prove by contrapositive. If x is reducible, and some a, b such that $x|ab$. This implies $x \nmid a$ and $x \nmid b$, which tells that x is not prime.

$\square$

Example 5. *Recall the ring $\mathbb{Z}/6\mathbb{Z}$. The element 2 is prime by nature, but reducible by $2 = 2 \cdot 4$.*

As a conclusion, we give the following diagram:

$$\begin{tikzcd}
x \textit{ prime} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & x \textit{ irreducible} \\
(x) \textit{ prime} & (x) \textit{ maximal} \arrow[l, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}$$

1.3 Decompositions

Given any element x in a noetherian ring R,

$$x = x_1 x_2 = x_1 x_2' x_3 = \cdots = x_1 x_2' \cdots x_{n-1}' x_n.$$

This process terminates since R is noetherian, so $(x) \subsetneq (x_2) \subsetneq (x_3) \subsetneq \cdots$ terminates.

However such decomposition of an element might not be unique. We want to know a sufficient condition for the decomposition to be unique.

Definition 5 (UFD)**.** *A commutative unital ring R is a* ***unique factorization domain (UFD)*** *if for element $x \in R$ with decompositions $x = x_1 \cdots x_n = y_1 \cdots y_m$, we have $n = m$ and x_i associate with y_i after arrangements.*

We say that every element has ***unique*** *decomposition in a UFD.*

We will connect noetherian rings with prime elements and give the uniqueness condition via the following proposition.

Proposition 5. *A noetherian domain R becomes a UFD iff irreducible elements of R are prime.*

Proof. We prove from both directions.

$\Rightarrow$: Given any irreducible element x, and $x|ab$, we want to show $x|a$ or $x|b$. Write $x|ab$ as $cx = ab$. Since R a UFD, there exist unique decomposition $c_1 \cdots c_k x = a_1 \cdots a_m b_1 \cdots b_n$. This tells x must associate with some a_i or b_j, proving that $x|a$ or $x|b$.

$\Leftarrow$: We want to show the factorization given at the top of this subsection is unique. Let $x = p_1 \cdots p_n = q_1 \cdots q_m$, with p_i, q_j irreducible. By hypothesis, p_i and q_j are prime, so $p_1 | q_1 \cdots q_m$, so $p_1 | q_j$ for some j. Since q_j irreducible, this implies p_1 and q_j associates, so we can cancel them. We repeat inductively and show that $n = m$ with p_i associates with q_i after permutations.

$\square$

Now we conclude with a refined diagram of inclusions:

$$\begin{tikzcd}
 & & & \textit{noetherian ring} \arrow[drr, hook] & & \\
\textit{field} \arrow[r, hook] & \textit{ED} \arrow[r, hook] & \textit{PID} \arrow[ur, hook] \arrow[dr, hook] & & & \textit{commutative unital ring} \\
 & & & \textit{domain} \arrow[r, hook] & \textit{UFD} \arrow[ur, hook] &
\end{tikzcd}$$

1.4 Radical ideals

Given ideals I, J, we have the following diagram of inclusions:

$$\begin{tikzcd} & & I \arrow[dr, hook] & \\ IJ \arrow[r, hook] & I \cap J \arrow[ur, hook] \arrow[dr, hook] & & I + J \\ & & J \arrow[ur, hook] & \end{tikzcd}$$

Proposition 6. *Two ideals are called* ***comaximal*** *if $I + J = R$. This is equivalent to $IJ = I \cap J$.*

Proof. $IJ \subset I \cap J$ always holds.

For the other direction, let $x \in I \cap J$, we want to show $x \in IJ$. Since $I + J = R$, there exist $a \in I, b \in J$ such that $a + b = 1$. Thus,

$$x = x \cdot 1 = x \cdot (a + b) = x \cdot a + x \cdot b \in IJ + IJ = IJ.$$

□

Now we introduce radical ideals.

Definition 6 (radical)**.** *Given an ideal I, the* ***radical*** *of I is defined as $\sqrt{I} := \{x \in R : x^n \in I\}$.*

An ideal I is called ***radical*** *if $I = \sqrt{I}$.*

Given a ring R, the ***nilradical*** *of R is defined as $\operatorname{nil}(R) := \sqrt{(0)} = \{x : x^n = 0\}$.*

From the definition, we immediately have the following properties.

Proposition 7. *Given commutative unital ring R and an ideal I,*

1. $I \leq \sqrt{I}$.
2. $\operatorname{nil}(R)$ *is composed by nilpotent elements, that is, some power of the element is* 0. *This means* $\operatorname{nil}(R) = 0$ *if R is a domain.*
3. *Prime ideals are radical, since $x^n \in \mathfrak{p}$ implies $x \in \mathfrak{p}$.*
4. *The radical of and ideal is a radical ideal.*

Example 6. *Consider the ring $\mathbb{Z}$ and $I = (12), J = (8)$. We then have:*

$$IJ = (96), \qquad I \cap J = (24), \qquad I + J = (4), \qquad \sqrt{I} = (6), \sqrt{J} = (2).$$

$\operatorname{nil}(\mathbb{Z}) = 0$.

This example tells that radical ideals may not be prime.

The next property is more sophisticated and links with prime ideals.

Proposition 8. *Given a commutative unital ring R and an ideal I,*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}.$$

Proof. We show both sides of inclusions. $\subset$ is immediate.

For the other direction $\sqrt{I} \supset \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$, we prove by contrapositive. Pick $x_0 \notin \sqrt{I}$, we want to find a prime ideal $\mathfrak{p} \supset I$ that does not contain x. Consider the set:

$$\Sigma := \{J \lhd R : I \subset J, x_0^n \notin J, \forall n \geq 1\}.$$

This set is nonempty since $I \in \Sigma$. We give an order on Σ by inclusion of ideals. By Zorn's Lemma, there is a maximal ideal $\mathfrak{p}_0 \in \Sigma$. Thus we could get a prime ideal $\mathfrak{p} := \mathfrak{p}_0 \supset I$ with $x_0 \notin \mathfrak{p}$. This tells $x_0 \notin \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$. □

Related topics:
Transporter ideal [I:J].
Jacobson radical, it is the intersection of all maximal ideals.

1.5 Local rings

2 Graded structure, projective space

3 Algebraic sets and varieties

We will discuss the affine and projective case separately. We start from the classic approach then generalize to the modern approach.

We want to study geometric objects algebraically. The subject of matter here are varieties. There are affine varieties and projective varieties.

3.1 Affine varieties

Affine varieties live in affine spaces.

We start with a ring $A = k[x_1, \cdots, x_n]$ where k algebraically closed, with reason to be explained later. The affine space corresponding to A is defined as follows.

Definition 7 (affine spaces)**.** *Given an algebraically closed field k, define the **affine space** as:*

$$\mathbb{A}_k^n = \mathbb{A}^n := k^n = \{a = (a_1, \cdots, a_n) : a_i \in k \; \forall 1 \leq i \leq n\}.$$

Note that this is different from vector spaces as they do not have linear structures. It is only a topological space with the Zariski topology.

The affine varieties can be

Definition 8 (affine variety)**.** *An affine algebraic set of the space $\mathbb{A}^n$ is defined by an ideal $I \lhd A$ as:*

$$Z(I) = \{a \in \mathbb{A}^n : f(a) = 0 \; \forall f \in I\}.$$

An affine algebraic variety is a irreducible component of an algebraic set.

As the ring $k[x_1, \cdots, x_n]$ is Noetherian, ideals are always finitely generated, thus:

$$Z[I] = Z[f_1, \cdots, f_m] = \{z \in \mathbb{A}^n : f_i = 0 \; \forall i = 1, \cdots, m\},$$

for some chosen generators $f_1, \cdots, f_m$.

Algebraic subsets satisfies properties ... , hence is inherited with Zariski topology.

We especially need the ring to be a polynomial ring over algebraically closed field, since they have strong properties.

Proposition 9. *Given a ring $k[x_1, \cdots, x_n]$ with k algebraically closed, its maximal ideals are of the form:*

$$\mathfrak{m}_a = (x_1 - a_1, \cdots, x_n - a_n), (a_i) \in \mathbb{A}^n.$$

Now that we have spaces and varieties as geometric objects, we would want to know the functions defined on these objects.

Definition 9 (coordinate ring)**.** *Given an affine variety $X \subset \mathbb{A}^n$, define the **ideal** of X as:*

$$I(X) = \{f \in k[x_1, \cdots, x_n] : f(X) = 0\}.$$

*Then, the **coordinate ring** of X is defined as:*

$$A(X) = A/I(X) = k[x_1, \cdots, x_n]/I(X).$$

The ring $A(X)$ would be the set of all functions, to be shown later.

3.2 Projective varieties

The projective varieties live in projective spaces.

Given algebraically closed field k, we would start with the graded ring $S = k[x_0, \cdots, x_n]$.

Definition 10 (projective space)**.** *Given a field k, algebraically closed, the **projective space** is defined as:*

$$\mathbb{P}^n_k = k\mathbb{P}^n := \{[a_n, \cdots, a_n] : a_i \in k, a_0 \neq 0\} = \{(a_n, \cdots, a_n) : a_i \in k, a_0 \neq 0\}/\sim,$$

where $[a_0, \cdots, a_n]$ is a equivalent class, and $(a_0, \cdots, a_n) \sim (b_0, \cdots, b_n)$ iff:

$$\frac{a_i}{a_0} = \frac{b_i}{b_0}, \qquad \forall i = 1, \cdots, n.$$

This is to say that two elements are viewed as the same if they have the same tangent.

Similarly, we have algebraic sets in $\mathbb{P}^n$, so we have **projective varieties** Y defined as irreducible projective sets.

Given a graded ring S, the homogeneous ideals are ideals generated by homogeneous functions vanishing at some point in Y. We also have projective coordinate rings:

$$S(Y) := S/I(Y) = k[x_0, \cdots, x_n]/I(Y).$$

In general, a variety can refer to any of the above cases.

Definition 11 (variety)**.** *Let k be an algebraically closed field. A **variety** over k is any affine, quasi-affine, projective, or quasi-projective variety defined above.*

3.3 Modern approach

For the modern approach, we start with any commutative unital ring A. The affine planes are generalized to affine schemes. But at first the schemes are only topological spaces.

Definition 12 (spectrum space)**.**

$$\mathbb{A}^n = \mathrm{Spec}(A) = X,$$

equipped with the Zariski topology.

The varieties are reflected by the closed points in the spectrum spaces.

Now given the topological space $X = \mathrm{Spec}(A)$, we want to turn it into a ringed topological space. That is by adding a sheaf structure to it.

By especially choosing a well-structured sheaf, and by gluing local to form global structure, the spectrum spaces becomes schemes.

4 Regular functions

Regular functions are important because they can help construct morphisms between varieties, especially to determine when are two varieties isomorphic to each other.

We will be working on rings $k[x_1, \cdots, x_n]$, where k an algebraically closed field.

4.1 Classic approach

The most natural functions to define on $\mathbb{A}^n$ are the regular functions.

Definition 13 (regular functions)**.** *Given an affine variety $X \subset \mathbb{A}^n$ with Zariski topology τ, an open set $U \in \tau$, and a point $p \in U$, we say that a function $f : U \to K$ is **regular at** p if:*

$$\exists V \in \tau, V \ni p \qquad \exists g, h \in k[x_1, \cdots x_n], h(p) \neq 0 \qquad f = \frac{g}{h}.$$

*We say that f is **regular on** U if f is regular at every point of U.*

Example 7 (polynomials)**.** *Every polynomial is regular everywhere on the affine plane, hence is always regular.*

This means that regular functions are natural generalizations of polynomials.

Example 8 (coordinate restrictions)**.** *Let $X = V(xy - 1) \subset \mathbb{A}^2$. Then the function:*

$$f(x, y) = \frac{1}{x}$$

is regular on the variety X, though not defined on all $\mathbb{A}^2$.

Example 9. *Consider $\mathbb{A}^1 = k$. The function*

$$f(x) = x$$

is not regular on the entire $\mathbb{A}^1$, but is regular on the open set $U := \mathbb{A}^2 \setminus \{x = 0\}$. This open set U forms a variety itself.

We want to study the family where all regular functions lives. We would show that the coordinate ring of X is exactly the ring of regular functions regular at every point p of X.

Proposition 10 (equivalence of regular function rings)**.**

1. *(affine) The regular functions on any open subset $U \subset X$ form a ring $\mathcal{O}_X(U)$, and in particular,*

$$\mathcal{O}_X(X) \simeq A(X).$$

2. *(projective) $\mathcal{O}_Y(Y) \simeq k$.*

Definition 14 (local ring)**.** *If for U and V as open subsets of X, the regular functions $\langle U, f\rangle$ and $\langle V, g\rangle$ agree on $U \cap V$, then we say f and g are equivalent and denote $\langle U, f\rangle \sim \langle V, g\rangle$. The **local ring** of p on X is defined as:*

$$\mathcal{O}_{X,p} := \{\langle U, f\rangle, p \in U\}/\sim .$$

The elements in $\mathcal{O}_{X,p}$ can be viewed as tangents at of X at the point p.
Similarly, we have the corresponding:

Proposition 11 (equivalence of localizations)**.**

1. *(affine) For each point $p \in X$ affine variety, let $m_p \lhd A(X)$ be the ideal of functions vanishing at p, then:*

$$\mathcal{O}_{X,p} \simeq A(X)_{m_p},$$

and $\dim \mathcal{O}_{X,p} = \dim X$.

2. *(projective) For each point $p \in Y$ projective variety, let $m_p \lhd S(Y)$ be the ideal generated by homogeneous functions vanishing at p, then:*

$$\mathcal{O}_{Y,p} \simeq S(Y)_{(m_p)}.$$

Definition 15 (function field)**.** *Similar to the local ring, the **function field** of X is defined as:*

$$K(X) := \{\langle U, f\rangle\}/\sim,$$

only that open sets U are not asked to contain certain points.

Through the process, we actually glued functions together and constructed a new function on $U \cup V$, called the **rational functions**.
$K(X)$ is indeed a field.
Rational functions are important because they help define birational maps, which can help classify varieties under birational equivalence.

Proposition 12 (equivalence of function field)**.**

1. *(affine)* $K(X) \simeq \operatorname{Frac} A(Y) = A(Y)_{(0)}$.
2. *(projective)* $K(Y) \simeq \operatorname{Frac} S(Y) = S(Y)_{((0))}$.

The three rings by nature have the following inclusion property.

Proposition 13. *Given a variety X,*

$$\mathcal{O}_X(X) \hookrightarrow \mathcal{O}_{X,p} \hookrightarrow K(X).$$

4.2 Modern approach

Given a scheme $(X, \mathcal{O}_X)$, the images of the equipped sheaf on open sets $\mathcal{O}_X(U)$ become rings of regular functions.

Regular functions correspond to sections:

Definition 16 (sections)**.** *A section s lives in some $\mathcal{O}_X(U)$.*

The formal regular functions could be defined as:

$$\forall f \in A \qquad f(p) := f + p$$

with images as cosets.

A localization of a section is a germ:

Definition 17 (germs of regular functions)**.** *Given affine variety X and coordinate rings $\mathcal{O}_X(U)$ of open subsets of X, the germ at a point p is defined as:*

$$\mathcal{O}_{X,p} := \varinjlim_{p \in U} \mathcal{O}_X(U).$$

*Here $\mathcal{O}_{X,p}$ is called the **stalk** of X at p. There are group homomorphisms:*

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,p} \qquad s \mapsto s_x.$$

*It maps a section to its **germ**.*

4.3 Morphisms

Morphisms here refer to maps between varieties. The definition of morphisms requires compatibility with regular functions on these varieties.

Definition 18 (morphism)**.** *Let k algebraically closed, and X, Y two varieties. Then a **morphism** $\phi : X \to Y$ is a continuous map such that:*

$$\forall V \in \tau_Y \quad \forall f \in \mathcal{O}_Y(V) \qquad f \circ \phi \in \mathcal{O}_X(\phi^{-1}(V)).$$

*An **isomorphism** is a morphism $\phi : X \to Y$ that admits an inverse morphism $\psi : Y \to X$ such that:*

$$\psi \circ \phi = \mathbb{1}_X, \qquad \phi \circ \psi = \mathbb{1}_Y.$$

Under morphisms, varieties becomes a category $\texttt{Var}_k$, and we can classify varieties under isomorphisms. A useful property for us to find equivalence classes is stated as follows. We first post a stronger proposition.

Proposition 14. *Given two varieties X, Y, if Y is affine, then there is a natural bijection of sets:*

$$\alpha : \operatorname{Hom}(X, Y) \xrightarrow{\sim} \operatorname{Hom}_k(A(Y), \mathcal{O}(X)).$$

Proof. We construct α as follows:

$$\alpha : (\phi : X \to Y) \mapsto (f \mapsto f \circ \phi).$$

Since for affine varieties, $\mathcal{O}(Y) \simeq A(Y)$, the latter can be induced to $\mathrm{Hom}_k(A(Y), \mathcal{O}(X))$. The inverse of α is given as the following β:

$$\beta : (\psi : X \to \mathbb{A}^n \quad P \mapsto (\xi_1(P), \cdots \xi_n(P))) \quad \leftarrow\!\shortmid \quad (h : \overline{x_i} \mapsto \xi_i).$$

The $\overline{x_i}$ presents the coordinates x_i in the ring $A(Y) = k[x_1, \cdots, x_n]/I(Y)$. This definition works since given any $f \in I(Y)$,

$$f(\psi(P)) = f(\xi_1(P), \cdots, \xi_n(P)) = h(f(\overline{x_1}), \cdots, f(\overline{x_n}))(P) = 0,$$

which satisfies that ψ is a morphism from X to Y. ☐

Corollary 1. *Given two affine varieties X and Y,*

$$X \simeq Y \qquad \Longleftrightarrow \qquad A(Y) \simeq A(X).$$

Classifying varieties under isomorphisms is actually one of the most important problem in algebraic geometry. We could consider a weaker classification using a generalized equivalence relation introduced as follows.

4.4 Rational maps

Definition 19 (rational map)**.** *Let X, Y be varieties. A **rational map** $\phi : X \to Y$ is a morphism defined on dense open subset on X. More concretely, given morphisms $\phi_U : U \to Y$, where $\emptyset \neq U \in \tau_X$, we give a rational map ϕ by gluing the ϕ_U under equivalence:*

$$\phi_U \sim \phi_V \quad \textit{if} \quad \phi_U(U \cap V) = \phi_V(U \cap V).$$

*We call the rational map ϕ **dominant** if its image is dense in Y.*

Proposition 15. *The rational map $\phi : X \to Y$ is dominant if the image of one U is dense.*

Proof. If two morphisms from X to Y agree on an open set, the two morphisms much be the same. ☐

Definition 20 (birational map)**.** *Let X, Y be varieties. We call a rational map $\phi : X \to Y$ **birational** if there exists a rational map $\psi : Y \to X$ such that:*

$$\psi \circ \phi = \mathbb{1}_X, \qquad \phi \circ \psi = \mathbb{1}_Y.$$

*If there exists a birational map between X and Y, we say that X and Y are **birational**.*

Birational is an equivalence relation.

Proposition 16. *If X and Y are birational, then they are isomorphic on open dense sets.*

This means that two varieties are isomorphic on dense open subsets

Corollary 2. *Let X and Y be two varieties. Then,*

$$X \simeq Y \qquad \Longrightarrow \qquad X, Y \textit{ birational}$$

This means that isomorphism is a stronger equivalence that gives finer classification.

Example 10. *We list two counterexamples here.*

1. *Consider smooth projective varieties of genus 0. They all birational to $\mathbb{P}^1$, but they are usually not isomorphic.*

2. *Consider smooth projective varieties of genus 1. They are all birational to each other, and their function field is the function field of elliptic curves $k(E) := \{\text{meromorphic functions on } E\}$. But they are usually not isomorphic.*

We return to the classification question.

Proposition 17. *Two varieties X and Y are birational if and only if their function fields $K(Y)$ and $K(X)$ are isomorphic k-algebras.*

Proposition 18. *The genus is a birational invariant.*

Example 11. *We explain the above example further.*

1. *For $g = 0$, there is only one birational equivalence class;*

2. *For each $g > 0$, there is a continuous family of birational equivalence class, parameterized by an irreducible algebraic variety. Curves with $g = 1$ are the elliptic curves.*

5 Plane Curves

This section is dedicated to algebraic varieties in spaces of dimension 2.

5.1 Local properties

Definition 21 (affine plane curve)**.** *An **affine plane curve** is an equivalence class $[f]$ of non-constant polynomials $f \in k[x, y]$ under the equivalence relation:*

$$f \sim g \iff f = \lambda g, \quad 0 \neq \lambda \in k.$$

*The **degree** of a curve is the degree of a defining polynomial for the curve. A curve of degree 1 is a **line**.*

Definition 22 (simple point)**.** *Let f be an affine plane curve and $P = (a, b) \in f$. We say that P is **simple** if:*

$$f_x(P) \neq 0 \quad \text{or} \quad f_y(P) \neq 0.$$

*A curve is said to be **nonsingular** if all its points are simple.*

*If the above fails, i.e. $f_x(P) = f_y(P) = 0$, we say that P is **multiple** or **singular**. A curve is said to be **singular** if it consists a singular point.*

Example 12 (tangent line)**.** *Given f an affine plane curve. If $P \in f$ is simple, we define the **tangent line** to f at the point P as:*

$$f_x(P)(x - a) + f_y(P)(y - b) = 0.$$

Definition 23 (multiplicity)**.** *Given a graded polynomial ring $k[x, y]$, fix $P = (0, 0)$. We discuss P as a multiple point in the following cases.*

1. *If a polynomial f can be decomposed into irreducible factors as:*

$$f = \prod f_i^{e_i},$$

*then we say that f_i are the **components** of f of **multiplicity** e_i. A component f_i is **simple** if its multiplicity e_i is 1.*

2. *If not, then we could first write its elements as addition of forms:*

$$f = f_m + f_{m+1} + \cdots + f_n,$$

*where f_i is a form of degree i, and $f_m \neq 0$. Then we say that $m = m_P(f)$ is the **multiplicity** of f at point P. Point P is called **simple**, **double**, or **triple** if $m = 1, 2,$ or 3. Then, we could decompose f_m into components as:*

$$f_m = \prod L_i^{r_i}.$$

*Here, the L_i are then the **tangent lines** to f at P, and r_i are then the **multiplicity** of each tangent line.*

If P is a multiple point of f, and the m tangents of f at P are distinct, then we say that P is an ***ordinary*** *multiple point. If $m = 2$, then we call P a* ***node****.*

Example 13 (elliptic curve)**.** *Elliptic curves are plane curves generally defined as $E : y^2 = ax^3 + bx^2 + c$, with coefficients in k. The followings are the 3 typical cases.*

1. *Consider the elliptic curve $y^2 = x(x-1)(x+1)$. It could be rewritten as the zeros of a polynomial:*

$$f(x, y) = y^2 - x^3 + x.$$

This polynomial is nonsingular, and all points on it are simple.

2. *The elliptic curve $y^2 = x^3$ rewritten as $g(x, y) = y^2 - x^3$ is singular at point $P = (0, 0)$. This point is the only multiple point, and it is a nonordinary double point.*

3. *The elliptic curve $y^2 = x^2(x-1)$ rewritten as $h(x, y) = y^2 - x^3 + x^2$ is singular at point $P = (0, 0)$. This point is the only multiple point, and it is a node.*

5.2 Discrete valuation ring

Proposition 19. *Let R be a domain but not a field. Then the followings are equivalent:*

1. *Ring R is noetherian and local, and its unique maximal ideal is principal;*

2. *There is an irreducible element $t \in R$ such that:*

$$\forall\, 0 \neq z \in R \qquad \exists!\, u \in R^\times, n \in \mathbb{N} \qquad z = ut^n.$$

The second property guarantees a unique valuation of any element as a form in the polynomial ring.

Definition 24 (discrete valuation ring)**.** *A R that is noetherian, local, and has maximal ideal to be principal is called a* ***discrete valuation ring*** *or abbreviated* ***DVR****.*

The irreducible element t in the valuation is called a ***uniformizing parameter*** *or* ***uniformizer*** *for R.*

Equivalently, a DVR is a local PID.
Given a uniformizer t, any other uniformizers are of the form ut, where $u \in R^\times$.

Example 14. *We list some examples of DVRs here.*

1. *The ring of power series $k[[t]]$. Its unique maximal ideal is (t), a uniformizer could be t, and it has valuation:*

$$v(f) = \text{the order of vanishing at}\, t = 0.$$

Every nonzero element in $k[[t]]$ has the form:

$$f(t) = t^n u(t), u(0) \neq 0.$$

2. *$\mathbb{Z}_{(p)}$. Its unique maximal ideal is (p), a uniformizer could be p.*

3. *$k[x]_{(x-a)}$. Its unique maximal ideal is $(x - a)$, a uniformizer could be $x - a$, and it has valuation as the order of vanishing at $x = a$.*

4. *The local ring of a smooth curve $\mathcal{O}_{C,P}$.*

5.3 Intersection numbers

5.4 Linear systems of curves

6 Intersection Theory

6.1 Dimension, hypersurfaces and linear system

6.2 Multiplicity and intersection multiplicity

6.3 Bezout's theorem and Chow ring

Given two varieties, their intersection might not be again a variety, but it must be an algebraic set.

We first look at intersection of algebraic curves in space of dimension 2.

The classical approach: geometric intuition tells that two varieties of order m and n would have $m \cdot n$ points of intersection. However there are counterexamples.

Example 15. *We give counterexamples and methods on encountering them, from simple to complicated.*

1. *parallel lines: we need projective curves*
2. *intersection of two circles: we need complex curves*
3. *tangent line of a conic: we need to count multiplicity*
4. $X = \{f(x,y)(x-1)\}, Y = \{g(x,y)(x-1)\}$*: we need without commune components*
5. *two circles with same center but different radius: we could have only up to the amount of intersection points*

Bézout's theorem provides a rigorous claim that works on the majority of cases.

Theorem 1 (Bézout, weak version)**.** *Given two complex projective algebraic curve X, Y without commune components of degree m and n, the set of intersection points $\{p : X(p) = Y(p)\}$ has finite degree of at most $m \cdot n$ counting multiplicity.*

The complex projective algebraic curves on $\mathbb{P}^2$ are also called **projective plane curves**.

7 References

1. Fulton, algebraic curves
2. Harris, GTM 133
3. Hartshorne, GTM 52
4. Liu, algebraic geometry and arithmetic curves