

Survey in Symmetric Group Representation Theory

Yuekun Feng

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1 Symmetric group

1.1 Basic languages

Definition 1 (symmetric group). *Given set X , the **symmetric group** S_X of X is defined as*

$$S_X := \{\sigma : X \simeq X\},$$

where the elements σ are called **permutations**. It has group structure with operation as composition of permutations, i.e.,

$$\circ : S_X \times S_X \rightarrow S_X \quad (\sigma, \tau) \mapsto \tau \circ \sigma =: \tau\sigma.$$

When $|X| = n$, we simply consider $X \simeq \{1, \dots, n\}$ and write $S_n := S_X$.

Note that there are three common ways to present a permutation:

$$\begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix}, \quad [\sigma(1), \dots, \sigma(n)], \quad (a_i, \sigma(a_i), \dots, \sigma^{k-1}(a_i)), \quad 1 \leq i \leq n$$

which are called **two-line form**, **one-line form**, and **cycle form** respectively. The **length** of a cycle is $k = \text{ord}_i \sigma$, and such cycles are called **k -cycle**.

There are many ways to decompose a permutation. In fact, any cycle can be decomposed uniquely into a degenerated composition of disjoint cycles, in the form of

$$[\sigma(1), \dots, \sigma(n)] = (a_1, \sigma(a_1), \dots, \sigma^{k_1-1}(a_1))(a_2, \sigma(a_2), \dots, \sigma^{k_2-1}(a_2)) \cdots (a_l, \sigma(a_l), \dots, \sigma^{k_l-1}(a_l)),$$

where $a_j \neq \sigma^h(a_i), \forall 1 \leq j \leq l, 1 \leq h \leq k_j, 1 \leq i \leq j$. These cycles are disjoint by construction, and this combination is unique up to 1-cycles since σ is bijective, and they are commutative as they are disjoint.

We could also decompose cycles into smaller pieces. The **transpositions** are cycles of length 2. In fact, all cycles can be written as compositions of transpositions, which can be stated as follows.

Proposition 1. *Given a cycle $(a_1, \dots, a_k) \in S_n$,*

$$\begin{aligned} (a_1, \dots, a_k) &= (a_1, a_k)(a_1, a_{k-1}) \cdots (a_1, a_2) \\ &= (a_1, a_2)(a_2, a_3) \cdots (a_{k-1}, a_k). \end{aligned}$$

Proof. We prove both equalities by induction. The conclusion is obvious when $k = 1$ or 2. Assume the equation holds for $k = m$, then

$$(a_1, \dots, a_{m+1}) = (a_1, a_{m+1})(a_1, \dots, a_m) = (a_1, a_{m+1})(a_1, a_m) \cdots (a_1, a_2).$$

Meanwhile,

$$(a_1, \dots, a_{m+1}) = (a_1, a_2)(a_2, \dots, a_{m+1}) = (a_1, a_2)(a_2, a_3) \cdots (a_m, a_{m+1}).$$

□

1.2 Class equations

Define a group action of a symmetric group on itself as

$$S_n \curvearrowright S_n : S_n \times S_n \rightarrow S_n \quad g(w) := (g, w) \mapsto gwg^{-1}.$$

The following result is useful when computing actions of symmetric groups.

Proposition 2. *Assume group action defined as above, with $w = (a_1, \dots, a_k)$, then*

$$g(w) = (g(a_1), \dots, g(a_k)).$$

Proof. Assume $g = (b_1, \dots, b_l)$. Then $g^{-1} = (b_l, \dots, b_1)$, and we see that

$$gwg^{-1} = g(a_1, a_2) \cdots (a_{k-1}, a_k)g^{-1} = g(a_1, a_2)g^{-1}g(a_2, a_3) \cdots (a_{k-1}, a_k)g^{-1},$$

and

$$\begin{aligned} g(a_i, a_{i+1})g^{-1} &= (b_1, b_2) \cdots (b_{l-1}, b_l)(a_i, a_{i+1})(b_l, b_{l-1}) \cdots (b_2, b_1) \\ &= (b_1, b_2) \cdots (b_{l-1}, b_l)(a_i, a_{i+1})(b_{l-1}, b_l) \cdots (b_1, b_2), \end{aligned}$$

hence we could simple consider g, w as transpositions.

With out loss of generality, consider $w = (a_1, a_2), g = (b_1, b_2)$, we need to consider if a_1 can be permuted by g . If none of the $b_1 \neq a_1, b_2 \neq a_2$, then $g(w) = gwg^{-1} = w$. If $b_1 = a_1$ and $b_2 = a_2$, then $gwg^{-1} = w^3 = w$. If $b_1 = a_1$ and $b_2 \neq a_2$, then

$$gwg^{-1} = (a_1, b_2)(a_1, a_2)(b_2, a_1) = (b_2, a_2) = (g(a_1), g(a_2)).$$

Since these covers all possibilities, the same result holds for arbitrary transpositions, and thus for arbitrary cycles. \square

By the above proposition, we could divide the symmetric group into multiple orbits. Elements from each orbit can be written in compositions of disjoint cycles with the same lengths respectively.

Definition 2 (cycle type). *A **cycle type** of $x \in S_n$ is an ordered tuple*

$$\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k, \quad \lambda_1 \geq \dots \geq \lambda_k,$$

where λ_i be the length of the i 'th cycle, $1 \leq i \leq k$.

Denote $z_\lambda = \#\{w \in S_n : w \text{ is of cycle type } \lambda\}$.

In the context of group actions, two permutations are of the same cycle type refers exactly to that they are in the same orbit of action. Hence being in the same cycle type is an equivalence relation.

Another method of presenting a cycle type is by relating λ with

$$m = (m_1, \dots, m_n) \in \mathbb{Z}^n, \quad m_i := \#\{\sigma \in S_n : \sigma \text{ be } i\text{-cycles, } \sigma \text{ appears in cycle type } \lambda\}.$$

With this presentation, we have the formula

$$m_1 + 2m_2 + \dots + nm_n = n.$$

This is since every number from 1 to n appears exactly once.

It is natural to ask the number of transpositions in each cycle type, which the following result gives an explicit formula.

Proposition 3. *Given a cycle type λ in S_n , the number of permutations is given by*

$$z_\lambda = \frac{n!}{\prod_{j=1}^n j^{m_j} m_j!}.$$

Proof. There are $n!$ ways to arrange n distinct elements in a sequence. The goal is to partition the sequence into disjoint cycles.

For cycles of length j , there are $(j-1)!$ distinct cycles, with each counted j times due to rotations. Thus, for each collection of j elements, we divide by j . Since there are m_j such cycles, the factor is j^{m_j} .

For distinct cycles of the same length, the m_j cycles can be permuted in $m_j!$ ways without changing the permutation. Thus, we divide by $m_j!$.

The final formula is thus deduced by combining the above two factors and multiplying through 1 to n . \square

1.3 Presentation of S_n

We now study the presentation of S_n by generators and relations. Denote $s_i := (i, i+1)$, and consider the following result.

Proposition 4. *In S_n ,*

$$(i, j) = s_{j-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{j-1}.$$

Proof. We prove by induction. If $j = i+1$, then

$$(i, i+1)(i+1, i+2)(i, i+1)(i+1, i+2)(i, i+1) = (i, i+2).$$

If it holds for arbitrary $j > i$, then by the above Proposition 2,

$$s_j(i, j)s_j = (i, s_j(j)) = (i, j+1),$$

which proves the case of $j+1$. \square

By this relationship, together with Proposition 1, we have the following presentation of S_n :

$$S_n = \langle s_1, \dots, s_{n-1} : s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \quad \forall |i-j| > 1 \rangle, \quad s_n := s_1 \quad (1)$$

We can see now that symmetric groups are examples of the **Coxeter groups**, without commutativity and with **braid relation** $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ as multiple or infinite products, which will be introduced later. Some references for further reading include [1] and [4].

1.4 Reflections

With the help of presentation of S_n by generators, we could try to present special elements in S_n and study their properties more explicitly. This subsection and the following study two special elements: the reflections and the inversions.

Definition 3 (reflection). *A reflection is a permutation $\sigma_0 \in S_n$ of the form*

$$\sigma_0 = (s_1 \cdots s_{k-1}) s_k (s_{k-1} \cdots s_1),$$

where $s_i \in S_n, \forall 1 \leq i \leq n-1$ are generators.

Reflections have the following useful property in the corollary. We first give a more general result as shown in the following proposition.

Proposition 5. *For permutations $\sigma_0, \dots, \sigma_n$ of S_n , we have*

$$\sigma_0(\sigma_1 \cdots \sigma_n) = \sigma_1 \cdots \hat{\sigma}_k \cdots \sigma_n \iff \sigma_0 = (\sigma_1 \cdots \sigma_{k-1}) \sigma_k^{-1} (\sigma_1 \cdots \sigma_{k-1})^{-1},$$

where $(\sigma_1 \cdots \hat{\sigma}_k \cdots \sigma_n)$ stands for $(\sigma_1 \cdots \sigma_{k-1} \sigma_k \cdots \sigma_n)$.

Proof. Denote $A = (\sigma_1 \cdots \sigma_{k-1}), C = (\sigma_{k+1} \cdots \sigma_n)$. Then observing the left equation, we have

$$\sigma_0 A \sigma_k C = AC \iff \sigma_0 A \sigma_k = A \iff \sigma_0 = A \sigma_k^{-1} A^{-1},$$

which is the right equation. Thus the result is proved since the above deduction is invertible. \square

Corollary 1. *For generators s_0, \dots, s_n of S_n , we have*

$$s_0(s_1 \cdots s_n) = s_1 \cdots \hat{s}_k \cdots s_n \iff s_0 = (s_1 \cdots s_{k-1}) s_k (s_{k-1} \cdots s_1).$$

This result follows immediately since the generators are all transpositions which have inverses as themselves.

Reflections are given their name due to their geometric nature, especially in Lie theory, where it will be used to define the **root systems**, where they perform as generators of the **Weyl groups**, which act on the maximal tori of certain Lie groups.

1.5 Inversions

Definition 4 (inversion). *Given a one line form of permutation $\sigma = [\sigma(1), \dots, \sigma(n)]$, the **inversions** of σ are ordered pairs ij , $1 \leq i < j \leq n$, where $\sigma(i) > \sigma(j)$.*

Denote the set of inversions as $I(\sigma) := \{ij\}$. Denote its cardinality as $\text{inv}(\sigma) := |I(\sigma)|$.

Inversions are closely related to reflections, due to the following presentation:

$$\underline{kn} = (s_{n-1} \cdots s_{k+1}) s_k (s_{k+1} \cdots s_{n-1}).$$

This can be seen by transposing the numbers one by one. With this presentation, there are always the same amount of reflections and inversions, and they are conjugate to each other as

$$\sigma_0 = w\sigma w^{-1}, \quad w = s_1 \cdots s_{n-1},$$

where σ_0 is a reflection and σ the corresponding inversion. This could help when calculating one from another.

We now describe a further result on the transposition decomposition for a permutation. Define the **sign** of a permutation σ as

$$\text{sgn}(\sigma) := (-1)^{\text{inv}(\sigma)}.$$

Proposition 6. *A permutation σ must be decomposed into a fixed parity of transpositions, of even or odd number. In other words, $\text{sgn}(\sigma)$ does not depend on the presentation of this cycle.*

Proof. First, observe that any transposition τ has sign $\text{sgn}(\tau) = -1$. This is because swapping two elements changes the number of inversions by an odd number.

Next, we prove that $\text{sgn} : S_n \rightarrow \{1, -1\}$ is a group homomorphism. That is to prove

$$\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau).$$

By definition, we also have the following formula for the sign function:

$$\text{sgn}(\sigma) = \prod_{1 \leq i < j \leq n} \frac{\sigma(j) - \sigma(i)}{j - i}.$$

Hence

$$\begin{aligned} \text{sgn}(\sigma\tau) &= \prod_{1 \leq i < j \leq n} \frac{\sigma\tau(j) - \sigma\tau(i)}{j - i} = \prod_{1 \leq i < j \leq n} \frac{\sigma(\tau(j) - \tau(i))}{\sigma(j - i)} \frac{\sigma(j - i)}{j - i} \\ &= \sigma \left(\prod_{1 \leq i < j \leq n} \frac{\tau(j) - \tau(i)}{j - i} \right) \prod_{1 \leq i < j \leq n} \frac{\sigma(j) - \sigma(i)}{j - i} = \prod_{1 \leq i < j \leq n} \frac{\sigma(j) - \sigma(i)}{j - i} \prod_{1 \leq i < j \leq n} \frac{\tau(j) - \tau(i)}{j - i} \\ &= \text{sgn}(\sigma)\text{sgn}(\tau). \end{aligned}$$

Thus sgn is well defined as a group homomorphism. Hence for a chosen $\sigma \in S_n$ and an arbitrary decomposition $\sigma = \tau_1 \cdots \tau_k$,

$$\text{sgn}(\sigma) = \text{sgn}(\tau_1 \cdots \tau_k) = \text{sgn}(\tau_1) \cdots \text{sgn}(\tau_k) = (-1)^k$$

is a fixed number. Thus the parity of σ is fixed. \square

Permutations are called **odd** if it is decomposed into an odd number of transpositions, i.e., k is even and $\text{sgn}(\sigma) = 1$, and called **even** if it is decomposed into an even number of transpositions, i.e., k is odd and $\text{sgn}(\sigma) = -1$.

The **alternating group** A_n is defined to be the collection of even permutations. One can show that S_n is not a simple group and is not Abelian whenever $n \geq 3$, since $A_n \triangleleft S_n$.

Inversions are useful due to the following result.

Proposition 7. *Given a permutation $\sigma \in S_n$, by decomposing σ into a degenerated composition of transpositions, we have*

$$\text{inv}(\sigma) = \#\{\text{transpositions of } \sigma\}.$$

Proof. To see this, consider what results after applying each transposition. The operation s_k swaps the values at point $(k, k + i)$, and changes the number by exactly ± 1 . Specifically, $\text{inv}(w)$ increases 1 when $w(k) < w(k + 1)$, and decreases 1 when $w(k) > w(k + 1)$. Since for identity $w_0 = (1, \dots, n)$, $\text{inv}(w_0) = 0$, we have that each transposition contributes to a inversion. \square

This result is reasonable due to the previous proposition, since letting $k = \#\{\text{transpositions of } \sigma\}$ gives

$$(-1)^{\text{inv}(\sigma)} = \text{sgn}(\sigma) = \text{sgn}(\tau_1 \cdots \tau_k) = (-1)^k,$$

which means that they have the same parity.

1.6 An example

Consider the permutation $w = (1342) \in S_5$. Its two line form and one line form are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}, \quad [3, 1, 4, 2, 5].$$

This permutation has length 4, hence it is a 4-cycle. It is already decomposed as composition of disjoint cycle, since it is already a single cycle. It can be further decomposed as composition of transpositions as

$$w = (1342) = (12)(14)(13) = (13)(34)(42).$$

As this permutation can also be written as $(1342)(5)$, it has cycle type $\lambda = (4, 1)$ or $m = (1, 0, 0, 1, 0)$. It follows that $1 \cdot 1 + 4 \cdot 1 = 5$, which is the cardinality of the set being permuted. By the given formula, the number of permutations in this cycle type is $5!/(1^1 \cdot 1!)(4^1 \cdot 1!) = 30$.

By observing the one line form $w = [3, 1, 4, 2, 5]$, this permutation has inversion 12, 14, 34, and hence $\text{inv}(w) = 3$. Thus $\text{sgn}(w) = -1$, and it is an odd permutation, hence does not lie in the alternating group A_5 . Due to the previous degenerate decomposition, the number of transpositions equals 3, which is the same with the number of inversions.

To find all its reflections, we try expressing them as conjugates of inversions under w . Since $w(12)w^{-1} = (13)$, $w(14)w^{-1} = (23)$, $w(14)w^{-1} = (24)$, these are all its reflections.

2 Representation theory

2.1 Representations

Definition 5 (representation). A *representation* of a group G is a F -vector space V , together with a module structure by ρ a group homomorphism as

$$\rho : G \rightarrow \text{GL}(V) \quad g \mapsto (\rho_g : V \rightarrow V \quad v \mapsto \rho_g(v)),$$

where $\text{GL}(V)$ is the set of linear isomorphisms from V to V . Namely, it satisfies the following conditions:

- $\rho_{gh} = \rho_g \rho_h$,
- $\rho_g(av + bw) = a\rho_g(v) + b\rho_g(w)$,
- $\rho_1 = \mathbb{1} : x \mapsto x$.

$\forall g, h \in G, v, w \in V, a, b \in \mathbb{C}$.

The **dimension** of the representation is the dimension of V as a vector space.

Representations on G are also called **G -representations**. Given a representation on G , we can view that the given G and V are equipped with an action of G on V under the rule of

$$G \curvearrowright V : g \cdot v = \rho_g(v).$$

Thus, we can also call V a **G -module**.

Now consider enlarging the action by elements of G to elements of the group algebra $\mathbb{C}[G]$. Given a G -representation ρ , define

$$\rho' : \mathbb{C}[G] \rightarrow \text{End}(V) \quad \sum z_i g_i \mapsto (v \mapsto \sum z_i (g_i \cdot v)), \quad z_i \in \mathbb{C}, g_i \in G,$$

where $g_i \cdot v$ refers to representing elements in G . It can be checked that ρ' is a $\mathbb{C}[G]$ -representation by

$$\begin{aligned} (\sum z_i g_i \sum w_i g_i) \cdot v &= (\sum_i \sum_{j+k=i} z_j w_k g_i) \cdot v = (\sum_j \sum_k z_j g_j w_k g_k) \cdot v \\ &= \sum_j (z_j g_j \sum_k w_k) (g_k \cdot v) = (\sum z_i g_i) \cdot \sum w_i (g_i \cdot v), \\ \sum z_i g_i \cdot (c_1 v + c_2 w) &= \sum z_i (g_i (c_1 v + c_2 w)) = \sum z_i (g_i \cdot c_1 v) + \sum z_i (g_i \cdot c_2 w) \\ &= c_1 \sum z_i (g_i \cdot v) + c_2 \sum z_i (g_i \cdot v), \end{aligned}$$

proves that linearity and group homomorphism are both satisfied, $w_i, c_1, c_2 \in \mathbb{C}$. Also, $\text{End}(V)$ in this context is moreover an \mathbb{C} -algebra, which means that it has multiplication as action composition. Thus this is moreover a group algebra representation.

We want to observe the relation of $\mathbb{C}[G]$ -representations with G -representations. We could simply restrict action from elements in $\mathbb{C}[G]$ to G , and consider $\text{GL}(V)$ as $\text{End}(V)$ without multiplicative structure, since an algebra is always a module. Thus, $\rho = \rho' \mid_V$. Therefore we have the following equivalence:

$$\{G\text{-representation } \rho\} \longleftrightarrow \{\mathbb{C}[G]\text{-representation } \rho'\}, \quad \{G\text{-module}\} \longleftrightarrow \{\mathbb{C}[G]\text{-module}\}. \quad (2)$$

Such understanding is necessary when it comes to introducing representations of **Hecke algebra** in later context, which could not be constructed from first establishing a group representation.

We mainly consider the case when group G is finite and V has finite dimension. In this context, a representation always has finite dimension. Thus we could consider ρ_g 's as invertible matrices and the group representation as matrix representation:

$$\text{GL}(V) = \text{GL}(n, F).$$

2.2 First examples

We give the following 5 examples at $F = \mathbb{C}$.

Example 1 (trivial representation). *Consider the representation*

$$\rho : G \rightarrow \text{GL}(V) \quad g \mapsto \mathbb{1}.$$

This means that every element of G acts on V trivially as identity. Hence $\dim \rho = 1$ since $\text{im } \rho \simeq 0 \simeq \{(1) \in \text{GL}(1, \mathbb{C})\}$, where 0 the trivial group.

Example 2 (C_n -representation of dim 1). *We denote C_n the cyclic group of order n , i.e., $C_n = \{1, g, \dots, g^{n-1}\}$. To see all its 1 dimensional representations, we must satisfy that*

$$\rho_g^n = \rho_{g^n} = \mathbb{1}.$$

Thus the representations form a cyclic group of order n . Therefore, one choice could be $\rho_g \simeq \zeta_n$, where $\zeta_n = e^{2\pi i/n}$ is the n -th root of unity. To list all the choices, we need to consider the factors of n . In fact, all the ρ 's satisfying

$$\rho_g \simeq \zeta_n^m, \quad \gcd(n, m) = 1$$

could be a C_n -representation of dim 1. Hence there are $\varphi(n)$ of them, φ the Euler function.

Example 3 (S_n -representation of dim 1). It suffices to study the image of every generator of S_n . We will nail down all S_n -representations of dim 1 with the help of their relations from the presentation Equation 1.

First recall the relation $s_i^2 = 1, \forall 1 < i < n - 1$. This means $\rho_{s_i}^2 = \rho_{s_i} = \mathbb{1}$, which gives

$$\rho_{s_i} = \pm \mathbb{1}, \quad \forall 1 < i < n - 1.$$

Next consider the relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. This means that either all s_i 's are mapped to $\mathbb{1}$, which is the trivial representation, or all even transpositions s_{2i} are mapped to $\mathbb{1}$ and all odd transpositions s_{2i-1} mapped to $-\mathbb{1}$. This is called the **signed representation**.

Indeed, these are the only two cases that satisfies this relation. And they both satisfy the third relation $s_i s_j = s_j s_i, \forall |i - j| > 1$.

From this example, we can further deduce that if we restrict the S_n -representation of dim 1 to A_n , then both trivial representation and signed representation give trivial representation.

Example 4 (permutation representation/defining representation of S_n). Since all elements in S_n are permutations on the set $\{1, \dots, n\}$, there is a natural action of S_n on $\{1, \dots, n\}$ as

$$S_n \curvearrowright \{1, \dots, n\} : \sigma \cdot i = \sigma(i).$$

To construct a vector space, we simply take $\{1, \dots, n\}$ as basis, then spanning over \mathbb{C} gives a \mathbb{C} -vector space $\mathbb{C}\langle 1, \dots, n \rangle \cong \mathbb{C}^n$, and the action becomes

$$S_n \curvearrowright \mathbb{C}^n : \sigma \cdot v = \bigoplus_{i=1}^n z_i \sigma(i), \quad v = (z_1, \dots, z_n).$$

This is called the **permutation representation** or the **defining representation** of S_n . Under this construction, every transposition (i, j) is mapped to a permutation matrix where it has 1's at entries $(i, \sigma(i)), (j, \sigma(j))$, $1 \leq i \leq n$ and 0's elsewhere. It performs as transformation of the basis.

Example 5 (regular representations). Notice that group algebras $\mathbb{C}[G] := \{\sum z_i g_i : g \in G\}$ are \mathbb{C} -vector spaces with G as their set of basis. They have dimension $\dim(\mathbb{C}[G]) = |G|$. If we are given an action of G on G as left translation, then it induces the **regular representation**

$$G \curvearrowright \mathbb{C}[G] : g \cdot x = gx,$$

or equivalently, $\mathbb{C}[G] \curvearrowright \mathbb{C}[G]$ with the same action.

2.3 Subrepresentations

Definition 6. A **subrepresentation** of a G -representation V is a G -invariant subspace $W \leq V$, i.e.,

- W a sub F -vector space of V ,
- W closed under action by G .

The second requirement also infers that W is a G -module, which carries the information that it is a representation. We could also call W a **G -submodule**.

Example 6 (trivial subrepresentation). These are subrepresentations W that are equal to the representation V themselves or the 0 space. That is, subrepresentation with $\dim W = \dim V$ or 0.

Example 7 (C_n -representation of dim 1, continuance of 2). Since these representations have dimension 1, it has only trivial subrepresentations.

Example 8 (defining representation, continuance of 4). We typically choose the element $1 + \dots + n \in \mathbb{C}\{1, \dots, n\}$ as the generator of a subspace of $\mathbb{C}\{1, \dots, n\}$. By action of S_n ,

$$S_n \curvearrowright \mathbb{C}\{1, \dots, n\} \quad \pi \cdot (1 + \dots + n) = \pi(1) + \dots + \pi(n) = 1 + \dots + n.$$

This spans a vector space of dim 1, which means it has only trivial subrepresentations. Since all defining representation can be viewed equivalent to the above definition after permuting basis, they all have trivial subrepresentations.

Example 9 (regular representation, continuance of 5). Define

$$G \curvearrowright \mathbb{C}[G] \quad g \cdot (c_1g_1 + \cdots + c_ng_n) = c_1g \cdot g_1 + \cdots + c_ng \cdot g_n.$$

In general, this could be a n dimensional representation and uneasy to study its subrepresentations. However, if we put $c_1 = \cdots = c_n = 1$, this is exactly the defining representation with S_n being G . It has dim 1 and has only trivial representations.

Consider on the other hand taking $G = S_n$ and

$$S_n \curvearrowright \mathbb{C}[S_n] \quad g \cdot (\sum c_i\pi_i) = \sum c_i\sigma(\pi_i).$$

We now assume $c_i = \text{sgn}(\pi_i)$. Recall that $\text{sgn}(\sigma \cdot \pi_i) = \text{sgn}(\sigma)\text{sgn}(\pi_i)$. If σ is even, $\text{sgn}(\sigma \cdot \pi_i) = \text{sgn}(\pi_i)$, and

$$\sigma \cdot (\sum \text{sgn}(\pi_i)\pi_i) = \sum \text{sgn}(\pi_i)(\sigma \cdot \pi_i) = \sum \text{sgn}(\sigma \cdot \pi_i)(\sigma \cdot \pi_i) = \sum \text{sgn}(\pi_i)\pi_i.$$

Similarly, if σ is odd, $\text{sgn}(\sigma \cdot \pi_i) = -\text{sgn}(\pi_i)$, and

$$\sigma \cdot (\sum \text{sgn}(\pi_i)\pi_i) = \sum \text{sgn}(\pi_i)(\sigma \cdot \pi_i) = -\sum \text{sgn}(\sigma \cdot \pi_i)(\sigma \cdot \pi_i) = -\sum \text{sgn}(\pi_i)\pi_i.$$

Hence the representation is restricted to the dim 1 vector space $\{\pm 1\}$, and has only trivial subrepresentations.

2.4 Decompositions

We begin the story from defining the basic components.

Definition 7 (irreducible/simple representation). A representation V of G is called **irreducible** or **simple**, if it has only trivial subrepresentations.

A representation that is not irreducible is called **reducible**.

From the discussion in the previous section, all representation V of dim 1 is irreducible.

If we are given two irreducible representations, we are able to construct combined representations. We could either perform direct sums or tensor products. Assume V_1, V_2 are G -modules, then $V_1 \oplus V_2$ and $V_1 \otimes V_2$ are G -modules with action

$$\begin{aligned} G \curvearrowright V_1 \oplus V_2 \quad g \cdot (x, y) &= (g \cdot x, g \cdot y), \\ G \curvearrowright V_1 \otimes V_2 \quad g \cdot (x \otimes y) &= (g \cdot x) \otimes (g \cdot y). \end{aligned}$$

It is natural to ask if the reverse holds. That is, given a representation V , if we could find two representations with their combination being V . We mainly focus on direct sums currently.

Theorem 1. Given W a subrepresentation of V of finite group G . There is a complementary subrepresentation W' of V , such that $V = W \oplus W'$.

Proof. (following Serre's book) Let W' be the complement subspace of V wrt. W . Let p be the **projection** from V to its subspace W , that is,

$$p : V \rightarrow W \quad v = (w, w') \mapsto (w, 0).$$

We construct another projection from V to W as follows. Define the mean projection

$$p' : V \rightarrow V \quad p' = \frac{1}{g} \sum_{t \in G} \rho_t \cdot p \cdot \rho_t^{-1}, \quad g = |G|.$$

Since $\rho_t^{-1} : V \rightarrow V, p : V \rightarrow W, \rho_t : W \rightarrow W$ all linear, we see that $p' \in \text{End}(V)$ restricting its image to W . Since $\forall x \in W \quad p \cdot \rho_t^{-1}x = \rho_t^{-1}x$, we see that $\rho_t \cdot p \cdot \rho_t^{-1}x = x$, hence p' is a projection from V to W . Also, by observing the conjugates of p , we see

$$\rho_s \cdot p' \cdot \rho_s^{-1} = \frac{1}{g} \sum_{t \in G} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{g} \sum_{t \in G} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p'.$$

Thus, p' lies in the centralizer $C_{\text{End}(V)}(\text{GL}(V))$. Therefore, $\forall x' \in W' \quad p'x' = 0$ gives $p' \cdot \rho_s x' = \rho_s \cdot p' x' = \rho_s(0) = 0$, which means $\rho_s x' \in W'$ and hence W' is G -invariant. This proves that W' is a subrepresentation of V . \square

The proof of this theorem underlines an idea for constructing symmetrization structures under group G . Another example would also concern vector spaces. In general, given F -vector space V and its subspace W , one way to construct its complement W' such that $V = W \oplus W'$ is by introducing an inner product structure. Given an inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow F$, we call it **G -invariant** if

$$\langle g \cdot x, g \cdot y \rangle = \langle x, y \rangle \quad \forall g \in G, x, y \in V.$$

With this given, being complementary W' of W as representation coincide with being subspace W^\perp having G -invariant inner product 0 with W , since $\forall u \in W^\perp, w \in W$, G -stability of W tells

$$\langle g \cdot u, w \rangle = \langle u, g^{-1} \cdot w \rangle = 0.$$

To define a proper G -invariant inner product on any representation V , we first set up a basis for V . Given a basis x_1, \dots, x_m for W and y_1, \dots, y_n for W' , V is then spanned by v_1, \dots, v_{m+n} , where $v_i = x_i, i = 1, \dots, m$, and $v_{m+j} = y_j, j = 1, \dots, n$. We build up G -invariance from the discrete form, i.e.,

$$\langle v_i, v_j \rangle := \delta_{ij}, \quad \langle v_i, v_j \rangle' := \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v_i, g \cdot v_j \rangle, \quad \forall v_i, v_j \in V. \quad (3)$$

This is defined as an inner product since it is summing over the inner product $\langle \cdot, \cdot \rangle$. The constant in front makes sure that $\langle v_i, v_i \rangle = 1 \forall i$, i.e. every vector has norm 1. The inner product $\langle \cdot, \cdot \rangle'$ is G -invariant since it sums over all possibilities. If $\langle \cdot, \cdot \rangle$ is already G -invariant, then $\langle \cdot, \cdot \rangle \sim \langle \cdot, \cdot \rangle'$. Due to its nice structure, it is often called **the Weyl's averaging trick**.

Through this theorem, we could present any member $\rho_t \in \text{GL}(V)$ with matrix R_t under basis v_1, \dots, v_{m+n} via the form

$$R_t \sim \begin{pmatrix} R_s & 0 \\ 0 & R'_s \end{pmatrix}, \quad (4)$$

where v_1, \dots, v_m give basis to $R_s \sim \rho_s \in \text{GL}(W)$, v_{m+1}, \dots, v_{m+n} give basis to $R'_s \sim \rho'_s \in \text{GL}(W')$, and $V = W \oplus W'$. Since these representations are first to be vector spaces, linear algebra tells

$$\dim V = \dim W + \dim W'. \quad (5)$$

Example 10 (reducing S_n). As shown previously, every symmetric group has the defining representation $W = \mathbb{C}\{1 + \dots + n\}$ of dimension one. Thus any n -dimensional S_n -representation $V = \mathbb{C}\{1, \dots, n\}$ could be of the form $V = W \oplus W'$, where W' is of dimension $n - 1$.

Example 11 (reducing S_3). From example 4 and the above equation Equation 5, we know that S_3 can be reduced to two subrepresentations W and W' , where they have dimensions $\dim W = 1$ and $\dim W' = 2$, and $W = \mathbb{C}\{1 + 2 + 3\}$, which is irreducible. By linear algebra, $W' = \mathbb{C}\{1 + 2 - 3, 1 - 2 - 3\}$.

We claim that W' is also irreducible, since every subspace of $\dim = 1$ cannot be a S_3 -subrepresentation. Indeed, $\forall w = c_1(1 + 2 - 3) + c_2(1 - 2 - 3) \in W', c_1, c_2 \in \mathbb{C}$,

$$\begin{aligned} (123)w &= z_1w, & (c_1 + c_2) &= z_1(c_1 - c_2), & c_1 &= c_2 = 0. \\ (132)w &= z_2w & z_1(c_2 - c_1) &= z_2(c_2 + c_1) \end{aligned}$$

Every S_3 -representation could thus be written as a matrix of the form

$$\begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix}.$$

The above result shows that every reducible representation V of $\dim > 1$ can be written as the direct sum two subrepresentations of smaller dimension. Further, we want to know if these representations can be decomposed into irreducible subrepresentations, that is, if the representation V is **decomposable** or **completely reducible**.

Corollary 2 (Maschke's theorem). Every representation is decomposable.

Proof. Assume V is a linear representation of group G . We prove by induction through $\dim(V)$. If $\dim(V) = 0$ or 1, then it is already irreducible. If $\dim(V) > 1$, and V is irreducible, then we are done as well. If $\dim(V) > 1$ and V is not irreducible, then it has at least one nontrivial subrepresentation W . By the previous theorem, we could decompose V into $W \oplus W'$, with $\dim(W), \dim(W') < \dim(V)$. By induction, the dimension of these subrepresentations decrease to 1. \square

By the Maschke's theorem, we could decompose any member $\rho_t \in \mathrm{GL}(V)$ described previously into smaller blocks

$$R_t \sim \begin{pmatrix} R_{s_1} & & \\ & \ddots & \\ & & R_{s_r} \end{pmatrix},$$

where $R_{s_j}, j = 1, \dots, r$ are invertible matrices mapping within irreducible subspaces $V_j, j = 1, \dots, r$, with $V = V_1 \oplus \dots \oplus V_r$.

2.5 The category $G\text{-Rep}$

Given finite group G , we now consider the relation between different G -representations. We call a category **C** as **semisimple** if all the objects in **C** can be decomposed into simple objects. For example, by Maschke's theorem 2, the category of G -representations $G\text{-Rep}$ is semisimple. Meanwhile, the category of groups **Grp** is not semisimple, since there exist counter example group \mathbb{Z} , which cannot be decomposed as direct sum of simple groups $\mathbb{Z}/k\mathbb{Z}$.

In the following context, we focus on the semisimple category $G\text{-Rep}$. Let V be an object in this category, we call a morphism $f : V \rightarrow V$ as **G -equivariant** or **G -linear** if

$$g \cdot f(v) = f(g \cdot v) \quad \forall g \in G, v \in V.$$

A G -equivariant map is further an **G -intertwiner** if it is F -linear. As we can see, the mean projection map p' defined above is a $\mathrm{GL}(V)$ -intertwiner.

We want to know if the $G\text{-Rep}$ category being semisimple helps with studying its morphisms. The following results show that it suffices to consider only the morphisms between simple representations.

Proposition 8. Given F -vector space $V, W_i, i = 1, \dots, m$ with $W = W_1 \oplus \dots \oplus W_m, V = V_1 \oplus \dots \oplus V_l$, we have

$$\mathrm{Hom}_G(V, \bigoplus_{i=1}^m W_i) \simeq \bigoplus_{i=1}^m \mathrm{Hom}_G(V, W_i), \quad \mathrm{Hom}_G(\bigoplus_{j=1}^l V_j, W) \simeq \bigoplus_{j=1}^l \mathrm{Hom}_G(V_j, W),$$

where the image of functor $\mathrm{Hom}_G(V, -) : G\text{-Rep} \rightarrow G\text{-Rep}$ is a F -vector space with G -module structure.

Proof. It suffices to proof the first equation. Consider an element

$$f : V \rightarrow \bigoplus_{i=1}^m W_i \quad v \mapsto (f_1(v), \dots, f_m(v)),$$

where $f_i(v) \in W_i \quad \forall i = 1, \dots, m$. If we further let $f \in \mathrm{Hom}(V, \bigoplus_{i=1}^m W_i)$, then f is G -equivariant, i.e.

$$(f_1(gv), \dots, f_m(gv)) = f(gv) = gf(v) = g(f_1(v), \dots, f_m(v)) = (gf_1(v), \dots, gf_m(v)).$$

This means that all f_i 's are also G -equivariant and $f_i \in \mathrm{Hom}(V, W_i) \quad \forall i = 1, \dots, m$.

Meanwhile, since f has F -linearity,

$$\begin{aligned} (af_1(v) + bf_1(w), \dots, af_m(v) + bf_m(w)) &= a(f_1(v), \dots, f_m(v)) + b(f_1(w), \dots, f_m(w)) \\ &= af(v) + bf(w) = f(av + bw) = (f_1(av + bw), \dots, f_m(av + bw)), \quad \forall a, b \in F, v, w \in V \end{aligned}$$

shows that the $f_i, i = 1, \dots, m$ has F -linearity. Hence the f_i 's, $i = 1, \dots, m$ are also G -intertwiners.

Now construct a mapping

$$\phi : \text{Hom}_G(V, \bigoplus_{i=1}^m W_i) \rightarrow \bigoplus_{i=1}^m \text{Hom}_G(V, W_i) \quad f \mapsto (f_1, \dots, f_m)$$

This map ϕ is well defined as a G -module homomorphism since every f and its corresponding f_i 's are G -intertwiners. We now show that ϕ is bijective.

Surjectivity: every element in $\bigoplus_{i=1}^m \text{Hom}_G(V, W_i)$ can be presented as (f_1, \dots, f_m) . We can define

$$F : V \rightarrow \bigoplus_{i=1}^m W_i \quad v \mapsto (f_1(v), \dots, f_m(v)).$$

Since f_i 's are G -equivariant, F is G -equivariant, hence $F \in \text{Hom}_G(V, \bigoplus_{i=1}^m W_i)$.

Injectivity: given $(f_1, \dots, f_m) = (0, \dots, 0)$, we have

$$(f_1(v), \dots, f_m(v)) = (0, \dots, 0), \quad \forall v \in V.$$

This shows $f_1 = \dots = f_m = 0$. Thus the above defined element F has $F(v) = 0 \quad \forall v \in V$. Hence $F = 0$.

Thus ϕ is defined as an isomorphism between V -vector spaces with structure as G -modules. \square

This proposition shows that $\text{Hom}_G(V, W)$ contains only G -intertwiners.

Corollary 3. Given representations with decompositions $V = \bigoplus_i U_i, W = \bigoplus_j U_j$,

$$\text{Hom}_G(V, W) = \text{Hom}_{\mathbb{C}[G]}(V, W) = \text{Hom}_{\mathbb{C}[G]}(\bigoplus_i U_i, \bigoplus_j U_j) = \bigoplus_{i,j} \text{Hom}_{\mathbb{C}[G]}(U_i, U_j).$$

Proof. This follows immediately from the previous proposition and the equivalence between G -modules and $\mathbb{C}[G]$ -modules. \square

We now study further results concerning morphisms between irreducible G -modules. Zooming from the categorical view to a more focused view, we simply call them G -module homomorphisms.

Proposition 9. Given G -module homomorphism $f : V \rightarrow W$,

- 1) $\ker f \leq V$ as subrepresentation,
- 2) $\text{im } f \leq W$ as subrepresentation.

Proof. For both statements, linear algebra tells that they are subspaces. We are left to show that they are G -invariant. Remind that G -module homomorphisms are necessary to be G -intertwiners.

1) $\forall v \in \ker f, g \in G$, since f intertwines, $f(g \cdot v) = g \cdot f(v) = g \cdot 0 = 0$. This tells that $g \cdot \ker f \subset \ker f$.

2) $\forall w \in \text{im } f, g \in G$, there exist a preimage $v \in V$ such that $f(v) = w$. Let $v' = g \cdot v$, since f intertwines, $g \cdot w = g \cdot f(v) = f(g \cdot v) = f(v') \in \text{im } f$. Thus $g \cdot \text{im } f \subset \text{im } f$. \square

Theorem 2 (Schur's lemma). Given a complete list of irreducible G -modules $\{U_i\}$ up to isomorphisms, given G -module homomorphisms $f_{ij} : U_i \rightarrow U_j \quad \forall i, j$. Then f_{ij} is either 0 or an isomorphism. Moreover,

$$\begin{cases} i \neq j & \implies f_{ij} = 0, \\ i = j & \implies f_{ij} = 0 \text{ or } f_{ij} = \lambda_{ij} \mathbb{1}, \end{cases}$$

where $\lambda_{ij} \in \mathbb{C} \quad \forall i, j$.

Proof. If $i \neq j$, then $\text{im } f_{ij} \leq U_j$. Since U_j is irreducible, $\text{im } f_{ij}$ is either 0 or U_j itself. If $\text{im } f_{ij} = U_j$, then $\ker f_{ij} \neq U_i$. Hence U_i is irreducible tells $\ker f_{ij} = 0$. Thus $U_i \simeq U_j$. However the collection $\{U_i\}$ is constructed up to isomorphisms, which contradicts. Thus $\text{im } f_{ij} = 0$, in other words $f_{ij} = 0$. Similarly, if $i = j$, then $f_{ij} = 0$ or $\lambda_{ij} \mathbb{1}$.

Assume now $i = j$ and f_{ij} is not 0. Then there is at least one eigenvalue denoted $\lambda_{ij} \in \mathbb{C}$. It has a corresponding eigenvector $v \in U_i$ with

$$f_{ij}(v) = \lambda_{ij} v \implies (f_{ij} - \lambda_{ij} \mathbb{1})v = 0.$$

Since f_{ij} and $\mathbb{1}$ are both G -intertwiners, $(f_{ij} - \lambda_{ij}\mathbb{1}) : U_i \rightarrow U_j$ is also an G -intertwiner. Notice that $(f_{ij} - \lambda_{ij}\mathbb{1})$ could not be injective, otherwise v could only be the zero vector. Hence $(f_{ij} - \lambda_{ij}\mathbb{1})$ cannot be an isomorphism. This forces it to be 0. Thus $f_{ij} = \lambda_{ij}\mathbb{1}$. \square

This result is powerful not only because it shows explicitly the homomorphisms between irreducible representations, but moreover it gives the nontrivial case a beautiful and computable form of a scalar matrix to be applied to interesting examples.

Example 12 (S_3 , continuance of 11). Recall that S_3 can be decomposed as $V = W \oplus W'$, with

$$W = \mathbb{C}\{1 + 2 + 3\}, \quad W' = \mathbb{C}\{1 + 2 - 3, 1 - 2 - 3\}.$$

Then all representation homomorphism $f : V \rightarrow V$ are direct sum of the homomorphisms of irreducible representations, which have the form

$$\begin{aligned} 0 : W \rightarrow W', \quad 0 : W' \rightarrow W, \\ 0 : W \rightarrow W, \quad 0 : W' \rightarrow W', \\ \lambda\mathbb{1}_1 : W \rightarrow W, \quad \lambda'\mathbb{1}_2 : W' \rightarrow W', \end{aligned}$$

where $\lambda, \lambda' \in \mathbb{C}$, and $\mathbb{1}_i$ the identity transformation between vector spaces of dimension i . Thus every S_3 representation could be written as a matrix of the form

$$\begin{pmatrix} \lambda & & \\ & \lambda' & \\ & & \lambda' \end{pmatrix}.$$

2.6 Density theorem

From previous discussions, we know that every representation is decomposable, moreover the decomposition into irreducible subrepresentations is unique. It remains unknown how many representations could a group have. As all of them can be decomposed, we first need to know the number of irreducible representations for each group.

Recall that a regular representation is given by the action

$$G \curvearrowright \mathbb{C}[G] \quad \mathbb{C}[G] = \bigoplus_i n_i U_i,$$

where U_i forms a complete list of irreducible G -subrepresentations of $\mathbb{C}[G]$, with $n_i \in \mathbb{N}$ its **multiplicity**, which is denoted by $n_i U_i = U_i \oplus \cdots \oplus U_i$ with U_i repeated n_i times.

Proposition 10 (Degree Theorem). Given a finite group G and the decomposition of its regular representation $\bigoplus_{i=1}^r n_i U_i$, there are equations

$$n_i = \dim U_i, \quad |G| = \sum_{i=1}^r \dim^2 U_i.$$

These equations are useful for checking if we have all the irreducible G -representations found. For example, we have given S_3 trivial representation of dimension 1, defining representation of dimension 1, and a complimentary irreducible representation of dimension 2. These gives $1^2 + 1^2 + 2^2 = 6 = |S_3|$. This means that these three are exactly all the irreducible subrepresentations of S_3 .

Proof. The second equation can be obtained directly from the first by recalling the structure of $\mathbb{C}[G]$ as a vector space, and summing up all dimensions with the help of Equation 5 as

$$|G| = \dim \mathbb{C}[G] = \dim \left(\bigoplus_{i=1}^r n_i U_i \right) = \sum_{i=1}^r n_i \dim U_i = \sum_{i=1}^r \dim^2 U_i.$$

Now we prove the first equation. With the equivalence Equation 2, we consider

$$\mathbb{C}[G] \curvearrowright \mathbb{C}[G] \quad \alpha \cdot x \mapsto \alpha x.$$

Now consider G -module homomorphism from U_i to $\mathbb{C}[G]$. From Schur's lemma we only need to care about the image isomorphic to U_i , that is the n_i copies of U_i 's. Thus $\text{Hom}_G(U_i, \mathbb{C}[G]) = \text{Hom}_G(U_i, n_i U_i)$. Taking the equivalence again and putting their dimension into consideration, we have

$$n_i = \dim \text{Hom}_G(U_i, n_i U_i) = \dim \text{Hom}_G(U_i, \mathbb{C}[G]) = \dim \text{Hom}_{\mathbb{C}[G]}(U_i, \mathbb{C}[G]).$$

By results from 3

$$\text{Hom}_{\mathbb{C}[G]}(U_i, \mathbb{C}[G]) \simeq \bigoplus_{i=1}^r \text{Hom}_{\mathbb{C}[G]}(U_j, U_i) \simeq \bigoplus_{j=1}^r \text{Hom}_{\mathbb{C}[G]}(U_j, U_i) \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i),$$

which gives $n_i = \dim \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i)$.

We now show that $U_i \simeq \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i) \forall i = 1, \dots, r$. Notice that if the image of 1 in $\mathbb{C}[G]$ is determined for an element $\sigma \in \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i)$, then σ is completely determined by this image. To see this, given

$$\sigma : \mathbb{C}[G] \rightarrow U_i \quad 1 \mapsto x,$$

then $\sigma : \alpha = \alpha \cdot 1 \mapsto \alpha \cdot x$. Denote $\sigma_x := \sigma$, where $\sigma_x(1) = x$. Letting x run through U_i , these σ_x 's are hence all the elements in $\text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i)$. Now construct a mapping

$$\phi : U_i \rightarrow \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i) \quad x \mapsto \sigma_x.$$

This is well defined since

$$\phi(0)(\alpha) = \sigma_0(\alpha) = \alpha \cdot 0 = 0, \quad \forall \alpha \in \mathbb{C}[G].$$

It is a $\mathbb{C}[G]$ -module homomorphism since

$$\begin{aligned} \phi(ax + by)(\alpha) &= \sigma_{ax+by}(\alpha) = \alpha \cdot (ax + by) = a\alpha \cdot x + b\alpha \cdot y = a\sigma_x(\alpha) + b\sigma_y(\alpha) = (a\phi(x) + b\phi(y))(\alpha), \\ \phi(\alpha x)(\beta) &= \phi(\alpha \cdot x)(\beta) = \sigma_{\alpha \cdot x}(\beta) = \beta \cdot (\alpha \cdot x) = (\beta\alpha) \cdot x = \alpha \cdot (\beta \cdot x) = \alpha \cdot \sigma_x(\beta) = \alpha\phi(x)(\beta), \\ \phi(1)(\alpha) &= \sigma_1(\alpha) = \alpha \cdot 1 = \alpha, \end{aligned}$$

$\forall x, y \in U_i, a, b \in \mathbb{C}, \alpha, \beta \in \mathbb{C}[G]$. It is injective since

$$\phi(x) = \sigma_x = 0 \implies \sigma_x(\alpha) = \alpha \cdot x = 0 \quad \forall \alpha \in \mathbb{C}[G] \implies x = 0.$$

It is surjective since

$$\forall \sigma_x \in \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i) \quad x \in U_i \implies \sigma_x \in \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i).$$

We have thus established an $\mathbb{C}[G]$ -module isomorphism $\phi : U \xrightarrow{\sim} \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i)$. Thus $\dim U_i = \dim \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i) = n_i \forall i = 1 \dots r$.

We complete the proof by adding a remark that r , the number of irreducible $\mathbb{C}[G]$ -modules up to isomorphisms, is finite, since $U_i \leq \mathbb{C}[G]$ is of $\dim \mathbb{C}[G] = |G| < \infty$, so r being infinite would lead to contradiction. \square

Based on this, the following results provide a method to quickly compute the number r of irreducible representations.

Proposition 11 (density theorem). *Given a finite group G and decomposition of its regular representation $\mathbb{C}[G] = \bigoplus_{i=1}^r n_i U_i$, there is a $\mathbb{C}[G]$ -algebra isomorphism*

$$\mathbb{C}[G] \simeq \text{End}(U_1) \oplus \dots \oplus \text{End}(U_r).$$

Proof. Since the left hand side has dimension $\dim \mathbb{C}[G] = |G|$. The right when viewed as direct sum of algebra of matrices, it has dimension $\sum_{i=1}^r \dim \text{End}(U_i) = \sum_{i=1}^r \dim^2 U_i$. Thus by 10, the two sides has the same dimension. Therefore, it suffices to show that there exists an \mathbb{C} -algebra homomorphism $f : \mathbb{C}[G] \rightarrow \bigoplus_{i=1}^r \text{End}(U_i)$ which is injective or surjective.

Here we prove its existence and its injectivity. Denote each regular action as $\mathbb{C}[G] \curvearrowright U_i$ as representation $\rho_i : \mathbb{C}[G] \rightarrow \text{End}(U_i)$. Define

$$f : \mathbb{C}[G] \rightarrow \bigoplus_{i=1}^r \text{End}(U_i) \quad x \mapsto (\rho_1(x), \dots, \rho_r(x)).$$

It is well defined since

$$f(0) = (\rho_1(0), \dots, \rho_r(0)) = (0, \dots, 0).$$

It is an $\mathbb{C}[G]$ -algebra homomorphism since

$$\begin{aligned} f(ax + by) &= (\rho_1(ax + by), \dots, \rho_r(ax + by)) = (a\rho_1(x) + b\rho_1(y), \dots, a\rho_r(x) + b\rho_r(y)) \\ &= a(\rho_1(x), \dots, \rho_r(x)) + b(\rho_1(y), \dots, \rho_r(y)) = af(x) + bf(y), \\ f(xy) &= (\rho_1(xy), \dots, \rho_r(xy)) = (\rho_1(x)\rho_1(y), \dots, \rho_r(x)\rho_r(y)) = f(x)f(y), \\ f(1) &= (\rho_1(1), \dots, \rho_r(1)) = (1, \dots, 1), \end{aligned}$$

$\forall a, b \in C, x, y \in \mathbb{C}[G]$. It is injective since

$$(\rho_1(x), \dots, \rho_r(x)) = 0 \implies \rho_i(x) = 0 \ \forall i \implies x \cdot v = 0 \ \forall v \in U_i \implies x = 0.$$

□

Corollary 4. Given a finite group G and decomposition of its regular representation $\mathbb{C}[G] = \bigoplus_{i=1}^r n_i U_i$, then

$$r = \#\{\text{conjugacy classes of } G\}.$$

Proof. From the previous proposition,

$$Z(\mathbb{C}[G]) \simeq Z\left(\bigoplus_{i=1}^r \text{End}(U_i)\right) = \bigoplus_{i=1}^r Z(\text{End}(U_i)),$$

where Z denotes the center of a group. By Schur lemma, $\text{End}(U_i) = \{c_i \mathbb{1}_i\}$, where $c_i \in \mathbb{C}, \mathbb{1}_i : U_i \rightarrow U_i$. Thus,

$$\dim Z(\mathbb{C}[G]) = \dim \bigoplus_{i=1}^r Z(\text{End}(U_i)) = r.$$

Hence we only need that $\dim Z(\mathbb{C}[G])$ equals the number of conjugacy classes of G .

An arbitrary element $z \in \mathbb{C}[G]$ has the form $z = \sum_{g \in G} c_g g$, $c_g \in \mathbb{C}$. If $z \in Z(\mathbb{C}[G])$, then $hz = zh \ \forall h \in G$, i.e.

$$\sum_{g \in G} c_g (gh) = zh = hz = h \sum_{g \in G} c_g g = \sum_{g \in G} c_g (hg).$$

By indexing the coefficients,

$$\sum_{k \in G} c_{h^{-1}k} k = \sum_{k \in G} c_{kh^{-1}} k.$$

This gives $c_{gk} = c_{kg} \ \forall g, k \in G$. Thus, if g' is a conjugate of g , then $c_{g'} = c_g$. Conversely, if the coefficients are constant on the conjugacy classes, then for $a, b \in G$, ab and ba are conjugate to each other by $ba = a^{-1}(ab)a$. Thus $c_{ab} = c_{ba}$, which is in the center.

To summarize, $z \in Z(\mathbb{C}[G])$ if and only if the coefficients c_g 's are constant on the conjugacy classes. Define $e_C := \sum_{g \in C} g$, where C is any conjugacy class, as they are not conjugate to each other, and each of them span a conjugate class. These perform as a basis for each conjugacy class. To check that every e_C is in the center,

$$he_C = he_C h^{-1}h = e_C h, \quad \forall h \in G.$$

Hence e_C commutes with every group element, and also with every element in $\mathbb{C}[G]$. The dimension of $\mathbb{C}[G]$ is the number of these basis, which equals to the number of conjugacy classes. □

There also exists another set of basis which spans the conjugate classes. Given the representation $\rho : \mathbb{C}[G] \rightarrow \bigoplus_{i=1}^r \text{End } U_i$, these basis are

$$p_i := \rho^{-1}(\text{diag}(0, \dots, 0, \mathbb{1}_i, 0, \dots, 0)), \quad (6)$$

where $\mathbb{1}_i := \mathbb{1}_{U_i}$ is at the position for the i 'th subspace. These basis will be useful as we approach to the next section.

3 Character theory

3.1 Basic concepts

Definition 8 (character). *Given a representation $\rho : G \rightarrow \text{GL}(V)$, its **character** is the function*

$$\chi_\rho = \chi_V : G \rightarrow \mathbb{C} \quad x \mapsto \text{tr } \rho(x),$$

where $\text{tr} : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ refers to taking the trace. We often write $\chi := \chi_\rho$ if the action ρ is clear.

Similarly, characters could be defined on $\mathbb{C}[G]$ -representations.

Since the action of the identity element of a group is always the identity matrix, we see that for any representation V ,

$$\chi(1_G) = \dim V. \quad (7)$$

Example 13 (regular representation). *Consider $\rho : \mathbb{C}[G] \curvearrowright \mathbb{C}[G]$ which is regular. If we take $g = 1_G$ then its character would be the dimension of $\mathbb{C}[G]$, which is $|G|$. Otherwise, if $g \neq 1_G$, then every basis element g_i are permuted after action by g , hence the diagonal elements of the matrix $\rho(g)$ would all be 0, giving 0 trace. In summary,*

$$\chi(g) = \begin{cases} |G|, & g = 1_G, \\ 0, & g \neq 1_G. \end{cases}$$

Proposition 12. 1. χ is well-defined;

2. Given a $\mathbb{C}[G]$ -module V , the character χ is a linear function on $\mathbb{C}[G]$;

3. Given G -modules V, W , we have the following relations:

$$\chi_{V \oplus W} = \chi_V + \chi_W, \quad \chi_{V \otimes W} = \chi_V \cdot \chi_W, \quad (8)$$

which means $\{\chi : G \rightarrow \mathbb{C}\}$ is a ring;

4. Given a G -module V ,

$$\chi(g) = \chi(hgh^{-1}) \quad \forall h \in G, \quad \text{i.e., } \chi \upharpoonright_C = \text{constant} \quad \forall \text{conjugate classes } C.$$

Proof. 1. Assume we have now an action $\rho : g \mapsto A \in \text{End}(V)$. We need to show if the trace of the transformation A is irrelevant to the chosen basis for V . Suppose we have chosen another basis such that $g \mapsto B \in \text{End}(V)$, then by equivalence of basis, there exists be a transformation matrix $P \in \text{GL}_n(\mathbb{C})$ such that $PAP^{-1} = B$.

Consider arbitrary matrix $X = (x_{ij}), Y = (y_{ij}) \in M_n(\mathbb{C})$. They have the relation $\text{tr}(XY) = \text{tr}(YX)$ since

$$\text{tr}(XY) = \sum_{i=1}^n \sum_{j=1}^n x_{ij}y_{ji} = \sum_{j=1}^n \sum_{i=1}^n y_{ji}x_{ij} = \sum_{k=1}^n \sum_{l=1}^n y_{kl}x_{lk} = \text{tr}(YX).$$

Setting $X = PA$ and $Y = P^{-1}$ gives

$$\text{tr}(A) = \text{tr}(P^{-1}PA) = \text{tr}(PAP^{-1}) = \text{tr}(B).$$

2. This requires checking $\chi(c_1g + c_2h) = c_1\chi(g) + c_2\chi(h)$, $g, h \in G$, $c_1, c_2 \in \mathbb{C}$. Indeed, let $g \mapsto A, h \mapsto B$, then

$$\chi(c_1g + c_2h) = \text{tr}(c_1A + c_2B) = c_1 \text{tr}(A) + c_2 \text{tr}(B) = c_1\chi(g) + c_2\chi(h).$$

3. From Equation 4, we see that $\forall \rho_1 : g \mapsto R_g^1 \in \text{End}(V), \rho_2 : g \mapsto R_g^2 \in \text{End}(W)$,

$$\begin{aligned} \chi_{V \oplus W}(g) &= \text{tr}\left(\begin{pmatrix} R_g^1 & 0 \\ 0 & R_g^2 \end{pmatrix}\right) = \text{tr}(R_g^1) + \text{tr}(R_g^2) = \chi_V(g) + \chi_W(g), \\ \chi_V(g) = \sum_{i_1} r_{i_1 i_1}(g), \quad \chi_W(g) = \sum_{i_2} r_{i_2 i_2}(g) \quad \Rightarrow \quad \chi_{V \otimes W}(g) &= \sum_{i_1, i_2} r_{i_1 i_1}(g) \cdot r_{i_2 i_2}(g) = \chi_V(g) \cdot \chi_W(g). \end{aligned}$$

4. Let $\rho : g \mapsto A, h \mapsto B$, then due to that ρ is a group homomorphism,

$$\chi(hgh^{-1}) = \text{tr}(BAB^{-1}), \quad \chi(g) = \text{tr}(A).$$

The rest follows from the proof for 1. \square

3.2 Class functions

From the 4th proposition above, we see that fixing a group G , every character gives constant values on conjugacy classes of G . It is natural to explore more functions satisfying such properties.

Definition 9 (class function). *Given a group G , a **class function** on G is a function $f : G \rightarrow \mathbb{C}$ giving constant values on conjugacy classes of G .*

The collection of class functions is denoted as $\mathcal{CF}(G)$.

We can extend every element $f \in \mathcal{CF}(G)$ to a function on $\mathbb{C}[G]$, which is defined as

$$f : \mathbb{C}[G] \rightarrow \mathbb{C} \quad \sum c_g g \mapsto \sum c_g f(g).$$

$\mathcal{CF}(G)$ form a vector space, as the conjugacy classes of G remains unchanged implies the constants are combined linearly as

$$f_1, f_2 \in \mathcal{CF}(G), c_1, c_2 \in \mathbb{C} \implies c_1 f_1 + c_2 f_2 \in \mathcal{CF}(G).$$

From the above discussion on characters, we see that for every G -representation, its character lives in $\mathcal{CF}(G)$. It is surprising that every class function is as well a character for some representation. That is,

Proposition 13. *Given a finite group G , then vector spaces*

$$\mathcal{CF}(G) = \{\chi : G \rightarrow \mathbb{C}\}.$$

Proof. It suffice to show $\mathcal{CF}(G) \subset \{\chi : G \rightarrow \mathbb{C}\}$. Denote the number of conjugacy classes of G as r . Notice that the class functions are completely determined by the r values on the conjugacy classes, thus $\dim \mathcal{CF}(G) = r$. If we are provided a basis g_1, \dots, g_r for the conjugacy classes, then a basis of $\mathcal{CF}(G)$ can be given as f_1, \dots, f_r where

$$f_j(g_i) = \delta_{ij}, \quad 1 \leq i, j \leq r.$$

Moreover, such a basis f_j is the dual basis for g_i .

Thus, it suffices to show there exists basis $f_j = \chi_j$ for some basis g_i . Recall from 4, that given G , the characters $\chi_j := \chi_{U_j}$ of the r irreducible representations U_j 's form a basis. Hence we pick basis $g_i = p_i$ as in Equation 6, and see that

$$\chi_j(p_i) = \dim U_i \cdot \delta_{ij}, \quad \text{i.e.,} \quad \chi_j((\dim U_i)^{-1} p_i) = \delta_{ij},$$

which means χ_j and $(\dim U_i)^{-1} p_i$ gives a dual basis. This shows that those characters span $\mathcal{CF}(G)$. \square

Taking a step forward, this proof tells that χ_j and $(\dim U_i)^{-1}p_i$ are dual basis. Since quantity matrices commute with any matrix, $(\dim U_i)^{-1}p_i \in Z(G)$, and since there are r of them, they form a basis. Thus

$$(Z(G))^* \simeq \mathcal{CF}(G), \quad \forall 1 \leq j \leq r.$$

This can be realized as a correspondence by giving each element a unique image as:

$$\begin{aligned}\Phi : (Z(G))^* &\rightarrow \mathcal{CF}(G) & f &\mapsto (g \mapsto |C|^{-1}f(e_C)) \\ \Psi : \mathcal{CF}(G) &\rightarrow (Z(G))^* & \phi &\mapsto (e_C \mapsto \phi(g)),\end{aligned}$$

where $e_C = \sum_{h \in C} h$ is the basis for $Z(G)$ as same as defined in 4, and C is the conjugacy class of G containing g .

3.3 Key property

The character is given its name due to its nice property in characterizing representations. We demonstrate the importance of characters in this subsection.

Proposition 14. *Given a finite group G and G -representations V and W ,*

$$\chi_V = \chi_W \iff V \simeq W.$$

Proof. By Maschke theorem page 10, V and W can be decomposed into direct sum of irreducible G -representations as:

$$V = \bigoplus_i n_i U_i, \quad W = \bigoplus_i m_i U_i,$$

where the U_i 's are irreducible G -representations, and $n_i, m_i \in \mathbb{N}$. As characters can also be uniquely written as linear combinations of characters which forms the basis of $\mathcal{CF}(G)$, which can be chosen as the χ_{U_i} 's due to Equation 8, we have the presentation:

$$\chi_V = \sum_i n_i \chi_{V_i}, \quad \chi_W = \sum_i m_i \chi_{V_i}.$$

Thus, $\chi_V = \chi_W \iff n_i = m_i \iff V \simeq W$. □

The above proposition tells that each character corresponds to a unique representation. This infers that given a finite group G , the information of a G -representation is completely encoded in its character. Hence, to study all G -representations, it suffices for us to study all basis characters of G . That is, to study the value of these basis characters on each conjugacy class of G .

Therefore, we introduce the character table of G of the following form:

G	C_1	C_2	\dots	C_r
χ_1	$\chi_1(C_1)$	$\chi_1(C_2)$	\dots	$\chi_1(C_r)$
χ_2	$\chi_2(C_1)$	$\chi_2(C_2)$	\dots	$\chi_2(C_r)$
\vdots	\vdots	\vdots	\ddots	\vdots
χ_r	$\chi_r(C_1)$	$\chi_r(C_2)$	\dots	$\chi_r(C_r)$

Example 14 (character table of S_3). *Recall that S_n contains the same number of transpositions decomposed for each element in a same conjugacy class. By filling out the table above for S_3 , we get:*

S_3	(1)	(12), (23), (13)	(123), (132)
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_3	2	0	-1

where Example 11 tells $\chi_3 + \chi_{triv} = \chi_{def}$, and the character table for χ_{def} is given as:

S_3	(1)	(12), (23), (13)	(123), (132)
χ_{def}	3	1	0

3.4 Orthonormal basis

Now that the characters are of such importance, it is motivating to study further the space $\mathcal{CF}(G)$ where they live.

Due to Proposition 13, we see that characters are already basis of $\mathcal{CF}(G)$. We further want them to be orthonormal. As a \mathbb{C} -vector space, this leads us to consider giving it a inner product structure. Define:

$$\langle \cdot, \cdot \rangle : \quad \mathcal{CF}(G) \times \mathcal{CF}(G) \rightarrow \mathbb{C} \quad (\chi_i, \chi_j) \mapsto \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)}. \quad (9)$$

Lemma 1. *Given the mapping $\langle \cdot, \cdot \rangle$ defined above,*

$$1. \quad \langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g^{-1}),$$

2. *$\langle \cdot, \cdot \rangle$ is an inner product.*

Proof. 1. This is to proof $\overline{\chi(g)} = \chi(g^{-1})$, for every character χ .

From the Weyl's averaging trick Equation 3, we see that every representation can be **unitary**. That is, $\rho(g)^{-1} = \rho(g)^\dagger$, where $U^\dagger := \overline{U}^T$, $\rho(g) = U \in \text{End}(V)$. Since ρ is a group homomorphism, $\rho(g)^{-1} = \rho(g^{-1})$. Thus,

$$\overline{\chi(g)} = \overline{\text{tr } \rho(g)} = \text{tr } \overline{\rho(g)} = \text{tr } \rho(g)^\dagger = \text{tr}(\rho(g)^{-1}) = \text{tr } \rho(g^{-1}) = \chi(g^{-1}).$$

2. We need to check the following 3 properties:

- positive definite:

$$\langle \chi_i, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_i(g)} = \frac{1}{|G|} \sum_{g \in G} |\chi_i(g)|^2.$$

If $\chi_i = 0$, then $\chi_i(g) = 0 \forall g \in G$, so $\langle \chi_i, \chi_i \rangle = 0$. On the other hand, if $\langle \chi_i, \chi_i \rangle = 0$, then $\sum_{g \in G} |\chi_i(g)|^2 = 0$, which implies $|\chi_i(g)|^2 = 0$ for all $g \in G$, hence $\chi_i = 0$.

- Hermitian symmetry:

$$\overline{\langle \chi_j, \chi_i \rangle} = \overline{\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)}} = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_j(g)} \chi_i(g) = \langle \chi_i, \chi_j \rangle.$$

- bilinear:

$$\begin{aligned} \langle \alpha \chi_i + \beta \chi_j, \chi_k \rangle &= \frac{1}{|G|} \sum_{g \in G} (\alpha \chi_i + \beta \chi_j)(g) \overline{\chi_k(g)} = \alpha \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_k(g)} + \beta \frac{1}{|G|} \sum_{g \in G} \chi_j(g) \overline{\chi_k(g)} \\ &= \alpha \langle \chi_i, \chi_k \rangle + \beta \langle \chi_j, \chi_k \rangle, \quad \forall \alpha, \beta \in \mathbb{C}. \end{aligned}$$

Linearity in the second term is done with the help of Hermitian symmetry.

□

Now comes the main theorem of this subsection. It claims that **irreducible characters**, i.e., these basis characters of $\mathcal{CF}(G)$ are orthogonal under such defined $\langle \cdot, \cdot \rangle$.

Theorem 3. *Given a finite group G and irreducible G -representations $(\rho_i, V_i), i = 1, \dots, r$,*

1. (the Great Orthogonality Theorem) $\forall 1 \leq i, j \leq r$, $\sum_{g \in G} \rho_i(g)_{kl} \overline{\rho_j(g)_{mn}} = \frac{|G|}{\dim \rho_i} \delta_{ij} \delta_{km} \delta_{ln}$,
2. the irreducible characters of G form an orthonormal basis under the inner product $\langle \cdot, \cdot \rangle$.

Proof. 1. It suffices to prove for arbitrarily picked representations, say $i = 1$ and $j = 2$. We construct a matrix $Y \in M(\dim \rho_1 \times \dim \rho_2, \mathbb{C})$ as:

$$Y := \sum_{g \in G} \rho_1(g) X \rho_2(g^{-1}), \quad X \in M(\dim \rho_1 \times \dim \rho_2, \mathbb{C}).$$

First consider computing Y term by term. Let $X = E_{ab}$ be the basis matrix where it is 1 at only the (a, b) position and 0's elsewhere, $1 \leq a, b \leq \dim V_1$. Then

$$Y_{km} = \sum_{g \in G} \sum_{m,n} \rho_1(g)_{kn} X_{nl} \rho_2(g^{-1})_{lm} = \sum_{g \in G} \rho_1(g)_{ka} \rho_2(g^{-1})_{bm}.$$

Since ρ_2 is unitary,

$$Y_{km} = \sum_{g \in G} \rho_1(g)_{ka} \overline{\rho_2(g)_{mb}}.$$

On the other hand, we try to make use of the fact these representations are irreducible. We can show that $Y : V_2 \rightarrow V_1$ is a G -intertwiner, since:

$$\begin{aligned} \rho_1(h)Y &= \rho_1(h) \sum_{g \in G} \rho_1(g) X \rho_2(g^{-1}) = \sum_{g \in G} \rho_1(gh) X \rho_2(g^{-1}) = \sum_{k \in G} \rho_1(k) X \rho_2(hk^{-1}) \\ &= \sum_{k \in G} \rho_1(k) X \rho_2(k^{-1}) \rho_2(h) = Y \rho_2(h), \quad \forall h \in G. \end{aligned}$$

Now then, since ρ_1, ρ_2 are irreducible, we could apply Schur's lemma Theorem 2 and say that:

- if $\rho_1 \simeq \rho_2$, then $Y = \lambda \mathbb{1}, \lambda \in \mathbb{C}$,
- otherwise, then $Y = 0$.

We care about the nontrivial case, so now we aim to compute λ when $i = j$. Consider taking the trace of Y . Schur's lemma and Equation 7 tells that $\text{tr } Y = \lambda \dim V_1$. Meanwhile,

$$\text{tr } Y = \sum_{g \in G} \text{tr}(\rho_1(g) X \rho_2(g^{-1})) = \sum_{g \in G} \text{tr } X = |G| \text{tr } X.$$

Thus, Y is given by:

$$Y = \frac{|G|}{\dim \rho_1} \text{tr } X \cdot I, \quad I \sim \mathbb{1} \in \text{End}(V_1).$$

We now let $X = E_{ab}$, return to general labeling i and j , and combine the two results above. These give the followings:

- if $\rho_1 \simeq \rho_2$, that is $i = j$, then $\text{tr } X = \delta_{ab}$. Thus:

$$Y_{km} = \frac{|G|}{\dim \rho_i} \delta_{ab} \delta_{km},$$

- if not, that is, $i \neq j$, then from the componentwise computation:

$$Y_{km} = \sum_{g \in G} \rho_i(g)_{ka} \overline{\rho_j(g)_{mb}} = 0 = \delta_{ij}.$$

Thus, we have the expression:

$$\sum_{g \in G} \rho_i(g)_{ka} \overline{\rho_j(g)_{mb}} = \frac{|G|}{\dim \rho_i} \delta_{ij} \delta_{km} \delta_{ab}, \quad \forall 1 \leq i, j \leq r.$$

By replacing $a \mapsto l, b \mapsto n$, we proved the wanted result.

2. We need to show orthogonality and normality.

First consider orthogonality. Let us take $k = l$ and $m = n$ in the previous result, then:

$$\sum_{g \in G} \rho_i(g)_{kk} \overline{\rho_j(g)_{mm}} = \frac{|G|}{\dim \rho_i} \delta_{ij} \delta_{km} \delta_{km}.$$

Since those summands are the diagonal elements, and by noticing that $\sum_{k,m} \delta_{km} \delta_{km} = \sum_k 1 = \dim \rho_i$, we sum over all possible choices and give:

$$\sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \sum_{g \in G} \left(\sum_k \rho_i(g)_{kk} \sum_m \overline{\rho_j(g)_{mm}} \right) = \frac{|G|}{\dim \rho_i} \delta_{ij} \sum_{k,m} \delta_{km} \delta_{km} = \frac{|G|}{\dim \rho_i} \delta_{ij} \dim \rho_i = |G| \delta_{ij}.$$

Dividing by $|G|$ gives:

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \delta_{ij}.$$

Next we proof normality. The above immediately gives:

$$\langle \chi_i, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi_i(g)|^2 = 1,$$

which ends the proof of this theorem. \square

3.5 Further properties

Based on the previous inner product structure, we give the following three properties for irreducible characters.

First comes with useful applications of the inner product.

Proposition 15. *Given a character χ of finite group G and its decomposition into irreducible characters $\chi = \sum_i m_i \chi_i$,*

$$1) \langle \chi, \chi_i \rangle = m_i, \quad 2) \langle \chi, \chi \rangle = \sum_i m_i^2, \quad 3) \langle \chi, \chi \rangle = 1 \iff \exists i \quad \chi = \chi_i.$$

Proof. By definition, 1) is given by:

$$\langle \chi, \chi_i \rangle = \frac{1}{|G|} \sum_g \chi(g) \overline{\chi_i(g)} = \frac{1}{|G|} \sum_g \sum_j m_j \chi_j(g) \overline{\chi_i(g)} = \sum_j m_j \frac{1}{|G|} \sum_g \chi_j(g) \overline{\chi_i(g)} = \sum_j m_j \delta_{ij} = m_i.$$

2) follows from 1) by linearity in the second term. ‘ \Rightarrow ’ of 3) follows from 2) since $m_i \in \mathbb{Z}$. ‘ \Leftarrow ’ of 3) follows from 1) by taking the decomposition of itself with $m_i = 1$. \square

We apply the above results to study the following property on symmetric group representations.

Proposition 16. *Given symmetric group S_n and its defining representation V_{def} , its decomposition into trivial subrepresentation W with its complement as:*

$$V_{\text{def}} = W \oplus W'$$

is a complete decomposition. In other words, W' is irreducible.

Proof. We prove it in a combinatoric way. Denote the character of V_{def} as χ_{def} , and its decomposition $\chi_{\text{def}} = m\chi + m'\chi'$. If such decomposition into the direct sum of W and W' is complete, then by 1) from the previous proposition we see that $\langle \chi_{\text{def}}, \chi_{\text{def}} \rangle = 2$. On the other hand, if $\langle \chi_{\text{def}}, \chi_{\text{def}} \rangle = 2$, then by partition of integers $m = m' = 1$. By 3) from the previous proposition, W and W' are irreducible. Thus it suffices to prove the following:

$$\langle \chi_{\text{def}}, \chi_{\text{def}} \rangle = 2.$$

From the definition of inner product,

$$\langle \chi_{\text{def}}, \chi_{\text{def}} \rangle = \frac{1}{|S_n|} \sum_{\pi} \chi_{\text{def}}(\pi) \chi_{\text{def}}(\pi^{-1}).$$

Recall that given any permutation π , its inverse π^{-1} and itself are in the same S_n -conjugacy class since its inverse can be written in the reverse order in the one-lined form, thus giving the same number of transpositions after decomposing them. Together with $|S_n| = n!$, we have the following:

$$\langle \chi_{\text{def}}, \chi_{\text{def}} \rangle = \frac{1}{n!} \sum_{\pi} \chi_{\text{def}}(\pi)^2.$$

Now reconsider $\chi_{\text{def}}(\pi)$. Since defining representation maps a permutation to permutation matrices, its character refers to the number of fixed indices. Thus we have the following:

$$\begin{aligned} \langle \chi_{\text{def}}, \chi_{\text{def}} \rangle &= \frac{1}{n!} \sum_{\pi} |\text{Fix}(\pi)|^2 = \frac{1}{n!} \sum_{\pi} |\text{Fix}(\pi) \times \text{Fix}(\pi)| = \frac{1}{n!} \sum_{\pi} |\{(i, j) : \pi(i) = i, \pi(j) = j\}| \\ &= \frac{1}{n!} \sum_{i,j} |\{\pi : \pi(i) = i, \pi(j) = j\}| = \frac{1}{n!} \left(\sum_i |\{\pi : \pi(i) = i\}| + \sum_{j \neq i} |\{\pi : \pi(i) = i, \pi(j) = j\}| \right) \\ &= \frac{1}{n!} (n(n-1)! + n(n-1)(n-2)!) = 2. \end{aligned}$$

□

The proof of this proposition relies on the fact that each permutation π and its inverse π^{-1} are in the same conjugacy class. This reduces the inner product to its squared norm.

The next property explores tensor product structures.

Proposition 17. *Given a finite group G ,*

1. *If V is a G -module and W is an H -module, then $V \otimes W$ is a $G \times H$ -module.*
2. *Moreover, if V and W are irreducible, their characters denoted χ and ϕ , then $\chi \otimes \phi$ is irreducible.*

Proof. The $G \times H$ -module structure can be realized by:

$$(g, h) \cdot (v \otimes w) = (g \cdot v) \otimes (h \cdot w).$$

Thus group homomorphism is given by group homomorphism of G and H , while linearity is given by bilinearity of the tensor product.

To show that $\chi \otimes \phi$ is irreducible, it suffices by 1) of the first property to show:

$$\langle \chi \otimes \phi, \chi \otimes \phi \rangle = 1.$$

Indeed, by the second equation of Equation 8 and definition,

$$\begin{aligned} \langle \chi \otimes \phi, \chi \otimes \phi \rangle &= \frac{1}{|G \times H|} \sum_{g,h} (\chi \otimes \phi)(g, h) (\chi \otimes \phi)(g^{-1}, h^{-1}) = \frac{1}{|G| \cdot |H|} \sum_{g,h} \chi(g)\phi(h) \cdot \chi(g^{-1})\phi(h^{-1}) \\ &= \frac{1}{|G|} \sum_g \chi(g)\chi(g^{-1}) \frac{1}{|H|} \sum_h \phi(h)\phi(h^{-1}) = \langle \chi, \chi \rangle \cdot \langle \phi, \phi \rangle = 1 \cdot 1 = 1. \end{aligned}$$

□

The last proposition tells that given complete lists of irreducible characters χ_i 's and ϕ_j 's of two groups G and H , the irreducible characters for the group $G \times H$ can be listed as all combinations of the tensor product $\chi_i \otimes \phi_j$'s, as an one to one correspondence. Thus,

$$\mathcal{CF}(G \times H) = \mathcal{CF}(G) \otimes \mathcal{CF}(H)$$

as vector spaces.

4 Induced representations

After studying $\rho : G \rightarrow \mathrm{GL}(V)$, and its subrepresentation regarding variation on V as well as their properties, we now turn to variation on G , which gives representations belonging to a different category. Some references here include [6] and [2]

4.1 Basic concepts

Definition 10 (restricted and induced representation). *Given a finite group G with a G -module V and a subgroup $H \leq G$, the **restricted representation** refers to $\mathrm{Res}_H^G V = V \downarrow_H^G = V \downarrow$ as the vector space V with module structure of actions only by elements from H .*

*On the other hand, given group G , its subgroup H , and an H -module W , the **induced representation** refers to the following as a G -module:*

$$\mathrm{Ind}_H^G W = W \uparrow_H^G = W \uparrow := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, \quad G \curvearrowright \mathrm{Ind}_H^G W : g \cdot (x \otimes w) = (g \cdot x) \otimes w.$$

This means viewing W as left $\mathbb{C}[H]$ -module while $\mathbb{C}[G]$ as a $\mathbb{C}[H]$ -bimodule, this gives a left $\mathbb{C}[G]$ -module.

Notice that restricted and induced representations are both functors, as follows:

$$\mathrm{Res}_H^G : G\text{-Rep} \rightarrow H\text{-Rep}, \quad \mathrm{Ind}_H^G : H\text{-Rep} \rightarrow G\text{-Rep}.$$

However, they are not dual to each other, since restrictions preserve the dimension of representations, while inductions elevate them.

Well defined: Now we prove that these definitions are well-defined. The restricted representation comes as a direct case, since G -actions are closed in vector space refers to H -actions are closed as well.

As for induced representations, we clarify its module bilinear structure as:

$$\begin{aligned} x \cdot h \otimes w &= x \otimes h \cdot w, \\ (x + y) \otimes w &= x \otimes w + y \otimes w, \\ x \otimes (w + v) &= x \otimes w + x \otimes v, \end{aligned}$$

where $x, y \in \mathbb{C}[G], h \in \mathbb{C}[H], w, v \in W$. Thus the action is admissible by checking the followings:

$$\begin{aligned} x \cdot (h_1 h_2) \otimes w &= x \cdot h_1 \otimes h_2 \cdot w = x \otimes (h_1 h_2) \cdot w, \\ g \cdot ((x \cdot h) \otimes w) &= (g \cdot (x \cdot h)) \otimes w = (gx) \otimes (h \cdot w) = g \cdot (x \otimes (h \cdot w)) = g \cdot ((x \cdot h) \otimes w), \\ (g_1 g_2) \cdot (x \otimes w) &= (g_1 g_2 \cdot x) \otimes w = (g_1 \cdot (g_2 \cdot x)) \otimes w = g_1 \cdot ((g_2 \cdot x) \otimes w) = g_1 \cdot (g_2 \cdot (x \otimes w)), \end{aligned}$$

where $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

4.2 Transversals

We discuss more on induced representations since they have more complex structures. We care about its dimension and basis. Given finite group G with its subgroup H , and an H -module W , consider elements of the form:

$$g_i \otimes w_j, \quad \forall g_i \in G, \mathbb{C}\{w_j\} = W.$$

As g_i 's are a basis for $\mathbb{C}[G]$ and w_j as selected to be a basis for W , the above $g_i \otimes w_j$'s clearly span Ind_H^G .

However these elements do not form a basis, since there exists overlapping elements due to the bimodule structure. The ones and the only ones that overlap satisfy the following relation:

$$g_i \cdot h \otimes w_j = g_i \otimes h \cdot w_j = g_i \otimes (\sum c_j w_j),$$

since each element in W can be written as a linear combination of its basis. Thus for each left coset g_iH , we care about the tensor of only one of its elements. Thus,

$$\text{Ind}_H^G = \mathbb{C}\{g_iH \otimes w_j : g_i \in G, w_j \in W\}, \quad \dim \text{Ind}_H^G W = \frac{|G|}{|H|} \dim W.$$

The g_i picked to form left cosets are called **transversal** of H in G .

Lets check that the transversals are well defined. Assume we have two transversals $g_1, g_2 \in g_iH$ for some i . This means $g_1H = g_2H = g_iH$. Thus the following holds:

$$\forall h_1 \in H \quad \exists h_2 \in H \quad g_1h_1 = g_2h_2.$$

Hence, for each pair $h_1, h_2 \in H$,

$$g_1 \otimes (\sum c_{1j}w_j) = g_1 \otimes h_1 \cdot w_j = g_1 \cdot h_1 \otimes w_j = g_2 \cdot h_2 \otimes w_j = g_2 \otimes h_2 \cdot w_j = g_2 \otimes (\sum c_{2j}w_j).$$

This tells that

Given representation $\rho : H \rightarrow \text{GL}(W)$, a basis $g_i \otimes w_j$ for $\text{Ind}_H^G W$, and a permutation of labels σ such that $g_i = g'_{\sigma(i)}h$, define:

$$\phi : \bigoplus_{i,j} (g_i \otimes w_j) \rightarrow \bigoplus_{ij} (g'_i \otimes w_j) \quad g_i \otimes w_j \mapsto g'_{\sigma(i)} \otimes \rho(h_i)w_j.$$

It follows that ϕ is well defined and bijective. Now lets check that ϕ is G -equivariant. On one side, $\forall x \in G$,

$$\phi(x \cdot (g_i \otimes w_j)) = \phi((xg_i) \otimes w_j) = \phi(g_{i'} \otimes h \cdot w_j) = g'_{\sigma(i')} \otimes (h_{i'}h) \cdot w_j,$$

where $xg_i = g_{i'}h$. On the other hand,

$$x \cdot \phi(g_i \otimes w_j) = x \cdot (g'_{\sigma(i)} \otimes h_i \cdot w_j) = (xg'_{\sigma(i)}) \otimes h_i \cdot w_j = g'_{\sigma(i')}h' \otimes h_i \cdot w_j = g'_{\sigma(i')} \otimes h_i \cdot w_j,$$

where $xg'_{\sigma(i)} = g'_{\sigma(i')}h'$. These two results coincide since

$$xg_i = x(g'_{\sigma(i)}h_i) = (xg'_{\sigma(i)})h_i = (g'_{\sigma(i')}h')h_i = g'_{\sigma(i')}(h'h_i),$$

and $g_{i'} = g'_{\sigma(i')}$ as well as $h = h'h_i$. Hence $h_{i'}h = h'h_i$, giving $\phi(x \cdot (g_i \otimes w_j)) = x \cdot \phi(g_i \otimes w_j)$. Thus ϕ is a change of basis intertwiner that changes two transversals, which shows equivalence of two collections of transversals.

4.3 Induced characters:

We now consider the characters for induced representations. Denote the representation as $\rho \uparrow : G \rightarrow \text{GL}(\text{Ind}_H^G V)$, with its **induced character** $\chi \uparrow = \text{tr}(\rho \uparrow)$. As G is completely divided by left cosets of H , there exists some l such that $g \cdot g_i \in g_lH$, and left action by g on g_i refers to right action by h on g_l , i.e.,

$$g \cdot (g_i \otimes w_j) = (g \cdot g_i) \otimes w_j = (g_l \cdot h) \otimes w_j = g_l \otimes (h \cdot w_j).$$

Now consider $g_l = g_i$, that is,

$$g \cdot g_i = g_i \cdot h.$$

This consideration helps focus on only the diagonal blocks of the general linear mappings, which have the chance to contribute to traces. From the above relation, $h = g_i^{-1} \cdot g \cdot g_i \in H$. From our definition, the induced character has value as the sum of characters of $g_i \otimes W$, where g_i is transversal. Thus,

$$\chi \uparrow (g) = \sum_{g_i^{-1} \cdot g \cdot g_i \in H} \chi(g_i^{-1} \cdot g \cdot g_i) = \sum_{g_i \text{ transversal}} \chi(g_i^{-1} \cdot g \cdot g_i) = \frac{1}{|H|} \sum_{x \in G} \chi(x^{-1}gx),$$

where the last '=' holds since all traces are the same in a single left coset.

The last to check is that the choice of our transversal element does not matter the character of g by its conjugate. It is shown as follows:

$$\chi(g_1^{-1}gg_1) = \chi((g_1h)^{-1}g(g_1h)) = \chi(h^{-1}g_1^{-1}gg_1h),$$

where g_1 is transversal and $h \in H$. Since characters are in correspondence with its preceding representation, this matches the result above that different choices of transversals do not effect our induced representation.

Example 15 ($\text{Ind}_{S_2}^{S_3}$). Recall $S_2 \simeq \{(1), (12)\}$. It has two 1 dimensional representations: trivial and sign. As $|S_3| = 6$, this means $\dim \text{Ind}_{S_2}^{S_3} \rho = 3$. Let us choose transversals $g_1 = (1), g_2 = (23), g_3 = (13) \in S_3$, and set $(1), (12), (123)$ as basis elements for S_3 . The induced representation $\text{Ind}_{S_2}^{S_3} \rho$ is constructed by:

$$\text{Ind}_{S_2}^{S_3} \rho(g) = \begin{pmatrix} \rho(g_1^{-1}gg_1) & \rho(g_1^{-1}gg_2) & \rho(g_1^{-1}gg_3) \\ \rho(g_2^{-1}gg_1) & \rho(g_2^{-1}gg_2) & \rho(g_2^{-1}gg_3) \\ \rho(g_3^{-1}gg_1) & \rho(g_3^{-1}gg_2) & \rho(g_3^{-1}gg_3) \end{pmatrix}.$$

First, we pick our S_2 -representation ρ_{triv} as the trivial representation $(1) \mapsto \mathbb{1}, (12) \mapsto 1$. Then, by evaluating at basis elements,

$$\rho_{triv} \uparrow (1) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \rho_{triv} \uparrow (12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_{triv} \uparrow (123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, they are of trace $(3, 1, 0)$.

We pick our S_2 -representation ρ_{sign} as the sign representation $(1) \mapsto \mathbb{1}, (12) \mapsto -\mathbb{1}$. Then, by evaluating at basis elements,

$$\rho_{sign} \uparrow (1) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \rho_{sign} \uparrow (12) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \rho_{sign} \uparrow (123) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, they are of trace $(3, -1, 0)$.

Hence the character table for $\rho \uparrow$ is:

	$\text{Ind}_{S_2}^{S_3}$	(1)	(12)	(123)
$\chi_{triv} \uparrow$	$\mathbb{3}$	1	0	
$\chi_{sign} \uparrow$	$\mathbb{3}$	-1	0	

From example 14, we see that:

$$\chi_{triv} \uparrow = \chi_{def} = \chi_{triv} + \chi_3, \quad \chi_{sign} \uparrow = \chi_{sign} + \chi_3.$$

4.4 Frobenius reciprocity

Recall that induced and restricted representations are not dual to each other. We then ask if something weaker can be deduced between them. That is, we want to know if they are connected computationally instead of structurally.

Theorem 4 (Frobenius reciprocity law). Given finite groups $H \leq G$, and given ϕ an H -character and χ a G -character,

$$\langle \phi \uparrow, \chi \rangle_G = \langle \phi, \chi \downarrow \rangle_H.$$

Proof. We prove by definition. The left hand side of the above equation becomes:

$$\langle \phi \uparrow, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \phi \uparrow(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H|} \sum_{x \in G} \phi(x^{-1}gx) \right) \chi(g^{-1}) = \frac{1}{|G| \cdot |H|} \sum_{g, x \in G} \phi(x^{-1}gx) \chi(g^{-1}).$$

We make the substitution $y = x^{-1}gx$. Since conjugation $g \rightarrow x^{-1}gx$ is bijective, and χ a class function, we have:

$$\langle \phi \uparrow, \chi \rangle_G = \frac{1}{|G| \cdot |H|} \sum_{y \in G} \sum_{x \in G} \phi(y) \chi(xy^{-1}x^{-1}) = \frac{1}{|G| \cdot |H|} \sum_{y \in G} \sum_{x \in G} \phi(y) \chi(y^{-1}) = \frac{1}{|H|} \sum_{y \in G} \phi(y) \chi(y^{-1}).$$

Since χ is defined as 0 in $G \setminus H$,

$$\langle \phi \uparrow, \chi \rangle_G = \frac{1}{|H|} \sum_{y \in G} \phi(y) \chi(y^{-1}) = \frac{1}{|H|} \sum_{y \in H} \phi(y) \chi(y^{-1}) = \langle \phi, \chi \downarrow \rangle_H.$$

□

A more general version is the Frobenius reciprocity law of module form as follows:

Corollary 5 (Strong Frobenius reciprocity law). *Given G, H, W, V defined as above, then:*

$$\text{Hom}_G(\text{Ind}_H^G W, V) \simeq \text{Hom}_H(W, \text{Res}_H^G V),$$

where W an H -module and V a G -module.

Proof. Recall that $\text{Ind}_H^G W = k[G] \otimes_{k[H]} W$, with G acting by left multiplication on the $k[G]$ -factor as:

$$g \cdot (a \otimes w) = (ga) \otimes w \quad (g, a \in G, w \in W).$$

Let us define:

$$\begin{aligned} \Phi : \text{Hom}_G(k[G] \otimes_{k[H]} W, V) &\rightarrow \text{Hom}_H(W, \text{Res}_H^G V), \\ F &\mapsto (w \mapsto F(1 \otimes w)). \end{aligned}$$

We check $\Phi(F)$ is H -equivariant: for $h \in H$,

$$\begin{aligned} \Phi(F)(h \cdot w) &= F(1 \otimes h \cdot w) = F((1h) \otimes w) = F(h \cdot (1 \otimes w)) \\ &= h \cdot F(1 \otimes w) = h \cdot \Phi(F)(w). \end{aligned}$$

Conversely, define:

$$\begin{aligned} \Psi : \text{Hom}_H(W, \text{Res}_H^G V) &\rightarrow \text{Hom}_G(k[G] \otimes_{k[H]} W, V), \\ \varphi &\mapsto ((g \otimes w) \mapsto g \cdot \varphi(w)). \end{aligned}$$

Well-definedness: In the tensor product we identify $(gh) \otimes w = g \otimes (h \cdot w)$. Then,

$$\Psi(\varphi)((gh) \otimes w) = (gh) \cdot \varphi(w) = g \cdot (h \cdot \varphi(w)) = g \cdot \varphi(h \cdot w) = \Psi(\varphi)(g \otimes h \cdot w),$$

where we used H -equivariance of φ . Hence $\Psi(\varphi)$ is well-defined.

G-linearity: For $x \in G$,

$$\Psi(\varphi)(x \cdot (g \otimes w)) = \Psi(\varphi)((xg) \otimes w) = (xg) \cdot \varphi(w) = x \cdot (g \cdot \varphi(w)) = x \cdot \Psi(\varphi)(g \otimes w).$$

Thus $\Psi(\varphi)$ is G -linear.

Mutually inverse: Finally, Φ and Ψ are inverses: for F ,

$$(\Psi \circ \Phi)(F)(g \otimes w) = g \cdot \Phi(F)(w) = g \cdot F(1 \otimes w) = F(g \cdot (1 \otimes w)) = F(g \otimes w),$$

so $\Psi(\Phi(F)) = F$. For φ ,

$$(\Phi \circ \Psi)(\varphi)(w) = \Psi(\varphi)(1 \otimes w) = 1 \cdot \varphi(w) = \varphi(w).$$

Hence Φ and Ψ are mutually inverse. □

Example 16 ($\text{Ind}_{S_3}^{S_4}$). Recall the character table of S_3 as:

S_3	1	(12)	(123)
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_3	2	0	-1

We now want to compute $\chi_3 \uparrow$. The traditional way is through evaluating at each of the four transversals the matrix:

$$\text{Ind}_{S_3}^{S_4} \rho(g) = (\rho(g_m^{-1} g g_n))_{m,n},$$

where $1 \leq m, n \leq 4$, and ρ the S_3 -representation with character χ_3 . Then calculate its trace. Now with the help of Frobenius reciprocity law, we can give the same result with computing $\chi \downarrow$, then using the formula:

$$\langle \chi_3 \uparrow, \phi \rangle_{S_4} = \langle \chi_3, \phi \downarrow \rangle_{S_3},$$

where ϕ runs through S_4 -characters.

Since reduced characters have the same value on the elements in the subgroup S_3 , we compute the character table of S_4 with elements in S_3 .

Recall that characters are class functions, so we need to understand conjugacy classes of S_4 . There are five of them in total:

1. identity, only one element (1), denote its conjugacy class as (1^4) ,
2. one two cycle, there are 6 elements, denote their conjugacy class as $(2, 1^2)$,
3. composition of two disjoint two cycles, there are 3 elements, denote their conjugacy class as (2^2) ,
4. one three cycle, there are 8 elements, denote their conjugacy class as $(3, 1)$,
5. one four cycle, there are 6 elements, denote their conjugacy class as (4) .

We check that there are indeed 24 elements in total.

Recall that $|S_4| = \sum_{i=1}^5 \dim(U_i)$, where U_i 's are irreducible S_4 -representations. Thus it gives the only possibility of characters as:

$$24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2.$$

We denote each of the corresponding character in the same notation, with the trivial representation corresponding to (1^4) and the sign representation corresponding to (4) . Then the character table of S_4 is:

S_4	(1^4)	$(2, 1^2)$	$(2, 2)$	$(3, 1)$	(4)
ϕ_{triv}	1	1	1	1	1
ϕ_{sign}	1	-1	1	-1	1
$(3, 1)$	3	1	-1	0	-1
$(2, 2)$	2	0	2	-1	0
$(2, 1^2)$	3	-1	-1	0	1

We only need classes $(1^4), (2, 1^2), (3, 1)$, which gives the table:

$\text{Res}_{S_3}^{S_4}$	(1^4)	$(2, 1^2)$	$(3, 1)$
ϕ_{triv}	1	1	1
ϕ_{sign}	1	-1	1
$(3, 1)$	3	1	0
$(2, 2)$	2	0	-1
$(2, 1^2)$	3	-1	0

Thus by Frobenius reciprocity law,

$$\begin{aligned}\langle \chi_{triv} \uparrow, \phi \rangle &= \langle \chi_{triv}, \phi \downarrow \rangle = \begin{cases} 1, & \phi = \phi_{triv}, (3, 1), \\ 0, & \text{otherwise,} \end{cases} \\ \langle \chi_{sign} \uparrow, \phi \rangle &= \langle \chi_{sign}, \phi \downarrow \rangle = \begin{cases} 1, & \phi = \phi_{sign}, (2, 1^2), \\ 0, & \text{otherwise,} \end{cases} \\ \langle \chi_3 \uparrow, \phi \rangle &= \langle \chi_3, \phi \downarrow \rangle = \begin{cases} 1, & \phi = (3, 1), (2, 2), (2, 1^2), \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Therefore we have the following structures of induced characters:

$$\text{Ind}_{S_3}^{S_4} \chi_{triv} \cong \phi_{triv} \oplus (3, 1), \quad \text{Ind}_{S_3}^{S_4} \chi_{sign} \cong \phi_{sign} \oplus (2, 1^2), \quad \text{Ind}_{S_3}^{S_4} \chi_3 \cong (3, 1) \oplus (2, 2) \oplus (2, 1^2).$$

4.5 More properties

Proposition 18. Given finite groups $H \leq K \leq G$, the functors can be composed as follows:

$$\text{Ind}_H^G = \text{Ind}_K^G \circ \text{Ind}_H^K.$$

Proof. Recall $\text{Ind}_L^M W = \mathbb{C}[M] \otimes_{\mathbb{C}[L]} W$, where $L \leq M$ are finite groups and W a L -module. For V an $\mathbb{C}[H]$ -module,

$$\text{Ind}_K^G(\text{Ind}_H^K V) = \mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} V).$$

Define the following linear map:

$$\Phi : \mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} V) \rightarrow \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, \quad g \otimes (k \otimes v) \mapsto (gk) \otimes v,$$

and

$$\Psi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \rightarrow \mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} V), \quad g \otimes v \mapsto g \otimes (1 \otimes v).$$

Well-definedness:

In $\mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\cdot)$ we have

$$(gk') \otimes (k \otimes v) = g \otimes (k'k \otimes v), \quad k' \in K.$$

Applying Φ ,

$$\Phi((gk') \otimes (k \otimes v)) = (gk'k) \otimes v = \Phi(g \otimes (k'k \otimes v)).$$

So Φ respects the $\mathbb{C}[K]$ -balancing. Inside $\mathbb{C}[K] \otimes_{\mathbb{C}[H]} V$, we have

$$k \otimes (hv) = (kh) \otimes v, \quad h \in H.$$

Applying Φ ,

$$\Phi(g \otimes (k \otimes (hv))) = (gk) \otimes hv,$$

$$\Phi(g \otimes ((kh) \otimes v)) = (gkh) \otimes v.$$

These are equal in $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ since $(gkh) \otimes v = (gk) \otimes hv$. Thus Φ is well-defined. Similarly, for Ψ : in $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$,

$$(gh) \otimes v = g \otimes (hv), \quad h \in H.$$

Applying Ψ ,

$$\Psi((gh) \otimes v) = gh \otimes (1 \otimes v),$$

$$\Psi(g \otimes (hv)) = g \otimes (1 \otimes hv).$$

But in $\mathbb{C}[K] \otimes_{\mathbb{C}[H]} V$, $1 \otimes (hv) = h \otimes v$. So

$$\Psi(g \otimes (hv)) = g \otimes (h \otimes v) = gh \otimes (1 \otimes v),$$

which matches. Hence Ψ is well-defined.

G-equivariance: Both sides are G -modules, with G acting by left multiplication on the first $\mathbb{C}[G]$ factor. For Φ :

$$\Phi(g_0 \cdot (g \otimes (k \otimes v))) = \Phi((g_0 g) \otimes (k \otimes v)) = (g_0 g k) \otimes v,$$

while

$$g_0 \cdot \Phi(g \otimes (k \otimes v)) = g_0 \cdot ((gk) \otimes v) = (g_0 g k) \otimes v.$$

So Φ is G -equivariant. The same check applies to Ψ .

Mutual inverses. For $g \in G$, $v \in V$,

$$(\Phi \circ \Psi)(g \otimes v) = \Phi(g \otimes (1 \otimes v)) = g \otimes v,$$

so $\Phi \Psi = \mathbb{1}$. For $g \in G$, $k \in K$, $v \in V$,

$$(\Psi \circ \Phi)(g \otimes (k \otimes v)) = \Psi((gk) \otimes v) = (gk) \otimes (1 \otimes v).$$

But in the balanced tensor product, $(gk) \otimes (1 \otimes v) = g \otimes (k \otimes v)$. Thus $\Psi \Phi = \mathbb{1}$.

One checks that Φ and Ψ are well-defined, G -equivariant, and mutual inverses on pure tensors, hence define inverse isomorphisms. This yields our desired result. \square

We realize that this proof is similar with the proof for the strong Frobenius reciprocity law. Indeed, they both require the defined map from one object to be related with G , as well as bijective. The difference lies at they concern different modules: one with Hom_G and the other with $\mathbb{C}[G] \otimes$, and they correspond to different compatibility with G : one as G -linearity and the other as G -equivariance.

5 Representations of some finite groups

In this section, we explore applications to representations of some finite groups, and finally leading to S_n -representations.

5.1 C_n

Let us first study the cyclic groups. Since all elements are powers of any generator, say x ,

$$ghg^{-1} = x^k x^l x^{-k} = x^l = h, \quad g = x^k, h = x^l \in C_n.$$

Thus for any element $h \in C_n$, the conjugacy class containing h must contain only h . Thus C_n has $|C_n| = n$ conjugacy classes, each with one element.

This conclusion can be generalized. In fact,

Proposition 19. *Every irreducible representation of abelian group is of dimension 1.*

Proof. Conjugation in abelian group is trivial. \square

This also implies that abelian groups has no nontrivial normal subgroups, and the number of its conjugacy classes equal to its cardinality.

By density theorem its corollary 4, we see that C_n must have n irreducible representations. By degree theorem 10,

$$n = |C_n| = \sum_{i=1}^n \dim^2 U_i \quad \Rightarrow \quad \dim U_i \equiv 1.$$

The above also shows that its character table is $n \times n$.

Now let us compute all these one dimensional irreducible representations ρ . Notice that for every element $g = x^k$, where $(x) = C_n$, we have $g^n = x^{kn} = 1^k = 1$. Thus,

$$\rho(g)^n = \rho(x^k)^n = \rho(x^n)^k = \rho(1)^k = 1^k = 1.$$

This means that every irreducible representation correspond to an n -th root of unity. Moreover, for $C_n = \langle x \rangle$, it suffices to know $\rho(x)$, which must generate all n -th roots of unity. Its character table can be written as follows:

C_n	1	x	x^2	\dots	x^{n-1}
$\chi_0 = 1$	1	1	1	\dots	1
χ_1	1	$e^{2\pi i/n}$	$e^{4\pi i/n}$	\dots	$e^{(2n-2)\pi i/n}$
χ_2	1	$e^{4\pi i/n}$	$e^{8\pi i/n}$	\dots	$e^{2(2n-2)\pi i/n}$
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots
χ_{n-1}	1	$e^{2(n-1)\pi i/n}$	$e^{4(n-1)\pi i/n}$	\dots	$e^{(2n-2)(n-1)\pi i/n}$

5.2 D_n

Irreducible representations Let $D_n = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{n-1} \rangle$. Then D_n consist $2n$ elements:

$$\begin{aligned} & 1, r, \dots, r^{n-1}, \\ & s, sr, \dots, sr^{n-1}. \end{aligned}$$

We could understand them as rotations and reflections. That is, for a regular n -gon, $1, r, \dots, r^{n-1}$ are rotations passing one vertex to another, of $2\pi i/n$ as angle, counterclockwise, while s, sr, \dots, sr^{n-1} are reflections, with the symmetric lines each perpendicular with an edge.

Let us now consider the 1 dimensional irreducible representations. Assume:

$$\rho : s \mapsto x, r \mapsto y, \quad x, y \in \mathbb{C}^\times.$$

Then we could deduce the following relations:

$$x^2 = \rho(s)^2 = \rho(s^2) = \rho(1) = 1,$$

which gives $x = \pm 1$. Similarly,

$$y^n = \rho(r)^n = \rho(r^n) = \rho(1) = 1,$$

which gives $y = e^{2\pi i/n}$. By their given relation $srs = r^{n-1}$, one has the following:

$$xyx = \rho(srs) = \rho(r^{n-1}) = y^{n-1}.$$

However, $xyx = x^2y = y$, so we have $y = y^{n-1}$, i.e., $y^2 = y^n = 1$. Thus $y = \pm 1$ as well.

The structure of the n -gon has more to tell. We embed it into the xy -plane, with the geometric center of the n -gon at the origin. If n were even, then a reflection along the y -axis would remain the graph unchanged. However, for every odd n , the nontrivial reflection along the y -axis would give the n -gon flipped along the y -axis, which means that the D_n group has non automorphic actions. Thus, even n 's permit both $y = 1$ and $y = \pm 1$, while odd n 's permit only $y = 1$.

To study the D_n -representations of other dimensions, we need the number of conjugacy classes of D_n . Define rotation and reflection by their angle variation as:

$$\text{rot}_\theta(x) = x + \theta, \quad \text{ref}_\psi(x) = 2\psi - x.$$

Then we have the following relations:

$$\begin{aligned} \text{ref}_\psi \circ \text{rot}_\theta \circ \text{ref}_{-\psi}(x) &= \text{ref}_\psi \circ \text{rot}_\theta \circ \text{ref}_\psi(x) = 2\psi - ((2\psi - x) + \theta) = 2\psi - 2\psi + x - \theta = \text{rot}_{-\theta}(x), \\ \text{rot}_\theta \circ \text{ref}_\psi \circ \text{rot}_{-\theta}(x) &= 2\psi - (x - \theta) + \theta = 2\psi + 2\theta - x = \text{ref}_{\psi+\theta}(x). \end{aligned}$$

The first identity tells us that conjugating a rotation by a reflection inverts the rotation angle, i.e. $\text{rot}_\theta \mapsto \text{rot}_{-\theta}$. The second shows that conjugating a reflection by a rotation simply shifts the axis of reflection by that rotation angle. From these two facts we can now deduce how many distinct conjugacy classes exist: all non-trivial rotations are paired with their inverses, while the reflections fall into either one or two conjugacy classes depending on the parity of n .

One more remark lands on the rotation and reflection. They can both be presented by matrices as:

$$\text{rot}_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \text{ref}_\psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, by degree theorem 10,

$$r = \begin{cases} 1 + \frac{n-2}{2} + 1 + 2 = \frac{n+6}{2}, & n \text{ even}, \\ 1 + \frac{n-1}{2} + 1 = \frac{n+3}{2}, & n \text{ odd}. \end{cases} \quad 2n = |D_n| = \sum_i^r \dim^2 U_i = \begin{cases} 4 \cdot 1^2 + \frac{n-2}{2} \cdot 2^2, & n \text{ even}, \\ 2 \cdot 1^2 + \frac{n-1}{2} \cdot 2^2, & n \text{ odd}. \end{cases}$$

To summarize up,

- If n is even: there are four 1-dimensional irreps ($r \mapsto \pm 1, s \mapsto \pm 1$), and $(n-2)/2$ two-dimensional irreps ρ_k defined as above.
- If n is odd: there are two 1-dimensional irreducible representations (trivial, and $s \mapsto -1$), and $(n-1)/2$ two-dimensional irreducible representations ρ_k ($1 \leq k \leq (n-1)/2$) given by

$$\rho_k(r) = \begin{pmatrix} \cos(2\pi k/n) & -\sin(2\pi k/n) \\ \sin(2\pi k/n) & \cos(2\pi k/n) \end{pmatrix}, \quad \rho_k(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Character tables for small n 's For now, we give results for $n = 2, 3, 4$.

$$D_2 = S_2, \quad D_3 = S_3.$$

Taking $n = 4$ into the summary above, as it is even, the degree theorem implies:

$$8 = |D_4| = 4 \cdot 1^2 + 1 \cdot 2^2.$$

Then have the following character table for D_4 :

D_4	1	r^2	$\{r, r^3\}$	$\{s, r^2s\}$	$\{rs, r^3s\}$
$\chi_{++} (r \mapsto 1, s \mapsto 1)$	1	1	1	1	1
$\chi_{-+} (r \mapsto -1, s \mapsto 1)$	1	1	-1	1	-1
$\chi_{+-} (r \mapsto 1, s \mapsto -1)$	1	1	1	-1	-1
$\chi_{--} (r \mapsto -1, s \mapsto -1)$	1	1	-1	-1	1
$\chi (r^m \mapsto 2 \cos(m\pi/2), s \mapsto 0)$	2	-2	0	0	0

5.3 V_4

Define the Klein 4-group as:

$$V_4 = \{1, a, b, ab : a^2 = b^2 = 1, ab = ba\}.$$

Since V_4 is abelian, the proposition 19 tells that all its irreducible representations are of dimension 1. From its defining relations, we have the following relations for its representations:

$$\begin{aligned} \rho(1) &= 1, \\ \rho(a)^2 &= 1 \implies \rho(a) = \pm 1, \\ \rho(b)^2 &= 1 \implies \rho(b) = \pm 1, \\ \rho(ab) &= \rho(a)\rho(b). \end{aligned}$$

There are 4 possible cases as above. This matches with the degree theorem 10 as:

$$1^2 + 1^2 + 1^2 + 1^2 = 4 = |V_4|.$$

Thus its character table is as follows:

	D_4	1	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	-1
χ_3	1	-1	1	-1	-1
χ_4	1	-1	-1	1	1

5.4 H_8

The Hamilton quaternion group H_8 is defined as:

$$H_8 = \langle 1, i, j, k \rangle / \begin{pmatrix} i^2 = j^2 = k^2 = -1, \\ ij = k, jk = i, ki = j, \\ ji = -1, kj = -i, ik = -j. \end{pmatrix} \simeq \{\pm 1, \pm i, \pm j, \pm k\}.$$

Notice $\forall x \in H_8 \quad x^{-1} = -x$.

We study all its conjugacy classes as follows.

$$\begin{aligned} x^{-1}1x &= 1, x^{-1}(-1)x = -1, \quad \forall x \in H_8, \\ i^{-1}ii &= i, i^{-1}(-i)i = -i \\ j^{-1}ij &= -i, j^{-1}(-i)j = i, \\ k^{-1}ik &= -i, k^{-1}(-i)k = i. \end{aligned}$$

The relations with negative conjugation gives the same results hence are put away. The relations with respect to j and k are similar since i, j, k are equivalently defined. Thus, there are 5 conjugacy classes:

$$\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}.$$

The degree theorem 10 then tells that:

$$8 = |H_8| = \sum_{i=1}^5 \dim^2 U_i,$$

where U_i 's are irreducible H_8 -representations. The only combination is that there are 4 irreducible representations of $\dim = 1$ and 1 irreducible representation of $\dim = 2$. As for the representation $\rho : H_8 \rightarrow \text{GL}(V)$, they also need to satisfy the relations:

$$\begin{aligned} \rho(-1) &= \pm \rho(1) = \pm \mathbb{1}_V, \\ \rho(i)\rho(-i) &= \rho(j)\rho(-j) = \rho(k)\rho(-k) = \rho(1) = \mathbb{1}_V, \\ \rho(k) &= \rho(i)\rho(j), \rho(j)\rho(i) = \rho(-k), \\ \rho(i) &= \rho(j)\rho(k), \rho(k)\rho(j) = \rho(-i), \\ \rho(j) &= \rho(k)\rho(i), \rho(i)\rho(k) = \rho(-j), \\ \chi(k) &= \chi(i)\chi(j) = \chi(j)\chi(i) = \chi(-k), \\ \chi(i) &= \chi(j)\chi(k) = \chi(k)\chi(j) = \chi(-i), \\ \chi(j) &= \chi(k)\chi(i) = \chi(i)\chi(k) = \chi(-j). \end{aligned}$$

Under the above constraints, we could list the possible choice of irreducible H_8 -representations as follows.

one dimensional. There are four 1-dimensional representations, where $\rho_n = \chi_n$:

$$\begin{aligned} \chi_1(g) &= 1 \quad \forall g \in H_8 \\ \chi_2(g) &= \begin{cases} 1 & \text{if } g \in \{1, -1, \pm i\} \\ -1 & \text{if } g \in \{\pm j, \pm k\} \end{cases} \\ \chi_3(g) &= \begin{cases} 1 & \text{if } g \in \{1, -1, \pm j\} \\ -1 & \text{if } g \in \{\pm i, \pm k\} \end{cases} \\ \chi_4(g) &= \begin{cases} 1 & \text{if } g \in \{1, -1, \pm k\} \\ -1 & \text{if } g \in \{\pm i, \pm j\} \end{cases} \end{aligned}$$

two dimensional. There is one 2-dimensional irreducible representation:

$$\rho_5(g) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & g = 1, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & g = -1, \\ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & g = i, \quad \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} & g = -i, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & g = j, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & g = -j, \\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & g = k, \quad \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & g = -k. \end{cases}$$

character table. The character table of H_8 is:

	\mathcal{C}_1 $\{1\}$	\mathcal{C}_2 $\{-1\}$	\mathcal{C}_3 $\{\pm i\}$	\mathcal{C}_4 $\{\pm j\}$	\mathcal{C}_5 $\{\pm k\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

verify that χ_1, \dots, χ_5 are irreducible. Recall the inner product formula as: Equation 9. We show they are irreducible by proving they form an orthonormal basis in $\mathcal{CF}(H_8)$.

The rows are orthogonal:

$$\begin{aligned} \langle \chi_1, \chi_2 \rangle &= \frac{1}{8}(1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot (-1) + 2 \cdot 1 \cdot (-1)) = 0, \\ \langle \chi_5, \chi_5 \rangle &= \frac{1}{8}(1 \cdot 2 \cdot 2 + 1 \cdot (-2) \cdot (-2) + 2 \cdot 0 \cdot 0 + 2 \cdot 0 \cdot 0 + 2 \cdot 0 \cdot 0) = \frac{1}{8}(4 + 4) = 1. \end{aligned}$$

The columns are orthogonal:

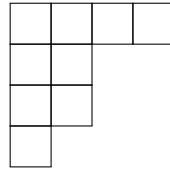
$$\begin{aligned} \langle \mathcal{C}_1, \mathcal{C}_2 \rangle &= 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 2 \cdot (-2) = 1 + 1 + 1 + 1 - 4 = 0, \\ \langle \mathcal{C}_3, \mathcal{C}_4 \rangle &= 1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 1 + 1 \cdot (-1) + 0 \cdot 0 = 1 - 1 + 1 - 1 = 0. \end{aligned}$$

6 Representations of symmetric groups: Young tabloids

In this section, we introduce combinatoric tools which helps in the study of symmetric group representations. Some references here includes [5] and [3].

6.1 Young diagrams

We start with the Young diagrams. A **Young diagram** is a collection of boxes in a plane of the form:



where the boxes are aligned to the left, and their number decreases from upper rows to bottom ones.

Young diagrams can be used to interpret the partition of integers. Consider $n \in \mathbb{N}$, and its partition $n = \lambda_1 + \dots + \lambda_k$, where $\lambda_1 \geq \dots \geq \lambda_k$, we denote such a partition as the following:

$$\lambda = (\lambda_1, \dots, \lambda_k) \vdash n.$$

Then, the corresponding Young diagram can be drawn as a collection of n boxes, with λ_i boxes at the i 'th row, $i = 1, \dots, k$.

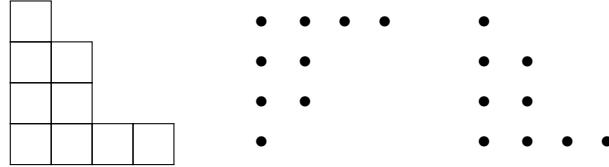
Example 17. The Young diagram drawn above corresponds to $\lambda = (4, 2, 2, 1) \vdash 9$.

As such a correspondence is bijective, the Young diagram can also be defined by a partition as the following.

Definition 11 (Young diagram as partition). *Given a natural number $n \in \mathbb{N}$, an n -Young diagram is a k -tuple $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$, satisfying:*

1. $0 \leq k \leq n$,
2. $\lambda_i \geq \lambda_{i+1} \quad \forall i = 1, \dots, k-1$,
3. $\sum_{i=1}^k \lambda_i = n$.

From this definition, we notice that the formal presentation of Young diagrams as boxes does not matter that much. Though we still take the convention as drawn in Example 17, there are other conventions of presenting Young diagrams which could be used elsewhere, such as reversing the second condition to $n_i \leq n_{i+1}$, or presenting by dots. For instance, Example 17 can be presented as the followings:



Now we relate this with the preceding contents. As we can see, the number of n -Young diagrams equals to the number of partitions of n . Meanwhile, given a symmetric group S_n , recall that every element in S_n can be written as a combination of cycles with sum of the length as n . We also have the result that each conjugacy class of S_n contains all elements of the same cycle type, that is, combination disjoint cycles of the same lengths. This amounts to find the number of partition of n . Thus, the number of conjugacy classes of S_n equals to the number of partitions of n , which is also the number of n -Young diagrams. On the other hand, Corollary 4 tells that this also equals to the number of irreducible representations of S_n . Therefore, there is the following equality:

$$\#\{\text{n-Young diagrams}\} = \#\{\text{irreducible S_n-representations}\}.$$

6.2 Young tableaux

A close relative of the Young diagrams is the **Young tableaux**. They are a complete filling of numbers for the Young diagrams. Take the boxed form as in Example 17, a filling from 1 to 9 could be something like the following:

2	1	8	4
5	6		
3	9		
7			

where each number is placed in exactly one box. In other words, if we view any n -Young diagram as labeled with 1 to n from the left top to the right bottom, we could define a Young tableau as follows.

Definition 12 (Young tableau). *Given a number $n \in \mathbb{N}$ and a n -Young diagram $\lambda \in \mathbb{N}^k$ for some k , a λ -Young tableau is a bijection:*

$$t^\lambda : \{1, \dots, n\} \xrightarrow{\sim} \{1, \dots, n\}.$$

This actually means that t^λ is a permutation of n numbers, hence lives in S_n . In fact the collection of λ -Young tableaux is exactly S_n , which means they form a group that we are familiar of.

We could also consider S_n acting on the set of all λ -Young tableaux, which is simply:

$$S_n \curvearrowright S_n : \quad \tau \cdot \sigma = \tau\sigma.$$

Proposition 20. *The action $S_n \curvearrowright S_n$ is well-defined and transitive.*

Proof. Well-definedness is natural since S_n is a group. Transitivity requires that the action has a single orbit, which means that for any element on the orbit, which is S_n itself, can be transited by any element in the set via action by some element in the acting group. Thus, we need to show that $\forall \sigma_1, \sigma_2 \in S_n \quad \exists \tau \in S_n \quad \tau \cdot \sigma_1 = \sigma_2$. Thus by defining $\tau := \sigma_1^{-1}\sigma_2$ gives the proof. \square

Example 18 (following 17). *If we take $\lambda = (4, 2, 2, 1)$, the λ -Young tableau drawn above is then:*

$$t^\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 8 & 4 & 5 & 6 & 3 & 9 & 7 \end{pmatrix} = (12)(3798) = (12)(38)(39)(37).$$

If we ask to act on t^λ by an element $\tau \in S_n$, then the result would be:

$$\tau \cdot t^\lambda = \tau(12)(3798) = (\tau(1)\tau(2))(\tau(3)\tau(7)\tau(9)\tau(8)), \quad \text{or}$$

$\tau(2)$	$\tau(1)$	$\tau(8)$	$\tau(4)$
$\tau(5)$	$\tau(6)$		
$\tau(3)$	$\tau(9)$		
$\tau(7)$			

For example let $\tau = t^\lambda$. Then,

$$\tau \cdot t^\lambda = (12)(3798)(12)(3798) = (39)(78).$$

This corresponds in the tableaux form as:

$$\tau \cdot t^\lambda = \begin{array}{|c|c|c|c|} \hline 2 & 1 & 8 & 4 \\ \hline 5 & 6 & & \\ \hline 3 & 9 & & \\ \hline 7 & & & \\ \hline \end{array} \quad . \quad \begin{array}{|c|c|c|c|} \hline 2 & 1 & 8 & 4 \\ \hline 5 & 6 & & \\ \hline 3 & 9 & & \\ \hline 7 & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 9 & 4 \\ \hline 5 & 6 & & \\ \hline 8 & 7 & & \\ \hline 3 & & & \\ \hline \end{array}.$$

6.3 Young tabloids

Given a n -Young diagram, we could classify the $n!$ λ -Young tableaux with into some classes. Define a relation \sim between any two λ -Young tableaux, such that if t_1 and t_2 satisfies \sim if and only if the each row has the same collection of numbers, when the λ -Young tableaux are presented by the boxed form. For example, let

$$t_1 = \begin{array}{|c|c|c|c|} \hline 2 & 1 & 8 & 4 \\ \hline 5 & 6 & & \\ \hline 3 & 9 & & \\ \hline 7 & & & \\ \hline \end{array}, \quad t_2 = \begin{array}{|c|c|c|c|} \hline 2 & 8 & 4 & 1 \\ \hline 5 & 6 & & \\ \hline 9 & 3 & & \\ \hline 7 & & & \\ \hline \end{array},$$

then $t_1 \sim t_2$.

If we view λ -Young tableaux as permutations in S_n with $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, then the above relation \sim can be defined as follows.

Definition 13. Given $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, two λ -Young tableaux t_1, t_2 are said to satisfy $t_1 \sim t_2$ if the following holds:

$$\{t_1(s) : s = \lambda_m + 1, \dots, \lambda_m + \lambda_{m+1}\} = \{t_2(s) : s = \lambda_m + 1, \dots, \lambda_m + \lambda_{m+1}\},$$

which holds $\forall m = 0, \dots, k-1$, and set $\lambda_0 := 0$.

Proposition 21. Such a relation \sim is an equivalence.

Proof. Given a partition $\lambda \vdash n$ into k integers, the k equations as above are equivalence. \square

Thus any set of λ -Young tableaux can be separated into equivalent classes. By the above relation, the number of equivalent classes will be:

$$|S_n / \sim| = \binom{n}{\lambda_1} \binom{n - \lambda_1}{\lambda_2} \binom{n - \lambda_1 - \lambda_2}{\lambda_3} \cdots \binom{\lambda_k}{\lambda_k} = \prod_{m=1}^k \binom{n - \sum_{j=0}^{m-1} \lambda_j}{\lambda_m} = \frac{n!}{\lambda_1! \cdots \lambda_k!}. \quad (10)$$

Further, we could pick a special element from each class, named the standard λ -Young tabloids, or simply the λ -tabloids.

Definition 14 (Young tabloid). Given an n -Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$, the **standard λ -tabloids** $\{t\}$'s are the equivalent set of λ -Young tableaux, where t is a special element in this λ -Young tableau such that the following holds for t :

$$i < j, \quad \forall \lambda_m + 1 \leq i, j \leq \lambda_m + \lambda_{m+1}, \quad \forall m = 0 \cdots k-1.$$

The above special element must be unique for each equivalent class, due to that the ordering is unique in finitely many collections of integers.

Example 19 (following 18). Recall the λ -Young tableau $t^\lambda = (12)(3798)$ mentioned before as the following left tableau. The corresponding unique λ -Young tabloid t can be interpreted as the middle, and denote equivalent class λ -Young tabloid $\{t\}$ containing t as the right:

2	1	8	4			
5	6					
3	9					
7						

 \sim

1	2	4	8			
5	6					
3	9					
7						

 \in

1	2	4	8			
5	6					
3	9					
7						

We could likewise consider S_n acting on the collection of all λ -Young tabloids. Define:

$$S_n \curvearrowright \{\text{Young tabloids}\} : \quad \pi \cdot \{t\} = \{\pi t\}.$$

Proposition 22. The action defined above is well-defined and transitive.

Proof. As for well-definedness, we need to check $\forall \pi \in S_n$ and $\forall \{t\}$ a λ -Young tabloid, $\pi \cdot \{t\}$ is still a λ -Young tabloid. This can be checked as the followings:

$$\pi \cdot \{t\} = \pi \cdot \{t'\} \implies \{\pi t\} = \{\pi t'\} \implies \pi t' \in \{\pi t\} \implies \pi t' \sim \pi t \implies t' \sim t \implies \{t'\} = \{t\}.$$

To check it is transitive, one needs to show:

$$\forall \{t\}, \{t'\} \quad \exists \pi \in S_n \quad \pi \cdot \{t\} = \{t'\}.$$

To prove this, let $\pi = t^{-1}t'$, and it holds the followings as we want:

$$\pi \cdot \{t\} = \{\pi t\} = \{t'\}.$$

\square

6.4 Young subgroups and permutation modules

Symmetric groups are generally complex in their structures. We wish to study the representations of them by taking the idea of doing induced representation from some simpler subgroups. A good candidate lies from the combinatoric object of Young tabloids previously introduced.

Definition 15 (Young subgroup). *Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ be a partition of n . The **Young subgroup** of S_n associated to λ is:*

$$S_\lambda = S_{\{1, \dots, \lambda_1\}} \times S_{\{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}} \times \dots \times S_{\{n - \lambda_k + 1, \dots, n\}},$$

where each factor permutes the entries inside the corresponding block of consecutive integers. It is straightforward to check that this is indeed a subgroup of S_n .

Example 20. Consider $\lambda = (4, 2, 2, 1) \vdash 9$. Then the associated Young subgroup is:

$$S_\lambda = S_{\{1, 2, 3, 4\}} \times S_{\{5, 6\}} \times S_{\{7, 8\}} \times S_{\{9\}}.$$

Here $S_{\{1, 2, 3, 4\}}$ permutes the first four entries, $S_{\{5, 6\}}$ permutes the next two, and so on. The subgroup S_λ thus has order $4! \cdot 2! \cdot 2! \cdot 1! = 96$.

This form of the order is seen before at Equation 10. There is indeed a relation between the order of λ -Young subgroups and the number of λ -Young tabloids, for every chosen λ .

Proposition 23 (correspondence between subgroups and tabloids). *Fix the standard tabloid $\{t^\lambda\}$ whose rows are the blocks used to define S_λ . The map:*

$$\Phi : S_n / S_\lambda \longrightarrow \{\lambda\text{-tabloids}\}, \quad \Phi(gS_\lambda) = \{g \cdot t^\lambda\}$$

is a well-defined bijection. In particular, the number of left cosets $[S_n : S_\lambda]$ equals the number of λ -tabloids.

Proof. Well-defined. If $gS_\lambda = g'S_\lambda$ then $g' = gh$ for some $h \in S_\lambda$. By construction h permutes entries within each row of t^λ , so:

$$\{g' \cdot t^\lambda\} = \{gh \cdot t^\lambda\} = \{g \cdot t^\lambda\}.$$

Hence Φ is well-defined on cosets.

Injective. If $\Phi(gS_\lambda) = \Phi(g'S_\lambda)$ then $\{g \cdot t^\lambda\} = \{g' \cdot t^\lambda\}$. Thus there is some row permutation $r \in R_{g \cdot t^\lambda}$ with $g' \cdot t^\lambda = r(g \cdot t^\lambda)$. Conjugating back to the standard tableau shows $g^{-1}g' \in S_\lambda$, so $gS_\lambda = g'S_\lambda$.

Surjective. Every tabloid is some $g \cdot \{t^\lambda\}$, so every tabloid is in the image.

Therefore Φ is a bijection. \square

Further, a Young subgroup S_μ is conjugate to the standard Young subgroup S_λ if and only if there exists a permutation $\pi \in S_n$ such that $\pi S_\lambda \pi^{-1} = S_\mu$. This happens precisely when the sets defining S_μ are $\pi(1, \dots, \lambda_1), \pi(\lambda_1 + 1, \dots, \lambda_1 + \lambda_2), \dots$, which are exactly the rows of the tabloid $\pi \cdot t^\lambda$.

Example 21. Continuing with $\lambda = (4, 2, 2, 1)$, the standard tabloid is:

1	2	3	4
5	6		
7	8		
9			

where rows are unordered. The proposition shows that cosets gS_λ correspond bijectively to tabloids of this shape. For instance, applying the permutation (1 5) to the standard tableau yields:

5	2	3	4
1	6		
7	8		
9			

whose tabloid is a different basis element in the induced module (see below).

Now, we wish to construct S_n -representations by inducing from S_λ -representation. We recall that such a induced module has basis as tensors of transversals with elements in S_λ . It is thus important to take a second look into the transversals of symmetric groups.

Definition 16 (transversal / coset representatives). *A (left) transversal for S_λ in S_n is a choice of a subset:*

$$\mathcal{T} = \{\pi_1, \dots, \pi_m\} \subset S_n$$

containing exactly one representative from each left coset of S_λ (so $m = [S_n : S_\lambda]$). Equivalently $S_n = \bigsqcup_{i=1}^m \pi_i S_\lambda$ and the π_i are pairwise in distinct cosets.

Example 22. For $\lambda = (4, 2, 2, 1)$, the number of cosets is:

$$[S_9 : S_\lambda] = \frac{9!}{4! 2! 2! 1!} = 3780.$$

One way to choose a transversal is to pick, for each tabloid of shape λ , a permutation π sending the standard tabloid:

1	2	3	4
5	6		
7	8		
9			

to that tabloid. For instance, the permutation $(1\ 5)$ is a representative of the coset corresponding to the tabloid:

5	2	3	4
1	6		
7	8		
9			

Proposition 24 (Basis of the induced module). *Let $\mathbf{1} \simeq \mathbb{C}$ denote the trivial $\mathbb{C}[S_\lambda]$ -module. Then the induced module:*

$$M^\lambda := \text{Ind}_{S_\lambda}^{S_9}(\mathbf{1}) = \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \mathbf{1}$$

has \mathbb{C} -basis $\{\pi_i \otimes 1 : \pi_i \in \mathcal{T}\}$ for any choice of transversal \mathcal{T} .

Via the coset/tabloid bijection, these basis elements may be identified with λ -tabloids. Hence M^λ is called the **permutation module** as it is the module spanned by λ -tabloids acted by S_n .

Note that permutation modules are generally reducible. Their irreducible components are of great importance and will be discussed later.

Example 23. In the case $\lambda = (4, 2, 2, 1)$, the induced module

$$M^\lambda = \text{Ind}_{S_\lambda}^{S_9}(\mathbf{1})$$

has dimension 3780. Its basis is indexed by the λ -tabloids, one of which is:

5	2	3	4
1	6		
7	8		
9			

Thus the representation space is the vector space spanned by all λ -tabloids, with S_9 acting by permuting the entries.

6.5 Symmetrizers

Here are more results following from above.

Definition 17 (stabilizers, symmetrizer, antisymmetrizer). *For a tableau $t = t^\lambda$, write:*

$$R_t = \prod_{\text{rows } R} S_R, \quad C_t = \prod_{\text{columns } C} S_C$$

for the subgroups that permute entries inside each row and column. We call them the **row** and **column stabilizers** of t . Inside the group algebra $\mathbb{C}[S_n]$ define the **row symmetrizer** and **column antisymmetrizer** as:

$$a_t = \sum_{\rho \in R_t} \rho, \quad \kappa_t = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma) \sigma.$$

Example 24. For the standard tableau of shape $\lambda = (4, 2, 2, 1)$ as follows:

1	2	3	4
5	6		
7	8		
9			

the row stabilizer is:

$$R_t = S_{\{1,2,3,4\}} \times S_{\{5,6\}} \times S_{\{7,8\}} \times S_{\{9\}},$$

and the column stabilizer C_t permutes entries within each column, e.g. $S_{\{1,5,7,9\}} \times S_{\{2,6,8\}} \times S_{\{3\}} \times S_{\{4\}}$. Thus,

$$a_t = \sum_{\rho \in R_t} \rho, \quad \kappa_t = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma) \sigma,$$

which sums over $|R_t| = 96$ elements and $|C_t| = 144$ elements respectively.

Proposition 25. For any $\pi \in S_n$ and tableau t ,

$$R_{\pi t} = \pi R_t \pi^{-1}, \quad C_{\pi t} = \pi C_t \pi^{-1}, \quad \kappa_{\pi t} = \pi \kappa_t \pi^{-1}.$$

Proof. If $r \in R_t$ permutes entries within the i -th row of t , then $\pi r \pi^{-1}$ permutes entries within the corresponding row of πt . Hence $\pi R_t \pi^{-1} \subseteq R_{\pi t}$, and equality follows by symmetry. The same argument applies to columns, so $C_{\pi t} = \pi C_t \pi^{-1}$. Finally,

$$\pi \kappa_t \pi^{-1} = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma) \pi \sigma \pi^{-1}.$$

Since conjugation preserves sign, the right-hand side is exactly $\kappa_{\pi t}$. □

6.6 Specht modules

The above property of κ_t as antisymmetrizers inspires us to give the following theory.

Definition 18 (Polytabloid). *Let $\lambda \vdash n$ be a partition and t a λ -tableau. Let $C_t \subseteq S_n$ be the column-stabilizer of t . The **polytabloid** associated to t is*

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi) \pi \{t\},$$

where $\{t\}$ denotes the tabloid of t .

If $\{t\}$ denotes the λ -tabloid associated to t , then

$$e_t = \kappa_t \{t\}.$$

In other words, a polytabloid is obtained by applying the column antisymmetrizer κ_t to the tabloid $\{t\}$.

Proposition 26 (Well-definedness of polytabloids). *Let t and t' be two λ -tableaux with the same tabloid:*

$$\{t\} = \{t'\}.$$

Then their associated polytabloids coincide:

$$e_t = e_{t'}.$$

Proof. By definition,

$$e_t = \sum_{\pi \in C_t} \text{sgn}(\pi) \pi\{t\}, \quad e_{t'} = \sum_{\pi \in C_{t'}} \text{sgn}(\pi) \pi\{t'\}.$$

Since $\{t\} = \{t'\}$, the two tabloids span the same row sets. Moreover, the column-stabilizers C_t and $C_{t'}$ are conjugate subgroups of S_n , and their action on the tabloid $\{t\}$ produces the same signed orbit. Thus e_t depends only on the tabloid $\{t\}$, not on the particular tableau t . \square

This fact allows us to think of a polytabloid e_t as being attached to a *tabloid* $\{t\}$, even though the definition is stated using a tableau.

Having this said, note that polytabloids are specially defined elements living in $\mathbb{C}[\{\{t\} : \{t\} \text{ a standard tabloid}\}]$. Since $\{t\}$ refers to equivalent classes, we could consider their span as a vector space, to study the elements presented by these polytabloids.

Definition 19 (Specht module). *Given $\lambda \vdash n$, the **Specht module** S^λ is the subspace of the permutation module M^λ spanned by all polytabloids e_t .*

We give several notes of Specht modules. First to say, more than \mathbb{C} -vector spaces, Specht modules are S_n -representations, since actions of S_n on M^λ preserves S^λ . That is, for any $\sigma \in S_n$ and any λ -polytabloid e_t , we have $\sigma \cdot e_t \in S^\lambda$.

Based on this, Specht modules S^λ are S_n -subrepresentations of M^λ . So we have the big picture:

$$\mathbb{C} \leq S^\lambda \leq M^\lambda \leq \mathbb{C}[S_n],$$

where \mathbb{C} is the trivial representation and $\mathbb{C}[S_n]$ is the regular representation.

Note that all e_t 's might not form a basis of S^λ .

6.7 Example: Specht modules for S_3

Partitions of 3. The partitions of 3 are

$$(3), \quad (2,1), \quad (1,1,1).$$

Their Young diagrams are

$$\begin{array}{c} \text{--- --- ---} \\ \text{--- --- ---} \end{array} \quad (3), \quad \begin{array}{c} \text{--- ---} \\ \text{---} \end{array} \quad (2,1), \quad \begin{array}{c} \text{--- ---} \\ \text{--- ---} \\ \text{---} \end{array} \quad (1,1,1).$$

The corresponding Specht modules and dimensions are:

- $S^{(3)}$: dimension 1 (trivial representation),
- $S^{(2,1)}$: dimension 2 (standard representation),
- $S^{(1,1,1)}$: dimension 1 (sign representation).

The trivial module. We want to show:

$$S^{(3)} \simeq \mathbf{1}.$$

That is, every permutation acts as $+1$ on $S^{(3)}$.

The partition (3) has Young diagrams consisting of a single row of length 3. A tableau t of shape (3) has only singleton columns, so its column-stabilizer C_t is trivial. Thus for any tableau t of shape (3) the polytabloid is

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi) \pi\{t\} = \{t\}.$$

Hence $S^{(3)}$ is the 1-dimensional subspace of $M^{(3)}$ spanned by the single tabloid $\{1\ 2\ 3\}$ (the unique tabloid of shape (3)). For every $\sigma \in S_3$ we have $\sigma \cdot \{1\ 2\ 3\} = \{1\ 2\ 3\}$, so σ acts as the identity on $S^{(3)}$. Therefore $S^{(3)} \cong \mathbf{1}$.

The sign module. We want to show:

$$S^{(1,1,1)} \simeq \operatorname{sgn}.$$

That is, every permutation acts as $\operatorname{sgn}(\sigma)$ on $S^{(1,1,1)}$.

The partition $(1, 1, 1)$ has a single column of length 3. For a tableau t of this shape the column-stabilizer is the whole S_3 . Hence:

$$e_t = \sum_{\pi \in S_3} \operatorname{sgn}(\pi) \pi\{t\}.$$

Compute the action of $\sigma \in S_3$ on e_t :

$$\sigma \cdot e_t = \sum_{\pi \in S_3} \operatorname{sgn}(\pi) (\sigma\pi)\{t\} = \sum_{\tau \in S_3} \operatorname{sgn}(\sigma^{-1}\tau) \tau\{t\} = \operatorname{sgn}(\sigma^{-1}) \sum_{\tau \in S_3} \operatorname{sgn}(\tau) \tau\{t\} = \operatorname{sgn}(\sigma) e_t.$$

Thus every permutation acts on e_t by the sign character, so $S^{(1,1,1)} \cong \operatorname{sgn}$.

The standard module $S^{(2,1)}$. Shape $(2, 1)$ admits 2 standard tableaux:

$$t_1 = \underline{\overline{1\ 2}}, \quad t_2 = \underline{\overline{1\ 3}}.$$

Inside the permutation module $M^{(2,1)}$ we use the tabloids:

$$T_{12} := \{1\ 2 \mid 3\}, \quad T_{13} := \{1\ 3 \mid 2\}, \quad T_{23} := \{2\ 3 \mid 1\}.$$

The column-stabilizers are:

$$C_{t_1} = \{\mathbb{1}, (13)\}, \quad C_{t_2} = \{\mathbb{1}, (12)\}.$$

Hence the corresponding polytabloids are:

$$e_{t_1} = T_{12} - T_{23}, \quad e_{t_2} = T_{13} - T_{23}.$$

Thus $\{e_{t_1}, e_{t_2}\}$ is a basis for $S^{(2,1)}$.

Character table of S_3 . With conjugacy classes labelled by cycle type:

	1 ³	2 · 1	3
$S^{(3)}$	1	1	1
$S^{(2,1)}$	2	0	-1
$S^{(1,1,1)}$	1	-1	1

6.8 Example: The Specht module $S^{(n-1,1)}$

Among the Specht modules, the case of the partition $(n-1, 1)$ is quite special. It generalizes the familiar example $S^{(2,1)}$ for S_3 to arbitrary n , and it corresponds to **the standard representation** of the symmetric groups.

The partition $(n-1, 1)$ corresponds to a Young diagram with $n-1$ boxes in the top row and one box in the second row. The Specht module $S^{(n-1,1)}$ has dimension $n-1$. One concrete realization is as the subrepresentation of the natural permutation representation of S_n on \mathbb{C}^n consisting of all vectors orthogonal to the all-ones vector:

$$S^{(n-1,1)} \cong \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \dots + x_n = 0\}.$$

This makes $S^{(n-1,1)}$ the standard representation of S_n . It refers to the important nontrivial action of S_n on n points after the trivial part is being factored.

Let us draw its Young diagram explicitly. For general n , the partition $(n-1, 1)$ corresponds to:

$$\begin{array}{c} \overline{1 \ 2 \ 3 \ \cdots \ n-1} \\ \hline \overline{n} \end{array}$$

with $n-1$ boxes in the first row and 1 box in the second row.

Each standard tableau of this shape arises by placing n in the second row and filling the remaining boxes in the top row with the numbers $1, \dots, n-1$. Hence there are exactly $n-1$ such tableaux, and thus

$$\dim S^{(n-1,1)} = n-1.$$

From the construction, one obtains a natural basis for $S^{(n-1,1)}$ indexed by the choice of where n is placed. In other words, if t_i denotes the tableau with i in the bottom row and the other entries in order across the top, then $\{e_{t_1}, \dots, e_{t_{n-1}}\}$ spans $S^{(n-1,1)}$.

Example 25. When $n=3$, the partition $(2, 1)$ gives the Specht module $S^{(2,1)}$. As computed earlier, this has dimension 2 and provides the standard representation of S_3 . The general case works in the same way, only in higher dimensions.

Example 26. When $n=4$, the partition $(3, 1)$ has diagram:

$$\begin{array}{c} \overline{1 \ 2 \ 3} \\ \hline \overline{4} \end{array}$$

There are three standard tableaux of this shape, giving a basis of three polytabloids. Thus $S^{(3,1)}$ is 3-dimensional, and inside \mathbb{C}^4 the module $S^{(3,1)}$ can be realized as the space of vectors (x_1, x_2, x_3, x_4) with $x_1 + x_2 + x_3 + x_4 = 0$.

7 Representations of symmetric groups: Specht modules

We now turn to some general structural results about Specht modules. Our goal is to show that the Specht modules provide all irreducible representations of S_n over \mathbb{C} .

7.1 The bilinear form

To show that $\{e_t\}$ is a basis of S^λ , we construct an inner product on M^λ .

Definition 20. Define a bilinear form $\langle \cdot, \cdot \rangle$ on the permutation module M^λ by

$$\langle \{t\}, \{s\} \rangle = \delta_{\{t\}, \{s\}},$$

where $\{t\}$ denotes a λ -tabloid.

Lemma 2. *The form $\langle \cdot, \cdot \rangle$ has the following properties:*

1. *Symmetry:* $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in M^\lambda$.
2. *S_n -invariance:* $\langle \sigma x, \sigma y \rangle = \langle x, y \rangle$ for all $\sigma \in S_n$.

Proof. Symmetry follows immediately from the fact that $\delta_{\{t\}, \{s\}}$ is symmetric in t and s . For invariance, note that $\sigma\{t\} = \{\sigma t\}$. Thus

$$\langle \sigma\{t\}, \sigma\{s\} \rangle = \delta_{\{\sigma t\}, \{\sigma s\}} = \delta_{\{t\}, \{s\}} = \langle \{t\}, \{s\} \rangle,$$

as required. \square

Since $S^\lambda \subseteq M^\lambda$, the form restricts to S^λ . We will show that this restriction is non-degenerate, and this will give us the independence of $\{e_t\}$.

7.2 Gram matrix and non-degeneracy

Given the polytabloid basis $\{e_t\}$, we may consider the **Gram matrix**:

$$G = (\langle e_t, e_s \rangle)_{t,s}.$$

If G is invertible, the form is non-degenerate.

Lemma 3 (Key identity). *For any tableau t and $y \in M^\lambda$, the coefficient $a_{\{t\}}$ of $\{t\}$ in $\kappa_t y$ is exactly*

$$a_{\{t\}} = \langle y, e_t \rangle,$$

where $\kappa_t = \sum_{\pi \in C_t} \text{sgn}(\pi) \pi$ is the column antisymmetrizer.

Proof. By definition of the form,

$$a_{\{t\}}(\kappa_t y) = \langle \kappa_t y, \{t\} \rangle.$$

Using S_n -invariance,

$$\langle \kappa_t y, \{t\} \rangle = \langle y, \kappa_t \{t\} \rangle.$$

But $\kappa_t \{t\} = e_t$ by definition of a polytabloid. Hence

$$a_{\{t\}}(\kappa_t y) = \langle y, e_t \rangle.$$

\square

Lemma 4 (Non-degeneracy). *The bilinear form $\langle \cdot, \cdot \rangle$ on S^λ is non-degenerate. It amounts to say that suppose $x \in S^\lambda$ satisfies $\langle x, y \rangle = 0$ for all $y \in S^\lambda$, then $x = 0$.*

Proof. Write $x = \sum_{\{s\}} a_{\{s\}} \{s\}$ as a linear combination of tabloids. Order the tableaux in some total order (for instance the *last-letter order*). Let t_0 be maximal such that $a_{\{t_0\}} \neq 0$.

Apply κ_{t_0} to x . On one hand, by the key identity,

$$a_{\{t_0\}}(\kappa_{t_0} x) = \langle x, e_{t_0} \rangle = 0.$$

On the other hand, expand:

$$\kappa_{t_0} x = \sum_{\{s\}} a_{\{s\}} \sum_{\pi \in C_{t_0}} \text{sgn}(\pi) \pi \{s\}.$$

For $\{t_0\}$ to appear in this expansion, we must have $\{s\} = \pi^{-1}\{t_0\}$ for some $\pi \in C_{t_0}$. By maximality of t_0 , the only such $\{s\}$ is $\{t_0\}$ itself (with $\pi = \text{id}$). Thus

$$a_{\{t_0\}}(\kappa_{t_0} x) = a_{\{t_0\}}.$$

Comparing both calculations gives $a_{\{t_0\}} = 0$, contradicting the choice of t_0 . Hence $x = 0$, proving non-degeneracy. \square

Example 27 (The case S_3). *For $\lambda = (3)$ (trivial representation), S^λ has basis $\{e_t\}$, and $G = (1)$.*

For $\lambda = (2, 1)$, there are three polytabloids, but they satisfy the relation $e_{t_1} + e_{t_2} + e_{t_3} = 0$. Choosing $\{e_{t_1}, e_{t_2}\}$ as a basis, the Gram matrix is 2×2 invertible.

For $\lambda = (1, 1, 1)$ (sign representation), again S^λ is one-dimensional, with $G = (1)$.

7.3 Basis theorem

To organize our arguments, it is convenient to introduce an order on tableaux.

Definition 21 (Dominance order). *Given two standard tableaux t and s of the same shape λ , we say $t \triangleright s$ if for every $k \leq n$,*

$$\lambda^{(k)}(t) := (|\text{entries} \leq k \text{ in row } 1 \text{ of } t|, |\text{entries} \leq k \text{ in rows } 1-2 \text{ of } t|, \dots),$$

we require

$$\lambda^{(k)}(t) \trianglerighteq \lambda^{(k)}(s)$$

in the dominance order of partitions.

Intuitively, in dominance order, the earlier numbers in t should be placed left and up compared to s .

Lemma 5 (Dominance triangularity). *If t and s are standard tableaux of the same shape, then*

$$\langle e_s, e_t \rangle = 0 \quad \text{unless } s \triangleright t.$$

In particular, the Gram matrix is block upper-triangular with respect to dominance order.

Proof. When expanding $\kappa_t\{s\}$, the only tabloids that survive correspond to tableaux not larger than t in dominance order. Thus, if s does not dominate t , the coefficient vanishes. \square

This triangularity is the reason we can pick the maximal tableau t_0 in the basis proof. It guarantees that only the e_{t_0} term survives when testing against e_{t_0} , forcing its coefficient to vanish.

We can now prove linear independence of polytabloids.

Theorem 5 (Basis Theorem for Specht Modules). *The set $\{e_t : t \text{ is a standard tableau of shape } \lambda\}$ forms a basis for S^λ .*

Proof. We already know that S^λ is spanned by $\{e_t\}$. It remains to prove independence.

Suppose there is a linear relation

$$\sum_t a_t e_t = 0.$$

Let t_0 be maximal in the dominance order with $a_{t_0} \neq 0$. Apply κ_{t_0} and then pair with e_{t_0} :

$$0 = \left\langle \kappa_{t_0} \cdot \left(\sum_t a_t e_t \right), e_{t_0} \right\rangle = \sum_t a_t \langle \kappa_{t_0} e_t, e_{t_0} \rangle.$$

By the dominance lemma, all terms vanish except when $t = t_0$, in which case we obtain $a_{t_0}c$ with $c \neq 0$. Thus $a_{t_0} = 0$, a contradiction. Hence the relation was trivial, and $\{e_t\}$ is independent. \square

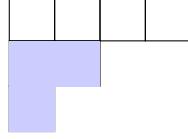
7.4 The hook-length formula

Once we know that the standard polytabloids form a basis, we can compute $\dim S^\lambda$ by counting standard tableaux. This is given by the **hook-length formula**.

Definition 22 (Hook length). *Let λ be a partition, and let (i, j) denote the box in row i , column j of the Young diagram of λ . The **hook** at (i, j) is the set of boxes consisting of:*

- the box (i, j) itself,
- all boxes to the right in the same row,
- all boxes below in the same column.

The **hook length** $\text{hook}(i, j)$ is the number of boxes in this set.



In the diagram above (shape $(4, 2, 1)$), the hook of the box $(2, 1)$ consists of $(2, 1), (2, 2), (3, 1)$. Thus $\text{hook}(2, 1) = 3$.

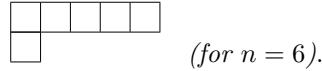
Theorem 6 (Hook-length formula). *For any partition $\lambda \vdash n$, the number of standard tableaux of shape λ is*

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} \text{hook}(i,j)}.$$

Hence

$$\dim S^\lambda = f^\lambda.$$

Example 28 (The $(n - 1, 1)$ representation). *Consider $\lambda = (n - 1, 1)$. Its diagram has one long first row of length $n - 1$, and one box in the second row:*



The hook lengths can be computed as:

- Top-left box $(1, 1)$ has hook length n .
- Boxes $(1, j)$ for $2 \leq j \leq n - 1$ each have hook length $n - j$.
- The bottom box $(2, 1)$ has hook length 1.

The product of hook lengths is

$$n \cdot (n - 2)! \cdot 1.$$

Thus

$$\dim S^{(n-1,1)} = \frac{n!}{n \cdot (n - 2)!} = n - 1.$$

This is exactly the standard $(n - 1)$ -dimensional representation of S_n .

7.5 Irreducibility and orthogonality

The non-degenerate form is the key to irreducibility.

Theorem 7 (Irreducibility of Specht modules). *Over \mathbb{C} , every Specht module S^λ is irreducible.*

Proof. Suppose $U \subset S^\lambda$ is a proper nonzero submodule. By invariance of the form, U^\perp is also a submodule. Thus

$$S^\lambda = U \oplus U^\perp.$$

But S^λ is indecomposable as a submodule of M^λ . This contradiction shows U must equal 0 or S^λ . Hence S^λ is irreducible. \square

Theorem 8 (Orthogonality of distinct Specht modules). *If $\lambda \neq \mu$, then $\langle S^\lambda, S^\mu \rangle = 0$.*

Proof. Let $x \in S^\lambda$, $y \in S^\mu$. One can find $\sigma \in S_n$ acting as +1 on S^λ and as -1 on S^μ . Then

$$\langle x, y \rangle = \langle \sigma x, \sigma y \rangle = \langle x, -y \rangle = -\langle x, y \rangle.$$

Thus $\langle x, y \rangle = 0$. \square

Theorem 9 (Classification). *Over \mathbb{C} , the irreducible representations of S_n are precisely the Specht modules S^λ for $\lambda \vdash n$.*

Example 29 (S_3). *Partitions: $(3), (2, 1), (1, 1, 1)$. Dimensions: $1, 2, 1$. Sum of squares: $6 = |S_3|$. These correspond to the trivial, standard, and sign representations.*

Example 30 (S_4). *Partitions: $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$. Dimensions: $1, 3, 2, 3, 1$. Sum of squares: $1 + 9 + 4 + 9 + 1 = 24 = |S_4|$. Thus these are all the irreducible ones of S_4 .*

7.6 A second method

We aim to prove that Specht modules S^λ are irreducible. Also, they are exactly all the irreducible S_n -representations. We will also try to find how S_n -representations are decomposed into Specht modules.

We will start with some preparations work, following by three lemmas.

Lemma 6. *Given $\lambda, \mu \vdash n$, $\{t\}$ as a λ -tabloid and $\{s\}$ as a μ -tabloid,*

1. $\kappa_t\{s\} \neq 0 \implies \lambda \trianglerighteq \mu$, in other words, $\lambda \triangleleft \mu \implies \kappa_t\{s\} = 0$.
2. $\lambda = \mu \implies \kappa_t\{s\} = \pm e_t$.

Proof. 1. Since $\lambda \triangleleft \mu$, there exists i, j as numbers between 1 to n , such that i and j lie in the same row of s while lie in the same column of t . This tells that $(ij) \in C_t$. Thus, there is a subgroup $H := \{1, (ij)\}$ of C_t , so C_t can be partitioned by cosets πH , where $\pi \in C_t$.

Now, we can compute the desired result by the followings:

$$\kappa_t\{s\} = \sum_{\pi \in C_t} \text{sgn}(\pi) \pi\{s\} = \sum_k (\text{sgn}(\pi_k) \pi_k + \text{sgn}(\pi_k(ij)) \pi_k(ij))\{s\}.$$

Since i and j are in the same row of $\{s\}$, we have $(ij)\{s\} = \{s\}$. Therefore, for each coset representative π_k ,

$$\begin{aligned} (\text{sgn}(\pi_k) \pi_k + \text{sgn}(\pi_k(ij)) \pi_k(ij))\{s\} &= (\text{sgn}(\pi_k) \pi_k + \text{sgn}(\pi_k) \text{sgn}((ij)) \pi_k(ij))\{s\} \\ &= \text{sgn}(\pi_k)(\pi_k\{s\} - \pi_k(ij)\{s\}) = \text{sgn}(\pi_k)(\pi_k\{s\} - \pi_k\{s\}) = 0. \end{aligned}$$

Thus the sum $\kappa_t\{s\}$ is 0.

2. If $\lambda = \mu$, then $\{t\}$ and $\{s\}$ are both λ -tabloids. Then, by the transitive property 22, there exists a permutation $\sigma \in S_n$, such that $\{s\} = \sigma\{t\}$. Thus,

$$\begin{aligned} \kappa_t\{s\} &= \kappa_t\sigma\{t\} = \left(\sum_{\pi \in C_t} \text{sgn}(\pi) \pi \right) \sigma\{t\} = \text{sgn}(\sigma) \sum_{\pi \in C_t} \text{sgn}(\pi\sigma) \pi\sigma\{t\} \\ &= \pm \sum_{\pi\sigma \in C_t} \text{sgn}(\pi\sigma) \pi\sigma\{t\} = \pm \kappa_t\{t\} = \pm e_t. \end{aligned}$$

□

Lemma 7. *Let $\lambda \vdash n$. Then $\forall x \in M^\lambda$ and $\forall t$ a λ -tableau, $\kappa_t x = f e_t$, where $f \in \mathbb{C}$.*

Proof. Write any element $x \in M^\lambda$ as $x = \sum_i c_i \{t_i\}$, where $t_i = t_i^\lambda$ is a λ -tableau. Then, given any λ -tableau t ,

$$\kappa_t x = \sum_i \kappa_t c_i \{t_i\} = \sum_i c_i \kappa_t \{t_i\} = \sum_i c_i (\pm e_t) = f e_t,$$

where $f = \pm \sum_i c_i \in \mathbb{C}$.

□

Recall that $S^\lambda \leq M^\lambda$ as \mathbb{C} -vector spaces and as S . The following lemma tells more than this.

Lemma 8. *Given $\lambda \vdash n$, denote $M^\lambda = S^\lambda \oplus S^{\lambda \perp}$ as \mathbb{C} -vector space decompositions. Then $S^{\lambda \perp} \leq M^\lambda$ as S_n -subrepresentations.*

Proof. We need to show that for all $w \in S_n, y \in S^{\lambda \perp}$, we have $w \cdot y \in S^{\lambda \perp}$. This amounts to prove that $\forall x \in S^\lambda$, we have $x \perp w \cdot y$. Using the inner product,

$$\langle x, w \cdot y \rangle = \langle w^{-1}x, y \rangle = 0,$$

since $w^{-1}x$ is still in S^λ .

□

This lemma tells that every S^λ is not only a submodule, but has the structure of a subrepresentation.

We have made enough preparations. Now comes the central theorem that describes the structure of S^λ in M^λ .

Theorem 10. *Given $\lambda \vdash n$, for any $U \leq M^\lambda$ an S_n -submodule, one of the following holds:*

$$i) U \geq S^\lambda \quad ii) U \leq S^{\lambda\perp}.$$

In other words, as a subset of M^λ , if U intersect with S^λ , then it has to contain all of S^λ .

Proof. We discuss in two cases.

If there exist $0 \neq x \in U$, and there also exists $t = t^\lambda$ a λ -tableau,

$$\kappa_t \cdot x = fe_t \neq 0.$$

In this case, $e_{w \cdot t} \in U$ for all $w \in S_n$. Thus $S^\lambda \leq U$.

The opposite of this case is when $\forall x \in U \forall t = t^\lambda \kappa_t \cdot x = 0$. This tells:

$$\langle e_t, x \rangle = \langle \kappa_t \{t\}, x \rangle = \left\langle \sum_{\pi \in C_t} \text{sgn}(\pi) \pi \{t\}, x \right\rangle = \langle \{t\}, \sum_{\pi \in C_t} \text{sgn}(\pi) \pi x \rangle = \langle \{t\}, \kappa_t \cdot x \rangle = 0.$$

This implies that x is perpendicular with S^λ . Hence $U \perp S^\lambda$ and thus $U < S^{\lambda\perp}$. \square

Corollary 6 (Main Theorem 1). *All S^λ are irreducible.*

Proof. This is because no proper submodule U is contained in S^λ . \square

Theorem 11 (Main Theorem 2). *S^λ forms a complete list of irreducible S_n -modules.*

Proof. By the definition of S^λ , there are the number of conjugacy classes of S_n of them. Thus, from the precedent result 4, there could be no irreducible representations other than the Specht modules. \square

We would also like to know how S_n -modules can be decomposed into Specht modules.

Theorem 12 (Main Theorem 3). *Let M^μ be the permutation module associated with the partition μ of n , with its decomposition*

$$M^\mu = \bigoplus_{\lambda} c_{\mu\lambda} S^\lambda,$$

where S^λ denotes the Specht module and $c_{\mu\lambda}$ are the **Kostka numbers**. Then:

1. If $c_{\mu\lambda} \neq 0$, then $\lambda \trianglerighteq \mu$ (dominance order).

2. For $\lambda = \mu$, we have $c_{\lambda\lambda} = 1$.

Proof. Recall from 10 that

$$c_{\mu\lambda} = \dim \text{Hom}_{S_n}(S^\lambda, M^\mu).$$

(1) Suppose $c_{\mu\lambda} \neq 0$. Then there exists a nonzero S_n -homomorphism

$$\theta : S^\lambda \longrightarrow M^\mu.$$

Choose a λ -tableau t . Since S^λ is generated by $e_t = \kappa_t \{t\}$, we have

$$\theta(e_t) = \theta(\kappa_t \{t\}) = \kappa_t \theta(\{t\}) = \kappa_t \left(\sum_i c_i \{t_i\} \right) = \sum_i c_i \kappa_t \{t_i\},$$

where each $\{t_i\}$ is a μ -tabloid.

If $\lambda \lhd \mu$, then by the dominance properties of tabloids,

$$\kappa_t \cdot \{t_i\} = 0 \quad \text{for all } i,$$

hence $\theta(e_t) = 0$ for all t . But θ is nonzero by assumption, a contradiction. Therefore, $\lambda \trianglerighteq \mu$.

(2) When $\lambda = \mu$, we claim that S^λ appears exactly once in M^λ .

Consider the natural S_n -invariant inner product on M^λ defined by

$$\langle \{t\}, \{s\} \rangle = \begin{cases} 1, & \text{if } \{t\} = \{s\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $S^\lambda \subset M^\lambda$ be the Specht submodule generated by the polytabloids e_t . We will show that any other copy of S^λ must be orthogonal to it, forcing the multiplicity to be one.

For $\{t\} \in M^\lambda$ and $e_s = \kappa_s \{s\}$, we have

$$\langle \{t\}, e_s \rangle = \langle \{t\}, \kappa_s \{s\} \rangle = \left\langle \sum_{\pi \in C_s} \operatorname{sgn}(\pi) \pi \{t\}, \{s\} \right\rangle.$$

The vector $\sum_{\pi \in C_s} \operatorname{sgn}(\pi) \pi \{t\}$ is alternating under the column stabilizer, while $\{s\}$ is fixed under it. Hence the inner product vanishes:

$$\langle \{t\}, e_s \rangle = 0.$$

Therefore, S^λ is orthogonal to its complement in M^λ , and since $e_t \neq 0$, this submodule appears with multiplicity one, i.e. $c_{\lambda\lambda} = 1$. \square

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