Lattice Models for Quantum Superalgebras

Colors and Supercolors

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Overview

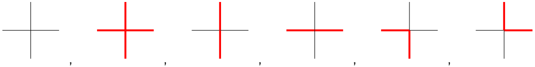
1. Introduction and Motivations
2. Strategy
3. Results and Conjecture

Lattice Model

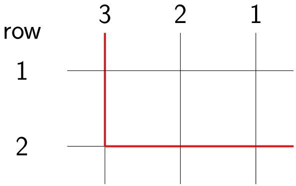
Definition

A lattice model $\mathcal{L}$ is an $n \times m$ grid with its edges filled according to vertex table.

- We generally think of them as paths going from the top and exiting to the right
- They are indexed by external edges. Typically, we fix edge colorings at the top row with a partition $\lambda = (\lambda_1, ..., \lambda_k)$.
- The example below uses this vertex table:



Example. Take $\lambda = (3)$

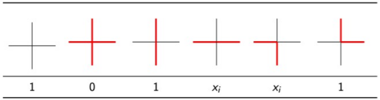


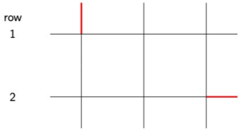
Partition Functions

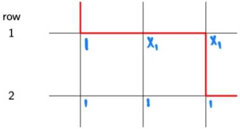
Definition (Partition Function)

Given a lattice model with fixed boundary conditions, the partition function $\mathcal{Z}(\mathcal{L})$ of the lattice is the sum of all admissible states, which are paths with non-zero weight.

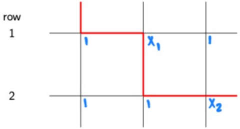
Note: Non-specified vertices have weight 0.



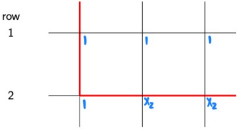




x_1^2



x_1x_2



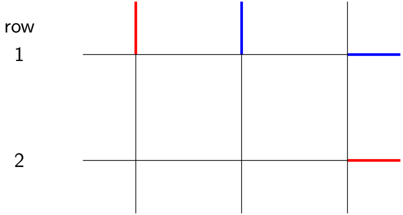
x_2^2

In this case: $\mathcal{Z}(\mathcal{L}) = x_1^2 + x_1x_2 + x_2^2$

Boundary Conditions

For models with multiple colors, we may also fix a permutation $w \in S_m$ on the side to indicate the order of colors from top to bottom.

Example. Take $\lambda = (3, 2)$ and $w = (12)$

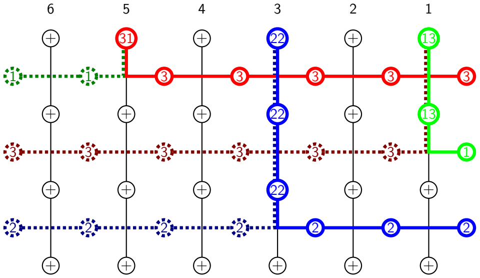


Super-Lattice Model

Definition

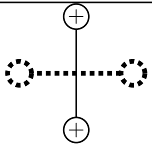
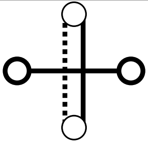
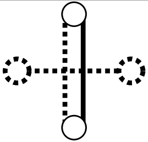
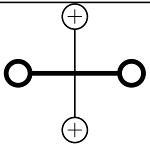
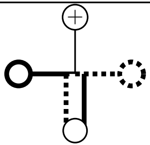
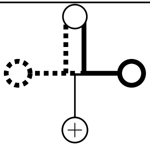
A super-lattice model is a lattice model indexed by one partition λ and two permutations $w, w' \in S_m$ such that it has both colored and dotted colored (supercolor) paths going in opposite directions.

Example. Below is an admissible state with $\lambda = (5, 3, 1)$, $w = (312)$, and $w' = (132)$



Weights for Super-Lattice Model

We are working with these weights [2]:

a_1	a_2	b_1	b_2	c_1	c_2
					

Research Questions

Motivating Questions

- What do the partition functions of these lattice $\mathcal{L}_{\lambda,w,w'}$ models look like?
- What combinatorial objects represent them?

Past work:

- Previous Polymath projects have "solved" models with one partition λ and one permutation w. [2]

For experts: The weights of the given model were chosen based on quantum superalgebra modules.

- Further Question: How can changing the weights in accordance with these superalgebras affect the partition functions?
- Connected to the supersymmetries between bosons and fermions

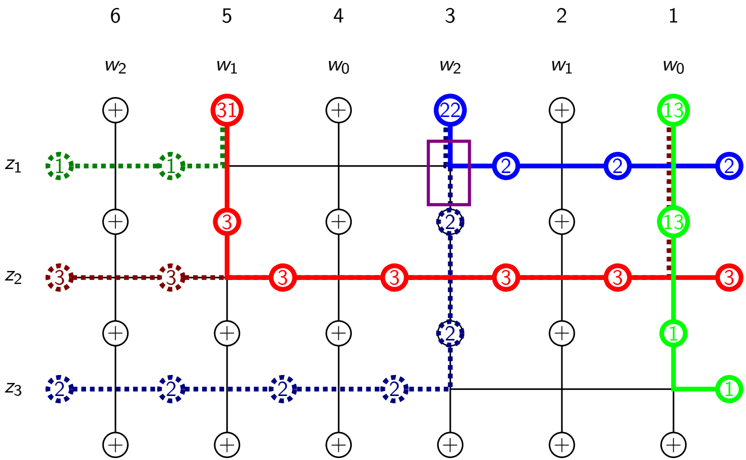
Research Strategy

- **Goal.** Compute the partition function of $\mathcal{L}_{\lambda,w,w'}$ for all $\lambda \vdash m$ and $w, w' \in S_n$.
- **First steps.**
  1. Identify w, w' with $\mathcal{L}_{\lambda,w,w'} = 0$.
  2. Identify w, w' such that $\mathcal{L}_{\lambda,w,w'}$ has a unique admissible state.
  3. Compute remaining partition functions recursively by relating permutation index pairs (train argument).
- **Dream.** Understand the quantum group module for super-lattice models

Question (rephrased).

- What is the minimal set of states we need to compute to know all partition functions?
- And what do their partition functions look like?

An Inadmissible (Vanishing) State



Vanishing Conjecture

Vanishing Conjecture

For boundary conditions $(w, w_0 u)$, if $u = w$ then there is only one state and if $u < w$ then there are no states, where $<$ indicates strong Bruhat order and w_0 indicates the longest word.

Strong (full) Bruhat order on S_3 [1]

For $1 \leq i < j \leq 3$ let $(i\,j)$ be the transposition exchanging i and j. Given $u \in S_3$ we declare

$$u \;<\; (i\,j)\,u \quad\Longleftrightarrow\quad \ell\big((i\,j)u\big) \;=\; \ell(u) + 1. \tag{$*$}$$

The *strong Bruhat order* is the reflexive–transitive closure of this relation $(*)$; i.e. for $u, v \in S_3$

$$u \;\leq v \;\Longleftrightarrow\; \text{there exists a chain } u = w_0 < w_1 < \cdots < w_k = v \text{ each step satisfying } (*).$$

There are $3! = 6$ elements, which we list by *length* $\ell(w) = \#\{\,(i < j) \mid w(i) > w(j)\,\}$ (the number of inversions).

Length Comparison

length	elements
0	$e = 123$
1	$213 = s_1, \quad 132 = s_2$
2	$231 = s_1 s_2, \quad 312 = s_2 s_1$
3	$321 = s_1 s_2 s_1 = s_2 s_1 s_2 = w_0.$

$e < s_1,\ s_2,$

$s_1 < s_1 s_2,\ s_2 s_1 < w_0,$ with $s_1,\ s_2$ incomparable and likewise $s_1 s_2,\ s_2 s_1$.

$s_2 < s_1 s_2,\ s_2 s_1 < w_0.$

Vanishing / one-state table

$$
\begin{array}{c|cccccc}
 & w_0 & s_2 s_1 & s_1 s_2 & s_1 & s_2 & e \\
\hline
e & 1 & * & * & * & * & * \\
s_2 & 0 & 1 & * & * & * & * \\
s_1 & 0 & * & 1 & * & * & * \\
s_1 s_2 & 0 & 0 & 0 & 1 & * & * \\
s_2 s_1 & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{*} & \color{red}{1} & \color{red}{*} \\
w_0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

1 exactly one state ($u = w$),

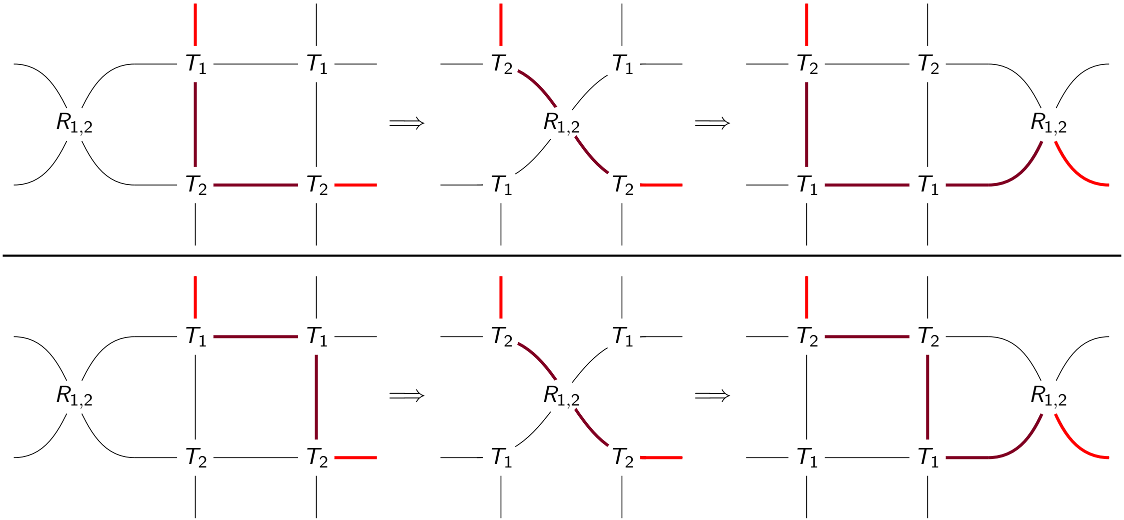
0 vanishes ($u < w$ in strong Bruhat order),

$*$ conjecture does not constrain this pair.

Observations

1. Anti-diagonal of ones.
2. Zeros lie strictly to the left of that anti-diagonal.
3. Row counts reflect the poset.
4. Almost–unitriangular shape.

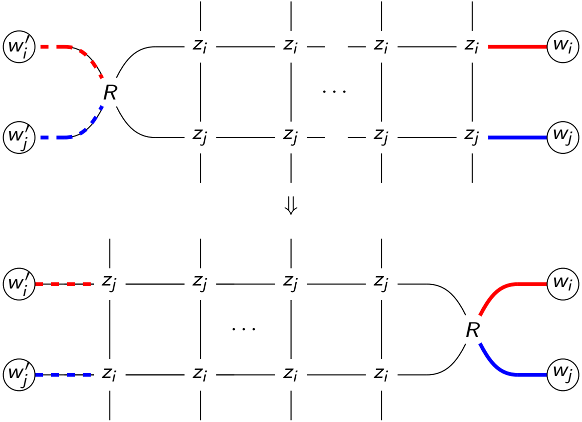
Train argument for single colored lattice model



Train argument for color/scolor model

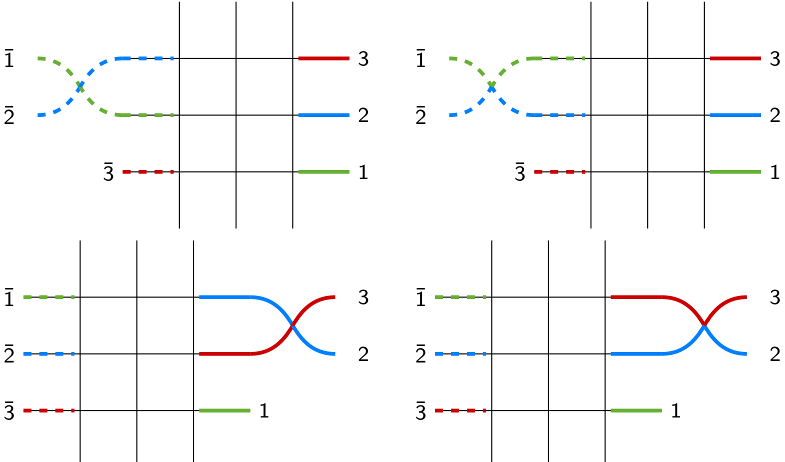
Notation:

- w_i represents the **color** decorated at row i under permutation of w, and w_i' represents the **scolor** decorated at row i under permutation of w'.
- z_i and z_j marks the row number before and after the run-through of R-vertex.



Example of train argument on 3×3 Super-lattice model

For simplification, it seems like for a 3×3 lattice model, but one can image the arbitrary rows above and below with arbitrary column in the middle. The lattice models would be, from left to right, $\mathcal{L}_{\lambda,w_0,s_1}, \mathcal{L}_{\lambda,w_0,e}, \mathcal{L}_{\lambda,s_1 s_2,e}, \mathcal{L}_{\lambda,w_0,e}$.



The relations from train argument on the super-lattice model

Case $(w = w_0, w' = e)$

$$q\,(z_1^3 - z_2^3)\,Z\big(\mathcal{L}_{\lambda,w_0,s_1}\big) + (1-q^2)\,z_2^2\,z_1\,Z\big(\mathcal{L}_{\lambda,w_0,e}\big)$$
$$= s_1\Big[(z_1^3 - z_2^3)\,Z\big(\mathcal{L}_{\lambda,s_1s_2,e}\big) + (1-q^2)\,z_1^3\,Z\big(\mathcal{L}_{\lambda,w_0,e}\big)\Big],$$

$$Z\big(\mathcal{L}_{\lambda,w_0,s_1}\big) = 0, \quad Z\big(\mathcal{L}_{\lambda,w_0,e}\big) = z_1^{\lambda_1-2}\,z_2^{\lambda_2-1}\,z_3^{\lambda_3}.$$

Upshot

By applying the vanishing partition function value and the already known partition function formula, we are only left with $Z(\mathcal{L}_{\lambda,s_1s_2,e})$ as a variable of the equation, which means then we can solve for $Z(\mathcal{L}_{\lambda,s_1s_2,e})$, one of the unknown partition functions according to the table.

Note: s_1 simply means to flip z_1 and z_2 in the context of spectral parameters. For example, if $f = z_1^2 + z_2$, then $s_1 f = z_2^2 + z_1$.

In the context of the table

$$
\begin{array}{c}
 \\ e \\ s_2 \\ s_1 \\ s_1 s_2 \\ s_2 s_1 \\ w_0
\end{array}
\begin{array}{c}
\begin{array}{cccccc} w_0 & s_2 s_1 & s_1 s_2 & s_1 & s_2 & e \end{array} \\
\begin{pmatrix}
1 & * & * & * & * & * \\
0 & 1 & * & * & * & * \\
0 & * & 1 & * & * & * \\
0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & * & 1 & * \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

$Z(\mathcal{L}_{\lambda, s_1 s_2, e})$ corresponds to the circled entry, which is one of the partition functions that we do not know yet (we only know the partition functions corresponding with entries of either 0 or 1).

Gelfand-Tsetlin Patterns

Definition.

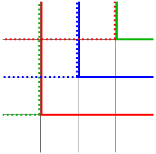
A **strict GT-pattern** is a triangular arrangement of non-negative integers

$$\begin{array}{ccccccc} x_{n,1} & & x_{n,2} & & \cdots & & x_{n,n} \\ & \ddots & & \ddots & & \iddots & \\ & & x_{2,1} & & x_{2,2} & & \\ & & & x_{1,1} & & & \end{array}$$

with the constraint that $x_{i+1,j} \leq x_{i,j} \leq x_{i+1,j+1}$ and $x_{i,j-1} < x_{i,j} < x_{i,j+1}$

Bijection. The numbers in each row record the columns with a color descending path

Example.



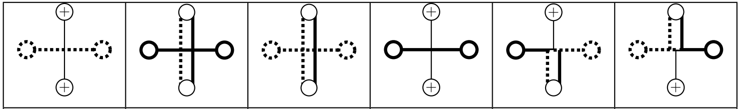
$$\Leftrightarrow \quad \begin{array}{ccccc} 3 & & 2 & & 1 \\ & 3 & & 2 & \\ & & 3 & & \end{array}$$

Alternating Sign Matrices

Definition.

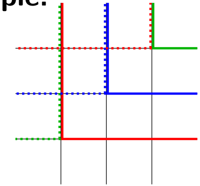
An **alternate sign matrix** is an $n \times n$ matrix with entries of $-1, 0, 1$ such that each column and row sum to 1, with the non-zero alternating sign entries.

Bijection between super-lattice vertices and ASM entries:

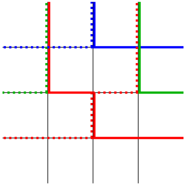


0	0	0	0	-1	1

Example.



$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



$$\Leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

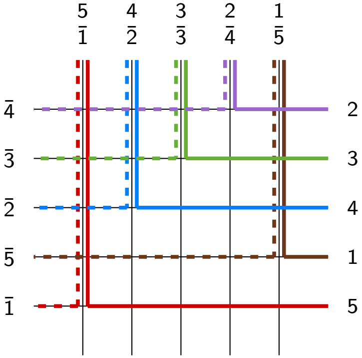
Simple Permutation Pairs

Proposition

For $w \in S_n$, $\mathcal{L}_{\lambda,w,w_0w}$ have a unique non-zero admissible state.

- These lattice models are in bijection with the permutation matrices.

Example.



$$\Longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

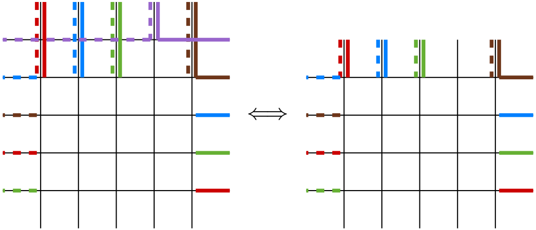
$(n-1) \times n$ and $n \times n$ lattices

Idea.

The color/scolor exiting at the first row of a square lattice model need to be coming from the same column.

Proposition.

Given top boundary λ, the number of admissible states of $n \times n$ lattice model $\mathcal{S}_\lambda$ equals to the number of admissible states of $(n-1) \times n$ lattice model $\mathcal{L}_{\lambda'}$ such that λ' has one pair of color and scolor decoration less then λ.



Criteria of non-zero admissible states

We use the previous ideas to make a final conjecture:

$$\begin{cases} \text{Alternating sign matrix} \implies \text{Uniqueness representation of each path} \\ \text{Simple permutation pairs} \implies \text{Foundation for train argument} \\ (n-1)\times n \text{ and } n\times n \text{ lattice} \implies \text{Enabled analysis on reduced dimension} \end{cases}$$

Remark: More specifically, we can take the idea from reducing n rows to $n-1$, and applied it more drastically and inductively, from n rows to lattice i rows with $i = n-1, ..., 2$; as given a $i \times n$ lattice model with an admissible states, we can always find a $j \times n$ lattice model such that $i \leq j$ which the states is contained by some admissible state of the bigger lattice model.

Thus with this in mind, we want to introduce our last result.

Criteria of non-zero admissible states

Idea: Given any lattice model with a fixed boundary condition (λ, w', w), we want to instantly justify whether it will or will not have any non-zero admissible state.

Define $s := \prod_{l=1}^{k} \sigma_l$ and $S_l := \{x \in \mathbb{Z}_n : \sigma_l(x) \neq x\}$, where σ_l are all disjoint permutations in S_n.

Conjecture

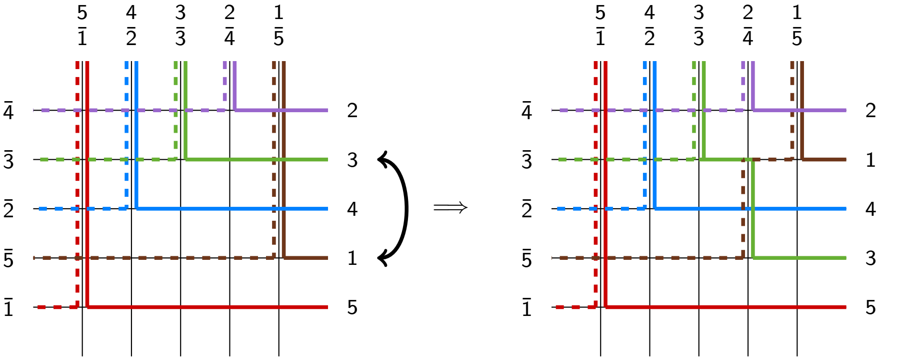
Given $\mathcal{L}_{(\lambda,w,w_0w)}$ has a non-zero admissible state, then $\mathcal{L}_{(\lambda,w,sw_0w)}$ has a non-zero admissible state if and only if

(i) $w_0w >_B sw_0w$ in strong Bruhat order; and

(ii) for all l, there exists $y \in S_l,\ y \neq \max(S_l)$ such that $y+1 \notin S_l$ and $y+1$ has exited.

Demonstration of the conjecture

	Simple permutation pair	**Extra transposition**
λ	$(5,4,3,2,1)$	$(5,4,3,2,1)$
$w'; sw$	$(\bar{1}\bar{5}\bar{4})(\bar{2}\bar{3}), e(1432)$	$(\bar{1}\bar{5}\bar{4})(\bar{2}\bar{3}), (13)(1432) = (12)(34)$



Recap

Questions.

- What is the partition function of the super-lattice Model?
- Are there any known combinatorial objects in bijection to these lattice models?

Results.

- Special cases with monostate
  - Gelfand-Tsetlin patterns, Alternating Sign Matrices
- Conjecture for non-vanishing states

Next Steps.

- Determine all boundary conditions with unique and multiple states in the square lattice model.
- Find the operator between two arbitrary partition functions for super-lattice models.

References

[1] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Vol. 231. Graduate Texts in Mathematics. Springer, 2005. DOI: 10.1007/3-540-27596-7.

[2] Ben Brubaker et al. *Kirillov's conjecture on Hecke-Grothendieck polynomials*. 2024. arXiv: 2410.07960 [math.CO]. URL: https://arxiv.org/abs/2410.07960.

Thank you!