Lattice Models for Quantum Superalgebras

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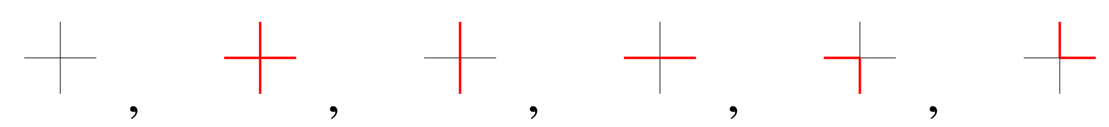
Introduction and Motivations

Lattice Model

Definition. A lattice model $\mathcal{L}$ is an $n \times m$ grid with its edges filled according to vertex table.

- We generally think of them as paths going from the top and exiting to the right
- They are indexed by external edges. Typically, we fix edge colorings at the top row with a partition $\lambda = (\lambda_1, ..., \lambda_k)$.
- The example below uses this vertex table:



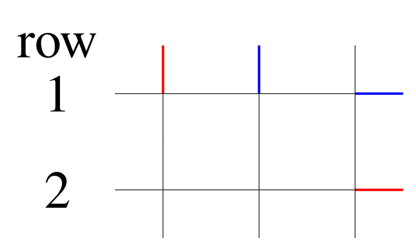
Partition Function

Definition. Given a lattice model with fixed boundary conditions, the partition function $\mathcal{Z}(\mathcal{L})$ of the lattice is the sum of all admissible states, which are paths with non-zero weight. Note: Non-specified vertices have weight 0.

Boundary Conditions

For models with multiple colors, we may also fix a permutation $w \in S_m$ on the side to indicate the order of colors from top to bottom.

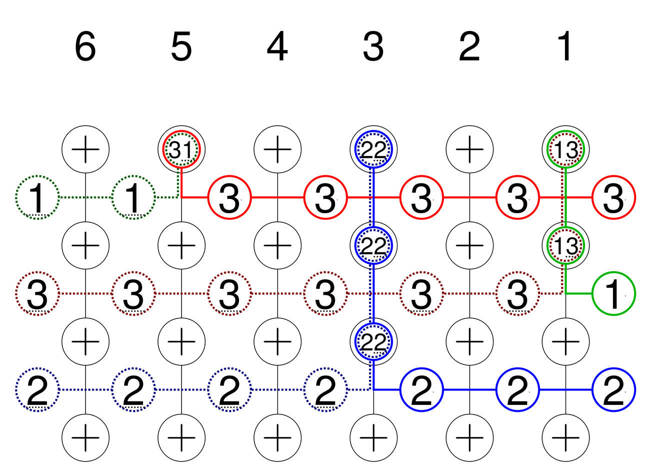
Example. Take $\lambda = (3, 2)$ and $w = (12)$



Super-Lattice Model

Definition: A super-lattice model is a lattice model indexed by one partition λ and two permutations $w, w' \in S_m$ such that it has both colored and dotted colored (supercolor) paths going in opposite directions.

Example. Below is an admissible state with $\lambda = (5, 3, 1)$, $w = (312)$, and $w' = (132)$



Motivating Questions

- What do the partition functions of these lattice $\mathcal{L}_{\lambda,w,w'}$ models look like?
- What combinatorial objects represent them?

Past work:

- Previous Polymath projects have "solved" models with one partition λ and one permutation w. [Bru+24]

Research Strategies

Goal. Compute the partition function of $\mathcal{L}_{\lambda,w,w'}$ for all $\lambda \vdash m$ and $w, w' \in S_n$.

First steps.

- Identify w, w' with $\mathcal{L}_{\lambda,w,w'} = 0$.
- Identify w, w' such that $\mathcal{L}_{\lambda,w,w'}$ has a unique admissible state.
- Compute remaining partition functions recursively by relating permutation index pairs (train argument).

Dream. Understand the quantum group module for super-lattice models

Question Rephrased

- What is the minimal set of states we need to compute to know all partition functions?
- And what do their partition functions look like?

Strategy

Vanishing Conjecture

For boundary conditions $(w, w_0 u)$, if $u = w$ then there is only one state and if $u < w$ then there are no states, where $<$ indicates strong Bruhat order and w_0 indicates the longest word.

Strong (full) Bruhat order on S_3 [BB05]

For $1 \le i < j \le 3$ let $(i\,j)$ be the transposition exchanging i and j. Given $u \in S_3$ we declare

$$u \;<\; (i\,j)\,u \quad\Longleftrightarrow\quad \ell\big((i\,j)u\big) \;=\; \ell(u)+1. \qquad (*)$$

The *strong Bruhat order* is the reflexive–transitive closure of this relation $(*)$; i.e. for $u, v \in S_3$

$$u \le v \iff \text{there exists a chain } u = w_0 < w_1 < \cdots < w_k = v,$$

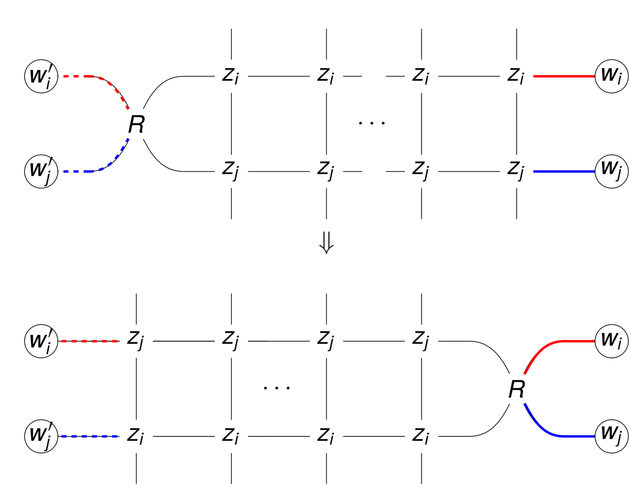
each step satisfying $(*)$.

There are $3! = 6$ elements, which we list by *length* $\ell(w) = \#\{\,(i<j) \mid w(i) > w(j)\,\}$ (the number of inversions).

Train argument for color/scolor model

Notation:

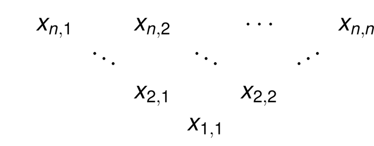
- w_i represents the **color** decorated at row i under permutation of w, and w'_i represents the **scolor** decorated at row i under permutation of w'.
- z_i and z_j marks the row number before and after the run-through of R-vertex.



Results and Conjectures

Gelfand-Tsetlin Patterns

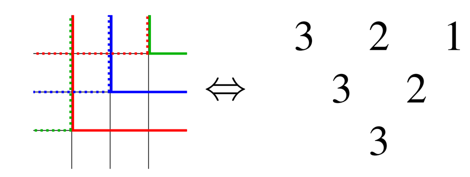
Definition. A **strict GT-pattern** is a triangular arrangement of non-negative integers:



with the constraint that $x_{i+1,j} \le x_{i,j} \le x_{i+1,j+1}$ and $x_{i,j-1} < x_{i,j} < x_{i,j+1}$.

Bijection. The numbers in each row record the columns with a color descending path.

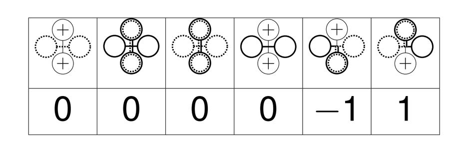
Example.



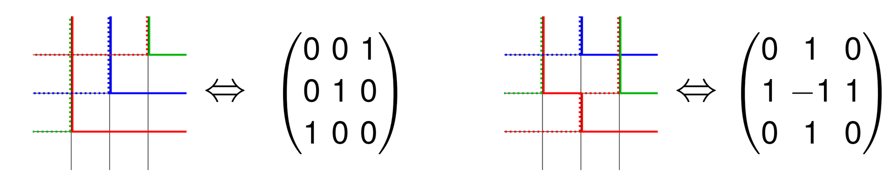
Alternating Sign Matrices

Definition. An **alternate sign matrix** is an $n \times n$ matrix with entries $-1, 0, 1$ s.t. each column and row sum to 1, with the non-zero alternating sign entries.

Bijection between super-lattice vertices and ASM entries:



Example.



Criteria of non-zero admissible states

Idea. Given lattice model with fixed boundary condition (λ, w', w), we wonder whether it has nonzero admissible state. Define $s := \prod_{l=1}^{k} \sigma_l$ and $S_l := \{x \in \mathbb{Z}_n : \sigma_l(x) \neq x\}$ of disjoint permutations in S_n.

Conjecture. Given $\mathcal{L}_{(\lambda,w,w_0w)}$ has a nonzero admissible state, then $\mathcal{L}_{(\lambda,w,sw_0w)}$ has a nonzero admissible state if and only if

(i) $w_0 w >_B s w_0 w$ in strong Bruhat order;
(ii) $\exists y \in S_l,\ y \neq \max(S_l)$ s.t. $y+1 \notin S_l$ and $y+1$ has exited.

Next Steps

- Determine all boundary conditions with unique and multiple states in the square lattice model.
- Find the operator between two arbitrary partition functions for super-lattice models.

References

[BB05] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Vol. 231. Graduate Texts in Mathematics. Springer, 2005. DOI: 10.1007/3-540-27596-7.

[Bru+24] Ben Brubaker et al. *Kirillov's conjecture on Hecke-Grothendieck polynomials*. 2024. arXiv: 2410.07960 [math.CO]. URL: https://arxiv.org/abs/2410.07960.

[Koj12] Takeo Kojima. "Diagonalization of transfer matrix of supersymmetry $U_q(\widehat{sl}(m+1|n+1))$ chain with a boundary". In: *arXiv: Exactly Solvable and Integrable Systems* (2012). URL: https://arxiv.org/abs/1202.2853.