Topics in Algebraic Geometry

November 5, 2025

Materials to be covered:

1. Lie algebra prerequisites
2. Representation of Lie algebras
3. Algebraic groups
4. Representation of algebraic groups
5. Algebraic geometry prerequisites
6. Flag varieties and Borel-Weil-Bott theorem

Contents

1 Localization, local rings

2 Graded structure, projective space

3 Zariski topology and algebraic varieties

We will discuss the affine and projective case separately. We start from the classical approach then generalize to the modern approach.

The subject of matter here are affine varieties. They live in affine spaces.

Definition 1 (affine spaces) *Given an algebraically closed field k, define the **affine space** as:*

$$\mathbb{A}_k^n := k^n = \{a = (a_1, \cdots, a_n) : a_i \in k \forall 1 \leq i \leq n\}.$$

Note that this is different from vector spaces as they do not have linear structures. It is only a topological space with the Zariski topology.

The affine varieties can be

Definition 2 (affine variety) *An affine algebraic subset of this space is defined by an ideal $I \lhd A$:*

$$Z(I) = \{a \in k^n : f(a) = 0 \forall f \in I\}.$$

Algebraic subsets satisfies properties ... , hence is inherited with Zariski topology.

For the modern approach, the varieties are generalized to affine schemes.

Definition 3 (affine schemes)

$$\mathbb{A}^n = \mathrm{Spec}(A).$$

The affine varieties are reflected by the closed points in affine schemes.

4 Algebraic sets and ring of regular functions

We will be working on rings $R = k[z_1, \cdots, z_n]$, where k an algebraically closed field.

On affine varieties.

Definition 4 (regular functions) *Given an affine variety $X \subset \mathbb{A}^n$ with Zariski topology τ, an open set $U \in \tau$, and a point $p \in U$, we say that a function $f : U \to K$ is **regular at** p if:*

$$\exists V \in \tau, V \ni p \qquad \exists g, h \in K[z_1, \cdots z_n], h(p) \neq 0 \qquad f = \frac{g}{h}.$$

*We say that f is **regular on** U if f is regular at every point of U.*

Example 1 (polynomials) *Every polynomial is regular everywhere on the affine plane, hence is always regular.*

Example 2 (coordinate restrictions) *Let $X = V(xy - 1) \subset \mathbb{A}^2$. Then the function:*

$$f(x, y) = \frac{1}{x}$$

is regular on the variety X, though not defined on all $\mathbb{A}^2$.

Example 3 *Consider $\mathbb{A}_k^1 = \mathrm{Spec}(k[z])$. The function*

$$f(z) = z$$

is not regular on the entire $\mathbb{A}_k^1$, but is regular on the open set $U := \mathbb{A}^2 \setminus \{x = 0\}$. This open set U forms a

We want to study the family where all regular functions lives.

Definition 5 *Given an affine variety $X \subset \mathbb{A}^n$, define the **ideal** of X as:*

$$I(X) = \{f \in K[z_1, \cdots, z_n] : f(X) = 0\}.$$

*Then, the **coordinate ring** of X is defined as:*

$$A(X) = K[z_1, \cdots, z_n]/I(X).$$

We would show that the coordinate ring of X is exactly the ring of regular functions regular at every point of X.

Proposition 1 *The regular functions on X form a ring, which is exactly the coordinate ring $A(X)$.*

5 Intersection Theory

5.1 Dimension, hypersurfaces and linear system

5.2 Multiplicity and intersection multiplicity

5.3 Bezout's theorem and Chow ring

6 Algebraic groups

7 References:

1. Humphreys, Lie algebras and representation theory, GTM9
2. Fulton, Harris, representation theory, GTM129
3. Milne, algebraic groups
4. Springer, linear algebraic groups
5. Lakshmibai, Brown, flag varieties
6. Brion, lectures on the geometry of flag varieties