2.1.1. Show that giving the quotient topology makes the natural surjection π an open mapping.

Pf: Recall $f: X \to Y$ open mapping refers to

$$\forall U \in \tau_X \quad f(U) \in \tau_Y .$$

Given $\pi: \mathcal{H} \to Y(\Gamma)$ the ~~quotient~~ natural surjection,

$$\tau \mapsto \Gamma\tau$$

$\mathcal{H}$ has ~~standard~~ Euclidean topology $\tau_{\mathcal{H}} = \langle \{(a_1,b_1)\times(a_2,b_2) : a_1, a_2 \in \mathbb{R},\ b_1, b_2 \in \mathbb{R}_+\} \rangle$

the quotient topology on $Y(\Gamma)$ is defined as

$$\mathcal{T}_{Y(\Gamma)} = \{ V \subset Y(\Gamma) : \pi^{-1}(V) \in \tau_{\mathcal{H}} \}.$$

which makes π into a quotient map

i.e. $V \in \mathcal{T}_{Y(\Gamma)}$ iff $\pi^{-1}(V) \in \tau_{\mathcal{H}}$.

Now π continuous surjective.

$\forall U \in \tau_{\mathcal{H}}$ ~~$\pi(U) \in \mathcal{T}_{Y(\Gamma)}$. hence $\pi^{-1}(\pi(U)) \in \tau_{\mathcal{H}}$.~~

to show $\pi(U) \in \mathcal{T}_{Y(\Gamma)}$ i.e. $\pi^{-1}(\pi(U)) \in \tau_{\mathcal{H}}$.

By definition, $\pi^{-1}(\pi(U)) = \{\tau : \pi(\tau) \in \pi(U)\}$

$= \{\tau : \Gamma\tau = \Gamma z \ \exists z \in U\} = \{\tau : \gamma\tau = z \ \exists z, \gamma\}$

$= \{\tau : \Gamma\tau \cap U \neq \emptyset\} = \bigcup_{\gamma \in \Gamma} \gamma U.$

We need $\gamma U \in \tau_{\mathcal{H}}$ $\forall \gamma \in \Gamma$.

Since $\gamma \in \Gamma \subset SL_2(\mathbb{Z})$ γ is a Möbius transformation, which is holomorphic and bijective. By open mapping theorem in complex analysis: nonconstant + holomorphic $\Rightarrow$ open

γ is open.

2.1.2 Establish equivalence (2.1):

$$\pi(U_1)\cap\pi(U_2)=\phi \text{ in } Y(\Gamma)^* \iff \Gamma(U_1)\cap U_2=\phi \text{ in } \mathcal{H}.$$

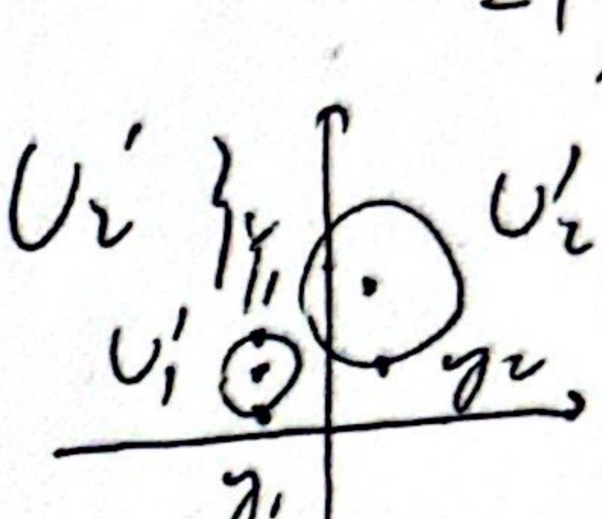
Pf: $\pi: \mathcal{H}\to Y(\Gamma)$. $\pi(U)=\{\Gamma\tau : \tau\in U\}$.

$$\pi(U_1)\cap\pi(U_2)=\phi \iff \{\Gamma\tau_1=\Gamma\tau_2 \quad \tau_1\in U_1,\ \tau_2\in U_2\}=\phi.$$

$$\iff \{\Gamma\tau_1=\tau_2 \quad \tau_1\in U_1,\ \tau_2\in U_2\}=\phi \iff \Gamma(U_1)\cap U_2=\phi.$$

2.1.3. (a) Establish ~~equivalence~~ inequality (2.2): for all but finitely gcd(c,d) = 1,

$$\sup\{\mathrm{Im}\,\gamma(\tau) : \gamma\in SL_2(\mathbb{Z}),\ \tau\in U_1'\} < \inf\{\mathrm{Im}\,\tau : \tau\in U_2'\}$$



Pf: Let $y_i=\inf\{\mathrm{Im}(\tau):\tau\in U_i'\}$, $Y_i=\sup\{\mathrm{Im}(\tau):\tau\in U_i'\}$

Then $\tau\in U_1' \Rightarrow \mathrm{Im}\,\gamma(\tau)=\dfrac{\mathrm{Im}\,\tau}{|c\tau+d|^2}\le\min\left\{\dfrac{1}{c^2y_1},\ \dfrac{Y_1}{(c\mathrm{Re}\,\tau+d)^2}\right\}$

$$=\frac{y}{(cx+d)^2+(cy)^2}$$

Examine $\dfrac{1}{c^2\underset{\text{const}}{y_1}} < \underset{\text{const}}{y_2}$ for all but finitely many c since $c\in\mathbb{Z}$.

$Y_1 \leftarrow$ const

$\dfrac{Y_1}{(c\mathrm{Re}\,\tau+d)^2}<\underset{\text{const}}{y_2}$ uniformly in τ ($\forall c\ \exists\{d\}$ ineq holds except for $\{d\}$) for all but finitely many d

(b) Show $\gamma=\begin{pmatrix}*&*\\c&d\end{pmatrix}\in SL_2(\mathbb{Z}) \Rightarrow \gamma=\begin{pmatrix}1&k\\0&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $ad-bc=1$.

Pf: If $\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}*&*\\c&d\end{pmatrix}$. Then $\begin{cases}\cancel{d}c'a+d'c=c\\ \cancel{b}c'b+d'd=d\end{cases}$

$\Rightarrow c'=0,\ d'=1.$

$\begin{vmatrix}a'&b'\\c'&d'\end{vmatrix}=1$ ~~$\in SL_2\mathbb{Z}$~~ $\Rightarrow a'=1.$

2.1.3 (*). Show only finitely many $\gamma \in SL_2(\mathbb{Z})$ $\gamma(U_i) \cap U_i \neq \emptyset$.

Pf: We study point i. (Step 1) To show there is a subgroup

$$SO_2(\mathbb{R}) \subset SL_2(\mathbb{R}).$$

By definition, $SO_2(\mathbb{R}) = \{ X \in M_2(\mathbb{R}) : XX^T = I, \det X = 1 \} \subset SL_2(\mathbb{R})$. ~~$\Rightarrow \det X^2 = 1 \Rightarrow \det X = 1$~~

(Step 2) To show this is the subgroup fixing i.

$\forall X \in SO_2(\mathbb{R})$ $\quad X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad Xi = \dfrac{ai+b}{ci+d}$

$$= \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{ac+bd+(ad-bc)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{1}{c^2+d^2} i.$$

$$XX^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} = I.$$

$\Rightarrow \begin{cases} a^2+b^2 = c^2+d^2 = 1. \\ ac+bd = 0. \end{cases}$ Thus $Xi = i$.

(Step 3) Note $\forall \tau \in \mathcal{H}$ $\exists \gamma \in SL_2(\mathbb{R})$ $\gamma i = \tau$.
$\tau = x+iy$

since $s: \mathcal{H} \to SL_2(\mathbb{R})$

$$\tau \mapsto \frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

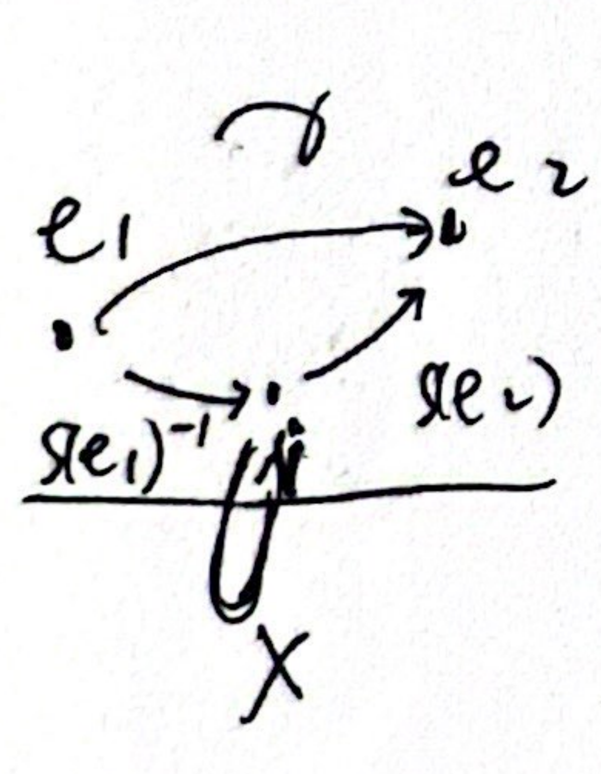
why $\frac{1}{\sqrt{y}}$: $s(\tau) \in SL_2(\mathbb{R})$, $\det s(\tau) = 1$

gives $s(\tau)\, i = \dfrac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = \begin{pmatrix} \frac{y}{\sqrt{y}} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} i$

$$= \frac{\frac{y}{\sqrt{y}} i + \frac{x}{\sqrt{y}}}{\frac{1}{\sqrt{y}}} = x + yi = \tau.$$

(On the other hand $s(\tau)^{-1}\tau = i$)

Thus $s: \mathcal{H} \cong SL_2(\mathbb{R}) / SO_2(\mathbb{R})$.



Thus $\forall \gamma \in SL_2(\mathbb{R})$ $\forall e_1, e_2 \in \mathcal{H}$.

$\gamma(e_1) = e_2 \Leftrightarrow \gamma = s(e_2) X s(e_1)^{-1} \Leftrightarrow \gamma \in s(e_2) SO_2(\mathbb{R}) s(e_1)^{-1}$

$X \in SO_2(\mathbb{R})$ X could be identity.

Let e_1 range over $\overline{U_i}$. Then $\gamma(U_i) \cap U_2 \neq \emptyset$

$\Rightarrow \gamma \in$ compact group, $K = \{ s(e_2) SO_2(\mathbb{R}) s(e_1)^{-1} ;\ e_1 \in \overline{U_i},\ e_2 \in \overline{U_i} \}$

Also, $\gamma \in SL_2(\mathbb{Z})$ discrete. Thus there are only finite many of them.