Draft of Survey on Question 2 (question 1 in the report)

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1 Question 2

How do models with colors/dotted-colors change when we change quantum group module? This question is a bit hard to tackle for now so I'll suggest one to get your hand dirty with Q1 and Q3 first before our next meeting/office hour.

Contents

1.1 Basic languages

References to be added on Bump, Kac, Kassel, Majid.

Contents: 1.1.1 lattice models, 1.1.2 Hopf algebras, 1.1.3 Drinfeld theory, 1.1.4 $U_q(\mathfrak{sl}(2))$-modules, 1.1.5 affine Lie algebras, 1.1.6 superalgebras, 1.1.7 Hecke algebras, 1.1.8 induced representations.

1.1.1 lattice models

Definition 1 (R-matrix)**.** *Given objects A, B in a monoidal category, an R-**matrix** $\mathcal{R}$ is a braid morphism $c_{A,B} : A \otimes B \to B \otimes A$ such that*

$$c_{A,(B\otimes C)} = (\mathbb{1}_B \otimes c_{A,C}) \circ (c_{A,B} \otimes \mathbb{1}_C),$$
$$c_{(A\otimes B),C} = (c_{A,C} \otimes \mathbb{1}_B) \circ (\mathbb{1}_A \otimes c_{B,C}).$$

The Yang-Baxter equation can be written in terms of R-matrices as

$$(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1}) = (\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R}).$$

Proposition 1. *The Yang-Baxter equation always holds in braided monoidal categories.*

1.1.2 Hopf algebras

The Hopf algebra is a old fashioned name for quantum groups. The reason why we study them is because under a Hopf algebra structure, we are able to decompose or invert interactions by Δ, ϵ, S to be introduced later, which helps solving the Yang-Baxter equation.

Definition 2 (Hopf algebra)**.** *A **Hopf algebra** is a bialgebra with an antipode satisfying the Hopf structure. To be specific, given a field k, a Hopf algebra H is a k-vector space, a k-algebra, and together with the following properties satisfied in the commutative diagrams:*

*coassociativity of **coproduct** Δ :*

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "\mathbb{1}\otimes\Delta"] \\ H \otimes H \arrow[r, "\Delta\otimes\mathbb{1}"] & H \otimes H \otimes H \end{tikzcd},$$

***counit** ϵ :*

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[dr, "\simeq"'] & H \otimes H \arrow[d, "\epsilon\otimes\mathbb{1}"] \\ & k \otimes H \end{tikzcd}, \quad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[dr, "\simeq"'] & H \otimes H \arrow[d, "\mathbb{1}\otimes\epsilon"] \\ & H \otimes k \end{tikzcd},$$

***antipode** S :*

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\eta"] & H \otimes H \arrow[r, "\mathbb{1}\otimes S"] & H \otimes H \arrow[d, "\mu"] \\ H \arrow[rr, "\epsilon"] & & H \end{tikzcd}, \quad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\eta"] & H \otimes H \arrow[r, "\mathbb{1}\otimes S"] & H \otimes H \arrow[d, "\mu"] \\ H \arrow[rr, "\epsilon"] & & H \end{tikzcd}$$

Hopf structure:

$$\begin{tikzcd} H \otimes H \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\mu"] & H \otimes H \otimes H \otimes H \arrow[r, "\mathbb{1}\otimes\tau\otimes\mathbb{1}"] & H \otimes H \otimes H \otimes H \arrow[d, "\mu\otimes\mu"] \\ H \arrow[rr, "\Delta"] & & H \otimes H \end{tikzcd}$$

where μ is the product, η is the unit, and τ is the flip, all perform as k-algebra homomorphisms.

*A k-vector space only with the properties being coassociative and counit is called a **coalgebra**, while a coalgebra that is also a k-algebra is called a **bialgebra**.*

Let H_1, H_2 be Hopf algebras. Given a bilinear form $\langle \cdot, \cdot \rangle : H_1 \times H_2 \to k$, we say that H_1 and H_2 are in **duality** if

$$\begin{aligned} \langle uv, x \rangle &= \sum_x \langle u, x' \rangle \langle v, x'' \rangle, \quad \langle 1, x \rangle = \epsilon(x), \\ \langle u, xy \rangle &= \sum_u \langle u', x \rangle \langle u'', y \rangle, \quad \langle u, 1 \rangle = \epsilon(u), \\ \langle S(u), x \rangle &= \langle u, S(x) \rangle. \end{aligned}$$

Proposition 2. *Consider arbitrary H and $U(\mathfrak{g})$ as Hopf algebras, define*

$$\begin{aligned} \varphi : U(\mathfrak{g}) \to H^* \quad & u \mapsto (x \mapsto \langle u, x \rangle), \\ \psi : H \to U(\mathfrak{g})^* \quad & x \mapsto (u \mapsto \langle u, x \rangle), \end{aligned}$$

then we have the following equivalence:

$$U(\mathfrak{g}), H \text{ are dual to each other} \iff \varphi, \psi \text{ are } k\text{-algebra homomorphisms.}$$

Commutativity, Cocommutativity

Example 1 (algebras over a finite group)**.** *Given a finite group G, $\mathbb{C}[G]$ and $\mathcal{O}(G)$ are dual to each other.*

Example 2 (Lie group G and universal enveloping algebra $U(\mathfrak{g})$)**.** *Under this definition,*

$$\Delta X = X \otimes 1 + 1 \otimes X.$$

Example 3 (q-deformed Lie group G_q and quantized enveloping algebra $U_q(\mathfrak{g})$)**.**

Example 4 (quantized positive Borel subalgebra $U_q(\mathfrak{b}_+)$)**.** *This Hopf algebra is generated by* $1, X, g, g^{-1}$ *with relations*

$$gg^{-1} = 1 = g^{-1}g, \quad gX = qXg,$$

where $q \in k$ *is fixed and invertible. Under this definition,*

$$\Delta X = X \otimes 1 + g \otimes X.$$

To relate with the previous example 3, this is a subalgebra of $U_q(\mathfrak{g})$*, and* $\mathfrak{b} = \boldsymbol{Lie}(B)$ *is the Borel subalgebra of* $\mathfrak{g}$*. Its dual Hopf algebra is the* ***quantum coordinate algebra*** $\mathcal{O}_q(B_-)$*, which contains functions on the negative Borel subgroup.*

We list some useful properties of Hopf algebras.

Proposition 3. *The category of Hopf algebras is monoidal. The category of modules on quantized enveloping algebras of finite dimension is braided.*

Proposition 4. *The dual vector space of a coalgebra is an algebra. The dual vector space of* H *has a coalgebra structure.*

1.1.3 Drinfeld theory

Drinfeld's insight is that Hopf algebras can generate R-matrices systematically.

Definition 3 (quasitriangularity)**.** *A Hopf algebra* H *is called* ***quasitriangular*** *if there is an element* $\mathcal{R} \in H \otimes H$ *sastisfying*

$$(\Delta \otimes \mathbb{1})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathbb{1} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12},$$

and the ***quantum Yang-Baxter equation:***

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Such an element $\mathcal{R}$ *is called the* ***universal*** R***-matrix****.*

These extra structures ensures that H-representation give rise to solutions of the YBE.
Quantization of Lie algebras.
Drinfeld-Jimbo construction.
Quantum groups $U_q(\mathfrak{g})$ are neither commutative, nor cocommutative. Δ is noncommutative.

1.1.4 $U_q(\mathfrak{sl}(2))$-modules

These are important for this project since in most cases we are working with six-vertex lattice models, in which we have the R-matrices $\mathcal{R} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, hence $\mathcal{R}$ comes from the quasitriangular structure of $U_q(\mathfrak{sl}(2))$.

Why studying H-modules is important.

Proposition 5. *Let* H *be a Hopf algebra, and* U, V *are* H*-modules, then*

1. $U \otimes V$ *is an* $H \otimes H$*-module.*
2. *if* H *is cocommutative, then the flip* $\tau_{U,V} : U \otimes V \xrightarrow{\sim} V \otimes U$ *is an* H*-module isomorphism.*

Definition 4 (quantum plane)**.** *Given* $q \in k^\times$*, the quantum plane is defined as a* k*-algebra*

$$k_q[x, y] := k\{x, y\}/(yx - qxy).$$

Quantum planes are noncommutative if $q \neq 1$.

Action of $\mathfrak{sl}(2)$ on the affine plane. Given an affine plane $\mathbb{A}^2 = k[x, y]$, we give a graded structure on it by setting $k[x, y]_n$ as the k-vector subspace of homogeneous polynomials of total degree n. Then there exist a duality that $k[x, y]_n^* \simeq V(n)$ is a $SL(2)$-module, where $V(n)$ denotes the simple $U(\mathfrak{sl}(2))$-module of $\dim = n + 1$, generated by a highest weight vector of weight n. Thus $k[x, y]_n^*$ is also a $U(\mathfrak{sl}(2))$-module.

Action of $U_q(\mathfrak{sl}(2))$ on the quantum plane.
Different cases of $U_q(\mathfrak{sl}(2))$-modules.
The Hopf algebras $U(\mathfrak{sl}(2))$ and $SL(2)$ are dual to each other.
Especially when $\mathfrak{g} = \mathfrak{sl}(2)$,

$$\begin{aligned} U_q(\mathfrak{sl}(2)) = k[E, F, K, K^{-1}]/(KK^{-1} - 1, \quad & K^{-1}K - 1, \\ & KEK^{-1} - q^2E, \quad KFK^{-1} - q^{-2}F, \\ & [E, F] - \frac{K - K^{-1}}{q - q^{-1}}). \end{aligned}$$

There exists a unique Cartan automorphism $\omega \in \mathrm{Aut}(U_q(\mathfrak{sl}(2)))$, such that

$$\omega(E) = F, \qquad \omega(F) = E, \qquad \omega(K) = K^{-1}.$$

1.1.5 affine Lie algebras

These algebras are given affine symmetries, so that they correspond to some symmetric properties in solvability of lattice models.

Definition 5. *Given a finite dimensional simple Lie algebra $\mathfrak{g}$, the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$ is constructed as doing central extension by $\mathbb{C}[t, t^{-1}]$, then composing with one dimensional center $\mathbb{C}c$. Explicitly,*

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \cdot c, \qquad [a \otimes t^n + \alpha c, b \otimes t^m + \beta c] := [a, b] \otimes t^{n+m} + \langle a|b \rangle n \delta_{m+n,0} c,$$

*where $\langle \cdot | \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \quad (x, y) \mapsto \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$ is the **Killing form** on $\mathfrak{g}$.*
In other words, the affine Lie algebras satisfy the following exact sequence:

$$0 \to \mathbb{C} \cdot c \to \widehat{\mathfrak{g}} \to \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \to 0.$$

More on Killing form: Dynkin index and the dual Coxeter number.

Example 5 ($\widehat{\mathfrak{sl}}(n)$)**.**

Example 6 (Heisenberg algebra $\widehat{\mathfrak{u}}(1)$)**.**

$$[a_m, a_n] := m\delta_{m+n,0}c.$$

Structure of affine Lie algebras: Cartan-Weyl basis, affine Killing form, affine root system.
The Weyl group of affine Lie algebras: Weyl-Kac character formula.
Representation of affine Lie algebras: Verma modules.

1.1.6 superalgebras

We study them to encounter cases when there are multiple scolors. The solid lines are bosonic (nonsuper), while the dotted are fermionic (super).

Definition 6 (Lie superalgebra)**.** *Given a Lie algebra $\mathfrak{g}$ with both commutators and anticommutators*

$$[X, Y] = XY - (-1)^{|X||Y|}YX,$$

where $|X| \in \{0, 1\}$ is the parity, giving $0 =$ bosonic and $1 =$ fermionic.

Lie superalgebras are $\mathbb{Z}_2$-graded.
The super R-matrices satisfies the **graded YBE**:

$$(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1}) = (-1)^{|\mathcal{R}|}(\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R}).$$

Example 7 ($\mathfrak{sl}(m|n)$)**.**

Definition 7 (realization)**.** *A matrix $A \in M_n(\mathbb{C})$ is called a* ***generalized Cartan matrix*** *if*

$$1)\ a_{ii} = 2, \qquad 2)\ a_{ij} \le 0, \qquad 3)\ a_{ij} = 0 \implies a_{ji} = 0, \qquad \forall 1 \le i \ne j \le n.$$

Given a generalized Cartan matrix A, a ***realization*** *of A is a $\mathbb{C}$-vector space $\mathfrak{h}$, together with sets $\Pi := \{\alpha_1, \cdots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{\alpha_1^\vee, \cdots, \alpha_n^\vee\} \subset \mathfrak{h}$, such that*

1. *Π and $\Pi^\vee$ are both linear independent,*
2. *$\langle \alpha_i^\vee, \alpha_i \rangle = a_{ij}$,*
3. *$n - \operatorname{rank} A = \dim \mathfrak{h} - n$.*

Two realizations $\mathfrak{h}_1, \mathfrak{h}_2$ are isomorphic if there exists a $\mathbb{C}$-vector space isomorphism $\phi : \mathfrak{h}_1 \xrightarrow{\sim} \mathfrak{h}_2$ such that $\phi(\Pi_1^\vee) = \Pi_2^\vee, \phi^*(\Pi_2) = \Pi_1$. Here α_i are the simple roots and $\alpha_i^\vee$ are the simple coroots.

Definition 8 (Kac-Moody algebra)**.** *The* ***Kac-Moody algebra*** *is the Lie algebra $\mathfrak{g}(A)$ with A a generalized Cartan matrix. Here, $\mathfrak{g}(A)$ is defined as*

$$\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r},$$

where $\mathfrak{r} \lhd \tilde{\mathfrak{g}}(A)$ is a maximal ideal, and $\tilde{\mathfrak{g}}(A)$ the ***auxiliary Lie algebra*** *with $A \in M_n(\mathbb{C})$ its* ***Cartan matrix*** *is defined as*

$$\tilde{\mathfrak{g}}(A) := \mathbb{C}(\{e_i, f_i\} \cup \mathfrak{h})/([e_i, f_j] - \delta_{ij}\alpha_i^\vee, \quad [h, h'], \quad [h, e_i] - \langle \alpha_i, h \rangle e_i, \quad [h, f_i] - \langle \alpha_i, h \rangle f_i).$$

These equivalent relations are called the ***Cartan-Kac*** *relations, the generators e_i, f_i are called the* ***Chevally generators****, while $\mathfrak{h}$ is called the* ***Cartan subalgebra*** *of $\mathfrak{g}(A)$.*

The Kac-Moody superalgebras are affinizations of Lie superalgebras.

Example 8 ($\widehat{\mathfrak{sl}}(m|n)$ and $U_q(\widehat{\mathfrak{sl}}(m|n))$)**.**

Proposition 6. *Given a Cartan matrix A,*

1. *(well-defined) $\mathfrak{g}(A)$ depends only on A,*
2. *(triangular decomposition) $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$,*
3. *(Poincaré-Birkhoff-Witt theorem) $\mathfrak{n}_-$ is freely generated by $f_1, \cdots, f_n$,*
4. *(root space decomposition) denote $Q := \sum_{i=1}^n \mathbb{Z}\alpha_i, Q_\pm = \sum_{i=1}^n \mathbb{Z}_\pm \alpha_i$ the root lattices,*

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha, \qquad \mathfrak{h} = \bigoplus_{\alpha : \mathrm{ht}\alpha = 0} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_\pm = \bigoplus_{j \ge 1} \bigoplus_{\alpha : \mathrm{ht}\alpha = \pm j} \mathfrak{g}_\alpha.$$

1.1.7 Hecke algebras

1.1.8 induced representations

Definition 9. *Given a finite group G and its subgroup H, let (π, V) be a representation of V, then the induced representation $(\mathrm{Ind}_H^G(\pi), V^G) = (\pi^G, V^G)$ is defined as*

$$V^G := \{f : G \to V : \quad f(hg) = \pi(h)f(g), \forall h \in H\},$$
$$\mathrm{Ind}_H^G(\pi) = \pi^G : G \to \mathrm{GL}(V^G) \qquad \pi^G(g)(f) : x \mapsto f(xg).$$

Some important results:
On Borel subgroups: $\mathrm{Ind}_B^G(\chi)$ gives classification of $\mathrm{GL}(2, \mathbb{F}_q)$-representations.
Frobenius reciprocity: $\mathrm{Hom}_G(U, V^G) \simeq \mathrm{Hom}_H(U_H, V), \mathrm{Hom}_G(V^G, U) \simeq \mathrm{Hom}_H(V, U)$.
Mackey's theorem: $\mathrm{Hom}_G(V_1^G, V_2^G) \simeq \{\Delta : G \to \mathrm{Hom}(V_1, V_2) : \quad \Delta(h_2 g h_1) = \pi_2(h_2) \circ \Delta(g) \circ \pi_1(h_1)\}$.

Branching rules Branching rules tells the multiplicity of an irreducible H-representation W in V.

For example $U(n) \downarrow U(n-1)$, $[\cdot : \cdot]$ denotes multiplicity:

$$\pi_\lambda^{U(n)} \restriction_{U(n-1)} = \bigoplus_\mu [\pi_\lambda^{U(n)} : \pi_\mu^{U(n-1)}] \pi_\mu^{U(n-1)},$$

$$[\pi_\lambda^{U(n)} : \pi_\mu^{U(n-1)}] = 1 \iff \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n.$$

Cauchy identity

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

Schur polynomials:

$$s_\lambda : (x_1, \cdots, x_n) \mapsto \operatorname{tr}(\pi_\lambda \operatorname{diag}(x_1, \cdots, x_n)),$$

where π_λ is holomorphic irreducible representation on $\mathrm{GL}_n(\mathbb{C})$.

Holomorphic representation: a $\mathrm{GL}_n(\mathbb{C})$-representation with $g \mapsto g \cdot v$ holomorphic on G for every v.

1.2 Question restatement

Given different Hopf algebras H, we want to find the H-modules, which inherit H-representations and R-matrices $\mathcal{R}$.

R-matrices $\mathcal{R}$ acts on $V \otimes V$ as an intertwiner, where $V = \mathbb{C}\{+,-\} \simeq \mathbb{C}^2$ a $\mathbb{C}$-vector space containing local states $+$ and $-$. Thus $V \otimes V \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$. This means $\mathcal{R}$ lives in some H-module, where for six-vertex models, $H = U_q(\mathfrak{sl}_2)$.

For lattice models with three colors (RGB) in two scolors (solid and dotted) as well as uncolored $(+)$, we would want to discuss $U_q(\mathfrak{sl}(4|3))$-modules that are actions on $V \otimes V \simeq \mathbb{C}^{4|3} \otimes \mathbb{C}^{4|3}$ of dimension 49, and discuss its graded intertwiner element $\mathcal{R}$.

1.3 Past results

We list some related results here.

Awata, Noumi, Odake, 1993, studied the Heisenberg realization of $U_q(\mathfrak{sl}_n)$. arXiv:hep-th/9306010

Awata, Odake, Shiraishi, 1993, studied the free Boson (Boson field) representation of $U_q(\widehat{\mathfrak{sl}_3})$. arXiv:hep-th/9305017

Awata, Odake, Shiraishi, 1993, studied the free Boson realization of $U_q(\widehat{\mathfrak{sl}_N})$. arXiv:hep-th/9305146

Awata, Odake, Shiraishi, 1997, studied the q-difference realization of $U_q(\mathfrak{sl}(M|N))$, and application to free Boson realization of $U_q(\widehat{\mathfrak{sl}}(2|1))$. arXiv:q-alg/9701032

Kojima, 2011, studied the free field realization of $U_q(\widehat{\mathfrak{sl}}(N|1))$. arXiv:1105.5772

Kojima, 2012, studied the bosonization of $U_q(\widehat{\mathfrak{sl}}(N|1))$ for level $k \neq -N+1$. arXiv:1211.2909

Kojima, 2014, studied the bosonization of $U_q(\widehat{\mathfrak{sl}}(N|1))$ at any level. arXiv:1404.5744

Kojima, 2017, studied a bosonization of $U_q(\widehat{\mathfrak{sl}}(M|N))$. arXiv:1701.03645

Kojima, 2018, studied the Wakimoto realization of $U_q(\widehat{\mathfrak{sl}}(M|N))$. arXiv:1807.02269

Other papers often used in the program, not yet sure how they are related to the above ones:

Kojima, 2018, DIAGONALIZATION OF TRANSFER MATRIX OF SUPERSYMMETRY $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$ CHAIN WITH A BOUNDARY, arXiv:1211.2912

Brubaker, Buciumas, Bump, Henrik, Gustafsson, 2020, METAPLECTIC IWAHORI WHITTAKER FUNCTIONS AND SUPERSYMMETRIC LATTICE MODELS, arXiv:2012.15778

Aggarwal, Borodin, Wheeler, 2021, Colored Fermionic Vertex Models and Symmetric Functions, arXiv:2101.01605

1.4 New results

Some thoughts:

Is the general bosonization of $U_q(\widehat{\mathfrak{sl}}(M|N))$ found?

Kojima's current focus in W-algebras, what are those.