Survey in Algebraic Groups

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1 Algebraic geometry revised

We want to study geometric objects algebraically. The subject of matter here are varieties, or schemes. This could then give geometric information for algebraic objects.

We start from the classic approach then generalize to the modern approach. In the classic approach that concerns varieties, we will discuss the affine and projective cases separately.

Some great textbooks here includes [2], [4], [6], and [3].

1.1 Affine varieties

Affine varieties live in affine spaces.

We start with a ring $A = k[x_1, \cdots, x_n]$ where k algebraically closed, with reason to be explained later. The affine space corresponding to A is defined as follows.

Definition 1 (affine spaces)**.** *Given an algebraically closed field k, define the **affine space** as:*

$$\mathbb{A}^n_k = \mathbb{A}^n := k^n = \{a = (a_1, \cdots, a_n) : a_i \in k \ \forall 1 \leq i \leq n\}.$$

Note that this is different from vector spaces as they do not have linear structures. It is only a topological space with the Zariski topology.

Definition 2 (affine variety)**.** *An affine algebraic set of the space $\mathbb{A}^n$ is defined by an ideal $I \lhd A$ as:*

$$Z(I) = \{a \in \mathbb{A}^n : f(a) = 0 \ \forall f \in I\}.$$

An affine algebraic variety is a irreducible component of an algebraic set.

As the ring $k[x_1, \cdots, x_n]$ is Noetherian, ideals are always finitely generated, thus:

$$Z[I] = Z[f_1, \cdots, f_m] = \{z \in \mathbb{A}^n : f_i = 0 \ \forall i = 1, \cdots, m\},$$

for some chosen generators $f_1, \cdots, f_m$.

Algebraic subsets satisfies properties ... , hence is inherited with Zariski topology.

We especially need the ring to be a polynomial ring over algebraically closed field, since they have strong properties.

Proposition 1. *Given a ring $k[x_1, \cdots, x_n]$ with k algebraically closed, its maximal ideals are of the form:*

$$\mathfrak{m}_a = (x_1 - a_1, \cdots, x_n - a_n), (a_i) \in \mathbb{A}^n.$$

Now that we have spaces and varieties as geometric objects, we would want to know the functions defined on these objects.

Definition 3 (coordinate ring)**.** *Given an affine variety $X \subset \mathbb{A}^n$, define the **ideal** of X as:*

$$I(X) = \{f \in k[x_1, \cdots, x_n] : f(X) = 0\}.$$

*Then, the **coordinate ring** of X is defined as:*

$$A(X) = A/I(X) = k[x_1, \cdots, x_n]/I(X).$$

The ring $A(X)$ would be the set of all functions, to be shown later.

1.2 Projective varieties

The projective varieties live in projective spaces.

Given algebraically closed field k, we would start with the graded ring $S = k[x_0, \cdots, x_n]$.

Definition 4. *Given a field k, algebraically closed, the projective space is defined as:*

$$\mathbb{P}_k^n = k\mathbb{P}^n := \{[a_n, \cdots, a_n] : a_i \in k, a_0 \neq 0\} = \{(a_n, \cdots, a_n) : a_i \in k, a_0 \neq 0\}/\sim,$$

where $[a_0, \cdots, a_n]$ is a equivalent class, and $(a_0, \cdots, a_n) \sim (b_0, \cdots, b_n)$ iff:

$$\frac{a_i}{a_0} = \frac{b_i}{b_0}, \qquad \forall i = 1, \cdots, n.$$

This is to say that two elements are viewed as the same if they have the same tangent.

Similarly, we have algebraic sets in $\mathbb{P}^n$, so we have **projective varieties** Y defined as irreducible projective sets.

Given a graded ring S, the homogeneous ideals are ideals generated by homogeneous functions vanishing at some point in Y. We also have projective coordinate rings:

$$S(Y) := S/I(Y) = k[x_0, \cdots, x_n]/I(Y).$$

1.3 Modern approach

For the modern approach, we start with any commutative unital ring A. The affine planes are generalized to affine schemes. But at first the schemes are only topological spaces.

Definition 5 (spectrum space)**.**

$$\mathbb{A}^n = \mathrm{Spec}(A) = X,$$

equipped with the Zariski topology.

The varieties are reflected by the closed subsets in the spectrum spaces.

Now given the topological space $X = \mathrm{Spec}(A)$, we want to turn it into a ringed topological space. That is by adding a sheaf structure to it. By especially choosing a well-structured sheaf, and by gluing local to form global structure, the spectrum spaces becomes **schemes**.

The collection of maximal ideals mSpec can also be denoted as Spm. This is a functor between algebras:

$$\mathrm{Spm} : \mathsf{Ring} \to \mathsf{Top} \qquad \mathfrak{m} \to P$$

where P is a closed point in the topological space $\mathrm{Spm}(A)$ under the Zariski topology.

With this functor, the schemes that correspond to varieties are of the form:

$$V(I) := \{\mathfrak{m} \in \mathrm{Spm}(A) : I \subset \mathfrak{m}\}.$$

A structure sheaf $\mathcal{O}_{\mathrm{Spm}(A)}$ is defined, making $\mathrm{Spm}(A)$ a locally ringed space.

The functor Spm will be useful when defining an algebraic group.

1.4 Regular function: Classic approach

Regular functions are important because they can help construct morphisms between varieties, especially to determine when are two varieties isomorphic to each other.

We will be working on rings $k[x_1, \cdots, x_n]$, where k an algebraically closed field.

The most natural functions to define on $\mathbb{A}^n$ are the regular functions.

Definition 6 (regular functions)**.** *Given an affine variety $X \subset \mathbb{A}^n$ with Zariski topology τ, an open set $U \in \tau$, and a point $p \in U$, we say that a function $f : U \to K$ is* ***regular at*** *p if:*

$$\exists V \in \tau, V \ni p \qquad \exists g, h \in k[x_1, \cdots x_n], h(p) \neq 0 \qquad f = \frac{g}{h}.$$

We say that f is ***regular on*** *U if f is regular at every point of U.*

Example 1 (polynomials)**.** *Every polynomial is regular everywhere on the affine plane, hence is always regular.*

This means that regular functions are natural generalizations of polynomials.

Example 2 (coordinate restrictions)**.** *Let $X = V(xy - 1) \subset \mathbb{A}^2$. Then the function:*

$$f(x, y) = \frac{1}{x}$$

is regular on the variety X, though not defined on all $\mathbb{A}^2$.

Example 3. *Consider $\mathbb{A}^1 = k$. The function*

$$f(x) = x$$

is not regular on the entire $\mathbb{A}^1$, but is regular on the open set $U := \mathbb{A}^2 \setminus \{x = 0\}$. This open set U forms a variety itself.

We want to study the family where all regular functions lives. We would show that the coordinate ring of X is exactly the ring of regular functions regular at every point p of X.

Proposition 2 (equivalence of regular function rings)**.**

1. *(affine) The regular functions on any open subset $U \subset X$ form a ring $\mathcal{O}_X(U)$, and in particular,*

$$\mathcal{O}_X(X) \simeq A(X).$$

2. *(projective) $\mathcal{O}_Y(Y) \simeq k$.*

Definition 7 (local ring)**.** *If for U and V as open subsets of X, the regular functions $\langle U, f\rangle$ and $\langle V, g\rangle$ agree on $U \cap V$, then we say f and g are equivalent and denote $\langle U, f\rangle \sim \langle V, g\rangle$. The* ***local ring*** *of p on X is defined as:*

$$\mathcal{O}_{X,p} := \{\langle U, f\rangle, p \in U\}/\sim.$$

The elements in $\mathcal{O}_{X,p}$ can be viewed as tangents at of X at the point p.
Similarly, we have the corresponding:

Proposition 3 (equivalence of localizations)**.**

1. *(affine) For each point $p \in X$ affine variety, let $m_p \lhd A(X)$ be the ideal of functions vanishing at p, then:*

$$\mathcal{O}_{X,p} \simeq A(X)_{m_p},$$

and $\dim \mathcal{O}_{X,p} = \dim X$.

2. *(projective) For each point $p \in Y$ projective variety, let $m_p \lhd S(Y)$ be the ideal generated by homogeneous functions vanishing at p, then:*

$$\mathcal{O}_{Y,p} \simeq S(Y)_{(m_p)}.$$

Definition 8 (function field)**.** *Similar to the local ring, the* ***function field*** *of X is defined as:*

$$K(X) := \{\langle U, f\rangle\}/\sim,$$

only that open sets U are not asked to contain certain points.

Through the process, we actually glued functions together and constructed a new function on $U \cup V$, called the **rational functions**.

$K(X)$ is indeed a field.

Rational functions are important because they help define birational maps, which can help classify varieties under birational equivalence.

Proposition 4 (equivalence of function field)**.**

1. *(affine)* $K(X) \simeq \operatorname{Frac} A(Y) = A(Y)_{(0)}$.
2. *(projective)* $K(Y) \simeq \operatorname{Frac} S(Y) = S(Y)_{((0))}$.

The three rings by nature have the following inclusion property.

Proposition 5. *Given a variety X,*

$$\mathcal{O}_X(X) \hookrightarrow \mathcal{O}_{X,p} \hookrightarrow K(X).$$

1.5 Regular function: Modern approach

Given a scheme $(X, \mathcal{O}_X)$, the images of the equipped sheaf on open sets $\mathcal{O}_X(U)$ become rings of regular functions.

Regular functions correspond to sections:

Definition 9 (sections)**.** *A section s lives in some $\mathcal{O}_X(U)$.*

The formal regular functions could be defined as:

$$\forall f \in A \qquad f(p) := f + p$$

with images as cosets.

A localization of a section is a germ:

Definition 10 (germs of regular functions)**.** *Given affine variety X and coordinate rings $\mathcal{O}_X(U)$ of open subsets of X, the germ at a point p is defined as:*

$$\mathcal{O}_{X,p} := \varinjlim_{p \in U} \mathcal{O}_X(U).$$

*Here $\mathcal{O}_{X,p}$ is called the **stalk** of X at p. There are group homomorphisms:*

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,p} \qquad s \mapsto s_x.$$

*It maps a section to its **germ**.*

1.6 Morphisms

Morphisms here refer to maps between varieties. The definition of morphisms requires compatibility with regular functions on these varieties.

Definition 11 (morphism)**.** *Let k algebraically closed, and X, Y two varieties. Then a **morphism** $\phi : X \to Y$ is a continuous map such that:*

$$\forall V \in \tau_Y \quad \forall f \in \mathcal{O}_Y(V) \qquad f \circ \phi \in \mathcal{O}_X(\phi^{-1}(V)).$$

*An **isomorphism** is a morphism $\phi : X \to Y$ that admits an inverse morphism $\psi : Y \to X$ such that:*

$$\psi \circ \phi = \mathbb{1}_X, \qquad \phi \circ \psi = \mathbb{1}_Y.$$

Under morphisms, varieties becomes a category Var_k, and we can classify varieties under isomorphisms. A useful property for us to find equivalence classes is stated as follows. We first post a stronger proposition.

Proposition 6. *Given two varieties X, Y, if Y is affine, then there is a natural bijection of sets:*

$$\alpha : \mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_k(A(Y), \mathcal{O}(X)).$$

Proof. We construct α as follows:

$$\alpha : (\phi : X \to Y) \mapsto (f \mapsto f \circ \phi).$$

Since for affine varieties, $\mathcal{O}(Y) \simeq A(Y)$, the latter can be induced to $\mathrm{Hom}_k(A(Y), \mathcal{O}(X))$. The inverse of α is given as the following β:

$$\beta : (\psi : X \to \mathbb{A}^n \quad P \mapsto (\xi_1(P), \cdots \xi_n(P))) \quad \leftarrow\!\shortmid \quad (h : \overline{x_i} \mapsto \xi_i).$$

The $\overline{x_i}$ presents the coordinates x_i in the ring $A(Y) = k[x_1, \cdots, x_n]/I(Y)$. This definition works since given any $f \in I(Y)$,

$$f(\psi(P)) = f(\xi_1(P), \cdots, \xi_n(P)) = h(f(\overline{x_1}), \cdots, f(\overline{x_n}))(P) = 0,$$

which satisfies that ψ is a morphism from X to Y. □

Corollary 1. *Given two affine varieties X and Y,*

$$X \simeq Y \qquad \Longleftrightarrow \qquad A(Y) \simeq A(X).$$

Classifying varieties under isomorphisms is actually one of the most important problem in algebraic geometry. We could consider a weaker classification using a generalized equivalence relation introduced as follows.

1.7 Rational maps

Definition 12 (rational map)**.** *Let X, Y be varieties. A* ***rational map*** *$\phi : X \to Y$ is a morphism defined on dense open subset on X. More concretely, given morphisms $\phi_U : U \to Y$, where $\emptyset \neq U \in \tau_X$, we give a rational map ϕ by gluing the ϕ_U under equivalence:*

$$\phi_U \sim \phi_V \quad \textit{if} \quad \phi_U(U \cap V) = \phi_V(U \cap V).$$

We call the rational map ϕ ***dominant*** *if its image is dense in Y.*

Proposition 7. *The rational map $\phi : X \to Y$ is dominant if the image of one U is dense.*

Proof. If two morphisms from X to Y agree on an open set, the two morphisms much be the same. □

Definition 13 (birational map)**.** *Let X, Y be varieties. We call a rational map $\phi : X \to Y$* ***birational*** *if there exists a rational map $\psi : Y \to X$ such that:*

$$\psi \circ \phi = \mathbb{1}_X, \qquad \phi \circ \psi = \mathbb{1}_Y.$$

If there exists a birational map between X and Y, we say that X and Y are ***birational.***

Birational is an equivalence relation.

Proposition 8. *If X and Y are birational, then they are isomorphic on open dense sets.*

This means that two varieties are isomorphic on dense open subsets

Corollary 2. *Let X and Y be two varieties. Then,*

$$X \simeq Y \quad \Longrightarrow \quad X, Y \textit{ birational}$$

This means that isomorphism is a stronger equivalence that gives finer classification.

Example 4. *We list two counterexamples here.*

1. *Consider smooth projective varieties of genus* 0*. They all birational to* $\mathbb{P}^1$*, but they are usually not isomorphic.*

2. *Consider smooth projective varieties of genus* 1*. They are all birational to each other, and their function field is the function field of elliptic curves* $k(E) := \{$*meromorphic functions on* $E\}$*. But they are usually not isomorphic.*

We return to the classification question.

Proposition 9. *Two varieties* X *and* Y *are birational if and only if their function fields* $K(Y)$ *and* $K(X)$ *are isomorphic* k*-algebras.*

Proposition 10. *The genus is a birational invariant.*

Example 5. *We explain the above example further.*

1. *For* $g = 0$*, there is only one birational equivalence class;*

2. *For each* $g > 0$*, there is a continuous family of birational equivalence class, parameterized by an irreducible algebraic variety. Curves with* $g = 1$ *are the elliptic curves.*

2 Topological groups

This is the prerequisite theory that gives groups a topological space structure.
A classical textbook here is [8].

2.1 Definitions and first examples

Definition 14 (topological group)**.** *A* topological group *is a group* G *equipped with a topology such that:*

1. *the multiplication map* $m : G \times G \to G \quad (g, h) \mapsto gh$

2. *the inversion map* $i : G \to G \quad g \mapsto g^{-1}$

are continuous.

Many naturally occurring groups (Lie groups, p-adic groups, real algebraic groups) are topological groups.
We could have the concepts as **connected** and **path-connected**, **compact** and **locally compact**, **morphisms** and **kernel**, **image**.

Example 6. *Consider* $k = \mathbb{R}$ *or* $\mathbb{C}$*,*

1. GL_n*;*

2. SL_n*,* O_n*,* SO_n*. They are all closed subgroups of* GL_n*.*

Theorem 1 (isomorphism theorem of topological group)**.** *Given two topological groups* G_1, G_2*, and a homomorphism of topological groups* $f : G_1 \to G_2$*, if* f *is open, then we have topological group isomorphism:*

$$G_1/\ker f \simeq \operatorname{im} f.$$

2.2 Representations of compact groups

Definition 15. *Given a topological group* G*, the homomorphism of topological group:*

$$\rho : G \to \mathrm{GL}_n$$

is called a ***representation*** *of* G*.*

We now study the reducibility of representations of topological groups.

Definition 16 (Haar measure)**.** *Given a topological group G, we denote the collection of all continuous real-valued functions on G as $C(G,\mathbb{R})$. It is naturally an $\mathbb{R}$-vector space. Define a functional:*

$$\int_G \bullet \mathrm{d}x : C(G,\mathbb{R}) \to \mathbb{R}.$$

*We call it a **Haar integral** if it is:*

1. *linear: $\int_G kf(x) + lg(x)\mathrm{d}x$.*
2. *positive definite: $f \geq 0 \implies \int_G f(x)\mathrm{d}x \geq 0$.*
3. *normal: $\int_G 1\mathrm{d}x = 1$.*
4. *left-invariant: $\forall g \in G \quad \int_G f(gx)\mathrm{d}x = \int_G f(x)\mathrm{d}x$.*
5. *right-invariant: $\forall g \in G \quad \int_G f(xg)\mathrm{d}x = \int_G f(x)\mathrm{d}x$.*
6. *inverse-invariant: $\int_G f(x^{-1})\mathrm{d}x = \int_G f(x)\mathrm{d}x$.*

Left-invariance, right invariance, and inverse-invariant are equivalent, so we only need one of them and call it the property of **invariance**.

Positive definite shows that if $\int_G f(x)\mathrm{d}x = 0$, then $f = 0$.

Theorem 2 (Haar)**.** *Any locally compact group exists a unique Haar integral.*

Example 7 (finite group)**.** *If G is a finite topological group, then all continuous functions on G are linear:*

$$C(G,\mathbb{R}) \simeq (\mathbb{R}G)^*, \qquad \int_G \bullet \mathrm{d}x \in ((\mathbb{R}G)^*)^* \simeq \mathbb{R}G.$$

Let a correspond to the Haar integral. Invariance tells that:

$$a(f) = \int_G f(x)\mathrm{d}x = \int_G f(gx)\mathrm{d}x = ga(f) \quad \forall f \in (\mathbb{R}G)^*.$$

This gives $ga = a \quad \forall g \in G$. Thus $a = c\sum_{h\in G} h$, where $c \in \mathbb{R}$. By normality,

$$1 = \int_G 1\mathrm{d}x = a(1) = (c\sum_{h\in G} h) \cdot 1 = c|G| = 1.$$

This tells that $c = \frac{1}{|G|}$. Thus:

$$((\mathbb{R}G)^*)^* \simeq \mathbb{R}G \qquad \int_G \bullet \mathrm{d}x \mapsto \frac{1}{|G|}\sum_{h\in G} h.$$

Example 8. $\mathrm{SO}_2 \simeq \mathbb{S}^1$*. Its Haar integral is $\frac{1}{2\pi}\int_{\mathbb{S}^1} \bullet \mathrm{d}s$.*

Haar theorem gives a simpler proof for the classical Masche theorem.

Theorem 3 (Masche, finite version)**.** *Given a G-representation V and a subrepresentation $V_1 \leq V$, then there exists a decomposition: $V \simeq V_1 \oplus V_1^{\perp}$.*

Proof. Claim 1: Let $V_1 \leq V$ be a subrepresentation. If we have a G-inv inner product, then $V_1^{\perp}$ is a subrepresentation.

$\forall v_1' \in V_1^{\perp}, g \in G$, it suffices to show $g \cdot v_1' \in V_1^{\perp}$. This can be shown by:

$$(v_1, g \cdot v_1') = (g^{-1}v_1, v_1') = 0 \implies g \cdot v_1' \in V_1^{\perp}.$$

Claim 2: If G is a finite group, then there exists G-inv inner product on V.

Given any inner product $(\cdot,\cdot)$ on V, then the following is an inner product, and it is G-inv:

$$\langle u, v\rangle := \frac{1}{|G|}\sum_{g\in G}(g \cdot u, g \cdot v).$$

□

Given finite group representation $\rho : G \to \mathrm{GL}_V$, since G-inv gives

$$(g \cdot u, g \cdot v) = (\rho(g)u, \rho(g)v) = u'\rho(g')\rho(g)v = u'v.$$

it infers that $\rho(g) \in \mathrm{O}_n$. We call such representation as **orthogonal representation**. Similarly, if representation gives G-inv complex inner product, then we call the representation as **unitary representation**.

The Masche theorem can be rephrased as:

All finite group representations are unitary.

Proposition 11. *All compact group representation exists inner product that is G-inv.*

Proof. Given any inner product $(\cdot,\cdot)$ on V, define:

$$f_{u,v} : G \to \mathbb{R} \qquad g \mapsto (g \cdot u, g \cdot v).$$
$$\langle u, v\rangle := \int_G f_{u,v}(x)\mathrm{d}x.$$

We have $f_{g\cdot u, g\cdot v}(x) = f_{u,v}(x, g)$. □

The Masche theorem on compact groups can be rephrased as:

All compact group representations are unitary.

This proves the Masche theorem on compact groups, that all representations of a compact group can be fully decomposed into irreducible representations.

Corollary 3. *SO_n is the maximal connected compact subgroup of GL_n*

2.3 Characters on compact groups

Definition 17. *Given compact group G and representation $\rho : G \to \mathrm{GL}_V$, we define the **character** of ρ as:*

$$\chi : G \to \mathbb{R} \qquad g \mapsto \operatorname{tr} \rho(g).$$

We could see that characters are continuous functions, i.e., $\chi \in C(G, \mathbb{R})$. They are also class functions.

The natural question is if characters on a compact group are orthogonal.

We now give a inner product for characters. Define:

$$C(G, \mathbb{R}) \times C(G, \mathbb{R}) \to \mathbb{R} \qquad (f_1, f_2) := \int_G f_1(x)f_2(x^{-1})\mathrm{d}x.$$

Let (ρ, V) and (ρ', U) are two representations that has $\rho(g) = (a_{ij})_{n\times n}$ and $\rho'(g) = (b_{ij})_{m\times m}$. The question is rephrased as whether $(a_{ij}(\cdot), b_{kl}(\cdot)) = 0$ holds.

Let $A(g) := (a_{ij}(g))_{i,j}$. Denote $\int_G A(g)\mathrm{d}g := (\int_G a_{ij}(g)\mathrm{d}g)_{i,j}$. Define:

$$f : U \to V \qquad f((\alpha_1, \cdots, \alpha_m)) = C_{m\times n}\begin{pmatrix}\beta_1 \\ \vdots \\ \beta_n\end{pmatrix},$$

where α_i is a basis of U and β_j is a basis of V. Claim that:

$$\widetilde{C} := \int_G \rho'(g^{-1})C_{m\times n}\rho(g)\mathrm{d}g \in \mathrm{Hom}_G(U, V), \qquad \text{or} \qquad \widetilde{C}\rho(h) = \rho'(h)\widetilde{C}, \qquad \textit{i.e.} \qquad \begin{array}{ccc} U & \xrightarrow{\widetilde{C}} & V \\ \big\downarrow{\scriptstyle \rho'(h)} & & \big\downarrow{\scriptstyle \rho(h)} \\ U & \xrightarrow{\widetilde{C}} & V \end{array}.$$

First prove that if U is not isomorphic to V then $(a_{ij}(\cdot), b_{kl}(\cdot)) = 0$. By Schur lemma, $\widetilde{C} = 0$ if $\mathrm{Hom}_G(U, V) = 0$. Let $C = E_{ki}$, then $b_{ij}(g^{-1})Ca_{ij}(g) = (b_{lk}(g^{-1})a_{ij}(g))_{l,j}$. Thus $\widetilde{C} = (\int_G b_{lk}(g^{-1})a_{ij}(g)\mathrm{d}g)$. Hence $\widetilde{C} = 0$ gives $(a_{ij}(\cdot), b_{lk}(\cdot)) = 0$.

On the other hand, if $\widetilde{C} \neq 0$, by Schur lemma, $\widetilde{C} = \lambda I_n = \frac{\delta_{ik}}{n} I_n$. Thus, $(a_{ij}(\cdot), b_{lk}(\cdot)) = \widetilde{C}_{ij} = \frac{\delta_{ik}}{n}\delta_{lj}$.

In all, we proved that:

$$(a_{ij}(\cdot), b_{kl}(\cdot)) = \begin{cases} \dfrac{1}{n}\delta_{ik}\delta_{jl}, & U \simeq V \\ 0, & \text{otherwise} \end{cases}$$

This gives:

$$(\chi_U, \chi_V) = (\sum_i a_{ii}(g), \sum_j b_{jj}(g)) = \sum_{i,j}(a_{ii}(\cdot), b_{jj}(\cdot)) = \delta_{U \simeq V}.$$

2.4 Peter-Weyl Theorem

The final question is how many irreducible characters to we have for compact groups.

Proposition 12. *$\mathbb{S}^1$ has infinitely many irreducible complex representations.*

Proof. Since $\mathbb{S}^1$ is abelian, every elements forms a conjugacy class. Since this group is infinite, we could not deduce its number of irreducible representation from this method. We now construct a family of irreducible representation as follows. $\forall n \in \mathbb{Z}$, define:

$$\rho_n : \mathbb{S}^1 \to \mathbb{C}^* \qquad e^{i\theta} \mapsto e^{in\theta}.$$

□

We study with another method. Given $f \in C(\mathbb{S}^1, \mathbb{R})$. Since f continuous, we have its Fourier expansion:

$$f(\theta) = \sum a_n(f)e^{in\theta}.$$

We wish all irreducible characters converges to any class functions.

Definition 18. *Let $\Delta \subset C(G, \mathbb{R})$ be a* ***complete*** *subset if:*

$$\forall f \in C(G, \mathbb{R})\ \forall \varepsilon > 0 \qquad \exists a_1(f), \cdots a_n(f) \in \mathbb{R}\ \exists f_1, \cdots f_n \in \Delta \qquad |f(g) - \sum_i a_i(f)f_i(g)| < \varepsilon \quad \forall g \in G.$$

Moreover, we call Δ ***orthogonal complete*** *if $\langle f_i, f_j \rangle = 0$ for every $f_i \neq f_j \in \Delta$.*

Theorem 4 (Peter, Weyl)**.** *Given a compact group G,*

1. *All element function of the matrix $\rho(g)$ forms a orthogonal complete subset of $C(G, \mathbb{R})$.*
2. *All irreducible characters forms a orthogonal complete subset of all class functions on G.*

Example 9. *Recall the previous example $\mathbb{S}^1$ and its irreducible representation ρ_n.*

Example 10 (SU_2)**.** *Define $\mathrm{SU}_2 := \{A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} : AA^\dagger = I_2, |A| = 1\}$, where $A^\dagger := \overline{A'}$.*

First notice that $z_3 = -\overline{z_2}, z_4 = \overline{z_1}$. This is since $z_1\overline{z_1} + z_2\overline{z_2} = z_3\overline{z_3} + z_4\overline{z_4} = 1$, and $z_1\overline{z_3} + z_2\overline{z_4} = z_3\overline{z_1} + z_4\overline{z_2} = 0$.

Now SU_2 is of dimension 4, with a basis:

$$e_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad e_2 = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Actually, $\mathrm{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_3\} \simeq \mathbb{H}$, the Hamiltonian group. We introduce an inner product on $\mathbb{H}$ as $(e_i, e_j) = \delta_{ij}$. Then we are choosing elements of norm 1 in SU_2. Thus as a topological space,

$$\mathrm{SU}_2 \simeq \mathbb{S}^3.$$

This gives $\mathbb{S}^3$ is a compact Lie group. In fact $\mathbb{S}^1$ and $\mathbb{S}^3$ are the only two spheres that has group structures.

Now we construct irreducible representations of SU_2. Consider $\mathbb{C}[x,y]$, pick the homogeneous space of degree n: $V_n := \mathrm{span}_{\mathbb{C}}\{y^n, y^{n-1}x^1, \cdots, x^n\}$, which is of dimension $n+1$. Define action:

$$\Phi_n : \mathrm{SU}_2 \to \mathrm{GL}_{n+1} \qquad A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \mapsto (x^i y^{n-i} \mapsto (z_1 x + z_3 y)^i (z_2 x + z_4 y)^{n-i})$$

We claim that Φ_n is irreducible, and prove as follows. Define matrix $A(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \mathrm{SU}_2$. Moreover, $\{A(\theta) : \theta \in \mathbb{C}\} \simeq \mathbb{S}^1$ is abelian. Check that:

$$A(\theta) \cdot (x^j y^{n-j}) = (e^{i\theta}x)^j (e^{-i\theta}y)^{n-j} = (e^{i\theta})^{2n-j}(x^j y^{n-j}).$$

Thus $x^j y^{n-j}$ is an eigenvector. As there are $n+1$ of them, they form a basis of V. Let $W \subset V$ be a subrepresentation, as $A(\theta)$ has eigenvector in W, W must contain some element $x^j y^{n-j}$. Thus any $(z_1 x + z_3 y)^j (z_2 x + z_4 y)^{n-j} \in W$. We can reconstruct elements $y^n, y^{n-1}x, \cdots x^n$ from them, which gives $W = V$.

Finally, we prove that Φ_n is all irreducible representations. From linear algebra, $A(\theta)$ gives $f(\theta) = f(-\theta)$. A complete set is $\{e^{i\theta} + e^{-i\theta}, \cdots, (e^{i\theta} + e^{-i\theta})^n\}$., which can be shown by

$$\chi_{V_n}(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}) = \sum_j (e^{i\theta})^{2j-n}.$$

The important idea during this process is to find an abelian subgroup of the infinite group. As for the example $\mathbb{S}^3$, we found $\mathbb{S}^1$.

3 Algebraic groups: Definitions and first examples

One reason of studying algebraic groups is that we want to understand the geometry of groups.

Standard textbooks here includes [5], [9], and [1].

3.1 Definition and first examples

Algebraic groups are varieties equipped with compatible group structures defined by regular maps. Let k be a number field.

Definition 19 (algebraic group)**.** *An **algebraic group** is a variety X with regular maps:*

$$m : X \times X \to X, \qquad i : X \to X,$$

that satisfy group axioms:

$$\begin{aligned} associative: &\quad m(m(x,y),z) = m(x, m(y,z)), \\ identity: &\quad m(e,x) = x = m(x,e), \\ inverse: &\quad m(x, i(x)) = e = m(i(x), x). \end{aligned}$$

With such group structure, we often denote an algebraic group as G or $G(k)$.

*An algebraic group G are called **abelian** if:*

$$m(x,y) = m(y,x) \qquad \forall x, y \in G$$

Algebraic groups can also be called as **group varieties** from its definition.

This is a group object in the category of varieties. More modern explanations to be introduced later.

Example 11. *We list some classic examples.*

1. $\mathbb{G}_a := (\mathbb{A}^1, +)$ *with the usual addition on k:*

$$m(x,y) = x + y, \qquad i(x) = -x.$$

2. $\mathbb{G}_m := (\mathbb{A}^1 \setminus \{(0)\}, \cdot)$ *with the usual multiplication on k:*

$$m(x,y) = x \cdot y, \qquad i(x) = x^{-1}.$$

3. $\mathbb{G}_a^n := (\mathbb{A}^n, +)$ *with the usual addition of vectors:*

$$m(x,y) = x + y = (x_i + y_i)_{1 \le i \le n}, \qquad i(x) = -x = (-x_i)_{1 \le i \le n}.$$

4. $M_n = (M_n(k), +) \simeq (\mathbb{A}^{n^2}, +)$ *with the usual addition of matrices:*

$$m(x,y) = x + y = (x_{ij} + y_{ij})_{1 \le i,j \le n}, \qquad i(x) = -x = (-x_{ij})_{1 \le i,j \le n}.$$

5. $\mathrm{GL}_n = (\mathrm{GL}_n(k), \cdot)$ *with the usual multiplication of matrices:*

$$m(x,y) = x \cdot y = (\sum_k x_{ik} y_{kj})_{i,j}, \qquad i(x) = x^{-1} = \det(x)^{-1} x^*.$$

The inversion is regular because $\det(x) \neq 0$ defines an open subset.

We introduce a large class of examples with the help of the following proposition.

Proposition 13. *If G is an algebraic group and X is any variety, then $\mathrm{Hom}(X, G)$ has a natural group structure given by:*

$$(f \cdot g)(x) = m(f(x), g(x)).$$

Example 12. *Let $\mathbb{G}_a$ and $\mathbb{G}_m$ be defined as in the above example. Let $\mathcal{O}(U) = \mathcal{O}_X(U)$ be the ring of regular functions on a variety X.*

1. *For any variety X,*

$$\mathrm{Hom}(X, \mathbb{G}_a) \simeq \mathcal{O}(X),$$

since any morphism $X \to \mathbb{A}^1$ is a regular function, so it lives in $\mathcal{O}(X)$.

2. *For any variety X,*

$$\mathrm{Hom}(X, \mathbb{G}_m) \simeq \mathcal{O}(X)^\times,$$

since any morphism $X \to \mathbb{G}_m$ is a regular function without zeros, so it lives in $\mathcal{O}(X)^\times$.

This gives characters and cocharacters to be introduced later.

3.2 Algebraic subgroups and morphisms

We will study the relations between algebraic groups in this subsection.

Definition 20 (algebraic subgroup)**.** *Given an algebraic group G over k, an **algebraic subgroup** H is a subvariety of G which is stable under the group law and inverse map of G.*

*An algebraic subgroup is **closed** if its underlying variety is a closed subvariety of G. We often abbreviate a closed algebraic subgroup as a **closed subgroup**.*

Closed subgroups will play an important role in the future.

To study further properties, we need the concept of morphisms between algebraic groups. From their definition, it should be compatible with morphisms between varieties and homomorphisms between groups.

Definition 21 (morphism)**.** *Given two algebraic groups G and H over field k, a **morphism** $\phi : G \to H$ is a regular map such that:*

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2), \qquad \phi(e_G) = e_H$$

This means exactly that ϕ is a morphism in the category of group objects in the category of varieties.

Given this definition, the definition of a closed subgroup can be rephrased as a subgroup $H \leq G$, whose underlying algebraic variety is a closed subvariety of G, such that the inclusion $H \hookrightarrow G$ is a morphism of algebraic groups.

Given a morphism, consequently, we have its **kernel** and **image**, as well as the followings:

1. ϕ is a **monomorphism** if $\ker \phi = 0$.

2. ϕ is a **epimorphism** if $\phi(G)$ is dense in H.

3. ϕ is a **isomorphism** if ϕ is bijective, regular, and has regular inverse.

4. ϕ is a **isogeny** if ϕ surjects and $\ker \phi$ is finite.

In the theory of Riemann surface, an isogeny can be defined as a holomorphic homomorphism between complex tori. Since complex tori are compact, the homomorphism must surjects, and its kernel is finite because it is discrete, otherwise complex analysis tells that it must be 0.

Proposition 14. *Given a morphism $\phi : G \to H$,*

1. *Its kernel $\ker \phi := \{g \in G : \phi(g) = e_H\}$ is a closed subgroup of G.*

2. *Its image $\operatorname{im} \phi := \phi(G)$ is a closed subgroup of H.*

Proof.

1. The kernel is the fiber of ϕ over the point e_H, hence closed.

2. The image of a morphism of algebraic varieties need not be closed, but for group morphisms, the image is constructible and stable under group operations. By Chevalley's theorem, it must be closed. □

This tells that morphisms between algebraic groups are much stronger than morphisms between varieties, since the image of morphisms between algebraic groups behave much better.

Example 13. *We revise previous examples and list some morphisms between them.*

1. *The inclusion map $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$ is a morphism. It is closed since SL_n is the kernel of the inclusion map.*

2. *For any $\Gamma \subset \mathbb{G}_a^n$, the algebraic group Γ is closed.*

3. *The determinant map:*
$$\det : \mathrm{GL}_n \to \mathbb{G}_m$$
is a morphism. It is a regular map and it is multiplicative.

4. *The trace map:*
$$\mathrm{tr} : M_n \to \mathbb{G}_a$$
is a morphism. It is a regular map and it is additive.

3.3 Quotient algebraic subgroups

We consider more examples of algebraic groups and their morphisms.

Example 14 (normal algebraic subgroup)**.** *A closed algebraic subgroup $N \subset G$ is **normal** if $gNg^{-1} = N$ for all $g \in G$. Equivalently, the conjugate map $c_g : G \to G \quad x \mapsto gxg^{-1}$ satisfies $c_g(N) \subset N$.*

1. *For any morphism $\phi : G \to H$, the kernel $\ker \phi$ is a normal closed subgroup.*

2. *The **center** $Z(G) := \{g \in G : m(g,x) = m(x,g) \forall x \in G\}$ is a normal closed subgroup.*

3. *In GL_n, the collection of scalar matrices $\{ae_{\mathrm{GL}_n} : a \in k\}$ form a normal subgroup. It is isomorphic to $\mathbb{G}_m$.*

Proposition 15 (quotient algebraic subgroup)**.** *Let $H \subset G$ be a closed normal subgroup. The quotient sheaf G/H is representable by a variety iff H is normal. In this case, the* ***quotient group*** *is an algebraic group with:*

$$\mathcal{O}(G/H) := \{f \in \mathcal{O}(G) : f(gh) = f(g)\ \forall h \in H\}.$$

Example 15. *We review previous examples.*

1. $\det : \mathrm{GL}_n / \mathrm{SL}_n \simeq \mathbb{G}_m$.
2. $\mathbb{G}_a/\{0\} \simeq \mathbb{G}_a$.
3. $\mathbb{G}_m/\mu_n \simeq \mathbb{G}_m$ *via* $t \mapsto t^n$.

Proposition 16 (homomorphism algebraic group)**.** *Given algebraic groups G, G_1, G_2 and H, H_1, H_2,*

$$\mathrm{Hom}(G_1 \times G_2, H) \simeq \mathrm{Hom}(G_1, H) \times \mathrm{Hom}(G_2, H),$$
$$\mathrm{Hom}(G, H_1 \times H_2) \simeq \mathrm{Hom}(G, H_1) \times \mathrm{Hom}(G, H_2).$$

as isomorphisms between algebraic groups.

An important example of morphisms is the characters and cocharacters.

Example 16 (character and cocharacter)**.** *Let G be an algebraic group.*

1. *A* ***character*** *of G is a morphism:*
$$\chi : G \to \mathbb{G}_m.$$
Their collection $X^*(G) := \mathrm{Hom}(G, \mathbb{G}_m)$ forms an abelian algebraic group.
2. *A* ***cocharacter*** *of G is a morphism:*
$$\lambda : \mathbb{G}_m \to G.$$
Their collection $X_*(G) := \mathrm{Hom}(\mathbb{G}_m, G)$ forms an abelian algebraic group.

For tori, $X^*(G)$ is a free abelian group of finite rank. This is the starting point for root systems.

3.4 Modern approach

The aim of introducing the modern definition is to replace varieties with schemes, so that group varieties becomes group schemes.

A good reference here is [7].

Definition 22 (algebraic group, modern viewpoint)**.** *Let G be a scheme over k, and let $m : G \to G$ be a morphism of schemes. Then G becomes an* ***algebraic group*** *if there exists scheme morphisms:*

$$e : \mathrm{Spm}(k) \to G, \qquad \mathrm{inv} : G \to G,$$

that is the identity and inverse, satisfying the following three properties:

$$\begin{aligned} \textit{associative}: &\quad (\mathbb{1} \times m) \circ m \simeq (m \times \mathbb{1}) \circ m, \\ \textit{identity}: &\quad m \circ (e \times \mathbb{1}) = \mathbb{1} = m \circ (\mathbb{1} \times e), \\ \textit{inverse}: &\quad m \circ (\mathrm{inv}, \mathbb{1}) = e = m \circ (\mathbb{1}, \mathrm{inv}). \end{aligned}$$

An algebraic group becomes a ***group scheme*** *if the algebraic group is smooth.*

We could also consider algebraic group over any commutative unital ring A, but it would lose much more properties. If A is an arbitrary field, then it could already tell a lot, and we study them in the section of F**-groups**34. It works the best if A is an algebraically closed field:

Proposition 17. *An algebraic group is always smooth if k is algebraically closed.*

We could also consider modern definition for the morphisms between algebraic groups. They are then morphisms between schemes:

$$\varphi : G \to H.$$

Its kernel is defined as the fiber product:

$$\ker\varphi := \begin{tikzcd} & \mathrm{Spm}(k) \arrow[dr, "e"] & \\ G \times_H \mathrm{Spm}(k) \arrow[ur] \arrow[dr] & & H \\ & G \arrow[ur, "\varphi"] & \end{tikzcd}.$$

4 Linear algebraic groups

4.1 Definitions and first examples

Definition 23 (linear algebraic group)**.** *Let G be an algebraic group. If the underlying variety is affine, then we call G **linear** or **affine**.*

We will first show that linear algebraic groups can be viewed as Hopf algebras.

Linear algebraic groups are affine varieties could give:

$$G \text{ is linear} \qquad \Longleftrightarrow \qquad G \simeq \operatorname{Spec} A, \text{ for } A \text{ a finitely generated } k\text{-algebra.}$$

By definition, the algebra is $A = \mathcal{O}(G)$. Since $k[G]$ is the finitely generated group algebra over k, we will denote $k[G] := \mathcal{O}(G)$ for the coordinate ring on G. Recall the following result for affine varieties.

Proposition 18. *Given A a finitely generated reduced k-algebra, $\mathcal{O}(\operatorname{Spec} A) \simeq A$ as k-algebras.*

Therefore, for a linear algebraic group G,

$$\mathcal{O}(\operatorname{Spec} k[G]) \simeq k[G], \qquad \text{hence } G \simeq \operatorname{Spec} k[G].$$

The algebra $k[G]$ has more structures induced by the group laws on G. Denote $A := k[G]$, since:

$$k[G \times G] \simeq k[G] \otimes_k k[G],$$

the group laws:

$$m : G \times G \to G, \qquad i : G \to G, \qquad e : \operatorname{Spec} k \to G$$

induce dual algebra homomorphisms in the opposite direction:

$$\begin{aligned} \textit{comultiplication}: &\qquad \Delta := m^{\#} : A \to A \otimes_k A \\ \textit{antipode}: &\qquad \iota := i^{\#} : A \to A \\ \textit{counit}: &\qquad \varepsilon := e^{\#} : A \to k \end{aligned}$$

The group axioms translate into the Hopf algebra identities:

coassociativity : $\quad (\Delta \otimes \mathbb{1}) \circ \Delta = (\mathbb{1} \otimes \Delta) \circ \Delta$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes_k A \arrow[d, "\mathbb{1}\otimes\Delta"] \\ A \otimes_k A \arrow[r, "\Delta\otimes\mathbb{1}"] & A \otimes_k A \otimes_k A \end{tikzcd}$$

counit : $\quad (\varepsilon \otimes \mathbb{1}) \circ \Delta = \mathbb{1} = (\mathbb{1} \otimes \varepsilon) \circ \Delta$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\mathbb{1}"] & A \otimes_k A \arrow[d, "\varepsilon\otimes\mathbb{1}"] \\ A \otimes_k A \arrow[r, "\mathbb{1}\otimes\varepsilon"] & A \end{tikzcd}$$

antipode : $\quad m \circ (\iota \otimes \mathbb{1}) \circ \Delta = \varepsilon = m \circ (\mathbb{1} \otimes \iota) \circ \Delta$

$$\begin{tikzcd} & A \otimes_k A \arrow[r, "\iota\otimes\mathbb{1}"] & A \otimes_k A \arrow[dr, "m"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rrr, "\varepsilon"] & & & A \\ & A \otimes_k A \arrow[r, "\mathbb{1}\otimes\iota"] & A \otimes_k A \arrow[ur, "m"'] & \end{tikzcd}$$

Definition 24 (Hopf algebra). *A **Hopf algebra** is a k-algebra equipped with maps $(\Delta, \varepsilon, \iota)$ satisfying the above identities.*

4.2 Identification with Hopf algebras

We now identify linear algebraic groups with certain Hopf algebras.

Theorem 5. *There exists a contravariant equivalence of categories:*

$$\mathsf{AffAlgGrp}_k \simeq \mathsf{CommHopfAlg}_{\text{f.g.}}$$

between affine algebraic groups and finitely generated commutative Hopf algebras, given by:

$$G \mapsto k[G], \qquad (\phi : G \to H) \mapsto (\phi^{\#} : k[H] \to k[G]).$$

Proof. Let k be a field. We check the functor from affine algebraic groups to Hopf algebras.

Let G be an affine algebraic group over k. By definition, G is an affine variety with a group structure such that the multiplication m, i, e are morphisms of varieties. Since G is affine, $G = \operatorname{Spec} k[G]$ for a finitely generated commutative k-algebra $k[G]$. By functoriality of Spec, the structure maps induce k-algebra homomorphisms

$$\Delta := m^{\#} : k[G] \to k[G] \otimes_k k[G], \qquad \varepsilon := e^{\#} : k[G] \to k, \qquad \iota := i^{\#} : k[G] \to k[G].$$

The axioms of a group translate exactly into the Hopf algebra axioms:

- Associativity of m corresponds to coassociativity of Δ: Associativity of the group law means that the diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathbb{1}"] \arrow[d, "\mathbb{1} \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

  commutes. Applying the contravariant functor Spec (or equivalently, pulling back regular functions) yields the commutative diagram

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & k[G] \otimes k[G] \arrow[d, "\Delta \otimes \mathbb{1}"] \\ k[G] \otimes k[G] \arrow[r, "\mathbb{1} \otimes \Delta"'] & k[G] \otimes k[G] \otimes k[G]. \end{tikzcd}$$

  This is precisely the coassociativity condition

$$(\Delta \otimes \mathbb{1}) \circ \Delta = (\mathbb{1} \otimes \Delta) \circ \Delta.$$

- The identity axiom corresponds to the counit axiom for ε: The identity element $e : \operatorname{Spec} k \to G$ satisfies

$$m \circ (e \times \mathbb{1}) = \mathbb{1}_G \qquad \text{and} \qquad m \circ (\mathbb{1} \times e) = \mathbb{1}_G.$$

  Dualizing, we obtain the commutative diagrams

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[dr, "\mathbb{1}"'] & k[G] \otimes k[G] \arrow[d, "\varepsilon \otimes \mathbb{1}"] \\ & k \otimes k[G] \cong k[G] \end{tikzcd} \qquad \begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[dr, "\mathbb{1}"'] & k[G] \otimes k[G] \arrow[d, "\mathbb{1} \otimes \varepsilon"] \\ & k[G] \otimes k \cong k[G]. \end{tikzcd}$$

  Thus

$$(\varepsilon \otimes \mathbb{1}) \circ \Delta = \mathbb{1} = (\mathbb{1} \otimes \varepsilon) \circ \Delta,$$

  which is exactly the counit axiom.

- The inverse axiom corresponds to the antipode axiom for ι: The inverse morphism $i : G \to G$ satisfies

  $$m \circ (\mathbb{1} \times i) \circ \Delta_G = e \circ \pi,$$

  where $\pi : G \to \operatorname{Spec} k$ is the structure morphism. Dualizing, we obtain

  $$m_k \circ (\mathbb{1} \otimes \iota) \circ \Delta = \varepsilon(\cdot)\, e,$$

  and similarly

  $$m_k \circ (\iota \otimes \mathbb{1}) \circ \Delta = \varepsilon(\cdot)\, e.$$

  Hence ι satisfies the antipode axiom, completing the Hopf algebra structure.

Thus $k[G]$ is a finitely generated commutative Hopf algebra.

We now check Functoriality. Let $\phi : G \to H$ be a morphism of affine algebraic groups. Since ϕ is a morphism of varieties, it induces a k-algebra homomorphism

$$\phi^{\#} : k[H] \to k[G].$$

Compatibility with multiplication means that the diagram

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "\phi\times\phi"'] & G \arrow[d, "\phi"] \\ H \times H \arrow[r, "m_H"'] & H \end{tikzcd}$$

commutes. Dualizing, we obtain

$$\Delta_G \circ \phi^{\#} = (\phi^{\#} \otimes \phi^{\#}) \circ \Delta_H.$$

Similarly, compatibility with identity and inverse implies

$$\varepsilon_G \circ \phi^{\#} = \varepsilon_H, \qquad S_G \circ \phi^{\#} = \phi^{\#} \circ S_H.$$

Thus $\phi^{\#}$ is a morphism of Hopf algebras. Moreover,

$$(\mathrm{id}_G)^{\#} = \mathrm{id}_{k[G]}, \qquad (\psi \circ \phi)^{\#} = \phi^{\#} \circ \psi^{\#},$$

so this assignment defines a contravariant functor:

$$\mathsf{AffAlgGrp}_k \longrightarrow \mathsf{CommHopfAlg}_{\mathrm{f.g.}}, \quad G \mapsto k[G].$$

Conversely, let A be a finitely generated commutative Hopf algebra over k. Set $G := \operatorname{Spec} A$, the Hopf algebra structure maps $\Delta, \varepsilon, \iota$ define morphisms:

$$m := \operatorname{Spec} \Delta : G \times G \to G, \quad e := \operatorname{Spec} \varepsilon : \operatorname{Spec} k \to G, \quad i := \operatorname{Spec} \iota : G \to G.$$

Similarly, the Hopf algebra axioms ensure that these morphisms satisfy the group axioms, so G is an affine algebraic group. Since A is finitely generated, G is an affine variety.

Now we check equivalence of categories. Starting from an affine algebraic group G, the canonical evaluation map

$$G \longrightarrow \operatorname{Spec} k[G]$$

is an isomorphism of algebraic groups. Conversely, for a finitely generated commutative Hopf algebra A, the natural map

$$A \longrightarrow k[\operatorname{Spec} A]$$

is an isomorphism of Hopf algebras. Moreover, morphisms of algebraic groups correspond exactly to morphisms of Hopf algebras via pullback. Hence the functor

$$G \mapsto k[G]$$

defines a contravariant equivalence of categories between $\mathsf{AffAlgGrp}_k$ and $\mathsf{CommHopfAlg}_{\mathrm{f.g.}}$.

$\square$

A **group object** in affine varieties corresponds exactly to a **Hopf algebra object** in commutative algebras.

Algebraic groups are either linear algebraic groups, or **abelian varieties**. We will mostly focus on linear algebraic groups is this section.

Example 17. *We revisit previous examples and and give some new examples.*

1. $\mathbb{G}_a$. *Its coordinate ring is* $k[\mathbb{G}_a] \simeq k[x]$. *We have extra group laws* $\Delta(x) = x \otimes 1 + 1 \otimes x, \iota(x) = -x, \varepsilon(x) = 0$.

2. $\mathbb{G}_m$. *Its coordinate ring is* $k[\mathbb{G}_m] \simeq k[x, x^{-1}]$. *We have extra group laws* $\Delta(x) = x \otimes x, \iota(x) = x^{-1}, \varepsilon(x) = 1$.

3. M_n. *Its coordinate ring is* $k[M_n] \simeq k[s_{ij}]$. *The determinant* $\det(x_{ij})$ *is a regular function on it.*

4. GL_n. *As a principal open subset* $\{X \in M_n : \det X \neq 0\}$,
$$\mathrm{GL}_n \simeq \mathrm{Spec}(k[x_{ij}, \det(x_{ij})^{-1}]) \implies k[\mathrm{GL}_n] \simeq k[x_{ij}, \det(x_{ij})^{-1}].$$
The induced Hopf operations are:
$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}, \qquad \iota(x_{ij}) = (x^{-1})_{ij} = \frac{1}{\det} C_{ji}, \qquad \varepsilon(x_{ij}) = \delta_{ij}.$$

5. T_n *of upper matrices;*

6. $\mathrm{SL}_n, SO_n, Sp_{2n}$.

Note that if these operations replaces k with arbitrary commutative unital ring R then the resulting (A, Δ, ι, e) is a **group scheme**.

Morphisms between linear algebraic groups are k-algebra homomorphisms between their coordinate Hopf algebras.

4.3 More examples

In general, the following property gives more example.

Proposition 19. *All subgroups of* GL_n *that is closed in the Zariski topology defines a linear algebraic group.*

Two other examples include the followings.

Example 18 (torus)**.** *A* ***torus*** T *over algebraically closed* k *is an algebraic group* T_k *such that* $T_k \simeq (\mathbb{G}_m)^r$.

Examples: The ***diagonal torus*** D_n, *which is the group of nonsingular diagonal matrices.*

Example 19 (unipotent group)**.** *Over algebraically closed* k, *the* ***unipotent groups*** *are closed subgroups of upper triangular matrices with all diagonal entries equal to 1.*

Equivalently, every element in a unipotent group satisfies $(g-1)^N = 0$ *for some* N. *That is, every element is* ***unipotent****.*

Example: The ***unipotent radical*** $U_n = \{X \in T_n : x_{ii} = 1\}$. *Also, strict upper triangular matrices are unipotent.*

We next give some counterexamples.

Example 20 (projective algebraic group)**.** *For the projective variety* $\mathbb{P}^n$,
$$\mathcal{O}(\mathbb{P}^n) = k,$$
so we could not reconstruct $\mathbb{P}^n$ *from its coordinate ring.*

Thus projective varieties, in particular abelian varieties, do not yield Hopf algebras in this way.

Example 21 (elliptic curve)**.** *An elliptic curve in* $\mathbb{P}^2$, *given by:*
$$x_0 x_2^2 = x_1^3 + a x_1 x_0^2 + b x_0^3, \qquad \Delta = 4a^3 + 27b^2 \neq 0,$$
carries out a group with identity point $[0, 0, 1]$.

This is an example of a non-affine algebraic group.

4.4 Rational representations

Let k be an algebraically closed field and let G be a linear algebraic group over k. We want to study linear actions of G on finite-dimensional vector spaces that are compatible with the algebraic structure of G. This leads to the notion of rational representations. Such tool will be useful in studying the structure of G, to be elaborated in the next section.

Definition 25 (rational representation)**.** *A* ***rational representation*** *of a linear algebraic group G on a finite-dimensional k-vector space V is a morphism of algebraic groups:*

$$\rho : G \longrightarrow \mathrm{GL}(V).$$

Equivalently, it is a regular action of G on V, i.e. a morphism

$$G \times V \longrightarrow V$$

that is linear in the second variable.

Equivalently, specifying a rational representation of G on V is the same as giving a structure of a $k[G]$-comodule on V, that is, a linear map:

$$\Delta_V : V \longrightarrow V \otimes k[G]$$

satisfying the usual coassociativity and counit conditions.

A morphism between rational G-modules V and W is a k-linear map:

$$f : V \longrightarrow W$$

such that:

$$f(g \cdot v) = g \cdot f(v) \quad \forall g \in G,\, v \in V.$$

A subspace $W \subset V$ is called a G-*submodule* if it is stable under the action of G. The category of rational G-modules is an abelian category, and kernels, cokernels, and direct sums are formed as in the category of vector spaces.

Example 22. *We give some examples of rational representation of previously introduced algebraic groups.*

1. *The* ***trivial representation****, where G acts trivially on V.*
2. *The* ***standard representation*** *of GL_n on k^n.*
3. *For a character $\chi : G \to \mathbb{G}_m$, the one-dimensional representation*

$$g \cdot v = \chi(g)v.$$

4. *The* ***adjoint representation*** *of G on its Lie algebra $\mathrm{Lie}(G)$.*

Now we introduce three important propositions.

Let T be a torus over k, i.e. $T \cong (\mathbb{G}_m)^r$. The group of characters:

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$$

is a free abelian group of rank r. To see this, since k is algebraically closed, every torus splits, hence $T \simeq (\mathbb{G}_m)^r$. A morphism of algebraic groups $\chi : (\mathbb{G}_m)^r \to \mathbb{G}_m$ is given by a Laurent monomial:

$$\chi(t_1, \cdots, t_r) := t_1^{n_1} \cdots t_r^{n_r}, \qquad n_i \in \mathbb{Z}.$$

Thus,

$$X^*((\mathbb{G}_m)^r) \simeq \mathbb{Z}^r.$$

This identification is canonical up to the choice of splitting, and shows that $X^*(T)$ is a free abelian group of rank k.

Proposition 20. *Every rational representation V of a torus T decomposes as a direct sum of weight spaces:*

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi,$$

where

$$V_\chi := \{v \in V \mid t \cdot v = \chi(t)v \text{ for all } t \in T\}.$$

Proof. We may assume $T \cong (\mathbb{G}_m)^r$. A rational representation $\rho : T \to \mathrm{GL}(V)$ corresponds to a $k[T]$-comodule structure:

$$\Delta_V : V \to V \otimes k[T].$$

Since $k[T] = k[t_1^{\pm 1}, \cdots, t_r^{\pm 1}]$, it has a basis of characters $\{\chi : \chi \in X^*(T)\}$.

For each $v \in V$, we may write:

$$\Delta_V(v) = \sum_\chi v_\chi \otimes \chi,$$

with finitely many nonzero terms. Define:

$$V_\chi := \{v \in V : \Delta_V(v) = v \otimes \chi\}.$$

This space coincides with the eigenspace defined above. Because characters are linearly independent in $k[T]$, this decomposition is direct, and every vector lies in a finite sum of weight spaces. Hence $V = \bigoplus_{\chi \in X^*(T)} V_\chi$. □

This decomposition plays a fundamental role in the structure theory of representations of reductive groups, to be introduced later.

Proposition 21. *If U is a unipotent algebraic group, then every rational representation of U has a nonzero fixed vector:*

$$V^U := \{v \in V \mid u \cdot v = v \text{ for all } u \in U\} \neq 0.$$

More generally, every rational representation of U admits a filtration whose successive quotients are trivial representations.

Proof. We proceed by induction on $\dim U$. Since U is unipotent it has a nontrivial center $Z(U)$, and the center contains a subgroup isomorphic to $\mathbb{G}_a$. Restricting the representation to $\mathbb{G}_a$, we obtain a rational representation of $\mathbb{G}_a$ on V.

Every rational representation of $\mathbb{G}_a$ is unipotent, hence has a nonzero fixed vector. Therefore $V^{\mathbb{G}_a} \neq 0$.

Because $\mathbb{G}_a \subset Z(U)$, the subspace $V^{\mathbb{G}_a}$ is stable under U. The induced action of U on $V^{\mathbb{G}_a}$ factors through the quotient $U/\mathbb{G}_a$, which is again unipotent of smaller dimension. By induction,

$$V^U = (V^{\mathbb{G}_a})^{U/\mathbb{G}_a} \neq 0.$$

□

This fact will be a key ingredient in the proof of the Lie–Kolchin theorem afterwards.

A rational G-module V is called **semisimple** if it is a direct sum of irreducible submodules.

Proposition 22. *If G is a connected reductive algebraic group over k of characteristic zero, then every finite-dimensional rational representation of G is semisimple.*

In contrast, representations of solvable or unipotent algebraic groups are rarely semisimple, as later illustrated by the Lie–Kolchin theorem.

5 Structure theory

The first fundamental question in structure theory:

What do closed subgroups of GL_n look like, and if they can be put into a simple canonical form.

Surprisingly, solvable groups can always be triangularized after base change. This is the content of the Lie–Kolchin theorem for algebraic groups, the starting point for the classification of linear algebraic groups.

5.1 Connected and solvable subgroups

We will first introduce the concept of an algebraic group being connected and solvable.

Definition 26 (connected)**.** *Let G be an algebraic group over algebraically closed k. We say that G is **connected** if its underlying variety is connected in the Zariski topology.*

*The **connected component** of G is a connected algebraic subgroup of G which is **open-closed**, i.e., both open and closed. Clearly that each element in G belongs to only one connected component. We call the connected component containing the identity e as **identity component** and is denoted as G°.*

Proposition 23. *Given an algebraic group G over algebraically closed k, G is connected is equivalent to any of the followings:*

1. *G has no nontrivial idempotent in $k[G]$.*
2. *$k[G]$ has no decomposition as direct sums of two Hopf algebras.*
3. *G cannot be written as disjoint union of two nonempty open-closed subsets.*

The identity component of G has the following properties:

1. *G° is a closed normal connected algebraic subgroup.*
2. *G/G° is a finite discrete group, called the **component group**.*
3. *G° is the unique largest connected subgroup.*

Example 23. *We review previous examples.*

1. *$\mathbb{G}_a, \mathbb{G}_m, \mathrm{GL}_n, \mathrm{SL}_n$, and all classical groups are connected.*
2. *A finite group viewed as a $\dim 0$ algebraic group is not connected.*
3. *Over algebraically closed field k, $D_n \simeq (\mathbb{G}_m)^n$ and is therefore connected.*
4. *For field that is not algebraically closed, for example $k = \mathbb{R}$, the group $\mathbb{R}^\times$ has two connected components.*

Definition 27 (solvable)**.** *Given algebraic group G over algebraically closed k, the **derived series** of G is:*

$$G^{(0)} := G, \qquad G^{(i+1)} = [G^{(i)}, G^{(i)}].$$

*We say that G is solvable if the derived series **terminates**, i.e., $G^{(r)} = \{e_G\}$ for some finite r. The derived subgroup $[G, G]$ is the Zariski closure of the subgroup generated by commutators.*

Example 24. *We review previous examples.*

1. *All tori $(\mathbb{G}_m)^r$.*
2. *All unipotent groups, for example U_n.*
3. *All connected upper triangular matrices $B_n \subset \mathrm{GL}_n$.*
4. *The Borel subgroup of GL_n.*

In fact, every connected solvable group is built from tori and unipotent groups, to be shown later.

5.2 Connected solvable group decomposition

The next theorem will answer what structures connected solvable linear algebraic groups have.

Theorem 6 (Lie, Kolchin). *Let k be an algebraically closed field and let*

$$G \subset \mathrm{GL}(V)$$

be a connected solvable linear algebraic group acting rationally on a finite-dimensional k-vector space V. Then there exists a basis of V, such that every element of G is represented by an upper triangular matrix. Equivalently,

$$G \subset B_n(k).$$

In other words, every connected solvable linear algebraic group is triangularizable.

Proof. We argue by induction on $n := \dim V$.

We first prove that there exists a common eigenvector. Since G is connected and solvable, its derived subgroup $[G, G]$ is connected and unipotent. Let

$$U := [G, G].$$

Then U is a connected unipotent algebraic group acting rationally on V.

We claim that There exists a nonzero vector $v \in V$ fixed by U. Indeed, since U is unipotent, every element of U acts as a unipotent linear transformation.Equivalently, the representation of U on V admits a nonzero fixed vector:

$$V^U := \{v \in V \mid u \cdot v = v \text{ for all } u \in U\} \neq 0.$$

Now we fix $0 \neq v \in V^U$, and construct a G-stable line. Since U is normal in G, the subspace V^U is G-stable. The induced action of G on V^U factors through the abelian quotient G/U. Because G is connected, G/U is a connected commutative linear algebraic group. Hence every rational representation of G/U decomposes into weight spaces. In particular, there exists a character:

$$\chi : G \to \mathbb{G}_m$$

such that

$$g \cdot v = \chi(g)\, v \quad \text{for all } g \in G.$$

Thus the line $kv \subset V$ is G-stable.

Now we do induction on dimension. Consider the short exact sequence of G-modules

$$0 \longrightarrow kv \longrightarrow V \longrightarrow V/kv \longrightarrow 0.$$

Since $\dim(V/kv) = n - 1$, the induction hypothesis applies to the G-action on V/kv. Therefore, there exists a basis of V/kv with respect to which the image of G in $\mathrm{GL}(V/kv)$ consists of upper triangular matrices.

We choose a basis of V lifting this basis of V/kv and extending v. With respect to this basis, every element of G acts by an upper triangular matrix on V. By induction on $\dim V$, the representation of G is simultaneously triangularizable. Equivalently, $G \subset B_n(k)$. This completes the proof. □

The connectedness of the algebraic group is essential in the proof.

If G is more over unipotent, then theorem strengthens to $G \subset U_n(k)$.

Theorem 7 (decomposition theorem for connected solvable groups). *If G is a connected solvable algebraic group over algebraically closed k, then:*

$$G \simeq T \ltimes U,$$

where T is a torus and U is a connected unipotent group. They can be picked as T maximal, i.e., the ***maximal torus****, the maximal connected diagonalizable subgroup, and U the unipotent radical U_n.*

5.3 Borel subgroups and maximal tori

The Lie-Kolchin triangulation theorem gives a canonical example of a solvable group, called the Borel group.

Definition 28 (Borel subgroup)**.** *Given a linear algebraic group G, a* Borel subgroup *$B \subset G$ is a maximal connected solvable closed subgroup.*

Sometimes we would directly call the Borel subgroups as *Borels*. Its existence follows from Lie-Kolchin theorem, Zorn lemma, and the fact that solvable groups embed in B_n.

Example 25. *We review previous examples.*

1. *In GL_n, B_n the upper triangular matrices is Borel.*
2. *In SL_n, the intersection with upper triangular matrices $\mathrm{SL}_n \cap B_n$ is Borel.*
3. *In a torus T, the group itself $B = T$ is Borel.*

Proposition 24. *Let G be a connected linear algebraic group. Then:*

1. *All Borel subgroups are conjugate.*
2. *The quotient G/B is a smooth projective variety, i.e., the flag variety of G.*

Borel subgroups contain important commutative subgroups called maximal tori.

Definition 29 (maximal torus)**.** *A **maximal torus** in G is a torus not contained in any larger torus of G.*

Maximal tori are closely related to Borel subgroups:

Proposition 25. *Every maximal torus is contained in some Borel subgroup B.*

This tells that maximal torus $T \subset G$ always exist, and are conjugate with each other. Their common dimension is called the **rank** of G.

Example 26. *We revise previous examples.*

1. *For GL_n, the maximal torus is $T = \{\mathrm{diag}(t_1, \cdots, t_n)\} \simeq \mathbb{G}_m^n$ with dimension n. Clearly the rank of GL_n is the dimension of $\mathrm{diag}(t_1, \cdots, t_n)$, so the concept of rank for linear algebraic groups is the same with the rank of matrices.*
2. *For SL_n, the maximal torus is $T = \{\mathrm{diag}(t_1, \cdots, t_n) : \prod_i t_i = 1\} \simeq G_m^{n-1}$. It is also $\dim n$.*
3. *For $T = \mathbb{G}_m^r$, the group itself is the unique maximal torus.*
4. *A unipotent group U_n has no nontrivial torus.*

5.4 Jordan decomposition

Definition 30 (semisimple and unipotent elements)**.** *Let $g \in \mathrm{GL}_n$ over k algebraically closed.*

- *g is **semisimple** if it is diagonalizable over k. Equivalently, the minimal polynomial of g has distinct roots in k.*
- *g is **unipotent** if all its eigenvalues are 1. Equivalently, $g - I$ is nilpotent.*

An element that is both semisimple and unipotent must be the identity: $g = I$.

Over $\mathbb{C}$, semisimple elements correspond to diagonalizable matrices (in some basis), while unipotent elements correspond to matrices with all eigenvalues 1. For a general field, the Jordan decomposition exists in $\mathrm{GL}_n(\overline{F})$ and descends to $\mathrm{GL}_n(F)$ when $g \in \mathrm{GL}_n(F)$.

Example 27. *We revise previous examples.*

1. *Any diagonal matrix* $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ *in* GL_n *is semisimple.*

2. *A rotation matrix in* $\mathrm{SO}_2(\mathbb{R})$ *is semisimple over* $\mathbb{C}$.

3. *A unipotent Jordan block* $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is not semisimple.*

Theorem 8 (Jordan decomposition theorem). *Let $G \subset \mathrm{GL}_n$ be a linear algebraic group and $g \in G$. Then there exist uniquely a semisimple element $g_s \subset G$ and a unipotent element $g_u \subset G$, such that:*

$$g = g_s g_u = g_u g_s.$$

Moreover, g_s and g_u are polynomial expressions in g, and g_s lies in some maximal torus of G.

Example 28. *For group GL_n, the semisimple element g_s is the diagonalizable part of the matrix, and the unipotent element g_u is the 1-eigenvalue Jordan blocks.*

5.5 Reductive and semisimple groups

Given a linear algebraic group G, recall that the unipotent radical $R_u(G)$ is the largest connected normal unipotent subgroup of G.

Definition 31 (reductive and semisimple subgroups). *An algebraic group G over algebraically closed field k is* ***reductive*** *if it has trivial unipotent radical $R_u(G) = \{e_G\}$. That is, there is no nontrivial connected normal unipotent subgroups. Equivalently, every representation of G is completely reducible*

A connected linear algebraic group G over an algebraically closed field k is ***semisimple*** *if it has trivial radical $R_u(G) = \{e\}$.*

Equivalently,

- *G has no nontrivial connected solvable normal subgroups.*
- *G is a product, up to isogeny, of simple algebraic groups.*

With such notion, the Jordan decomposition theorem can be rephrased as: For every linear algebraic group G over algebraically closed k,

$$G \simeq S \times U,$$

where $g_s \in S$ a semisimple subgroup, and $g_u \in U$ a unipotent subgroup.

To see its relation with the connected solvable group decomposition, notice that every reductive group G can be decomposed as:

$$G \simeq [G, G] \cdot Z(G)^\circ,$$

where $[G, G]$ is the semisimple part, and $Z(G)^\circ$ is the central torus. Semisimple elements act like diagonalizable matrices, so they should live inside tori. This tells that:

Jordan decomposition induces connected solvable group decomposition.

Example 29 (semisimple subgroups). *We revise previous examples and introduce a new example.*

1. SL_n, SO_n, Sp_{2n} *are semisimple, and simple when n is in the appropriate range.*

2. *A maximal torus $T \cong \mathbb{G}_m^n$ in GL_n is reductive but not semisimple.*

3. *The subgroup* $\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in k^\times \right\}$ *is isomorphic to $\mathbb{G}_m$, which is reductive but not semisimple.*

Definition 32 (Simple algebraic group). *A connected linear algebraic group G over k is* ***simple*** *if:*

1. *G is non-commutative.*

2. *G has no proper nontrivial connected normal subgroups.*

Notice that we allow finite center $Z[G]$, thus a simple algebraic group need not be simple as a group, but only as an algebraic group.

Example 30 (simple subgroups)**.** *We revise previous examples.*

1. SL_n *for* $n \geq 2$ *is simple.*
2. $\mathrm{SO}_3 \cong \mathrm{PGL}_2$ *is simple.*

So far, we are concerning with reductive groups, semisimple groups, and simple groups. We clarify the relations between these notions here:

- Simple $\Rightarrow$ semisimple $\Rightarrow$ reductive
- A semisimple group is an almost direct product of simple groups
- GL_n is reductive but not semisimple, as its center $\mathbb{G}_m$ is a nontrivial torus
- A torus is reductive but not semisimple, unless trivial

Example 31. *We revise the previous examples.*

1. *For* GL_n*, its unipotent radical is trivial, and* $Z(G) \simeq \mathbb{G}_m$*, so it is reductive but not semisimple.*
2. *For* $\mathbb{G}_m^r$*, it is completely reductive, but not semisimple since the center is itself.*
3. *The groups* $\mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n$ *are all semisimple.*
4. *The Borel subgroups* B_n *is not reductive, since it contains the unipotent* U_n*.*
5. *Any unipotent group* U_n *is not reductive since it contains itself as the unipotent radical.*

5.6 Levi decomposition

Recall that for connected solvable groups we already have:

$$G \simeq T \ltimes U.$$

The general Levi decomposition applies to all connected linear algebraic groups.

Theorem 9 (Levi decomposition)**.** *Let* G *be a connected linear algebraic group over algebraically closed* k*, then:*

$$G \simeq L \ltimes R_u(G),$$

where $R_u(G)$ *is the unipotent radical, and* L *is a reductive subgroup, unique up to conjugacy. We call such a reductive subgroup the* ***Levi subgroup*** *of* G*.*

To see its relation with the connected solvable group decomposition, compare them as:

$$\textit{connected solvable}\, G \simeq T \ltimes U, \qquad \textit{connected}\, G \simeq L \ltimes R_u(G),$$

so if we give the linear group G a connected and solvable structure, then the reductive group can be picked as the maximal torus, and by loosening the conditions for maximal torus and unipotent radical, we could decompose G into some chosen torus T with corresponding unipotent subgroup U. This tells that:

Levi decomposition induces connected solvable group decomposition.

Example 32. *We revise previous examples.*

1. GL_n *has trivial unipotent, so the Levi subgroup is itself,* $L = G$*.*
2. *The Borel subgroups* B_n *has decomposition:*

$$B_n \simeq T \ltimes U_n,$$

so it has Levi subgroup $L \simeq T$*.*

6 Algebraic groups over general fields

In the subsection when introducing $\mathrm{Spm}(A)$ subsection 1.3 we already mentioned that A could be any commutative unital ring. Here we study $k = F$ to be arbitrary fields.

6.1 F-groups

To study algebraic groups over non-algebraically closed fields, we need the notion of an F-variety.

Definition 33 (F-variety)**.** *Let F an arbitrary field.*

*An F**-variety** is a reduced separated scheme of finite type over* $\mathrm{Spec}(F)$*.*

*An **affine** F**-variety** is the spectrum* $\mathrm{Spec}\, A$ *of a finitely generated reduced F-algebra A, with structure morphism induced by the F-algebra structure $F \to A$.*

The category of F-varieties Var_F has products defined by fiber product over $\mathrm{Spec}\, F$, denoted $X \times_F Y$.

Definition 34 (F-group)**.** *An **algebraic group over** F or an F**-group** is a group object in the category of F-varieties. More explicitly, it is an F-variety G together with F-morphisms:*

$$\textit{multiplication}: \quad m : G \times_F G \to G, \qquad \textit{inverse}: i : G \to G, \qquad \textit{identity}: \quad e : \mathrm{Spec}\, F \to G,$$

satisfying the usual group axioms.

If the field F is not algebraically closed, then we need more concepts in realizing their structures.

Definition 35 (F-rational points)**.** *Let G be an F-group. We call the set:*

$$G(F) := \mathrm{Hom}_F(\mathrm{Spec}\, F, G)$$

*as the set of F**-rational points** or F**-points**. It inherits a group structure from the morphisms m, i, and e. The group operation on $G(F)$ is given by $(g, h) \mapsto m(g, h)$ for $g, h \in G(F)$.*

Example 33. *We generalize previous examples.*

- $\mathbb{G}_a(F) \cong (F, +)$ *as groups.*
- $\mathbb{G}_m(F) \cong F^\times$ *as groups.*
- $\mathrm{GL}_n(F) = \{A \in M_n(F) \mid \det A \neq 0\}$*, the usual general linear group over F.*

6.2 Affine F-groups and Hopf algebras

Most algebraic groups encountered in practice are affine. In this case the group structure can be dualized to the structure of a Hopf algebra on the coordinate ring.

Coordinate ring as a Hopf algebra Let G be an affine F-group, and let $A_0 = F[G]$ be its coordinate ring (a finitely generated reduced F-algebra with $G = \mathrm{Spec}\, A_0$). The group structure on G translates to the following structure on A_0:

$$\textit{comultiplication}: \quad \Delta : A_0 \to A_0 \otimes_F A_0, \qquad \textit{counit}: \quad \varepsilon : A_0 \to F, \qquad \textit{antipode}: \quad \iota : A_0 \to A_0.$$

These maps satisfy the same axioms of a commutative Hopf algebra: coassociativity, counit, and antipode.

Conversely, any commutative Hopf algebra over F that is finitely generated as an F-algebra gives rise to an affine F-group.

Example 34. *Here are the open examples introduced previously.*

1. *$\mathbb{G}_a = \operatorname{Spec} F[t]$ as an affine F-variety. The Hopf algebra structure is given by:*

$$\Delta(t) = t \otimes 1 + 1 \otimes t, \qquad \varepsilon(t) = 0, \qquad \iota(t) = -t.$$

   The comultiplication encodes addition: $t \mapsto t + t'$ in $F[t] \otimes F[t'] \cong F[t, t']$. The group of F-points is $\mathbb{G}_a(F) \cong (F, +)$.

2. *$\mathbb{G}_m = \operatorname{Spec} F[t, t^{-1}]$. The Hopf maps are:*

$$\Delta(t) = t \otimes t, \qquad \varepsilon(t) = 1, \qquad \iota(t) = t^{-1}.$$

   The comultiplication encodes multiplication: $t \mapsto t \cdot t'$. We have $\mathbb{G}_m(F) \cong F^{\times}$.

3. *As an affine F-variety:*

$$\mathrm{GL}_n = \operatorname{Spec} F[x_{ij}, \det(x)^{-1}],$$

   where $\det(x)$ denotes the determinant polynomial in the variables x_{ij}. Write $X = (x_{ij})$ for the "generic" $n \times n$ matrix. The Hopf structure is:

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}, \qquad \varepsilon(x_{ij}) = \delta_{ij}, \qquad S(x_{ij}) = (X^{-1})_{ij},$$

   Thus $\mathrm{GL}_n(F)$ is the usual group of invertible $n \times n$ matrices over F.

Many important F-groups arise as closed subgroups of GL_n, defined by polynomial equations with coefficients in F:

Example 35. *Here are the closed linear examples introduced previously.*

1. *$\mathrm{SL}_n, \mathrm{O}_n, \mathrm{Sp}_{2n}$.*
2. *Torus T, for example the diagonal subgroups defined by equations like $x_{ij} = 0$ for $i \neq j$.*
3. *Unipotent subgroups U, for example U_n of upper-triangular matrices with ones on the diagonal.*

Each such subgroup is an affine F-group whose coordinate ring is the quotient of $F[\mathrm{GL}_n]$ by the ideal generated by the defining equations.

7 Further topics

7.1 Nonlinear algebraic groups

Theorem 10. *A smooth cubic curve $E \subset \mathbb{P}^2$ is a projective algebraic group, with group law defined by the chord-tangent construction. Its identity is a chosen inflection point.*

Example 36.
1. *Nonsingular cubic: elliptic curve is projective algebraic group;*
2. *Cuspidal cubic: isomorphic to $\mathbb{G}_a$;*
3. *Nodal cubic: isomorphic to $\mathbb{G}_m$.*

Proposition 26. *Nonlinear algebraic groups exists, but only in dimension 1.*

7.2 Representation theory

Studying the actions could help understand the geometry of the varieties. Actions allow us to study varieties via symmetry. Many geometric properties come from analyzing: orbits and orbit closures, stabilizers, quotients, invariants.

Definition 36 (action)**.** *An action of an algebraic group G on a variety X is a regular map:*

$$\varphi : G \times X \to X$$

such that X becomes a G-module, that is:

$$\varphi(g, \varphi(h, x)) = \varphi(gh, x).$$

7.3 Root systems and Weyl groups

The classical simple algebraic groups over algebraically closed fields are:

- SL_{n+1} (type A_n, $n \geq 1$)
- SO_{2n+1} (type B_n, $n \geq 2$)
- Sp_{2n} (type C_n, $n \geq 3$)
- SO_{2n} (type D_n, $n \geq 4$)
- The five exceptional groups: G_2, F_4, E_6, E_7, E_8

Fix a maximal torus $T \subset G$ contained in a Borel subgroup B. The adjoint action of T on the Lie algebra $\mathfrak{g}$ of G yields a weight space decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{t} = \mathrm{Lie}(T)$ and $\Phi \subset X^*(T)$ is a finite set of nonzero characters called the **roots** of G.

The structure $(\Phi, X^*(T))$ forms a **root system**. Associated to it is the **Weyl group**:

$$W = N_G(T)/T,$$

which acts faithfully on Φ and is generated by reflections corresponding to simple roots.

The classification of semisimple algebraic groups over algebraically closed fields corresponds to the classification of irreducible root systems via their Dynkin diagrams: A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2.

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