Report on Summer Research Project

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Contents:

The project

The Polymath Junior 2025 is an online math program funded by National Science Foundation (NSF). It has 12 projects in many realms, with mentors and participants from all over the world. Front page: https://geometrynyc.wixsite.com/polymathreu

I selected the project: Lattice Models and Representation Theory. The mentors are Professor Ben Brubaker at the University of Minnesota, and 2 PhD's. We have 14 participants.

The official period of this program is from late June to mid August. I joined regular meetings twice a week, and several extra meetings with the mentors or students, to discuss on progresses and thoughts.

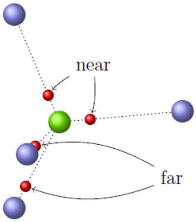
The results are open ended, and the mentors wish to further develop a paper in the coming school year. Most of the current works are shared in Overleaf: https://www.overleaf.com/project/6876baa866c2d95fd788ba19.

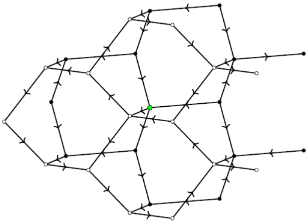
1. Lattice models

Lattice models arise from the structure of ice molecules.

Example

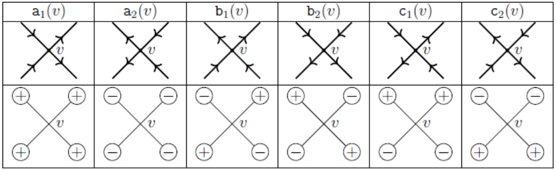
The left is a partial graph of mutually related H_2O's. Regarding to the hydrogen being near or far, we draw it as a directed graph as the right:





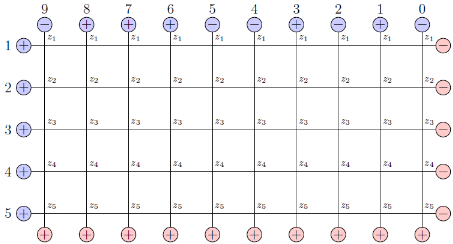
1. Lattice models

By focusing on any oxygen, and viewing entry arrows as $+$ and exit arrows as $-$, we simplify each molecule as one of the followings:



A **lattice model** is a set of vertices and edges in $\mathbb{Z}^n$, together with boundary conditions of colors.

Example

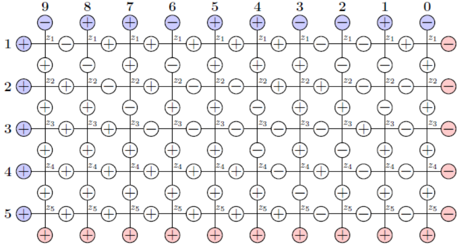


This is a lattice model in $\mathbb{Z}^2$ with two colors $+$ and $-$. The actual color blue and red are explained later.

1.1. Admissible states

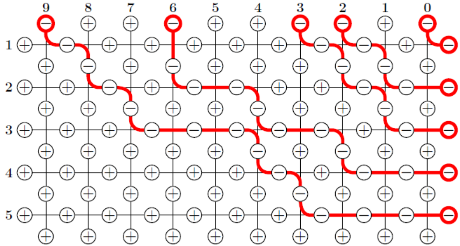
A **solution / admissible state** to a lattice model is a filling of colors inside the model.

Example



This amount to find paths that connect $+$'s and $-$'s respectively. The blues are entries and the reds are exists, required for those paths.

Example

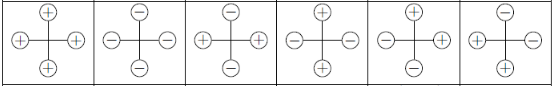


1.2. R-matrices and partition functions

In each admissible state, every vertex gives an R-**matrix**, referring to the weight of this vertex.

Example

a_1	a_2	b_1	b_2	c_1	c_2
1	z_i	$-q$	z_i	$z_i(1-q)$	1



where the bottom line refers to weights, depending on variables z_i's and parameter q.

The **partition function** Z for this lattice model S is defined as:

$$Z(S) = \sum_{\text{all admissible states,}} \prod_{\text{all vertices in the lattice}} \text{weight(vertex)}$$

As it generalizes, there can be more than two colors, and some special ones would enter from north or east, and exit from south or west.

2. Questions

The project separated into three groups working on different problems. I mostly worked on questions 1 and 2.

1. List all possible choices of R-matrices, so that the lattice model has admissible states.
—— quantum group representations.

2. Find stategies in reducing partition functions to be easier presented.
—— use ABW proposition in reducing colors.
(—— use Alternating Sign Matrices in reducing admissible states. Less focused by myself.)

(3. Find different kinds of functions that arise as partition functions of some lattice models.
—— this is also the main focus for the project last year, with the paper Kirillov's Conjecture on Hecke–Grothendieck Polynomials, arXiv:2410.07960v1)

2.1. Quantum group representations

Every R-matrix $R_{\mathbf{z}}$ lives in $\mathrm{End}(V \otimes W)$, where. Here V, W are vector spaces spanned by color sets.

Every vector space V spanned by color sets rises as quantum group modules, where the quantum group often picked as $U_q(\widehat{\mathfrak{sl}}(m|n))$.

Example

As in the previous slides, assume we are working in 2 dimensional six-vertex models with two colors, then:

$$\begin{aligned} V = W = \mathbb{C}\{+,-\} \simeq \mathbb{C}^2 &\implies R_{\mathbf{z}} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \simeq \mathrm{End}(\mathbb{C}^4) \\ &\implies R_{\mathbf{z}} \in U_q(\mathfrak{sl}(2)) \curvearrowright \mathbb{C}^4. \end{aligned}$$

Hence R-matrices are special elements in quantum group representations, namely an intertwiner satisfying quantized Yang-Baxter Equation. More to be found in the draft of survey.

Finding all quantum group representations is still active in research.

2.1. Quantum group Representations

Realizations and bosonizations can be viewed as more general concepts in studying representations.

Past results

Awata, Noumi, Odake, 1993, studied the Heisenberg realization of $U_q(\mathfrak{sl}_n)$. arXiv:hep-th/9306010

Awata, Odake, Shiraishi, 1993, studied the free Boson realization of $U_q(\widehat{\mathfrak{sl}_N})$. arXiv:hep-th/9305146

Awata, Odake, Shiraishi, 1997, studied the q-difference realization of $U_q(\mathfrak{sl}(M|N))$. arXiv:q-alg/9701032

Kojima, 2011, studied the free field realization of $U_q(\widehat{\mathfrak{sl}}(N|1))$. arXiv:1105.5772

Kojima, 2014, studied the bosonization of $U_q(\widehat{\mathfrak{sl}}(N|1))$ at any level. arXiv:1404.5744

Kojima, 2017, studied a bosonization of $U_q(\widehat{\mathfrak{sl}}(M|N))$. arXiv:1701.03645

Some thoughts

1. What is the general bosonization of $U_q(\widehat{\mathfrak{sl}}(M|N))$.
2. Kojima is currently focusing on W-algebras. What is the theory behind and if it helps with this question.

2.2. Reducing colors

Construct a lattice model in domain $\mathcal{D} = \mathcal{D}(\mathbf{p}, \mathbf{p}') \subset \mathbb{Z}^2$ by filling with grids, where $\mathbf{p}, \mathbf{p}'$ its boundaries.

Given boundary conditions $\mathfrak{E}$ on $\mathbf{p}$ and $\mathfrak{F}$ on $\mathbf{p}'$ and R-matrices $R_{\mathbf{z}}$, the partition function is:

$$Z_{\mathcal{D}}(\mathfrak{E}; \mathfrak{F} | \mathbf{z}) = \sum_{\text{states}} \prod_{v \in \mathcal{D}} R_{\mathbf{z}(v)}.$$

Proposition (Aggarwal, Borodin, Wheeler)

If $\mathcal{D}$ is a south-east domain, then the partition function can be reduced with less colors in $\mathfrak{E}$ and $\mathfrak{F}$. To be specific,

$$\sum_{\breve{\mathfrak{E}}, \breve{\mathfrak{F}}} \pm Z_{\mathcal{D}}(\breve{\mathfrak{E}}; \breve{\mathfrak{F}}) = Z_{\mathcal{D}}(\theta\mathfrak{E}; \theta\mathfrak{F}),$$

where θ reassigns colors such that $\theta\breve{\mathfrak{E}} = \theta\mathfrak{E}, \theta\breve{\mathfrak{F}} = \theta\mathfrak{F}$.

2.2. Reducing colors

Here we outline the main idea of the prove for the proposition, where its ideas can be used in computations elsewhere:

Step one: reduce $\mathcal{D}$ to $\mathbf{p} = \mathbf{p}'$.
Step two: reduce $\mathcal{D}$ in length.

Its key induction process is of the form:

$$\sum_{\mathfrak{e},\mathfrak{f},\mathfrak{E},\mathfrak{F}} \pm Z_{\{v\}}(\breve{\mathfrak{e}};\breve{\mathfrak{f}}) Z_{\mathcal{D}\setminus\{v\}}(\breve{\mathfrak{E}};\breve{\mathfrak{F}}) = Z_{\{v\}}(\theta\mathfrak{e};\theta\mathfrak{f}) Z_{\mathcal{D}}(\theta\mathfrak{E};\theta\mathfrak{F}),$$

where $\mathfrak{e},\mathfrak{f}$ are boundary conditions for vertex v. This process inspires me to make the following conjectures.

Conjecture 1

The proposition holds for arbitrary domain $\mathcal{D}$.

Conjecture 2

This proposition holds simultaneously with different choice of $\mathbf{z}$.

References

Besides what is mentioned, some more papers I studied on:

Aggarwal, Borodin, Wheeler: Colored Fermionic Vertex Models and Symmetric Functions, arXiv:2101.01605v2

Brubaker, Buciumas, Bump, Gustafsson: Metaplectic Iwahori Whittaker Functions And Suppersymmetric Lattice Models, arXiv:2012.15778v4

Kojima: Diagonalization of Transfer Matrix of Supersymmetry $U_q(\widehat{\mathfrak{sl}}(M+1|N+1))$, arXiv:1211.2912v1

Some textbooks I studied on:

Bump: Lie Groups
Bump, Schilling: Solvable Lattice Models
Fulton, Harris: Representation Theory
Kac: Infinite-Dimensional Lie Algebras
Kassel: Quantum Groups
Majid: A Quantum Groups Primer

Most of the graphics in this report is taken from Bump, Schilling: Solvable Lattice Models, or jointly made with other members.