Final Exam Recitation

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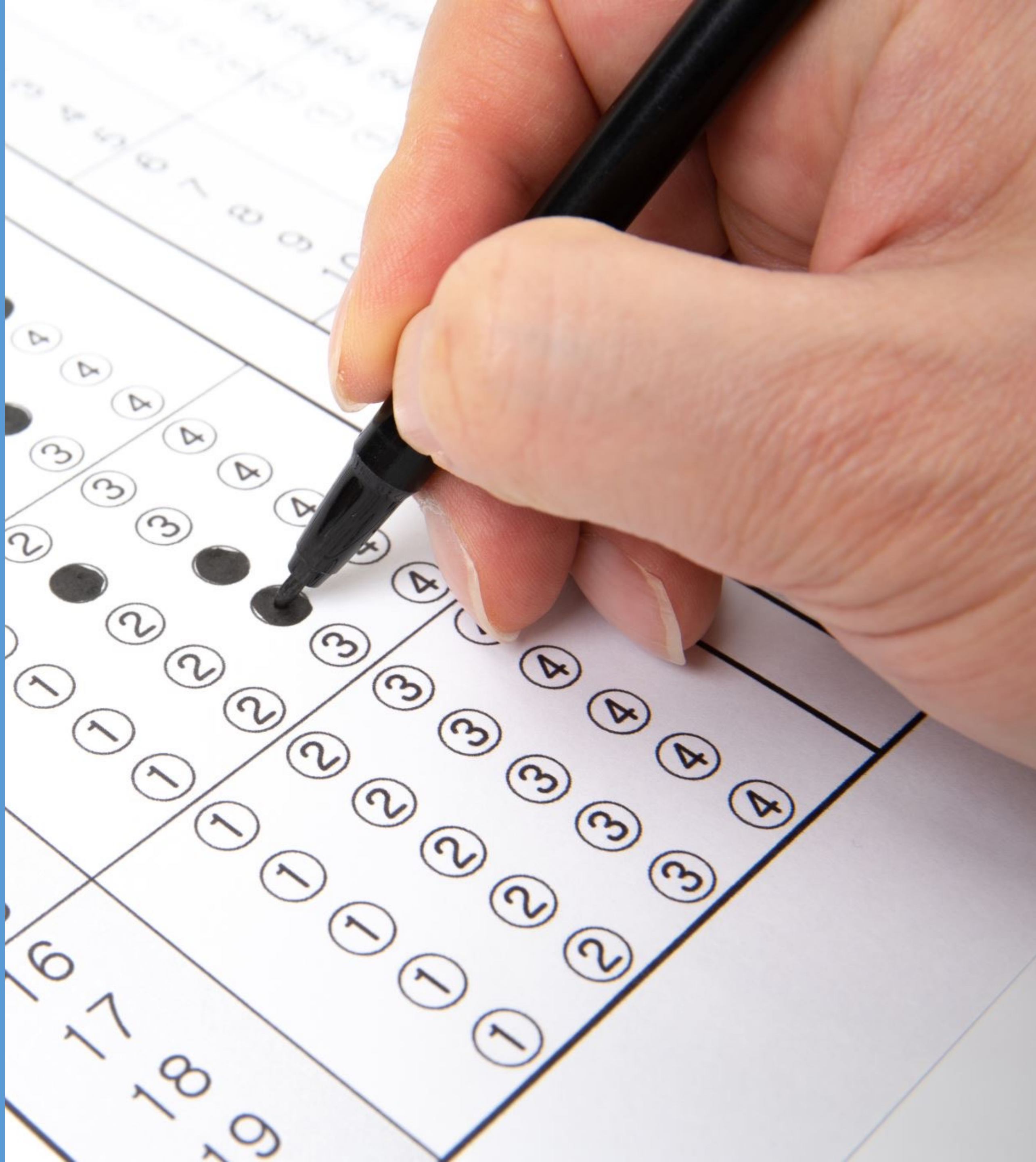

Stochastic Exponential

If $\alpha \in \mathcal{L}^1$ and $\beta \in \mathcal{L}^2$ are processes of dimension 1×1 and $1 \times K$, respectively, define a one-dimensional process $\eta[\alpha, \beta]$ by

$$\eta[\alpha, \beta](t) = \exp\left[\int_0^t \left(\alpha - \frac{1}{2}\beta\beta^\top\right) ds + \int_0^t \beta \, dW\right]$$

A process of this form is called a *stochastic exponential*.

It is an Itô process, by Itô's lemma.

Stochastic exponential

- Let X be a continuous local martingale.
- The **stochastic exponential** (also called **Doléans-Dade exponential**) of X, denoted by $\mathcal{E}(X)$, is defined as the solution Z to the SDE

$$dZ(t) = Z(t)dX(t).$$

  with initial condition $Z(0) = 1$.
- *Can define stochastic exponential for continous semi-martingales.
- What is $\mathcal{E}(X)$ when X is a constant multiple of BM?
- Find a formula for $\mathcal{E}(X)$ when X is a general continuous local martingale. (By hand)

$$\mathcal{E}(X)(t) = \exp\left(X(t) - X(0) - \frac{1}{2}[X,X](t)\right).$$

- What's nice about stochastic exponential?
- Simplify $\int_0^t \frac{1}{Z(s)} d[Y,Z](s)$ when $Z = \mathcal{E}(X)$. (By hand)

Girsanov's Theorem

Theorem

Girsanov's theorem.

Suppose $E_P \eta[0, -\lambda](T) = 1$.

Let Q *be the probability measure which has density* $\eta[0, -\lambda](T)$ *with respect to* P.

Then the process W^λ *is a standard* K*-dimensional Wiener process with respect to* F *and* Q.

Girsanov's Theorem

Girsanov theorem

Theorem (Girsanov; Cameron and Martin)

Let W be a BM defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{F}(t)\}$ be the natural (augmented) filtration generated by W. Let X be a continuous $\mathbb{P}$-local martingale. Suppose $\tilde{\mathbb{P}}$ is another measure on $(\Omega, \mathcal{F})$ whose density process w.r.t. $\mathbb{P}$ is $\mathcal{E}(X)$, i.e.

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}(t)} = \mathcal{E}(X)(t).$$

Then $\widetilde{W} := W - [W, X]$ is a $\tilde{\mathbb{P}}$-BM.

- When $X(t) = \int_0^t \theta(s) dW(s)$, $\widetilde{W}(t) = W(t) - \int_0^t \theta(s) ds$.
- In other words, under $\tilde{\mathbb{P}}$, W is a drifted BM with drift θ.
- Sketch proof by hand. (Use Girsanov-Meyer and Levy)

$$\mathcal{E}(X)(t) = \exp\left(X(t) - X(0) - \frac{1}{2}[X, X](t)\right).$$

Novikov Condition

Proposition

The process $\eta[0, -\lambda]$ *is a martingale on* $[0, T]$ *if and only if* $E\eta[0, -\lambda](T) = 1$.

A sufficient condition for this is the so-called Novikov condition:

$$E \exp\left[\frac{1}{2}\int_0^T \lambda\lambda^\top ds\right] < \infty$$

Novikov condition

- $\mathcal{E}(X)$ needs to be a martingale to serve as a density process.
- In general, we only know $\mathcal{E}(X)$ is a local martingale.
- When is it a true martingale?
- **Novikov condition** says $\mathcal{E}(X)$ is a martingale on $[0, T]$ if

$$\mathbb{E}\exp\left(\frac{1}{2}[X,X](T)\right) < \infty.$$

- When $X(t) = \int_0^t \theta(s)dW(s)$, the condition reads

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T \theta^2(s)ds\right) < \infty.$$

Conditional Gaussian One Factor

A *conditionally Gaussian one-factor processes* is an Itô process r whose stochastic differential has the form

$$dr = (\alpha - ar)\, dt + \sigma\, dW$$

where α, a, and σ are deterministic processes with $\sigma \in \mathcal{L}^2$ and $a, \alpha \in \mathcal{L}^1$.

There exists a unique Itô process r such that

$$dr = (\alpha - ar)\, dt + \sigma\, dW$$

and $r(0) = r_0$. It is given by

$$r(t) = e^{-K(t)} \left[r_0 + \int_0^t e^K \alpha\, ds + \int_0^t e^K \sigma\, dW \right]$$

OU Process

Assume that α, a, and σ are deterministic constants.

Assume W is a one-dimensional Wiener process relative to F.

If $a = 0$, then the equation

$$dr = (\alpha - ar)\,dt + \sigma\,dW$$

reduces to

$$dr = \alpha\,dt + \sigma\,dW$$

whose solution is the generalized Wiener process

$$r(t) = r_0 + \alpha t + \sigma W(t)$$

The process

$$r(t) = e^{-at} r_0 + \left(1 - e^{-at}\right) b + \sigma e^{-at} \int_0^t e^{au} \, dW(u)$$

is the unique Itô process r such that

$$dr = a(b - r) \, dt + \sigma \, dW$$

and $r(0) = r_0$.

For $0 \leq s \leq t$, we can express $r(t)$ in terms of $r(s)$ as follows:

$$r(t) = e^{-a(t-s)}r(s) + (1 - e^{-a(t-s)})b + \sigma e^{-at} \int_s^t e^{au}\, dW(u)$$

The conditional mean function of r is

$$E(r(t) \mid \mathcal{F}_s) = e^{-a(t-s)}r(s) + (1 - e^{-a(t-s)})b$$

for $0 \leq s \leq t$.

If r_0 is integrable, then r is integrable, and the unconditional mean of $r(t)$ is

$$Er(t) = e^{-at}Er_0 + \left(1 - e^{-at}\right) b$$

for $0 \leq t$.

The conditional covariance function is

$$\begin{aligned} \mathrm{Cov}(r(t), r(u) \mid \mathcal{F}_s) &= e^{-K(u)-K(t)} \int_s^t e^{2K} \sigma^2 \, dv \\ &= e^{-a(u-t)} \frac{\sigma^2}{2a} (1 - e^{-2a(t-s)}) \end{aligned}$$

for $0 \leq s \leq t \leq u$.

In particular, the conditional variance is

$$\mathrm{Var}(r(t) \mid \mathcal{F}_s) = \frac{\sigma^2}{2a} (1 - e^{-2a(t-s)})$$

In particular, the unconditional variance is

$$\mathrm{Var}(r(t)) = e^{-2at} \mathrm{Var}(r_0) + \frac{\sigma^2}{2a} \left(1 - e^{-2at}\right)$$

Strong Existence

Strong Existence

Let ν be a probability measure on $\mathbb{R}^N$.

Say that *strong existence* holds for (μ, σ, ν) if for every K dimensional setup $(\Omega, \mathcal{F}, P, F, W)$ and for every K dimensional random vector ξ which is measurable with respect to $\mathcal{F}_0$ and has distribution ν, there exists a solution X of (μ, σ) on $(\Omega, \mathcal{F}, P, F, W)$ with $X(0) = \xi$.

Say that *strong existence* holds for (μ, σ) if for every probability distribution ν on $\mathbb{R}^N$, strong existence holds for (μ, σ, ν).

Weak Existence

Weak Existence

Let ν be a probability measure on $\mathbb{R}^N$.

Say that *weak existence* holds for (μ, σ, ν) if there exists a K dimensional setup $(\Omega, \mathcal{F}, P, F, W)$ and a solution X of (μ, σ) on $(\Omega, \mathcal{F}, P, F, W)$ such that $X(0)$ has distribution ν.

Say that *weak existence* holds for (μ, σ) if for every probability distribution ν on $\mathbb{R}^N$, weak existence holds for (μ, σ, ν).

So, strong existence means that a solution exists on any setup and for any initial value with distribution ν.

By contrast, weak existence means only that a solution exists for some setup and some initial value with distribution ν.

Pathwise Uniqueness

Pathwise Uniqueness

Say that *pathwise uniqueness* holds for (μ, σ, ν) if for every K dimensional setup $(\Omega, \mathcal{F}, P, F, W)$ and any two solutions X and Y of (σ, μ) on $(\Omega, \mathcal{F}, P, F, W)$ such that $X(0) = Y(0)$ with probability one and such that $X(0)$ and $Y(0)$ have distribution ν, X and Y are indistinguishable.

Say that *pathwise uniqueness* holds for (μ, σ) if for every probability distribution ν on $\mathbb{R}^N$, pathwise uniqueness holds for (μ, σ, ν).

Indistinguishable

- Given stochastic processes X and Y, we say Y is **indistinguishable** from X if their paths coincide a.s., i.e. $\mathbb{P}(X(t) = Y(t)\,\forall t \in \mathbb{T}) = 1$.
- Remark: there is also a weaker notion called modification. We say Y is a **modification** of X if $\mathbb{P}(X(t) = Y(t)) = 1\ \forall t \in \mathbb{T}$. Clearly, indistinguishable $\Rightarrow$ modification.

Let W be a Brownian motion and τ be its first passage time to level 1. Define another stochastic process Y by

$$Y(t) := \begin{cases} W(t), & \tau \neq t, \\ -W(t), & \tau = t. \end{cases}$$

Is Y a modification of W? Is Y indistinguishable from W? Justify your answers.

Uniqueness in Distribution

Uniqueness in Distribution

Say that *uniqueness in distribution* (or in law) holds for (μ, σ, ν) if for any two solutions X and $\tilde{X}$ on K dimensional setups $(\Omega, \mathcal{F}, P, F, W)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{F}, \tilde{W})$ such that $X(0)$ and $\tilde{X}(0)$ both have distribution ν, X and $\tilde{X}$ have the same distribution.

Say that *uniqueness in distribution* (or in law) holds for (μ, σ) if for every probability distribution ν on $\mathbb{R}^N$, uniqueness in distribution holds for (μ, σ, ν).

Here we are talking about distributions on path space.

Q

Theorem

Weak existence and pathwise uniqueness for (μ, σ, ν) imply strong existence and uniqueness in distribution for (μ, σ, ν).

This theorem is often used for establishing strong existence.

More

- Markov & Martingale
- BS formula derivation
- Vasicek Model
- right-continuous filtrations

Appendix

Stochastic Exp

Girsanov-Meyer when Z is a stochastic exponential

Proposition

Suppose X, Y are continuous $\mathbb{P}$-local martingales, and $Z = \mathcal{E}(X)$ is the density process of $\tilde{\mathbb{P}}$ w.r.t. $\mathbb{P}$. Then

$$Y - [Y, X]$$

is a $\tilde{\mathbb{P}}$-local martingale.

$$\mathcal{E}(X)(t) = \exp\left(X(t) - X(0) - \frac{1}{2}[X, X](t)\right).$$

Change of measure

- In many financial applications, we have two probability measures: the actual probability measure $\mathbb{P}$ and the risk-neutral probability measure $\tilde{\mathbb{P}}$
- How are different measures related?
- In a finite probability space, can try to find $Z(\omega)$ such that

$$\tilde{\mathbb{P}}(\omega) = Z(\omega)\mathbb{P}(\omega)$$

  - If $Z > 1$, probability is revised upward; if $Z < 1$, probability is revised downward.
  - For any r.v. X, have that

$$\tilde{\mathbb{E}}X = \sum_{\omega\in\Omega} X(\omega)\tilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega} X(\omega)Z(\omega)\mathbb{P}(\omega) = \mathbb{E}[XZ].$$

  - Z exists if and only if $\tilde{\mathbb{P}}(\omega) = 0$ whenever $\mathbb{P}(\omega) = 0$.

Equivalent measures

- Let $\mathbb{P}$, $\tilde{\mathbb{P}}$ be probability measures on $(\Omega, \mathcal{F})$. We say $\tilde{\mathbb{P}}$ is *absolutely continuous* with respect to $\mathbb{P}$, written as $\tilde{\mathbb{P}} \ll \mathbb{P}$, if for any $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ implies $\tilde{\mathbb{P}}(A) = 0$.
  (Anything impossible under $\mathbb{P}$ is also impossible under $\tilde{\mathbb{P}}$.)
- $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are said to be *equivalent*, written as $\mathbb{P} \sim \tilde{\mathbb{P}}$, if $\tilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \tilde{\mathbb{P}}$, i.e. $\mathbb{P}$ and $\tilde{\mathbb{P}}$ agree on what is possible and what is impossible.
- Example by hand.

Radon-Nikodým derivative

Theorem (Radon-Nikodým)

Let $\mathbb{P}$, $\tilde{\mathbb{P}}$ be equivalent probability measures on $(\Omega, \mathcal{F})$. Then there exists an almost surely positive random variable Z such that $\mathbb{E}Z = 1$ and

$$(*) \qquad \tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for every } A \in \mathcal{F}.$$

- Z in the above theorem is called the *Radon-Nikodým derivative* of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, denoted by $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$.
- Can you derive $\mathbb{E}Z = 1$ from $(*)$?
- If only $\tilde{\mathbb{P}} \ll \mathbb{P}$ is assumed, then Z is only nonnegative.

New measures from the old

- Given a reference measure $\mathbb{P}$, any $\tilde{\mathbb{P}} \ll \mathbb{P}$ can be specified via its Radon-Nikodým derivative Z.

$$\tilde{\mathbb{P}}(A) := \int_A Z(\omega) d\mathbb{P}(\omega), \; A \in \mathcal{F}.$$

- To guarantee the new measure is a probability measure, we need $Z \geq 0$ ($\mathbb{P}$-)a.s. and $\mathbb{E}Z = 1$.
  (Exercise. Verify $\tilde{\mathbb{P}}$ is a probability measure.)
- It's clear that $\tilde{\mathbb{P}} \ll \mathbb{P}$. To guarantee $\tilde{\mathbb{P}} \sim \mathbb{P}$, need $Z > 0$ a.s.
- Assuming $Z \geq 0$ a.s. and $\mathbb{E}Z = 1$, we have for any nonnegative or $\tilde{\mathbb{P}}$-integrable random variable X,

$$\tilde{\mathbb{E}}X = \mathbb{E}[XZ].$$

Example. Change of measure for a normal random variable

Let X be a standard normal random variable defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $Y = X + \theta$ for some constant θ.

- Compute the moment-generating function $\mathbb{E}e^{tX}$, $t \in \mathbb{R}$.
- Construct another measure $\tilde{\mathbb{P}} \sim \mathbb{P}$ under which Y is standard normal. (Fact: the moment-generating function of a random variable determines its distribution.)

Change of measure

Change of measure

- Given $\mathbb{P}$, and a $\mathbb{P}$-a.s. non-negative r.v. Z satisfying $\mathbb{E}Z = 1$, we can construct a new measure $\tilde{\mathbb{P}} \ll \mathbb{P}$ by $d\tilde{\mathbb{P}}/d\mathbb{P} = Z$, i.e.

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

- If $Z > 0$ a.s., we have $\tilde{\mathbb{P}} \sim \mathbb{P}$.
- $\tilde{\mathbb{E}}[X] = \mathbb{E}[XZ]$.
- We have seen two examples:
  - If $X \sim \mathcal{N}(0,1)$ under $\mathbb{P}$, and $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{\theta X - \frac{1}{2}\theta^2}$, then $X \sim \mathcal{N}(\theta, 1)$ under $\tilde{\mathbb{P}}$.
  - If $X \sim \exp(\lambda)$ under $\mathbb{P}$, and $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)X}$, then $X \sim \exp(\tilde{\lambda})$ under $\tilde{\mathbb{P}}$.

Strong & Weak Solution

Strong vs. weak solution

- A **strong solution** of $(*)$ is a progressively measurable process X satisfying

$$\int_0^T |b(u, X(u))| du + \int_0^T |\sigma(u, X(u))|^2 du < \infty$$

and

$$X(t) = x_0 + \int_0^t b(u, X(u)) du + \int_0^t \sigma(u, X(u)) dW(u),\ t \in [0, T].$$

- A **weak solution** of $(*)$ is a process X defined on *some* probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for *some* BM W and *some* filtration $\mathbb{F}$ for the BM, all requirements in the previous definition hold true.
- The probabilistic structure $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$ is given for a strong solution and is part of the unknown for a weak solution.

Existence, uniqueness and properties of strong solution

Theorem

Suppose that there exists a constant K such that the following conditions are satisfied for all x, y and t.

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|,$$
$$|b(t,x)| + |\sigma(t,x)| \leq K(1+|x|).$$

Then there exists a unique strong solution to $(*)$, up to indistinguishability. The solution has the following properties:

(i) X is adapted and has continuous paths.

(ii) X is a Markov process.

(iii) There exists a constant C_T such that

$$\mathbb{E}[\sup_{0\leq t\leq T} |X(t)|^2] \leq C_T(1+|x_0|^2).$$

Existence, uniqueness and properties of strong solution

- Existence is ensured by Lipschitz and linear growth condition.
- Uniqueness is pathwise.
- Existence can be proved using Picard's iteration: $X_t^{(0)} \equiv x_0$,

$$X_t^{(n+1)} = x_0 + \int_0^t b(u, X^{(n)}(u))du + \int_0^t \sigma(u, X^{(n)}(u))dW(u).$$

- For a detailed proof, see Section 9.4 of Steele.
- What is the intuition behind the Markov property?

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t)$$

The distribution of $dX(t)$ only depends on $X(t)$ but not on past information.