Set Theory

Relations II

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Content

- Equivalence relation (等价关系)
- Order relation (序关系)

Equivalence Relations

等价关系满足：自反、对称、可传递性。

Equivalence Relations

- Equivalence relations are used to relate objects that are similar in some way.
- **Definition**: A relation on a set A is called an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.
- Two elements that are related by an equivalence relation R are called equivalent. 通过等价关系R相关的两个元素称为等价。

Equivalence Relations

- Since R is symmetric, a is equivalent to b whenever b is equivalent to a.
- Since R is reflexive, every element is equivalent to itself.
- Since R is transitive, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.
- Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

- **Example**: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a)=l(b)$,where $l(x)$ is the length of the string x. Is R an equivalence relation?
- **Solution**:
  - R is reflexive, because $l(a) = l(a)$ and therefore aRa for any string a.
  - R is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa.
  - R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc.
- R is an equivalence relation.

等价类

Equivalence Classes

- **Definition**: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.
- The equivalence class of a with respect to R is denoted by $[a]_R$.
- When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class. 当只有一种关系R的时候可以把下标省略直接写[a]
- If $b \in [a]_R$, b is called a representative of this equivalence class.

b是 $[a]_R$ 这个等价类的代表

Equivalence Classes

- **Example**: In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?
- **Solution**:
  - [mouse] is the set of all English words containing five letters.
  - For example, 'horse' would be a representative of this equivalence class.

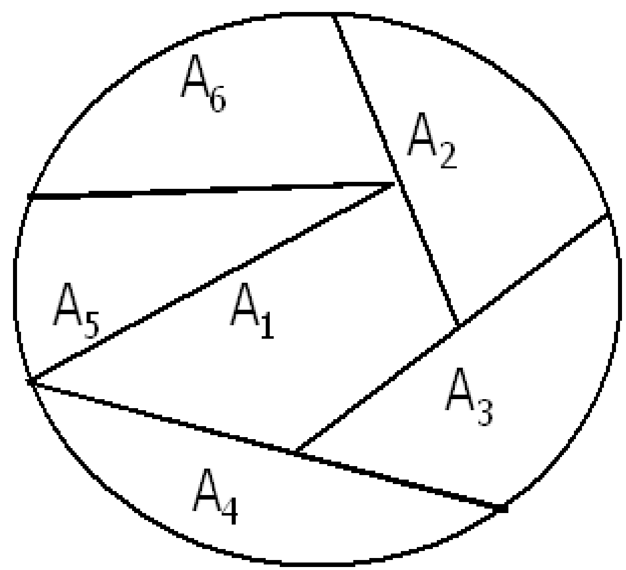
等价类

Equivalence Classes

- **Theorem**: Let R be an equivalence relation on a set A. The following statements are equivalent:
  - 1. aRb
  - 2. [a] = [b]
  - 3. $[a] \cap [b] \neq \emptyset$ 集合的划分

集合S的分区是S的不相交非空子集的集合，这些子集以S为联合。换句话说，子集Ai，i∈I构成了S iff的分区

- **Definition**: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$, forms a partition of S iff
  - 1. $A_i \neq \emptyset$ for $i \in I$
  - 2. $A_i \cap A_j = \emptyset$, if $i \neq j$
  - 3. $\cup_{i \in I} A_i = S$



Equivalence Classes

Examples: Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition S ? 是集合的划分吗?

- {{m, o, c, k}, {r, u, b, s}} yes.
- {{c, o, m, b}, {u, s}, {r}} no (k is missing).
- {{b, r, o, c, k}, {m, u, s, t}} no (t is not in S).
- {{u, m, b, r, o, c, k, s}} yes.
- {{b, o, o, k}, {r, u, m}, {c, s}} yes ({b,o,o,k} ={b,o,k}). 同一个集里 可重复
- {{u, m, b}, {r, o, c, k, s}, ∅} no (∅ not allowed).

Equivalence Classes

定理：令R是集合S上的等价关系。然后R的等价类形成S的一个划分。相反，给定一个分区{A i I集合S的i∈I}，存在一个等价关系R，它具有集合A i i∈I作为其等价类。

- **Theorem**: Let R be an equivalence relation on a set S. Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.
- **Example**:
  - Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney.

Equivalence Classes

- Let R be the equivalence relation {(a, b) | a and b live in the same city} on the set P = {Frank, Suzanne, George, Stephanie, Max, Jennifer}.
- Then R = { (Frank, Frank), (Frank, Suzanne), (Frank,George), (Suzanne,Frank), (Suzanne,Suzanne), (Suzanne,George), (George,Frank), (George, Suzanne), (George,George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max Max), (Jennifer, Jennifer) }.

Equivalence Classes

- Then the equivalence classes of R are:
  - {{Frank, Suzanne, George}, {Stephanie, Max}, {Jennifer}}.
  - This is a partition of P.
- The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to exactly one of the equivalence classes.

在集合S上定义的任何等价关系R的等价类都构成S的分区，因为S中的每个元素都精确地分配给其中一个等价类。

Equivalence Classes

(考点)

- Example:

  模3同余 → 模等价

  - Let R be the relation {(a,b)|a≡b(mod 3)} on the set of integers.
  - Is R an equivalence relation?

    模等价

    - Yes, R is reflexive, symmetric, and transitive.
  - What are the equivalence classes of R ?
    - { {…, -6, -3, 0, 3, 6, …}, {…, -5, -2, 1, 4, 7, …}, {…, 4, 1, 2, 5, 8, …} }

$\{(a,b) \mid a,b \in R \wedge \frac{a-b}{3}$ 为整数$\}$

验证： R= (1,1), (1,4), (1,7), (2,2), (2,5), (3,3), (3,6), (4,1), (4,4), (4,7), (5,5), (5,2), (6,6), (6,3), (7,7), (7,4), (7,1)

R的关系矩阵 $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

关系图

Equivalence relations

(y, 4)

- **Example:**
  - Consider set X = {1,2,…,13}. Define xRy as 5 divides x – y (i.e., x – y = 5k, for some int k). We can verify that R is reflexive, symmetric, and transitive. Here is how.

    x-y 被5整除

    $\frac{x-y}{5} = k \quad x = 5k+y \quad (k=0/1)$
  - The equivalence class [1] consists of all x with xR1. Thus:
    - [1] = {x ∈ X | 5 divides x – 1} = {1, 6, 11}
  - Similarly:
    - [2] = {2, 7, 12}
    - [3] = {3, 8, 13}
    - [4] = {4, 9}
    - [5] = {5, 10}

Equivalence relations

- These 5 sets partition X. Note that:
- [1] = [6] = [11]
- [2] = [7] = [12]
- [3] = [8] = [13]
- [4] = [9]
- [5] = [10]
- For this relation, equivalence is “has the same reminder when divided by 5”.

偏序关系

Partial Orders Relations

Order relations

- **Definition**: Let X be a set and R a relation on X, R is a partial order on X if R is **reflexive**, **antisymmetric** and **transitive**. A set X together with a partial ordering R is called a partially ordered set, or poset, or PO, and is denoted by (X, R).

  偏序集合

- **Example**: Is $(x, y) \in R$ in partial order if $x >= y$?
  - Yes, since:
    - Reflexive: $(x, x) \in R$
    - Anti-symmetric: If $(x, y) \in R$ and $x \neq y$, then $(y, x) \notin R$
    - Transitive: If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$

Order relations

包含关系是个偏序关系吗?

- Example: Is the "inclusion relation" $\subseteq$ a partial ordering on the power set of a set S?
  - $\subseteq$ is reflexive, because $A \subseteq A$ for every set $A \in S$.
  - $\subseteq$ is antisymmetric, because if $A \neq B$, then $A \subseteq B \wedge B \subseteq A$ is false.
  - $\subseteq$ is transitive, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- Consequently, $(P(S), \subseteq)$ is a partially ordered set or poset.

全序关系

Order relations

- Let $x,y \in X$,
  - If (x,y) or (y,x) are in R, then x and y are **comparable**. 可比较的
  - If $(x,y) \notin R$ and $(y,x) \notin R$, then x and y are **incomparable**. 不可比较的
  - Definition: If every pair of elements in X are comparable, then R is a **total order** on X.
    - In this case, X is called a totally ordered or linearly ordered set, and $\leq$ is called a total order or linear order. A totally ordered set is also called a **chain**.

Order relations

- **Example**: Is $(Z, \leq)$ a totally ordered poset?
  - Yes, because $a \leq b$ or $b \leq a$ for all integers a and b.
- **Example**: Is $(Z^+$, division) a totally ordered poset?
  - No, because it contains incomparable elements such as 5 and 7.

    整除不是全序关系。(5,7)、(7,5)都不属于R

Order relations

- In a poset the notation $a \leq b$ denotes that $(a, b) \in R$.
- Note that the symbol $\leq$ is used to denote the relation in any poset, not just the "less than or equal" relation.
- The notation $a < b$ denotes that $a \leq b$, but $a \neq b$.
- If $a < b$ we say "a is less than b" or "b is greater than a".

字典序

Lexicographic Order

- How can we define a lexicographic ordering on the set of English words?

这是从集合上的部分排序构造的集合上字符串排序的一种特殊情况

- This is a **special case** of an ordering of strings on a set constructed from a partial ordering on the set.
- We already have an ordering of letters (such as a < b, b < c, …), and from that we want to derive an ordering of strings.我们要导出字符串的顺序。
- Let us take a look at the general case, that is, how the construction works in any poset.

Lexicographic Order

笛卡尔积

- **First step**: Construct a partial ordering on the Cartesian product of two posets, $(A_1, \leq_1)$ and $(A_2, \leq_2)$: $A_1 \times A_2$
- $(a_1,a_2) < (b_1,b_2)$ if $(a_1 <_1 b_1) \vee [(a_1=b_1) \wedge (a_2 <_2 b_2)]$
- $(a_1,a_2) \leq (b_1, b_2)$ if $(a_1 <_1 b_1) \vee [(a_1 = b_1) \wedge (a_2 \leq_2 b_2)]$

先比较第一个元素是否满足$<_1$
再看第2个是否满足$<_2$

- **Examples**:
  - In the poset $(Z \times Z, \leq)$, …
    - is $(5, 5) < (6, 4)$? YES
    - is $(6, 5) < (6, 4)$? NO
    - is $(3, 3) < (3, 3)$? NO

Lexicographic Order

- **Second step**: Extend the previous definition to the Cartesian product of n posets $(A_1, \leq_1), (A_2, \leq_2), \ldots, (A_n, \leq_n)$:
- $(a_1, a_2, \ldots, a_n) < (b_1, b_2, \ldots, b_n)$ if $(a_1 <_1 b_1) \vee \exists i>0 \; (a_1 = b_1, a_2 = b_2, \ldots, a_i = b_i, a_{i+1} <_{i+1} b_{i+1})$
- Examples:
  - Is (1,1,1,2,1) < (1,1,1, 1, 2)? No
  - Is (1, 1, 1, 1, 1) < (1, 1, 1, 1, 2)? Yes

Lexicographic Order

We can now define lexicographic ordering of strings. Consider the strings $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$ on a partially ordered set S.

Suppose these strings are not equal. Let t be the minimum of m and n. The definition of lexicographic ordering is that the string $a_1a_2 \ldots a_m$ is less than $b_1b_2 \ldots b_n$ if and only if

- $(a_1, a_2, \ldots, a_t) < (b_1, b_2, \ldots, b_t)$, or
- $(a_1, a_2, \ldots, a_t) = (b_1, b_2, \ldots, b_t)$ and $m < n$,

where $<$ in this inequality represents the lexicographic ordering of S^t.

Lexicographic Order

- In other words, to determine the ordering of two different strings, the longer string is truncated to the length of the shorter string, namely, to $t = \min(m, n)$ terms.
- Then the t-tuples made up of the first t terms of each string are compared using the lexicographic ordering on S^t .
- One string is less than another string if the t-tuple corresponding to the first string is less than the t-tuple of the second string, or if these two t-tuples are the same, but the second string is longer.

- $(a_1a_2 \dots a_{m)}$ R $(b_1b_2 \dots b_n)$ if
  - $(a_1, a_2, \dots, a_t) = (b_1, b_2, \dots, b_t)$ and $t= m = n$,
  - $(a_1, a_2, \dots, a_t) < (b_1, b_2, \dots, b_t)$, or $(a_1, a_2, \dots, a_t) = (b_1, b_2, \dots, b_t)$ and $m < n$,
- R is a partial ordering

Hasse Diagram (哈斯图)

- Hasse diagram is a graphical display of a poset.
- A point is drawn for each element of the poset, and line segments are drawn between these points according to the following two rules:
  - 1. If $x < y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y.
  - 2. The line segment between the points corresponding to any two elements x and y of the poset is included in the drawing iff x covers y or y covers x.

盖住关系 (a、b之间没有其他元素)

Cover Relation

- Let $(S, \leq)$ be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x < y$ and there is no element $z \in S$ such that $x < z < y$. The set of pairs (x, y) such that y covers x is called **the covering relation** of $(S, \leq)$.

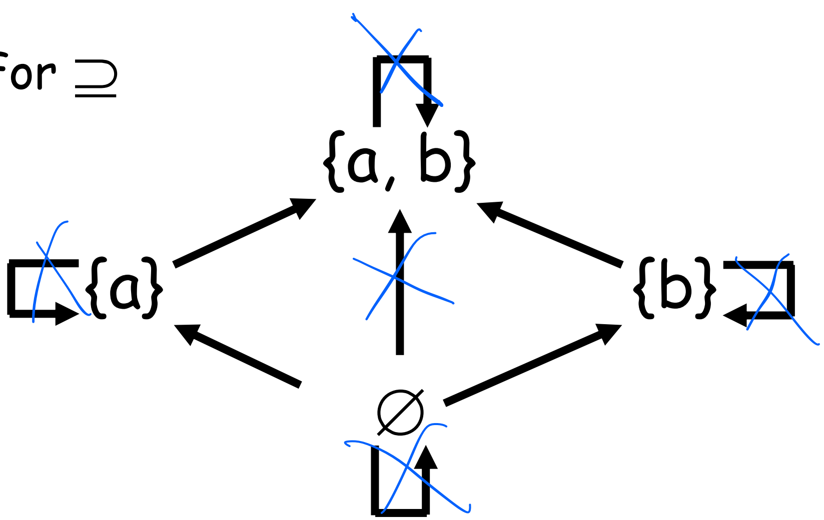
Hasse Diagrams

作图方法

We produce Hasse Diagrams from directed graphs of relations by doing a **transitive reduction** plus a **reflexive reduction** (if weak) and (usually) **dropping arrowheads** (using, instead, "above" to give direction)

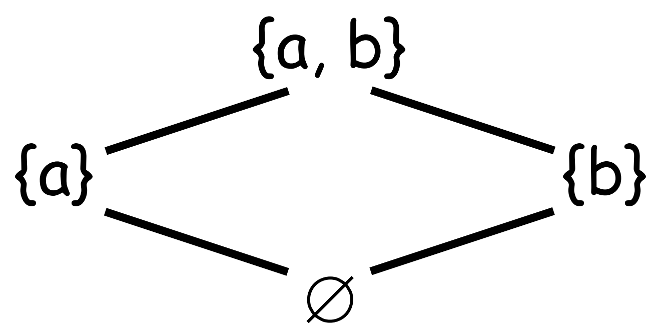
1) Transitive reduction — discard all arcs except those that "directly cover" an element. 去掉可传递的
2) Reflexive reduction — discard all self loops. 去掉自反
3) 最小的在下面，大的在上面
4) 把箭头去掉

For $\supseteq$



$\equiv$

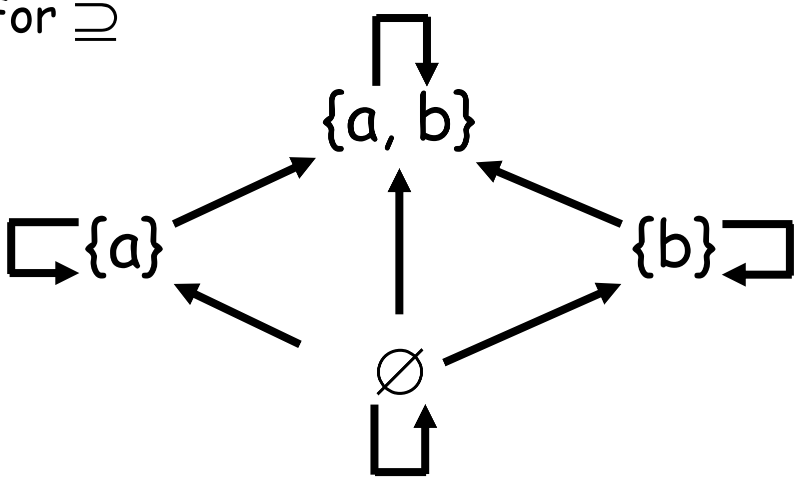
we write:



The Procedure Summary

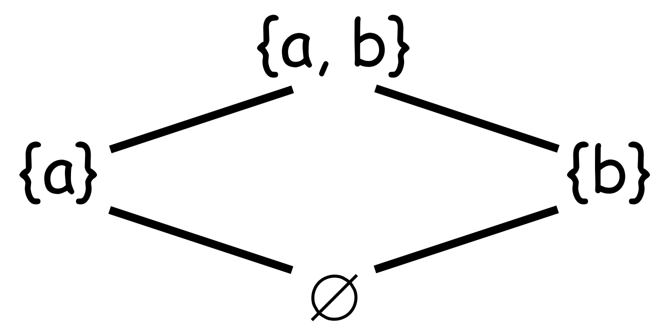
- Start with the directed graph for this relation.
- Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops.
- Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x < z$ and $z < x$.
- Finally, arrange each edge so that its initial vertex is below its terminal vertex (as it is drawn on paper). Remove all the arrows on the directed edges, because all edges point “upward” toward their terminal vertex.

For $\supseteq$



$\equiv$

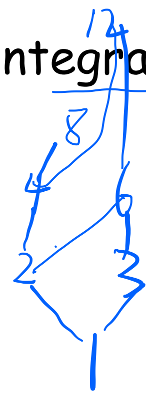
we write:

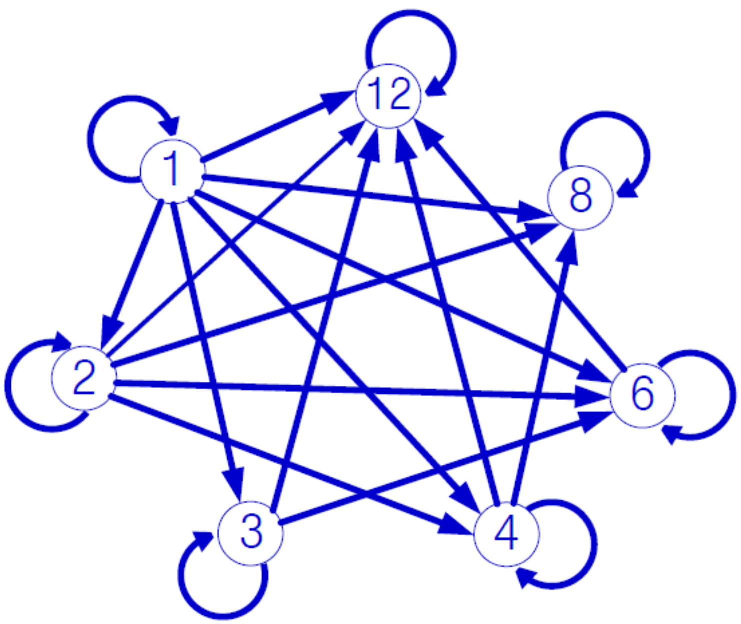


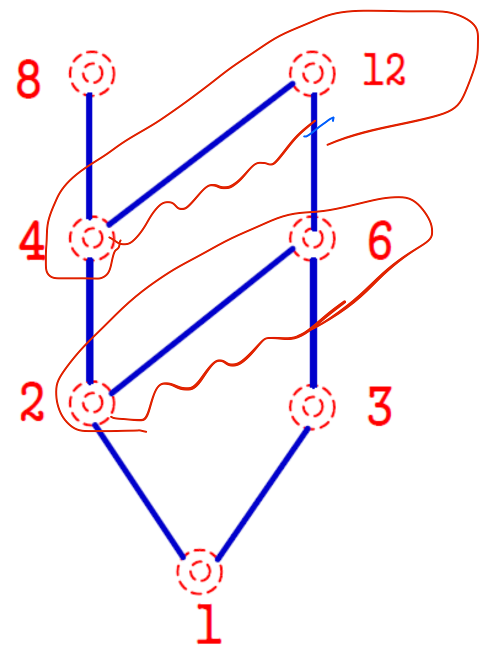
(考试把有向图画出作哈斯图)

Hasse Diagram

- **Example**: A={1,2,3,4,6,8,12}, integral division relation.



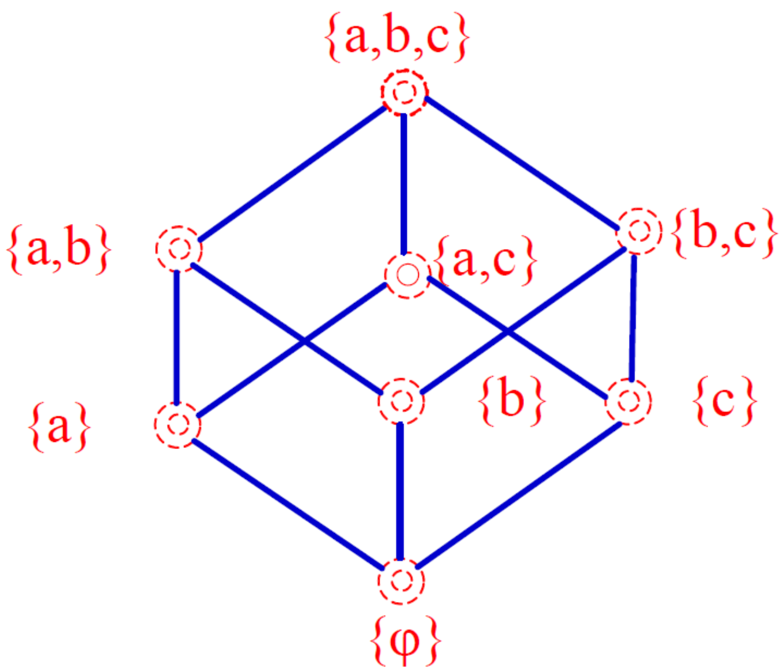


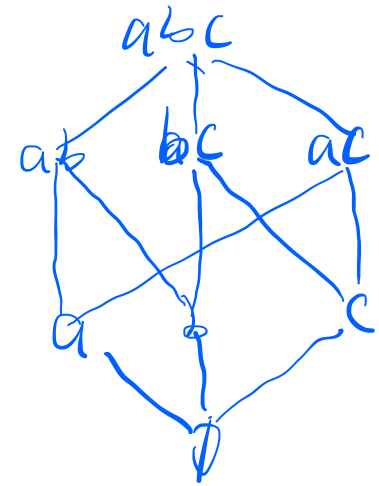


Hasse Diagram

幂集的集合

- **Example**: S={a, b, c}, (P(S), ⊆)

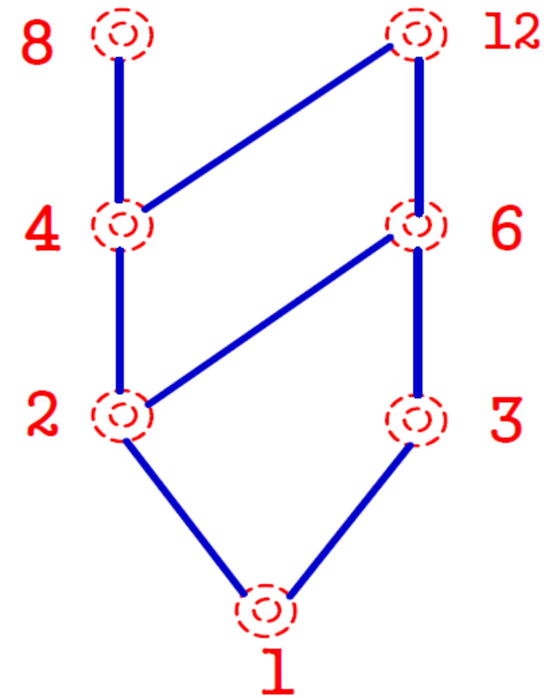




Maximum/Minimum/Greatest/Least

只考

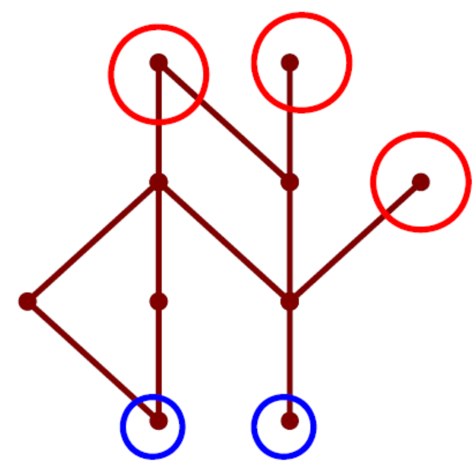
- Maximum/Minimum element
- 极大、极小
- Greatest/Least element
- 最大、最小
- Upper/Lower bound
- 上界、下界
- Least upper/Greatest lower bound
- 最小上界、最大下界



极小 Minimum and 极大 Maximum

没有能够cover的元素

- **Definition:** In a poset S, an element z is a **minimum** element if there is no element $b \in S$, thus $b \leq z$ and $b \neq z$.
- How about definition for **maximum** element?
  没有元素能cover它
- **Example:**
  - Reds are maximal.
  - Blues are minimal.



最小 / 最大

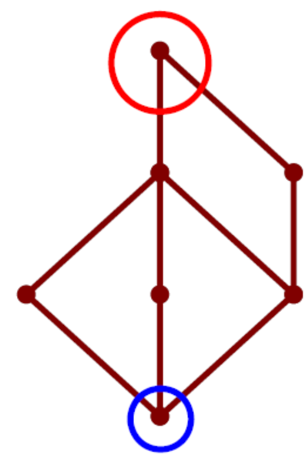
Least and Greatest

能被任何一个元素 cover

- **Definition**: In a poset S, an element z is a **Least** element if ∀b∈S, z ≤ b.
- How about definition for **Greatest** element.

能cover每个元素

- **Example**:
  - Reds are greatest.
  - Blues are least.
- Greatest/Least may not exist.



Least and Greatest

在每个偏序集合中，如果存在最大元素，那么它是唯一的。最少也一样

- **Theorem:** In every poset, if the **greatest** element exists, then it is **unique**. Similarly for the **least**.
- **Proof**:
  - Suppose there are two greatest elements, a_1 and a_2, with $a_1 \neq a_2$. Then $a_1 \leq a_2$, and $a_2 \leq a_1$, by defn of greatest. So $a_1 = a_2$, a contradiction. Thus, our assumption was incorrect, and the greatest element, if it exists, is unique.
  - Similar proof for least.

The End