Discrete Mathematics

Review

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Course Schedule

Topics	Courses	note
Logic	**Propositional Logic**	****
	Predicate Logic	****
Set Theory	**Sets**	****
	Functions	***
	Relations	*****
Graph Theory	**Graphs**	***
	Trees	****
Review	All	*****

Propositional logic (命题逻辑)

Key Points

- Negation (否定) (NOT, ¬)
- Conjunction (合取) (AND, ∧)
- Disjunction (析取) (OR, ∨)
- Conditional statement (条件) (Implication) (IF THEN→)

- Logical equivalence (逻辑等价)
  - Prove logical equivalence
- Normal forms (范式)*
  - Disjunctive normal form (析取范式)
  - Conjunctive normal form (合取范式)

Translating English to Logical Expressions

Problem: Translate the following sentence into propositional logic:

- "You can access the Internet from campus only if you are a computer science major or you are not a freshman."
- One solution:
  - p: "You can access the internet from campus,"
  - q: "You are a computer science major,"
  - r: "You are a freshman."

$$p \rightarrow (q \vee \neg r)$$

Different ways of expressing p→q

- if p, then q
- if p, q
- **q unless ¬p**
- **q if p**
- q whenever p
- q follows from p
- p implies q
- **p only if q**
- q when p
- p is sufficient for q
- q is necessary for p

- A necessary condition for p is q
- A sufficient condition for q is p

Common mistake for p→q

- Correct: p only if q
- Mistake to think "q only if p"

Equivalence proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{array}{rcll}
\neg(p \vee (\neg p \wedge q)) & \equiv & \neg p \wedge \neg(\neg p \wedge q) & \text{by the second De Morgan law} \\
& \equiv & \neg p \wedge [\neg(\neg p) \vee \neg q] & \text{by the first De Morgan law} \\
& \equiv & \neg p \wedge (p \vee \neg q) & \text{by the double negation law} \\
& \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \text{by the second distributive law} \\
& \equiv & F \vee (\neg p \wedge \neg q) & \text{because } \neg p \wedge p \equiv F \\
& \equiv & (\neg p \wedge \neg q) \vee F & \text{by the commutative law} \\
& & & \text{for disjunction} \\
& \equiv & (\neg p \wedge \neg q) & \text{by the identity law for } \mathbf{F}
\end{array}
$$

Simplifying statement

We can use logical rules to simplify a logical formula.

$$\neg(\neg p \wedge q) \wedge (p \vee q)$$

$$\equiv (\neg\neg p \vee \neg q) \wedge (p \vee q)$$ De Morgan

$$\equiv (p \vee \neg q) \wedge (p \vee q)$$

$$\equiv p \vee (\neg q \wedge q)$$ Distributive law

$$\equiv p \vee \textsf{False}$$

$$\equiv p$$

The De Morgan's Law allows us to always "**move the NOT inside**".

Principal disjunctive normal form (PDNF):

- **求主析取范式的方法：**
  - 先化成与其等价的析取范式；
  - 若析取范式的基本积中同一命题变元出现多次，则将其化成只出现一次；
  - 去掉析取范式中所有为永假式的基本积，即去掉基本积中含有形如

  $p \wedge \neg p$的子公式的那些基本积；
  - 若析取范式中缺少某一命题变元如p，则可用公式$(p \vee \neg p) \wedge q \Leftrightarrow q$将命题变元P补充进去，并利用分配律展开，然后合并相同的基本积

Principal conjunctive normal form (PCNF)

- **求主合取范式的方法：**
  - 先化成与其等价的合取范式；
  - 若合取范式的基本积中同一命题变元出现多次，则将其化成只出现一次；
  - 去掉合取范式中所有为永真式的基本积，即去掉基本和中含有形如

  $p \vee \neg p$的子公式的那些基本和；
  - 若合取范式中缺少某一命题变元如p，则可用公式$(p \wedge \neg p) \vee q \Leftrightarrow q$将命题变元P补充进去，并利用分配律展开，然后合并相同的基本和

Minterm vs Maxterm

- The relations between m_i and M_i are

$$M_i \Leftrightarrow \neg m_i \qquad m_i \Leftrightarrow \neg M_i$$

p,q,r	maxterms	p∧q∨r		p,q,r	minterms	p∧q∨r	
0,0,0	$p \vee q \vee r$	0	M_0	0,0,0	$\neg p \wedge \neg q \wedge \neg r$	0	m_0
0,0,1	$p \vee q \vee \neg r$	1	M_1	0,0,1	$\neg p \wedge \neg q \wedge r$	1	m_1
0,1,0	$p \vee \neg q \vee r$	0	M_2	0,1,0	$\neg p \wedge q \wedge \neg r$	0	m_2
0,1,1	$p \vee \neg q \vee \neg r$	1	M_3	0,1,1	$\neg p \wedge q \wedge r$	1	m_3
1,0,0	$\neg p \vee q \vee r$	0	M_4	1,0,0	$p \wedge \neg q \wedge \neg r$	0	m_4
1,0,1	$\neg p \vee q \vee \neg r$	1	M_5	1,0,1	$p \wedge \neg q \wedge r$	1	m_5
1,1,0	$\neg p \vee \neg q \vee r$	1	M_6	1,1,0	$p \wedge q \wedge \neg r$	1	m_6
1,1,1	$\neg p \vee \neg q \vee \neg r$	1	M_7	1,1,1	$p \wedge q \wedge r$	1	m_7

Translate CNF to DNF

Let CNF of A be

$$(P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$

Find the DNF of A.

Solution:

The CNF of A is $M_1 \wedge M_{3,}$ so the DNF can be written as

$$\sum (0, 2, 4, 5, 6, 7)$$

And thus we have

$$(\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$$
$$\vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$$

Find the normal forms of $(p \rightarrow \neg q) \rightarrow \neg r$

Solution:

$(p \rightarrow \neg q) \rightarrow \neg r$

$\Leftrightarrow \neg(\neg p \vee \neg q) \vee \neg r$

$\Leftrightarrow (p \wedge q) \vee \neg r$

$\Leftrightarrow (p \vee \neg r) \wedge (q \vee \neg r)$

$\Leftrightarrow (p \vee \neg r \vee (q \wedge \neg q)) \wedge (q \vee \neg r \vee (p \wedge \neg p))$

$\Leftrightarrow (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$

$\Leftrightarrow \prod(1,3,5)$

/*其中$\prod$表示求合取*/

$\Leftrightarrow \sum(0,2,4,6,7)$

/*即该公式是可满足的，应存在与其等价的主析取范式*/

Predicate Logic (谓词逻辑)

Key Points

- Predicates (谓词)

- Quantifiers (量词)*
  - Universal quantifier (全称量词)
  - Existential quantifier (存在量词)

- Nested quantifiers (嵌套量词)*

Logically Equivalent

Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.

Solution:

- By De Morgan's law for universal quantifiers, we know that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(\neg(P(x) \rightarrow Q(x)))$ are logically equivalent.
- It is also known that $\neg(P(x) \rightarrow Q(x))$ and $P(x) \land \neg Q(x)$ are logically equivalent for every x.
- Because we can substitute one logically equivalent expression for another in a logical equivalence, it follows that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.

Nested quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $$\forall x \, \exists y(x + y = 0)$$

  where the domains of x and y are the real numbers.
- We can also think of nested propositional functions:

  $\forall x \, \exists y(x + y = 0)$ can be viewed as $\forall x \, Q(x)$ where $Q(x)$ is $\exists y \, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Translating nested quantifiers into English

Example 1: Translate the statement

$$\forall x \ (C(x) \vee \exists y \ (C(y) \wedge F(x, y)))$$

where $C(x)$ is "x has a computer," and $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \forall y \ \forall z \ ((F(x, y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: There is a student none of whose friends are also friends with each other.

Translating English into predicate logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables x and y, and specify the domain, to obtain:

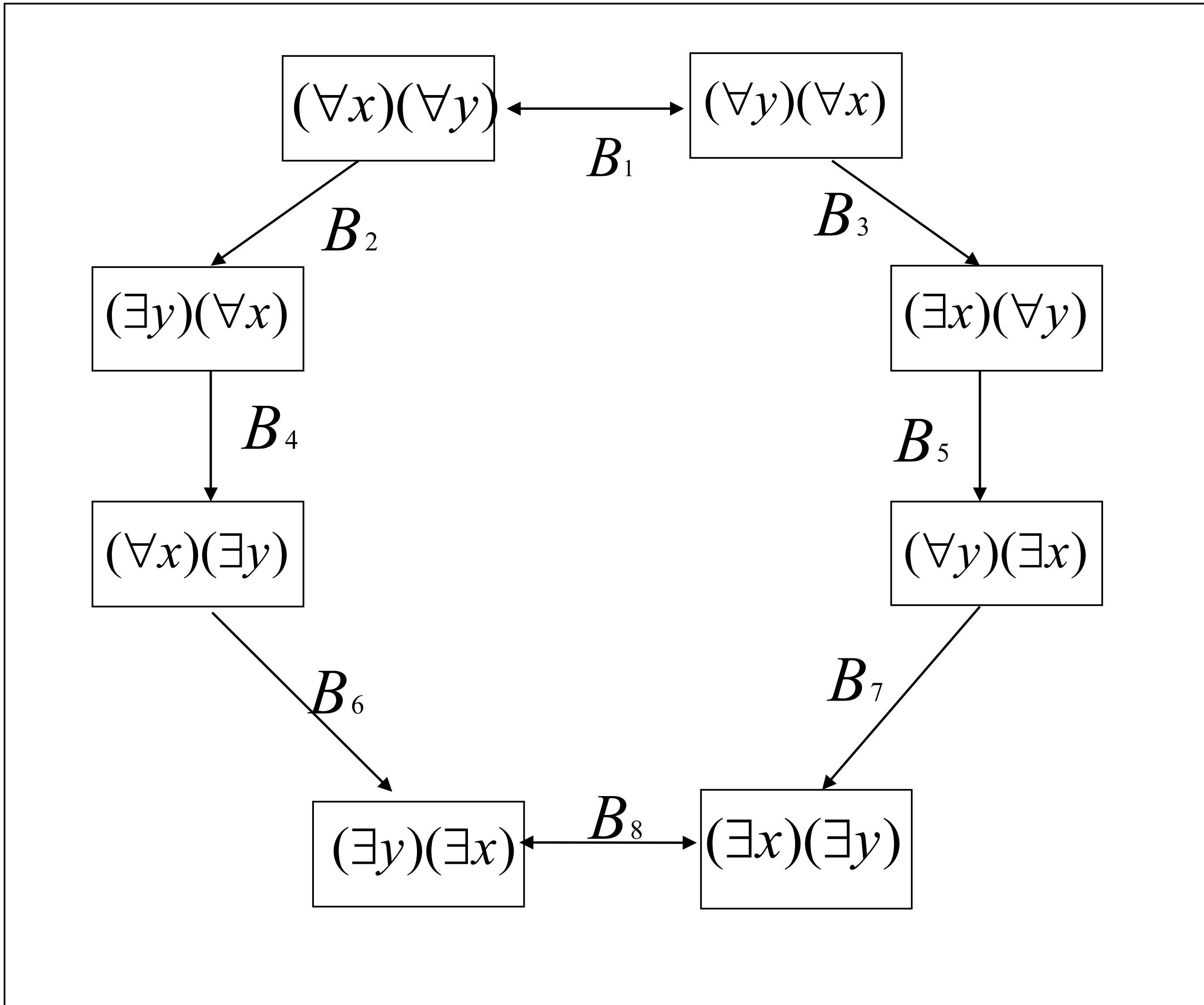
   "For all positive integers x and y, x + y is positive."

3. The result is:

$$\forall x \ \forall \ y \ ((x > 0) \land \ (y > 0) \rightarrow \ (x + y > 0))$$

   where the domain of both variables consists of all integers

记忆规律



Set

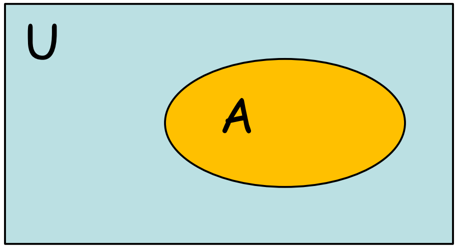
Key Points

- set, subset (proper subset 真子集)
- Cardinality (基数：Size of a Set)
- Set Operations
  - Union (并)
  - Intersection (交)
  - Difference (差)
  - Complement (补)
  - Symmetric Difference (对称差) (Option)

Sets

- Set = a collection of distinct **unordered** objects
- Members of a set are called **elements**
- How to determine a set
  - **Listing**
    - Example: $A = \{1,3,5,7\} = \{7, 5, 3, 1, 3\}$
  - **Description**
    - Example: $B = \{x \mid x=2k+1, 0 < k < 30\}$
  - **Venn Diagrams**

- A Venn diagram provides a graphic view of sets
- Venn diagrams are useful in representing sets and set operations which can be easily and visually identified.
- Various sets are represented by circles inside a big rectangle representing the universal set.

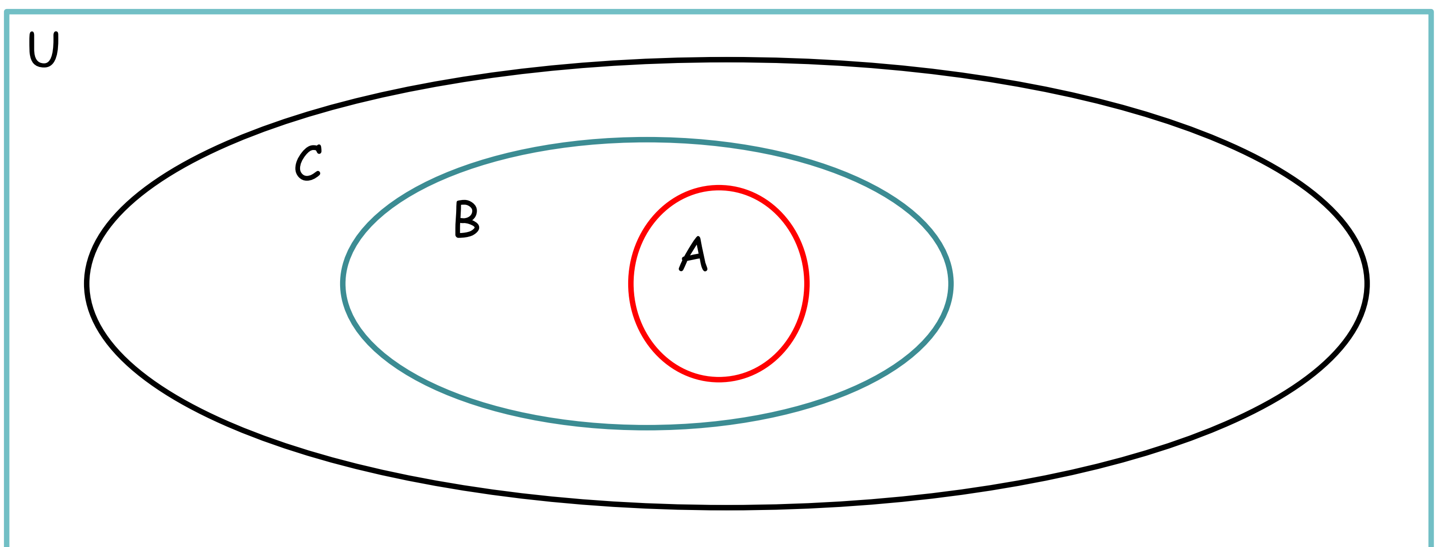


Subsets

- Useful rules:
- $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
- $(A \subseteq B) \wedge (B \subseteq C) \Rightarrow A \subseteq C$ (next Venn Diagram)

When "$\subset$" is used instead of "$\subseteq$", proper containment is meant. subset A of B is said to be a **proper subset** if A is not equal to B.
Notationally:

$$A \subset B \quad \Leftrightarrow \quad A \subseteq B \wedge \exists x \ (x \notin A \wedge x \in B)$$
$$\Leftrightarrow \quad \forall x \ (x \in A \rightarrow x \in B) \wedge \exists x \ (x \in B \wedge x \notin A)$$



Cardinality

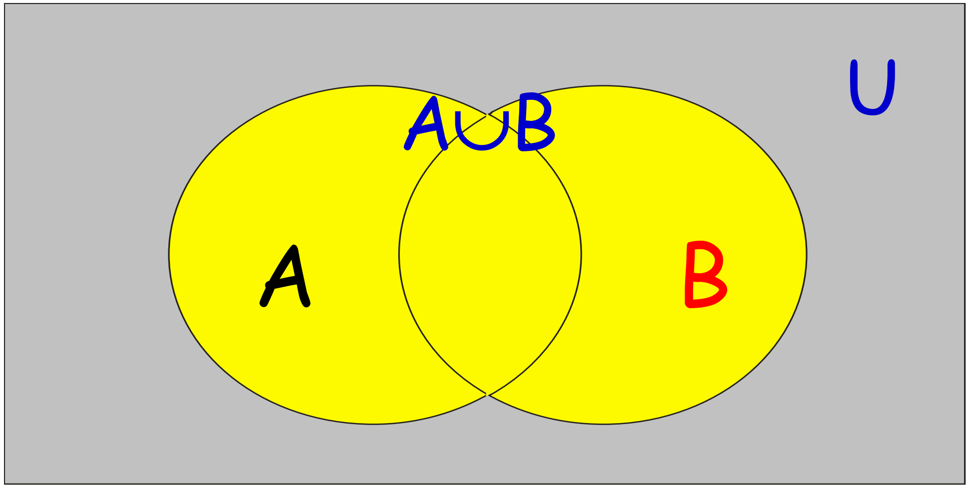
Hint: After eliminating the redundancies just look at the number of top level commas and add 1 (except for the empty set).

A:

1. $|\{1, -13, 4, -13, 1\}| = |\{1, -13, 4\}| = 3$
2. $|\{3, \{1,2,3,4\}, \varnothing\}| = 3$. To see this, set $S = \{1,2,3,4\}$. Compute the cardinality of $\{3, S, \varnothing\}$
3. $|\{\}| = |\varnothing| = 0$
4. $|\{ \{\}, \{\{\}\}, \{\{\{\}\}\} \}| = |\{ \varnothing, \{\}, \{\varnothing\}, \{\{\varnothing\}\}| = 3$

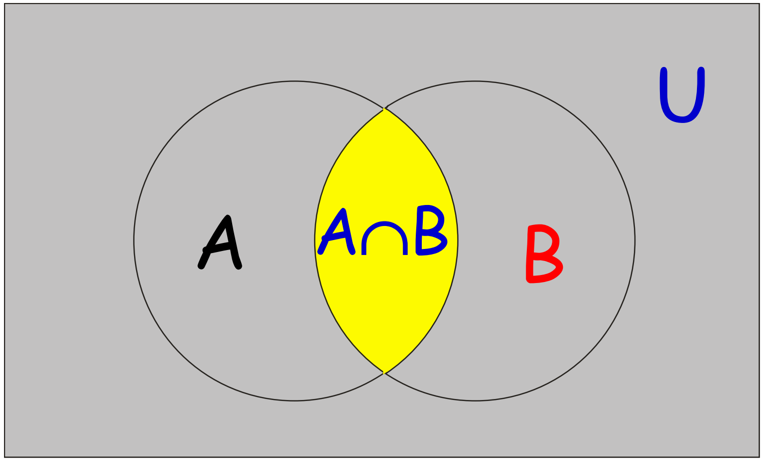
Set Operations

- Union: Elements in at least one of the two sets.
  - $A \cup B = \{x \mid x \in A \vee x \in B\}$
  - Example:
    - $A = \{a, b\}$, $B = \{b, c, d\}$
    - $A \cup B = \{a, b, c, d\}$



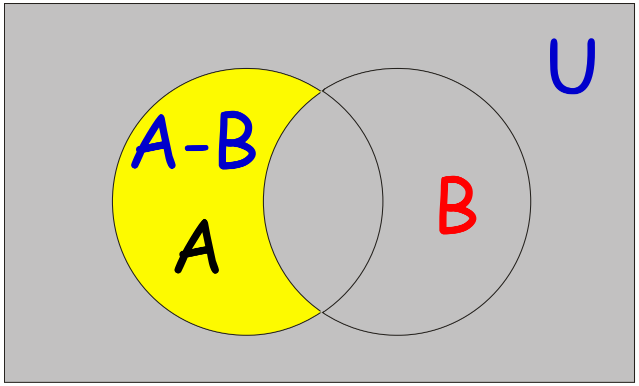
Set Operations

- Intersection: Elements in exactly one of the two sets.
  - $A \cap B = \{x \mid x \in A \wedge x \in B\}$
  - Example:
    - $A = \{a, b\}$, $B = \{b, c, d\}$
    - $A \cap B = \{b\}$



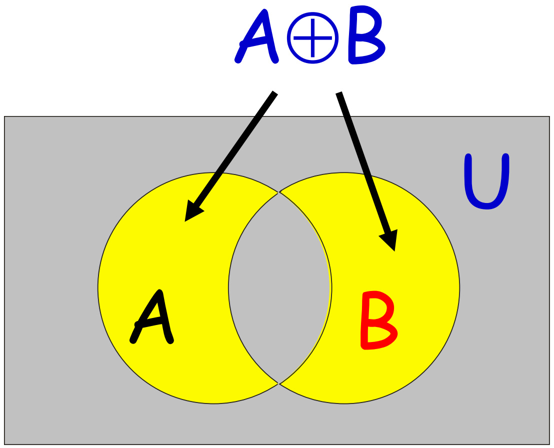
Set Operations

- Difference: Elements in first set but not second. Difference is also called the relative complement （相对补） of B in A.
  - $A-B = \{x \mid x \in A \wedge x \notin B\} = A \cap B^c$
  - Example
    - $A = \{a, b\}$, $B = \{b, c, d\}$
    - $A-B = \{a\}$



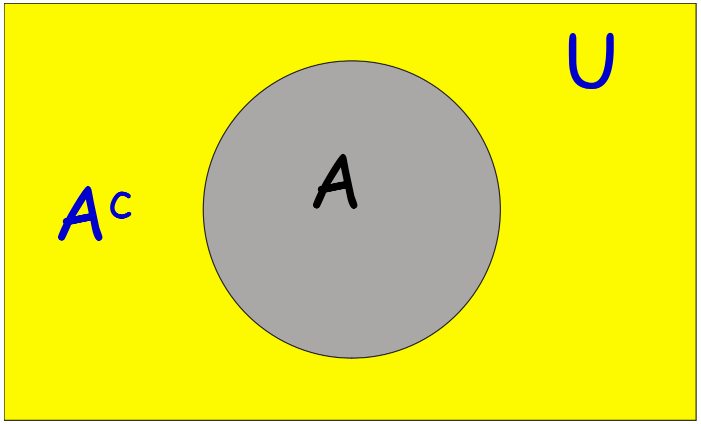
Set Operations

- Symmetric Difference: Elements in exactly one of the two sets.
  - $A \oplus B = \{ x \mid x \in A \oplus x \in B \} = (A-B) \cup (B-A) = (A \cup B)-(A \cap B)$
  - Example:
    - $A = \{a, b\}$, $B = \{b, c, d\}$
    - $A \oplus B = \{a,c,d\}$



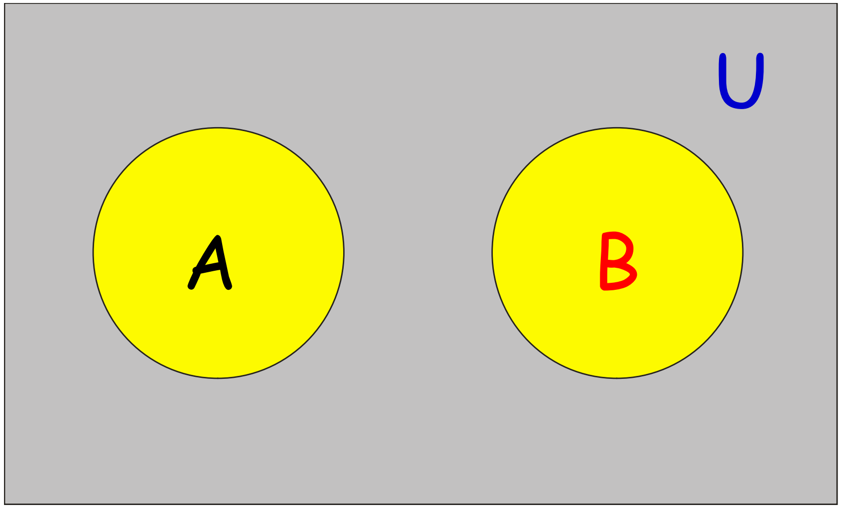
Set Operations

- Complement: Elements not in the set (unary operator).
  - $A^c = \{ x \mid x \notin A \}$
  - Example:
    - $U = N$, $A = \{250, 251, 252, \ldots\}$
    - $A^c = \{0, 1, 2, \ldots, 248, 249\}$



Disjoint Sets

- Disjoint: If A and B have no common elements, they are said to be disjoint.
  - $A \cap B = \varnothing$



Using Properties of Set Operations

- How can we prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?
- Method I:
- $x \in A \cup (B \cap C)$
- $\Leftrightarrow x \in A \vee x \in (B \cap C)$
- $\Leftrightarrow x \in A \vee (x \in B \wedge x \in C)$
- $\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
- (distributive law for logical expressions)
- $\Leftrightarrow x \in (A \cup B) \wedge x \in (A \cup C)$
- $(A \cup B) \cap (A \cup C)$

Using Properties of Set Operations

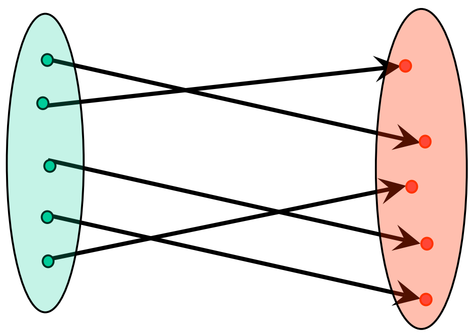
- Method II: Membership table
  - 1 means "x is an element of this set"
  - 0 means "x is not an element of this set"

A B C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0 0 0	0	0	0	0	0
0 0 1	0	0	0	1	0
0 1 0	0	0	1	0	0
0 1 1	1	1	1	1	1
1 0 0	0	1	1	1	1
1 0 1	0	1	1	1	1
1 1 0	0	1	1	1	1
1 1 1	1	1	1	1	1

Functions

More formally, we write $f : A \to B$

to represent that f is a function from set A to set B, which associates an element $f(a) \in B$ with an element $a \in A$.



The *domain (input)* of f is A.

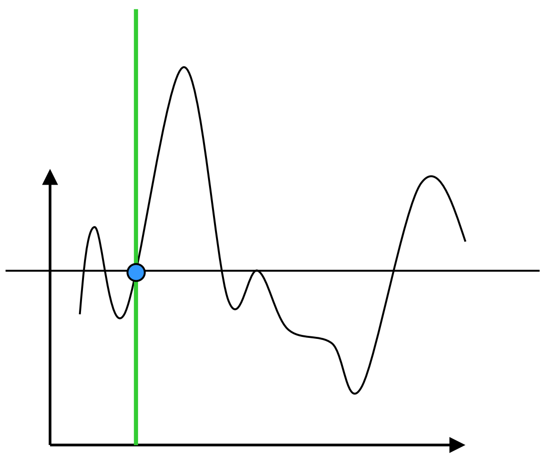
The *codomain (output)* of f is B.

Definition: For every input there is exactly one output.

Note: the input set can be the same as the output set, e.g. both are integers.

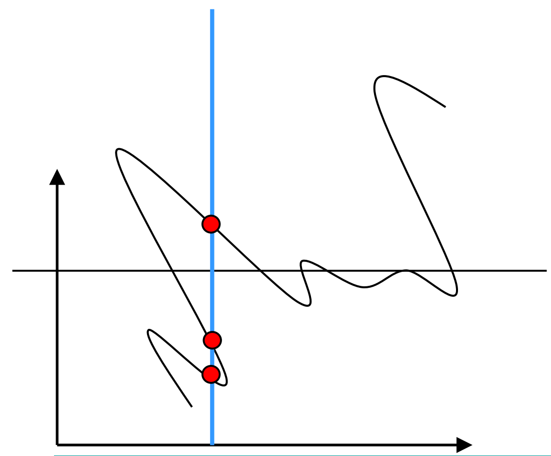
Algebraically speaking

- Note that such definitions on functions are consistent with what you have seen in your Calculus courses.



function

→ 1 intersection



not a function

→ violations when > 1

Key Points

- Properties of functions
- Composition of functions

Properties of Functions

- A function $f: A \to B$ is said to be one-to-one (or **injective**（单射）), if and only if $\forall x, y \in A\ (f(x) = f(y) \to x = y)$

- In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

- How can we prove that a function f is one-to-one?
  - Whenever you want to prove something, first take a look at the relevant definition(s):

    $\forall x, y \in A\ (f(x) = f(y) \to x = y)$

- Example:

  $f: \mathbf{R} \to \mathbf{R}$

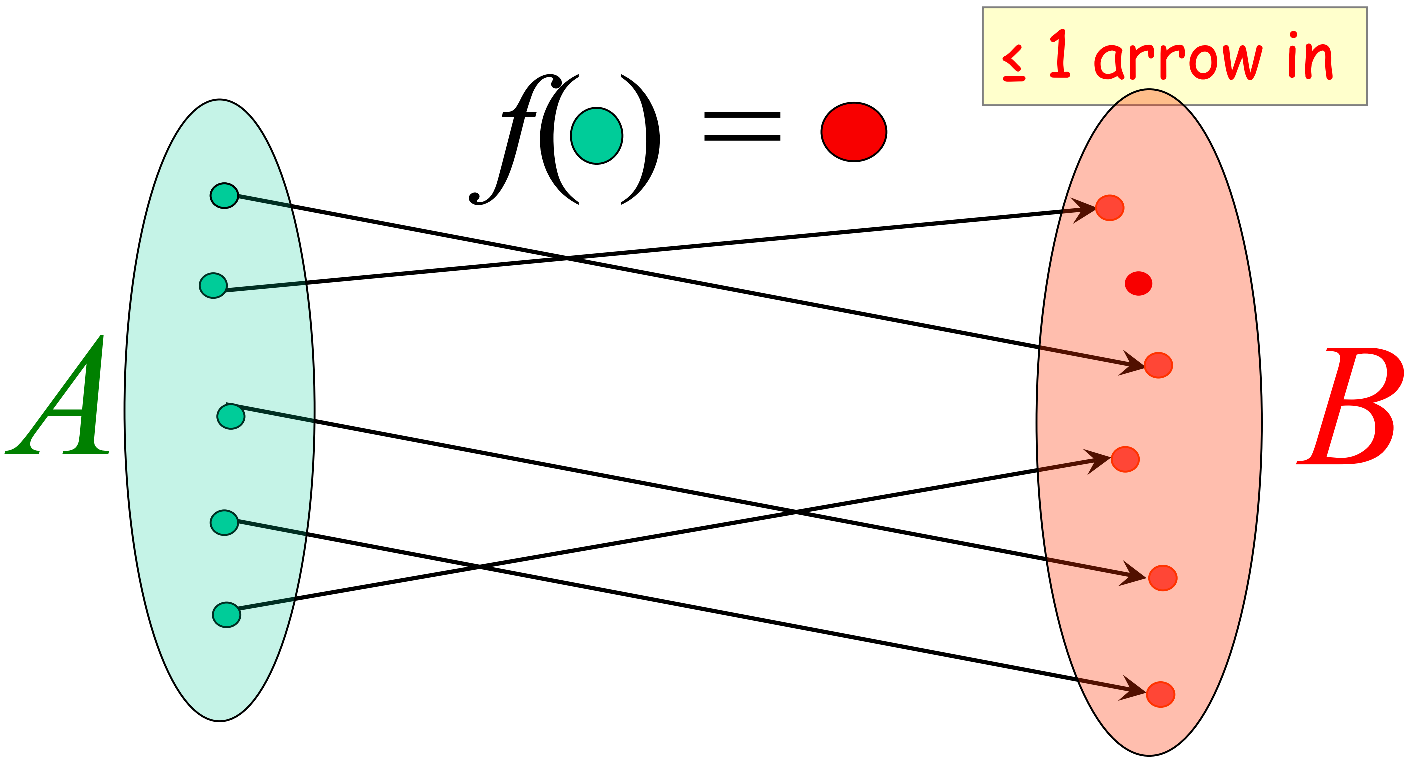
  $f(x) = x^2$

- Disproof by counterexample:

$f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

Injections

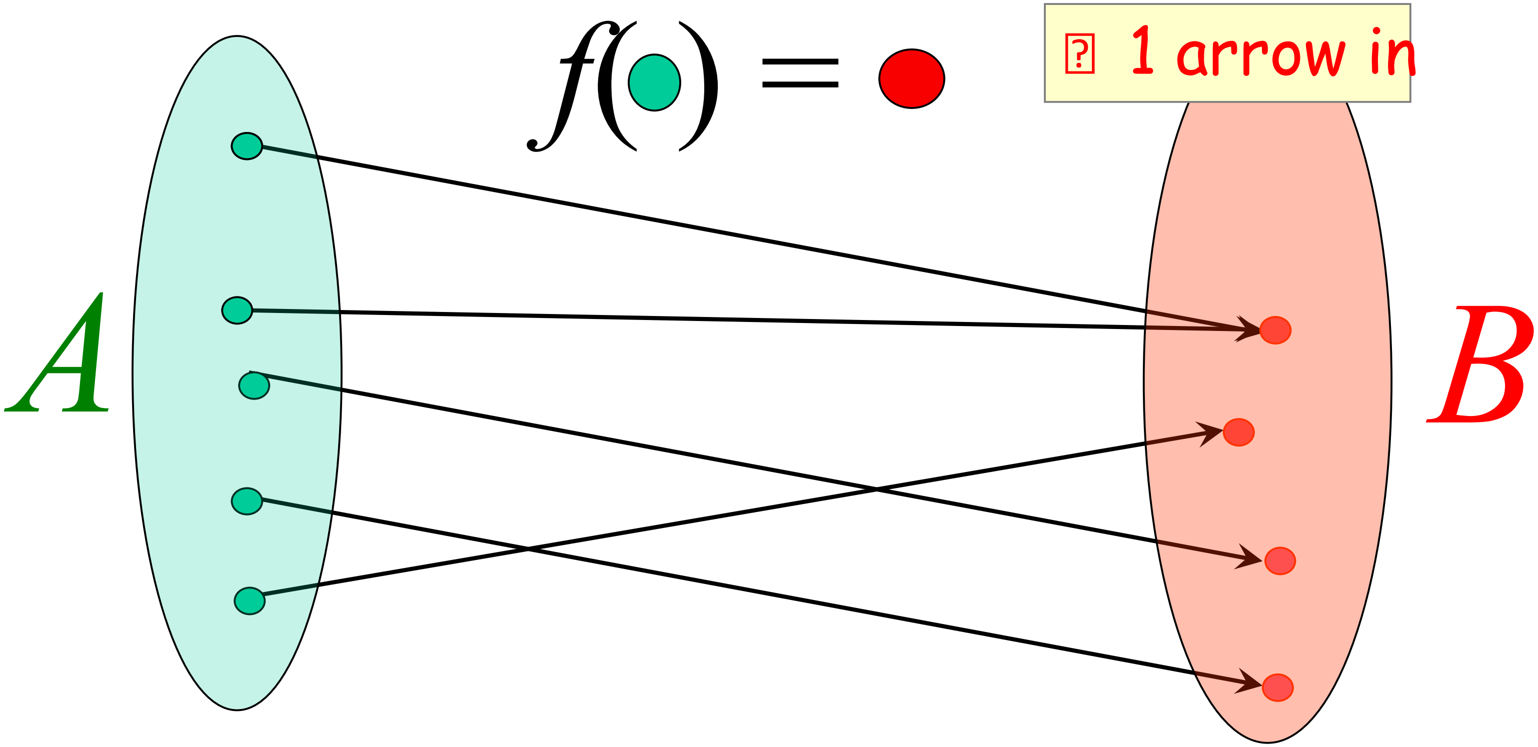
$f : A \to B$ is an *injection* if no two inputs have the same output.



$$|A| \leq |B|$$

Surjections

$f : A \to B$ is a *surjection* if every output is possible.

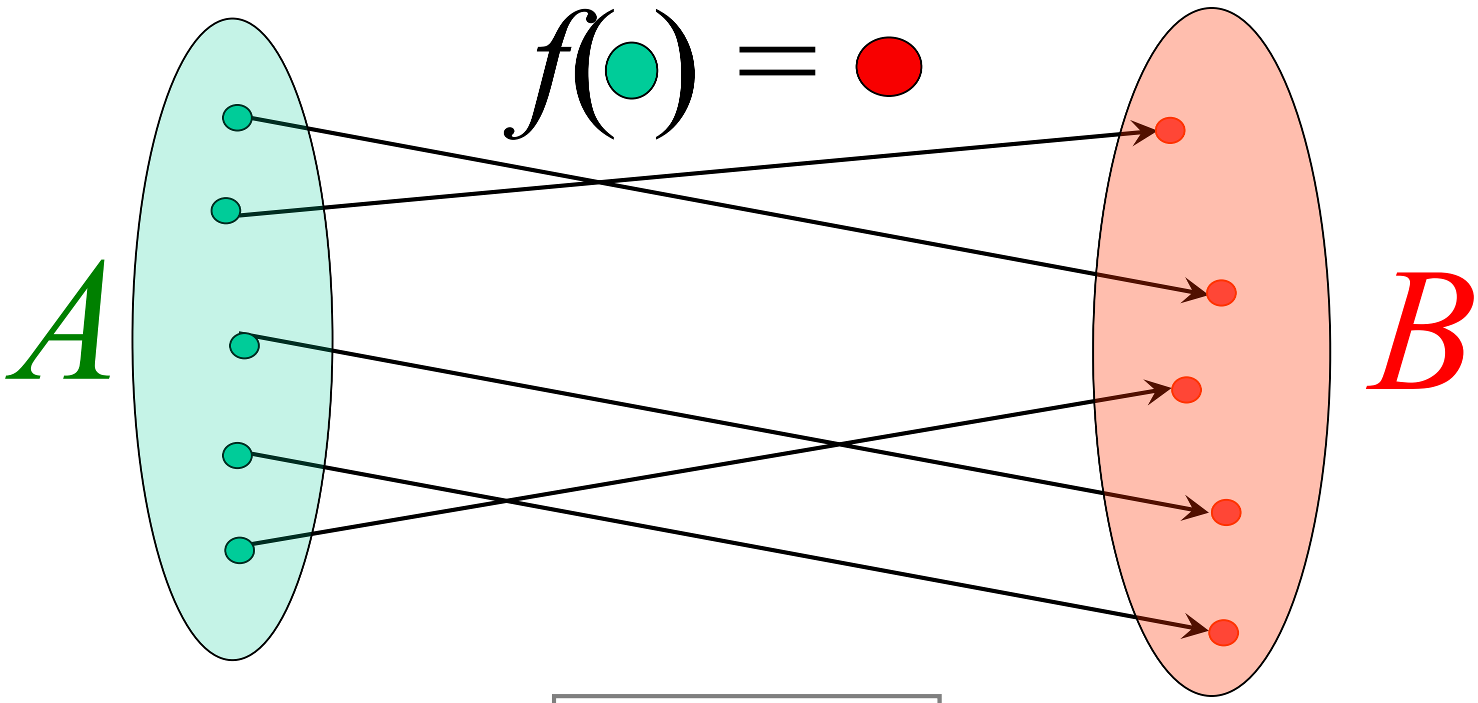


$$\boxed{|A| \geq |B|}$$

Bijections

$f : A \to B$ is a bijection if it is surjection and injection.

exactly one arrow in

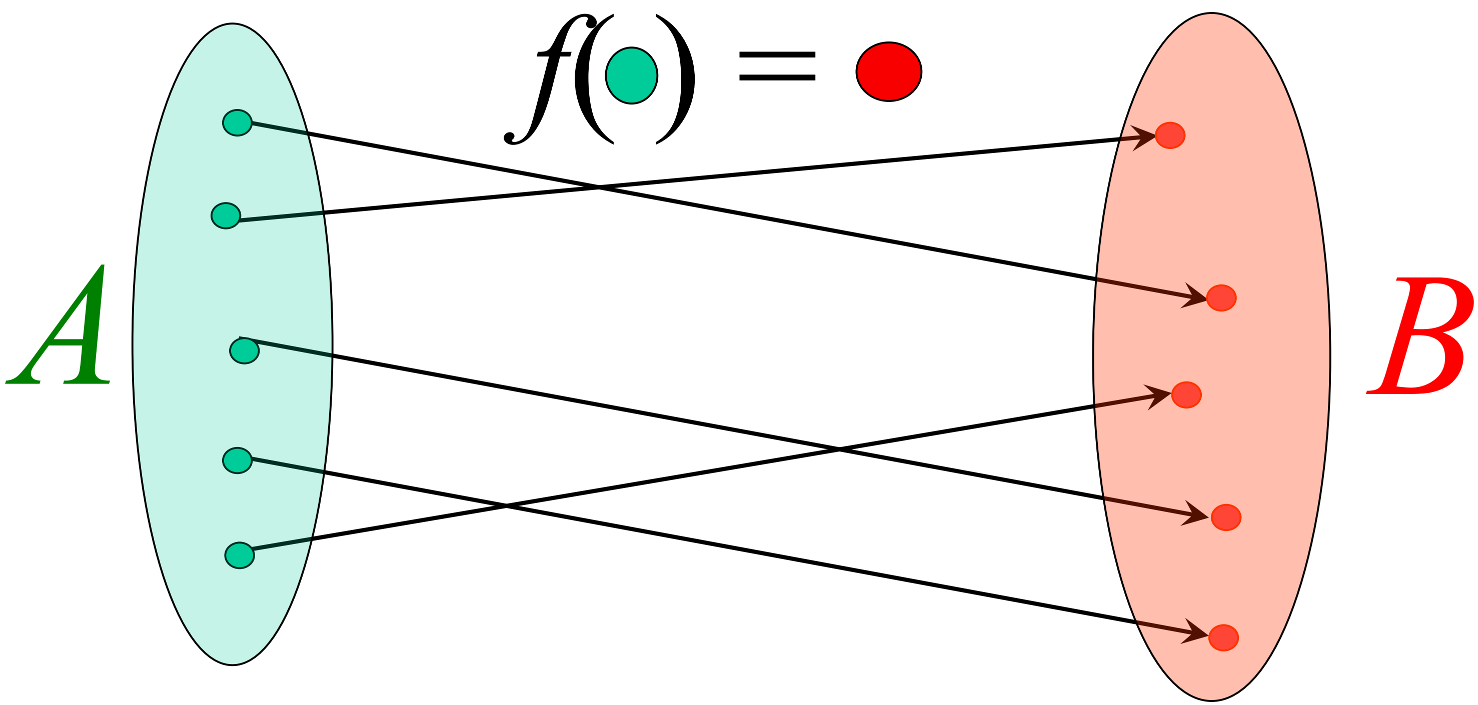


$$\boxed{|A| = |B|}$$

Inverse Function

Informally, an inverse function f^{-1} is to "undo" the operation of function f.

exactly one arrow in



There is an inverse function f^{-1} for f if and only if f is a bijection.

Composition

- The composition of two functions $g:A\to B$ and $f:B\to C$, denoted by $f\circ g$, is defined by $(f\circ g)(a) = f(g(a))$
- This means that
  - first, function g is applied to element $a\in A$, mapping it onto an element of B,
  - then, function f is applied to this element of B, mapping it onto an element of C.
  - Therefore, the composite function maps from A to C.

- Example:

$f(x) = 7x - 4$, $g(x) = 3x$,

$f:\mathbf{R}\to\mathbf{R}$, $g:\mathbf{R}\to\mathbf{R}$

$(f\circ g)(5) = f(g(5)) = f(15) = 105 - 4 = 101$

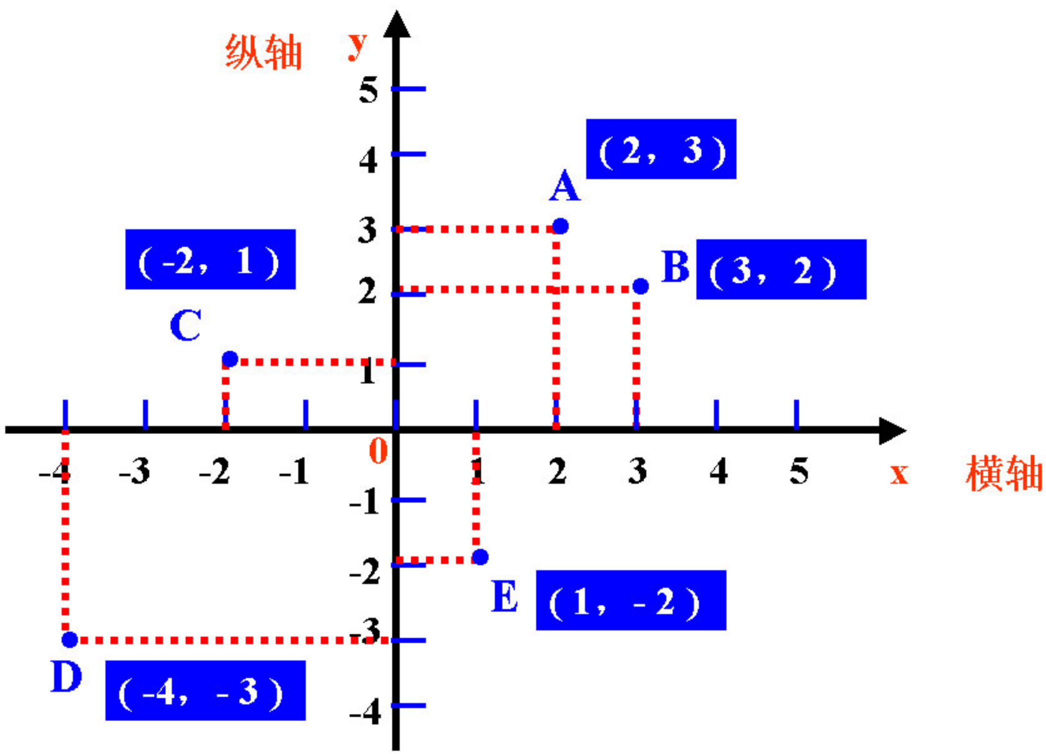
$(f\circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Key Points

- Cartesian Product, Relations and Binary Relations
- Properties of relations
  - reflexive (自反) , irreflexive (反自反), symmetric (对称), antisymmetric (反对称)，transitive (传递)
- Representing Binary Relations
- Operations of relations
- Closure (闭包)

Cartesian product (笛卡尔积)

- If $A_1, A_2, \ldots, A_m$ are nonempty sets, then the **Cartesian Product** of them is the set of all ordered m-tuples $(a_1, a_2, \ldots, a_m)$, where $a_i \in A_i$, $i = 1, 2, \ldots m$.
- Denoted $A_1 \times A_2 \times \ldots \times A_m = \{(a_1, a_2, \ldots, a_m) \mid a_i \in A_i, i = 1, 2, \ldots m\}$



$\{2,3\}=\{3,2\}$

$(2,3) \neq (3,2)$

Using matrices to denote Cartesian product

- If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, find $A \times B$
- $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)\}$

- For Cartesian Product of two sets, you can use a matrix to find the sets.
- Example: Assume $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. The table below represents $A \times B$.

	a	b	c
1	(1, a)	(1, b)	(1, c)
2	(2, a)	(2, b)	(2, c)
3	(3, a)	(3, b)	(3, c)

The cardinality of the Cartesian Product equals the product of the cardinality of all of the sets:

$$| A_1 \times A_2 \times \ldots \times A_m | = | A_1 | \cdot | A_2 | \cdot \ldots \cdot | A_m |$$

Relations: Subsets of Cartesian products

A:

1. Database $\subseteq$ {Names}$\times${Foods}$\times${Colors}$\times${Jobs}

Definition: Let A_1, A_2, … , A_n be sets. An n-**ary relation** on these sets (in this order) is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

Most of the time we consider n = 2 in which case have a **binary relation** and also say the relation is "**from** A_1 to A_2".

With this terminology, all functions are relations, but not vice versa.

Functions as specific relations

- Recall that a function f from a set A to a set B assigns **exactly one** element of B to each element of A
- The graph of f is the set of ordered pairs (a, b) such that b=f(a)
- Because the graph of f is a subset of A x B, it is a relation from A to B
- A relation can be used to express one-to-many relationship between the elements of the sets A and B where an element of A may be related to more than one element of B
- A function represents a relation where exactly one element of B is related to each element of A
- Relations are a generalization of functions

Binary relations *

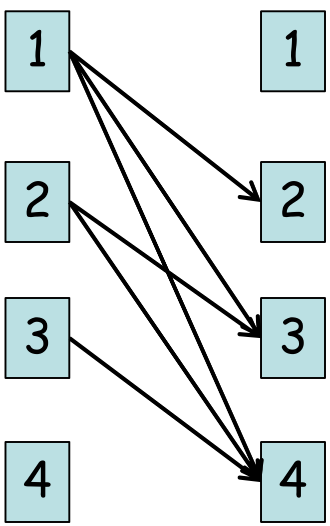
- Given two sets A and B, its Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$
  - In symbols $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- **Definition**: Let A and B be sets. A **binary relation** R from a set A to a set B is a subset of the Cartesian product $A \times B$.
- In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a \not R b$ to denote $(a, b) \notin R$.

Binary relations

- When (a, b) belongs to R, a is said to be related to b by R.
- **Example**: A = {1, 2, 3} and B = {a, b}
  - R = {(1, a), (1, b), (2, b), (3, a)} is a relation between A and B. 3 is related to a by R.
- **Example**: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).

Binary relations on a set

- Solution: R={(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

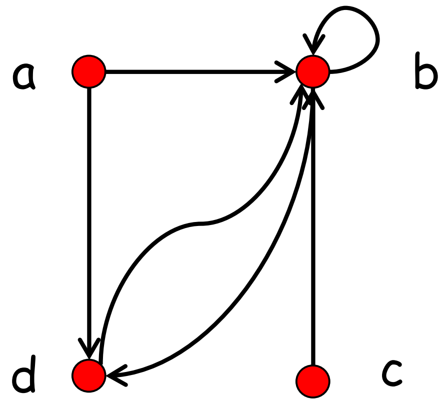


R	1	2	3	4
1		x	x	x
2			x	x
3				x
4				

Representing binary relations

- We have many ways of representing binary relations. We now take a closer look at two ways of representation: **Boolean (zero-one) matrices** and **directed graphs**.

	C1	C2	C3	C4	C5
P1	0	0	0	1	1
P2	1	0	0	1	0
P3	0	0	0	0	0



The matrix of a relation

- If R is a relation from a set X to itself and M_R is the matrix of R, then M_R is a square matrix.
- Example: Let X = {2,3,4,5} and R = {(x,y) | x+y divides by 3}. Then :

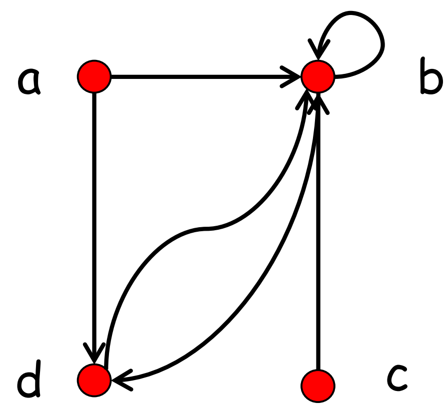
M_R =

	2	3	4	5
2	0	0	1	0
3	0	1	0	0
4	1	0	0	1
5	0	0	1	0

fill it

Representing relations - Digraphs

- **Example**:
  - Display the digraph with V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



- An edge of the b called a loop form (b, b) is loop.

Properties of relations

Special properties for relation on a set A:

- **Reflexive** : every element is self-related. I.e. aRa for all a $\in A$
- **Symmetric** : order is irrelevant. I.e. for all a,b $\in A$ aRb iff bRa
- **Transitive** : when a is related to b and b is related to c, it follows that a is related to c. I.e. for all a,b,c $\in A$ aRb and bRc implies aRc

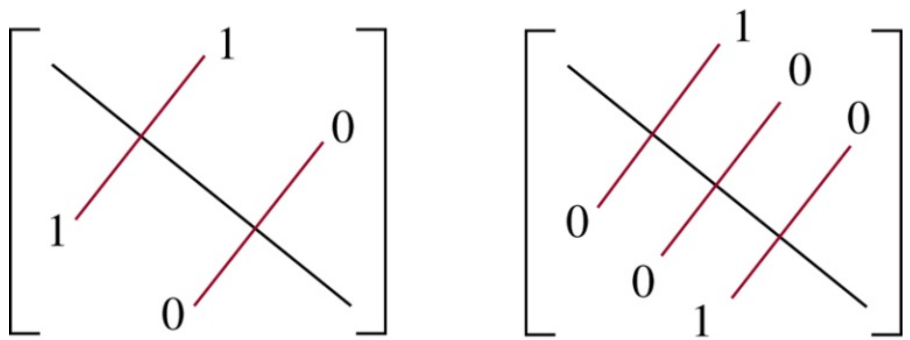
Q: Which of these properties hold for:

1) "Siblinghood" 2) "<" 3) "≤"

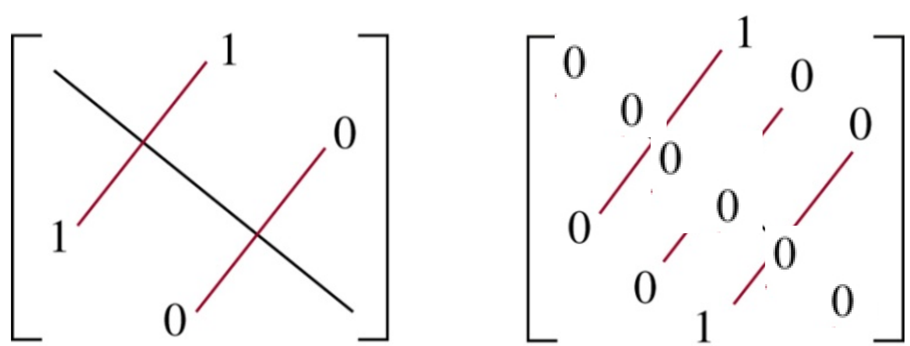
Properties of relations - Warnings

- Warnings: there are additional concepts with confusing names
  - **Antisymmetric**: not equivalent to "not symmetric". Meaning: it's never the case for $a \neq b$ that both aRb and bRa hold.
  - **Asymmetric**: also not equivalent to "not symmetric". Meaning: it's never the case that both aRb and bRa hold.
  - **Irreflexive**: not equivalent to "not reflexive". Meaning: it's never the case that aRa holds.

Visualization



(a) Symmetric (b) Antisymmetric



(a) Symmetric (b) Asymmetric

Symmetric, Asymmetric and Antisymmetric

- 对称的(symmetric)：对所有的aRb,都有bRa
- 非对称的(not symmetric)：存在一些aRb,满足b$\cancel{R}$a
- 不对称的(asymmetric)：对所有的aRb,都有b$\cancel{R}$a
- 非不对称的(not asymmetric)：存在一些aRb,满足bRa
- 反对称的(antisymmetric)：对所有的aRb和bRa,都有a=b
- 非反对称的(not antisymmetric)：存在一些a≠b,满足aRb和bRa

可见：

- (1) asymmetric→not symmetric,而not symmetric不能得出asymmetric
- (2) asymmetric→antisymmetric,而antisymmetric不能得出asymmetric

举例1：A={1,2,3,4},R={(1,2),(2,2),(3,4),(4,1)},则：

- R是非对称的(not symmetric),因为(1,2)属于R,而(2,1)不属于R;
- R是非不对称的(not asymmetric),因为(2,2)属于R
- R是反对称的(antisymmetric),因为对于任意a≠b,不存在(a,b)和(b,a)都属于R

Useful summary

- Let R be a relation on a set A, i.e. R is a subset of the Cartesian product $A \times A$
  - R is **reflexive** if $(x, x) \in R$ for every $x \in A$
  - R is **irreflexive** if $(x, x) \notin R$ for every element $x \in A$.
  - R is **symmetric** if for all $x, y \in A$ such that $(x,y) \in R$ then $(y, x) \in R$
  - R is **antisymmetric** if for all $x, y \in A$ such that $x \neq y$, if $(x, y) \in R$ then $(y, x) \notin R$
  - R is **transitive** if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$

Operators on Relations

- Operators on Sets
- Inversion
- Composite *

Combining Relations

- Let R_1 be a relation from X to Y
- Let R_2 be a relation from Y to Z
- **Definition**: The composition of R_1 and R_2, denoted $R_2 \bullet R_1$ (or $R_2 \odot R_1$), is a relation from X to Z defined by $\{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$
- In other words, if relation R_1 contains a pair (x, y) and relation R_2 contains a pair (y, z), then $R_2 \bullet R_1$ contains a pair (x, z).

$R_2 \bullet R_1$ contains a pair (x, z) : $x \, R_2 \bullet R_1 \, z$

Definition: If R is a relation from A to B, and S is a relation from B to C then the **composite** of R and S is the relation $S \bullet R$ (or just SR) from A to C defined by setting $a (S \bullet R) c$ if and only if there is some b such that aRb and bSc.

Notation is weird because generalizing functional composition: $f \bullet g (x) = f (g (x))$.

Combining Relations

Composite relation: $S \circ R$

$$(a, b) \in S \circ R \leftrightarrow \exists x : (a, x) \in R \land (x, b) \in S$$

Note:

$$(a, b) \in R \land (b, c) \in S \rightarrow (a, c) \in S \circ R$$

Example:

$$R = \{ (1,1), (1,4), (2,3), (3,1), (3,4) \}$$

$$S = \{ (1,0), (2,0), (3,1), (3,2), (4,1) \}$$

$$S \circ R = \{ (1,0), (1,1), (2,1), (2,2), (3,0), (3,1) \}$$

Examples

- Let X and Y be relations on $A = \{1, 2, 3, \ldots\}$.
- $X = \{(a, b) \mid b = a + 1\}$ "b equals a plus 1"

  ($X = \{(b, c) \mid c = b + 1\}$ "c equals b plus 1")

- $Y = \{(a, b) \mid b = 3a\}$ "b equals 3 times a"

  ($Y = \{(b, c) \mid c = 3b\}$ "c equals 3 times b")

Y maps an element a to the element 3a, and afterwards X maps 3a to 3a + 1), resulting in $X \bullet Y = \{(a,b) \mid b = 3a + 1\}$

$X \bullet Y = \{(a,c) \mid c = 3a + 1\} = \{(a,b) \mid b = 3a + 1\}$

$Y \bullet X = \{(a, b) \mid b = 3a + 3\}$

Power of relations

Power of relation: R^n

$$R^0 = I_A \quad R^1 = R \quad R^{n+1} = R^n \circ R$$

Example: $R = \{(1,1),(2,1),(3,2),(4,3)\}$

$$R^2 = R \circ R = \{(1,1),(2,1),(3,1)\ (4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1)\ (4,1)\}$$

$$R^4 = R^3 \circ R = R^3$$

Summary

Theorem: Let R be the relation on a set A. Then we have

R is reflexive iff $I_A \subseteq R$

R is irreflexive iff $I_A \cap R = \emptyset$

R is symmetric iff $R = R^{-1}$

R is antisymmetric iff $R \cap R^{-1} \subseteq I_A$

R is transitive iff $R \bullet R \subseteq R$

Theorem: A relation R is transitive if an only if $R^n \subseteq R$ for all $n = 1,2,3,\ldots$

Closures of Relations

R: X→X

关系**S**满足

(1) S 是自 可传递的

(2)R⊆ S

(3)对任何 可传递 关系**S′**

R ⊆ S′ ⇒ **S⊆S′**

R的 可传递 闭包

r(R) s t(R)

Reflexive Closures

- **Example**: Find the reflexive closure of relation R = {(1, 1),(1, 2), (2, 1), (3, 2)} on the set A={1,2,3}.
- **Solution**:
  - We know that any reflexive relation on A must contain the elements (1,1), (2,2),and (3, 3).
  - By adding (2, 2) and (3, 3) to R, we obtain the reflexive relation S, which is given by S = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 3)}.
  - S is reflexive, contains R, and is contained within every reflexive relation that contains R.
  - Therefore S is reflexive closure of R.

Symmetric Closures

- **Example**: Find the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers.
- **Solution**:
  - The symmetric closure of R is given by $R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}$

Transitive Closures

- **Example:** Find the transitive closure of the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set A = {1, 2, 3, 4}.
- **Solution**:
  - R would be transitive, if for all pairs (a, b) and (b, c) in R there were also a pair (a, c) in R.
  - If we add the missing pairs (1, 2), (2, 3), (2, 4), and (3, 1), will R be transitive?

  - No, because the extended relation R contains (3, 1) and (1, 4), but does not contain (3, 4).
  - By adding new elements to R, we also add new requirements for its transitivity. We need to look at paths in digraphs to solve this problem.
  - Imagine that we have a relation R that represents all train connections in the US.

Transitive Closures

- Therefore, R* is the union of R^n across all positive integers n:

$$R^* = \bigcup_{n=1}^{\infty} R^{\mathrm{n}} = R^1 \bigcup R^2 \bigcup R^3 \cdots$$

Theorem: For a relation R on a set *A* with n elements, the transitive closure R* is given by:

$$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$$

For matrices representing relations we have:

$$M_{R^*} = M_R \vee M_R{}^{[2]} \vee M_R{}^{[3]} \vee \ldots \vee M_R{}^{[n]}$$

Equivalence Relations

- Equivalence relations are used to relate objects that are similar in some way.
- **Definition**: A relation on a set *A* is called an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.
- Two elements that are related by an equivalence relation R are called equivalent.

Equivalence Relations

- **Example**: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a)=l(b)$,where $l(x)$ is the length of the string x. Is R an equivalence relation?
- **Solution**:
  - R is reflexive, because $l(a) = l(a)$ and therefore aRa for any string a.
  - R is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa.
  - R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc.
- R is an equivalence relation.

Equivalence relations

- **Example**:
  - Consider set $X = \{1,2,\ldots,13\}$. Define xRy as 5 divides $x - y$ (i.e., $x - y = 5k$, for some int k). We can verify that R is reflexive, symmetric, and transitive. Here is how.
  - The equivalence class [1] consists of all x with $xR1$. Thus:
    - $[1] = \{x \in X \mid 5 \text{ divides } x - 1\} = \{1, 6, 11\}$
  - Similarly:
    - $[2] = \{2, 7, 12\}$
    - $[3] = \{3, 8, 13\}$
    - $[4] = \{4, 9\}$
    - $[5] = \{5, 10\}$

Partial Order relations

- **Definition**: Let X be a set and R a relation on X, R is a partial order on X if R is **reflexive**, **antisymmetric** and **transitive**. A set X together with a partial ordering R is called a partially ordered set, or poset, or PO, and is denoted by (X, R).
- **Example**: Is $(x, y) \in R$ in partial order if $x >= y$?
  - Yes, since:
    - Reflexive: $(x, x) \in R$
    - Anti-symmetric: If $(x, y) \in R$ and $x \neq y$, then $(y, x) \notin R$
    - Transitive: If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$

Partial Order relations

- **Example**: Is the "inclusion relation" $\subseteq$ a partial ordering on the power set of a set S?
  - $\subseteq$ is reflexive, because $A \subseteq A$ for every set $A \in S$.
  - $\subseteq$ is antisymmetric, because if $A \neq B$, then $A \subseteq B \wedge B \subseteq A$ is false.
  - $\subseteq$ is transitive, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- Consequently, $(P(S), \subseteq)$ is a partially ordered set or poset.

Partial Order relations

- Let $x,y \in X$,
  - If (x,y) or (y,x) are in R, then x and y are **comparable**.
  - If $(x,y) \notin R$ and $(y,x) \notin R$, then x and y are **incomparable**.
  - **Definition**: If every pair of elements in X are comparable, then R is a **total order** on X.
    - In this case, X is called a totally ordered or linearly ordered set, and $\leq$ is called a total order or linear order. A totally ordered set is also called a **chain**.

Partial Order relations

- **Example**: Is $(Z, \leq)$ a totally ordered poset?
  - Yes, because $a \leq b$ or $b \leq a$ for all integers a and b.
- **Example**: Is $(Z^+$, division) a totally ordered poset?
  - No, because it contains incomparable elements such as 5 and 7.

Partial Order relations

- In a poset the notation $a \leq b$ denotes that $(a, b) \in R$.
- Note that the symbol $\leq$ is used to denote the relation in any poset, not just the "less than or equal" relation.
- The notation $a < b$ denotes that $a \leq b$, but $a \neq b$.
- If $a < b$ we say "a is less than b" or "b is *greater* than a".

Lexicographic Order

- How can we define a lexicographic ordering on the set of English words?
- This is a **special case** of an ordering of strings on a set constructed from a partial ordering on the set.
- We already have an ordering of letters (such as a < b, b < c, …), and from that we want to derive an ordering of strings.
- Let us take a look at the *general* case, that is, how the construction works in any poset.

Lexicographic Order

- **First step**: Construct a partial ordering on the Cartesian product of two posets,$(A_1, \leq_1)$ and $(A_2, \leq_2)$:
- $(a_1,a_2) < (b_1,b_2)$ if $(a_1 <_1 b_1) \vee [(a_1=b_1) \wedge (a_2 <_2 b_2)]$
- $(a_1,a_2) \leq (b_1, b_2)$ if $(a_1 <_1 b_1) \vee [(a_1 = b_1) \wedge (a_2 \leq_2 b_2)]$
- **Examples**:
  - In the poset $(Z \times Z, \leq)$, …
    - is (5, 5) < (6, 4) ? YES
    - is (6, 5) < (6, 4) ? NO
    - is (3, 3) < (3, 3) ? NO

Lexicographic Order

- **Second step**: Extend the previous definition to the Cartesian product of n posets $(A_1, \leq_1), (A_2, \leq_2), \ldots, (A_n, \leq_n)$:
- $(a_1, a_2, \ldots, a_n) < (b_1, b_2, \ldots, b_n)$ if $(a_1 <_1 b_1) \vee \exists i>0\ (a_1 = b_1, a_2 = b_2, \ldots, a_i = b_i, a_{i+1} <_{i+1} b_{i+1})$
- $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ if $(a_1 <_1 b_1) \vee \exists i>0\ (a_1 = b_1, a_2 = b_2, \ldots, a_i = b_i, a_{i+1} <_{i+1} b_{i+1}) \vee (a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n)$

- Examples:
  - Is $(1,1,1,2,1) < (1,1,1,1,2)$? No
  - Is $(1, 1, 1, 1, 1) < (1, 1, 1, 1, 2)$? Yes

Hasse Diagram (哈斯图)

- Hasse diagram is a graphical display of a poset.
- A point is drawn for each element of the poset, and line segments are drawn between these points according to the following two rules:
  - 1. If $x < y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y.
  - 2. The line segment between the points corresponding to any two elements x and y of the poset is included in the drawing iff x covers y or y covers x.

Cover Relation

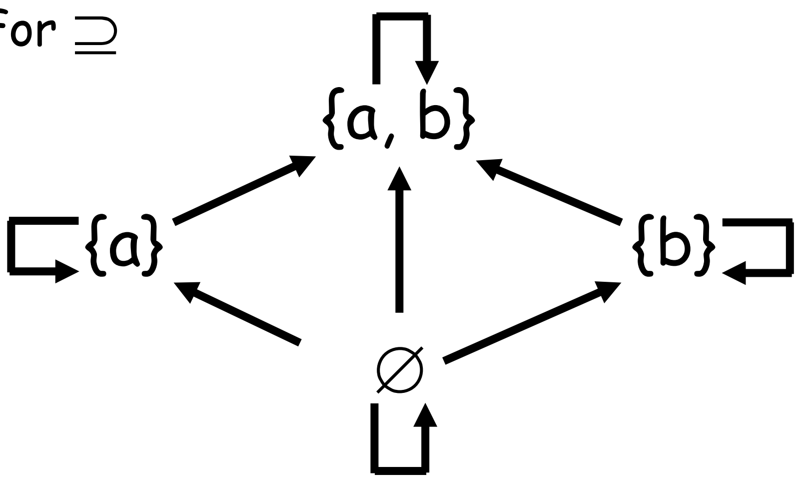
- Let $(S, \leq)$ be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x < y$ and there is no element $z \in S$ such that $x < z < y$. The set of pairs (x, y) such that y covers x is called **the covering relation** of $(S, \leq)$.

Hasse Diagrams

We produce Hasse Diagrams from directed graphs of relations by doing a **transitive reduction** plus a **reflexive reduction** (if weak) and (usually) **dropping arrowheads** (using, instead, "above" to give direction)

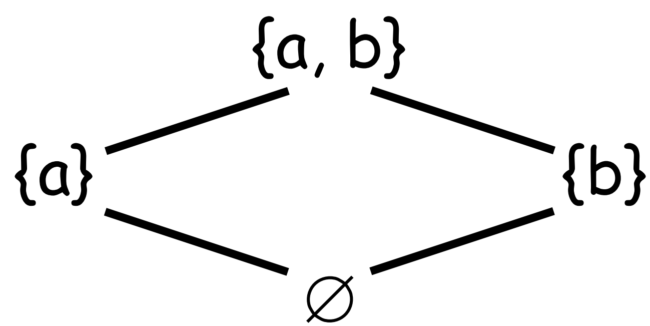
1) Transitive reduction — discard all arcs except those that "directly cover" an element.
2) Reflexive reduction — discard all self loops.

For $\supseteq$



$\equiv$

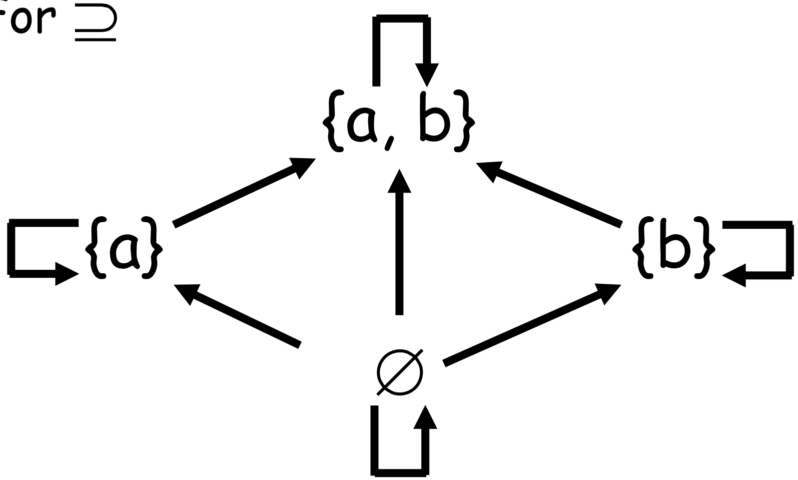
we write:



The Procedure Summary

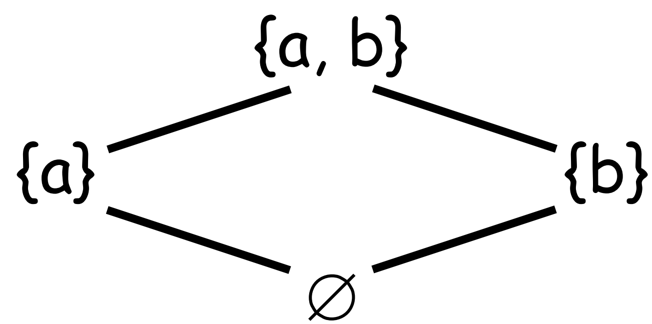
- Start with the directed graph for this relation.
- Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops.
- Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x < z$ and $z < x$.
- Finally, arrange each edge so that its initial vertex is below its terminal vertex (as it is drawn on paper). Remove all the arrows on the directed edges, because all edges point “upward” toward their terminal vertex.

For $\supseteq$



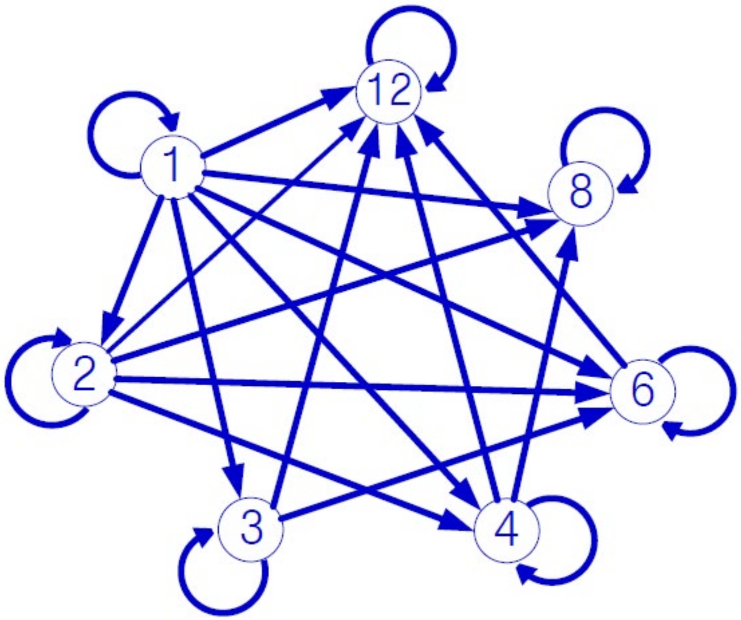
$\equiv$

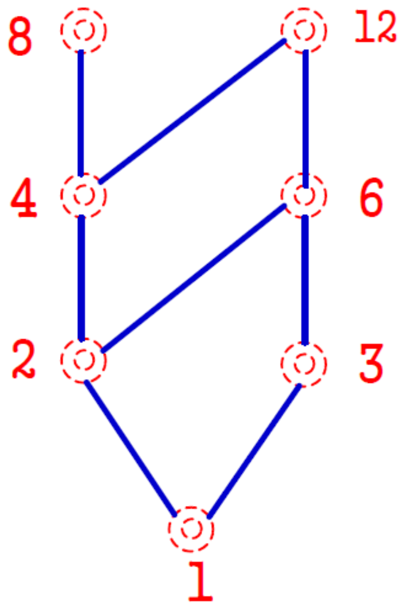
we write:



Hasse Diagram

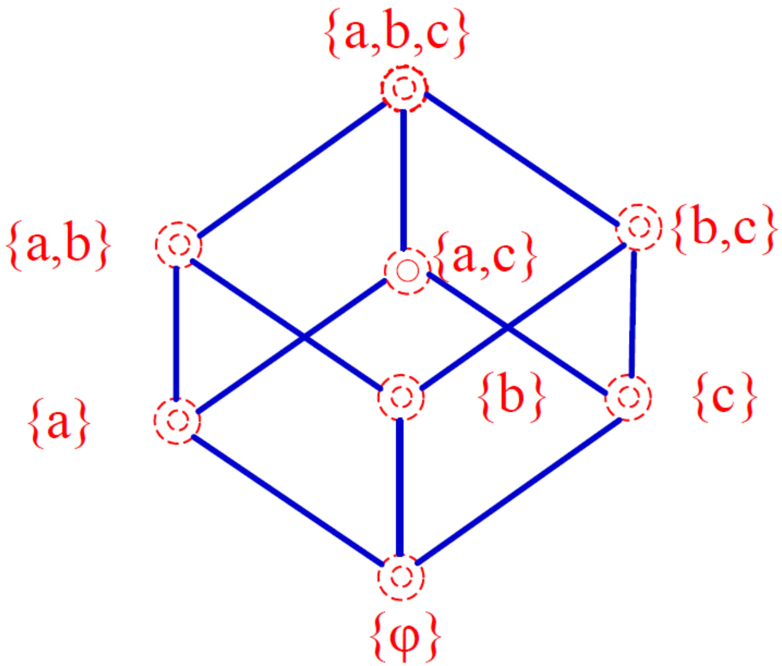
- **Example**: A={1,2,3,4,6,8,12}, integral division relation.





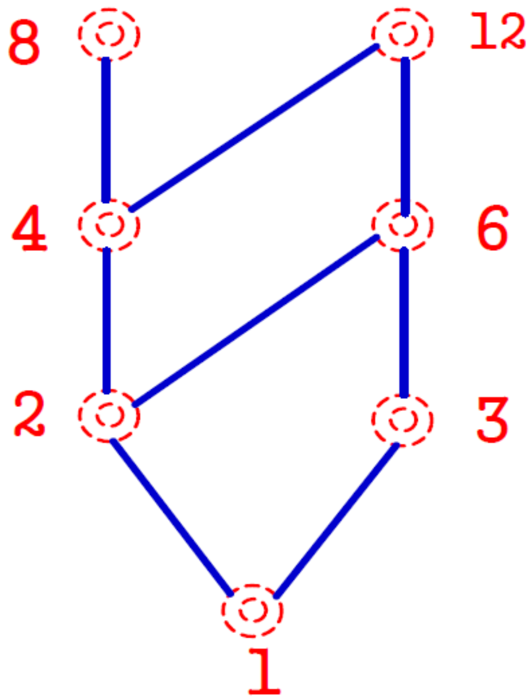
Hasse Diagram

- **Example**: S={a, b, c}, (P(S), ⊆)



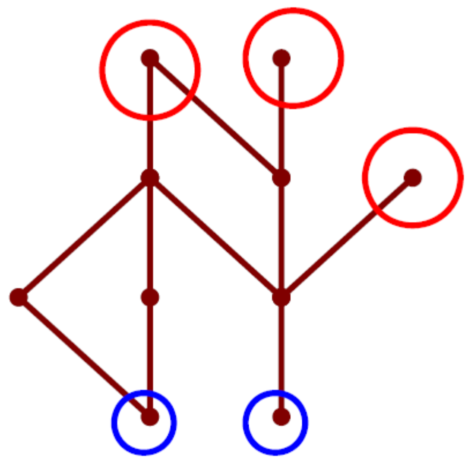
Maximum/Minimum/Greatest/Least

- Maximum/Minimum element
- 极大、极小
- Greatest/Least element
- 最大、最小
- Upper/Lower bound
- 上界、下界
- Least upper/Greatest lower bound
- 最小上界、最大下界



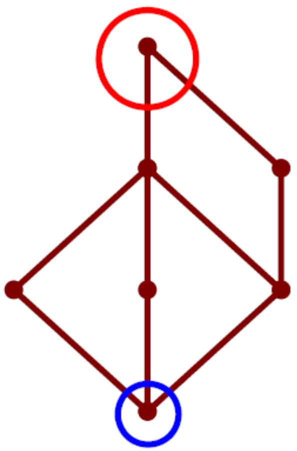
Minimum and Maximum

- **Definition**: In a poset S, an element z is a **minimum** element if there is no element b∈S, thus b ≤ z and b ≠ z.
- How about definition for **maximum** element?
- **Example**:
  - Reds are maximal.
  - Blues are minimal.



Least and Greatest

- **Definition**: In a poset S, an element z is a **Least** element if $\forall b \in S$, $z \leq b$.
- How about definition for **Greatest** element.
- **Example**:
  - Reds are greatest.
  - Blues are least.
- Greatest/Least may not exist.



Least and Greatest

- **Theorem**: In every poset, if the **greatest** element exists, then it is **unique**. Similarly for the **least**.
- **Proof**:
  - Suppose there are two greatest elements, a_1 and a_2, with $a_1 \neq a_2$. Then $a_1 \leq a_2$, and $a_2 \leq a_1$, by defn of greatest. So $a_1 = a_2$, a contradiction. Thus, our assumption was incorrect, and the greatest element, if it exists, is unique.
  - Similar proof for least.