7.6 向量值函数的微分法与多元函数的Taylor公式

本节内容仅限于了解

7.6.1 向量值函数的基本概念

默认向量为列向量.

定义　　设$D^n \subset \mathbb{R}^n, \boldsymbol{x} = (x_1, x_2, \dots, x_n)^T, \boldsymbol{y} = (y_1, y_2, \dots, y_m)^T$是两个变量，如果对应于每个$\boldsymbol{x} \in D^n$，变量$\boldsymbol{y}$按照一定法则总有确定的向量值和它对应，则称$\boldsymbol{y}$是$\boldsymbol{x}$的向量值函数. 记作

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in D^n$$

或者

$$\boldsymbol{f}: D^n \to \mathbb{R}^m$$

$$\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T \to \boldsymbol{y} = (y_1, y_2, \dots, y_m)^T$$

D^n为向量值函数$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$的定义域，

$V^m := \{\boldsymbol{y} \in \mathbb{R}^m | \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{x} \in D^n\}$是函数的值域.

默认向量为列向量.

向量值函数$\boldsymbol{f}: D^n \to \mathbb{R}^m$也称为从$n$维空间$\mathbb{R}^n$到$m$维空间$\mathbb{R}^m$的映射.

它在$\boldsymbol{x}$处的函数值是$\boldsymbol{y} = (y_1, y_2, \dots, y_m)^T$, 可见$\boldsymbol{y}$的每个坐标$y_i$都依赖于

函数值是向量的函数

$\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$，它们是$\boldsymbol{x}$的函数,即

$$y_i = f_i(\boldsymbol{x}), \qquad \boldsymbol{x} \in D^n \ (i = 1,2, \dots, m)$$

称$f_i(\boldsymbol{x})$是$\boldsymbol{f}$的坐标函数（或分量函数）.

默认向量为列向量.

用列向量书写

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{pmatrix}$$

默认向量为列向量.

例 $\boldsymbol{f}(x,y)=(x+2y,3x-y)^T$

其坐标函数为

$$f_1(x,y)=x+2y, f_2(x,y)=3x-y$$

函数可写成

$$\boldsymbol{f}(x,y)=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x+2y\\ 3x-y\end{pmatrix}$$

默认向量为列向量.

若向量值函数$\boldsymbol{f}(\boldsymbol{x}) = \left(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})\right)^T$在点$\boldsymbol{x}^0 = (x_1^0, x_2^0, \dots, x_n^0)^T$的某个去心邻域内有定义，如果每个分量函数$f_i(\boldsymbol{x})$ $(i = 1,2, \dots, m)$在$\boldsymbol{x} \to \boldsymbol{x}^0$时有极限，即

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}^0} f_i(\boldsymbol{x}) = a_i \quad (i = 1,2, \dots, m)$$

则称向量值函数$\boldsymbol{f}(\boldsymbol{x})$在$\boldsymbol{x} \to \boldsymbol{x}^0$时有极限，并称$\boldsymbol{a} = (a_1, a_2, \dots, a_m)^T$为$\boldsymbol{x} \to \boldsymbol{x}^0$时$\boldsymbol{f}(\boldsymbol{x})$的极限，记作

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}^0} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{a} = (a_1, a_2, \dots, a_m)^T$$

换句话说:

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}^0} \boldsymbol{f}(\boldsymbol{x}) \triangleq \begin{pmatrix} \lim\limits_{\boldsymbol{x} \to \boldsymbol{x}^0} f_1(\boldsymbol{x}) \\ \lim\limits_{\boldsymbol{x} \to \boldsymbol{x}^0} f_2(\boldsymbol{x}) \\ \vdots \\ \lim\limits_{\boldsymbol{x} \to \boldsymbol{x}^0} f_m(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

如果$f_i(\boldsymbol{x})$ $(i=1,2,\dots,m)$ 在点$\boldsymbol{x}^0=(x_1^0,x_2^0,\dots,x_n^0)^T$均连续，即

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}^0} f_i(\boldsymbol{x})=f_i(\boldsymbol{x}^0)\quad(i=1,2,\dots,m),$$

则称向量值函数$\boldsymbol{f}(\boldsymbol{x})$在点$\boldsymbol{x}^0$连续，　即

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}^0}\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x}^0)=\left(f_1(\boldsymbol{x}^0),f_2(\boldsymbol{x}^0),\dots,f_m(\boldsymbol{x}^0)\right)^T.$$

换句话说:

$$\text{连续}\Longleftrightarrow\lim_{\boldsymbol{x}\to\boldsymbol{x}^0}\boldsymbol{f}(\boldsymbol{x})=\begin{pmatrix}\lim_{\boldsymbol{x}\to\boldsymbol{x}^0} f_1(\boldsymbol{x})\\ \lim_{\boldsymbol{x}\to\boldsymbol{x}^0} f_2(\boldsymbol{x})\\ \vdots\\ \lim_{\boldsymbol{x}\to\boldsymbol{x}^0} f_m(\boldsymbol{x})\end{pmatrix}=\begin{pmatrix}f_1(\boldsymbol{x}^0)\\ f_2(\boldsymbol{x}^0)\\ \vdots\\ f_m(\boldsymbol{x}^0)\end{pmatrix}=\boldsymbol{f}(\boldsymbol{x}^0)$$

默认向量为列向量.

关于连续向量值函数的运算有如下性质：

(1) 若n 元m维向量值函数$\boldsymbol{f}(\boldsymbol{x}) = \left(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})\right)^T$和

$\boldsymbol{g}(\boldsymbol{x}) = \left(g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), \dots, g_m(\boldsymbol{x})\right)^T$) 在点$\boldsymbol{x}^0$均连续， 则对于任意实数$\alpha, \beta$

向量值函数$\alpha \boldsymbol{f}(\boldsymbol{x}) + \beta \boldsymbol{g}(\boldsymbol{x})$和$\boldsymbol{f}(\boldsymbol{x})^T \cdot \boldsymbol{g}(\boldsymbol{x})$在点$\boldsymbol{x}^0$均连续， 即

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}^0} [\alpha \boldsymbol{f}(\boldsymbol{x}) + \beta \boldsymbol{g}(\boldsymbol{x})] = \alpha \boldsymbol{f}(\boldsymbol{x}^0) + \beta \boldsymbol{g}(\boldsymbol{x}^0),$$

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}^0} \boldsymbol{f}(\boldsymbol{x})^T \cdot \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}^0)^T \cdot \boldsymbol{g}(\boldsymbol{x}^0).$$

(2) 若n元数量值函数$u=\varphi(\boldsymbol{x})$在点$\boldsymbol{x}^0$连续， 向量值函数

$$\boldsymbol{f}(u)=\begin{pmatrix} f_1(u) \\ f_2(u) \\ \vdots \\ f_m(u) \end{pmatrix}$$

在相应的点$u_0=\varphi(\boldsymbol{x}^0)$处连续， 则复合向量值函数

$$\boldsymbol{f}(\varphi(\boldsymbol{x}))=\begin{pmatrix} f_1(\varphi(\boldsymbol{x})) \\ f_2(\varphi(\boldsymbol{x})) \\ \vdots \\ f_m(\varphi(\boldsymbol{x})) \end{pmatrix}$$

在点$\boldsymbol{x}^0$连续.

7.6.2 向量值函数的微分法

默认向量为列向量.

设$\boldsymbol{f}(\boldsymbol{x}) = \left(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})\right)^T$, 如果$f_i(\boldsymbol{x})$ $(i = 1,2, \dots, m)$ 在 $\boldsymbol{x}^0$ 都存在偏导数,

记$\boldsymbol{f}(\boldsymbol{x})$关于自变量$x_j$的偏导数为$\frac{\partial \boldsymbol{f}(\boldsymbol{x}^0)}{\partial x_j}$或$\boldsymbol{f}_{x_j}(\boldsymbol{x}^0)$，且

$$\frac{\partial \boldsymbol{f}(\boldsymbol{x}^0)}{\partial x_j} = \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_j}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^0} = \left(\frac{\partial f_1}{\partial x_j}, \frac{\partial f_2}{\partial x_j}, \dots, \frac{\partial f_m}{\partial x_j}\right)^T \quad (j = 1,2, \dots, n)$$

换句话说:

$$\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_j} = \frac{\partial}{\partial x_j}\begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{pmatrix}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}^0} \triangleq \begin{pmatrix} \frac{\partial f_1}{\partial x_j} \\ \frac{\partial f_2}{\partial x_j} \\ \vdots \\ \frac{\partial f_m}{\partial x_j} \end{pmatrix}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}^0} \quad (j = 1,2, \dots, n)$$

默认向量为列向量.

如果每个分量函数$f_i(\boldsymbol{x})$ $(i=1,2,\dots,m)$在 $\boldsymbol{x}^0$ 都可微，则称$\boldsymbol{f}(\boldsymbol{x})$在 $\boldsymbol{x}^0$ 可微，

并称列向量

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}^0) \triangleq \left(df_1(\boldsymbol{x}^0), df_2(\boldsymbol{x}^0), \dots, df_m(\boldsymbol{x}^0)\right)^T$$

为向量值函数$\boldsymbol{f}(\boldsymbol{x})$在 $\boldsymbol{x}^0$ 的微分， 记为$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}^0)$，其中

$$df_i(\boldsymbol{x}^0) = \frac{\partial f_i}{\partial x_1}dx_1 + \frac{\partial f_i}{\partial x_2}dx_2 + \cdots + \frac{\partial f_i}{\partial x_n}dx_n \quad = \nabla f_i(\boldsymbol{x}^0)d\boldsymbol{x}$$

$$d\boldsymbol{x} = (dx_1, dx_2, \dots dx_n)^T$$

换句话说:

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}^0) = \mathrm{d}\begin{pmatrix} f_1(\boldsymbol{x}^0) \\ f_2(\boldsymbol{x}^0) \\ \vdots \\ f_m(\boldsymbol{x}^0) \end{pmatrix} \triangleq \begin{pmatrix} \mathrm{d}f_1(\boldsymbol{x}^0) \\ \mathrm{d}f_2(\boldsymbol{x}^0) \\ \vdots \\ \mathrm{d}f_m(\boldsymbol{x}^0) \end{pmatrix}$$

$$df_i(\boldsymbol{x}^0) = \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \dots \frac{\partial f_i}{\partial x_n}\right)\begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix}$$

∇f(x)是行向量.

若设

$$Jf(x)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}=\begin{pmatrix}\nabla f_1(\boldsymbol{x})\\ \nabla f_2(\boldsymbol{x})\\ \vdots\\ \nabla f_m(\boldsymbol{x})\end{pmatrix}=:\nabla\begin{pmatrix}f_1(\boldsymbol{x})\\ f_2(\boldsymbol{x})\\ \vdots\\ f_m(\boldsymbol{x})\end{pmatrix}=:\nabla\boldsymbol{f}(\boldsymbol{x})$$

向量值函数$f(x)$的梯度

则$J\boldsymbol{f}(\boldsymbol{x})$称矩阵为向量值函数$\boldsymbol{f}(\boldsymbol{x})$的Jacobi矩阵.

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}^0)=\begin{pmatrix}df_1(\boldsymbol{x}^0)\\ df_2(\boldsymbol{x}^0)\\ \vdots\\ df_m(\boldsymbol{x}^0)\end{pmatrix}=\begin{pmatrix}\nabla f_1(\boldsymbol{x}^0)d\boldsymbol{x}\\ \nabla f_2(\boldsymbol{x}^0)d\boldsymbol{x}\\ \vdots\\ \nabla f_m(\boldsymbol{x}^0)d\boldsymbol{x}\end{pmatrix}=J\boldsymbol{f}(\boldsymbol{x}^0)d\boldsymbol{x}$$

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}^0)\triangleq\left(df_1(\boldsymbol{x}^0),df_2(\boldsymbol{x}^0),\ldots,df_m(\boldsymbol{x}^0)\right)^T$$

$$df_i(\boldsymbol{x}^0)=\frac{\partial f_i}{\partial x_1}dx_1+\frac{\partial f_i}{\partial x_2}dx_2+\cdots+\frac{\partial f_i}{\partial x_n}dx_n=\nabla f_i(\boldsymbol{x}^0)d\boldsymbol{x}$$

例　　设向量值函数

$$\boldsymbol{f}(\boldsymbol{x}) = \left(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\right)^T, f_1(\boldsymbol{x}) = e^{2x_1+x_2}\cos x_1, f_2(\boldsymbol{x}) = (x_1+x_2)e^{-x_1-x_2}.$$

求$\boldsymbol{f}(\boldsymbol{x})$在点$O(0,0)$处的微分$d\boldsymbol{f}(\boldsymbol{x})$

解:

$$d\boldsymbol{f}(\boldsymbol{x}) = d\begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} df_1(\boldsymbol{x}) \\ df_2(\boldsymbol{x}) \end{pmatrix}$$

$$df_1(\boldsymbol{x}) = [2e^{2x_1+x_2}\cos x_1 - e^{2x_1+x_2}\sin x_1]dx_1 + e^{2x_1+x_2}\cos x_1\, dx_2$$

$$df_2(\boldsymbol{x}) = (1-x_1-x_2)e^{-x_1-x_2}(dx_1+dx_2)$$

$$d\boldsymbol{f}(\boldsymbol{x})\Big|_{(0,0)} = \begin{pmatrix} df_1(\boldsymbol{x}) \\ df_2(\boldsymbol{x}) \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 2dx_1 + dx_2 \\ dx_1 + dx_2 \end{pmatrix}$$

这页f是数量值函数.

$\nabla f(x)$是行向量.

默认向量为列向量.

n元函数$y=f(\boldsymbol{x})=f(x_1,x_2,\dots,x_n)$的梯度$\nabla f(\boldsymbol{x})$是向量值函数,

$$\nabla f=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots\frac{\partial}{\partial x_n}\right)f=\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\cdots\frac{\partial f}{\partial x_n}\right)$$

将$\nabla f(\boldsymbol{x})$写成∇f更简便

若$\nabla f(\boldsymbol{x})$在$\boldsymbol{x}$点可微， 则定义$f(\boldsymbol{x})$在$\boldsymbol{x}$点处的Hessian阵

$$\nabla^2 f\triangleq\begin{pmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n}\\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^2 f}{\partial x_n\partial x_1} & \frac{\partial^2 f}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}\end{pmatrix}=\begin{pmatrix}\nabla\frac{\partial f}{\partial x_1}\\ \nabla\frac{\partial f}{\partial x_2}\\ \vdots\\ \nabla\frac{\partial f}{\partial x_n}\end{pmatrix}$$

7.6.3 多元函数的Taylor公式

$\nabla f(x)$是行向量.

默认向量为列向量.

定理 设n元函数$f(\boldsymbol{x})$在点$\boldsymbol{x}^0$的某邻域内二阶偏导连续, 则

$$f(\boldsymbol{x}) = f(\boldsymbol{x}^0) + \nabla f(\boldsymbol{x}^0)(\boldsymbol{x}-\boldsymbol{x}^0) + \frac{1}{2!}(\boldsymbol{x}-\boldsymbol{x}^0)^T \nabla^2 f(\boldsymbol{x}^0)(\boldsymbol{x}-\boldsymbol{x}^0) + R_2$$

$$R_2 = o(\| \boldsymbol{x}-\boldsymbol{x}^0 \|^2) \quad (\boldsymbol{x} \to \boldsymbol{x}^0)$$

$$= f(\boldsymbol{x}^0) + \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots \frac{\partial f}{\partial x_n}\right)\begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ \vdots \\ x_n - x_n^0 \end{pmatrix}$$

Hessian阵

二次型

$$+\frac{1}{2!}(x_1 - x_1^0, x_2 - x_2^0, \ldots x_n - x_n^0)\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}\begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ \vdots \\ x_n - x_n^0 \end{pmatrix} + R_2$$

二元函数$f(x_1,x_2)$

$$f(x_1,x_2)=f(x_1^0,x_2^0)+\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}\right)\Big|_{(x_1^0,x_2^0)}\begin{pmatrix}x_1-x_1^0\\x_2-x_2^0\end{pmatrix}$$

$$+\frac{1}{2!}(x_1-x_1^0,x_2-x_2^0)\begin{pmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2}\\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2}\end{pmatrix}\Big|_{(x_1^0,x_2^0)}\begin{pmatrix}x_1-x_1^0\\x_2-x_2^0\end{pmatrix}+o(\rho^2)$$

二次型

(半)正定，(半)负定…

定义

对称阵A正定 $\Longleftrightarrow$ $\boldsymbol{x}^T A\boldsymbol{x}>0\ \ (\forall \boldsymbol{x}\neq\boldsymbol{0})$

$\Longleftrightarrow$ 顺序主子式都大于0

小结

向量值函数：取值为向量的“函数”
$$\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}=\begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{pmatrix}$$

向量值函数的极限
$$\lim_{\boldsymbol{x}\to\boldsymbol{x}^0}\boldsymbol{f}(\boldsymbol{x})\triangleq\begin{pmatrix} \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}^0} f_i(\boldsymbol{x}) \\ \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}^0} f_i(\boldsymbol{x}) \\ \vdots \\ \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}^0} f_i(\boldsymbol{x}) \end{pmatrix}$$

向量值函数的连续
$$\lim_{\boldsymbol{x}\to\boldsymbol{x}^0}\boldsymbol{f}(\boldsymbol{x})=\begin{pmatrix} \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}^0} f_i(\boldsymbol{x}) \\ \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}^0} f_i(\boldsymbol{x}) \\ \vdots \\ \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}^0} f_i(\boldsymbol{x}) \end{pmatrix}=\begin{pmatrix} f_1(\boldsymbol{x}^0) \\ f_2(\boldsymbol{x}^0) \\ \vdots \\ f_m(\boldsymbol{x}^0) \end{pmatrix}=\boldsymbol{f}(\boldsymbol{x}^0)$$

向量值函数的偏导数

$$\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_j} = \frac{\partial}{\partial x_j}\begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{pmatrix}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}^0} \triangleq \begin{pmatrix} \frac{\partial f_1}{\partial x_j} \\ \frac{\partial f_2}{\partial x_j} \\ \vdots \\ \frac{\partial f_m}{\partial x_j} \end{pmatrix}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}^0} (j = 1,2,\ldots,n)$$

向量值函数的微分

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}^0) = \mathrm{d}\begin{pmatrix} f_1(\boldsymbol{x}^0) \\ f_2(\boldsymbol{x}^0) \\ \vdots \\ f_m(\boldsymbol{x}^0) \end{pmatrix} \triangleq \begin{pmatrix} \mathrm{d}f_1(\boldsymbol{x}^0) \\ \mathrm{d}f_2(\boldsymbol{x}^0) \\ \vdots \\ \mathrm{d}f_m(\boldsymbol{x}^0) \end{pmatrix}$$

向量值函数的梯度

向量值函数的Jacobi

$$\nabla \boldsymbol{f}(\boldsymbol{x}) = \nabla \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \nabla f_1(\boldsymbol{x}) \\ \nabla f_2(\boldsymbol{x}) \\ \vdots \\ \nabla f_m(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = J\boldsymbol{f}(\boldsymbol{x})$$

二元函数$f(x_1,x_2)$的二阶泰勒展开

$$f(x_1,x_2)=f(x_1^0,x_2^0)+\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}\right)\begin{pmatrix}x_1-x_1^0\\x_2-x_2^0\end{pmatrix}$$

$$+\frac{1}{2!}(x_1-x_1^0,x_2-x_2^0)\begin{pmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2}\\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2}\end{pmatrix}\begin{pmatrix}x_1-x_1^0\\x_2-x_2^0\end{pmatrix}+o(\rho^2)$$