Set Theory

Functions

Shengfa Wang
sfwang@dlut.edu.cn

Content

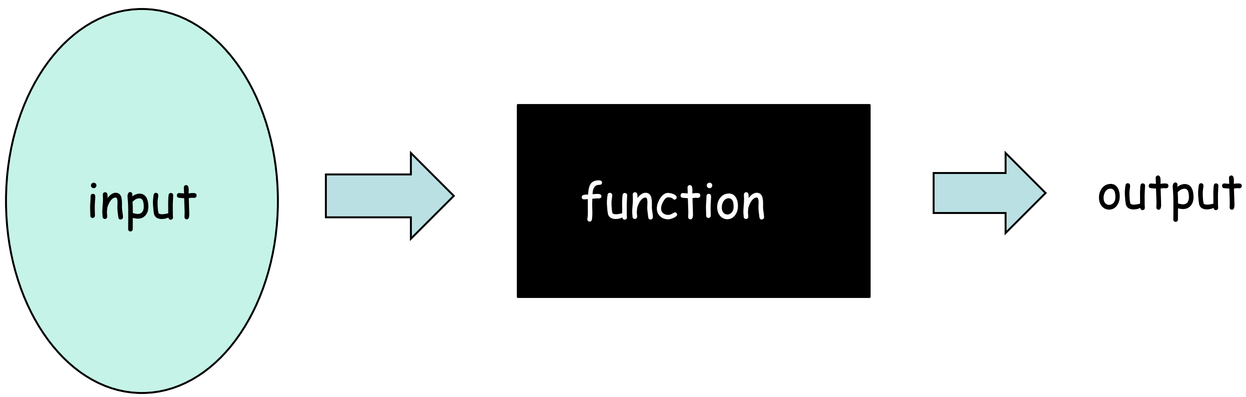
- Concepts
- Properties of functions

构成

- Composition of functions

Functions

Informally, we are given an "input set",
and a function gives us an output for each possible input.



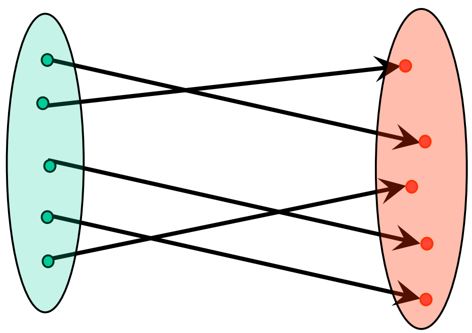
The important point is that there is only one output for each input.

We say a function f "maps" the element of an input set A
to the elements of an output set B.

Functions

More formally, we write $f : A \rightarrow B$

to represent that f is a function from set A to set B, which associates an element $f(a) \in B$ with an element $a \in A$.



The *domain (input)* of f is A.

上域/对应域

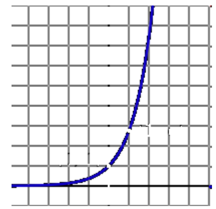
The *codomain (output)* of f is B.

Definition: For every input there is exactly one output.

Note: the input set can be the same as the output set, e.g. both are integers.

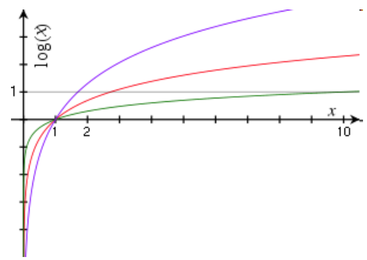
Examples of Functions

$$f(x) = e^x$$



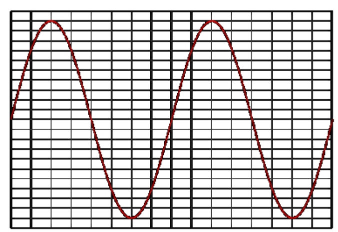
domain = R
codomain = R>0

$$f(x) = \log(x)$$



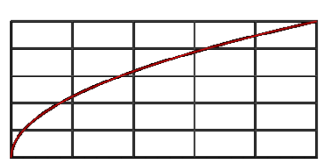
domain = R>0
codomain = R

$$f(x) = \sin(x)$$



domain = R
codomain = [-1,1]

$$f(x) = \sqrt{x}$$



domain = R>=0
codomain = R>=0

Examples of Functions

$f(S) = |S|$

所有有限集的集合

domain = the set of all finite sets

codomain = non-negative integers

$f(\text{string}) = \text{length}(\text{string})$

domain = the set of all finite strings

codomain = non-negative integers

?

$f(\text{student-name}) = \text{student-ID}$

not a function,
since one input could have
more than one output

是素数

$f(x) = \text{is-prime}(x)$

domain = positive integers

codomain = {T,F}

Functions

- A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.
- We write f(a) = b if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write f: A→B
- (note: Here, "→" has nothing to do with if… then)

Functions

上域

- If $f{:}A \to B$, we say that A is the domain of f and B is the codomain of f.

如果f（a）＝b，我们说b是a的像，而a是b的原像。

- If $f(a) = b$, we say that b is the image of a and a is the pre-image of b.

值域

- The range of $f{:}A \to B$ is the set of all images of elements of A.

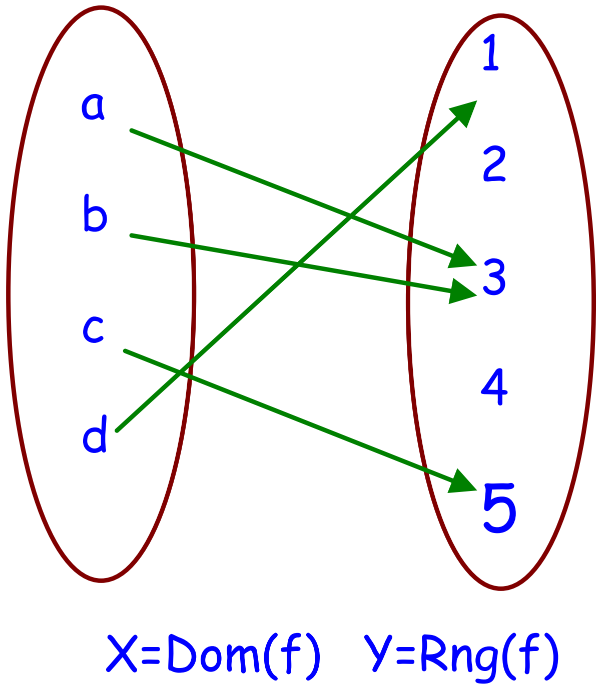
我们说f：A→B将A映射到B。

- We say that $f{:}A \to B$ maps A to B.

codomain $\neq$ range

Functions

- A function f from X to Y (in symbols $f : X \to Y$) is a relation from X to Y such that Dom(f) = X and if two pairs (x,y) and (x,y') $\in$ f, then y = y'
- **Example:**
  - Dom(f) = X = {a, b, c, d},
  - Rng(f) = {1, 3, 5}
  - f(a) = f(b) = 3, f(c) = 5, f(d) = 1.



Functions

- **Example**: Let us take a look at the function $f:P \rightarrow C$ with
  - P = {Linda, Max, Kathy, Peter}
  - C = {Boston, New York, Hong Kong, Moscow}
  - f(Linda) = Moscow
  - f(Max) = Boston
  - f(Kathy) = Hong Kong
  - f(Peter) = New York
  - Here, the range of f is C.

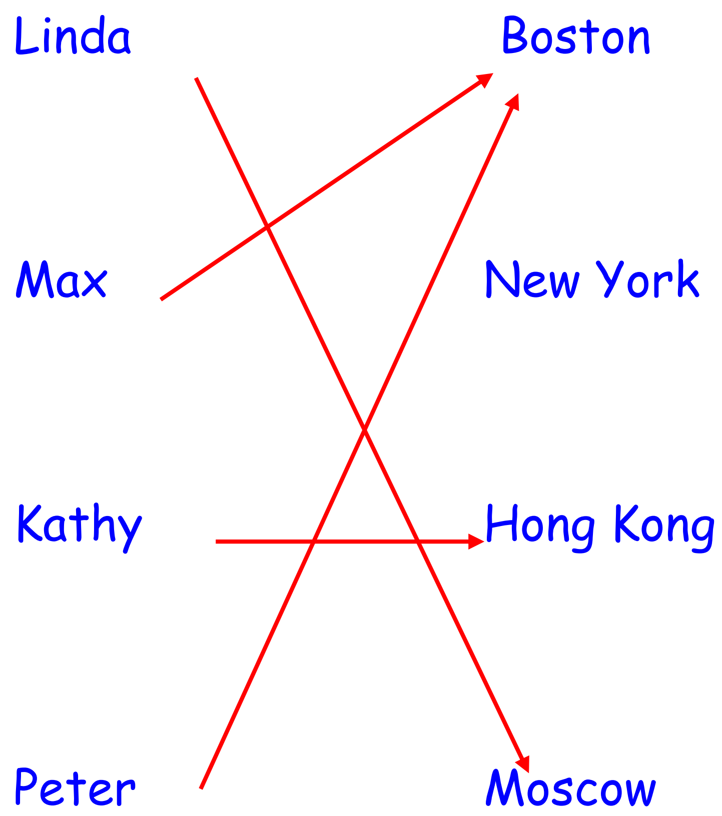
Functions

- Let us re-specify f as follows:
  - f(Linda) = Moscow
  - f(Max) = Boston
  - f(Kathy) = Hong Kong
  - f(Peter) = Boston
  - Is f still a function?
    - yes
  - What is its range?
    - {Moscow, Boston, Hong Kong}

Functions

- Other ways to represent f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



Functions

- If the domain of our function f is large, it is convenient to specify f with a formula, e.g.:

  公式

  $f: \mathbf{R} \to \mathbf{R}$

  $f(x) = 2x$

- This leads to:

  $f(1) = 2$

  $f(3) = 6$

  $f(-3) = -6$

Functions

- Let f_1 and f_2 be functions from A to **R**. Then the sum and the product of f_1 and f_2 are also functions from A to **R** defined by:

  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ 和复合函数不同

  $(f_1f_2)(x) = f_1(x)\, f_2(x)$

- **Example:**

  $f_1(x) = 3x,\ f_2(x) = x + 5$

  $(f_1 + f_2)(x)=f_1(x)+f_2(x)=3x + x + 5 = 4x + 5$

  $(f_1f_2)(x)=f_1(x)\, f_2(x)=3x(x + 5) = 3x^2 + 15x$

Functions

- We already know that the range of a function $f{:}A \rightarrow B$ is the set of all images of elements $a \in A$.

子集

- If we only regard a subset $S \subseteq A$, the set of all images of elements $s \in S$ is called the image of S.
- We denote the image of S by $f(S)$:
- $f(S) = \{f(s) \mid s \in S\}$

Functions

- Let us look at the following well-known function:

  f(Linda) = Moscow
  f(Max) = Boston
  f(Kathy) = Hong Kong
  f(Peter) = Boston

- What is the image of S = {Linda, Max} ?

  f(S) = {Moscow, Boston}

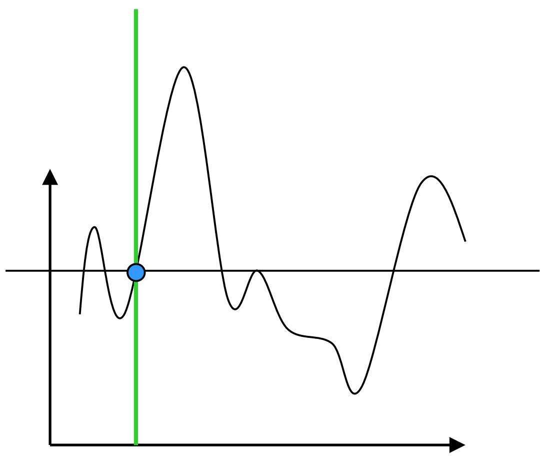
- What is the image of S = {Max, Peter} ?

  f(S) = {Boston}

Algebraically speaking

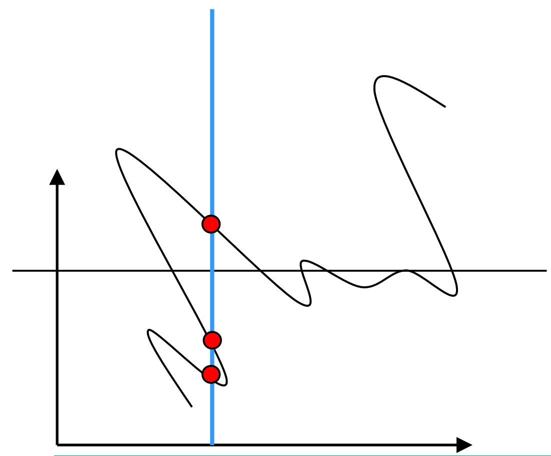
与……一致

- Note that such definitions on functions are consistent with what you have seen in your Calculus courses.微积分



function

→ 1 intersection



not a function

→ violations when > 1

性质

Properties of Functions

- A function f:A→B is said to be one-to-one (or **injective（单射）**), if and only if $\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$

$\forall x,y \in B\ (x \neq y \rightarrow f(x) \neq f(y))$ X　B 是 codomain 不是 range（值域），可能 B 中的元素找不到 A 中对应的像

- In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

injective 单射
surjective 满射
bijective 双射

Properties of Functions

And again…

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

Is f one-to-one?

g(Linda) = Moscow

g(Max) = Boston

g(Kathy) = Hong Kong

g(Peter) = New York

Is g one-to-one?

No, Max and Peter are mapped onto the same element of the image.

Yes, each element is assigned a unique element of the image.

Properties of Functions

- How can we prove that a function f is one-to-one?
  - Whenever you want to prove something, first take a look at the relevant definition(s):

    $\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$

- Example:

  $f: \mathbf{R} \rightarrow \mathbf{R}$

  $f(x) = x^2$

- Disproof by counterexample:

$f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

Properties of Functions

- … and yet another example:

$f: \mathbf{R} \to \mathbf{R}$

$f(x) = 3x$

One-to-one: $\forall x, y \in A\ (f(x) = f(y) \to x = y)$

To show: $f(x) \neq f(y)$ whenever $x \neq y$

$x \neq y$

$3x \neq 3y$

$f(x) \neq f(y)$,

so if $x \neq y$, then $f(x) \neq f(y)$, that is, f is one-to-one.

Properties of Functions

严格单增

- A function $f:A \to B$ with $A,B \subseteq R$ is called strictly increasing, if

  $$\forall x,y \in A\ (x < y \to f(x) < f(y)),$$

严格单减

and strictly decreasing, if

  $$\forall x,y \in A\ (x < y \to f(x) > f(y)).$$

- Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one 单射

Properties of Functions

range = codomain

- A function $f: A \rightarrow B$ is called onto, or **surjective（满射）**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
  - In other words, f is onto if and only if its range is its entire codomain.

- A function $f: A \rightarrow B$ is a one-to-one correspondence, or a **bijection（双射）**, if and only if it is both one-to-one and onto.
  - Obviously, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

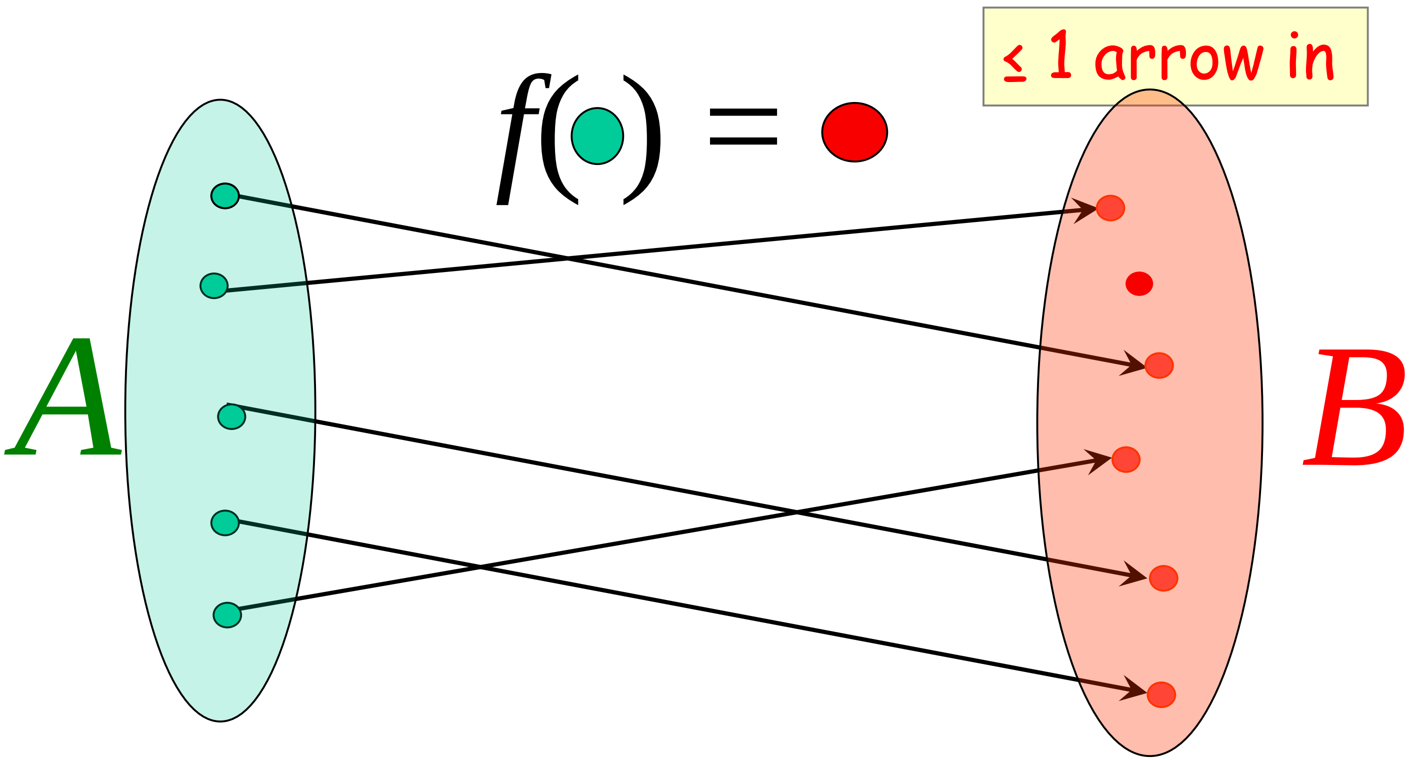
显然，如果f是双射，而A和B是有限集，则| A | = | B |。

双射：同时为单射和满射

单射
Injections

两个输入是同样的输出

$f : A \rightarrow B$ is an *injection* if no two inputs have the same output.

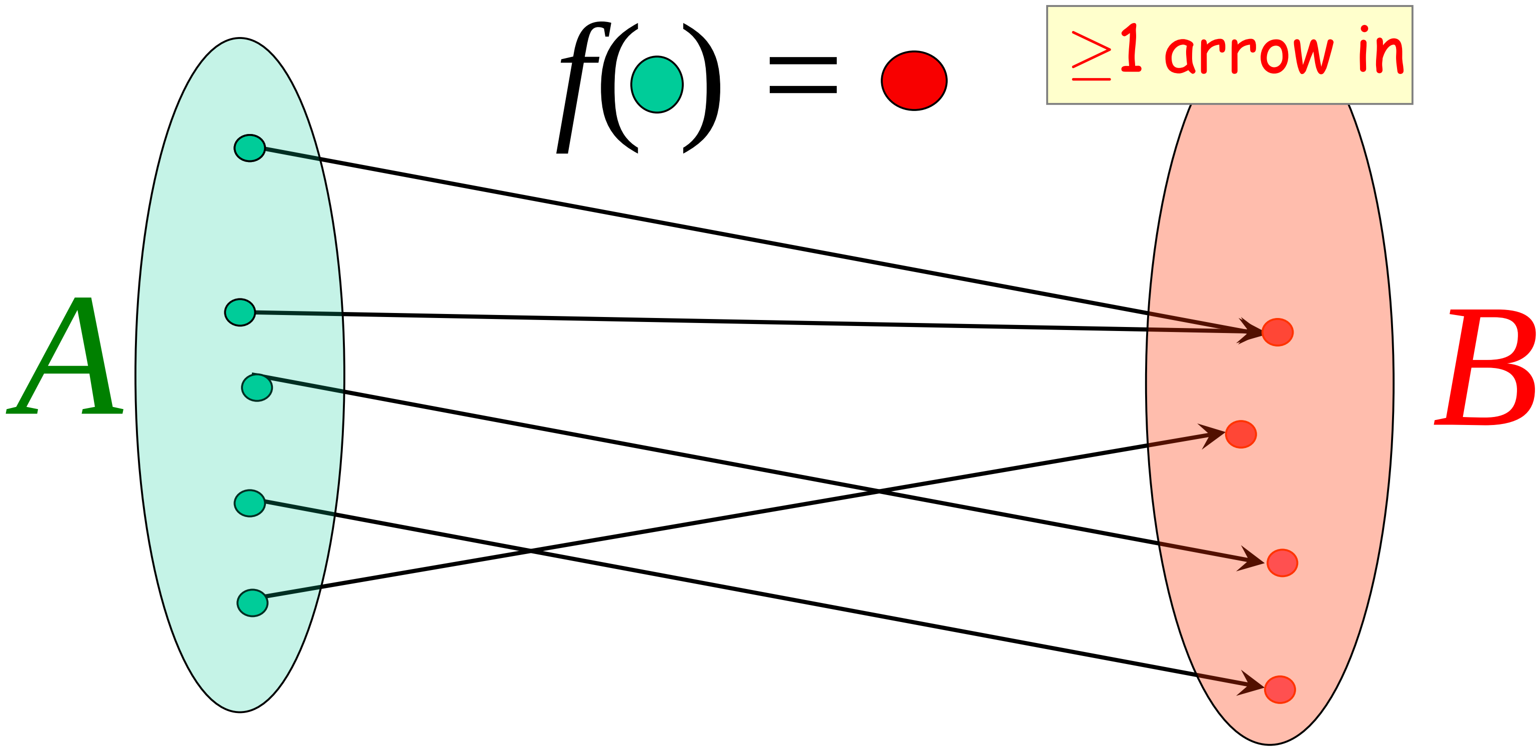


$$\boxed{|A| \le |B|}$$

满射

Surjections

$f : A \to B$ is a *surjection* if every output is possible.



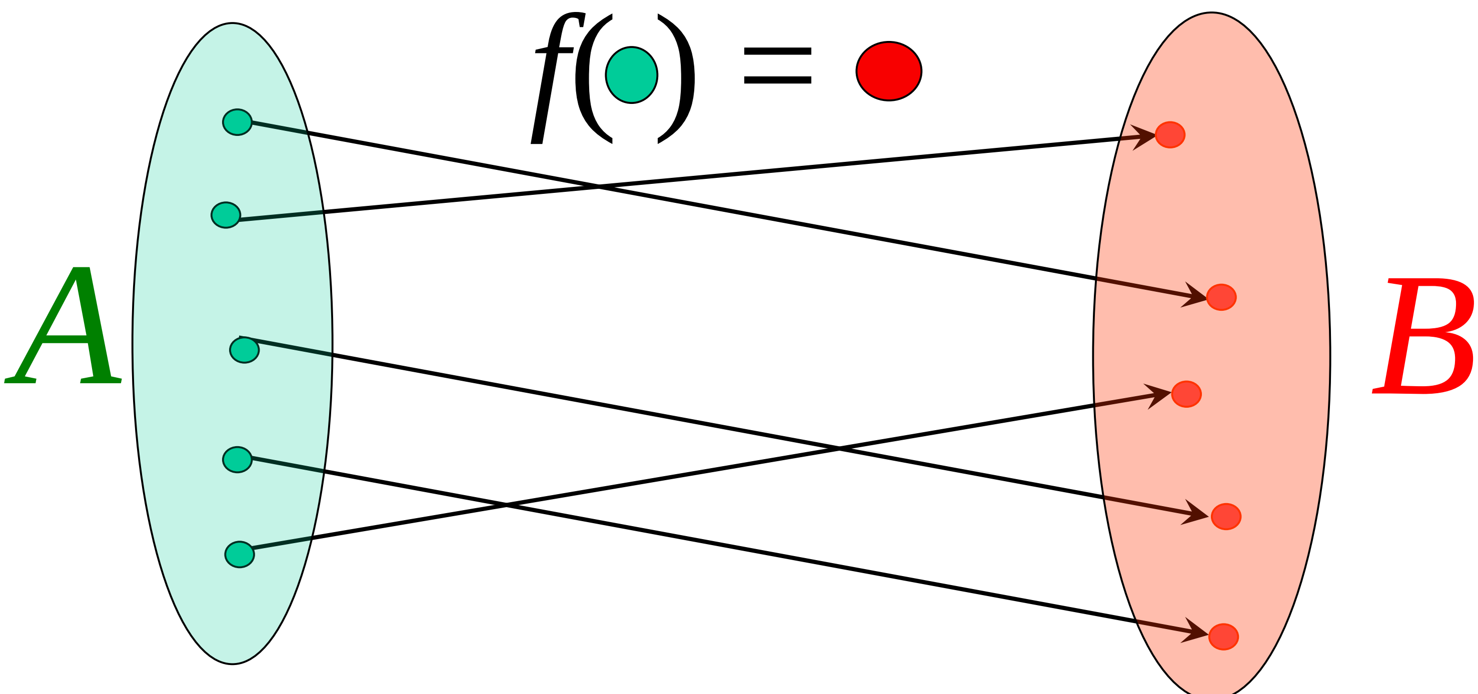
$$\boxed{|A| \geq |B|}$$

双射

Bijections

$f : A \to B$ is a *bijection* if it is surjection and injection.

exactly one arrow in

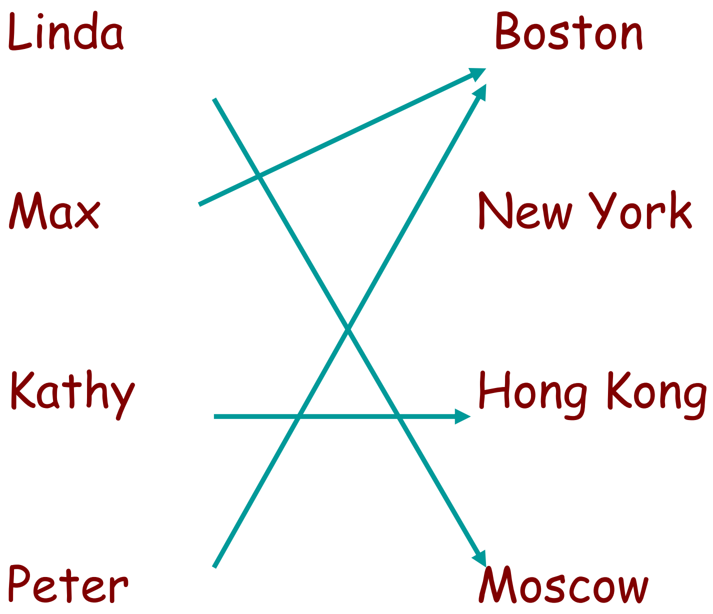


$\boxed{|A| = |B|}$

Properties of Functions

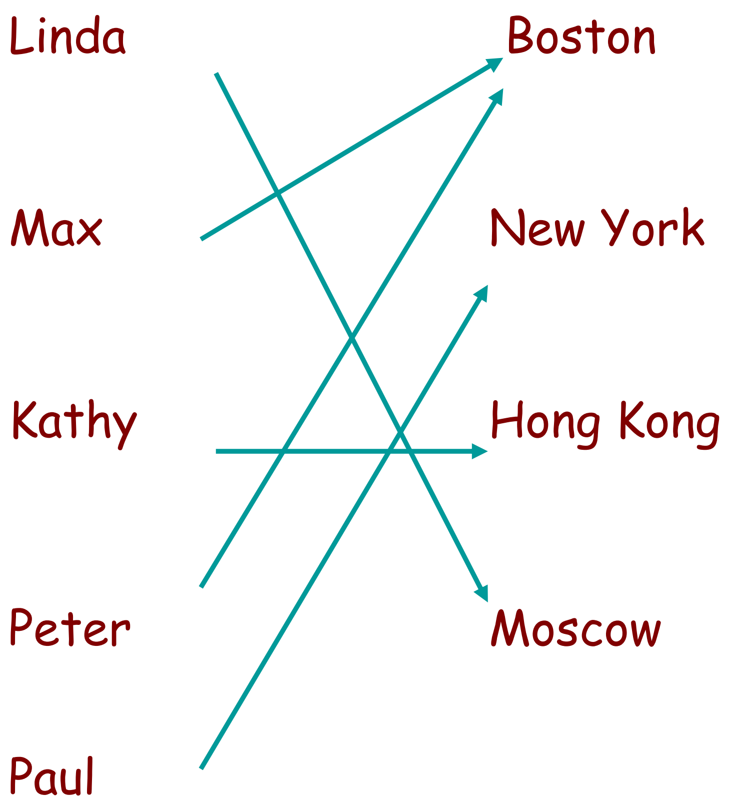
- **Examples:**
  - In the following examples, we use the arrow representation to illustrate functions $f{:}A \rightarrow B$.
  - In each example, the complete sets A and B are shown.

Properties of Functions



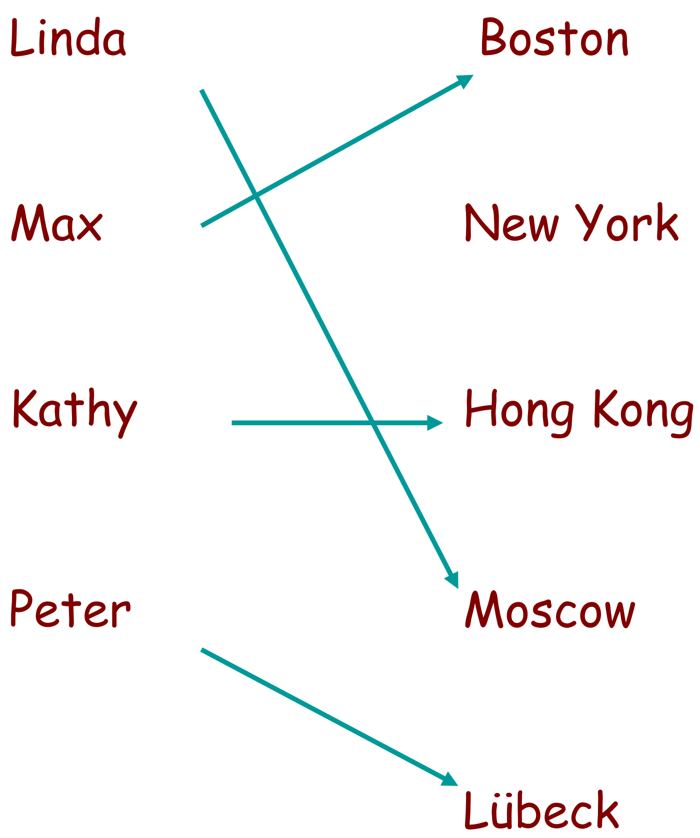
Is f injective?
No.
Is f surjective?
No.
Is f bijective?
No.

Properties of Functions



Is f injective?
No.
Is f surjective?
Yes.
Is f bijective?
No.

Properties of Functions



Is f injective?

Yes.

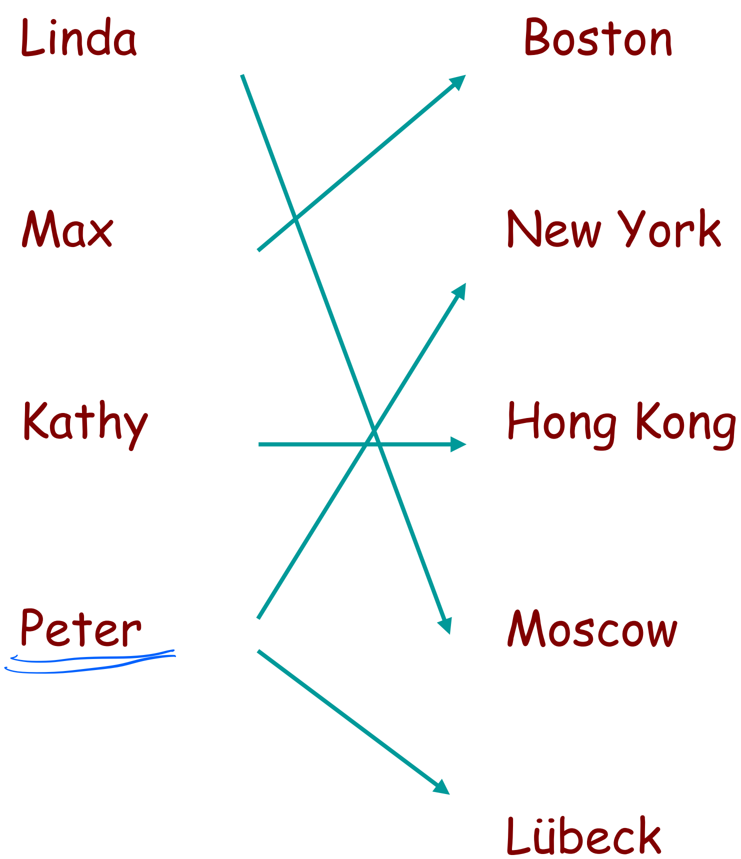
Is f surjective?

No.

Is f bijective?

No.

Properties of Functions

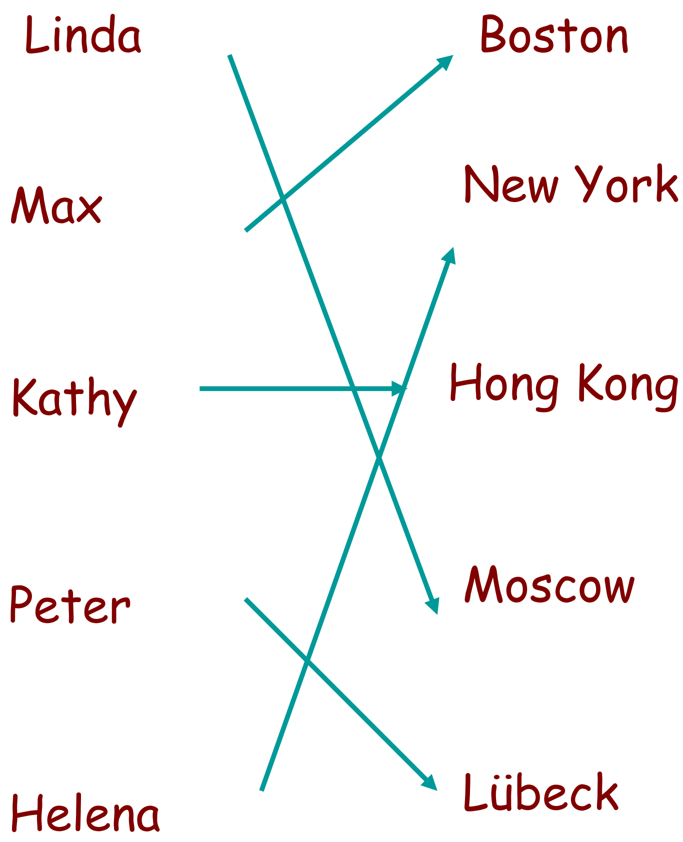


不是函数

Is f injective?

No! f is not even a function!

Properties of Functions



Is f injective?
Yes.
Is f surjective?
Yes.
Is f bijective?
Yes.

Operators on Functions

- Inversion
- Composition

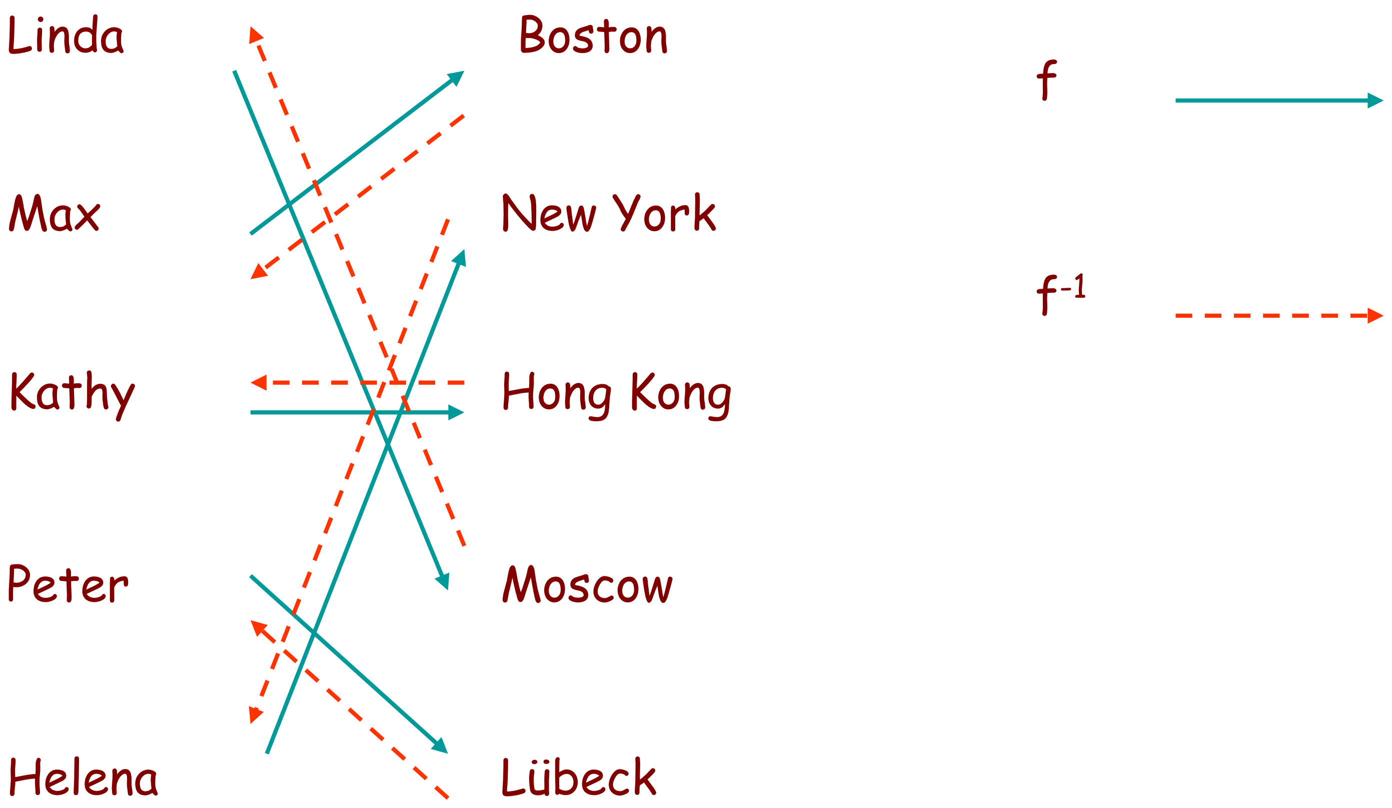
Inversion

反函数

- An interesting property of bijections is that they have an inverse function.

- The inverse function of the bijection $f:A \rightarrow B$ is the function $f^{-1}:B \rightarrow A$ with $f^{-1}(b) = a$ whenever $f(a) = b$.

Inversion



Linda
Max
Kathy
Peter
Helena
Boston
New York
Hong Kong
Moscow
Lübeck
f
f^{-1}

Inversion

Example:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Lübeck

f(Helena) = New York

Clearly, f is bijective.

The inverse function f^{-1} is given by:

f^{-1}(Moscow) = Linda

f^{-1}(Boston) = Max

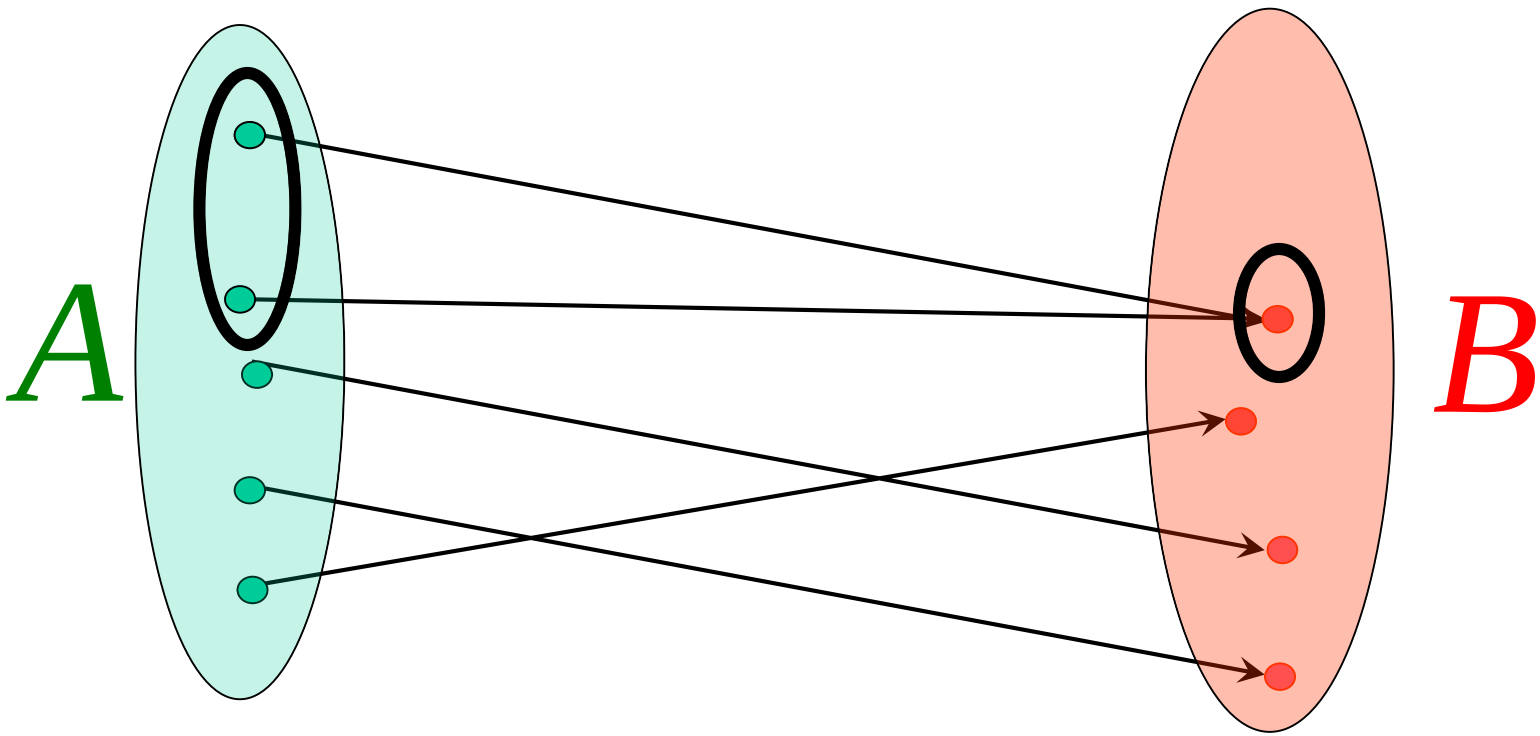
f^{-1}(Hong Kong) = Kathy

f^{-1}(Lübeck) = Peter

f^{-1}(New York) = Helena

Inversion is only possible for bijections
(= invertible functions)

Inverse Sets



Given an element y in B, the inverse set of y := $f^{-1}(y) = \{x \text{ in } A \mid f(x) = y\}$.

In words, this is the set of inputs that are mapped to y.

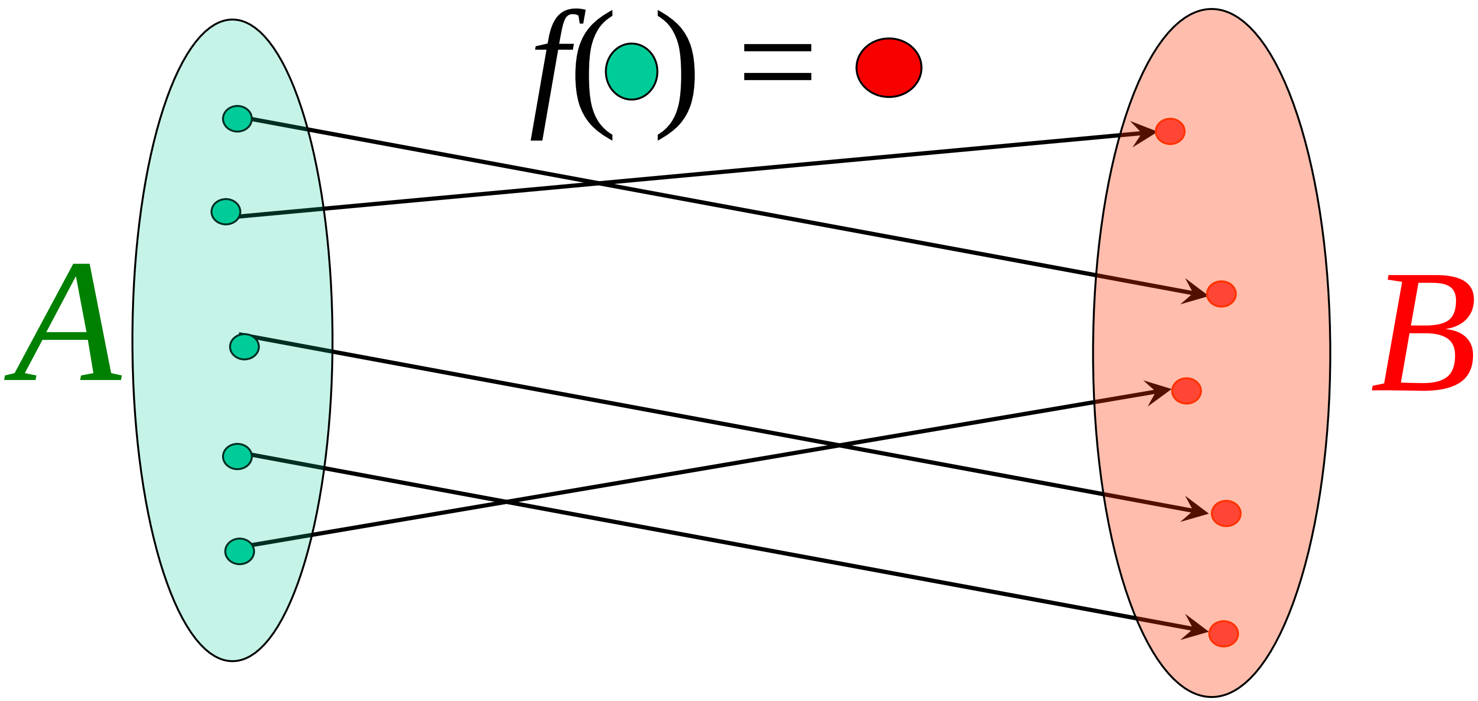
More generally, for a subset Y of B,

the inverse set of Y := $f^{-1}(Y) = \{x \text{ in } A \mid f(x) \text{ in } Y\}$.

Inverse Function

Informally, an inverse function f^{-1} is to "undo" the operation of function f.

exactly one arrow in



There is an inverse function f^{-1} for f if and only if f is a bijection.

Composition

复合函数 注意顺序:

- The composition of two functions $g:A \rightarrow B$ and $f:B \rightarrow C$, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$
- This means that
  - first, function g is applied to element $a \in A$, mapping it onto an element of B,
  - then, function f is applied to this element of B, mapping it onto an element of C.
  - Therefore, the composite function maps from A to C.

Composition

- Example:

  $f(x) = 7x - 4$, $g(x) = 3x$,

  $f:\mathbf{R}\to\mathbf{R}$, $g:\mathbf{R}\to\mathbf{R}$

  $(f\circ g)(5) = f(g(5)) = f(15) = 105 - 4 = 101$

  $(f\circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Composition

复合函数及其反函数

- Composition of a function and its inverse:
  - $(f^{-1} \circ f)(x) = \boxed{f^{-1}(f(x)) = x}$

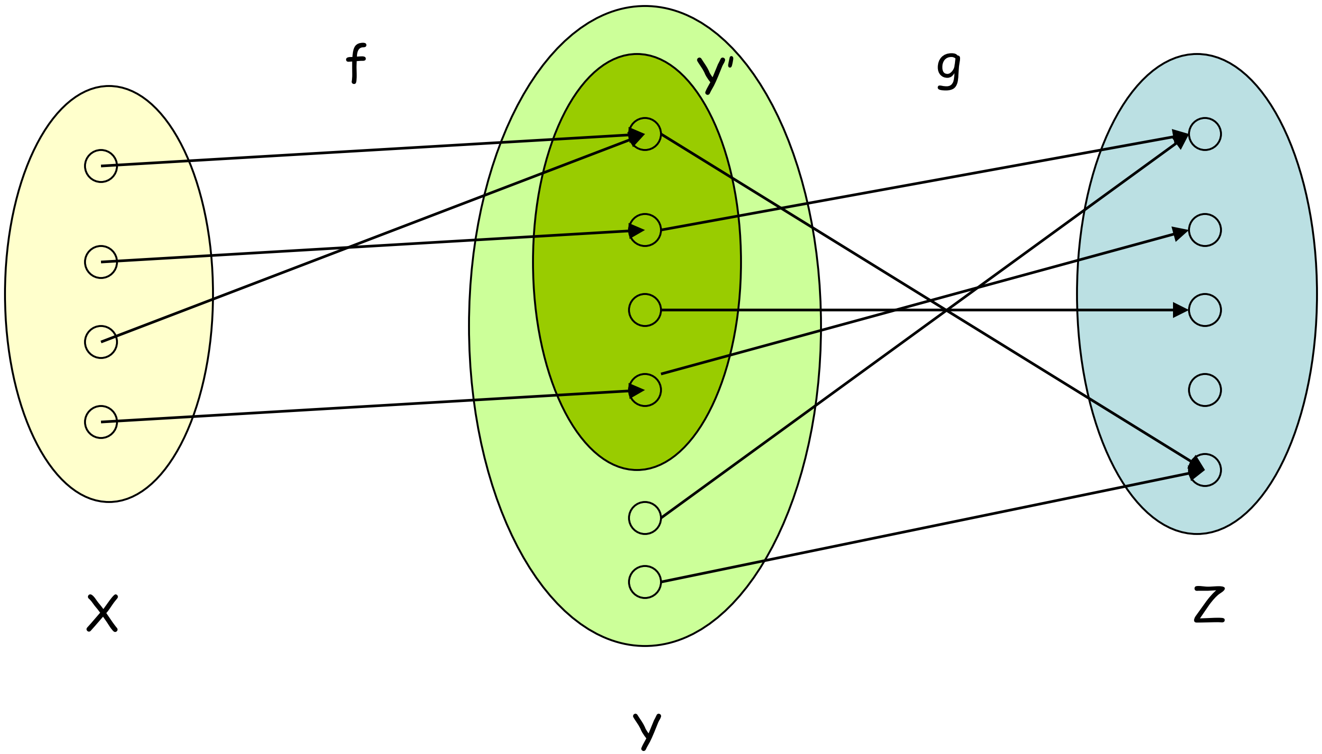
恒等函数

- The composition of a function and its inverse is the identity function $i(x) = x$.

Composition of Functions

Two functions f:X->Y', g:Y->Z so that Y' is a subset of Y, then the composition of f and g is the function $g \circ f$: X->Z, where

$$g \circ f(x) = g(f(x)).$$



Review: One-to-one functions

- A function $f : X \rightarrow Y$ is one-to-one$\Leftrightarrow$for each $y \in Y$ there exists at most one $x \in X$ with $f(x) = y$. (therefore, $f(x) = c$ is out of play)
- Alternative definition: $f : X \rightarrow Y$ is one-to-one $\Leftrightarrow$ for each pair of distinct elements $x_1, x_2 \in X$ there exist two distinct elements $y_1, y_2 \in Y$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Review: One-to-one functions

- **Examples:**
  1. The function $f(x) = 2^x$ from the set of real numbers to itself is one-to-one.
  2. The function $f : R \rightarrow R$ defined by $f(x) = x^2$ is <u>not</u> one-to-one, since for every real number x, $f(x)=f(-x)$.

Review: Onto functions

- A function $f : X \to Y$ is onto (or, subjective) $\Leftrightarrow$ for each $y \in Y$ there exists at least one $x \in X$ with $f(x) = y$, i.e. $Rng(f) = Y$.
  - **Example:** The function $f=\{(1,a),(2,c),(3,b)\}$ from $X=\{1,2,3\}$ to $Y=\{a,b,c\}$ is 1-to-1 and onto. If $Y=\{a,b,c,d\}$, then still 1-to-1, but not onto.

Review: Onto functions

- **Example:** The function $f(x) = e^x$ from the set of real numbers to itself is not onto Y(= the set of all real numbers). However, if Y is restricted to Rng(f) = R^+, the set of positive real numbers, then f(x) is onto. Why?
- You need to look at the visual examples

Review: Bijective functions

- A function $f : X \to Y$ is bijective $\Leftrightarrow$ f is one-to-one and onto.
  - **Examples:**
    1. Is A linear function $f(x) = ax + b$ a bijective function from the set of real numbers to itself. Why?

    2. Is the function $f(x) = x^3$ a bijective from the set of real numbers to itself. Why?

Review: Inverse function

- Given a function $y = f(x)$, the inverse f^{-1} is the set $\{(y, x) \mid y = f(x)\}$.
- The inverse f^{-1} of f is not necessarily a function.
  - **Example:** if $f(x) = x^2$, then $f^{-1}(4) = \sqrt{4} = \pm 2$, not a unique value and therefore f^{-1} is not a function.
- However, if f is a bijective function, it can be shown that f^{-1} is a function.

Review: Composition of functions

- Given two functions $g : X \to Y$ and $f : Y \to Z$, the composition $f \circ g$ is defined as follows:

  $f \circ g\ (x) = f(g(x))$ for every $x \in X$.

  - **Example**: $g(x) = x^2 - 1$, $f(x) = 3x + 5$. Then

    $f \circ g(x) = f(g(x)) = f(3x + 5) = (3x + 5)^2 - 1$

- Composition of functions is associative:

  $f \circ (g \circ h) = (f \circ g) \circ h$,

- But, in general, it is not commutative:

  $\boxed{f \circ g \neq g \circ f.}$

Summary

- f,g injective, f∘g injective
- f,g surjective, f∘g surjective
- f, g bijective, f∘g bijective

指数和对数函数

Exponential and logarithmic functions

- Let $f(x) = 2^x$ and $g(x) = \log_2 x = \lg x$
  - $f \circ g(x) = f(g(x)) = f(\lg x) = 2^{\lg x} = x$
  - $g \circ f(x) = g(f(x)) = g(2^x) = \lg 2^x = x$
- Therefore, the exponential and logarithmic functions are inverses of each other.

Unary operators (一元运算)

- A unary operator on a set X associates each single element of X to one element of X.
  - **Examples**:
    1. Let $X = U$ be a universal set and $\mathcal{P}(U)$ the power set of U. Define $f : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ the function defined by $f(A) = A'$, the set complement of A in U, for every $A \subseteq U$. Then f defines a unary operator on $\mathcal{P}(U)$. (The operator here is the "complement" itself).

Binary operators (二元运算)

- A binary operator on a set X is a function f that associates a single element of X to every pair of elements in X, i.e. $f : X \times X \rightarrow X$ and $f(x_1, x_2) \in X$ for every pair of elements x_1, x_2.
  - Examples of binary operators are addition, subtraction and multiplication of real numbers, taking unions or intersections of sets, concatenation of two strings over a set X, etc.

Modulus operator (模运算)

- Let x be a nonnegative integer and y a positive integer
r = x mod y是x除以y的余数
- r = x mod y is the remainder when x is divided by y
  - Examples:
    1 = 13 mod 3
    6 = 234 mod 19
    4 = 2002 mod 111
  - Basically, remove the complete y's and count what's left
  - mod is called the modulus operator 模除

Exercises

Function	Domain	Codomain	Injective?	Surjective?	Bijective?
$f(x)=\sin(x)$	Real	Real	✗	✓	✗
$f(x)=2^x$	Real	Positive real	✓	✓	✓
$f(x)=x^2$	Real	Non-negative real	✓	✓	✓
Reverse string	Bit strings of length n	Bit strings of length n			

Whether a function is injective, surjective, bijective depends on its domain (i.e. input) and the codomain (i.e. output).

Exercises

$X \to Z$

Function f	Function g	$g \circ f$ injective?	$g \circ f$ surjective?	$g \circ f$ bijective?
$f:X \to Y$ f surjective 满	$g:Y \to Z$ g injective 单	✗	✗	✗
$f:X \to Y$ f surjective	$g:Y \to Z$ g surjective	✗	✓	✗
$f:X \to Y$ f injective	$g:Y \to Z$ g surjective	✗	✗	✗
$f:X \to Y$ f bijective	$g:Y \to Z$ g bijective	✓	✓	✓
$f:X \to Y$	$f^{-1}:Y \to X$	✓ $f(f^{-1})$ $= X$	✓	✓

The End