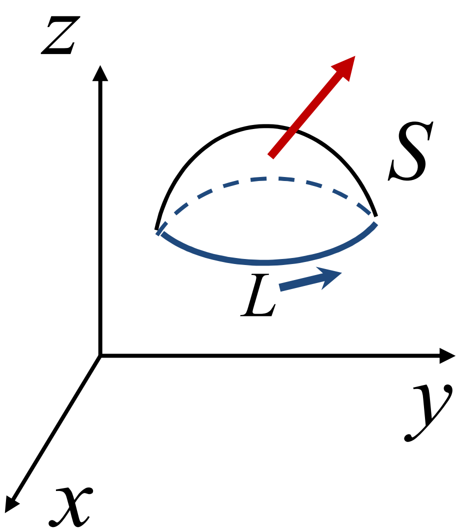
9.4.2 Stokes 公式

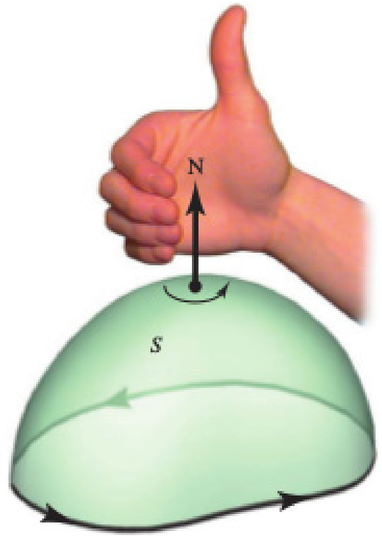
定理. 设光滑曲面 S 的边界 L 是分段光滑曲线, S 的侧与 L 的方向满足**右手定则**, P,Q,R 在包含 S 在内的一个空间区域内具有一阶连续偏导数, 则有

Stokes 公式

$$\iint_S \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$=\oint_L P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$



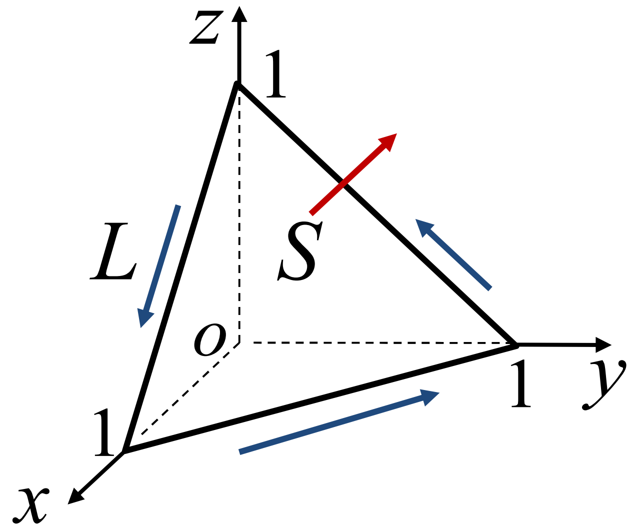


为便于记忆, Stokes 公式也可写作:

$$\iint_S \begin{vmatrix} \mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \oint_L P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

$$\boxed{\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y}$$

例. 计算积分$\oint_L z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z$, 其中$L$为平面$x+y+z=1$被三个坐标面所截三角形的整个边界, 方向如图所示.



解: 记三角形区域为S, 取上侧, 则

$$\oint_L z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z$$

$$= \iint_S \begin{vmatrix} \mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \iint_S \mathrm{d}y\,\mathrm{d}z + \mathrm{d}z\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}y = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

9.4.3 空间曲线积分与路径无关的条件

定理. 设 G 是空间一维单连通区域, 函数 P, Q, R 在G内具有一阶连续偏导数, 则下列四个命题相互等价:

(1) 对G内任一分段光滑闭曲线 L, 有

$$\oint_L P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = 0$$

(2) 对G内任一分段光滑曲线 L, $\int_L P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ 与路径无关

(3) 在G内存在某一函数 u, 使 $\mathrm{d}u = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$

(4) 在G内处处有

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

内容小结

1. Gauss 公式

$$\oiint_S P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}z\,\mathrm{d}x + R\,\mathrm{d}x\,\mathrm{d}y = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}V$$

计算第二型曲面积分, 非闭曲面时注意添加**辅助面**.

2. Stokes 公式

$$\oint_L P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

应用: (1) 计算空间中第二型曲线积分

(2) 推出空间曲线积分与路径无关的条件:

$\int_L P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ 在 G 内与路径无关

$\rightleftarrows$ 在 G 内处处有

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$