1. $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$都减第一行可化成箭形行列式

2. $$\begin{vmatrix} a^2+\frac{1}{a^2} & a & \frac{1}{a} & 1 \\ b^2+\frac{1}{b^2} & b & \frac{1}{b} & 1 \\ c^2+\frac{1}{c^2} & c & \frac{1}{c} & 1 \\ d^2+\frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix} = \begin{vmatrix} a^2 & a & \frac{1}{a} & 1 \\ b^2 & b & \frac{1}{b} & 1 \\ c^2 & c & \frac{1}{c} & 1 \\ d^2 & d & \frac{1}{d} & 1 \end{vmatrix} + \begin{vmatrix} \frac{1}{a^2} & a & \frac{1}{a} & 1 \\ \frac{1}{b^2} & b & \frac{1}{b} & 1 \\ \frac{1}{c^2} & c & \frac{1}{c} & 1 \\ \frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$$

$$\overset{\text{往外提公因式}}{=} \frac{1}{abcd}\begin{vmatrix} a^3 & a^2 & 1 & a \\ b^3 & b^2 & 1 & b \\ c^3 & c^2 & 1 & c \\ d^3 & d^2 & 1 & d \end{vmatrix} + \frac{1}{(abcd)^2}\begin{vmatrix} 1 & a^3 & a & a^2 \\ 1 & b^3 & b & b^2 \\ 1 & c^3 & c & c^2 \\ 1 & d^3 & d & d^2 \end{vmatrix}$$

3. $\boldsymbol{B} = (a_1, a_2, a_3, \cdots, a_{n-1}, a_n)\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 2 \\ 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 2 & 1 \end{pmatrix}$

4. $$\begin{vmatrix} a & b & b & \cdots & b & b \\ 0 & a & b & \cdots & b & b \\ 0 & 0 & a & \cdots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ b & 0 & 0 & \cdots & 0 & a \end{vmatrix} \overset{\text{按第1列展开}}{=} a^n + b\cdot(-1)^{n+1} \begin{vmatrix} b & b & b & \cdots & b & b \\ a & b & b & \cdots & b & b \\ 0 & a & b & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & b \\ 0 & 0 & 0 & \cdots & a & b \end{vmatrix}$$

$$5.\ D_n=\begin{vmatrix}k&1&1&\cdots&1\\2&k&1&\cdots&1\\2&2&k&\cdots&1\\\vdots&\vdots&\vdots&&\vdots\\2&2&2&\cdots&k\end{vmatrix}=\begin{vmatrix}1+(k-1)&1+0&1+0&\cdots&1+0\\2&k&1&\cdots&1\\2&2&k&\cdots&1\\\vdots&\vdots&\vdots&&\vdots\\2&2&2&\cdots&k\end{vmatrix}$$

$$=\begin{vmatrix}1&1&1&\cdots&1\\2&k&1&\cdots&1\\2&2&k&\cdots&1\\\vdots&\vdots&\vdots&&\vdots\\2&2&2&\cdots&k\end{vmatrix}+\begin{vmatrix}k-1&0&0&\cdots&0\\2&k&1&\cdots&1\\2&2&k&\cdots&1\\\vdots&\vdots&\vdots&&\vdots\\2&2&2&\cdots&k\end{vmatrix}$$

$$=\begin{vmatrix}1&1&1&\cdots&1\\0&k-2&-1&\cdots&-1\\0&0&k-2&\cdots&-1\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&k-2\end{vmatrix}+(k-1)D_{n-1}$$

$$=(k-2)^{n-1}+(k-1)D_{n-1}$$

$$D_n\overset{\text{转置}}{=}\begin{vmatrix}k&2&2&\cdots&2\\1&k&2&\cdots&2\\1&1&k&\cdots&2\\\vdots&\vdots&\vdots&&\vdots\\1&1&1&\cdots&k\end{vmatrix}=\begin{vmatrix}2+(k-2)&2+0&2+0&\cdots&2+0\\1&k&2&\cdots&2\\1&1&k&\cdots&2\\\vdots&\vdots&\vdots&&\vdots\\1&1&1&\cdots&k\end{vmatrix}$$

$$=\begin{vmatrix}2&2&2&\cdots&2\\1&k&2&\cdots&2\\1&1&k&\cdots&2\\\vdots&\vdots&\vdots&&\vdots\\1&1&1&\cdots&k\end{vmatrix}+\begin{vmatrix}k-2&0&0&\cdots&0\\1&k&2&\cdots&2\\1&1&k&\cdots&2\\\vdots&\vdots&\vdots&&\vdots\\1&1&1&\cdots&k\end{vmatrix}$$

$$=\begin{vmatrix}2&2&2&\cdots&2\\0&k-1&1&\cdots&1\\0&0&k-1&\cdots&1\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&k-1\end{vmatrix}+(k-2)D_{n-1}$$

$$=2(k-1)^{n-1}+(k-2)D_{n-1}$$

$$\begin{cases}D_n=(k-2)^{n-1}+(k-1)D_{n-1}\\D_n=2(k-1)^{n-1}+(k-2)D_{n-1}\end{cases}$$

消去 D_{n-1},得 $D_n=2(k-1)^n-(k-2)^n$

6. $\left|\boldsymbol{E}+\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}\right|=\begin{vmatrix} 1+k_1 & k_2 & k_3 & k_4 \\ 2k_1 & 1+2k_2 & 2k_3 & 2k_4 \\ 3k_1 & 3k_2 & 1+3k_3 & 3k_4 \\ 4k_1 & 4k_2 & 4k_3 & 1+4k_4 \end{vmatrix}$

都减第一行的倍数可化成箭形行列式

7.证： 记 $D_n=\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & x+a_1 \end{vmatrix}$

用 数 学 归 纳 法 证 明.

8. $\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_n & x \\ a_1^2 & a_2^2 & \cdots & a_n^2 & x^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} & x^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} & x^{n-1} \\ a_1^n & a_2^n & \cdots & a_n^n & x^n \end{vmatrix}$

$$\overset{\text{按第}n+1\text{列展开}}{=} A_{1,n+1}+xA_{2,n+1}+x^2A_{3,n+1}+\cdots+x^{n-2}A_{n-1,n+1}+x^{n-1}A_{n,n+1}+x^nA_{n+1,n+1}$$

其中$A_{n,n+1}=(-1)^{n+(n+1)}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix}$

可见算出开头那个行列式的值，找到x^{n-1}的项就可求出要求的行列式的值。

9. $$\begin{vmatrix} 2+x_1 & 2+x_1^2 & \cdots & 2+x_1^n \\ 2+x_2 & 2+x_2^2 & \cdots & 2+x_2^n \\ \vdots & \vdots & & \vdots \\ 2+x_n & 2+x_n^2 & \cdots & 2+x_n^n \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2+x_1 & 2+x_1^2 & \cdots & 2+x_1^n \\ 2 & 2+x_2 & 2+x_2^2 & \cdots & 2+x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2+x_n & 2+x_n^2 & \cdots & 2+x_n^n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 2 & x_1 & x_1^2 & \cdots & x_1^n \\ 2 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = -2 \begin{vmatrix} -\frac{1}{2} & 1 & 1 & \cdots & 1 \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1+(-\frac{3}{2}) & 1+0 & 1+0 & \cdots & 1+0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$