CS 476/676 Assignment 1

Winter 2022

Instructor: Yuying Li	Office: DC3623	*yuying@uwaterloo.ca*
Lecture Times:	MW 4-5:20pm.	
Yuying Li:	OH: Tues 3-4pm	.
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Chendi Ni	OH: Jan 26 (Wed) 1:00pm - 2:00pm	
Chendi Ni	OH: Jan 28 (Fri) 3:00pm - 4:00pm	

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Due: 4pm Jan 31, 2022

1. [(12 marks)] (Imply a Binomial Lattice from Option Prices)

Assume that a stock XYZ pays no dividend and is currently priced at $S_0 = \$12$. Assume that, at the expiry time $T > 0$, the stock price goes up to uS_0 with probability $0 < p < 1$ and down to dS_0 with probability $1 - p$. We know that $d < 1 < u$ but do not know d or u. Assume that there is no arbitrage and the interest rate is zero. Consider the following three options with the same expiry T on stock XYZ.

Assume that a European call option with strike price \$12 is priced at \$2 while another European call option with strike price \$13.5 is priced \$1.5.

(a) What is the fair value of a European put option with a strike price of \$15? Explain your answer.

(b) How many units of the underlying is required at $t = 0$ to hedge a short position in the put option specified in (a)? Explain your answer.

(c) Using the actual probability p, what is the expected option payoff for the European put in (a)? What is wrong with pricing this put option at this expected payoff value? If this European put option is priced at the expected payoff using p which is different from the fair value computed in (a), how can you construct an arbitrage?

2. [(8 marks)] (Lattice properties)

Given the binomial lattice over the time interval Δt as specified as below,

$$S(t + \Delta t) = \begin{cases} S(t)u, & \text{with probability } q \\ S(t)d, & \text{with probability } (1-q) \end{cases} \tag{1}$$

where

$$\begin{cases} u = e^{\sigma\sqrt{\Delta t}} \\ d = 1/u \\ q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}. \end{cases} \tag{2}$$

(a) Show that

$$E[S(t + \Delta t)|S(t)] \quad = \quad S(t)e^{r\Delta t} \ , \tag{3}$$

where $E[\cdot]$ is the expectation using (1) and (2).

(b) Show that

$$Var[S(t + \Delta t)|S(t)] \quad = \quad S(t)^2 \left[\sigma^2 \Delta t + O(\Delta t)^2\right] \ , \tag{4}$$

where Var is the variance using (1) and (2).

3. [(5 marks)] (**Lattice Property**)

Consider two European put options with the same expiry T but different strike K and $\bar{K}$, where $K < \bar{K}$. Denote P_t and $\bar{P}_t$ the corresponding fair put values at time t for $0 \le t \le T$.

(a) Assume that there is no arbitrage. Prove by constructing an arbitrage strategy that

$$\bar{P}_t - P_t \le e^{-r(T-t)} \left(\bar{K} - K\right). \tag{5}$$

(b) Let P_j^n and $\bar{P}_j^n$ be the values of European puts given by a no-arbitrage lattice with $T = N\Delta t$ and $d < e^{r\Delta t} < u$. Prove by induction that

$$\bar{P}_j^n - P_j^n \le e^{-r(N-n)\Delta t} \left(\bar{K} - K\right)$$

for $0 \le n \le N$ and $0 \le j \le n$.

4. [(7 marks)] (Bound on Lattice Put Solution).

Consider the no-arbitrage lattice with parameters below

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = e^{-\sigma\sqrt{\Delta t}}, \tag{6}$$

$$q^* = \frac{e^{r\Delta t} - d}{u - d}, \qquad S_{j+1}^{n+1} = uS_j^n, \qquad S_j^{n+1} = dS_j^n \tag{7}$$

Let P_j^n denote the put value from the binomial lattice at the node S_j^n, $j = 0, \ldots, n$.

(a) Show that, for a European put with strike price K,

$$P_0^N \quad \to \quad K \quad \text{as} \quad \Delta t \to 0\ .$$

(b) Show that, if Δt is sufficiently small, and $\sigma > 0$, then the up-move risk neutral probability q^* satisfies $0 \le q^* \le 1$. In addition, using (7), explicitly verify that

$$S_j^n = e^{-r\Delta t}(q^* S_{j+1}^{n+1} + (1 - q^*)S_j^{n+1})\ . \tag{8}$$

(c) Assuming Δt is sufficiently small, using induction, show that the put option value from the binomial lattice $0 \le P_j^n \le K$, $\forall\ 0 \le j \le n, 0 \le n \le N$.

5. [(6 marks)] (European Binomial Option Values)

Consider European option pricing under a binomial lattice. For an 1-period model $N = 1$, we have shown in class that the option value $V_0^0 = e^{-r\Delta t}(q^* V_1^1 + (1 - q^*)V_0^1)$. Consider a N-period binomial model, $N > 0$, and let V_j^n denote option value at time $t = n\Delta t$ and node S_j^n, $0 \le n \le N$, $0 \le j \le n$. Prove by induction on the number of periods N that, for any N-period model,

$$V_0^0 = e^{-rN\Delta t} \sum_{k=0}^{N} \left(\begin{array}{c} N \\ k \end{array}\right) (q^*)^k (1 - q^*)^{N-k} V_k^N$$

where $(q^*)^k$ denotes the kth power of q^* and $\left(\begin{array}{c} N \\ k \end{array}\right)$ denotes the number k-combinations from N. Hint: Firstly, show that the formula holds for $N = 1$. Then , assuming the formula is valid for N periods, show that the formula holds for (N+1) periods. Recall Pascal's rule

$$\left(\begin{array}{c} n \\ k \end{array}\right) + \left(\begin{array}{c} n \\ k+1 \end{array}\right) = \left(\begin{array}{c} n+1 \\ k+1 \end{array}\right)$$

6. [(6 marks)] (Properties of a Standard Brownian Motion)

Let $Z(t)$ be a standard Brownian motion and $Z(0) = 0$. Let $Z(t)^2$ denote the square of $Z(t)$.

What is the stochastic differential equation (SDE) satisfied by $G(Z(t), t) = \frac{1}{2}Z(t)^2 - \frac{t}{2}$? Using the SDE to determine an explicit expression for the integral, in terms of $Z(t)$ and t,

$$\int_0^T Z(t)dZ(t)$$