YING KIT HUI 20933849 Graduate Level

Q1a. To sum up, we get

$$C_0 = 2 = q^*(12u - 12) \quad (1)$$
$$\tilde{C}_0 = 1.5 = q^*(12u - 13.5) \quad (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{2}{1.5} = \frac{12u-12}{12u-13.5}$$

$$\Rightarrow \frac{4}{3}(12u - 13.5) = 12u - 12$$

$$\Rightarrow 4u = \frac{4}{3}13.5 - 12 = 6$$
$$\Rightarrow u = 3/2$$

Thus, by (1), $2 = q^*(12 \cdot \frac{3}{2} - 12)$

$$\Rightarrow q^* = \frac{1}{3}$$

Also $q^* = \frac{1-d}{u-d} = \frac{1-d}{\frac{3}{2}-d} = \frac{1}{3}$

$$\Rightarrow 3(1-d) = \frac{3}{2} - d$$
$$\Rightarrow d = 3/4$$

Then, binomial model for P_t is

P_0 → $\max(15 - 12u, 0) = \max(15 - 12 \cdot \frac{3}{2}, 0) = \max(15-18, 0) = 0$

P_0 → $\max(15 - 12d, 0) = \max(15 - 12 \cdot \frac{3}{4}, 0) = \max(15 - 9, 0) = 6$

By risk-neutral valuation,

$$P_0 = e^{-rT}(q^* \cdot 0 + (1 - q^*) \cdot 6)$$
$$= (1 - q^*) \cdot 6 = \frac{2}{3} \cdot 6 = 4.$$

Hence, the fair value of a European put option with a strike price of \$15 is \$4.

Q1(b) Let Π_t denote our portfolio at time t.
Let η_t and δ_t to denote the amount of bond and stock that we hold at time t.
To hedge a short position in the put option specified in (a) amounts to find η_0, δ_0 s.t. the portfolio

$$\Pi_0 = \underbrace{-P_0}_{\text{short put}} + \underbrace{\eta_0}_{\text{bond}} + \underbrace{\delta_0 S_0}_{\text{stock}}$$

have the value $\Pi_T = 0$ at expiry time T.

So we want $\Pi_T = -P_T + \eta_0 e^{rT} + \delta_0 S_T = 0$

$$\iff \eta_0 e^{rT} + \delta_0 S_T = P_T.$$

That is, we want to replicate the payoff P_T, this amounts to solve the following (based on our binomial model)

$$(*) \quad \begin{bmatrix} 1 & uS_0 \\ 1 & dS_0 \end{bmatrix} \begin{bmatrix} \eta_0 \\ \delta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

noting that $e^{rT} = e^0 = 1$.

$$(*) \Rightarrow \quad \eta_0 + \tfrac{3}{2} \cdot 12 \cdot \delta_0 = 0$$
$$\eta_0 + 3/4 \cdot 12 \cdot \delta_0 = 6$$
$$\Rightarrow \quad 12\left(\tfrac{3}{2} - 3/4\right)\delta_0 = -6$$
$$\delta_0 = -2/3$$
$$\Rightarrow \quad \eta_0 = -\tfrac{3}{2} \cdot 12 \cdot \delta_0 = -\tfrac{3}{2} \cdot 12 \cdot (-2/3) = 12$$

Thus, $\delta_0 = -2/3$ unit of the underlying is required at $t=0$ to hedge. That is, we need to short $2/3$ unit of the stock and long 12 dollars of bond to hedge.

Q1c We use the notation $E^P(\cdot)$ and $E^Q(\cdot)$ for expectation under actual probability p and under risk neutral expectation q^* respectively.

Then, using actual probability p, the expected option payoff for the European put in (a) is

$$\mathbb{E}^P(P_T) = p\cdot 0 + (1-p)\cdot 6 = 6(1-p)$$

Pricing this put option at this expected payoff value will cause arbitrage if $p \neq q^*$.

Now we demonstrate how to construct arbitrage assuming $p \neq q^*$ and suppose P_0 is given by $\mathbb{E}^P(P_T)$

Case 1: $p > q^*$, equivalently $1-p < 1-q^*$.

Then
$$P_0 = \mathbb{E}^P(P_T) = (1-p)6 < (1-q^*)6 = \mathbb{E}^Q(P_T) = 4$$

So, intuitively, this put is underpriced.

Construct portfolio by

$$\Pi_0 = \underbrace{P_0}_{<4} - \underbrace{(\eta_0 + \delta_0 S_0)}_{=4} < 0$$

(note that $\eta_0 + \delta_0 S_0 = 12 + -\frac{2}{3}\cdot 12 = 4.$)

and
$$\Pi_T = P_T - (\eta_0 + \delta_0 S_T)$$
$$= P_T - P_T = 0$$

since η_0, δ_0 are chosen s.t. $\eta_0 + \delta_0 S_T = P_T$ under each scenario in binomial model).

This is an arbitrage since $\Pi_0 < 0$, $\Pi_T = 0$.

Case 2: $p < q^*$, equivalently $1-p > 1-q^*$.

So $P_0 = \mathbb{E}^P(P_T) = 6(1-p) > 6(1-q^*) = 4 = \mathbb{E}^Q(P_T)$

So, intuitively, this put is overpriced.

Construct portfolio by

$$\Pi_0 = \underbrace{-P_0}_{<-4} + \underbrace{(\eta_0 + \delta_0 S_0)}_{=4} < 0$$

Then $\Pi_T = -P_T + (\eta_0 + \delta_0 S_T)$

$= -P_T + P_T = 0$

So $\Pi_0 < 0$, $\Pi_T = 0$, this is an arbitrage.

Hilroy

2a) $E[S(t+\Delta t) | S(t)]$

$$= S(t) u \cdot q + S(t) \cdot d \, (1-q)$$

Note $uq + d(1-q)$

$$= e^{\sigma\sqrt{\Delta t}} \left(\frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \right) + e^{-\sigma\sqrt{\Delta t}} \left(\frac{e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \right)$$

$$= \frac{e^{\sigma\sqrt{\Delta t} + r\Delta t} - e^{-\sigma\sqrt{\Delta t} + r\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$= e^{r\Delta t}$$

Hence $E[S(t+\Delta t) | S(t)]$

$$= S(t) u \cdot q + S(t) \cdot d \, (1-q)$$

$$= S(t) e^{r\Delta t}$$

b) $Var[S(t+\Delta t) | S(t)]$

$$= E[S(t+\Delta t)^2 | S(t)] - E[S(t+\Delta t) | S(t)]^2$$

Note $E[S(t+\Delta t)^2 | S(t)]$

$$= S(t)^2 u^2 \cdot q + S(t)^2 d^2 (1-q)$$

and $u^2 q + d^2 (1-q)$

$$= e^{2\sigma\sqrt{\Delta t}} \cdot \left(\frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \right) + e^{-2\sigma\sqrt{\Delta t}} \cdot \left(\frac{e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \right)$$

$$= \frac{-e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}} + e^{r\Delta t}\left(e^{2\sigma\sqrt{\Delta t}} - e^{-2\sigma\sqrt{\Delta t}}\right)}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$= -1 + e^{r\Delta t}\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right)$$

using $a^2 - b^2 = (a+b)(a-b)$

Hence, $Var[S(t+\Delta t)^2 | S(t)]$
$$= S(t)^2 (-1 + e^{r\Delta t}(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}))$$
$$- S(t)^2 e^{2r\Delta t}$$

note
$$e^{r\Delta t} = 1 + r\Delta t + \frac{(r\Delta t)^2}{2} + \cdots$$
$$= 1 + r\Delta t + O(\Delta t^2)$$
(by Taylor expansion)

similarly
$$e^{\sigma\sqrt{\Delta t}} = 1 + \sigma\sqrt{\Delta t} + \frac{(\sigma\sqrt{\Delta t})^2}{2} + \cdots$$
$$= 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2\Delta t}{2} + \frac{\sigma^3(\Delta t)^{3/2}}{6} + O(\Delta t^2)$$
$$e^{-\sigma\sqrt{\Delta t}} = 1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2\Delta t}{2} - \frac{\sigma^3(\Delta t)^{3/2}}{6} + O(\Delta t^2)$$

$$\Rightarrow \quad e^{r\Delta t}(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})$$
$$= (1 + r\Delta t + O(\Delta t^2))(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})$$
$$= e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}$$
$$+ (r\Delta t + O(\Delta t^2))(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})$$
$$= e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}} + (r\Delta t + O(\Delta t^2))(2 + O(\Delta t))$$
$$= e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}} + 2r\Delta t + O(\Delta t^2)$$

So,
$$Var[S(t+\Delta t)^2 | S(t)] = S(t)^2(-1 + e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}} + 2r\Delta t + O(\Delta t^2)) - S(t)^2 e^{2r\Delta t}$$

(by (*))
$$= S(t)^2(-1 + 2 + \sigma^2\Delta t + 2r\Delta t + O(\Delta t^2))$$
$$- S(t)^2 e^{2r\Delta t}$$

(Taylor expansion for $e^{2r\Delta t}$)
$$= S(t)^2(1 + 2r\Delta t + \sigma^2\Delta t + O(\Delta t^2))$$
$$- S(t)^2(1 + 2r\Delta t + O(\Delta t^2))$$
$$= S(t)^2(\sigma^2\Delta t + O(\Delta t^2)) = S(t)^2[\sigma^2\Delta t + O(\Delta t)^2]$$

noting that
$$e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}} = 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2\Delta t}{2} + \frac{\sigma^3(\Delta t)^{3/2}}{6} + O(\Delta t^2)$$
$$+ 1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2\Delta t}{2} - \frac{\sigma^3(\Delta t)^{3/2}}{6} + O(\Delta t^2)$$

(*)
$$= 2 + \sigma^2\Delta t + O(\Delta t^2)$$

and $O(\Delta t^2) = O(\Delta t)^2$

Q3a Construct a portfolio by

$$\Pi_0 = e^{-rT}(\bar{K}-K) - (\bar{P}_0 - P_0)$$
$$= \underbrace{e^{-rT}(\bar{K}-K)}_{\text{hold bond}} - \underbrace{\bar{P}_0}_{\text{short put}} + \underbrace{P_0}_{\text{long put}}$$

so $\Pi_t = e^{-r(T-t)}(\bar{K}-K) - \bar{P}_t + P_t$ for $0 \le t \le T$.

Note $\Pi_T = \bar{K} - K - \bar{P}_T + P_T$.

Let $S(T)$ be the price of underlying at expiry T.

If $S(T) \ge \bar{K}$, then

$\bar{P}_T = \max(\bar{K} - S(T), 0) = 0$,

$P_T = \max(K - S(T), 0) = 0$

and so $\Pi_T = \bar{K} - K > 0$.

If $K \le S(T) < \bar{K}$, then

$\bar{P}_T = \bar{K} - S(T)$,

$P_T = 0$

and so $\Pi_T = \bar{K} - K - (\bar{K} - S(T)) + 0$

$= S(T) - K \ge 0$.

If $S(T) < K$, then

$\bar{P}_T = \bar{K} - S(T)$,

$P_T = K - S(T)$

and so $\Pi_T = \bar{K} - K - (\bar{K} - S(T)) + K - S(T)$

$= 0$.

Thus, we see that in any case,

$\Pi_T \ge 0$.

Thus, no arbitrage implies that

$\Pi_t \ge 0$, for all $0 \le t \le T$.

Otherwise, if $\Pi_t < 0$ for some $0 \le t < T$, then at time t we can long this portfolio (obtaining cash $-\Pi_t$) and short this portfolio at time T, obtaining $\Pi_T \ge 0$, this is an arbitrage.

$\Pi_t \geq 0$ for $0 \leq t \leq T$ implies

$$\Pi_t = e^{-r(T-t)}(\bar{K}-K) - (\bar{P}_t - P_t) \geq 0$$

$$\Rightarrow \quad \bar{P}_t - P_t \leq e^{-r(T-t)}(\bar{K}-K).$$

Q3b) We use backwards induction. Employ S_j^N notation for underlying as in lecture 4.

<u>Base case</u>:

$$\bar{p}_j^N - p_j^N = \max(\bar{K}-S_j^N, 0) - \max(K - S_j^N, 0)$$

$$= \begin{cases} \bar{K}-K & \text{if } S_j^N < K \\ \bar{K}-S_j^N & \text{if } K \leq S_j^N < \bar{K} \\ 0 & \text{if } K \leq S_j^N \end{cases}$$

$$\leq \bar{K} - K = e^{-r(N-N)\Delta t}(\bar{K}-K)$$

noting that if $K \leq S_j^N < \bar{K}$, then

$$\bar{K} - S_j^N \leq \bar{K} - K.$$

(which is equivalent to $S_j^N \geq K$.)

<u>Inductive case</u>: Let n s.t. $0 \leq n < N$, $n \in \mathbb{Z}$ and $0 \leq j \leq n$.

Assume we have

$$\bar{p}_j^{n+1} - p_j^{n+1} \leq e^{-r(N-(n+1))\Delta t}(\bar{K}-K)$$

for $0 \leq j \leq n+1$.

Let $q^* = \dfrac{e^{r\Delta t}-d}{u-d}$. (We use risk-neutral valuation.)

Then for $0 \leq j \leq n$,

$$\bar{P}_j^n - P_j^n = e^{-r\Delta t}\left(q^* \bar{P}_{j+1}^{n+1} + (1-q^*)\bar{P}_j^{n+1} - q^* P_{j+1}^{n+1} - (1-q^*)P_j^{n+1}\right)$$

$$= e^{-r\Delta t}\left(q^*(\bar{P}_{j+1}^{n+1} - P_{j+1}^{n+1}) + (1-q^*)(\bar{P}_j^{n+1} - P_j^{n+1})\right)$$

(inductive hypothesis)

$$\leq e^{-r\Delta t}\left(q^* e^{-r(N-n-1)\Delta t}(\bar{K}-K) + (1-q^*)e^{-r(N-n-1)\Delta t}(\bar{K}-K)\right)$$

$$= e^{-r(N-n)\Delta t}(\bar{K}-K).$$

This finishes the induction.

3b) To be precise, the statement that we are inducting is (denoted by $P(n)$, for $0 \leq n \leq N$)

$P(n)$: for any $0 \leq j \leq n$, $\bar{p}_j^n - p_j^n \leq e^{-r(N-n)\Delta t}(\bar{K} - K)$

4a)

$$S_j^n \quad \begin{cases} S_{j+1}^{n+1} = u S_j^n \\ S_j^{n+1} = d S_j^n \end{cases}$$

Note that
$$S_0^N = d^N S_0 = \left(e^{-\sigma\sqrt{\Delta t}}\right)^N S_0$$
$$= S_0 \, e^{-\sigma N \sqrt{\Delta t}}$$
$$= S_0 \, e^{-\sigma N \sqrt{\frac{T}{N}}}$$

$$= S_0 \, e^{-\sigma \sqrt{N} \sqrt{T}}$$

noting that $\Delta t = \frac{T}{N}$,

Thus, $\Delta t = \frac{T}{N} \to 0$ iff $N \to \infty$.

Thus we see that
$$S_0^N = S_0 e^{-\sigma\sqrt{N}\sqrt{T}} \to 0 \quad \text{as} \quad N \to \infty$$

Thus $p_0^N = \max(K - S_0^N, 0) \to K$ as $N \to \infty$.

since $S_0^N \to 0$ and $K \geq 0$.

b) Recall that $q^* = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$, Assume $\sigma > 0$.

Then, we see $0 \leq q^* \leq 1$

iff $$0 \leq \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \leq 1$$

(*) iff $\left(e^{-\sigma\sqrt{\Delta t}} \leq e^{r\Delta t} \text{ and } e^{r\Delta t} \leq e^{\sigma\sqrt{\Delta t}} \right)$

Note $\lim_{\Delta t \to 0} (\sigma + r\sqrt{\Delta t}) = \sigma$.

and $\lim_{\Delta t \to 0} (\sigma - r\sqrt{\Delta t}) = \sigma$.

Hence if Δt is sufficiently small and $\sigma > 0$, then we have

$$\sigma + r\sqrt{\Delta t} > 0 \quad \text{and} \quad \sigma - r\sqrt{\Delta t} > 0,$$

and so $\sqrt{\Delta t}(\sigma + r\sqrt{\Delta t}) > 0$ and $\sqrt{\Delta t}(\sigma - r\sqrt{\Delta t}) > 0$.

4b) Thus, under the assumption that Δt is sufficiently small and $\sigma > 0$,

$$\frac{e^{r\Delta t}}{e^{-\sigma\sqrt{\Delta t}}} = e^{r\Delta t + \sigma\sqrt{\Delta t}} = e^{\sqrt{\Delta t}(\sigma + r\sqrt{\Delta t})} > 1 \quad \text{since } \sqrt{\Delta t}(\sigma + r\sqrt{\Delta t}) > 0$$

$$\frac{e^{\sigma\sqrt{\Delta t}}}{e^{r\Delta t}} = e^{\sigma\sqrt{\Delta t} - r\Delta t} = e^{\sqrt{\Delta t}(\sigma - r\sqrt{\Delta t})} > 1 \quad \text{since } \sqrt{\Delta t}(\sigma - r\sqrt{\Delta t}) > 0$$

so we have $e^{r\Delta t} > e^{-\sigma\sqrt{\Delta t}}$ and $e^{\sigma\sqrt{\Delta t}} > e^{r\Delta t}$.

Thus, by $(*)$, we have $0 \le q^* \le 1$, if Δt is sufficiently small and $\sigma > 0$.

$$q^* S_{j+1}^{n+1} + (1-q^*) S_j^{n+1}$$

$$= \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} S_j^n e^{\sigma\sqrt{\Delta t}} + \frac{e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} S_j^n e^{-\sigma\sqrt{\Delta t}}$$

$$= S_j^n \left(\frac{e^{\sigma\sqrt{\Delta t} + r\Delta t} - 1}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} + \frac{1 - e^{r\Delta t - \sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \right)$$

$$= S_j^n \frac{e^{\sigma\sqrt{\Delta t} + r\Delta t} - e^{r\Delta t - \sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$= e^{r\Delta t} S_j^n$$

$$\Rightarrow e^{-r\Delta t}\left(q^* S_{j+1}^{n+1} + (1-q^*) S_j^{n+1} \right)$$

$$= S_j^n$$

Hilroy

4c) We use backwards induction.

Base case: Note that since $S_0^0 \geq 0$, and that $u, d > 0$, we have $S_j^n \geq 0$ for any $0 \leq j \leq n$, $0 \leq n \leq N$.
Thus, for any $0 \leq j \leq N$,

$$0 \leq P_j^N = \max(K - S_j^N, 0) \leq K - S_j^N \leq K.$$

Note that we assume Δt is sufficiently small and $\sigma > 0$, so that $0 \leq q^* \leq 1$. We can use risk-neutral valuation for P_j^n, $\forall\, 0 \leq j \leq n$, $0 \leq n \leq N$.

Inductive case: Let n s.t. $0 \leq n < N$ and $n \in \mathbb{Z}$.
Assume we have

$$0 \leq P_j^{n+1} \leq K \quad \text{for} \quad 0 \leq j \leq n+1.$$

Let $0 \leq j \leq n$. Then $P_j^n = e^{-r\Delta t}(P_{j+1}^{n+1} q^* + P_j^{n+1}(1-q^*))$

(inductive hypothesis) $\leq e^{-r\Delta t}(K q^* + K(1-q^*))$

$= e^{-r\Delta t} K \leq K$

also $P_j^n = e^{-r\Delta t}(P_{j+1}^{n+1} q^* + P_j^{n+1}(1-q^*))$

(inductive hypothesis) $\geq e^{+r\Delta t}(0 \cdot + 0) = 0.$

This is true for any $0 \leq j \leq n$
This finishes induction.

To be precise, the statement that we are inducting is this (denoted by $P(n)$, where $0 \leq n \leq N$)

$$P(n): \text{ for any } 0 \leq j \leq n, \quad 0 \leq P_j^n \leq K.$$

Q5) The statement we want to induct is, where $N \in \mathbb{Z}^+$,

$P(N)$: In a N-period model,

$$V_0^0 = e^{-rN\Delta t} \sum_{k=0}^{N} \binom{N}{k} (q^*)^k (1-q^*)^{N-k} V_k^N$$

Base Case: Want to show $P(1)$ holds.

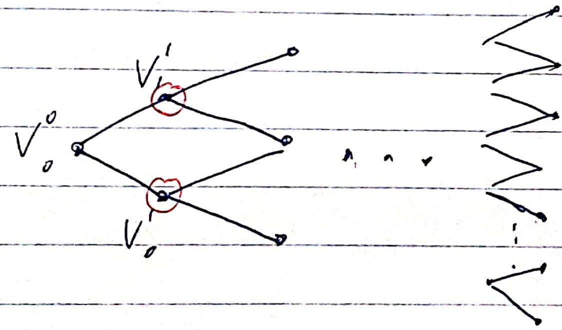
Recall in class that for a 1-period model,

$$V_0^0 = e^{-r\Delta t} (q^* V_1^1 + (1-q^*) V_0^1)$$

$$= e^{-r\Delta t} \sum_{k=0}^{1} \binom{1}{k} (q^*)^k (1-q^*)^{1-k} V_k^1$$

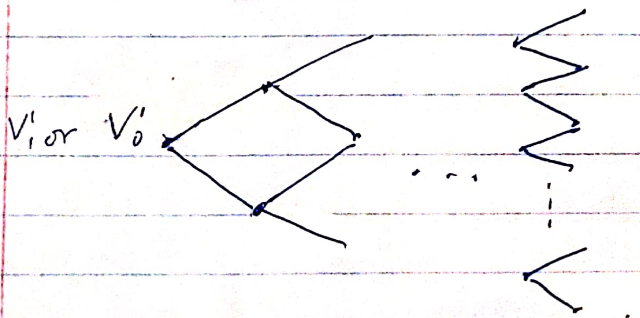
Inductive step: Assume $P(N)$ holds, where $N \in \mathbb{Z}^+$.

Want to show $P(N+1)$ holds.



a $(N+1)$-period model

If we start from V_1^1 or V_0^1 and treat them as the starting node of the binomial lattice, then we obtain a N-period model



a N-period model

(**)

Q5) Hence, by $P(N)$, we have

$$V_1^1 = e^{-rN\Delta t} \sum_{k=1}^{N+1} \binom{N}{k-1} (q^*)^{k-1} (1-q^*)^{N+1-k} V_k^{N+1}$$

(remark: the end nodes of (**) for the case V_1^1 will be $V_{N+1}^{N+1}, V_N^{N+1}, \dots, V_1^{N+1}$,

The summation formula looks a little bit different since the starting node is V_1^1.)

Similarly, $V_0^1 = e^{-rN\Delta t} \sum_{k=0}^{N} \binom{N}{k} (q^*)^k (1-q^*)^{N-k} V_k^{N+1}$

Thus,

$$V_0^0 = e^{-r\Delta t} (q^* V_1^1 + (1-q^*) V_0^1)$$

$$= e^{-r(N+1)\Delta t} \Big(q^* \sum_{k=1}^{N+1} \binom{N}{k-1} (q^*)^{k-1} (1-q^*)^{N+1-k} V_k^{N+1}$$

$$+ (1-q^*) \sum_{k=0}^{N} \binom{N}{k} (q^*)^k (1-q^*)^{N-k} V_k^{N+1} \Big)$$

$$= e^{-r(N+1)\Delta t} \Big((q^*)^{N+1} V_{N+1}^{N+1} + \sum_{k=1}^{N} \binom{N}{k-1} (q^*)^k (1-q^*)^{N+1-k}$$

$$+ (1-q^*)^{N+1} V_0^{N+1} + \sum_{k=1}^{N} \binom{N}{k} (q^*)^k (1-q^*)^{N+1-k} \Big)$$

$$= e^{-r(N+1)\Delta t} \Big((q^*)^{N+1} V_{N+1}^{N+1} + \sum_{k=1}^{N} \binom{N+1}{k} (q^*)^k (1-q^*)^{N+1-k}$$

$$+ (1-q^*)^{N+1} V_0^{N+1} \Big)$$

$$= e^{-r(N+1)\Delta t} \sum_{k=0}^{N+1} \binom{N+1}{k} (q^*)^k (1-q^*)^{N+1-k} V_k^{N+1}.$$

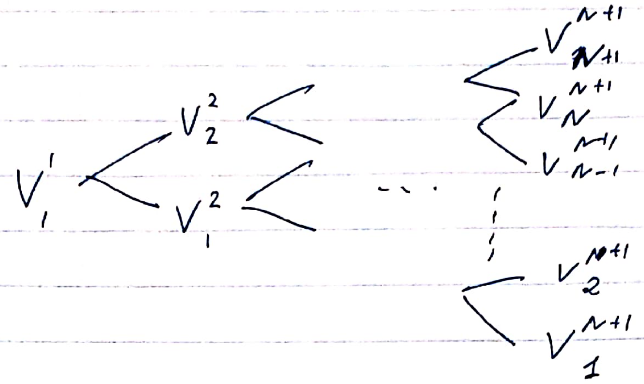
This finishes induction.

Hilroy

Q5) Further explanation on

$$V_1^1 = e^{-rN\Delta t} \sum_{k=1}^{N+1} \binom{N}{k-1} (q^*)^{k-1} (1-q^*)^{N+1-k} V_k^{N+1}$$

note



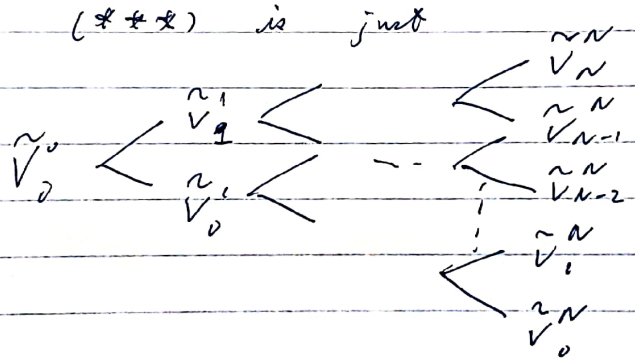
(***)

If we rename $V_1^1, V_2^2, V_1^2, \ldots$ by

$$V_1^1 = \tilde{V}_0^0$$

$$V_j^n = \tilde{V}_{j-1}^{n-1} \quad \text{for } 1 \le j \le n, \quad 1 \le n \le N+1,$$

then (***) is just



so that

$$V_1^1 = \tilde{V}_0^0 = e^{-rN\Delta t} \sum_{k=0}^{N} \binom{N}{k} (q^*)^k (1-q^*)^{N-k} \tilde{V}_k^N$$

$$= e^{-rN\Delta t} \sum_{k=1}^{N+1} \binom{N}{k-1} (q^*)^{k-1} (1-q^*)^{N+1-k} \tilde{V}_{k-1}^N$$

$$= e^{-rN\Delta t} \sum_{k=1}^{N+1} \binom{N}{k-1} (q^*)^{k-1} (1-q^*)^{N+1-k} V_k^{N+1}$$

Q6. Consider $G(z,t) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$G(z,t) = \frac{1}{2} z^2 - t/2.$$

Then $$\frac{\partial G}{\partial z} = z \qquad \frac{\partial G}{\partial t} = -\frac{1}{2}$$

$$\frac{\partial^2 G}{\partial z^2} = 1$$

By Ito's Lemma,

$$dG(Z(t),t) = \frac{\partial G}{\partial z}(Z(t),t) \cdot dZ(t) + \frac{\partial G}{\partial t}(Z(t),t)\,dt$$

$$+ \frac{1}{2} \frac{\partial^2 G}{\partial z^2}(Z(t),t)\,(dZ(t))^2$$

(note $(dZ(t))^2 = dt$)

$$= Z(t)\,dZ(t) - \frac{1}{2}dt + \frac{1}{2}dt$$

$$= Z(t)\,dZ(t)$$

Hence $G(Z(T),T) - G(Z(0),0) = \int_0^T Z(t)\,dZ(t)$

$$\Rightarrow \int_0^T Z(t)\,dZ(t) = G(Z(T),T)$$

$$= \frac{1}{2} Z(T)^2 - \frac{T}{2}$$

since $G(Z(0),0) = \frac{1}{2} \cdot 0^2 - \frac{0}{2} = 0.$

Hilroy