# 0.1 Preliminary remarks on set theory

**References.** Literature for this chapter:

- Sophie Morel Homological Algebra I.1,
- Daniel Murfet Foundations for Category Theory,
- Saunders MacLane Categories for the Working Mathematician I.6.

In this course we always assume a model of set theory that satisfies the Zermelo-Fraenkel axioms + the axiom of choice (ZFC).

**Definition** (Grothendieck universe; we assume ZFC). A universe  $\mathcal{U}$  is a set which has the following properties:

- (i)  $\emptyset$ ,  $\mathbb{N} \in \mathcal{U}$ ,
- (ii)  $X \in \mathcal{U}$  and  $y \in X \implies y \in \mathcal{U}$ ,
- (iii)  $X \in \mathcal{U} \implies \{X\} \in \mathcal{U}$ ,
- (iv)  $X \in \mathcal{U} \implies \mathcal{P}(X) \in \mathcal{U}$ ,
- (v) If  $I \in \mathcal{U}$  and  $\{X_i\}_{i \in I}$  is a family of members  $X_i \in \mathcal{U}$ , then  $\bigcup_{i \in I} X_i \in \mathcal{U}$ .

The existence of a universe is equivalent to the existence of a strongly inaccessible cardinal. (Thomas Jech - Set Theory)

**Axiom** (Axiom of universes (Grothendieck)). Every set lies in a universe. (We will assume this)

**Definition.** If  $\mathcal{U}$  is our chosen universe, then:

- A  $\mathcal{U}$ -set is an element in  $\mathcal{U}$ .
- A  $\mathcal{U}$ -class is a subset of  $\mathcal{U}$ .
- A  $\mathcal{U}$ -group is a group  $(G, e, \cdot)$  with  $G \in \mathcal{U}$  and  $\cdot : G \times G \to G \in \mathcal{U}$ .
- A  $\mathcal{U}$ -ring is a ring  $(R, 0, 1, +, \cdot)$  with  $R \in \mathcal{U}$  and also  $+, \cdot$
- etc.

Convention. We fix a  $\mathcal{U}$  and drop  $\mathcal{U}$ - in all terms.

# 0.2 Categories

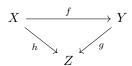
- **Definition 0.1.** (a) A directed graph (a diagram scheme) is a tuple (O, A, dom, cod) consisting of  $\mathcal{U}$ -classes O and A and maps dom,  $\text{cod}: A \to O$ . We call elements of O objects (or vertices) and elements of A arrows (or directed edges). For an arrow  $f \in A$  call dom(f) the source (or domain) of f and cod(f) the target (or codomain) of f.
- (b) For a graph as in (a) call  $A \times_O A := \{(g, f) \in A \times A \mid \text{dom}(g) = \text{cod}(f)\}$  set of composable arrow pairs.

(c) A subgraph of (O, A, dom, cod) is a graph (O', A', dom', cod') such that  $O' \subseteq O, A' \subseteq A, \text{dom}' = \text{dom} \mid_{A'}$  and  $\text{cod}' = \text{cod} \mid_{A'}$ .

**Example 0.2.** Let  $O = \{X, Y, Z\}, A = \{f, g, h\}, \text{dom}, \text{cod} : A \to O \text{ given by the table}$ 

$$\begin{array}{c|cccc} & f & g & h \\ \hline \text{dom} & X & Y & X \\ \hline \text{cod} & Y & Z & Z \end{array}$$

Illustration:



**Definition 0.3.** A category  $\mathcal{C}$  is a tuple  $(Ob \mathcal{C}, Mor \mathcal{C}, dom, cod, \circ, 1)$  consisting of a graph  $(Ob \mathcal{C}, Mor \mathcal{C}, dom, cod)$  and maps

$$1: \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}, X \mapsto 1_X$$

and

$$\circ : \operatorname{Mor} \mathcal{C} \times_{\operatorname{Ob} \mathcal{C}} \operatorname{Mor} \mathcal{C} \to \operatorname{Mor} \mathcal{C}, (g, f) \mapsto g \circ f$$

such that:

- (i)  $dom(1_X) = cod(1_X) = X, \forall X \in Ob \mathcal{C},$
- (ii)  $dom(g \circ f) = dom(f)$  and  $cod(g \circ f) = cod(g)$ ,
- (iii)  $\forall f \in \operatorname{Mor} \mathcal{C} \text{ with } X = \operatorname{dom}(f), Y = \operatorname{cod}(f)$

$$f \circ 1_X = 1_Y \circ f = f$$

(iv)  $\forall$  arrows  $f, g, h \in \text{Mor } \mathcal{C}$  such that (h, g) and (g, f) are composable aarrow pairs we have

$$h \circ (q \circ f) = (h \circ q) \circ f$$

Call elements of  $\mathrm{Ob}\,\mathcal{C}$  the objects of  $\mathcal{C}$  and elements of  $\mathrm{Mor}\,\mathcal{C}$  the morphisms of  $\mathcal{C}$ .

**Notation 0.4.** For a category C as in definition 3

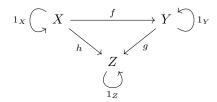
- (a) (often) write  $X, Y \in \mathcal{C}$  to mean  $X, Y \in \text{Ob } \mathcal{C}$
- (b) For  $X, Y \in \mathcal{C}$  write

$$\mathcal{C}(X,Y) := \operatorname{Mor}_{\mathcal{C}}(X,Y) := \{ f \in \mathcal{C} \mid \operatorname{dom} f = X, \operatorname{cod} f = Y \}$$

**Definition 0.5.** (a) Call a category  $\mathcal{C}$  locally small if  $\mathcal{C}(X,Y)$  is a set  $\forall X,Y \in \mathcal{C}$ .

(b) Call  $\mathcal{C}$  small if  $Ob \mathcal{C}$ ,  $Mor \mathcal{C}$  are sets.

**Remark 0.6** (Extension of example 2 to a category). Let  $O = \{X, Y, Z\}, A = \{f, g, h\} \cup \{1_X, 1_Y, 1_Z\}$ , cod, dom as before on  $\{f, g, h\}$  and uniquely extended to  $\{1_X, 1_Y, 1_Z\}$  by axiom (i) and  $\circ$  the only possible composition satisfying the axioms



composable arrow pairs:

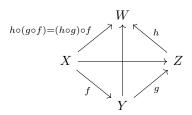
$$(1_X, 1_X), (f, 1_X), (1_Y, 1_Y), (1_Y, f), (g, 1_Y), (1_Z, 1_Z), (1_Z, g), (1_Z, h), (h, 1_X), (g, h)$$

Canonical universal extension would contain a second arrow  $X \to Z$  since it would not want to impose the condition  $g \circ f = h$ .

**Definition 0.7.** (a) A diagram in  $\mathcal{C}$  is a subgraph  $\Gamma$  of  $(Ob \mathcal{C}, Mor \mathcal{C}, dom, cod)$ .

(b) A diagram is commutative if for all objects X, Y of  $\Gamma$  and all chains of arrows from X to Y, their composition is the same (i.e. it only depends on X and Y).

Example (For associativity).



Examples (Examples of categories).

- Set (category of *U*-sets): where
- Ob Set = class of all  $\mathcal{U}$ -sets,
- Mor $\mathsf{Set} = \mathsf{class}$  of all  $\mathcal{U}$ -maps between sets,
- dom, cod are the domain and codomain (range) of a map. (Think of a map as a triple  $(X, Y, \text{graph map in } X \times Y)$ )
- $-\circ =$ composition of maps,
- $-1_X = id_X$  the identity map.
- Grp (category of abelian groups)
- Ring
- CRing
- Top
- RMod

- Mod<sub>B</sub>
- Vec<sub>K</sub>
- $Ab =_{\mathbb{Z}} Mod$

**Examples** (Abstract examples). 1. Ob  $C = \text{Mor } C = \emptyset$  (empty category)

- 2. Ob  $\mathcal{C} = \{X\}$ , Mor  $\mathcal{C} = \{1_X\}$  (1 arrow category)
- 3. Let G be a group, define a category  $\underline{G}$  by  $Ob \underline{G} = \{*\}$  (singleton set) and  $Mor \underline{G} = G$ , dom, cod the unique map  $G \to \{*\}$ ,  $1_* = e$  (unit element of G).  $\circ =$  composition in G:

$$\operatorname{Mor} \underline{G} \times \operatorname{Mor} \underline{G} = G \times G \to G = \operatorname{Mor} \underline{G}$$

4. Let  $\underline{A} = (M, \leq)$  be a partially ordered set. Define the associated category  $\operatorname{Ord} \underline{M}$  with  $\operatorname{Ob} \operatorname{Ord} \underline{M} = \text{elements of } M$ , morphisms are determined by

$$\operatorname{Ord} \underline{M}(X,Y) = \begin{cases} \text{singleton set,} & X \leq Y, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Unit is clear. composition dictated by  $Mor(Ord \underline{M})$  (i.e. by  $\leq$ )

**Definition 0.8.** For a category  $\mathcal{C} = (\operatorname{Ob} \mathcal{C}, \operatorname{Mor} \mathcal{C}, \operatorname{dom}, \operatorname{cod}, \circ, 1)$  define the tuple  $\mathcal{C}^{\operatorname{op}} = (\operatorname{Ob} \mathcal{C}, \operatorname{Mor} \mathcal{C}, \operatorname{cod}, \operatorname{dom}, \circ^{\operatorname{op}}, 1)$  with

$$\circ^{\mathrm{op}}: \{(f,g) \in \operatorname{Mor} \mathcal{C} \times \operatorname{Mor} \mathcal{C} \mid \operatorname{cod} f = \operatorname{dom} g\} \to \operatorname{Mor} \mathcal{C}$$
 
$$(f,g) \mapsto f \circ^{\mathrm{op}} g := g \circ f$$

(change the direction of arrows!)

**Proposition 0.9** (Exercise).  $C^{op}$  is a category, the opposite category to C.

$$\textbf{Example.} \ \ (\underline{G})^{\operatorname{op}} = \underline{(G^{\operatorname{op}})}, \ (G^{\operatorname{op}} = (G, e, \circ^{\operatorname{op}}) \ \text{with} \ g \circ^{\operatorname{op}} h = h \circ g).$$

**Warning 0.10.**  $\operatorname{Vec}_K^{\operatorname{op}}(V,W) \neq \operatorname{not}$  the set of maps  $V \to W$ , it is  $\{f: W \to V \mid f \text{ is } K\text{-linear}\}$ 

**Definition 0.11.** A subcategory of  $\mathcal{C} = (\operatorname{Ob} \mathcal{C}, \operatorname{Mor} \mathcal{C}, \operatorname{dom}, \operatorname{cod}, \circ, 1)$  is a category  $\mathcal{C}' = (\operatorname{Ob} \mathcal{C}', \operatorname{Mor} \mathcal{C}', \operatorname{dom}', \operatorname{cod}', \circ', 1')$  such that  $\operatorname{Ob} \mathcal{C}' \subseteq \operatorname{Ob} \mathcal{C}, \operatorname{Mor} \mathcal{C}' \subseteq \operatorname{Mor} \mathcal{C}, \operatorname{dom}' = \operatorname{dom}|_{\operatorname{Mor} \mathcal{C}'}, \operatorname{cod}' = \operatorname{cod}|_{\operatorname{Mor} \mathcal{C}'}, \circ' = \circ|_{\operatorname{Mor} \mathcal{C}' \times \operatorname{Ob} \mathcal{C} \operatorname{Mor} \mathcal{C}'}, 1' = 1|_{\operatorname{Ob} \mathcal{C}'}.$  We write  $\mathcal{C}' \subseteq \mathcal{C}$ .

**Example.** Ab  $\subseteq$  Grp and CRing  $\subseteq$  Ring, etc.

**Definition 0.12** (Product of categories). The product of two categories C and C' is the six-tuple:

$$(\operatorname{Ob} \mathcal{C} \times \operatorname{Ob} \mathcal{C}', \operatorname{Mor} \mathcal{C} \times \operatorname{Mor} \mathcal{C}', \operatorname{dom} \times \operatorname{dom}', \operatorname{cod} \times \operatorname{cod}', \circ, 1)$$

where  $\circ$  is componentwise composition  $(g,g')\circ (f,f')=(g\circ f,g'\circ f')$  and  $1_{X\times X'}=(1_X,1_{X'})$ 

**Definition 0.13** (Concepts inside categories). Let  $X,Y\in\mathcal{C}$ , then call  $f\in\mathcal{C}(X,Y)$ 

- (a) an isomorphism  $\iff \exists g \in \mathcal{C}(Y, X)$  such that  $g \circ f = 1_X, f \circ g = 1_Y,$
- (b) an endomorphism  $\iff X = Y$ ,
- (c) an  $automorphism \iff$  it is an isomorphism and an endomorphism

Moreover C is called a groupoid category  $\iff$  all morphisms are isomorphisms.

**Example.** Let G be a group, then G is a groupoid category. C a groupoid category  $\Longrightarrow C(X,X)$  is a group (under  $\circ, \forall X \in \mathrm{Ob}\,\mathcal{C}$ ).

**Definition 0.14.** Let  $X, Y \in \mathcal{C}$ , then call  $f \in \mathcal{C}(X, Y)$ :

(a) a monomorphism  $\iff$  f is left cancellable  $\iff$   $\forall W \in \mathcal{C}$  the map  $f_*: \mathcal{C}(W,X) \to \mathcal{C}(W,Y), g \mapsto f \circ g$  is injective.

$$W \stackrel{g_1}{\underset{g_2}{\Longrightarrow}} X \stackrel{f}{\underset{g_2}{\longleftrightarrow}} Y : f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

(b) an epimorphism  $\iff$  f is right cancellable  $\iff$   $\forall Z \in \mathcal{C}$  the map  $f^*$ :  $\mathcal{C}(Y,Z) \to \mathcal{C}(X,Z), h \mapsto h \circ f$  is injective.

$$X \xrightarrow{f} Y \xrightarrow{h_1} Z : h_1 \circ f = h_2 \circ f \implies h_1 = h_2.$$

(c) a split monomorphism  $\iff \exists g \in \mathcal{C}(Y,X)$  such that  $g \circ f = 1_X$ 

$$X \xrightarrow{\xi \xrightarrow{g}} Y$$

(d) a split epimorphism  $\iff \exists h \in \mathcal{C}(Y,X) \text{ such that } f \circ h = 1_Y$ 

$$X \xrightarrow{k \xrightarrow{h}} Y$$

Facts 0.15. (a) f split mono-/epimorphism  $\implies f$  mono-/epimorphism.

- (b) f (split) mono-/epimorphism in  $\mathcal{C} \implies f$  (split) mono-/epimorphism in  $\mathcal{C}^{\text{op}}$ .
- (c) (Exercise) For  $f \in \mathcal{C}(X,Y), (X,Y) \in \mathcal{C}$  the following are equivalent:
  - (i) f is an isomorphism
  - (ii)  $\forall W \in \mathcal{C} : f_* : \mathcal{C}(W, X) \to \mathcal{C}(W, Y), g \mapsto f \circ g$  is bijective.
  - (iii)  $\forall Z \in \mathcal{C} : f^* : \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z), h \mapsto h \circ f$  is bijective.
- (b') f is an isomorphism in  $\mathcal{C} \iff f$  is an isomorphism in  $\mathcal{C}^{op}$ .

*Proof.* (b), (b') are exercises. (c) "serious" exercise.

(a) For an epimorphism (check right cancellability) consider

$$X \xrightarrow{f} Y \xrightarrow{h_1} Z : h_1 \circ f \stackrel{(1)}{=} h_2 \circ f$$

By f a split epimorphism, we have  $h:Y\to X$  such that  $f\circ h=1_Y$  (2). Apply  $-\circ h$  to (1):

$$(h_1\circ f)\circ h=(h_2\circ f)\circ h$$
 
$$\parallel$$
 
$$\parallel$$
 
$$h_1=h_1\circ 1_Y=h_1\circ (f\circ h)=h_2\circ (f\circ h)=h_2\circ 1_Y=h_2$$

**Examples.** In Set, Grp, Ring the monomorphisms are the injective maps and in Set, Grp the epimorphisms are the surjective maps. But  $\mathbb{Z} \to \mathbb{Q}$  (inclusion) is an epimorphism in Ring. If  $K \subseteq E$  is purely inseperable, then it's an epimorphism in the category of fields.

**Definition 0.16.** (a)  $X \in \mathcal{C}$  is called an *initial object*  $\iff \forall Y \in \mathcal{C}: \#\mathcal{C}(X,Y) = 1$ 

- (b)  $X \in \mathcal{C}$  is called a terminal object  $\iff \forall Z \in \mathcal{C} : \#\mathcal{C}(Z,X) = 1$
- (c)  $X \in \mathcal{C}$  is called a *null object*  $\iff$  X is initial and terminal.

**Example.** •  $\emptyset$  is initial in Set, Top,

- {\*} is terminal in Set, Top
- $0 = \{0\}$  is a null object in  ${}_{R}\mathsf{Mod}, \mathsf{Ab}, \mathsf{Vec}_{K}$

#### 0.3 Functors

Let  $\mathcal{C}, \mathcal{C}', \mathcal{D}$  be categories.

**Definition 0.17.** A functor F from C to D  $(F: C \to D)$  is a pair of maps

$$F: \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}, X \mapsto F(X),$$
  
$$F: \mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}, f \mapsto F(f).$$

that "preserve sources, targets, units and composition", i.e.

- (i)  $\forall f \in \operatorname{Mor} \mathcal{C} : \operatorname{dom}(Ff) = F(\operatorname{dom} f) \text{ and } \operatorname{cod}(Ff) = F(\operatorname{cod} f)$
- (ii)  $\forall X \in \mathrm{Ob}\,\mathcal{C} : F(1_X) = 1_{FX}$
- (iii)  $\forall$  composable pairs (g, f) in  $\operatorname{Mor} \mathcal{C} \times_{\operatorname{Ob} \mathcal{C}} \operatorname{Mor} \mathcal{C} : F(g \circ f) = F(g) \circ F(f)$ . (other notation  $F(X \xrightarrow{f} Y) = FX \xrightarrow{Ff} FY$ )

Examples. (a) Powerset:

$$\mathcal{P}:\mathsf{Set} o \mathsf{Set}$$
 
$$X \mapsto \mathcal{P}(X)$$
 
$$f:X o Y \mapsto \mathcal{P}f:\mathcal{P}X o \mathcal{P}Y,$$
 
$$(U \subseteq X) \mapsto (f(U) \subseteq Y)$$

(b) Forgetful functor (it forgets structure)

$$\begin{split} V: \mathsf{Grp} &\to \mathsf{Set}, (G, e, \circ) \mapsto G \\ V: \mathsf{Top} &\to \mathsf{Set}, (X, \mathcal{T}) \mapsto X \\ V: {}_R\mathsf{Mod} &\to \mathsf{Ab}, (M, 0, +, \cdot) \mapsto (M, 0, +) \end{split}$$

(c)  $_R\mathsf{Mod} \to \mathsf{Mod}_{R^{\mathrm{op}}}$ , left R-modules  $\mapsto$  right R-modules.

Remark. Functors in definition 17 are also called covariant functors.

**Definition 0.18.** A contravariant functor from  $\mathcal{C} \to \mathcal{D}$  is a functor  $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ , i.e.

$$F(X \xrightarrow{f} Y) = (FX \xleftarrow{Ff} FY)$$

 $=(Y \xrightarrow{f} X)$  in  $\mathcal{C}^{\text{op}}$  and

$$F((Y \xrightarrow{g} Z) \circ (X \xrightarrow{f} Y)) = F(X \xrightarrow{f} Y) \circ F(Y \xrightarrow{g} Z)$$

Visually:

$$\begin{array}{cccc} X & X & FX \\ \downarrow^f & f \uparrow & Ff \uparrow \\ Y & Y & FY \\ \downarrow^g & g \uparrow & Fg \uparrow \\ Z & Z & FZ \end{array}$$

$$\mathrm{in}\; \mathcal{C} \qquad \mathrm{in}\; \mathcal{C}^{\mathrm{op}}$$

**Remark** (Exercise). Let  $F: \mathcal{C} \to \mathcal{D}$  be any functor  $\implies F$  maps isomorphisms to isomorphisms.

**Examples** (Contravariant functors). (a) Passage to the dual vector space

$$\begin{array}{ccc} D: \mathsf{Vec}_K^{\mathrm{op}} & \longrightarrow & \mathsf{Vec}_K \\ V & \longmapsto & V^* = \mathrm{Hom}_K(V,K) \\ f & & \uparrow Df = f^* \\ W & \longmapsto & W^* \end{array}$$

linear algebar:  $(f \circ g)^* = g^* \circ f^*$  for (f, g) a composable pair.

(b) Let Poset be the category of partially ordered sets, then we have a contravariant functor

$$\begin{array}{cccc} \mathcal{O}: \mathsf{Top}^{\mathrm{op}} & \longrightarrow \mathsf{Poset} \\ (X,\mathcal{T}) & \longmapsto & (\mathcal{T},\subseteq) \ni f^{-1}(V) \underset{\mathrm{open}}{\subseteq} X \\ \downarrow & & \uparrow & & \uparrow \\ (Y,\mathcal{T}') & \longmapsto & (\mathcal{T},\subseteq) & \ni & V \underset{\mathrm{open}}{\subseteq} Y \end{array}$$

(c) The contravariant powerset functor:

$$\begin{array}{ccc} \mathcal{P}^*: \mathsf{Set} & \longrightarrow & \mathsf{Set} \\ X & \longmapsto & \mathcal{P}^*(X) = \mathcal{P}(X) \\ f & & & \uparrow \mathcal{P}^*f \\ Y & \longmapsto & \mathcal{P}^*(Y) = \mathcal{P}(Y) \end{array}$$

**Definition 0.19.** Let C, C', D be categories, a functor  $C \times C' \to D$  is called a *bifunctor*.

**Example 0.20** (Important example). Let  $\mathcal{C}$  be any category

$$\begin{array}{cccc} \mathcal{C}(-,-):\mathcal{C}^{\mathrm{op}}\times\mathcal{C} & \longrightarrow & \mathsf{Set} \\ & (X,Y) & \longmapsto & \mathcal{C}(X,Y) & \ni & g \\ & & & \downarrow & & \downarrow \\ & & & \downarrow & & \downarrow \\ & & & (W,Z) & \longmapsto & \mathcal{C}(W,Z) \ni h \circ g \circ f \end{array}$$

If we fix a first argument X, we get

$$h_X := \mathcal{C}(X, -) \to \mathsf{Set}$$

If we fix a second argument Y, we get

$$h^Y := \mathcal{C}(-,Y) \to \mathsf{Set}$$

Soon: we will also have another important bifunctor

$$-\otimes -: \mathsf{Mod}_R \times_R \mathsf{Mod} \to \mathsf{Ab}$$

**Definition 0.21.** Let  $F: \mathcal{C} \to \mathcal{D}$  be a functor, F is called

- (a)  $faithful \iff \forall X, Y \in \mathcal{C} : \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$  is injective.
- (b)  $full \iff \forall X, Y \in \mathcal{C} : \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$  is surjective.
- (c) fully faithful F is full and faithful.
- (d) essentially surjective  $\iff \forall Y \in \mathcal{D} \exists X \in \mathcal{C} \exists \text{ isomorphism } FX \xrightarrow{\cong} Y.$
- (e)  $conservative \iff \forall f \in \operatorname{Mor} \mathcal{C} : f \text{ is an isomorphism } \iff Ff \operatorname{1} \operatorname{Mor} \mathcal{D} \text{ is an isomorphism. } (\implies \text{always holds})$
- (f) an isomorphism  $\iff \exists G: \mathcal{D} \to \mathcal{C}$  functor such that  $F \circ G = \mathrm{id}_{\mathcal{D}}$  and  $G \circ F = \mathrm{id}_{\mathcal{C}}$ .

Examples. (a) Forgetful functors are "often" faithful but not full

$$V:\mathsf{Grp}\to\mathsf{Set},\mathsf{Ab}\to\mathsf{Set},{}_R\mathsf{Mod}\to\mathsf{Set},\mathsf{Ring}\to\mathsf{Set}$$

are conservative.

- (b) The forgetful functor  $V:\mathsf{Top}\to\mathsf{Set}$  is not conservative and not full but essentially surjective.
- (c) The inclusion of a subcategory  $\mathcal{C}'$  into its ambient category  $\mathcal{C}$  is always faithful. Call  $\mathcal{C}'$  a full subcategory  $\iff \forall X,Y \in \mathcal{C}': \mathcal{C}'(X,Y) = \mathcal{C}(X,Y)$  ( $\iff i:\mathcal{C}' \to \mathcal{C}$  is full)

### 0.4 Natural transforations

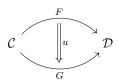
They are morphisms between functors.

**Definition 0.22.** Let  $F, G : \mathcal{C} \to \mathcal{D}$  be functors.

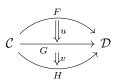
(a) A morphism from F to G (or a natural transforation) is a family  $u = (u_X : FX \to GX)_{X \in Ob \mathcal{C}}$  of morphisms in  $\mathcal{D}$ , such that forall  $f : X \to Y$  in  $\mathcal{C}$  we have the commutative diagram:

$$\begin{array}{ccc} FX & \xrightarrow{u_X} & GX \\ Ff \downarrow & & \downarrow Gf \\ FY & \xrightarrow{u_Y} & GY \end{array}$$

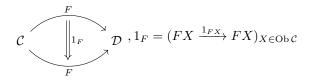
Notation:



(b) Composition: Let  $F, G, H : \mathcal{C} \to \mathcal{D}$  be functors and  $u : F \Rightarrow G, v : G \Rightarrow H$  natural transformations. The composition  $v \circ u : F \Rightarrow H$  is the natural transformation (check)  $(v_X \circ u_X : FX \xrightarrow{u_X} GX \xrightarrow{v_X} HX)_{X \in \mathrm{Ob}\,\mathcal{C}}$ 



(c) The category  $\mathcal{D}^{\mathcal{C}}$  (or Fun( $\mathcal{C}, \mathcal{D}$ )) whose objects are the functors  $\mathcal{C} \to \mathcal{D}$  and whose morphisms are the natural transformations  $(F: \mathcal{C} \to \mathcal{D}) \Rightarrow (G: \mathcal{C} \to \mathcal{D})$ . The composition is from (b), and the unit natural transforation is



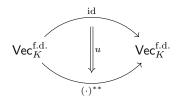
(dom, cod are clear). Remark: One can also define 2-categories (and the category of categories is an example of such, objects:  $\mathcal{C}, \mathcal{D}, \dots$  and morphisms are  $F: \mathcal{C} \rightrightarrows \mathcal{D}$  2-morphisms = natural transformations)

(d) A natural transformation  $u: F \Rightarrow G$  is called a *natural isomorphism*  $\iff \forall X \in \text{Ob}\,\mathcal{C}: u_X: FX \to GX$  is an isomorphism  $\iff \exists$  natural transformation  $v: G \Rightarrow F: v \circ u = \text{id}_F, u \circ v = \text{id}_G.$ 

**Example** (Famous linear algebra example of a natural transformation). Let  $(\cdot)^{**}: \mathsf{Vec}_K \to \mathsf{Vec}_K, V \mapsto V^{**}, f \mapsto f^{**}$  be the (covariant) bidual functor. id:  $\mathsf{Vec}_K \to \mathsf{Vec}_K$  denotes the identity, we set  $u_V: V \to V^{**}, v \mapsto (b_v: V^* \to V^*)$ 

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 $K, \xi \mapsto \xi(v)$ ) then  $u = (u_V)_{V \in \mathsf{Vec}_K}$  is a natural transformation  $u : \mathrm{id} \Rightarrow (\cdot)^{**}$  and restricted to the full subcategory  $\mathsf{Vec}_K^{\mathrm{f.d.}} \subseteq \mathsf{Vec}_K$  on finite dimensional K-vector spaces, it gives a natural isomorphism  $u : \mathrm{id} \Rightarrow (\cdot)^{**}$ 



**Definition 0.23** (important concept). A functor  $F: \mathcal{C} \to \mathcal{D}$  is called an equivalence of categories  $\iff \exists$  functor  $G: \mathcal{D} \to \mathcal{C}$  such that one has natural transforations  $\mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$  and  $\mathrm{id}_{\mathcal{D}} \Rightarrow F \circ G$ . Call  $\mathcal{C}$  and  $\mathcal{D}$  equivalent categories  $\iff \exists$  equivalence of categories  $F: \mathcal{C} \to \mathcal{D}$ .

**Remark.** The notion of "equivalence of categories" is far more important than the notion of isomorphism of categories.

**Example** (linear algebra). Let  $\mathsf{Vec}_K^{\mathsf{std.}}$  be the full subcategory of the  $\mathsf{Vec}_K^{\mathsf{f.d.}}$  on the object set  $\{K^n \mid n \in \mathbb{N}_0\}$ . Then the inclusion  $\iota : \mathsf{Vec}_K^{\mathsf{std.}} \to \mathsf{Vec}_K^{\mathsf{f.d.}}$  is an equivalence of categories. For  $G : \mathsf{Vec}_K^{\mathsf{std}} \to \mathsf{Vec}_K^{\mathsf{std}}$  take  $V \mapsto K^{\dim_K V}$ , choose a basis  $\underline{B}_V$  for any  $V \in \mathsf{Vec}_K^{\mathsf{f.d.}}$  then we get an isomorphism  $K^{\dim_K V} \xrightarrow{\alpha_V} V$ . Define:

$$V \longmapsto K^{\dim_K V}$$

$$f \downarrow \qquad \qquad \downarrow^{\alpha_W^{-1} \circ f \circ \alpha_V}$$

$$W \longmapsto K^{\dim_K W}$$

Find natural isomorphism  $G \circ \iota \Leftarrow id \Rightarrow \iota \circ G$ .

**Remark.** One also calls  $\mathsf{Vec}_K^{\mathrm{std.}}$  a  $\mathit{skeleton}$  of  $\mathsf{Vec}_K^{\mathrm{f.d.}}$ 

**Theorem 0.24.** For a functor  $F: \mathcal{C} \to \mathcal{C}'$  the following are equivalent:

- (i) F is an equivalence of categories.
- (ii) F is fully faithful and essentially surjective.

*Proof.* • (i)  $\Longrightarrow$  (ii): Exercise.

• (ii) 
$$\Longrightarrow$$
 (i): Standard textbook.

**Definition 0.25.** The *essential image* of a functor  $F: \mathcal{C} \to \mathcal{D}$  in  $\mathcal{D}$  is the full subcategory  $\mathcal{D}'$  of  $\mathcal{D}$  on objects isomorphic to FX for some  $X \in \text{Ob } \mathcal{C}$ .

**Corollary 0.26** (of 24 and the definition). Suppose  $F: \mathcal{C} \to \mathcal{D}$  is fully faithful. Let  $\mathcal{D}' \subseteq \mathcal{D}$  be the essential image of F, then  $F: \mathcal{C} \to \mathcal{D}'$  is an equivalence of categories.

### 0.5 The Yoneda lemma and presheaves

Let  $\mathcal{C}, \mathcal{C}', \mathcal{D}$  be categories.

**Definition 0.27.** (a) A  $\mathcal{D}$ -valued presheaf on  $\mathcal{C}$  is a functor

$$\mathcal{F}:\mathcal{C}^{\mathrm{op}}\to\mathcal{D}$$

- (b) The category of  $\mathcal{D}$ -valued presheaves on  $\mathcal{C}$  is  $PSh(\mathcal{C}, \mathcal{D}) = \mathcal{D}^{\mathcal{C}^{op}}$
- (c) If  $\mathcal{D} = \mathsf{Set}$ , then we omit it from the notation, so  $\mathsf{PSh}(\mathcal{C}) = \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ Note that if  $\mathcal{D}$  is a small category, then  $\mathcal{D}^{\mathcal{C}^{\mathsf{op}}}$  is a category.

**Remark** (On the terminoloy). (Pre-)sheaves come from topology/geometry. Example: Let  $(X, \mathcal{T})$  be a topological space (e.g.  $\mathbb{C}$  with the metric topology), For  $U \subseteq X$  define  $O_X(U) := \{f : U \to \mathbb{C} \text{ continuous}\}$  or  $(O_{\mathbb{C}}(U) := \{f : U \to \mathbb{C} \text{ holomorphic }\})$ . Check:

$$\begin{array}{ccc} O_X: \mathrm{ord}(T,\subseteq) & \longrightarrow & \mathsf{Set} \\ & U & \longmapsto & O_X(U) \\ & & & \downarrow^{\mathrm{restriction}} \\ & V & \longmapsto & O_X(V) \end{array}$$

this is a presheaf.

**Definition 0.28.** The Yoneda embedding is the functor  $h: \mathcal{C} \to \mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathsf{Set}), X \mapsto h_X := \mathcal{C}(-,X): \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ 

$$\begin{array}{cccc} h: \mathcal{C} & \longrightarrow \mathsf{PSh}(\mathcal{C}) \\ X & \longmapsto & h_X & \mathrm{Hom}(-,X) \\ f \downarrow & & & \downarrow h_f & & \downarrow f \circ - \\ Y & \longmapsto & h_Y & \mathrm{Hom}(-,Y) \end{array}$$

**Lemma 0.29.** For  $X \in \mathcal{C}, F \in PSh(\mathcal{C}) = Fun(\mathcal{C}^{op}, Set)$ 

$$\operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}(h_X, \mathcal{F}) \xrightarrow{\Phi} \mathcal{F}X$$
$$u := (u_Y : h_X Y \to \mathcal{F}Y)_{Y \in \operatorname{Ob}\mathcal{C}} \longmapsto u_X 1_X$$

is a bijection. (Hom<sub>PSh(C)</sub>( $h_x$ ,  $\mathcal{F}$ ) is a set.)

*Proof.* Reconstruct a natural transformation  $u^{\alpha}$  from  $\alpha \in \mathcal{F}X$ , first consider what  $u \in \operatorname{Mor}_{PSh(\mathcal{C})}(h_x, \mathcal{F})$  gives us

Define  $\psi : \mathcal{F}(X) \to \operatorname{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_X, \mathcal{F}), \alpha \mapsto (u_Y^{\alpha})_{Y \in \mathrm{Ob}\,\mathcal{C}}$  by setting  $u_Y^{\alpha} : h_X(Y) \to \mathcal{F}Y, g \mapsto \mathcal{F}g(\alpha)$ . Check  $u_Y^{\alpha}$  is a natural transforation: For any  $f : Z \to Y$  in  $\mathcal{C}$  we get TODO

**Corollary 0.30.** The functor  $h: \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$  is fully faithful, i.e.  $\mathcal{C}(X,Y) \leftrightarrow \mathrm{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y)$ .

Proof. We need to show  $\forall X, Y \in \text{Ob } \mathcal{C}$  the map  $\mathcal{C}(X,Y) \to \text{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y), f \mapsto h(f) = f \circ - \text{ is bijective. Observe: Yoneda } \Phi : \text{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y) \to h_y(X) = \mathcal{C}(X,Y), u \mapsto u_X(1_X) \text{ is a bijection, so it suffices to show } \Phi \circ h \text{ is a bijection.}$  For this:  $\Phi \circ h(f:X \to Y) = f \circ 1_X = f \implies \Phi \circ h = \text{id.}$ 

**Definition 0.31.** (a) Call  $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$  representable  $\iff \exists X \in \mathcal{C}$  such that  $h_X \cong \mathcal{F}$ .

(b) A presentation of a (representable)  $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$  is a pair  $(X, \alpha)$  with  $X \in \mathsf{Ob}\,\mathcal{C}, \alpha \in \mathcal{F}X$  such that  $\Psi(\alpha) : h_X \Rightarrow \mathcal{F}$  from the proof of lemma 29 is a natural isomorphism.

**Proposition 0.32.** Suppose  $(X, \alpha)$  and  $(Y, \beta)$  are presentations of  $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$ , then  $\exists !$  isomorphism  $f : X \to Y$  such that  $\mathcal{F}(f)(\beta) = \alpha$ 

*Proof.* Exercise.  $\Box$ 

#### 0.6 Conatravariant Yoneda

Proposition 0.33. The functor

$$\begin{array}{ccc} h^{\mathrm{op}}: \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathsf{Set}) \\ X & \longmapsto & \mathcal{C}(X, -) \\ f & & & \uparrow h^{\mathrm{op}}(f) \\ Y & \longmapsto & \mathcal{C}(Y, -) \end{array}$$

is fully faithful and for  $X \in \mathrm{Ob}\,\mathcal{C}$  and  $\mathcal{F}: \mathcal{C} \to \mathsf{Set}$  a functor, the map  $\Phi': \mathrm{Mor}_{\mathsf{Set}^{\mathcal{C}}}(h_X^{\mathrm{op}}, \mathcal{F}) \to \mathcal{F}(X), u \mapsto u_X(1_X)$  is bijective.

*Proof.* (Exercise) Apply Yoneda to  $C^{op}$ .

**Definition 0.34.** (a) A covariant functor  $F: \mathcal{C} \to \mathsf{Set}$  is *corepresentable*  $\iff$   $F \cong h_X^{\mathrm{op}}$  for some  $X \in \mathcal{C}$ .

(b) A presentation of a (corepresentable) functor F is a pair  $(X, \alpha)$  such that  $(\Phi')^{-1}(\alpha)$  is an isomorphism  $h_X^{\text{op}} \to \mathcal{F}$ 

**Proposition 0.35** (analog of 32). If  $(X, \alpha)$  and  $(Y, \beta)$  are 2 presentations of  $F: \mathcal{C} \to \mathsf{Set}$ , then  $\exists !$  isomorphism  $f: X \to Y$  such that  $F(f)(\alpha) = \beta$ .

**Remark.** We mostly drop co- in corepresentable because the functor dictates if it is representable or corepresentable (if F co- or contravariant)

**Remark.** For  $f: X \to Y$  we have

$$f \circ -: \mathcal{C}(Z, X) \to \mathcal{C}(X, Y)$$
 bij.  $\iff h(f) \cong \iff f \cong \iff h^{\mathrm{op}}(f) \cong \iff -\circ f : \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$  bij.

because h is fully faithful.

### 0.7 Universal pairs

- **Definition 0.36.** (a) Let  $F: \mathcal{C} \to \mathcal{D}$  be a functor and  $B \in \text{Ob } \mathcal{D}$ . A pair  $(U,\beta)$  with  $U \in \text{Ob } \mathcal{C}$  and  $\beta: B \to F(U)$  (in  $\mathcal{D}$ ) is (co-)universal for  $(F,\beta): \iff (U,\beta)$  (co-)represents  $\mathcal{D}(B,F(-)) = h_B^{\text{op}} \circ F: \mathcal{C} \to \mathsf{Set}$ .
- (b) Let  $G: \mathcal{D} \to \mathcal{C}$  be a functor and  $A \in \mathrm{Ob}\,\mathcal{C}$ . A pair  $(V, \alpha)$  with  $V \in \mathrm{Ob}\,\mathcal{D}$  and  $\alpha: G(V) \to A$  (in  $\mathcal{C}$ ) is universal for  $(G, A) \iff (V, \alpha)$  represents  $(\mathcal{C}(G(-), A)) = h_A \circ G: \mathcal{D}^{\mathrm{op}} \to \mathsf{Set}$ . TODO: interpretation

Examples 0.37. TODO

#### 0.8 Limits and colimits

Let  $\mathcal{C}$  be a category.

**Definition 0.38.** A diagram in  $\mathcal{C}$  is a functor  $F: J \to \mathcal{C}$  for J a small category (call J the *index category* of the diagram.)

**Remark** (Relation to previous notions of diagrams). Let  $V:\mathsf{Cat}\to\mathsf{Diag}$ , [MacLane II.7]:  $\exists$  functor TODO

**Definition 0.39.** A diagram  $F: J \to \mathcal{C}$  commutes  $\iff \forall i, j \in J$ :

$$\underbrace{F(J(i,j))}_{\text{is a singleton}} \subseteq \mathcal{C}(Fi,Fj)$$

(naive diagram commutes  $\iff F\varphi$  commutes.)

**Definition 0.40.** Let  $F: J \to \mathcal{C}$  denote a diagram in  $\mathcal{C}$ .

- (a) The constant functor from J to  $\mathcal{C}$  for  $X \in \text{Ob } \mathcal{C}$  is  $\Delta X : J \to \mathcal{C}$  with  $\Delta X(i) = X, \forall i \in J \text{ and } \Delta X(h) = 1_X, \forall h \in \text{Mor } J.$
- (b) The diagonal  $\Delta: \mathcal{C} \to \mathcal{C}^J = \operatorname{Fun}(J,\mathcal{C})$  with  $\Delta(X) := \Delta X$  from (a) and  $\Delta(f) :=$  the natural transformation  $\Delta X \Rightarrow \Delta Y$  given for any  $i \in J$  by  $\Delta X(i) = X \xrightarrow{f} \Delta Y(i) = Y$ .
- (c) A cone to  $F: J \to \mathcal{C}$  (any fixed F) with apex  $X \in \mathcal{C}$  is a natural transforation  $\Delta X \Rightarrow F$ . A cocone from F with vertex  $X \in \mathrm{Ob}\,\mathcal{C}$  is a natural transforation  $F \Rightarrow \Delta X$ .
- (d) Cones and cocones give rise to the following functors:
  - Cone $(-,F): \mathcal{C}^{\text{op}} \to \mathsf{Set}$  defined by Cone $(X,F) = \mathsf{set}$  of cones  $\Delta X \Rightarrow F$ , Cone $(f:X\to Y,F)$  maps Cone $(Y,F)\to \mathsf{Cone}(X,F), (\Delta Y \Rightarrow F) \mapsto (\Delta X \Rightarrow \Delta Y \Rightarrow F)$ .
  - Similarly Cocone(F, -) :  $\mathcal{C} \to \mathsf{Set}$  is the functor defined by Cocone(F, X) = set of cocones  $F \Rightarrow \Delta X$  etc.

Observe:

$$Cone(-, F) = \mathcal{C}^{J}(\Delta(-), F)$$
$$Cocone(F, -) = \mathcal{C}^{J}(F, \Delta(-))$$

Visualization: a natural transforation  $u:\Delta X\Rightarrow F$  (where  $X\in\mathcal{C},F:J\to\mathcal{C}$ ) is for any  $i\in J$  a morphism  $X=\Delta X(i)\stackrel{u_i}{\longrightarrow} F(i)$  such that  $\forall h:i\to j$  in J the following diagram commutes:

$$\Delta X(i) = X \xrightarrow{1_X} X = \Delta X(j)$$

$$\downarrow u_i \qquad \qquad \downarrow u_j \qquad \downarrow u_j$$

$$Fi \xrightarrow{Fh} Fj$$

for instance if TODO

**Remark 0.41.** (a) The cones to F form a full subcategory F-cones  $\subseteq \mathcal{C}^J/F$  on objects  $\Delta X \Rightarrow F$  ( $X \in \mathcal{C}$ ).

- (b) Similarly cocones from F form a full subcategory F-cocones  $\subseteq F/\mathcal{C}^J$  on objects  $F\Rightarrow \Delta X$ .
- **Definition 0.42.** (a) If  $Cone(-,F): \mathcal{C}^{op} \to Set$  is representable, the representing object is called a *limit* over F. Notation:  $\lim F$  or  $\lim_J F$  for the representing universal object.
- (b) If  $\operatorname{Cocone}(F, -) : \mathcal{C} \to \mathsf{Set}$  is representable, the representing object is called a *colimit* over F. Notation:  $\operatorname{colim} F$  or  $\operatorname{colim}_J F$

More explicitly:  $\lim_J F$  is an object  $L \in \mathcal{C}$  together with a (universal) ocne  $\Delta L \Rightarrow F$  such that  $\forall$  cones  $\varphi : \Delta X \Rightarrow F$  in  $\mathcal{C}^J \exists !$  morphism  $\psi : X \to L$  such that the diagram commutes



Yoneda implies that  $\lim F$  (if it exists) is unique up to unique isomorphism (similarly for colim F).

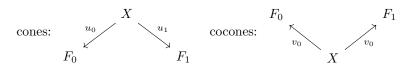
**Exercise 0.43.** (a)  $\lim F$  exists  $\iff$  category of F-cones has a terminal object.

(b)  $\operatorname{colim} F$  exists  $\iff$  category of F-cocones has an initial object.

**Proposition 0.44** (Exercise). Let  $\Delta : \mathcal{C} \to \mathcal{C}^J$  be the diagonal from above and F any diagram in  $\mathcal{C}^J$ . Then

- (a)  $\lim F$  is a universal object for the pair  $(\Delta, F)$  i.e.  $\mathcal{C}^J(\Delta(-), F) \leftrightarrow \mathcal{C}(-, \lim_J F)$ .
- (b) colim F is a couniveral object for the pair  $(\Delta, F)$  i.e.  $\mathcal{C}^J(F, \Delta(-)) = \mathcal{C}(\operatorname{colim}_J F, -)$ .

**Examples 0.45.** (a) J := the discrete category on the set  $\{0,1\}$ , i.e. (Ob =  $\{0,1\}$ , Mor  $J = \{1_0,1_1\}$ ,...). A functor  $F: J \to \mathcal{C}$  is given by the datum of a pair  $(F_0 = F(0), F_1 = F(1))$  of objects of  $\mathcal{C}$ ,



any pair of morphisms  $(u_0: X \to F_0, u_1: X \to F_1)$  resp.  $(v_0: F_0 \to X, v_1: F_1 \to X)$  defines a cone  $\lim F$  resp. colim F and satisfies

$$C(Y, \lim F) = C(Y, F_0) \times C(Y, F_1)$$

resp.

$$C(\operatorname{colim}(F, Z)) = C(F_0, Z) \times C(F_1, Z)$$

If  $\lim F$  exists write formally  $\lim F = F_0 \prod F_1$  (product), and if  $\operatorname{colim} F$  exists write  $F_0 \coprod F_1$  (coproduct). Concretely

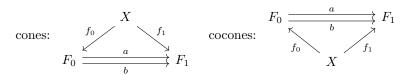
- $C = \mathsf{Set}, F_0 \times F_1 = F_0 \sqcup F_1$
- $C = {}_{R}\mathsf{Mod}, F_0 \times F_1 = F_0 \oplus F_1$
- $C = \mathsf{Grp}, F_0 \times F_1 = F_0 * F_1 \text{ (free product)}$
- (b) We can generalize to arbitrary discrete (small) categories with unterlying set I. Names for universal objects

$$\prod_{i \in I} F_i, \qquad \coprod_{i \in I} F_i$$

defining property:

$$C(\prod_{i \in I} F_i, Y) \stackrel{!}{=} \prod_{i \in I} C(F_i, Y) \text{ and } C(Z, \prod_{i \in I} F_i) \stackrel{!}{=} \prod_{i \in I} C(Z, F_i)$$

- Set,  $\prod_{i \in I} F_i = \bigsqcup_{i \in I} F_i$  (disjoint union)
- $_{R}$ Mod,  $\prod_{i \in I} F_{i} = \bigoplus_{i \in I} F_{i}$  (direct sum)
- Grp,  $\prod_{i \in I} F_i = \bigstar_{i \in I} F_i$  (free product)
- (c)  $J = \text{category on 2 objects } 0, 1 \text{ with 2 morphisms } 0 \Rightarrow 1 \text{ (besides } 1_0, 1_1).$   $F: J \to \mathcal{C}$  is determined by  $F_0 \stackrel{a}{\Rightarrow} F_1$



•  $\mathcal{C}(X, \lim F) = \{ f_0 \in \mathcal{C}(X, F_0) \mid a \circ f_0 = b \circ f_0 = f_1 \}. \lim F \text{ if it exists}$  is called the equalizer eq $(F_0 \stackrel{a}{\Longrightarrow} F_1).$ 

- $\mathcal{C}(\operatorname{colim} F_1, X) = \{ f_1 \in \mathcal{C}(F_1, X) \mid f_1 \circ a = f_1 \circ b \}.$  colim F if it exists is called the  $\operatorname{coequalizer} \operatorname{coeq}(F_0 \overset{a}{\underset{b}{\Longrightarrow}} F_1).$  TODO
- (d) Pullback and pushout:

**Definition 0.46.** C is  $(co\text{-})complete \iff C$  contains all (co)-limits.

**Theorem 0.47.** (a) If C contains all products and equalizers, then C is complete.

(b) If C contains all coproducts and coequalizers, then C is cocomplete.

Corollary 0.48. Set and <sub>R</sub>Mod are complete and cocomplete.

# 0.9 Adjoint Functors

**Definition 0.49.** (a) Functors  $F: \mathcal{C} \to \mathcal{D}$  and  $G: \mathcal{D} \to \mathcal{C}$  form an *adjoint* pair, when one has a natural isomorphism  $\alpha$  of bifunctors

$$\mathcal{C}^{\mathrm{op}} \times \overset{\mathcal{D}(F(-),-)}{\underset{\mathcal{C}(-,G(-))}{\longleftarrow}} \mathsf{Set}$$

In this situation one says that F is a left adjoint for G and G is a right adjoint for F. We write  $F \dashv G$  or  $C \xrightarrow{F} \mathcal{D}$ .

- (b) The tuple  $(F, G, \alpha)$  is called an adjunction.
- (c) We say  $F: \mathcal{C} \to \mathcal{D}$  has a right adjoint if  $\exists G: \mathcal{D} \to \mathcal{C}: F \dashv G$  (similarly  $G: \mathcal{D} \to \mathcal{C}$  has left adjoint.)

**Theorem 0.50.** For functors  $F; \mathcal{C} \to \mathcal{D}$  and  $G: \mathcal{D} \to \mathcal{C}$  we have:

- (a) F has a right adjoint  $\iff \forall B \in \mathcal{D} \exists$  universal pair  $(A, v : FA \to B)$  for (F, B) such that  $\operatorname{Hom}_{\mathcal{D}}(F(-), B) \cong \operatorname{Hom}_{\mathcal{C}}(-, A)$ .
- (b) G has a left adjoint  $\iff \forall A \in \mathcal{C} \exists \ universal \ pair \ (B, u : A \to GB) \ for \ (G, A) \ such \ that \ \operatorname{Hom}_{\mathcal{C}}(A, G(-)) \cong \operatorname{Hom}_{\mathcal{D}}(B, -).$