0.1 Exact functors

Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor.

Definition 0.1. F is called

(a) *left exact* $\iff$ F commutes with finite limits.

(b) *right exact* $\iff$ F commutes with finite colimits.

(c) *exact* $\iff$ F is left and right exact.

Remark. Since $\mathcal{A}, \mathcal{A}'$ are additive categories, all finite limits and colimits exist in $\mathcal{A}$ and $\mathcal{A}'$. So if $D : J \to \mathcal{A}$ is a finite diagram, we have $\lim_J D$ exists in $\mathcal{A}$, $\lim_J F \circ D$ exists in $\mathcal{A}'$ and we have a natural morphism

$$F(\lim_J D) \to \lim_J F \circ D$$

in $\mathcal{A}'$. F is left exact if this morphism is an isomorphism $\forall D$.

Proposition 0.2. *Let $F : \mathcal{A} \to \mathcal{A}'$ be additive, then:*

(a) The following are equivalent:

- *(i) F is left exact.*
- *(ii) F commutes with the formation of kernels, i.e. $\forall f : X \to Y$ in $\mathcal{A}$, the natural morphism*
  $$F(\ker f) \to \ker F(f)$$
  is an isomorphism.
- *(iii) $\forall$ exact sequence $0 \to X' \to X \to X''$ in $\mathcal{A}$, the sequence $0 \to FX' \to FX \to FX''$ is exact in $\mathcal{A}'$.*
- *(iv) $\forall$ exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, the sequence $0 \to FX' \to FX \to FX''$ is exact in $\mathcal{A}'$.*

(b) The following are equivalent:

- *(i) F is right exact.*
- *(ii) F commutes with the formation of cokernels.*
- *(iii) $\forall$ exact sequence $X' \to X \to X'' \to 0$ in $\mathcal{A}$, the sequence $FX' \to FX \to FX'' \to 0$ is exact in $\mathcal{A}'$.*
- *(iv) $\forall$ exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, the sequence $FX' \to FX \to FX'' \to 0$ is exact in $\mathcal{A}'$.*

(c) F is exact $\iff$ $\forall$ exact sequences $X' \xrightarrow{f} X \xrightarrow{g} X''$, the sequence

$$FX' \xrightarrow{Ff} FX \xrightarrow{Fg} FX''$$

is exact.

Proof. TODO. $\square$

Proposition 0.3. *$\forall X \in \mathcal{A}$ the co- and contravariant Hom functors* $\mathrm{Hom}_{\mathcal{A}}(X, -), \mathrm{Hom}_{\mathcal{A}}(-, X)$ *are left exact.*

Proof. TODO. $\square$

Proposition 0.4. *Let $F : \mathcal{A} \to \mathcal{A}'$ and $G : \mathcal{A}' \to \mathcal{A}$ be additive functors with $F \dashv G$. Then F is right exact and G is left exact.*

Proof. TODO. $\square$

Example 0.5. By Hom-$\otimes$ Adjunction (roughly $\mathrm{Hom}(M\otimes -, N) = \mathrm{Hom}(M, \mathrm{Hom}(-, N))$) $M \otimes -$ is left adjoint to $\mathrm{Hom}(-,) \implies M \otimes_R - : {}_R\mathsf{Mod} \to \mathsf{Ab}$ is right exact, as is $- \otimes_R M : \mathsf{Mod}_R \to \mathsf{Ab}$.

$$(- \otimes_R M : \mathsf{Mod}_R \to Ab) = (M \otimes_{R^{\mathrm{op}}} - : {}_{R^{\mathrm{op}}}\mathsf{Mod} \to \mathsf{Ab})$$

Remark. Next small goal: J any small index category. Are $\lim_J : \mathcal{A}^J \to \mathcal{A}, \operatorname{colim}_J \mathcal{A}^J \to \mathcal{A}$ exact?

Proposition 0.6. $\mathcal{A}^J = \mathrm{Fun}(J, \mathcal{A})$ *is an abelian category.*

Proposition 0.7. *Suppose $\mathcal{A}$ contains all limits (colimits) for a given small index category J, then $\lim_J : \mathcal{A}^J \to \mathcal{A}$ is left exact. ($\operatorname{colim}_J : \mathcal{A}^J \to \mathcal{A}$ is right exact.)*

Corollary 0.8. *Let I be a set and $\underline{I}$ the discrete category associated to I. Then:*

(a) $\prod_{i \in I} : A^I \to A, (A_i)_{i\in I} \mapsto \prod_{i\in I} A_i$

(b) $\bigoplus_{i\in I} : A^I \to A, (A_i)_{i\in I} \mapsto \bigoplus_{i\in I} A_i)$ *assuming existance, are exact functors.*

Definition 0.9. (a) A non-empty category J is called *filtered* iffy

(i) $\forall i, j \in J : \exists$ diagram
$$\begin{tikzcd} i \arrow[dr] & \\ j \arrow[r] & k \end{tikzcd} \in J.$$

(ii) $\forall f, g : i \rightrightarrows j \exists h : j \to k$ such that

$$h \circ f = h \circ g : i \to k$$

(b) A directed poset $(I, \subseteq)$ is a poset such that $\forall i, j \in I \exists h \in I$ such that $h \geq i, h \geq j$ ($\implies (I, \subseteq)$ is directed $\implies \mathrm{ord}(I, \subseteq)$ is filtered.)

Definition 0.10. $\mathcal{A}$ has exact filtered colimits $\iff$ for each filtered index category J, the functor $\operatorname{colim} \mathcal{A}^J \to \mathcal{A}$ is exact (and defined).

Theorem 0.11. ${}_R\mathsf{Mod}$ *has exact filtered colimits.*

Fundamental question: how to investigate the non-exactness of functors (that are left or right exact). Answer of homological algebra: e.g. $F : A \to B$ left exact, define "higher right derived functors" $(R^iF)_{i\geq 0}$ from F such that $\forall$ s.e.s.

$$0 \to X' \to X \to X'' \to 0$$

in A

$$\begin{tikzcd}
0 \arrow[r] & FX' \arrow[r] & FX \arrow[r] & FX'' \arrow[dll] \\
& R^1FX' \arrow[r] & R^1FX \arrow[r] & R^1FX'' \arrow[dll] \\
& R^2FX' \arrow[r] & R^2FX \arrow[r] & R^2FX'' \arrow[r] & \cdots
\end{tikzcd}$$

and $R^0F = F$. Study R^iF to understand the nonexactness of F, or to gain insight into some invariants of $\mathcal{A}$. Some R^iF (typically $i \leq 3$) have concrete meanings.

0.1.1 To define R^iF (or L_iF)

One wants "enough" injectives (projectives) in $\mathcal{A}$.

Theorem-Definition 0.12. *For $I \in \mathcal{A}$ the following are equivalent:*

(i) $\forall$ diagrams with ι monomorphism $\exists$ extension $g : B \to I$ such that the following commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "f"'] & B \arrow[dl, dashed, "g"] \\
& I &
\end{tikzcd}$$

(ii) The functor $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ *is exact.*

(iii) Every s.e.s. $0 \to I \xrightarrow{h} C \xrightarrow{k} D \to 0$ *in $\mathcal{A}$ is split.*

If any of these hold then I is called an injective object.

Theorem-Definition 0.13. *For $P \in \mathcal{A}$ the following are equivalent:*

(i) $\forall$ diagrams with π epimorphism $\exists$ lifting $g : P \to B$ such that $\pi \circ g = f$

$$\begin{tikzcd}
& P \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\
B \arrow[r, "\pi"'] & C \arrow[r] & 0
\end{tikzcd}$$

(ii) The functor $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathsf{Ab}$ *is exact.*

(iii) Every s.e.s. $0 \to A \to B \to P \to 0$ *is split.*

If these hold, then P is called a projective object in $\mathcal{A}$.

Remark. I injective in $\mathcal{A}$ $\iff$ I projective in $\mathcal{A}^{\mathrm{op}}$.

Proposition 0.14. *(a) If $(I_j)_{j \in J}$ (J a set) is a family of injectives in $\mathcal{A}$, such that $\prod_J I_j$ exist, then $\prod_J I_j$ is injective.*

(b) If $(P_j)_{j \in J}$ (J a set) is a family of projectives in $\mathcal{A}$, such that $\bigoplus_{j \in J} P_j$ exist, then $\bigoplus_J P_j$ is projective.

Example 0.15. (a) R is a projective R-module (hence so is $R^{(I)}$)

(b) $\mathbb{Q}/\mathbb{Z}$ is an injective in ${}_{\mathbb{Z}}\mathsf{Mod}$.

Definition 0.16. (a) $\mathcal{A}$ has enough injective $\iff \forall X \in \mathcal{A} \exists$ monomorphism $X \to I$ with $I \in \mathcal{A}$ injective.

(b) $\mathcal{A}$ has enough projectives $\iff \forall X \in \mathcal{A} \exists$ epimorphism $P \to X$ with $P \in \mathcal{A}$ projective.

Definition 0.17. $Q \in \mathcal{A}$ is called a

(a) *generator* $\iff \mathrm{Hom}_{\mathcal{A}}(Q, -) : \mathcal{A} \to \mathsf{Ab}$ is faithful.

(b) *cogenerator* $\iff \mathrm{Hom}_{A}(-, Q) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ is faithful.

Remark 0.18. For $Q \in \mathcal{A}$ the following are equivalent:

1. Q is a generator.

2. $\forall X, Y \in \mathcal{A} \forall f, g \in \mathcal{A}(X, Y)$:

$$f \neq g \implies \exists h \in \mathcal{A}(Q, X) : f \circ h \neq g \circ h$$

3. $\forall X, Y \in \mathcal{A} \forall f \in \mathcal{A}(X, Y)$:

$$f \neq 0 \implies \exists h \in \mathcal{A}(Q, X) : f \circ h = 0$$

Examples 0.19. 1. R is a generator for ${}_R\mathsf{Mod}$: Suppose $f : M \to N$ in ${}_R\mathsf{Mod}$ is nonzero. Then $\exists m \in M : f(m) \neq 0$. Define $h : R \to M, r \mapsto rm \implies f \circ h : R \to N$ is nonzero.

2. $\mathbb{Q}/\mathbb{Z}$ is a cogenerator in $\mathsf{Ab} = {}_{\mathbb{Z}}\mathsf{Mod}$ let $f : M \to N$ be non-zero in Ab. Let $X \in \mathrm{im}(f) \setminus \{0\}$. TODO

Definition 0.20. $\mathcal{A}$ is called a *Grothendieck abelian category* (GAC) iffy

1. $\mathcal{A}$ is cocomplete.

2. For any filtered small category J : colim : $A^J \to A$ is exact.

3. $\mathcal{A}$ possesses a cogenerator.

Theorem 0.21. *For a GAC $\mathcal{A}$ the following hold:*

1. *The subobjects and quotient objects of any $A \in \mathcal{A}$ form a set.*

2. *$\mathcal{A}$ has enough injectives.*

3. *$\mathcal{A}$ has an injective cogenerator.*

4. *$\mathcal{A}$ is complete.*

Example 0.22. 1. ${}_R\mathsf{Mod}$ is a GAC by examples 19(a) Thm 11 Cor II48

2. If $\mathcal{A}$ is a GAC and J is a small category then $\mathsf{PSh}(J, A)$ is a GAC

Lemma 0.23. *Suppose $F : \mathcal{A} \to \mathcal{A}'$ and $G : \mathcal{A}' \to \mathcal{A}$ are functors such that $F \dashv G$, then:*

1. *F exact $\implies$ G maps injectives in $\mathcal{A}'$ to injectives in $\mathcal{A}$.*
2. *G exact $\implies$ F maps projectives in $\mathcal{A}$ to projectives in $\mathcal{A}'$.*
3. *F faithful $\implies$ G maps cogenerators in $\mathcal{A}'$ to cogenerators in $\mathcal{A}$.*
4. *G faithful $\implies$ F maps generators in $\mathcal{A}$ to generators in $\mathcal{A}'$.*

Theorem 0.24. *Let $\mathcal{A}$ be a GAC with generator Q, then:*

Corollary 0.25. *$\mathbb{Q}/\mathbb{Z}$ is injective in* Ab.

Corollary 0.26. *For any ring R, $\mathrm{Hom}(R, \mathbb{Q}/\mathbb{Z})$ is an injective R-module and a cogenerator.*

Proposition 0.27. *Let $Q, J \in \mathcal{A}$, then*

1. *Suppose $\mathcal{A}$ contains all coproducts over index sets. Then the following are equivalent:*
   - *(a) Q is a generator.*
   - *(b) $\forall X \in \mathcal{A} \exists$ set $I, \exists$ epimorphism $Q^{(I)} := \coprod_{i \in I} Q \to X$. Moreover, if Q is a rojective generator then $\mathcal{A}$ has enough projectievs of the form $Q^{(I)}$, where I is a set.*
2. *Suppose $\mathcal{A}$ contains all products over index sets then the following are equivalent:*
   - *(a) J is a cogenerator.*
   - *(b) $\forall Y \in \mathcal{A} \exists$ set I, monomorphism $Y \to J^I := \prod_{i \in I} J$. Moreover if J is an injective cogenerrator then $\mathcal{A}$ has enough injectives of the form J^I where I is a set.*

Corollary 0.28. *${}_R\mathsf{Mod}$ has enough injectives of the form $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})^I$, I a set.*

Remark 0.29. TODO

0.2 (Co-)chain complexes, (co-)homology

Let $\mathcal{A}$ be an additive category.

Definition 0.30. (a) A *chain complex* (C_*, ∂_*) over $\mathcal{A}$ is a sequence $(\partial_i : C_i \to C_{i-1})_{i \in \mathbb{Z}}$ of morphisms in $\mathcal{A}$ such that $\partial_i \circ \partial_{i+1} = 0, \forall i \in \mathbb{Z}$. The map ∂_i is called the i-th *differential* or i-th *boundary map* of the complex. A morphism $f_* : C_* \to D_*$ of chain complexes is a sequence of morphisms $f_* = (f_i : C_i \to D_i)_{i \in \mathbb{Z}}$ such that the diagram commutes $\forall i \in \mathbb{Z}$

$$\begin{tikzcd} C_{i-1} \arrow[d, "f_{i-1}"'] & C_i \arrow[l, "\partial_i^C"'] \arrow[d, "f_i"] \\ D_{i-1} & D_i \arrow[l, "\partial_i^D"] \end{tikzcd}$$

The *category of chain complexes* over $\mathcal{A}$ is denoted $\mathsf{Ch}_*(\mathcal{A})$.

(b) A *cochain complex* (C^*, δ^*) over $\mathcal{A}$ is a sequence $(\delta^i : C^i \to C^{i+1})_{i\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$, such that $\delta^{i+1} \circ \delta^i = 0, \forall i \in \mathbb{Z}$. A morphism $f^* : C^* \to D^*$ of cochain complexes is a sequence $f^* = (f^i : C^i \to D^i)_{i\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$ such that the following diagram commutes $\forall i \in \mathbb{Z}$

$$\begin{tikzcd} D_i \arrow[r, "\delta_D^i"] & D_{i+1} \\ C_i \arrow[u, "f_i"] \arrow[r, "\delta_C^i"] & C_{i+1} \arrow[u, "f_{i+1}"'] \end{tikzcd}$$

The *category of cochain complexes* over $\mathcal{A}$ is denoted $\mathsf{Ch}^*(\mathcal{A})$

Exercise 0.31. TODO

In the following we do most things only for cochain complexes.

Definition 0.32. (a) The *support* of a cochain complex $C^* \in \mathsf{Ch}^*(\mathcal{A})$ is $\operatorname{supp} C = \{i \in \mathbb{Z} \mid C^i \neq 0\}$

(b) The full subcategory of $\mathsf{Ch}^*(\mathcal{A})$ on complexes supported on $\mathbb{N}_0$ (or on $-\mathbb{N}_0$) is denoted $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$ (or $\mathsf{Ch}^*_{\leq 0}(\mathcal{A})$).

Proposition 0.33. *If $\mathcal{A}$ is additive (or abelian), then so are* $\mathsf{Ch}^*(\mathcal{A}), \mathsf{Ch}^*_{\geq 0}(\mathcal{A}), \mathsf{Ch}^*_{\leq 0}(\mathcal{A})$.

Definition 0.34. For $i \in \mathbb{Z}$ define left shift by i by

$$\begin{tikzcd} \mathsf{Ch}^*(\mathcal{A}) \arrow[r] & \mathsf{Ch}^*(\mathcal{A}) \\ C \arrow[r, mapsto] \arrow[d, "f"'] & C[i] \arrow[d, "f[i]"] \\ D \arrow[r, mapsto] & D[i] \end{tikzcd}$$

where $C[i]^n = C^{n+i}, \delta^n_{C[i]} = \delta^{n+i}_C$ and $f[i]^n = f^{n+1}$.

Convention 0.35. We regard $\mathcal{A}$ as a subcategory of $\mathsf{Ch}^*(\mathcal{A})$, as the subcategory of complexes C^* with $\operatorname{supp} C^* \subseteq \{0\}$. Identify $X \in \mathcal{A}$ with complex

$$cdots \to 0 \to 0 \to X \to 0 \to 0 \to \cdots$$

Example 0.36. TODO

0.3 Double complexes

Can iterate the formation of $\mathcal{A} \to \mathsf{Ch}^*(\mathcal{A})$.

Definition 0.37. The category of double (cochain) complexes is $\mathsf{Ch}^{**}(\mathcal{A}) := \mathsf{Ch}^*(\mathsf{Ch}^*(\mathcal{A}))$, so objects of $\mathsf{Ch}^{**}(\mathcal{A})$ are complexes of complexes

Definition 0.38. $C \in \mathsf{Ch}^{**}(\mathcal{A})$ is called *bounded* $\iff \forall k \in \mathbb{Z} : \#\{(i,j) \in \mathbb{Z}^2 \mid i + j = k, C^{ij} \neq 0\} < \infty$. Write $\mathsf{Ch}^{**}_b(\mathcal{A}) \subseteq \mathsf{Ch}^{**}(\mathcal{A})$ for the full subcategory on bounded double complexes.

Exercise 0.39. If $\mathcal{A}$ is additive or abelian, then so is $\mathsf{Ch}^{**}_b(\mathcal{A})$.

Definition 0.40. The *total complex* $\mathrm{Tot}(C)$ of $C \in \mathsf{Ch}_b^{**}(\mathcal{A})$ is the complex $\overline{C} \in \mathsf{Ch}^*(\mathcal{A})$ defined as follows:

$$\overline{C}^k := \bigoplus_{\substack{(i,j)\in\mathbb{Z}^2\\ i+j=k}} C^{ij}$$

$\delta^k_{\overline{C}}$ is constructed as follows:

$$\delta^{ij} : C^{ij} \xrightarrow{(\delta_1^{ij},(-1)^i\delta_2^{ij})} C^{i+1,j} \oplus C^{i,j+1} \hookrightarrow \bigoplus_{i'+j'=k+1} C^{i'j'} = \overline{C}^{k+1}$$

Use the universal property of the direct sum to define

$$\delta^k_{\overline{C}} = \bigoplus_{i+j=k} \delta^{ij} : \overline{C}^k \to \overline{C}^{k+1}$$

TODO

Exercise 0.41. TODO

Definition 0.42. For $C = (C^*, \delta^*) \in \mathsf{Ch}^*(\mathcal{A})$ define:

- $Z^i(C) := \ker(\delta^i)$ as the i-th cocycle object.
- $B^i(C) := \mathrm{im}(\delta^{i-1})$ as the i-th coboundary object.
- $u^i(C) : B^i(C) \to Z^i(C)$ the canonical monomorphism.
- $H^i(C) := \mathrm{coker}(u^i(C)) = Z^i(C)/B^i(C)$ as the i-th cohomology object.

(Co-)homology measures the non-exactness of the complex C.

Lemma 0.43 (Alternative description of cohomology object)**.** *For $C \in \mathsf{Ch}^*(\mathcal{A})$ consider TODO*

Lemma 0.44 (exer)**.** *TODO*

Theorem 0.45. *(a) Given a s.e.s.*

$$\mathcal{E} : 0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$$

one obtains a long exact sequence

$$\cdots \to H^i(C) \xrightarrow{H^i(f)} H^i(D) \xrightarrow{H^i(g)} H^i(E) \xrightarrow{d^i_{\mathcal{E}}} H^{i+1}(C) \xrightarrow{H^{i+1}(f)} \cdots$$

for $i \in \mathbb{Z}$, where the connecting homomorphism $d^i_{\mathcal{E}}$ is defined by the snake lemma and $\mathcal{E} \mapsto d^i_{\mathcal{E}}$ is "functorial".

(b) Given a morphism of short exact sequences $\varphi : \mathcal{E} \to \mathcal{E}'$ in $\mathsf{Ch}^*(\mathcal{A})$:

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] \arrow[d, "\varphi"'] & C \arrow[r, "f"] \arrow[d, "\alpha"] & D \arrow[r, "g"] \arrow[d, "\beta"] & E \arrow[r] \arrow[d, "\gamma"] & 0 \\
\mathcal{E}' : 0 \arrow[r] & C' \arrow[r, "f'"'] & D' \arrow[r, "g'"'] & E' \arrow[r] & 0
\end{tikzcd}$$

one obtains a commutative ladder of long exact sequences from (a)

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(C) \arrow[r, "H^i(f)"] \arrow[d, "H^i(\alpha)"] & H^i(D) \arrow[r, "H^i(g)"] \arrow[d, "H^i(\beta)"] & H^i(E) \arrow[r, "d^i_{\mathcal{E}}"] \arrow[d, "H^i(\gamma)"] & H^{i+1}(C) \arrow[r] \arrow[d, "H^{i+1}(\alpha)"] & \cdots \\
\cdots \arrow[r] & H^i(C') \arrow[r, "H^i(f')"'] & H^i(D') \arrow[r, "H^i(g')"'] & H^i(E') \arrow[r, "d^i_{\mathcal{E}'}"'] & H^{i+1}(C') \arrow[r] & \cdots
\end{tikzcd}$$

Definition 0.46. $C \in \mathsf{Ch}^*(\mathcal{A})$ is called *acyclic* if C is exact $\iff$ $H^i(C) = 0, \forall i \in \mathbb{Z}$.

Corollary 0.47 (To theorem 45)**.** *Let $\mathcal{E} : 0 \to C' \to C \to C'' \to 0$ be a s.e.s. in $\mathsf{Ch}^*(\mathcal{A})$, then if any two of C', C, C'' are acyclic, then so is the third.*

Theorem 0.48 (Acyclicity criterion for total complexes)**.** *Let $C \in \mathsf{Ch}_b^{**}(\mathcal{A})$ such that*

(a) Each row $(C^{ij}, \delta_1^{ij})_{i\in\mathbb{Z}}$ is acyclic $\forall j \in \mathbb{Z}$, or

(b) Each column $(C^{ij}, \delta_1^{ij})_{j\in\mathbb{Z}}$ is acyclic $\forall i \in \mathbb{Z}$

then $\mathrm{Tot}(C)$ *is acyclic.*

Definition 0.49. An arrow $f : C \to D$ in $\mathsf{Ch}^*(\mathcal{A})$ is called a *quasi-isomorphism* (quism) $\iff$ $\forall i \in \mathbb{Z} : H^i(f) : H^i(C) \to H^i(D)$ is an isomorphism.

Lemma 0.50. *TODO*

Corollary 0.51. *For $f : C \to D$ in $\mathsf{Ch}^*(\mathcal{A})$ the following are equivalent:*

(a) f is a quasi-isomorphism.

(b) $\mathrm{Cone}\, f$ *is acyclic.*

Definition 0.52. Let $f, g : C \to D$ in $\mathsf{Ch}^*(\mathcal{A})$.

(a) A *homotopy* from f to g is a sequence of morphisms $(s^i : C^i \to D^{i-1})_{i\in\mathbb{Z}}$ such that $\forall i \in \mathbb{Z} : f^i - g^i = \delta_D^{i-1} \circ s^i + s^{i+1} \circ \delta_C^i$, i.e.

$$\begin{tikzcd}
 & C^i \arrow[r, "\delta_C^i"] \arrow[dl, "s^i"'] \arrow[d, "f^i - g^i"] & C^{i+1} \arrow[dl, "s^{i+1}"] \\
D^{i-1} \arrow[r, "\delta_D^{i-1}"'] & D^i &
\end{tikzcd}$$

(b) f is called *homotopic* to g if $\exists$ homotopy from f to g. Write $f \sim g$.

(c) f is called *nullhomotopic* if $f \sim 0$.

Note: This definition only requires that $\mathcal{A}$ is additive.

Proposition 0.53. *(a) For $C, D \in \mathsf{Ch}^*(\mathcal{A})$, homotopy defines an equivalence relation on* $\mathrm{Hom}_{\mathsf{Ch}^*(\mathcal{A})}(C, D)$.

(b) For $f, f' : C \to D$ and $g, g' : D \to E$ in $\mathsf{Ch}^*\mathcal{A}$ one has:

$$f \sim f', g \sim g' \implies g \circ f \sim g' \circ f'$$

(c) *Suppose $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, then one has a functor $F : \mathsf{Ch}^*\mathcal{A} \to \mathsf{Ch}^*\mathcal{B}$ by*

$$F(\delta^n : C^n \to C^{n+1})_{n\in\mathbb{Z}} = (F\delta^n : FC^n \to FC^{n+1})_{n\in\mathbb{Z}}$$

This functor preserves homotopy, i.e. $f \sim g \implies Ff \sim Fg$.

Proposition 0.54. *Suppose $f, g : C \to D$ in $\mathsf{Ch}^*\mathcal{A}$ are homotopic, then*

$$H^i(f) = H^i(g) : H^i(C) \to H^i(D), \forall i \in \mathbb{Z}$$

Definition 0.55. A morphism $f : C \to D$ in $\mathsf{Ch}^*\mathcal{A}$ is called a *homotopy equivalence* from C to D if $\exists g : D \to C$ in $\mathsf{Ch}^*\mathcal{A}$ such that

$$g \circ f \sim 1_C, \quad f \circ g \sim 1_D$$

and in this case C and D are called homotopy equivalent.

Proposition 0.56. *Suppose $f : C \to D$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Example. TODO

0.4 Injective and projective resolutions

Notation. Inj or $\mathrm{Inj}_{\mathcal{A}}$ and Proj or $\mathrm{Proj}_{\mathcal{A}}$ are the full subcategories of $\mathcal{A}$ on injective or projective objects respectively. Note that $\mathrm{Inj}_{\mathcal{A}}^{\mathrm{op}} = \mathrm{Proj}_{\mathcal{A}^{\mathrm{op}}}$.

Definition 0.57. (a) An *injective resolution* of $A \in \mathcal{A}$ is a quism $f : \underline{A} \to I$ in $\mathsf{Ch}^*_{\geq 0}\mathcal{A}$ with $I \in \mathsf{Ch}^*_{\geq 0}(\mathrm{Inj}_{\mathcal{A}})$ i.e. $\mathrm{Cone}\, f$, which is

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

is acyclic and all the I^j are injective.

(b) A *projective resolution* of $A \in \mathcal{A}$ is a quism $g : P \to \underline{A}$ in $\mathsf{Ch}^*_{\leq 0}\mathcal{A}$ with $P \in \mathsf{Ch}^*_{\leq 0}(\mathrm{Proj}\,\mathcal{A})$.

Proposition 0.58. *Consider the functor*

$$\begin{tikzcd} \widehat{\cdot}: \mathsf{Ch}^*(\mathcal{A})^{\mathrm{op}} \arrow[r] & \mathsf{Ch}^*(\mathcal{A}^{\mathrm{op}}) \\ (C^n, \delta^n_C) \arrow[r, "\widehat{\cdot}"] \arrow[d, "f"'] & (\widehat{C}^n, \widehat{\delta}^n_C) \\ (D^n, \delta^n_D) \arrow[r, "\widehat{\cdot}"] & (\widehat{D}^n, \widehat{\delta}^n_D) \arrow[u, "\widehat{f}"'] \end{tikzcd}$$

where $\widehat{C}^n = C^{-n}$, $\widehat{\delta}^n_C = \delta^{-n-1} \in \mathcal{A}(C^{-n-1}, C^{-n}) = \mathcal{A}^{\mathrm{op}}(\widehat{C}^n, \widehat{C}^{n+1})$ and $\widehat{f}^n := f^{-n} \in \mathcal{A}(C^{-n}, D^{-n}) = \mathcal{A}^{\mathrm{op}}(\widehat{D}^n, \widehat{C}^n)$. Then

(a) $\widehat{\cdot}$ is well defined, satisfies $\widehat{\cdot} \circ \widehat{\cdot} = \mathrm{id}$ and $\widehat{\cdot}$ is an isomorphism of categories.

(b) $\widehat{\ }(\mathsf{Ch}^*_{\geq 0/\leq 0}(\mathcal{A})^{\mathrm{op}}) = \mathsf{Ch}^*_{\leq 0/\geq 0}(\mathcal{A}^{\mathrm{op}})$.

(c) $\widehat{\ }(\mathsf{Ch}^*_{\geq 0/\leq 0}(\mathrm{Inj}_{\mathcal{A}} / \mathrm{Proj}_{\mathcal{A}}))^{\mathrm{op}} = \mathsf{Ch}^*_{\leq 0/\geq 0}(\mathrm{Proj}_{\mathcal{A}^{\mathrm{op}}} / \mathrm{Inj}_{\mathcal{A}^{\mathrm{op}}})$

(d) If $\underline{A} \to I$ *is an injective resolution in* $\mathsf{Ch}^* \geq 0(\mathcal{A})$*, then* $\widehat{I} \to \underline{A}$ *is a projective resolution in* $\mathsf{Ch}^*_{\leq 0}(\mathcal{A}^{\mathrm{op}})$

Theorem 0.59. *Let* $\mathcal{A}$ *be an abelian category with enough injectives, then*

(a) Each $A \in \mathcal{A}$ *possesses an injective resolution.*

(b) Let $h : A \to B$ *be a morphism, let* $f : \underline{B} \to I^\bullet$ *be an injective resolution and* $g : \underline{A} \to C^\bullet$ *be a quism in* $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$ *(*$C^\bullet$ *is a resolution of* A*), then there is a commutative diagram in* $\mathsf{Ch}^*_{\geq 0}\mathcal{A}$

$$\begin{tikzcd} \underline{A} \arrow[r, "g"] \arrow[d, "h"'] & C^\bullet \arrow[d, dashed, "H"] \\ \underline{B} \arrow[r, "f"] & I^\bullet \end{tikzcd}$$

(c) If diagram in (b) commutes with $H, H' : C^\bullet \to I^\bullet$*, then* $H' \sim H$*.*

Corollary 0.60. *Suppose* $\underline{A} \xrightarrow{g} I^\bullet$ *and* $\underline{A} \xrightarrow{g} J^\bullet$ *are inj resolutions of* A*. Then:*

(a) $\exists H : I^\bullet \to J^\bullet$ *such that*

$$\begin{tikzcd} & \underline{A} \arrow[dl, "g"'] \arrow[dr, "f"] & \\ J^\bullet & & I^\bullet \arrow[ll, dashed, "H"] \end{tikzcd}$$

commutes.

(b) H *in (a) is always a homotopy equivalence.*

Lemma 0.61 (Horseshoe 2)**.** *Let* $0 \to A' \to A \to A'' \to 0$ *be a s.e.s. in* $\mathcal{A}$*, let* $\underline{A}' \xrightarrow{f'} I^\bullet$ *and* $\underline{A}'' \xrightarrow{f''} J^\bullet$ *be injective resolutions. Then* $\exists$ *commutative diagram:*

$$\begin{tikzcd} 0 \arrow[r] & \underline{A}' \arrow[r] \arrow[d, "f'"] & \underline{A} \arrow[r] \arrow[d, dashed, "f"] & \underline{A}'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & I^\bullet \arrow[r] & K^\bullet \arrow[r] & J^\bullet \arrow[r] & 0 \end{tikzcd}$$

in $\mathsf{Ch}^*_{\geq 0}\mathcal{A}$ *with exact rows and an injective resolution* $f : \underline{A} \to K^\bullet$*. Moreover* $\forall n \geq 0$ *the following commutative diagram has exact rows and columns:*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathrm{im}(\delta_I^{n-1}) \arrow[r] \arrow[d] & \mathrm{im}(\delta_K^{n-1}) \arrow[r] \arrow[d] & \mathrm{im}(\delta_J^{n-1}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^n \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & J^n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{im}(\delta_I^n) \arrow[r] \arrow[d] & \mathrm{im}(\delta_K^n) \arrow[r] \arrow[d] & \mathrm{im}(\delta_J^n) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

So $f' = \delta_I^{-1}, f = \delta_K^{-1}$ *and* $f'' = \delta_J^{-1}$.

Definition 0.62. Define $\mathsf{Ex}_{\mathcal{A}}$ as the category of s.e.s. in $\mathcal{A}$ with objects:

$$\mathcal{E}_A : 0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, and morphisms are commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$ with exact rows represented by $\underline{f} = (f', f, f'')$ in $\mathsf{Ex}_{\mathcal{A}}(\mathcal{E}_A, \mathcal{E}_B)$. Composition of arrows is componentwise. TODO

Proposition 0.63. *(a)* $\mathsf{Ex}_{\mathcal{A}}$ *is an additive category.*

(b) We have additive functors $\mathrm{pr}_i : \mathsf{Ex}_{\mathcal{A}} \to \mathcal{A}$, *mapping* $0 \to A_1 \to A_2 \to A_3 \to 0$ *to* A_i.

(c) $\mathsf{Ex}_{\mathcal{A}}$ *is not abelian.*

Definition 0.64. An arrow $\underline{f} = (f', f, f'')$ in $\mathsf{Ex}_{\mathcal{A}}$ is called

(a) a *strict monomorphism (strict epimorphism)* $\iff$ f', f, f'' are monics (epics) in $\mathcal{A}$.

(b) *strict* $\iff$

$$0 \to \ker f' \to \ker f \to \ker f'' \to 0$$

is exact which is (by the snake lemma) equivalent to exactness of

$$0 \to \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f'' \to 0$$

Proposition 0.65. *If* $\underline{f}$ *is strict in* $\mathrm{Mor}(\mathsf{Ex}_{\mathcal{A}})$ *then* $\ker \underline{f}, \operatorname{coker} \underline{f}, \operatorname{im} \underline{f}, \operatorname{coim} \underline{f}$ *exist in* $\mathsf{Ex}_{\mathcal{A}}$ *and the canonical map* $\operatorname{coim} \underline{f} \to \operatorname{im} \underline{f}$ *is an isomorphism.*

Remark. $\mathsf{Ex}_{\mathcal{A}}$ is an exact category.

Definition 0.66. (a) A complex $\mathcal{E}^\bullet = (\underline{\delta}^i :)$ TODO

Theorem 0.67. *TODO*

0.5 Derived Functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories.

Definition 0.68. (a) A homological (resp. cohomological) δ-functor $(T_n, \delta_n)_{n \geq 0}$ (resp $(T^n, \delta^n)_{n \geq 0}$) from $\mathcal{A}$ to $\mathcal{B}$ consists of

(i) a sequence of additive functors $(T_n : \mathcal{A} \to \mathcal{B})_{n \geq 0}$ resp. $(T^n : \mathcal{A} \to \mathcal{B})_{n \geq 0}$.

(ii) A sequence of natural transformations

$$\begin{tikzcd} \mathsf{Ex}_{\mathcal{A}} \arrow[r, bend left, "T_{n+1}\circ \mathrm{pr}_3"] \arrow[r, bend right, "T_n\circ \mathrm{pr}_1"'] \arrow[r, phantom, "\Downarrow\delta_n"] & \mathcal{B} \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} \mathsf{Ex}_{\mathcal{A}} \arrow[r, bend left, "T^{n}\circ \mathrm{pr}_3"] \arrow[r, bend right, "T^{n+1}\circ \mathrm{pr}_1"'] \arrow[r, phantom, "\Downarrow\delta^n"] & \mathcal{B} \end{tikzcd}$$

such that this data assigns to each $\underline{\mathcal{E}} : 0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ in $\mathsf{Ex}_{\mathcal{A}}$ a long exact sequence in $\mathcal{B}$

$$\cdots \to T_{n+1} \xrightarrow{\delta_n^{\mathcal{E}}} T_n A_1 \xrightarrow{T_n f} T_n A_2 \xrightarrow{T_n g} T_n A_3 \xrightarrow{\delta_{n-1}^{\mathcal{E}}} T_{n-1} A_1 \to \cdots$$

resp.

$$\cdots \to T^{n-1} A_3 \xrightarrow{\delta_{\mathcal{E}}^{n-1}} T^n A_1 \xrightarrow{T^n f} T^n A_2 \xrightarrow{T^n g} T^n A_3 \xrightarrow{\delta_{\mathcal{E}}^{n}} T^{n+1} A_1 \to \cdots$$

Moreover the assignmennt $\mathcal{E} \to$ l.e.s. is functorial in $\mathsf{Ex}_{\mathcal{A}}$

(iii) TODO WTF IS THIS?

Example 0.69. TODO

Construction 0.70. Suppose $\mathcal{A}$ has enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor.

1. $\forall A \in \mathcal{A}$ choose an injective resolution $\iota_A : \underline{A} \to I_A^\bullet$ in $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$ and define $R^n F(A) := H^n(F I_A^\bullet)$ choice in $\mathcal{B}$ of n-th cohomology object.

2. $\forall f : A \to A'$ morphism in $\mathcal{A}$ choose an arrow ι_f such that

$$\begin{tikzcd} \underline{A'} \arrow[r, "\iota_{A'}"] & I_{A'}^\bullet \\ \underline{A} \arrow[u, "f"] \arrow[r, "\iota_A"'] & I_A^\bullet \arrow[u, "\iota_f"'] \end{tikzcd}$$

commutes in $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$ which implies that $F(\iota_f) : F I_A^\bullet \to F I_{A'}^\bullet$ morphism in $\mathsf{Ch}^*_{\geq 0}(\mathcal{B})$. Define: $R^n F(f) := H^n(F(\iota_f)) : R^n F(A) \to R^n F(A')$ ...

Lemma 0.71. *(a) $R^n F$ is a functor $\mathcal{A} \to \mathcal{B}$ (additive.)*

(b) If one makes other choices of injective resolutions $\widetilde{\iota}_A : A \to \widetilde{I}_A^\bullet$ and $\widetilde{\iota}_f : \widetilde{I}_A^\bullet \to \widetilde{I}_{A'}^\bullet$ then we get a natural isomorphism $R^n F \cong \widetilde{R}^n F$

3. Given $\mathcal{E} : 0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ in $\mathsf{Ex}_{\mathcal{A}}$ and an injective resolution $\iota_{\mathcal{E}} : \mathcal{E} \to J_{\mathcal{E}}^\bullet$ in $\mathsf{Ch}^*_{\geq 0}(\mathsf{Ex}_{\mathcal{A}})$

Lemma 0.72. *(a) $F J_{\mathcal{E}}^\bullet$ lies in $\mathsf{Ch}^*_{\geq 0}(\mathsf{Ex}_{\mathcal{B}})$, in particular, it is a s.e.s. of complexes in $\mathsf{Ch}^*_{\geq 0}(\mathcal{B})$.*

(b) Write $J^\bullet_{\mathcal{E}} = 0 \to I^\bullet_1 \to I^\bullet_2 \to I^\bullet_3 \to 0$ as a s.e.s. in $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$, then the I^n_i are injective and by (a) we have a s.e.s.

$$0 \to FI^\bullet_1 \to FI^\bullet_2 \to FI^\bullet_3 \to 0 \quad (*)$$

in $\mathsf{Ch}^*_{\geq 0}(\mathcal{B})$, then the following diagram commutes and the top row is a l.e.s. in $\mathcal{B}$, and the vertical maps are isomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(FI^\bullet_j) \arrow[r, "\delta^{n-1}_{FJ^\bullet_{\mathcal{E}}}"] & H^n(FI^\bullet_1) \arrow[r] & H^n(FI^\bullet_2) \arrow[r] & H^n(FI^\bullet_3) \arrow[r, "\delta^n_{FJ^\bullet_{\mathcal{E}}}"] & \cdots \\
\cdots \arrow[r] & R^{n-1}F(A_3) \arrow[r] \arrow[u, "u^{n-1}_{A_3}"] & R^nF(A_1) \arrow[r, "R^nF(f)"'] \arrow[u, "u^n_{A_1}"] & R^nF(A_2) \arrow[r, "R^nF(g)"'] \arrow[u, "u^n_{A_2}"] & R^nF(A_3) \arrow[r] \arrow[u, "u^n_{A_3}"] & \cdots
\end{tikzcd}$$

This is not even funny anymore. Define $\delta^n_{RF} : R^nF \circ \mathrm{pr}_3 \Rightarrow R^nF \circ \mathrm{pr}_1$ for $\mathcal{E}$ as $(u^{n+1}_{A_1})^{-1} \circ \delta^n_{FJ^\bullet_{\mathcal{E}}} \circ u^n_{A_3}$.

4. Show that δ^n_{RF} is well defined.

Lemma 0.73. *TODO*

Theorem 0.74. *Suppose $\mathcal{A}$ has enouguh injectives and F is an additive left exact functor, then $RF := (R^nF, \delta^n_{RF})_{n\geq 0}$ is a* universal cohomological δ-functor *and R^nF is called the n-*th right derived functor *of F. It satisfies:*

(a) $R^0F = F$.

(b) $I \in \mathrm{Inj}_{\mathcal{A}} \implies \forall n \geq 1, R^nF(I) = 0$.

Suppose $\mathcal{A}$ has enough projectives, let $G : \mathcal{A} \to \mathcal{B}$ be right exact, then

(a) $\exists$ homological δ-functor $LG = (L_iG, \delta_i)_{i\geq 0}$ such that $(*)$

  (i) $L_0G = G$

  (ii) $P \in \mathrm{Proj}_{\mathcal{A}} \implies \forall n \geq 1, L_nG(P) = 0$.

(b) LG with $(*)$ is universal.

(c) LG with $(*)$ is unique up to unique isomorphism.

(d) LG can be computed via projective resolutions, i.e. $\forall A \in \mathcal{A}$ with projective resolution $P^\bullet \to A$ in $\mathsf{Ch}^*_{\leq 0}(\mathcal{A})$ we have

$$L_iG(A) = H^{-i}(GP^\bullet)$$

Lemma. *If $T = (T_n, \delta_n)_{n\geq 0}$ is a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and if $\mathcal{A}$ has enough projectives and $T_iP = 0, \forall P \in \mathrm{Proj}_{\mathcal{A}}, i \geq 1$, then T is universal.*

0.6 The Ext Functor

Let $\mathcal{A}$ be abelian, $M, N \in \mathcal{A}$ and suppose $\mathcal{A}$ has enough projectives/injectives if needed.

Definition 0.75. Define

$$\operatorname{Ext}^i_{\mathcal{A}}(-, N) := R^i \operatorname{Hom}_{\mathcal{A}}(-, N)$$

with the natural transformations δ^i.

Example. To compute: if

$$\cdots \to P^{-2} \to P^1 \to P^0 \to M \quad (*)$$

is a projective resolution, then we have the complex $P^\bullet$, apply $\operatorname{Hom}_{\mathcal{A}}(-, N)$ and get

$$0 \to \operatorname{Hom}_{\mathcal{A}}(P^0, N) \to \operatorname{Hom}_{\mathcal{A}}(P^{-1}, N) \to \operatorname{Hom}_{\mathcal{A}}(P^{-2}, N) \to \cdots$$

and $\operatorname{Ext}^i_{\mathcal{A}}(M, N) = H^i(\operatorname{Hom}_{\mathcal{A}}(P^\bullet, N))$

Proposition 0.76. *(a) M projective $\implies \operatorname{Ext}^i_{\mathcal{A}}(M, N) = 0, \forall i \geq 1$.*

(b) N injective $\implies \operatorname{Ext}^i_{\mathcal{A}}(M, N) = 0, \forall i \geq 1$.

Definition 0.77. Define

$$\overline{\operatorname{Ext}}^i_{\mathcal{A}}(M, -) := R^i(\operatorname{Hom}_{\mathcal{A}}(M, -))$$

with the natural transformations δ^i.

Example. Compute $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(M, -)$ via injective resolutions of the 2nd argument:

$$0 \to N \to I^0 \to I^1 \to I^2 \to \cdots$$

and we get

$$\operatorname{Hom}_{\mathcal{A}}(M, I^0) \to \operatorname{Hom}_{\mathcal{A}}(M, I^1) \to \operatorname{Hom}_{\mathcal{A}}(M, I^2) \to \cdots$$

so $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(M, -) = H^i(\mathcal{A}(M, I^\bullet))$.

Proposition 0.78. *For M projective or N injective we have $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(M, N) = 0, \forall i \geq 1$.*

Remark. If $\mathcal{A}$ has enough projectives and injectives, $\operatorname{Ext}^i_{\mathcal{A}}(-, -)$ and $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(-, -)$ turn out to be isomorphic!

Remark. $\operatorname{Ext}^i_{\mathcal{A}}(-, -)$ and $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(-, -)$ are bifunctors.

Theorem 0.79. *Suppose $\mathcal{A}$ has enough injectives and projectives, then $\exists$ natural isomorphisms as bifunctors*

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$$
$$u^i_{M,N} : \operatorname{Ext}^i_{\mathcal{A}}(M, N) \to \overline{\operatorname{Ext}}^i_{\mathcal{A}}(M, N)$$

0.6.1 Classical interpretation of Ext^1 as extension objects

Definition 0.80. (a) An extension of M by N is a s.e.s. in $\mathcal{A}$:

$$\mathcal{E} : 0 \to N \to E \to M \to 0$$

(b) Extensions $\mathcal{E}$ and

$$\mathcal{E}' : 0 \to N \to E' \to M \to 0$$

are called *equivalent* (write $\sim$), iff $\exists$ commutative diagram:

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}_N"] & E \arrow[r] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
\mathcal{E}' : 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

(c) $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ denotes the set of equivalence classes of extensions of M by N.

Remark. (a) By the snake lemma: $\mathcal{E} \sim \mathcal{E}' \implies E \cong E'$.

(b) $\sim$ is an equivalence relation.

Proposition 0.81. *(a)* $\mathrm{Ext}_{\mathcal{A}}(M, N)$ *is an abelian group for addition* $\mathcal{E} + \mathcal{E}'$ *defined by the* Baer sum*:*

$$\begin{tikzcd}
0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, "\Sigma"'] & E \oplus E' \arrow[r] \arrow[d] & M \oplus M \arrow[r] \arrow[d, "\mathrm{id}_{M\oplus M}"] & 0 \\
0 \arrow[r] & N \arrow[r] & E \oplus E' \amalg_{N\oplus N} N \arrow[r] & M \oplus M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u, "\mathrm{id}_N"] & E + E' \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd}$$

where $E \oplus E' \amalg_{N\oplus N} N$ *is the pushout by the sum map* $+ : N \oplus N \to N, (n, m) \mapsto n + m$ *and* $E + E'$ *is the pullback by the diagonal map* $\Delta : M \to M \oplus M, m \mapsto (m, m)$*. And the zero object is given by the split s.e.s*

$$0 \to N \to N \oplus M \to M \to 0$$

(b) One has a natural isomorphism as bifunctors

$$u : \mathrm{Ext}_{\mathcal{A}}(-, -) \Rightarrow \mathrm{Ext}^1_{\mathcal{A}}(-, -)$$

if $\mathcal{A}$ *has enough projectives (or to* $\overline{\mathrm{Ext}}^1_{\mathcal{A}}(-, -)$ *if* $\mathcal{A}$ *has enough injectives).*

Remark. In fact, Yoneda also considered higher Ext-groups (in the absence of innjectives/projectives), e.g. $\mathrm{Ext}^2(M, N)$ is the "group" of exact sequences

$$0 \to N \to E_1 \to E_2 \to M \to 0$$

modulo a suitable equivalence relation.

Notation. For ${}_R\mathsf{Mod}$ one usually abbreviates

$$\mathrm{Ext}^i_R(-, -) := \mathrm{Ext}^i_{{}_R\mathsf{Mod}}(-, -)$$

For $R = \mathbb{Z}[G]$ and a group G, one also considers the group cohomology for $M \in {}_{\mathbb{Z}[G]}\mathsf{Mod}$ as

Definition 0.82. Define

$$H^i(G, M) := \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

such that $\mathbb{Z}$ is the trivial G-action.

Proposition. *In face $H^i(G, M)$ is the i-th right derived functor of*

$$\begin{aligned} M \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) &\overset{exer.}{=} M^G := \{m \in M \mid \forall g \in G : gm = m\} \\ \varphi_m : n \mapsto nm &\leftarrow\!\shortmid m \in M^G \end{aligned}$$

So far we had $\mathrm{Ext}^i_{\mathcal{A}}$ valued in Ab. But recall that $\mathrm{Hom}_R(-,-)$ is also a bifunctor

$$\begin{aligned} \mathrm{Hom}_R(-,-) : {}_R\mathsf{Mod}_{R'} \times {}_R\mathsf{Mod}_{R''} &\to {}_{R'}\mathsf{Mod}_{R''} \\ (M, N) &\mapsto \mathrm{Hom}_R(M, N) \end{aligned}$$

Fact: ${}_R\mathsf{Mod}_{R'}$ and ${}_R\mathsf{Mod}_{R''}$ have enough injectives and projectives because ${}_R\mathsf{Mod}_{R'}$ is isomorphic to ${}_{R\otimes(R')^{\mathrm{op}}}\mathsf{Mod}$.

Proposition 0.83 (Exer)**.** *This Bifunctor induces*

$$\mathrm{Ext}^i_R(-,-) : {}_R\mathsf{Mod}_{R'} \times {}_R\mathsf{Mod}_{R''} \to {}_{R'}\mathsf{Mod}_{R''}$$

0.7 The Tor Functor

Let R be a ring, $M \in \mathsf{Mod}_R$ and $N \in {}_R\mathsf{Mod}$. We had the bifunctor

$$- \otimes - : \mathsf{Mod}_R \times {}_R\mathsf{Mod} \to \mathsf{Ab}$$

and it is right exact, and also Mod_R and ${}_R\mathsf{Mod}$ have enough projectives.

Definition 0.84. For $i \geq 0$ define:

$$\begin{aligned} \mathrm{Tor}^R_i(M, -) &:= L_i(M \otimes_R -) \\ \overline{\mathrm{Tor}}^R_i(-, N) &:= L_i(- \otimes_R N) \end{aligned}$$

Proposition 0.85. *If $P^\bullet \to M$ (or $Q^\bullet \to N$) are projective resolutions, then one has:*

$$\begin{aligned} \overline{\mathrm{Tor}}^R_i(M, N) &= H^{-i}(P^\bullet \otimes_R N) \\ \mathrm{Tor}^R_i(M, N) &= H^{-i}(M \otimes_R Q^\bullet) \end{aligned}$$

Moreover, if M or N are projective, then

$$\mathrm{Tor}^R_i(M, N) \cong 0 \cong \overline{\mathrm{Tor}}^R_i(M, N)$$

Theorem 0.86. *One has natural isomorphism of bifunctors*

$$\mathrm{Tor}^R_i(-,-) \cong \overline{\mathrm{Tor}}^R_i(-,-)$$

Other ways to compute Tor_i, not using projective resolutions.

Definition 0.87. Let $T = (T_n, \delta_n)$ be a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$. Call $A \in \mathcal{A}$ T-acyclic if $T_iA = 0, \forall i \geq 1$ (with T_0 right exact) For example M projective $\implies$ M is $L(- \otimes_R N)$-acyclic

Facts 0.88. Let $0 \to A' \to A \to A'' \to 0$ be a s.e.s. in $\mathcal{A}$, then

(a) A'' is T-acyclic then

$$0 \to T_0A' \to T_0A \to T_0A'' \to 0$$

is a s.e.s. (because $T_1A'' = 0$ and l.e.s. from δ-functor).

(b) If A and A'' are T-acyclic, then so is A' (because of the l.e.s. from being a δ-functor).

$$\to T_2A' \to \underbrace{T_2A}_{=0} \to \underbrace{T_2A''}_{=0} \to T_1A' \to \underbrace{T_1A}_{=0} \to \underbrace{T_1A''}_{=0} \to T_0A' \to T_0A \to T_0A''$$

(c) If $0 \to A_0 \to A_1 \to \cdots \to A_n \to 0$ is exact in $\mathcal{A}$ and if $A_1, \cdots, A_n$ are T-acyclic then A_0 is also T-acyclic. This follows from (b) by ind. using the exact sequences $0 \to A_0 \to A_1 \to X \to 0$ and

$$0 \to X \to A_2 \to \cdots \to A_n \to 0$$

Lemma 0.89. *If $C^\bullet \in \mathsf{Ch}^*_{\leq 0}(\mathcal{A})$ is acyclic with all C^i T-acyclic, then $T_0C^\bullet$ is acyclic (we assume T_0 is right exact)*

Corollary 0.90. *Suppose $\mathcal{A}$ has enough projectives, $G : \mathcal{A} \to \mathcal{B}$ is right exact and $T = LG$. If $Q^\bullet \xrightarrow{g} \underline{A}$ is a resolution by T-acyclic objects. Then:*

$$L_iG(A) \cong H^{-i}(GQ^\bullet)$$

Definition 0.91. $M \in \mathsf{Mod}_R$ is called *flat* if $M \otimes_R - : {}_R\mathsf{Mod} \to \mathsf{Ab}$ is exact.

Proposition 0.92. *For $M \in \mathsf{Mod}_R$ the following are equivalent:*

(a) M is flat.

(b) $\mathrm{Tor}_1^R(M, -) = 0$.

(c) $\mathrm{Tor}_i^R(M, -) = 0 \forall i \geq 1$.

Theorem 0.93. *For $M \in \mathsf{Mod}_R$ the following are equivalent:*

(a) M is flat.

(b) $\forall N \in {}_R\mathsf{Mod} : M$ is $L(- \otimes N)$-acyclic.

And thus Tor_i^R can be computed via flat resolutions.