Definition 0.1. Let R be a commutative ring.

(a) A category $\mathcal{A}$ is called R-linear if

  (i) $\forall X, Y \in \mathcal{A}, \mathcal{A}(X,Y)$ is an R-module $(\mathcal{A}(X,Y), 0_{X,Y}, +_{X,Y}, \cdot_{X,Y})$.

  (ii) $\forall X, Y, Z \in \mathcal{A}$ the composition map

$$\begin{aligned} \mathcal{A}(X,Y) \times \mathcal{A}(Y,Z) &\to \mathcal{A}(X,Z) \\ (\varphi, \psi) &\mapsto \psi \circ \varphi \end{aligned}$$

  is R-bilinear. (in particular $r(\psi \circ \varphi) = (r \cdot \psi) \circ \varphi = \psi \circ (r\varphi)$).

(b) A functor $F : \mathcal{A} \to \mathcal{A}'$ between R-linear categories is called R-linear if $\forall X, Y \in \mathcal{A}$ the map $F : \mathcal{A}(X,Y) \to \mathcal{A}'(FX, FY)$ is R-linear.

(c) For $\mathbb{Z}$-linear (preadditive) categories $\mathcal{A}$ and $\mathcal{A}'$, the full subcategory $\mathrm{Add}(\mathcal{A}, \mathcal{A}') \subseteq \mathrm{Fun}(\mathcal{A}, \mathcal{A}')$ of $\mathbb{Z}$-linear (additive) functors between them is again a $\mathbb{Z}$-linear category, i.e. there is a natural addition on natural transforations.

Examples. (a) ${}_R\mathsf{Mod}$ is $\mathbb{Z}$-linear and in fact (R is commutative) R-linear.

(b) $\mathcal{A}$ is R-linear $\iff$ $\mathcal{A}^{\mathrm{op}}$ is also R-linear.

(c) If $\mathcal{A}$ and $\mathcal{A}'$ are R-linear, then $\mathcal{A} \times \mathcal{A}'$ is R-linear.

(d) Let S be a (not necessarily commutative) ring and $\underline{S}$ the category

$$(\{*\}, S, \mathrm{dom} = \{*\}, \mathrm{cod} = \{*\}, \mathrm{id}_* = \mathrm{id}_S, \circ = \cdot_S)$$

associated to S, then $\underline{S}$ is $\mathbb{Z}$-linear ($\mathrm{Hom}_{\underline{S}}(*, *) = S$).

Lemma 0.2. *If $\mathcal{A}$ is $\mathbb{Z}$-linear, then for $X \in \mathcal{A}$ the following are equivalent:*

(i) X is initial.

(ii) X is terminal.

(iii) $\mathcal{A}(X,X) = 0$ (in particular $\mathrm{id}_X = 0$)

Proof. Exercise. □

Definition 0.3. An $X \in \mathcal{A}$ satisfying conditions of Lemma 2 is called a *zero object*

Notation. The zero object of $\mathcal{A}$ (if it exists) is unique up to unique isomorphism so we just write 0 or 0_A for this object.

Lemma 0.4. *Let $\mathcal{A}$ be $\mathbb{Z}$-linear with a 0-object and $X, Y \in \mathcal{A}$, then the zero map $0_{X,Y} \in \mathcal{A}(X,Y)$ (remember $\mathcal{A}(X,Y)$ is an abelian group) is equal to the composition $X \xrightarrow{\exists!} 0 \xrightarrow{\exists!} Y$*

Proof. Exercise. □

Proposition 0.5. *Let $\mathcal{A}$ be $\mathbb{Z}$-linear, then for $X_1, X_2 \in \mathcal{A}$ the following are equivalent:*

(i) The product $X_1 \amalg X_2$ exists.

(ii) *The coproduct* $X_1 \amalg X_2$ *exists.*

(iii) $\exists Y \in \mathcal{A}, p_1, p_2 : Y \to X_i$ *and* $\iota_1, \iota_2 : X_i \to Y$ *such that*

$$p_i \circ \iota_j = \begin{cases} 1_{X_i}, & i = j \\ 0, & i \neq j \end{cases}$$

Proof. TODO. □

Remark. In (iii) (Y, ι_1, ι_2) is the coproduct and (Y, p_1, p_2) is the product.

Definition 0.6. In a $\mathbb{Z}$-linear category we deonte $X \Pi Y = X \amalg Y$ by $X \oplus Y$ and call it the *direct sum* of X and Y.

Lemma 0.7. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be* $\mathbb{Z}$*-linear and* $F : \mathcal{A} \to \mathcal{A}'$ *an additive functor, then*

(i) *If* $0_{\mathcal{A}}$ *exists then* $0_{\mathcal{A}'}$ *exists and* $F(0_{\mathcal{A}}) \cong 0_{\mathcal{A}'}$.

(ii) *If* $X, Y \in \mathcal{A}$ *and* $X \oplus Y$ *exists, then* $FX \oplus FY$ *exists and* $FX \oplus FY \cong F(X \oplus Y)$.

Proof. Exercise. □

Definition 0.8. A category $\mathcal{A}$ is called *additive* (or R-linear additive) if $\mathcal{A}$ is $\mathbb{Z}$-linear (or R-linear), $0_{\mathcal{A}}$ exists and $\forall X, Y \in \mathcal{A} : X \oplus Y$ exists.

Remark. If $\mathcal{A}$ is additive then the addition on $\mathcal{A}(X, Y)$ is determined by the composition map! (Morel II 1.2.4)

0.1 Kernels and cokernels

Recall that the equalizer of two morphisms $f, g : X \to Y$ is the limit of the diagram $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, so $Z \xrightarrow{u} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ such that for every $W \xrightarrow{v} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ with $f \circ v = g \circ v$, v factors through u:

$$\begin{tikzcd} Z \arrow[r, "u"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ W \arrow[u, dashed] \arrow[ur, "v"'] & & \end{tikzcd}$$

similarly the coequalizer is the colimit of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$.

In the category of R-modules, $\mathrm{eq}(f, g) = \ker(f - g)$ and $\mathrm{coeq}(f, g) = \mathrm{coker}(f - g)$. In particular $\ker f = \mathrm{eq}(f, 0)$.

Definition 0.9. Let $\mathcal{A}$ be a $\mathbb{Z}$-linear category and $f \in \mathcal{A}(X, Y)$.

(a) define $\ker f := \mathrm{eq}(f, 0)$ and $\mathrm{coker}\, f = \mathrm{coeq}(f, 0)$ (if they exist)

(b) If $\ker f$ exists, define the coimage as $X \to \mathrm{coim}\, f = \mathrm{coker}(\ker f \to X)$ (it might not exist).

(c) If $\operatorname{coker} f$ exists, define the image as $\operatorname{im} f \to Y = \ker(Y \to \operatorname{coker} f)$.

Example 0.10. Let $X_1, X_2 \in \mathcal{A}$ and $(X_1 \oplus X_2, \iota_1, \iota_2, p_1, p_2)$ the direct sum. Then $\iota_1 = \ker p_2, \iota_2 = \ker p_1, p_1 = \operatorname{coker} \iota_2, p_2 = \operatorname{coker} \iota_1$

Lemma 0.11. *Kernels are monomorphisms and cokernels are epimorphisms.*

Proof. We only prove the statement for kernels. Let $\ker f \xrightarrow{\iota} X \underset{0}{\overset{f}{\rightrightarrows}} Y$, now assume $\iota \circ \varphi = \iota \circ \varphi'$

$$\begin{tikzcd} Z \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\varphi'"'] & \ker f \arrow[r, "\iota"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

By the definition of equalizer

$$\begin{aligned} \mathcal{A}(Z, \ker f) &\simeq \{\psi : Z \to X \mid f \circ \psi = 0\} \\ \varphi &\mapsto \iota \circ \varphi \end{aligned}$$

is bijective, so $\varphi = \varphi'$. □

Theorem 0.12. *Let $f \in \mathcal{A}(X, Y)$ and assume* $\ker f, \operatorname{coker} f, \operatorname{im} f, \operatorname{coim} f$ *exist. Then there exists a unique morphism $u : \operatorname{coim} f \to \operatorname{im} f$ such that f is equal to the composition*

$$X \xrightarrow{c} \operatorname{coim} f \xrightarrow{\exists! u} \operatorname{im} f \xrightarrow{d} Y$$

This is called the canonical factorization *of f (or epi-mono factorization).*

Remark. Note that by lemma 11, c is an epimorphism and d is a monomorphism.

Proof. TODO. □

Note that in the category of R-modules, u is an isomorphism.

Definition 0.13. A category $\mathcal{A}$ is called *abelian* if

(i) $\mathcal{A}$ is additive ($\mathbb{Z}$-linear, 0 exists and $X \oplus Y$ exists)

(ii) $\forall f \in \mathcal{A}(X, Y) : \ker f, \operatorname{coker} f$ exist.

(iii) $\forall f \in \mathcal{A}(X, Y)$ with canonical factorization $f = d \circ u \circ c$, u is an isomorphism.

Example 0.14.
- For any ring R the category ${}_R\mathsf{Mod}$ is abelian.
- For rings R, R' the category ${}_R\mathsf{Mod}_{R'}$ is abelian ($\cong {}_{R \otimes_{\mathbb{Z}} (R')^{\mathrm{op}}}\mathsf{Mod}$).

Example 0.15.

Let $\mathcal{A} \subseteq \mathsf{Ab}$ be the fullsubcategory of finitely generated free $\mathbb{Z}$-modules, then $\mathcal{A}$ is additive.

For $f : X \to Y$ in $\mathcal{A}$ the usual $\ker_{\mathsf{Ab}} f$ as abelian group is again a finitely generated free abelian group and is also the kernel in $\mathcal{A}$.

The cokernel however might have torsion. In fact

$$\operatorname{coker}_{\mathcal{A}} f = {\operatorname{coker}_{\mathsf{Ab}} f}\Big/{\operatorname{Tor}(\operatorname{coker}_{\mathsf{Ab}} f)}$$

So kernels and cokernels exists in $\mathcal{A}$, now let $f : \mathbb{Z} \to \mathbb{Z}, n \mapsto 2n$, the canonical factorization is

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \xrightarrow{u} & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ & & n & \longmapsto & 2n & & \end{array}$$

since u is not an isomorphism, $\mathcal{A}$ is not abelian.

Proposition 0.16. *Let $\mathcal{C}$ be a category, then*

(i) *$\mathcal{C}$ is $\mathbb{Z}$-linear $\iff$ $\mathcal{C}^{\text{op}}$ is $\mathbb{Z}$-linear.*

(ii) *$\mathcal{C}$ is additive $\iff$ $\mathcal{C}^{\text{op}}$ is additive.*

(iii) *$\mathcal{C}$ is abelian $\iff$ $\mathcal{C}^{\text{op}}$ is abelian.*

Moreover (if they exist) $\ker_{\mathcal{C}} f = \operatorname{coker}_{\mathcal{C}^{\text{op}}} f$ *and* $\operatorname{im}_{\mathcal{C}} f = \operatorname{coim}_{\mathcal{C}^{\text{op}}} f$.

0.2 Abelian categories

From now on let $\mathcal{A}$ be an abelian category.

Remark. Let $f : X \to Y$ and $X \twoheadrightarrow \operatorname{coim} f \underset{u}{\overset{\cong}{\to}} \operatorname{im} f \hookrightarrow Y$ be the canonical factorization. Then either $X \twoheadrightarrow \operatorname{coim} f \hookrightarrow Y$ or $X \twoheadrightarrow \operatorname{im} f \hookrightarrow Y$ (or anything else isomorphic to them) is called "the" canonical factorization for f.

Proposition 0.17. *Let $X, Y \in \mathcal{A}$*

(a) $\operatorname{coker}(0 \to X) = X \xrightarrow{\text{id}} X$ *and* $\ker(X \to 0) = X \xrightarrow{\text{id}} X$.

(b) *$f : X \to Y$ is a monomorphism* $\iff$ $\ker f = 0$ $\iff$ *the canonical factorization of f is $X \xrightarrow{\text{id}} X \overset{f}{\hookrightarrow} Y$ $\iff$ f is a kernel.*

(c) *$g : C \to Y$ is an epimorphism* $\iff$ $\operatorname{coker} g = 0$ $\iff$ *the canonical factorization of g is $X \xrightarrow{g} Y \xrightarrow{\text{id}} Y$ $\iff$ g is a cokernel.*

(d) *u is an isomorphism $\iff$ u is a monomorphism and an epimorphism.*

Corollary 0.18. *$X \xrightarrow{a} Z \xrightarrow{b} Y$ is the canonical factorization for $f = b \circ a$ $\iff$ a ia a monomorphism and b an epimorphism.*

Definition 0.19. In an abelian category $f \in \mathcal{A}(X, Y)$ is called injective if $\ker f = 0$ and surjective if $\operatorname{coker} f = 0$.

Corollary 0.20. *Let $f \in \mathcal{A}(X, Y)$, then*

(a) *f is injective $\iff$ f is a monomorphism.*

(b) *f is surjective $\iff$ f is a epimorphism.*

(c) *f is injective and surjective $\iff$ f is a isomorphism.*

0.3 Exactness

Lemma 0.21. *Let* (g, f) *be a composable pair of morphisms in* $\mathcal{A}$ *such that* $g \circ f = 0$*, then* $\exists$ *a canonical injection* $\operatorname{im} f \hookrightarrow \ker g$*.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ and write the epi-mono factorization of f:

$$X \xrightarrow{c} \operatorname{coim} f \xrightarrow{\cong} \operatorname{im} f \xhookrightarrow{d} Y \xrightarrow{g} Z$$

and call the isomorphism in the middle $u : \operatorname{coim} f \cong \operatorname{im} f$. Now $g \circ d \circ u \circ c = 0 \implies g \circ d \circ u = 0$ since c is an epimorphism and u isomorphism $\implies g \circ d = 0$. By the definition of the kernel d must factor through $\ker g$:

$$\operatorname{im} f \xrightarrow{e} \ker g \xhookrightarrow{\iota} Y \xrightarrow{g} Z$$

Since d is injective: if $e \circ \varphi = c \circ \varphi' \implies \iota \circ c \circ \varphi = \iota \circ c \circ \varphi' \implies d \circ \varphi = d \circ \varphi' \implies \varphi = \varphi' \implies e$ is a monomorphism $\implies \operatorname{im} f \hookrightarrow \ker g$. □