0.1 Preliminary remarks on set theory

References. Literature for this chapter:

- Sophie Morel - Homological Algebra I.1,
- Daniel Murfet - Foundations for Category Theory,
- Saunders MacLane - Categories for the Working Mathematician I.6.

In this course we always assume a model of set theory that satisfies the Zermelo-Fraenkel axioms + the axiom of choice (ZFC).

Definition (Grothendieck universe; we assume ZFC)**.** A *universe* $\mathcal{U}$ is a set which has the following properties:

(i) $\emptyset, \mathbb{N} \in \mathcal{U}$,

(ii) $X \in \mathcal{U}$ and $y \in X \implies y \in \mathcal{U}$,

(iii) $X \in \mathcal{U} \implies \{X\} \in \mathcal{U}$,

(iv) $X \in \mathcal{U} \implies \mathcal{P}(X) \in \mathcal{U}$,

(v) If $I \in \mathcal{U}$ and $\{X_i\}_{i \in I}$ is a family of members $X_i \in \mathcal{U}$, then $\bigcup_{i \in I} X_i \in \mathcal{U}$.

The existence of a universe is equivalent to the existence of a strongly inaccessible cardinal. (Thomas Jech - Set Theory)

Axiom (Axiom of universes (Grothendieck))**.** *Every set lies in a universe. (We will assume this)*

Definition. If $\mathcal{U}$ is our chosen universe, then:

- A $\mathcal{U}$-set is an element in $\mathcal{U}$.
- A $\mathcal{U}$-class is a subset of $\mathcal{U}$.
- A $\mathcal{U}$-group is a group $(G, e, \cdot)$ with $G \in \mathcal{U}$ and $\cdot : G \times G \to G \in \mathcal{U}$.
- A $\mathcal{U}$-ring is a ring $(R, 0, 1, +, \cdot)$ with $R \in \mathcal{U}$ and also $+, \cdot$
- etc.

Convention. We fix a $\mathcal{U}$ and drop $\mathcal{U}$- in all terms.