0.1 Preliminary remarks on set theory

References. Literature for this chapter:

- Sophie Morel - Homological Algebra I.1,
- Daniel Murfet - Foundations for Category Theory,
- Saunders MacLane - Categories for the Working Mathematician I.6.

In this course we always assume a model of set theory that satisfies the Zermelo-Fraenkel axioms + the axiom of choice (ZFC).

Definition (Grothendieck universe; we assume ZFC)**.** A *universe* $\mathcal{U}$ is a set which has the following properties:

(i) $\emptyset, \mathbb{N} \in \mathcal{U}$,

(ii) $X \in \mathcal{U}$ and $y \in X \implies y \in \mathcal{U}$,

(iii) $X \in \mathcal{U} \implies \{X\} \in \mathcal{U}$,

(iv) $X \in \mathcal{U} \implies \mathcal{P}(X) \in \mathcal{U}$,

(v) If $I \in \mathcal{U}$ and $\{X_i\}_{i \in I}$ is a family of members $X_i \in \mathcal{U}$, then $\bigcup_{i \in I} X_i \in \mathcal{U}$.

The existence of a universe is equivalent to the existence of a strongly inaccessible cardinal. (Thomas Jech - Set Theory)

Axiom (Axiom of universes (Grothendieck))**.** *Every set lies in a universe. (We will assume this)*

Definition. If $\mathcal{U}$ is our chosen universe, then:

- A $\mathcal{U}$-set is an element in $\mathcal{U}$.
- A $\mathcal{U}$-class is a subset of $\mathcal{U}$.
- A $\mathcal{U}$-group is a group $(G, e, \cdot)$ with $G \in \mathcal{U}$ and $\cdot : G \times G \to G \in \mathcal{U}$.
- A $\mathcal{U}$-ring is a ring $(R, 0, 1, +, \cdot)$ with $R \in \mathcal{U}$ and also $+, \cdot$
- etc.

Convention. We fix a $\mathcal{U}$ and drop $\mathcal{U}$- in all terms.

0.2 Categories

Definition 0.1. (a) A *directed graph* (a diagram scheme) is a tuple $(O, A, \mathrm{dom}, \mathrm{cod})$ consisting of $\mathcal{U}$-classes O and A and maps $\mathrm{dom}, \mathrm{cod} : A \to O$. We call elements of O *objects* (or vertices) and elements of A *arrows* (or directed edges). For an arrow $f \in A$ call $\mathrm{dom}(f)$ the *source* (or domain) of f and $\mathrm{cod}(f)$ the target (or codomain) of f.

(b) For a graph as in (a) call $A \times_O A := \{(g, f) \in A \times A \mid \mathrm{dom}(g) = \mathrm{cod}(f)\}$ set of composable arrow pairs.

(c) A subgraph of $(O, A, \mathrm{dom}, \mathrm{cod})$ is a graph $(O', A', \mathrm{dom}', \mathrm{cod}')$ such that $O' \subseteq O, A' \subseteq A, \mathrm{dom}' = \mathrm{dom}|_{A'}$ and $\mathrm{cod}' = \mathrm{cod}|_{A'}$.

Example 0.2. Let $O = \{X, Y, Z\}, A = \{f, g, h\}, \mathrm{dom}, \mathrm{cod} : A \to O$ given by the table

	f	g	h
dom	X	Y	X
cod	Y	Z	Z

Illustration:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

Definition 0.3. A *category* $\mathcal{C}$ is a tuple $(\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C}, \mathrm{dom}, \mathrm{cod}, \circ, 1)$ consisting of a graph $(\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C}, \mathrm{dom}, \mathrm{cod})$ and maps

$$1 : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}, X \mapsto 1_X$$

and

$$\circ : \mathrm{Mor}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}, (g, f) \mapsto g \circ f$$

such that:

(i) $\mathrm{dom}(1_X) = \mathrm{cod}(1_X) = X, \forall X \in \mathrm{Ob}\,\mathcal{C}$,

(ii) $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$,

(iii) $\forall f \in \mathrm{Mor}\,\mathcal{C}$ with $X = \mathrm{dom}(f), Y = \mathrm{cod}(f)$

$$f \circ 1_X = 1_Y \circ f = f$$

(iv) $\forall$ arrows $f, g, h \in \mathrm{Mor}\,\mathcal{C}$ such that (h, g) and (g, f) are composable aarrow pairs we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Call elements of $\mathrm{Ob}\,\mathcal{C}$ the objects of $\mathcal{C}$ and elements of $\mathrm{Mor}\,\mathcal{C}$ the morphisms of $\mathcal{C}$.

Notation 0.4. For a category $\mathcal{C}$ as in definition 3

(a) (often) write $X, Y \in \mathcal{C}$ to mean $X, Y \in \mathrm{Ob}\,\mathcal{C}$

(b) For $X, Y \in \mathcal{C}$ write

$$\mathcal{C}(X, Y) := \mathrm{Mor}_{\mathcal{C}}(X, Y) := \{f \in \mathcal{C} \mid \mathrm{dom}\, f = X, \mathrm{cod}\, f = Y\}$$

Definition 0.5. (a) Call a category $\mathcal{C}$ locally small if $\mathcal{C}(X, Y)$ is a set $\forall X, Y \in \mathcal{C}$,

(b) Call $\mathcal{C}$ small if $\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C}$ are sets.

Remark 0.6 (Extension of example 2 to a category)**.** Let $O = \{X, Y, Z\}$, $A = \{f, g, h\} \cup \{1_X, 1_Y, 1_Z\}$, $\mathrm{cod}, \mathrm{dom}$ as before on $\{f, g, h\}$ and uniquely extended to $\{1_X, 1_Y, 1_Z\}$ by axiom (i) and $\circ$ the only possible composition satisfying the axioms

$$\begin{tikzcd}
X \arrow[loop left, "1_X"] \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[loop right, "1_Y"] \arrow[dl, "g"] \\
& Z \arrow[loop below, "1_Z"] &
\end{tikzcd}$$

composable arrow pairs:

$(1_X, 1_X), (f, 1_X), (1_Y, 1_Y), (1_Y, f), (g, 1_Y), (1_Z, 1_Z), (1_Z, g), (1_Z, h), (h, 1_X), (g, h)$

Canonical universal extension would contain a second arrow $X \to Z$ since it would not want to impose the condition $g \circ f = h$.

Definition 0.7. (a) A diagram in $\mathcal{C}$ is a subgraph Γ of $(\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C}, \mathrm{dom}, \mathrm{cod})$.

(b) A diagram is commutative if for all objects X, Y of Γ and all chains of arrows from X to Y, their composition is the same (i.e. it only depends on X and Y).

Example (For associativity)**.**

$$\begin{tikzcd}
& W & \\
X \arrow[ur, "h\circ(g\circ f)=(h\circ g)\circ f"] \arrow[rr] \arrow[dr, "f"'] & & Z \arrow[ul, "h"'] \\
& Y \arrow[uu] \arrow[ur, "g"'] &
\end{tikzcd}$$

Examples (Examples of categories)**.**

- Set (category of $\mathcal{U}$-sets): where
  - $\mathrm{Ob}\,\mathsf{Set}$ = class of all $\mathcal{U}$-sets,
  - $\mathrm{Mor}\,\mathsf{Set}$ = class of all $\mathcal{U}$-maps between sets,
  - $\mathrm{dom}, \mathrm{cod}$ are the domain and codomain (range) of a map. (Think of a map as a triple (X, Y, graph map in $X \times Y$))
  - $\circ$ = composition of maps,
  - $1_X = \mathrm{id}_X$ the identity map.
- Grp (category of abelian groups)
- Ring
- CRing
- Top
- ${}_R\mathsf{Mod}$

- Mod_R
- Vec_K
- $\mathsf{Ab} =_{\mathbb{Z}} \mathsf{Mod}$

Examples (Abstract examples)**.** 1. $\operatorname{Ob}\mathcal{C} = \operatorname{Mor}\mathcal{C} = \emptyset$ (empty category)

2. $\operatorname{Ob}\mathcal{C} = \{X\}, \operatorname{Mor}\mathcal{C} = \{1_X\}$ (1 arrow category)

3. Let G be a group, define a category $\underline{G}$ by $\operatorname{Ob}\underline{G} = \{*\}$ (singleton set) and $\operatorname{Mor}\underline{G} = G$, dom, cod the unique map $G \to \{*\}$, $1_* = e$ (unit element of G). $\circ =$ composition in G:
$$\operatorname{Mor}\underline{G} \times \operatorname{Mor}\underline{G} = G \times G \to G = \operatorname{Mor}\underline{G}$$

4. Let $\underline{A} = (M, \leq)$ be a partially ordered set. Define the associated category $\operatorname{Ord}\underline{M}$ with $\operatorname{Ob}\operatorname{Ord}\underline{M} =$ elements of M, morphisms are determined by
$$\operatorname{Ord}\underline{M}(X,Y) = \begin{cases} \text{singleton set}, & X \leq Y, \\ \emptyset, & \text{otherwise.} \end{cases}$$
Unit is clear. composition dictated by $\operatorname{Mor}(\operatorname{Ord}\underline{M})$ (i.e. by $\leq$)

Definition 0.8. For a category $\mathcal{C} = (\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}, \operatorname{dom}, \operatorname{cod}, \circ, 1)$ define the tuple $\mathcal{C}^{\mathrm{op}} = (\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}, \operatorname{cod}, \operatorname{dom}, \circ^{\mathrm{op}}, 1)$ with
$$\begin{aligned} \circ^{\mathrm{op}} : \{(f,g) \in \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C} \mid \operatorname{cod} f = \operatorname{dom} g\} &\to \operatorname{Mor}\mathcal{C} \\ (f,g) &\mapsto f \circ^{\mathrm{op}} g := g \circ f \end{aligned}$$
(change the direction of arrows!)

Proposition 0.9 (Exercise)**.** *$\mathcal{C}^{\mathrm{op}}$ is a category, the* opposite category *to $\mathcal{C}$.*

Example. $(\underline{G})^{\mathrm{op}} = \underline{(G^{\mathrm{op}})}$, $(G^{\mathrm{op}} = (G, e, \circ^{\mathrm{op}})$ with $g \circ^{\mathrm{op}} h = h \circ g)$.

Warning 0.10. $\mathsf{Vec}_K^{\mathrm{op}}(V,W) \neq$ not the set of maps $V \to W$, it is $\{f : W \to V \mid f \text{ is } K\text{-linear}\}$

Definition 0.11. A subcategory of $\mathcal{C} = (\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}, \operatorname{dom}, \operatorname{cod}, \circ, 1)$ is a category $\mathcal{C}' = (\operatorname{Ob}\mathcal{C}', \operatorname{Mor}\mathcal{C}', \operatorname{dom}', \operatorname{cod}', \circ', 1')$ such that $\operatorname{Ob}\mathcal{C}' \subseteq \operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}' \subseteq \operatorname{Mor}\mathcal{C}, \operatorname{dom}' = \operatorname{dom}|_{\operatorname{Mor}\mathcal{C}'}, \operatorname{cod}' = \operatorname{cod}|_{\operatorname{Mor}\mathcal{C}'}, \circ' = \circ|_{\operatorname{Mor}\mathcal{C}' \times_{\operatorname{Ob}\mathcal{C}} \operatorname{Mor}\mathcal{C}'}, 1' = 1|_{\operatorname{Ob}\mathcal{C}'}$. We write $\mathcal{C}' \subseteq \mathcal{C}$.

Example. $\mathsf{Ab} \subseteq \mathsf{Grp}$ and $\mathsf{CRing} \subseteq \mathsf{Ring}$, etc.

Definition 0.12 (Product of categories)**.** The product of two categories $\mathcal{C}$ and $\mathcal{C}'$ is the six-tuple:
$$(\operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C}', \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C}', \operatorname{dom} \times \operatorname{dom}', \operatorname{cod} \times \operatorname{cod}', \circ, 1)$$
where $\circ$ is componentwise composition $(g,g') \circ (f,f') = (g \circ f, g' \circ f')$ and $1_{X \times X'} = (1_X, 1_{X'})$

Definition 0.13 (Concepts inside categories)**.** Let $X, Y \in \mathcal{C}$, then call $f \in \mathcal{C}(X,Y)$

(a) an *isomorphism* $\iff \exists g \in \mathcal{C}(Y,X)$ such that $g \circ f = 1_X, f \circ g = 1_Y$,

(b) an *endomorphism* $\iff X = Y$,

(c) an *automorphism* $\iff$ it is an isomorphism and an endomorphism

Moreover $\mathcal{C}$ is called a groupoid category $\iff$ all morphisms are isomorphisms.

Example. Let G be a group, then $\underline{G}$ is a groupoid category. $\mathcal{C}$ a groupoid category $\implies \mathcal{C}(X,X)$ is a group (under $\circ, \forall X \in \operatorname{Ob}\mathcal{C}$).

Definition 0.14. Let $X, Y \in \mathcal{C}$, then call $f \in \mathcal{C}(X,Y)$:

(a) a *monomorphism* $\iff$ f is left cancellable $\iff$ $\forall W \in \mathcal{C}$ the map $f_* : \mathcal{C}(W,X) \to \mathcal{C}(W,Y), g \mapsto f \circ g$ is injective.

$$W \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y : f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

(b) an *epimorphism* $\iff$ f is right cancellable $\iff$ $\forall Z \in \mathcal{C}$ the map $f^* : \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z), h \mapsto h \circ f$ is injective.

$$X \xrightarrow{f} Y \underset{h_2}{\overset{h_1}{\rightrightarrows}} Z : h_1 \circ f = h_2 \circ f \implies h_1 = h_2.$$

(c) a *split monomorphism* $\iff \exists g \in \mathcal{C}(Y,X)$ such that $g \circ f = 1_X$

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[l, bend right, dashed, "g"'] \end{tikzcd}$$

(d) a *split epimorphism* $\iff \exists h \in \mathcal{C}(Y,X)$ such that $f \circ h = 1_Y$

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[l, bend right, dashed, "h"'] \end{tikzcd}$$

Facts 0.15. (a) f split mono-/epimorphism $\implies$ f mono-/epimorphism.

(b) f (split) mono-/epimorphism in $\mathcal{C}$ $\implies$ f (split) mono-/epimorphism in $\mathcal{C}^{\mathrm{op}}$.

(c) (Exercise) For $f \in \mathcal{C}(X,Y), (X,Y) \in \mathcal{C}$ the following are equivalent:

  (i) f is an isomorphism

  (ii) $\forall W \in \mathcal{C} : f_* : \mathcal{C}(W,X) \to \mathcal{C}(W,Y), g \mapsto f \circ g$ is bijective.

  (iii) $\forall Z \in \mathcal{C} : f^* : \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z), h \mapsto h \circ f$ is bijective.

(b') f is an isomorphism in $\mathcal{C}$ $\iff$ f is an isomorphism in $\mathcal{C}^{\mathrm{op}}$.

Proof. (b), (b') are exercises. (c) "serious" exercise.

(a) For an epimorphism (check right cancellability) consider

$$X \xrightarrow{f} Y \underset{h_2}{\overset{h_1}{\rightrightarrows}} Z : h_1 \circ f \overset{(1)}{=} h_2 \circ f$$

By f a split epimorphism, we have $h : Y \to X$ such that $f \circ h = 1_Y$ (2). Apply $- \circ h$ to (1):

$$\begin{array}{c} (h_1 \circ f) \circ h = (h_2 \circ f) \circ h \\ \| \qquad\qquad\quad \| \\ h_1 = h_1 \circ 1_Y = h_1 \circ (f \circ h) = h_2 \circ (f \circ h) = h_2 \circ 1_Y = h_2 \end{array}$$

$\square$

Examples. In $\mathsf{Set}, \mathsf{Grp}, \mathsf{Ring}$ the monomorphisms are the injective maps and in $\mathsf{Set}, \mathsf{Grp}$ the epimorphisms are the surjective maps. But $\mathbb{Z} \to \mathbb{Q}$ (inclusion) is an epimorphism in Ring. If $K \subseteq E$ is purely inseperable, then it's an epimorphism in the category of fields.

Definition 0.16. (a) $X \in \mathcal{C}$ is called an *initial object* $\iff$ $\forall Y \in \mathcal{C} : \#\mathcal{C}(X, Y) = 1$

(b) $X \in \mathcal{C}$ is called a *terminal object* $\iff \forall Z \in \mathcal{C} : \#\mathcal{C}(Z, X) = 1$

(c) $X \in \mathcal{C}$ is called a *null object* $\iff$ X is initial and terminal.

Example.

- $\emptyset$ is initial in $\mathsf{Set}, \mathsf{Top}$,
- $\{*\}$ is terminal in $\mathsf{Set}, \mathsf{Top}$
- $0 = \{0\}$ is a null object in ${}_R\mathsf{Mod}, \mathsf{Ab}, \mathsf{Vec}_K$

0.3 Functors

Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories.

Definition 0.17. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ ($F : \mathcal{C} \to \mathcal{D}$) is a pair of maps

$$\begin{aligned} F &: \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}, X \mapsto F(X), \\ F &: \operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{D}, f \mapsto F(f). \end{aligned}$$

that "preserve sources, targets, units and composition", i.e.

(i) $\forall f \in \operatorname{Mor}\mathcal{C} : \operatorname{dom}(Ff) = F(\operatorname{dom} f)$ and $\operatorname{cod}(Ff) = F(\operatorname{cod} f)$

(ii) $\forall X \in \operatorname{Ob}\mathcal{C} : F(1_X) = 1_{FX}$

(iii) $\forall$ composable pairs (g, f) in $\operatorname{Mor}\mathcal{C} \times_{\operatorname{Ob}\mathcal{C}} \operatorname{Mor}\mathcal{C} : F(g \circ f) = F(g) \circ F(f)$. (other notation $F(X \xrightarrow{f} Y) = FX \xrightarrow{Ff} FY$)

Examples. (a) Powerset:

$$\begin{aligned} \mathcal{P} &: \mathsf{Set} \to \mathsf{Set} \\ & X \mapsto \mathcal{P}(X) \\ f : X \to Y &\mapsto \mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y, \\ & \qquad (U \subseteq X) \mapsto (f(U) \subseteq Y) \end{aligned}$$

(b) Forgetful functor (it forgets structure)

$$V : \mathsf{Grp} \to \mathsf{Set}, (G, e, \circ) \mapsto G$$
$$V : \mathsf{Top} \to \mathsf{Set}, (X, \mathcal{T}) \mapsto X$$
$$V : {}_R\mathsf{Mod} \to \mathsf{Ab}, (M, 0, +, \cdot) \mapsto (M, 0, +)$$

(c) ${}_R\mathsf{Mod} \to \mathsf{Mod}_{R^{\mathrm{op}}}$, left R-modules $\mapsto$ right R-modules.

Remark. Functors in definition 17 are also called covariant functors.

Definition 0.18. A *contravariant* functor from $\mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, i.e.

$$F(X \xrightarrow{f} Y) = (FX \xleftarrow{Ff} FY)$$

$= (Y \xrightarrow{f} X)$ in $\mathcal{C}^{\mathrm{op}}$ and

$$F((Y \xrightarrow{g} Z) \circ (X \xrightarrow{f} Y)) = F(X \xrightarrow{f} Y) \circ F(Y \xrightarrow{g} Z)$$

Visually:

$$\begin{array}{ccc} X & X & FX \\ \downarrow f & f\uparrow & Ff\uparrow \\ Y & Y & FY \\ \downarrow g & g\uparrow & Fg\uparrow \\ Z & Z & FZ \end{array}$$

in $\mathcal{C}$ in $\mathcal{C}^{\mathrm{op}}$

Remark (Exercise)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be any functor $\implies$ F maps isomorphisms to isomorphisms.

Examples (Contravariant functors)**.** (a) Passage to the dual vector space

$$\begin{tikzcd} D : \mathsf{Vec}_K^{\mathrm{op}} \arrow[r] & \mathsf{Vec}_K \\ V \arrow[r, mapsto] \arrow[d, "f"'] & V^* = \mathrm{Hom}_K(V, K) \\ W \arrow[r, mapsto] & W^* \arrow[u, "Df = f^*"'] \end{tikzcd}$$

linear algebar: $(f \circ g)^* = g^* \circ f^*$ for (f, g) a composable pair.

(b) Let Poset be the category of partially ordered sets, then we have a contravariant functor

$$\begin{tikzcd} \mathcal{O} : \mathsf{Top}^{\mathrm{op}} \arrow[r] & \mathsf{Poset} & \\ (X, \mathcal{T}) \arrow[r, mapsto] \arrow[d, "f"'] & (\mathcal{T}, \subseteq) \ni f^{-1}(V) \underset{\text{open}}{\subseteq} X & \\ (Y, \mathcal{T}') \arrow[r, mapsto] & (\mathcal{T}, \subseteq) \ni V \underset{\text{open}}{\subseteq} Y \arrow[u, "\mathcal{O}(f)"'] & \end{tikzcd}$$

(c) The contravariant powerset functor:

$$\begin{tikzcd}[row sep=small] \mathcal{P}^* : \mathsf{Set} \arrow[r] & \mathsf{Set} \\ X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{P}^*(X) = \mathcal{P}(X) \\ Y \arrow[r, mapsto] & \mathcal{P}^*(Y) = \mathcal{P}(Y) \arrow[u, "\mathcal{P}^* f"'] \end{tikzcd}$$

Definition 0.19. Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories, a functor $\mathcal{C} \times \mathcal{C}' \to \mathcal{D}$ is called a *bifunctor*.

Example 0.20 (Important example)**.** Let $\mathcal{C}$ be any category

$$\begin{tikzcd}[row sep=small] \mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r] & \mathsf{Set} & \\ (X,Y) \arrow[r, mapsto] \arrow[d, shift left, "h"] & \mathcal{C}(X,Y) \arrow[d] & \ni \quad g \arrow[d, mapsto] \\ (W,Z) \arrow[r, mapsto] \arrow[u, shift left, "f"] & \mathcal{C}(W,Z) & \ni h \circ g \circ f \end{tikzcd}$$

If we fix a first argument X, we get

$$h_X := \mathcal{C}(X, -) \to \mathsf{Set}$$

If we fix a second argument Y, we get

$$h^Y := \mathcal{C}(-, Y) \to \mathsf{Set}$$

Soon: we will also have another important bifunctor

$$- \otimes - : \mathsf{Mod}_R \times {}_R\mathsf{Mod} \to \mathsf{Ab}$$

Definition 0.21. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, F is called

(a) *faithful* $\iff \forall X, Y \in \mathcal{C} : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$is injective.

(b) *full* $\iff \forall X, Y \in \mathcal{C} : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$is surjective.

(c) *fully faithful* F is full and faithful.

(d) *essentially surjective* $\iff \forall Y \in \mathcal{D} \exists X \in \mathcal{C} \exists$ isomorphism $FX \xrightarrow{\cong} Y$.

(e) *conservative* $\iff \forall f \in \operatorname{Mor} \mathcal{C} : f$ is an isomorphism $\iff Ff\imath \operatorname{Mor} \mathcal{D}$ is an isomorphism. ($\implies$ always holds)

(f) *an isomorphism* $\iff \exists G : \mathcal{D} \to \mathcal{C}$ functor such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$.

Examples. (a) Forgetful functors are "often" faithful but not full

$$V : \mathsf{Grp} \to \mathsf{Set}, \mathsf{Ab} \to \mathsf{Set}, {}_R\mathsf{Mod} \to \mathsf{Set}, \mathsf{Ring} \to \mathsf{Set}$$

are conservative.

(b) The forgetful functor $V : \mathsf{Top} \to \mathsf{Set}$ is not conservative and not full but essentially surjective.

(c) The inclusion of a subcategory $\mathcal{C}'$ into its ambient category $\mathcal{C}$ is always faithful. Call $\mathcal{C}'$ a *full subcategory* $\iff \forall X, Y \in \mathcal{C}' : \mathcal{C}'(X,Y) = \mathcal{C}(X,Y)$ ($\iff i : \mathcal{C}' \to \mathcal{C}$ is full)

0.4 Natural transforations

They are morphisms between functors.

Definition 0.22. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors.

(a) A morphism from F to G (or a *natural transforation*) is a family $u = (u_X : FX \to GX)_{X \in \operatorname{Ob}\mathcal{C}}$ of morphisms in $\mathcal{D}$, such that forall $f : X \to Y$ in $\mathcal{C}$ we have the commutative diagram:

$$\begin{tikzcd} FX \arrow[r, "u_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "u_Y"'] & GY \end{tikzcd}$$

Notation:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "u"] \end{tikzcd}$$

(b) Composition: Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors and $u : F \Rightarrow G, v : G \Rightarrow H$ natural transformations. The composition $v \circ u : F \Rightarrow H$ is the natural transformation (check) $(v_X \circ u_X : FX \xrightarrow{u_X} GX \xrightarrow{v_X} HX)_{X \in \operatorname{Ob}\mathcal{C}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=50, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=M, Rightarrow, "u"] \arrow[from=M, to=D, Rightarrow, "v"] \end{tikzcd}$$

(c) The category $\mathcal{D}^{\mathcal{C}}$ (or $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$) whose objects are the functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are the natural transformations $(F : \mathcal{C} \to \mathcal{D}) \Rightarrow (G : \mathcal{C} \to \mathcal{D})$. The composition is from (b), and the unit natural transforation is

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "1_F"] \end{tikzcd}, 1_F = (FX \xrightarrow{1_{FX}} FX)_{X \in \operatorname{Ob}\mathcal{C}}$$

(dom, cod are clear). Remark: One can also define 2-categories (and the category of categories is an example of such, objects: $\mathcal{C}, \mathcal{D}, ...$ and morphisms are $F : \mathcal{C} \rightrightarrows \mathcal{D}$ 2-morphisms = natural transformations)

(d) A natural transformation $u : F \Rightarrow G$ is called a *natural isomorphism* $\iff \forall X \in \operatorname{Ob}\mathcal{C} : u_X : FX \to GX$ is an isomorphism $\underset{\text{Exerc.}}{\iff} \exists$ natural transformation $v : G \Rightarrow F : v \circ u = \mathrm{id}_F, u \circ v = \mathrm{id}_G$.

Example (Famous linear algebra example of a natural transformation)**.** Let $(\cdot)^{**} : \mathsf{Vec}_K \to \mathsf{Vec}_K, V \mapsto V^{**}, f \mapsto f^{**}$ be the (covariant) bidual functor. $\mathrm{id} : \mathsf{Vec}_K \to \mathsf{Vec}_K$ denotes the identity, we set $u_V : V \to V^{**}, v \mapsto (b_v : V^* \to$

$K, \xi \mapsto \xi(v))$ then $u = (u_V)_{V \in \mathsf{Vec}_K}$ is a natural transformation $u : \mathrm{id} \Rightarrow (\cdot)^{**}$ and restricted to the full subcategory $\mathsf{Vec}_K^{\text{f.d.}} \subseteq \mathsf{Vec}_K$ on finite dimensional K-vector spaces, it gives a natural isomorphism $u : \mathrm{id} \Rightarrow (\cdot)^{**}$

$$\begin{tikzcd} \mathsf{Vec}_K^{\text{f.d.}} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "(\cdot)^{**}"'] \arrow[r, phantom, "\Downarrow u"] & \mathsf{Vec}_K^{\text{f.d.}} \end{tikzcd}$$

Definition 0.23 (important concept)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is called an equivalence of categories $\iff$ $\exists$ functor $G : \mathcal{D} \to \mathcal{C}$ such that one has natural transforations $\mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\mathrm{id}_{\mathcal{D}} \Rightarrow F \circ G$. Call $\mathcal{C}$ and $\mathcal{D}$ *equivalent categories* $\iff$ $\exists$ equivalence of categories $F : \mathcal{C} \to \mathcal{D}$.

Remark. The notion of "equivalence of categories" is far more important than the notion of isomorphism of categories.

Example (linear algebra)**.** Let $\mathsf{Vec}_K^{\text{std.}}$ be the full subcategory of the $\mathsf{Vec}_K^{\text{f.d.}}$ on the object set $\{K^n \mid n \in \mathbb{N}_0\}$. Then the inclusion $\iota : \mathsf{Vec}_K^{\text{std.}} \to \mathsf{Vec}_K^{\text{f.d.}}$ is an equivalence of categories. For $G : \mathsf{Vec}_K^{\text{std}} \to \mathsf{Vec}_K^{\text{std}}$ take $V \mapsto K^{\dim_K V}$, choose a basis $\underline{B}_V$ for any $V \in \mathsf{Vec}_K^{\text{f.d.}}$ then we get an isomorphism $K^{\dim_K V} \xrightarrow{\alpha_V} V$. Define:

$$\begin{tikzcd} V \arrow[r, mapsto] \arrow[d, "f"'] & K^{\dim_K V} \arrow[d, "\alpha_W^{-1} \circ f \circ \alpha_V"] \\ W \arrow[r, mapsto] & K^{\dim_K W} \end{tikzcd}$$

Find natural isomorphism $G \circ \iota \Leftarrow \mathrm{id} \Rightarrow \iota \circ G$.

Remark. One also calls $\mathsf{Vec}_K^{\text{std.}}$ a *skeleton* of $\mathsf{Vec}_K^{\text{f.d.}}$

Theorem 0.24. *For a functor $F : \mathcal{C} \to \mathcal{C}'$ the following are equivalent:*

(i) F is an equivalence of categories.

(ii) F is fully faithful and essentially surjective.

Proof.
- (i) $\implies$ (ii): Exercise.
- (ii) $\implies$ (i): Standard textbook. $\square$

Definition 0.25. The *essential image* of a functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathcal{D}$ is the full subcategory $\mathcal{D}'$ of $\mathcal{D}$ on objects isomorphic to FX for some $X \in \operatorname{Ob}\mathcal{C}$.

Corollary 0.26 (of 24 and the definition)**.** *Suppose $F : \mathcal{C} \to \mathcal{D}$ is fully faithful. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the essential image of F, then $F : \mathcal{C} \to \mathcal{D}'$ is an equivalence of categories.*

0.5 The Yoneda lemma and presheaves

Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories.

Definition 0.27. (a) A $\mathcal{D}$-valued presheaf on $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$$

(b) The category of $\mathcal{D}$-valued presheaves on $\mathcal{C}$ is $\mathrm{PSh}(\mathcal{C}, \mathcal{D}) = \mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$

(c) If $\mathcal{D} = \mathsf{Set}$, then we omit it from the notation, so $\mathrm{PSh}(\mathcal{C}) = \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$
Note that if $\mathcal{D}$ is a small category, then $\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$ is a category.

Remark (On the terminoloy)**.** (Pre-)sheaves come from topology/geometry. Example: Let $(X, \mathcal{T})$ be a topological space (e.g. $\mathbb{C}$ with the metric topology), For $U \subseteq X$ define $O_X(U) := \{f : U \to \mathbb{C} \text{ continuous}\}$ or $(O_{\mathbb{C}}(U) := \{f : U \to \mathbb{C} \text{ holomorphic }\})$. Check:

$$\begin{tikzcd}
O_X : \mathsf{ord}(T, \subseteq) \arrow[r] & \mathsf{Set} \\
U \arrow[r, mapsto] & O_X(U) \arrow[d, "\text{restriction}"] \\
V \arrow[u, "\cup\!\mathrm{I}"] \arrow[r, mapsto] & O_X(V)
\end{tikzcd}$$

this is a presheaf.

Definition 0.28. The Yoneda embedding is the functor $h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}), X \mapsto h_X := \mathcal{C}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$

$$\begin{tikzcd}
h : \mathcal{C} \arrow[r] & \mathsf{PSh}(\mathcal{C}) & \\
X \arrow[r, mapsto] \arrow[d, "f"] & h_X \arrow[d, Rightarrow, "hf"] & \mathrm{Hom}(-, X) \arrow[d, "f\circ -"] \\
Y \arrow[r, mapsto] & h_Y & \mathrm{Hom}(-, Y)
\end{tikzcd}$$

Lemma 0.29. *For $X \in \mathcal{C}, F \in \mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$*

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_X, \mathcal{F}) &\xrightarrow{\quad\Phi\quad} \mathcal{F}X \\
u := (u_Y : h_X Y \to \mathcal{F}Y)_{Y \in \mathrm{Ob}\mathcal{C}} &\longmapsto u_X 1_X
\end{aligned}$$

is a bijection. ($\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_x, \mathcal{F})$ *is a set.)*

Proof. Reconstruct a natural transformation u^α from $\alpha \in \mathcal{F}X$, first consider what $u \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_x, \mathcal{F})$ gives us

$$\begin{tikzcd}
X & h_X X = \mathrm{Hom}_{\mathcal{C}}(X, X) \arrow[r, "u_X"] \arrow[d, "-\circ g"] & \mathcal{F}X \arrow[d, "\mathcal{F}g"] & 1_X \arrow[r, mapsto] \arrow[d, mapsto] & u_X(1_X) = \alpha \arrow[d, mapsto] \\
Y \arrow[u, "g"] & h_X Y = \mathrm{Hom}_{\mathcal{C}}(Y, X) \arrow[r, "u_Y"] & \mathcal{F}Y & g \arrow[r, mapsto] & u_Y(g) = \mathcal{F}g(\alpha)
\end{tikzcd}$$

Define $\psi : \mathcal{F}(X) \to \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_X, \mathcal{F}), \alpha \mapsto (u_Y^\alpha)_{Y \in \mathrm{Ob}\,\mathcal{C}}$ by setting $u_Y^\alpha : h_X(Y) \to \mathcal{F}Y, g \mapsto \mathcal{F}g(\alpha)$. Check u_Y^α is a natural transforation: For any $f : Z \to Y$ in $\mathcal{C}$ we get TODO $\square$

Corollary 0.30. *The functor $h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is fully faithful, i.e. $\mathcal{C}(X,Y) \leftrightarrow \mathrm{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y)$.*

Proof. We need to show $\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$ the map $\mathcal{C}(X,Y) \to \mathrm{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y), f \mapsto h(f) = f \circ -$ is bijective. Observe: Yoneda $\Phi : \mathrm{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y) \to h_y(X) = \mathcal{C}(X,Y), u \mapsto u_X(1_X)$ is a bijection, so it suffices to show $\Phi \circ h$ is a bijection. For this: $\Phi \circ h(f : X \to Y) = f \circ 1_X = f \implies \Phi \circ h = \mathrm{id}$. □

Definition 0.31. (a) Call $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$ *representable* $\iff \exists X \in \mathcal{C}$ such that $h_X \cong \mathcal{F}$.

(b) A presentation of a (representable) $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$ is a pair (X, α) with $X \in \mathrm{Ob}\,\mathcal{C}, \alpha \in \mathcal{F}X$ such that $\Psi(\alpha) : h_X \Rightarrow \mathcal{F}$ from the proof of lemma 29 is a natural isomorphism.

Proposition 0.32. *Suppose (X, α) and (Y, β) are presentations of $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$, then $\exists!$ isomorphism $f : X \to Y$ such that $\mathcal{F}(f)(\beta) = \alpha$*

Proof. Exercise. □

0.6 Conatravariant Yoneda

Proposition 0.33. *The functor*

$$\begin{tikzcd}
h^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathsf{Set}) \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{C}(X, -) \\
Y \arrow[r, mapsto] & \mathcal{C}(Y, -) \arrow[u, "h^{\mathrm{op}}(f)"']
\end{tikzcd}$$

is fully faithful and for $X \in \mathrm{Ob}\,\mathcal{C}$ and $\mathcal{F} : \mathcal{C} \to \mathsf{Set}$ a functor, the map $\Phi' : \mathrm{Mor}_{\mathsf{Set}^{\mathcal{C}}}(h_X^{\mathrm{op}}, \mathcal{F}) \to \mathcal{F}(X), u \mapsto u_X(1_X)$ is bijective.

Proof. (Exercise) Apply Yoneda to $\mathcal{C}^{\mathrm{op}}$. □

Definition 0.34. (a) A covariant functor $F : \mathcal{C} \to \mathsf{Set}$ is *corepresentable* $\iff F \cong h_X^{\mathrm{op}}$ for some $X \in \mathcal{C}$.

(b) A presentation of a (corepresentable) functor F is a pair (X, α) such that $(\Phi')^{-1}(\alpha)$ is an isomorphism $h_X^{\mathrm{op}} \to \mathcal{F}$

Proposition 0.35 (analog of 32)**.** *If (X, α) and (Y, β) are 2 presentations of $F : \mathcal{C} \to \mathsf{Set}$, then $\exists!$ isomorphism $f : X \to Y$ such that $F(f)(\alpha) = \beta$.*

Remark. We mostly drop co- in corepresentable because the functor dictates if it is representable or corepresentable (if F co- or contravariant)

Remark. For $f : X \to Y$ we have

$$\begin{aligned}
f \circ - : \mathcal{C}(Z, X) \to \mathcal{C}(X, Y) \text{ bij.} &\iff h(f) \cong \iff f \cong \iff h^{\mathrm{op}}(f) \cong \\
&\iff - \circ f : \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z) \text{ bij.}
\end{aligned}$$

because h is fully faithful.

0.7 Universal pairs

Definition 0.36. (a) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $B \in \operatorname{Ob}\mathcal{D}$. A pair (U, β) with $U \in \operatorname{Ob}\mathcal{C}$ and $\beta : B \to F(U)$ (in $\mathcal{D}$) is *(co-)universal* for $(F, \beta) :\iff (U, \beta)$ (co-)represents $\mathcal{D}(B, F(-)) = h_B^{\mathrm{op}} \circ F : \mathcal{C} \to \mathsf{Set}$.

(b) Let $G : \mathcal{D} \to \mathcal{C}$ be a functor and $A \in \operatorname{Ob}\mathcal{C}$. A pair (V, α) with $V \in \operatorname{Ob}\mathcal{D}$ and $\alpha : G(V) \to A$ (in $\mathcal{C}$) is universal for $(G, A) \iff (V, \alpha)$ represents $(\mathcal{C}(G(-), A)) = h_A \circ G : \mathcal{D}^{\mathrm{op}} \to \mathsf{Set}$.
TODO: interpretation

Examples 0.37. TODO

0.8 Limits and colimits

Let $\mathcal{C}$ be a category.

Definition 0.38. A diagram in $\mathcal{C}$ is a functor $F : J \to \mathcal{C}$ for J a small category (call J the *index category* of the diagram.)

Remark (Relation to previous notions of diagrams)**.** Let $V : \mathsf{Cat} \to \mathsf{Diag}$, [MacLane II.7]: $\exists$ functor TODO

Definition 0.39. A diagram $F : J \to \mathcal{C}$ *commutes* $\iff \forall i, j \in J$:

$$\underbrace{F(J(i,j))}_{\text{is a singleton}} \subseteq \mathcal{C}(Fi, Fj)$$

(naive diagram commutes $\iff F\varphi$ commutes.)

Definition 0.40. Let $F : J \to \mathcal{C}$ denote a diagram in $\mathcal{C}$.

(a) The *constant functor* from J to $\mathcal{C}$ for $X \in \operatorname{Ob}\mathcal{C}$ is $\Delta X : J \to \mathcal{C}$ with $\Delta X(i) = X, \forall i \in J$ and $\Delta X(h) = 1_X, \forall h \in \operatorname{Mor} J$.

(b) The *diagonal* $\Delta : \mathcal{C} \to \mathcal{C}^J = \operatorname{Fun}(J, \mathcal{C})$ with $\Delta(X) := \Delta X$ from (a) and $\Delta(f) :=$ the natural transformation $\Delta X \Rightarrow \Delta Y$ given for any $i \in J$ by $\Delta X(i) = X \xrightarrow{f} \Delta Y(i) = Y$.

(c) A *cone* to $F : J \to \mathcal{C}$ (any fixed F) with apex $X \in \mathcal{C}$ is a natural transforation $\Delta X \Rightarrow F$. A *cocone* from F with vertex $X \in \operatorname{Ob}\mathcal{C}$ is a natural transforation $F \Rightarrow \Delta X$.

(d) Cones and cocones give rise to the following functors:

- $\operatorname{Cone}(-, F) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ defined by $\operatorname{Cone}(X, F) =$ set of cones $\Delta X \Rightarrow F$, $\operatorname{Cone}(f : X \to Y, F)$ maps $\operatorname{Cone}(Y, F) \to \operatorname{Cone}(X, F), (\Delta Y \Rightarrow F) \mapsto (\Delta X \Rightarrow \Delta Y \Rightarrow F)$.
- Similarly $\operatorname{Cocone}(F, -) : \mathcal{C} \to \mathsf{Set}$ is the functor defined by $\operatorname{Cocone}(F, X) =$ set of cocones $F \Rightarrow \Delta X$ etc.

Observe:

$$\begin{aligned} \mathrm{Cone}(-,F) &= \mathcal{C}^J(\Delta(-),F) \\ \mathrm{Cocone}(F,-) &= \mathcal{C}^J(F,\Delta(-)) \end{aligned}$$

Visualization: a natural transforation $u : \Delta X \Rightarrow F$ (where $X \in \mathcal{C}, F : J \to \mathcal{C}$) is for any $i \in J$ a morphism $X = \Delta X(i) \xrightarrow{u_i} F(i)$ such that $\forall h : i \to j$ in J the following diagram commutes:

$$\begin{tikzcd} \Delta X(i) = X \arrow[r, "1_X"] \arrow[d, "u_i"] \arrow[dr, "u_j"] & X = \Delta X(j) \arrow[d, "u_j"] \\ Fi \arrow[r, "Fh"'] & Fj \end{tikzcd}$$

for instance if TODO

Remark 0.41. (a) The cones to F form a full subcategory F-cones $\subseteq \mathcal{C}^J/F$ on objects $\Delta X \Rightarrow F$ $(X \in \mathcal{C})$.

(b) Similarly cocones from F form a full subcategory F-cocones $\subseteq F/\mathcal{C}^J$ on objects $F \Rightarrow \Delta X$.

Definition 0.42. (a) If $\mathrm{Cone}(-,F) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is representable, the representing object is called a *limit* over F. Notation: $\lim F$ or $\lim_J F$ for the representing universal object.

(b) If $\mathrm{Cocone}(F,-) : \mathcal{C} \to \mathsf{Set}$ is representable, the representing object is called a *colimit* over F. Notation: $\mathrm{colim}\, F$ or $\mathrm{colim}_J\, F$

More explicitly: $\lim_J F$ is an object $L \in \mathcal{C}$ together with a (universal) ocne $\Delta L \Rightarrow F$ such that $\forall$ cones $\varphi : \Delta X \Rightarrow F$ in $\mathcal{C}^J \exists!$ morphism $\psi : X \to L$ such that the diagram commutes

$$\begin{tikzcd} & \Delta X \arrow[dl, dashed, "\psi"'] \arrow[d, Rightarrow, "\varphi"] \\ \Delta L \arrow[r, Rightarrow] & X \end{tikzcd}$$

Yoneda implies that $\lim F$ (if it exists) is unique up to unique isomorphism (similarly for $\mathrm{colim}\, F$).

Exercise 0.43. (a) $\lim F$ exists $\iff$ category of F-cones has a terminal object.

(b) $\mathrm{colim}\, F$ exists $\iff$ category of F-cocones has an initial object.

Proposition 0.44 (Exercise)**.** *Let $\Delta : \mathcal{C} \to \mathcal{C}^J$ be the diagonal from above and F any diagram in $\mathcal{C}^J$. Then*

(a) $\lim F$ *is a universal object for the pair* (Δ, F) *i.e.* $\mathcal{C}^J(\Delta(-),F) \leftrightarrow \mathcal{C}(-,\lim_J F)$.

(b) $\mathrm{colim}\, F$ *is a couniveral object for the pair* (Δ, F) *i.e.* $\mathcal{C}^J(F,\Delta(-)) = \mathcal{C}(\mathrm{colim}_J\, F,-)$.

Examples 0.45. (a) $J :=$ the discrete category on the set $\{0,1\}$, i.e. $(\mathrm{Ob} = \{0,1\}, \mathrm{Mor}\, J = \{1_0, 1_1\}, ...)$. A functor $F : J \to \mathcal{C}$ is given by the datum of a pair $(F_0 = F(0), F_1 = F(1))$ of objects of $\mathcal{C}$,

cones:
$$\begin{tikzcd} & X \arrow[dl, "u_0"'] \arrow[dr, "u_1"] & \\ F_0 & & F_1 \end{tikzcd}$$
cocones:
$$\begin{tikzcd} F_0 \arrow[dr, "v_0"'] & & F_1 \\ & X \arrow[ur, "v_0"'] & \end{tikzcd}$$

any pair of morphisms $(u_0 : X \to F_0, u_1 : X \to F_1)$ resp. $(v_0 : F_0 \to X, v_1 : F_1 \to X)$ defines a cone $\lim F$ resp. $\operatorname{colim} F$ and satisfies

$$\mathcal{C}(Y, \lim F) = \mathcal{C}(Y, F_0) \times \mathcal{C}(Y, F_1)$$

resp.

$$\mathcal{C}(\operatorname{colim}(F, Z)) = \mathcal{C}(F_0, Z) \times \mathcal{C}(F_1, Z)$$

If $\lim F$ exists write formally $\lim F = F_0 \Pi F_1$ (product), and if $\operatorname{colim} F$ exists write $F_0 \amalg F_1$ (coproduct). Concretely

- $\mathcal{C} = \mathsf{Set}, F_0 \times F_1 = F_0 \sqcup F_1$
- $\mathcal{C} = {}_R\mathsf{Mod}, F_0 \times F_1 = F_0 \oplus F_1$
- $\mathcal{C} = \mathsf{Grp}, F_0 \times F_1 = F_0 * F_1$ (free product)

(b) We can generalize to arbitrary discrete (small) categories with unterlying set I.Names for universal objects

$$\prod_{i \in I} F_i, \qquad \coprod_{i \in I} F_i$$

defining property:

$$\mathcal{C}(\prod_{i\in I} F_i, Y) \overset{!}{=} \prod_{i \in I} \mathcal{C}(F_i, Y) \text{ and } \mathcal{C}(Z, \coprod_{i\in I} F_i) \overset{!}{=} \prod_{i\in I} \mathcal{C}(Z, F_i)$$

- $\mathsf{Set}, \prod_{i\in I} F_i = \bigsqcup_{i \in I} F_i$ (disjoint union)
- ${}_R\mathsf{Mod}, \prod_{i \in I} F_i = \bigoplus_{i\in I} F_i$ (direct sum)
- $\mathsf{Grp}, \prod_{i\in I} F_i = \Asterisk_{i \in I} F_i$ (free product)

(c) $J =$ category on 2 objects $0, 1$ with 2 morphisms $0 \rightrightarrows 1$ (besides $1_0, 1_1$). $F : J \to \mathcal{C}$ is determined by $F_0 \overset{a}{\underset{b}{\rightrightarrows}} F_1$

cones:
$$\begin{tikzcd} & X \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\ F_0 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] & & F_1 \end{tikzcd}$$
cocones:
$$\begin{tikzcd} F_0 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] & & F_1 \\ & X \arrow[ul, "f_0"] \arrow[ur, "f_1"'] & \end{tikzcd}$$

- $\mathcal{C}(X, \lim F) = \{f_0 \in \mathcal{C}(X, F_0) \mid a \circ f_0 = b \circ f_0 = f_1\}$. $\lim F$ if it exists is called the *equalizer* $\mathrm{eq}(F_0 \overset{a}{\underset{b}{\rightrightarrows}} F_1)$.

- $\mathcal{C}(\operatorname{colim} F_1, X) = \{f_1 \in \mathcal{C}(F_1, X) \mid f_1 \circ a = f_1 \circ b\}$. $\operatorname{colim} F$ if it exists is called the *coequalizer* $\operatorname{coeq}(F_0 \overset{a}{\underset{b}{\rightrightarrows}} F_1)$. TODO

(d) Pullback and pushout:

Definition 0.46. $\mathcal{C}$ is *(co-)complete* $\iff$ $\mathcal{C}$ contains all (co)-limits.

Theorem 0.47. *(a) If $\mathcal{C}$ contains all products and equalizers, then $\mathcal{C}$ is complete.*

(b) If $\mathcal{C}$ contains all coproducts and coequalizers, then $\mathcal{C}$ is cocomplete.

Corollary 0.48. Set *and* ${}_R\mathsf{Mod}$ *are complete and cocomplete.*

0.9 Adjoint Functors

Definition 0.49. (a) Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ form an *adjoint pair*, when one has a natural isomorphism α of bifunctors

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\mathcal{D}(F(-),-)"] \arrow[r, bend right, "\mathcal{C}(-,G(-))"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{Set} \end{tikzcd}$$

In this situation one says that F is a *left adjoint* for G and G is a *right adjoint* for F. We write $F \dashv G$ or $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \end{tikzcd}$.

(b) The tuple (F, G, α) is called an *adjunction.*

(c) We say $F : \mathcal{C} \to \mathcal{D}$ has a *right adjoint* if $\exists G : \mathcal{D} \to \mathcal{C} : F \dashv G$ (similarly $G : \mathcal{D} \to \mathcal{C}$ has left adjoint.)

Theorem 0.50. *For functors $F ; \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ we have:*

(a) F has a right adjoint $\iff$ $\forall B \in \mathcal{D} \exists$ universal pair $(A, v : FA \to B)$ for (F, B) such that $\operatorname{Hom}_{\mathcal{D}}(F(-), B) \cong \operatorname{Hom}_{\mathcal{C}}(-, A)$.

(b) G has a left adjoint $\iff$ $\forall A \in \mathcal{C} \exists$ universal pair $(B, u : A \to GB)$ for (G, A) such that $\operatorname{Hom}_{\mathcal{C}}(A, G(-)) \cong \operatorname{Hom}_{\mathcal{D}}(B, -)$.