1 Normal subgroups

Introduction. In the last talk we looked at induced representations and developed a nice way (Mackey's criterion) to check when the induced representation is also irreducible. Today we will continue this by showing some results on

Proposition 1 (Prop 24). *Let $A \trianglelefteq G, \rho : G \to \mathrm{GL}(V)$ irreducible, then either (a) $\exists H \lneq G$ containing A and $\sigma : H \to \mathrm{GL}(W)$ irred. which induces ρ, or (b) the restriction $\rho|_A$ is isotypical.*

Proof.

(1) Let $V = \bigoplus_{i\in I} V_i$ be the canonical decomposition of $\rho|_A$ into isotypical components.

(2) $G \curvearrowright \{V_i\}_{i\in I}$ by $s \cdot V_i := \rho_s(V_i)$:

(i) $A \trianglelefteq G \implies \forall s \in G, a \in A :$

$$\rho_{s^{-1}as}(V_i) = V_i \implies \rho_{s^{-1}}\rho_a\rho_s(V_i) = V_i \implies \rho_a(\rho_s(V_i)) = \rho_s(V_i) \implies \rho_s(V_i) \quad A\text{-stable}$$

$$\implies V = \rho_s(V) = \bigoplus_{i\in I} \rho_s(V_i)$$

(ii) $V_i = \bigoplus V_{i,j}$ irreducible. Same argument gives us $\rho_a(\rho_s(V_{i,j})) = \rho_s(V_{i,j})$, so $\rho_s(V_{i,j})$ is A-stable and $\rho_s(V_i) = \bigoplus_j \rho_s(V_{i,j})$, we get an equivalence of A-representations:

$$\begin{tikzcd} V_{i,j} \arrow[r, "\rho_{s^{-1}as}"] \arrow[d, "\rho_s"'] & V_{i,j} \arrow[d, "\rho_s"] \\ \rho_s(V_{i,j}) \arrow[r, "\rho_a"'] & \rho_s(V_{i,j}) \end{tikzcd}$$

This means that $V_{i,j}$ irred. $\implies \rho_s(V_{i,j})$ irred.

$$\rho_s(V_i) = \bigoplus_j \rho_s(V_{i,j})$$

So the $\rho_s(V_i)$ are also isotypical. Since $\bigoplus_{i\in I} V_i$ is the canonical decomposition, $\exists j : \rho_s(V_i) \leq V_j$, and since $\bigoplus_{i\in I} \rho_s(V_i) = V$. This must be equality $\rho_s(V_i) = V_j$

(3) The action $G \curvearrowright \{V_i\}$ is transitive by remark 1.4 of talk 9.

- *Why?* Choose some V_{i_0} and write $H := \mathrm{Stab}(V_{i_0})$, the orbit stabilizer theorem tells us that $G/H \simeq G \cdot V_{i_0}$. Take the direct sum over the orbit

$$\bigoplus_{sH\in G/H} \rho_s(V_{i_0}) = V$$

This is clearly G-stable, and since ρ is irreducible, it has to be V.

(4) Now fix a V_{i_0}, if $V_{i_0} = V$ then (b) holds.

(5) If not, then $A \leq H := \mathrm{Stab}_G(V_{i_0}) \lneq G$, and by proposition 1.3 from Talk 9 we have that $V = \mathrm{Ind}_H^G(V_{i_0})$.

- *Why is H a proper subgroup?* Because $\mathrm{Stab}_G(V_{i_0}) = G$ means that the orbit is V_{i_0} which contradicts ρ being irreducible.
- *Why is V_{i_0} irreducible?* Since $V = \mathrm{Ind}_H^G(V_{i_0})$ is irreducible, Mackey's Criterion tells us that V_{i_0} has to be irred.

□

Remark. If A is abelian, then (b) $\iff \mathrm{Res}_A^G(\rho)$ is a homothety for each a.

Corollary 2. *Let $A \trianglelefteq G$ be abelian, ρ an irred. repr. von G, then* $\deg\rho \mid [G : A]$.

Proof. Induction on the order of G in both cases of the proposition. **What is the base case?** $H = A$**?**

Case (a): (1) We show that $\deg\sigma \mid [H : A] \implies \deg\rho \mid [G : A]$

(2) $\deg\rho \overset{\text{Talk 5}}{\underset{\text{Rem. 1.2}}{=}} [G : H]\deg\sigma \mid [G : H] \cdot [H : A] = [G : A]$.

Case (b): (1) If ρ is faithful, then $\rho(G) \cong G$. We know that $\rho(A)$ are homotheties by the remark, so $\rho(A) \leq Z(\rho(G)) \implies A \leq Z(G)$.

(2) By proposition 4.5 from talk 6, since ρ irred.

$$\deg \rho \mid [G : Z(G)] \mid [G : Z(G)][Z(G) : A] = [G : A]$$

(3) If ρ is not faithful, we can mod out the kernel $K := \ker \rho$ and get a faithful irred. representation $\widetilde{\rho}$

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\pi"'] & \mathrm{GL}(V) \\ G/K \arrow[ur, hook, "\widetilde{\rho}"'] & \end{tikzcd}$$

Note that $\deg \widetilde{\rho} = \deg \rho$.

(4) The image $\pi(A) \cong \ker \pi|_A = A/A \cap K \cong AK/K$ so the subgroup $AK/K \trianglelefteq G/K$ is normal abelian.

(5) Notice that if

$$\deg \rho = \deg \widetilde{\rho} \mid \underbrace{[G/K : AK/K]}_{=[G:AK]} \mid [G : AK][AK : A] = [G : A] \tag{1}$$

□

Remark 3. If A is abelian, but not necessarily normal, this doesn't hold in general, but we have the upper bound $\deg \rho \leq [G : A]$ like we have seen in corollary 1.3. from talk 4.

2 $\rtimes$ by an abelian group

Let $A \trianglelefteq G$ abelian, $H \leq G$ so that $G = A \rtimes H$. We want to find a way to construct the irred. reps of G from irred. reps of certain subgroups of H. This is *Wigner and Mackey's little groups method.*

1. A is abelian $\implies$ irred. characters have degree 1 and form a group $X = \mathrm{Hom}(A, \mathbb{C}^\times)$. Let $G \curvearrowright X$ by

$$(s \cdot \chi)(a) = \chi(s^{-1}as), \quad s \in G, \chi \in X, a \in A$$

2. Let $(\chi_i)_{i \in X/H}$ be a system of representatives for the orbits of H in X and write $H_i := \mathrm{Stab}_H(\chi_i) \leq H$ and $G_i = A \rtimes H_i \leq G$.

3. Extend χ_i to a degree 1 character $\widetilde{\chi}_i$ of G_i by

$$\begin{tikzcd} A \rtimes H_i \arrow[r, "\pi_A"] \arrow[rr, bend right, "\widetilde{\chi}_i"'] & A \arrow[r, "\chi_i"] & \mathbb{C}^\times \end{tikzcd}$$

setting $\widetilde{\chi}_i(a \cdot h) = \chi_i(a)$ for $a \in A, h \in H_i$.

- *Why is this a character of degree 1 of G_i:* Character of degree 1 is the same as a representation of degree 1 ($\mathrm{Hom}(G_i, \mathbb{C}^\times)$),
- Let $g_1, g_2 \in G_i = A \rtimes H_i$. Then

$$\begin{aligned} \widetilde{\chi}_i(g_1 \cdot g_2) = \widetilde{\chi}_i(a_1 h_1 \cdot a_2 h_2) = \widetilde{\chi}_i(\underbrace{a_1 h_1 a_2 h_1^{-1}}_{\in A} \cdot \underbrace{h_1 h_2}_{\in H_i}) = \chi_i(a_1 h_1 a_2 h_1^{-1}) \\ = \chi_i(a_1)\chi_i(h_1 a_2 h_1^{-1}) \underset{(*)}{=} \chi_i(a_1) \cdot (h_1^{-1} \cdot \chi_i)(a_2) \\ \underset{h \in \mathrm{Stab}}{=} \chi_i(a_1)\chi_i(a_2) = \widetilde{\chi}_i(g_1) \cdot \widetilde{\chi}_i(g_2) \end{aligned}$$

4. Choose an irred. representation $\rho : H_i \to \mathrm{GL}(V)$ of H_i. Similarly, we can extend it to a representation of G_i:

$$\begin{tikzcd} A \rtimes H_i \arrow[r, "\pi_{H_i}"] \arrow[rr, bend right, "\widetilde{\rho}"'] & H_i \arrow[r, "\rho"] & \mathrm{GL}(V) \end{tikzcd}$$

This is now irreducible, because they have the same image.

- *$\widetilde{\rho}$ is irreducible (LONG ANSWER):* Suppose $\exists W \lneq V$ which is G_i-stable, then $\forall s \in G_i$ we would have $\widetilde{\rho}_s(W) = W$, but

$$\widetilde{\rho}_s(W) = (\rho \circ \pi)(s)(W)$$
$$= (\rho \circ \pi)(a \cdot h)(W) = \rho_{\pi(ah)}(W) = \rho_h(W) = W$$

and since π is surjective this holds $\forall h \in H_i$ so that W is H_i-stable which contradicts the irreducibility of ρ.

5. Now tensor by $\widetilde{\chi}_i$ and we get an irred. representation $\widetilde{\chi}_i \otimes \widetilde{\rho}$ of G_i:
   - *Why is $\widetilde{\chi}_i \otimes \widetilde{\rho}$ irred.?* This is because we are tensoring with a representation of degree 1. To see this let ψ be the character of $\widetilde{\rho}$. Then the character of $\widetilde{\chi}_i \otimes \widetilde{\rho}$ would be the product $\widetilde{\chi}_i\psi$. Look at

$$\begin{aligned}(\widetilde{\chi}_i\psi \mid \widetilde{\chi}_i\psi) &= \frac{1}{|G|}\sum_{s\in G}(\widetilde{\chi}_i\psi)(s)(\widetilde{\chi}_i\psi)(s^{-1}) \\ &= \frac{1}{|G|}\sum_{s\in G}\widetilde{\chi}_i(s)\psi(s)\underbrace{\widetilde{\chi}_i(s^{-1})}_{\widetilde{\chi}_i(s)^{-1}}\psi(s^{-1}) \\ &= \frac{1}{|G|}\sum_{s\in G}\psi(s)\psi(s^{-1}) \underset{\psi \text{ irr.}}{=} (\psi \mid \psi) = 1\end{aligned}$$

6. Write $\theta_{i,\rho} := \operatorname{Ind}_{G_i}^G(\widetilde{\chi}_i \otimes \widetilde{\rho})$.

Proposition 4 (Proposition 25)**.**

(a) $\theta_{i,\rho}$ is irred.

(b) $\theta_{i,\rho} \cong \theta_{i',\rho'} \implies i = i', \rho \cong \rho'$.

(c) Every irred. representation of G is $\cong$ to some $\theta_{i,\rho}$.

Proof. (a) Recall Mackey's irreducibility criterion (Theorem 3.2 from talk 9): $\theta_{i,\rho} := \operatorname{Ind}_{G_i}^G(\widetilde{\chi}_i \otimes \widetilde{\rho})$ is irreducible if and only if

(i) $\varphi = \chi_i \otimes \widetilde{\rho}$ is irred. (we have already seen this)

(ii) $\forall s \in G \setminus G_i : \langle \varphi^s, \operatorname{Res}_{K_s}\varphi\rangle_{K_s} = 0$, where we write $K_s := G_i \cap sG_is^{-1} = A \cdot H_i \cap A \cdot sH_is^{-1}$. This is equivalent to φ^s and $\operatorname{Res}_{K_s}\varphi$ having no irred. component in common.
   - If the restrictions to A are disjoint then φ^s and $\operatorname{Res}_{K_s}\varphi$ are disjoint. *Why?* Because if $V = \bigoplus V_i$ and $V' = \bigoplus V_i'$ shared an irred. component, then the restriction would also share this component.
   - $\operatorname{Res}_A \varphi = (\chi_i \otimes \widetilde{\rho})|_A$, recall that

$$\widetilde{\rho} : G_i = A \rtimes H_i \xrightarrow{\pi} H_i \xrightarrow{\rho} \mathrm{GL}(V)$$

     so $\widetilde{\rho}|_A = \mathrm{id}$. So we get $\operatorname{Res}_A\varphi = \chi_i \otimes \mathrm{id} = \chi_i \cdot \mathrm{id}$
   - Similarly for φ^s we have $\widetilde{\rho}|_A = \mathrm{id}$ since $sas^{-1} \in A$ and $\chi_i(sas^{-1}) = s \cdot \chi_i$ so we get $s\chi_i \cdot \mathrm{id}$.
   - Since $s \notin G_i = A \rtimes H_i = A \rtimes \operatorname{Stab}_H(\chi_i)$ we have $s \cdot \chi_i \neq \chi_i$ therefore $\varphi^s|_A$ and $\operatorname{Res}_A\varphi$ are disjoint $\implies \varphi^s$ and $\operatorname{Res}_{K_s}\varphi$ are disjoint.

So mackey's criterion $\implies \theta_{i,\rho} = \operatorname{Ind}_{G_i}^G(\varphi)$ is irred.

(b) (i) $\theta_{i,\rho} \cong \theta_{i',\rho} \implies i = i'$:

i. Consider the action of A on one of the sW from $\bigoplus_{\overline{s}\in G/G_i} sW$

$$(\chi_i \otimes \widetilde{\rho})\underbrace{(as)}_{s\cdot s^{-1}as}(W) = (\chi_i \otimes \widetilde{\rho})(s)((\chi_i \otimes \widetilde{\rho})(\underbrace{s^{-1}as}_{\in A})(W)) \tag{2}$$
$$= s\chi_i(a)(sW) \tag{3}$$

so this only depends on the orbit χ_i and thus determines i.

1. Look at the character of $\theta_{i,\rho}$ restricted to A, recall that the induced character is given by

$$\chi(a) = \frac{1}{|G_i|}\sum_{\substack{t\in G,\\ t^{-1}at\in G_i}}\chi_{\chi_i\otimes\widetilde{\rho}}(t^{-1}at) \tag{4}$$
$$= \frac{1}{|G_i|}\sum_{t\in G}\chi_i(t^{-1}at)\underbrace{\chi_{\widetilde{\rho}}(t^{-1}at)}_{=\dim W} \tag{5}$$
$$= \frac{\dim W}{|G_i|}\sum_{a'h\in A\rtimes H=G}\chi_i(h^{-1}\underbrace{a'^{-1}aa'}_{=a}h) \tag{6}$$
$$= \frac{|A|\dim W}{|G_i|}\sum_{h\in H}h\chi_i(a) \tag{7}$$

This means that $\theta_{i,\rho}|_A$ is a direct sum of irred. representations corresponding to the $h\chi_i$ which are in the orbit $H \cdot \chi_i$. Since orbits are disjoint we have $H\chi_i = H\chi_{i'} \implies i = i'$.

(ii) $\theta_{i,\rho}$ determines ρ up to isomorphism:

- For fixed i let $W_i := \{x \in W \mid \theta_{i,\rho}(a)(x) = \chi_i(a)(x), \forall a \in A\}$, then W_i is H_i-stable:
- So we want to check that $\theta_{i,\rho}(s)(x) \in W_i, \forall s \in H_i$ which is by definition equivalent to $\theta_{i,\rho}(a)(\theta_{i,\rho}(s)(x)) = \chi_i(a)(\theta_{i,\rho}(s)(x)), \forall a \in A$:
- Look at:

$$\theta_{i,\rho}(a)(\theta_{i,\rho}(s)(x)) = \theta_{i,\rho}(as)(x) = \theta_{i,\rho}(ss^{-1}as)(x) \tag{8}$$

$$\theta_{i,\rho}(s)\theta_{i,\rho}(\underbrace{s^{-1}as}_{\in A})(x) = \theta_{i,\rho}(s)s\chi_i(a)(x) \tag{9}$$

$$= \chi_i(a)(\theta_{i,\rho}(s)(x)) \tag{10}$$

So W_i is H_i-stable.

- One can check, that $\theta_{i,\rho}|_{H_i} \cong \rho$ TODO: HOW TO CHECK THIS??

(c) Let $a := |A|, h = |H|, h_i = |H_i|$,

1. First we show that $a = \sum_i h/h_i$:

$$\sum_i \frac{h}{h_i} = \sum_i [H : H_i] = \sum_i H \cdot \chi_i = |X| = |A| = a \tag{11}$$

2. Now for fixed i look at

$$\sum_\rho (\deg \theta_{i,\rho})^2 = \sum_\rho ([G : G_i] \deg(\widetilde{\chi}_i \otimes \widetilde{\rho}))^2 \tag{12}$$

$$= [H : H_i]^2 \sum_{\rho \text{ irr. of } H_i} (\deg \rho)^2 \tag{13}$$

$$= \frac{h^2}{h_i^2} \cdot h_i = \frac{h^2}{h_i} \tag{14}$$

3. Now sum over all $\theta_{i,\rho}$

$$\sum_{i \in X/H} \sum_{\rho \text{ irr.}} \deg(\theta_{i,\rho})^2 = \sum_{i \in X/H} \frac{h^2}{h_i} \tag{15}$$

$$= h \sum_i \frac{h}{h_i} = h \cdot a = |G| \tag{16}$$

So these are all of the irred. representations of G.

□

Example 5. Application to D_n, A_4, S_4.

(a) $D_n \cong A \rtimes H := C_n \rtimes C_2 \cong \langle r, s \mid r^n = s^2 = e, srs = r^{-1}\rangle$

1. let $X = \mathrm{Hom}(C_n, \mathbb{C}^\times)$, we saw before that these are $\chi_i : r \mapsto \zeta^i$ for $0 \le i < n$, where ζ is an n-th root of unity.
2. Let $H \curvearrowright X$ by $s \cdot \chi_i(r) = \chi_i(srs) = \chi_i(r^{-1}) = \chi_i(r)^{-1} = \zeta^{-i}$.
3. Let $(\chi_i)_{i \in X/H}$ be a representative system for the orbits.
4. If n is even, then χ_0 and $\chi_{n/2}$ will be fixed by H (because $n/2 \equiv -n/2 \mod n$) and so will have trivial orbit, and other orbits will look like $\{\chi_i, \chi_{-1}\}$.
5. For $0 < i < n/2$ we have $H_i = \mathrm{Stab}_H(\chi_i) = \{1\}$ and $G_i = A \rtimes H_i = A$, so the only possible $\rho : H_i \to \mathbb{C}^\times$ is trivial and hence $\widetilde{\rho}$ is also trivial. And therefore

$$\theta_{i,\rho} = \mathrm{Ind}_{C_n}^{D_n}(\chi_i) = \chi_i \oplus s\chi_i = \chi_i \oplus \chi_{-i}$$

6. For $i = 0$ and $i = n/2$ we have $H_i = H$ and $G_i = D_n$ so we won't have to induce, so then we get

$$\widetilde{\chi}_i : D_n \xrightarrow{\pi_A} A \xrightarrow{\chi_i} \mathbb{C}^\times \qquad \widetilde{\rho}_\pm : D_n \xrightarrow{\pi_H} H \xrightarrow{\rho_\pm} \mathbb{C}^\times$$

where $\rho_\pm$ are the irred. representations of H given by $s \mapsto \pm 1$. Notice $\widetilde{\chi}_0$ and $\widetilde{\rho}_+$ are trivial. We get the following:

$$\psi_1 = \widetilde{\chi}_0 \otimes \widetilde{\rho}_+ : r^k \mapsto 1, sr^k \mapsto 1$$
$$\psi_2 = \widetilde{\chi}_0 \otimes \widetilde{\rho}_- : r^k \mapsto 1, sr^k \mapsto -1$$
$$\psi_3 = \widetilde{\chi}_{n/2} \otimes \widetilde{\rho}_+ : r^k \mapsto \zeta^{n/2k} = (-1)^k$$
$$\psi_4 = \widetilde{\chi}_{n/2} \otimes \widetilde{\rho}_- : r^k \mapsto (-1)^k, sr^k \mapsto (-1)(-1)^k = (-1)^{k+1}$$

7. If n is odd, we don't have ψ_3 and ψ_4, and the same as above for $i \neq 0$. these are the same as the ones in talk 5.

(b) $A_4 \cong K \rtimes C_3$ where $K = \langle x, y \mid x^2, y^2, (xy)^2 \rangle \cong C_2 \oplus C_2$ and write $z := xy$.

1. Write χ_0 for the trivial char. on C_2 and $\chi_1 : s \mapsto -1$.
2. Irred. chars of K are tensors of irred. chars $\chi_{ij} = \chi_i \otimes \chi_j$ for $\chi_i \in \operatorname{Hom}(C_2, \mathbb{C}^\times)$
3. $X = \operatorname{Hom}(K, \mathbb{C}^\times) = \{\chi_1, \chi_x, \chi_y, \chi_z\}$.
4. In A_4 we have

$$rxr^{-1} = z, \quad ryr^{-1} = x, \quad rzr^{-1} = y$$

5. And therefore $C_3 \curvearrowright X$ by

$$r\chi_x = \chi_z, \quad r\chi_y = \chi_x, \quad r\chi_z = \chi_y$$

One checks that only χ_1 gets fixed by C_3, and the others are in the same orbit.

6. Lets look at χ_1, here $H_i = C_3$ and $G_i = K \rtimes C_3 = A_4$ is the whole group.
7. Extending χ_1 to $\widetilde{\chi}_1$ is still the trivial character.

$$\begin{tikzcd} A_4 = K \rtimes C_3 \arrow[r, "\pi_K"] \arrow[rr, bend right, "\widetilde{\chi}_1"'] & K \arrow[r, "\chi_1"] & \mathbb{C}^\times \end{tikzcd}$$

8. Now we take an irred. repr. of C_3

$$\begin{tikzcd} A_4 = K \rtimes C_3 \arrow[r, "\pi_{C_3}"] \arrow[rr, bend right, "\widetilde{\rho}_i"'] & C_3 \arrow[r, "\rho_i"] & \mathbb{C}^\times \end{tikzcd}$$

There are 3 irred. reps of C_3, write $\rho_i, 1 \leq i \leq 3$. And finally we get

$$\theta_{1,\rho_i} := \operatorname{Ind}_{A_4}^{A_4}(\widetilde{\chi}_1 \otimes \widetilde{\rho}_i) = \widetilde{\rho}_i, \quad 1 \leq i \leq 3$$

9. Now take any character in the other orbit χ_x, now the stabilizer is trivial and $K \rtimes 1 \cong K$, so $\widetilde{\chi}_x = \chi_x$.
10. Now we have only one choice for an irred. rep of the trivial group. So we get $\widetilde{\rho}$ trivial. And hence:

$$\theta_x, \rho = \operatorname{Ind}_K^{A_4}(\widetilde{\chi}_x) = \bigoplus_{gK \in A_4/K} g\widetilde{\chi}_x = \widetilde{\chi}_x \oplus \widetilde{\chi}_y \oplus \widetilde{\chi}_z$$

(c) $S_4 \cong K \rtimes S_3$ where $S_3 \cong D_3 \cong C_3 \rtimes C_2$

1. $X = \operatorname{Hom}(K, \mathbb{C}^\times)$ like above. $S_3 \curvearrowright X$.
2. The trivial character χ_1 will be ofc. stabilized by the whole group S_3, so $G_1 = K \rtimes S_3 = S_4$ (no need to induce), this gives us 3 irred. representations for each irred. representation of $S_3 \cong D_3 \cong C_3 \rtimes C_2$, so

$$\operatorname{Ind}_{S_4}^{S_4}(\chi_1 \otimes \widetilde{\rho}_i) = \widetilde{\rho}_i, \quad 1 \leq i \leq 3$$

3. The other orbit $\{\chi_x, \chi_y, \chi_z\}$ has stabilizer isomorphic to C_2, so $G_i = K \rtimes C_2$.
4. So we get two representations of degree 3:

$$\operatorname{Ind}_{K \rtimes C_2}^{K \rtimes S_3}(\chi_x \otimes \widetilde{\rho}_0) \quad \text{and} \quad \operatorname{Ind}_{K \rtimes C_2}^{K \rtimes S_3}(\chi_x \otimes \widetilde{\rho}_1)$$

3 Review on classes of groups

Definition 6. A group G is called

(a) *solvable*, if $\exists$ a series

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

and G_i/G_{i-1} are abelian $\forall 1 \leq i \leq n$.

(b) *supersolvable*, if G is solvable, all $G_i \trianglelefteq G$ are normal and all G_i/G_{i-1} are cyclic.

(c) *nilpotent*, if G has a normal series with $G_{i+1}/G_i \leq Z(G/G_i)$

(d) *a p-group*, if $|G| = p^k$ for some prime.

Remark 7. (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d)

Definition 8. A p-subgroup $H \leq G$ is called a *Sylow p-subgroup* if it's maximal.

Theorem 9 (Sylow)**.** *Let G be a group, then for each prime factor of $|G|$*

(a) Sylow p-subgroups exist, $|\operatorname{Syl}_p(G)| \equiv 1 \mod p$.

(b) They are are conjugates.

(c) Each p-subgroup is contained in a Sylow p-subgroup.

4 Representations of supersolvable groups

Definition 10. A group G is called *monomial*, if every irreducible representation is induced by a linear character of a subgroup.

Lemma 11 (Lemma 4)**.** *Let G be supersolvable and nonabelian. Then $\exists A \trianglelefteq G$ abelian not contained in $Z(G)$.*

Theorem 12 (Theorem 16)**.** *Supersolvable $\implies$ monomial.*

Proof. Induction on the order of G:

1. If G abelian, then all irreducible representations have deg 1 and we are done.
2. If G is not abelian, we can consider only faithful irred. reps ρ (i.e. $\ker\rho = \{1\}$)
   - *Why only faithful?* Short answer: We can factor through the kernel and get a faithful irred. representation $\widetilde{\rho}$

     $$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\pi"'] & \mathrm{GL}(V) \\ \widetilde{G} := G/\ker\rho \arrow[ur, hook, "\widetilde{\rho}"'] & \end{tikzcd}$$

     and we can show that if $\widetilde{\rho}$ is monomial then so is ρ.

     Proof. (1) Let $\widetilde{\rho}$ be monomial, i.e. $\widetilde{\rho} = \mathrm{Ind}_{\widetilde{H}}^{\widetilde{G}}(\widetilde{\varphi})$ for some subgroup $\widetilde{H} \leq \widetilde{G}$ and $\widetilde{\varphi} : \widetilde{H} \to \mathrm{GL}(W)$ ($\dim_{\mathbb{C}} W = 1$).

     (2) A subgroup $\widetilde{H} \leq G/\ker\rho$ must be of the form $H/\ker\rho$ for some $H \leq G$ **TODO: Why?**

     (3) So $\widetilde{\rho} = \mathrm{Ind}_{H/\ker\rho}^{G/\ker\rho}(\widetilde{\varphi})$ and we get

     $$\begin{aligned} V &= \bigoplus_{\widetilde{s}\widetilde{H} \in {}^{G/\ker\rho}\big/_{H/\ker\rho}} \widetilde{\rho}_{\widetilde{s}}(W) \\ &= \bigoplus_{sH \in G/H} (\widetilde{\rho}\circ\pi)(s)(W) = \bigoplus_{sH\in G/H} \rho_s(W) \end{aligned}$$

     (4) Write $\varphi : H \xrightarrow{\pi} H/\ker\rho \xrightarrow{\widetilde{\varphi}} \mathrm{GL}(W)$, so we have $\rho = \mathrm{Ind}_H^G(\varphi)$. ◻

3. let $A \trianglelefteq G$ be abelian not contained in $Z(G)$ (lemma 4)
4. ρ faithful $\implies$ $\rho(A)$ not in $Z(\rho(G))$ (because $G \cong \rho(G)$)
5. $\implies$ $\exists a \in A : \rho_a$ is not a homothety $\implies$ $\mathrm{Res}_A^G(\rho)$ is not isotypical (Remark after prop 24 since A abelian).
6. Prop 24 $\implies$ ρ is induced by an irred. rep. of a subgroup $H \lneq G$ containing A. Since induction is transitive, we can apply induction to H proves the theorem.

◻

Extension of Theorem 16 to semidirect products of supersolvable groups by an abelian normal subgroup.

Proposition 13. *$G = A \rtimes H$, where $A \trianglelefteq G$ is abelian and H is supersolvable $\implies$ G monomial.*

Proof. 1. $H \curvearrowright X := \mathrm{Hom}(A, \mathbb{C}^\times)$ by conjugation, (χ_i) a system of representatives for the orbits. $H_i := \mathrm{Stab}_H(\chi_i) \leq H$.

2. Subgroups of a supersolvable group are supersolvable **Why?**, so from the previous proposition H_i is monomial.
3. If we take an irred. $\rho : H_i \to \mathrm{GL}(W)$, then this is induced by a degree one σ rep. of some subgroup H_i'.
4. The extension $\widetilde{\rho}$ to $A \rtimes H_i$ is then also monomial

   $$\begin{tikzcd} A \rtimes H_i \arrow[r] \arrow[rr, bend right, "\widetilde{\rho}"'] & H_i \arrow[r, "\rho"] & \mathrm{GL}(W) \end{tikzcd}$$

   and is induced by $\begin{tikzcd} A \rtimes H_i' \arrow[r] \arrow[rr, bend right, "\widetilde{\sigma}"'] & H_i' \arrow[r, "\sigma"] & \mathbb{C}^\times \end{tikzcd}$

5. Since $\deg\widetilde{\chi}_i = 1$, we have that $\widetilde{\chi}_i \otimes \widetilde{\rho}$ is monomial and since induction is transitive also $\theta_{i,\rho} = \mathrm{Ind}_{G_i}^G(\widetilde{\chi}_i \otimes \widetilde{\rho})$.
6. By proposition 25 all irred. reps of G arise this way, so G is monomial.

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