Let R be a ring.

Definition 1. Let $M \in \mathsf{Mod}_R$ and $N \in {}_R\mathsf{Mod}$ and A an abelian group,

(a) A map $f : M \times N \to A$ is called *R-balanced* if

- its left $\mathbb{Z}$-linear, i.e. $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$.
- its right $\mathbb{Z}$-linear, i.e. $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$
- $\forall r \in R : f(mr, n) = f(m, rn)$.

(b) $\mathrm{Bal}^R_{M,N}(A) = \{f : M \times N \to A \mid f \text{ is } R\text{-balanced}\}$ is an abelian group.

(c) $\mathrm{Bal}^R_{M,N}(-) : \mathsf{Ab} \to \mathsf{Ab}$ is a functor via

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[dr, dashed, "\mathrm{Bal}^R_{M,N}(\varphi)"'] & A \arrow[d, "\varphi"] \\ & A' \end{tikzcd}$$

Idea: R-balaned (bilinear) maps appear naturally, but one needs to treat them seperately (they don't live in Ab). To fix this we want to turn these R-balanced maps $M \times N \xrightarrow{f} A$ into a usual group homomorphism

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "-\otimes-"'] & A \\ M \otimes N \arrow[ur, dashed, "\in\mathsf{Ab}"'] & \end{tikzcd}$$

Theorem 2. *With the notation from definition 1, the functor* $\mathrm{Bal}^R_{M,N} : \mathsf{Ab} \to \mathsf{Ab}$ *is representable, we denote the universal pair by*

$$(M \otimes_R N, - \otimes - : M \times N \to M \otimes N)$$

More concretely, $- \otimes - : M \times N \to M \otimes N$ *is an R-balanced map, such that*

$$\begin{aligned} \mathrm{Bal}^R_{M,N}(A) &\cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, A) \\ \varphi \circ (- \otimes -) &\leftarrow\!\shortmid \varphi \end{aligned}$$

Definition 3. $M \otimes_R N$ is called the *tensor product* of M and N and elements $m \otimes n$ in the $\mathrm{im}(- \otimes - : M \times N \to M \otimes_R N)$ are called *tensors*.

Remark. Its easy to see from the universal property that $m \otimes n$'s generate the group $M \otimes_R N$ (exercise)

$$\begin{tikzcd} M \times N \arrow[rr, "-\otimes-"] \arrow[dr] & & M \otimes N \arrow[dl, shift right, "q"'] \arrow[dl, shift left, "0"] \\ & M \otimes N \big/ \langle \mathrm{im}(- \otimes -) \rangle & \end{tikzcd} \implies q = 0$$

we have

$$M \otimes_R N = \bigoplus_{(m,n) \in M \times N} \mathbb{Z}(m \otimes n) \Bigg/ \left\langle \begin{array}{l} (m_1 + m_2) - m_1 \otimes n - m_2 \otimes n \\ m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2 \\ mr \otimes n - m \otimes rn \end{array} \;\middle|\; \begin{array}{l} m, m_1, m_2 \in M \\ n, n_1, n_2 \in N \\ r \in R \end{array} \right\rangle$$

Proposition 4. *$-\otimes_R - : \mathsf{Mod}_R \times {}_R\mathsf{Mod} \to \mathsf{Ab}$ is a bifunctor. More explicitly, for $M, M', M'' \in \mathsf{Mod}_R$ and $N, N', N'' \in {}_R\mathsf{Mod}$, the following holds:*

(a) For any $\varphi \in \mathrm{Hom}_{\mathsf{Mod}_R}(M, M')$ and $\psi \in \mathrm{Hom}_{RMod}(N, N')\exists!$ homomorphism $\varphi \otimes \psi \in \mathrm{Hom}_{\mathsf{Ab}}(M \otimes_R N, M' \otimes_R N')$ such that $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$

(b) If additionally we have $\varphi' \in \mathrm{Hom}_{ModR}(M', M'')$ and $\psi' \in \mathrm{Hom}_{R\mathsf{Mod}}(N', N'')$, then $(\varphi' \circ \varphi) \otimes (\psi' \circ \psi) = (\varphi' \otimes \psi') \circ (\varphi \otimes \psi)$.

(c) $\mathrm{id}_M \otimes \mathrm{id}_N = \mathrm{id}_{M \otimes N}$.

Proof. TODO. □

Proposition 5. *For $M \in \mathsf{Mod}_R$ and $N \in {}_R\mathsf{Mod}$*

one has:

(a) $M \to M \otimes_R R$ given by $m \mapsto m \otimes 1_R$ is an isomorphism in Ab.

(b) $N \to R \otimes_R N$ given by $n \mapsto 1_R \otimes n$ is an isomorphism in Ab.

Proof. Only (a): This map is clearly $\mathbb{Z}$-linear, we construct the inverse map by TODO. □

Proposition 6. *Let I be a set and $(M_i)_{i \in I}$ a family $M_i \in \mathsf{Mod}_R$ and $N \in {}_R\mathsf{Mod}$ (or the opposite), then $\exists!$ isomorphism*

$$\psi : \left(\bigoplus_{i \in I} M_i \right) \otimes N \xrightarrow{\cong} \bigoplus_{i \in I} (M_i \otimes N), (m_i)_i \otimes n \mapsto (m_i \otimes n)_i.$$

Proof. TODO.

□

Corollary 7. *For sets I and J, $R^{(I)} \otimes_R R^{(J)} \cong R^{(I \times J)}$ given by $e_i \otimes f_j \mapsto e_{(i,j)}$ on the basis.*

0.1 Tensor products over commutative rings

When R is commutative, $R \cong R^{\mathrm{op}}$ and $\mathsf{Mod}_R \cong {}_R\mathsf{Mod}$, in this casse $M \otimes_R N$ admits further structures:

Proposition 8. *Suppose that R is commutative and M, N are two R-modules, then*

(a) $M \otimes_R N$ is an R-module with scalar multiplication given by

$$r(m \otimes n) := rm \otimes n = m \otimes rn$$

on pure tensors. For $A \in {}_R\mathsf{Mod}$ define

$$\mathrm{Bil}^R_{M \times N}(A) := \left\{ f : M \times N \to A \;\middle|\; \begin{array}{l} f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \\ f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \\ f(rm, n) = f(m, rn) = rf(m, n) \end{array} \right\}$$

to be the set of R-bilinear maps from $M \times N$ to A.

(b) The functor $\mathrm{Bil}^R_{M\times N} : {}_R\mathsf{Mod} \to \mathsf{Set}$ *is representable by* $(M \otimes_R N, - \otimes -)$.

(c) $- \otimes - : {}_R\mathsf{Mod} \times {}_R\mathsf{Mod} \to {}_R\mathsf{Mod}$ *is a bifunctor.*

Proof. (a) Let $\ell_r : M \to M$ be given by $m \mapsto rm$, this gives $\ell_r \otimes \mathrm{id}_N : M \otimes_R N \to M \otimes_R N$ by proposition 4. We define scalar multiplication by r on $M \otimes_R N$ to be the above $r \cdot - := \ell_r \otimes \mathrm{id}$. Check that this gives $M \otimes_R N$ on ${}_R\mathsf{Mod}$ structure.

(b) and (c) exercises.

□

Remark. Note that we have less bilinear maps than balanced maps:

$$\mathrm{Hom}_R(M \otimes_R N, A) \cong \mathrm{Bil}^R_{M\times N}(A) \subseteq \mathrm{Bal}^R_{M\times N}(A) \cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, A)$$

0.2 Tensor product of algebras

Let A be a commutative ring and R, R' two A-algebras via $\varphi : A \to R$ and $\varphi' A \to R'$ (where $\varphi(A) \subseteq Z(R), \varphi'(A) \subseteq Z(R')$).

Proposition 9. *(a)* $\exists! A$*-bilinear multiplication* $- \cdot - : (R \otimes_A R') \times (R \otimes_A R') \to R \otimes_A R'$ *given by*

$$(r \otimes r') \cdot (s \otimes s') := rs \otimes r's'$$

on pure tensors.

(b) $(R \otimes_A R', +, \cdot, 0_R \otimes 0_{R'}, 1_R \otimes 1_{R'})$ *is a ring.*

(c) $R \otimes_A R'$ *is an* A*-algebra via* $\varphi_\otimes : A \to R \otimes_A R', a \mapsto a \otimes 1 = 1 \otimes a = a(1 \otimes 1)$.

Proof. TODO. □

Examples. (a) If R is an A-algebra then $M_{n\times n}(A) \otimes_A R \cong M_{n\times n}(R)$.

(b) $M_{n\times n}(A) \otimes_A M_{m\times m}(A) = M_{nm\times nm}(A)$.

(c) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$.

(d) $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_{2\times 2}(\mathcal{C})$ where $\mathbb{H}$ is Hamilton's quaternion algebra.

Definition 10. An (R, R')-*bimodule* is a tuple $(M, 0, +, \cdot, \cdot')$, where $(M, 0, +, \cdot) \in {}_R\mathsf{Mod}$ and $(M, 0, +, \cdot) \in \mathsf{Mod}_{R'}$ such that $\forall r \in R, r' \in R', m \in M$ we have

$$r \cdot (m \cdot' r') = (r \cdot m) \cdot' r'$$

We denote the category of bimodules by ${}_R\mathsf{Mod}_{R'}$.

Remark. (a) If $M \in {}_R\mathsf{Mod}_{R'}$ then one has ring homomorphisms

$$R \to \mathrm{End}_{\mathsf{Mod}_{R'}}(M), r \mapsto r \cdot -$$
$$(R')^{\mathrm{op}} \to \mathrm{End}_{RMod}(M), r' \mapsto - \cdot r'$$

(b) We have an equivalence of categories ${}_R\mathsf{Mod}_{R'} \cong {}_{R\otimes_{\mathbb{Z}}(R')^{\mathrm{op}}}\mathsf{Mod}$. The $R \otimes_{\mathbb{Z}} (R')^{\mathrm{op}}$-module structure comes from

$$R \times (R')^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(M), (r, r') \mapsto r \cdot - \cdot r', r \otimes r' \in R \otimes_{\mathbb{Z}} (R')^{\mathrm{op}}$$

notice that this is bilinear.

Proposition 11. *The bifunctor $- \otimes -$ extends to a bifunctor*

$$- \otimes_{R'} - : {}_R\mathsf{Mod}_{R'} \times {}_{R'}\mathsf{Mod}_{R''} \to {}_R\mathsf{Mod}_{R''}$$

More explicitly for $M \in {}_R\mathsf{Mod}_{R'}$ and $N \in {}_{R'}\mathsf{Mod}_{R''}$ we define the (R, R'')-bimodule on $M \otimes_{R'} N$ by

$$r \cdot (m \otimes n) \cdot r'' := rm \otimes nr''$$

Proof. Exercise. □

Remark. Similarly one has $- \otimes - : {}_R\mathsf{Mod}_{R'} \times {}_{R'}\mathsf{Mod} \to {}_R\mathsf{Mod}$ and $- \otimes - : \mathsf{Mod}_{R'} \times {}_{R'}\mathsf{Mod}_{R''} \to \mathsf{Mod}_{R''}$.

Examples.

1. **Base change:** Let $\varphi : R \to S$ be a ring homomorphism, then S is an (S, R)-bimodule. The functors ${}_R\mathsf{Mod} \to {}_S\mathsf{Mod}, M \mapsto S \otimes_R M$ and $\mathsf{Mod}_R \to \mathsf{Mod}_S, N \mapsto N \otimes_R S$ are called *base change* or base extension from R to S. These are left adjoints to restriction of scalars, for example:

$$S \otimes_R R = S, \quad S \otimes_R R^{(I)} = S^{(I)}$$

2. **Associativity of $\otimes$:** There exists a natural isomorphism

$$(- \otimes_R -) \otimes_S - \cong - \otimes_R (- \otimes_S -) : {}_T\mathsf{Mod}_R \times {}_R\mathsf{Mod}_S \times {}_S\mathsf{Mod}_Q \to {}_T\mathsf{Mod}_Q$$

Remark. Let R be commutative and $M_1, \dots, M_n \in R{-}\mathsf{Mod}$, then $\bigotimes_R^{1\le i\le n} M_i = M_1 \otimes_R \cdots \otimes_R M_n$ represents the functor $\mathrm{Multi}^R_{M_1\times\cdots\times M_n}(-)$ of R-multilinear maps on $\bigtimes_{1\le i\le n} M_i$.

Proposition 12. *Another functor on bimodules:*

(a) For $M \in {}_S\mathsf{Mod}_R$ and $N \in {}_T\mathsf{Mod}_R$ the abelian group $\mathrm{Hom}_R(M, N)$ carries a natural (T, S)-bimodule structure defined by

$$(t \cdot f \cdot s)(x) = t \cdot f(sx)$$

for $f \in \mathrm{Hom}_R(M, N), s \in S, t \in T, x \in M$, this gives a bifunctor

$${}_S\mathsf{Mod}_R \times {}_T\mathsf{Mod}_R \to {}_T\mathsf{Mod}_S$$

(b) Similarly one has a bifunctor ${}_R\mathsf{Mod}_S \times {}_R\mathsf{Mod}_T \to {}_S\mathsf{Mod}_T$

Proof. Exercise. □

Theorem 13 (Hom, $\otimes$ adjunction, Jacobson Prop. 3.8)**.** *Let Let R, S, T, U be rings and $M \in {}_R\mathsf{Mod}_S, N \in {}_S\mathsf{Mod}_T, P \in {}_U\mathsf{Mod}_T$. There exists a natural isomorphism:*

$$\begin{tikzcd} {}_R\mathsf{Mod}_S^{\mathrm{op}} \times {}_U\mathsf{Mod}_T \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & {}_U\mathsf{Mod}_R \mathit{TODO} \end{tikzcd}$$