# Talk 10: Examples of induced representations

**Remark.** In this talk all groups are finite and all representations are finite dimensional over $\mathbb{C}$.

## 1 Normal subgroups

**Definition 1** (Normal subgroup)**.** Recall that $N \trianglelefteq G :\iff gNg^{-1} = N, \forall g \in G$.

**Definition 2** (Isotypical representation)**.** A representation is called *isotypical* if its a direct sum of isomorphic irred. representations.

**Proposition 3.** *Let $A \trianglelefteq G, \rho : G \to \mathrm{GL}(V)$ be an irreducible representation, then either*

*(a) $\exists H \lneq G$ containing $A$ and an irred. $\sigma : H \to \mathrm{GL}(W)$ so that $\rho = \mathrm{Ind}_H^G(\sigma)$, or*

*(b) the restriction $\rho|_A$ is isotypic.*

**Remark 4.** If $A$ is abelian, then (b) $\iff \rho(a)$ is a homothety $\forall a \in A$.

**Corollary 5.** *If $A \trianglelefteq G$ is abelian, then $\forall \rho$ irred. representation of $G$ we have $\deg \rho \mid [G : N]$.*

**Remark 6.** If $A \leq G$ is an abelian subgroup (not necessarily normal), this doesn't hold in general, but we still have the bound $\deg \rho \leq [G : A]$ like we have seen in corollary 1.3. from talk 4.

## 2 Semidirect products by an abelian group

**Construction 7.** Let $G = A \rtimes H$, where $A \trianglelefteq G$ is an abelian normal subgroup.

(1) Let $X := \mathrm{Hom}(A, \mathbb{C}^\times)$ be the set of irred. characters of $A$, and let $G \curvearrowright X$ by $s \cdot \chi(a) = \chi(s^{-1}as)$.

(2) Let $(\chi_i)_{i \in X/H}$ be representatives for the orbits of the action $H \curvearrowright X$, and write $H_i := \mathrm{Stab}_H(\chi_i)$ and $G_i := A \rtimes H_i$.

(3) Extend $\chi_i$ to $G_i$ by $\widetilde{\chi}_i(a \cdot h) = \chi_i(a)$.

(4) Let $\rho : H_i \to \mathrm{GL}(W)$ be irreducible and extend it to an irred. representation of $G_i$ by $\widetilde{\rho}(a \cdot h) = \rho(h)$.

(5) The tensor $\widetilde{\chi}_i \otimes \widetilde{\rho}$ will be also irreducible.

(6) Finally define $\theta_{i,\rho} = \mathrm{Ind}_{G_i}^G(\widetilde{\chi}_i \otimes \widetilde{\rho})$.

**Proposition 8.** *(a) $\theta_{i,\rho}$ is irred.*

*(b) $\theta_{i,\rho} \cong \theta_{i',\rho'} \implies i = i'$ and $\rho \cong \rho'$.*

*(c) These are all the irred. representations of $G$.*

**Example 9.** Application to $D_n, A_4$ and $S_4$.

## 3 Review on classes of groups

**Definition 10.** A group $G$ is called

(a) *solvable*, if $G$ has a normal series

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

where all factors $G_i/G_{i-1}$ are abelian.

(b) *supersolvable*, if $G$ is solvable, all $G_i \trianglelefteq G$ are normal in $G$ and all factors $G_i/G_{i-1}$ are cyclic.

(c) *nilpotent*, if $G$ has a normal series with $G_{i+1}/G_i \leq Z(G/G_i)$

(d) *a $p$-group*, if $|G| = p^k$ for some prime.

**Remark 11.** (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d)

**Definition 12.** A $p$-subgroup $H \leq G$ is called a *Sylow $p$-subgroup* of $G$ if it's maximal.

**Theorem 13** (Sylow)**.** *Let $G$ be a group, then for each prime factor of $|G|$*

*(a) Sylow $p$-subgroups exist, and their number is congruent to $1 \mod p$.*

*(b) They are conjugates.*

*(c) Each $p$-subgroup is contained in a Sylow $p$-subgroup.*

# 4 Representations of supersolvable groups

**Definition 14** (Monomial)**.** A group $G$ is *monomial*, if $\forall$ irred. $\rho$ is induced by a degree 1 rep. of a subgroup.

**Lemma 15.** *Let $G$ be supersolvable and non-abelian. Then $\exists A \trianglelefteq G$ abelian not contained in $Z(G)$.*

**Theorem 16.** *Every supersolvable group is monomial.*

**Corollary 17.** *If $G = A \rtimes H$, where $A \trianglelefteq G$ is abelian and $H$ is supersolvable, then $G$ is monomial.*

# References

[Ser77] Jean-Pierre Serre: *Linear Representations of Finite Groups*, Springer-Verlag, New York, 1977.