Definition 0.1. Let R be a commutative ring.

(a) A category $\mathcal{A}$ is called R-linear if

  (i) $\forall X, Y \in \mathcal{A}, \mathcal{A}(X,Y)$ is an R-module $(\mathcal{A}(X,Y), 0_{X,Y}, +_{X,Y}, \cdot_{X,Y})$.

  (ii) $\forall X, Y, Z \in \mathcal{A}$ the composition map

$$\begin{aligned} \mathcal{A}(X,Y) \times \mathcal{A}(Y,Z) &\to \mathcal{A}(X,Z) \\ (\varphi, \psi) &\mapsto \psi \circ \varphi \end{aligned}$$

  is R-bilinear. (in particular $r(\psi \circ \varphi) = (r \cdot \psi) \circ \varphi = \psi \circ (r\varphi)$).

(b) A functor $F : \mathcal{A} \to \mathcal{A}'$ between R-linear categories is called R-linear if $\forall X, Y \in \mathcal{A}$ the map $F : \mathcal{A}(X,Y) \to \mathcal{A}'(FX, FY)$ is R-linear.

(c) For $\mathbb{Z}$-linear (preadditive) categories $\mathcal{A}$ and $\mathcal{A}'$, the full subcategory $\mathrm{Add}(\mathcal{A}, \mathcal{A}') \subseteq \mathrm{Fun}(\mathcal{A}, \mathcal{A}')$ of $\mathbb{Z}$-linear (additive) functors between them is again a $\mathbb{Z}$-linear category, i.e. there is a natural addition on natural transforations.

Examples. (a) ${}_R\mathsf{Mod}$ is $\mathbb{Z}$-linear and in fact (R is commutative) R-linear.

(b) $\mathcal{A}$ is R-linear $\iff$ $\mathcal{A}^{\mathrm{op}}$ is also R-linear.

(c) If $\mathcal{A}$ and $\mathcal{A}'$ are R-linear, then $\mathcal{A} \times \mathcal{A}'$ is R-linear.

(d) Let S be a (not necessarily commutative) ring and $\underline{S}$ the category

$$(\{*\}, S, \mathrm{dom} = \{*\}, \mathrm{cod} = \{*\}, \mathrm{id}_* = \mathrm{id}_S, \circ = \cdot_S)$$

associated to S, then $\underline{S}$ is $\mathbb{Z}$-linear ($\mathrm{Hom}_{\underline{S}}(*, *) = S$).

Lemma 0.2. *If $\mathcal{A}$ is $\mathbb{Z}$-linear, then for $X \in \mathcal{A}$ the following are equivalent:*

(i) X is initial.

(ii) X is terminal.

(iii) $\mathcal{A}(X,X) = 0$ (in particular $\mathrm{id}_X = 0$)

Proof. Exercise. □

Definition 0.3. An $X \in \mathcal{A}$ satisfying conditions of Lemma 2 is called a *zero object*

Notation. The zero object of $\mathcal{A}$ (if it exists) is unique up to unique isomorphism so we just write 0 or 0_A for this object.

Lemma 0.4. *Let $\mathcal{A}$ be $\mathbb{Z}$-linear with a 0-object and $X, Y \in \mathcal{A}$, then the zero map $0_{X,Y} \in \mathcal{A}(X,Y)$ (remember $\mathcal{A}(X,Y)$ is an abelian group) is equal to the composition $X \xrightarrow{\exists!} 0 \xrightarrow{\exists!} Y$*

Proof. Exercise. □

Proposition 0.5. *Let $\mathcal{A}$ be $\mathbb{Z}$-linear, then for $X_1, X_2 \in \mathcal{A}$ the following are equivalent:*

(i) The product $X_1 \Pi X_2$ exists.

(ii) *The coproduct* $X_1 \amalg X_2$ *exists.*

(iii) $\exists Y \in \mathcal{A}, p_1, p_2 : Y \to X_i$ *and* $\iota_1, \iota_2 : X_i \to Y$ *such that*

$$p_i \circ \iota_j = \begin{cases} 1_{X_i}, & i = j \\ 0, & i \neq j \end{cases}$$

Proof. TODO. □

Remark. In (iii) (Y, ι_1, ι_2) is the coproduct and (Y, p_1, p_2) is the product.

Definition 0.6. In a $\mathbb{Z}$-linear category we deonte $X \Pi Y = X \amalg Y$ by $X \oplus Y$ and call it the *direct sum* of X and Y.

Lemma 0.7. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be* $\mathbb{Z}$*-linear and* $F : \mathcal{A} \to \mathcal{A}'$ *an additive functor, then*

(i) *If* $0_{\mathcal{A}}$ *exists then* $0_{\mathcal{A}'}$ *exists and* $F(0_{\mathcal{A}}) \cong 0_{\mathcal{A}'}$.

(ii) *If* $X, Y \in \mathcal{A}$ *and* $X \oplus Y$ *exists, then* $FX \oplus FY$ *exists and* $FX \oplus FY \cong F(X \oplus Y)$.

Proof. Exercise. □

Definition 0.8. A category $\mathcal{A}$ is called *additive* (or R-linear additive) if $\mathcal{A}$ is $\mathbb{Z}$-linear (or R-linear), $0_{\mathcal{A}}$ exists and $\forall X, Y \in \mathcal{A} : X \oplus Y$ exists.

Remark. If $\mathcal{A}$ is additive then the addition on $\mathcal{A}(X, Y)$ is determined by the composition map! (Morel II 1.2.4)

0.1 Kernels and cokernels

Recall that the equalizer of two morphisms $f, g : X \to Y$ is the limit of the diagram $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, so $Z \xrightarrow{u} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ such that for every $W \xrightarrow{v} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ with $f \circ v = g \circ v$, v factors through u:

$$\begin{tikzcd} Z \arrow[r, "u"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ W \arrow[u, dashed] \arrow[ur, "v"'] & & \end{tikzcd}$$

similarly the coequalizer is the colimit of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$.
In the category of R-modules, $\mathrm{eq}(f, g) = \ker(f - g)$ and $\mathrm{coeq}(f, g) = \mathrm{coker}(f - g)$. In particular $\ker f = \mathrm{eq}(f, 0)$.

Definition 0.9. Let $\mathcal{A}$ be a $\mathbb{Z}$-linear category and $f \in \mathcal{A}(X, Y)$.

(a) define $\ker f := \mathrm{eq}(f, 0)$ and $\mathrm{coker}\, f = \mathrm{coeq}(f, 0)$ (if they exist)

(b) If $\ker f$ exists, define the coimage as $X \to \mathrm{coim}\, f = \mathrm{coker}(\ker f \to X)$ (it might not exist).

(c) If $\operatorname{coker} f$ exists, define the image as $\operatorname{im} f \to Y = \ker(Y \to \operatorname{coker} f)$.

Example 0.10. Let $X_1, X_2 \in \mathcal{A}$ and $(X_1 \oplus X_2, \iota_1, \iota_2, p_1, p_2)$ the direct sum. Then $\iota_1 = \ker p_2, \iota_2 = \ker p_1, p_1 = \operatorname{coker} \iota_2, p_2 = \operatorname{coker} \iota_1$

Lemma 0.11. *Kernels are monomorphisms and cokernels are epimorphisms.*

Proof. We only prove the statement for kernels. Let $\ker f \xrightarrow{\iota} X \overset{f}{\underset{0}{\rightrightarrows}} Y$, now assume $\iota \circ \varphi = \iota \circ \varphi'$

$$Z \overset{\varphi}{\underset{\varphi'}{\rightrightarrows}} \ker f \xrightarrow{\iota} X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

By the definition of equalizer

$$\begin{aligned} \mathcal{A}(Z, \ker f) &\simeq \{\psi : Z \to X \mid f \circ \psi = 0\} \\ \varphi &\mapsto \iota \circ \varphi \end{aligned}$$

is bijective, so $\varphi = \varphi'$. □

Theorem 0.12. *Let* $f \in \mathcal{A}(X, Y)$ *and assume* $\ker f, \operatorname{coker} f, \operatorname{im} f, \operatorname{coim} f$ *exist. Then there exists a unique morphism* $u : \operatorname{coim} f \to \operatorname{im} f$ *such that* f *is equal to the composition*

$$X \xrightarrow{c} \operatorname{coim} f \xrightarrow{\exists! u} \operatorname{im} f \xrightarrow{d} Y$$

This is called the canonical factorization *of* f *(or epi-mono factorization).*

Remark. Note that by lemma 11, c is an epimorphism and d is a monomorphism.

Proof. TODO. □

Note that in the category of R-modules, u is an isomorphism.

Definition 0.13. A category $\mathcal{A}$ is called *abelian* if

(i) $\mathcal{A}$ is additive ($\mathbb{Z}$-linear, 0 exists and $X \oplus Y$ exists)

(ii) $\forall f \in \mathcal{A}(X, Y) : \ker f, \operatorname{coker} f$ exist.

(iii) $\forall f \in \mathcal{A}(X, Y)$ with canonical factorization $f = d \circ u \circ c$, u is an isomorphism.

Example 0.14.

- For any ring R the category ${}_R\mathsf{Mod}$ is abelian.
- For rings R, R' the category ${}_R\mathsf{Mod}_{R'}$ is abelian ($\cong {}_{R \otimes_{\mathbb{Z}} (R')^{\mathrm{op}}}\mathsf{Mod}$).

Example 0.15.

Let $\mathcal{A} \subseteq \mathsf{Ab}$ be the fullsubcategory of finitely generated free $\mathbb{Z}$-modules, then $\mathcal{A}$ is additive.

For $f : X \to Y$ in $\mathcal{A}$ the usual $\ker_{\mathsf{Ab}} f$ as abelian group is again a finitely generated free abelian group and is also the kernel in $\mathcal{A}$.

The cokernel however might have torsion. In fact

$$\operatorname{coker}_{\mathcal{A}} f = {\operatorname{coker}_{\mathsf{Ab}} f}\Big/{\operatorname{Tor}(\operatorname{coker}_{\mathsf{Ab}} f)}$$

So kernels and cokernels exists in $\mathcal{A}$, now let $f : \mathbb{Z} \to \mathbb{Z}, n \mapsto 2n$, the canonical factorization is

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{u} & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} \\ & & n & \longmapsto & 2n & & \end{array}$$

since u is not an isomorphism, $\mathcal{A}$ is not abelian.

Proposition 0.16. *Let $\mathcal{C}$ be a category, then*

(i) $\mathcal{C}$ *is* $\mathbb{Z}$*-linear* $\iff$ $\mathcal{C}^{\mathrm{op}}$ *is* $\mathbb{Z}$*-linear.*

(ii) $\mathcal{C}$ *is additive* $\iff$ $\mathcal{C}^{\mathrm{op}}$ *is additive.*

(iii) $\mathcal{C}$ *is abelian* $\iff$ $\mathcal{C}^{\mathrm{op}}$ *is abelian.*

Moreover (if they exist) $\ker_{\mathcal{C}} f = \operatorname{coker}_{\mathcal{C}^{\mathrm{op}}} f$ *and* $\operatorname{im}_{\mathcal{C}} f = \operatorname{coim}_{\mathcal{C}^{\mathrm{op}}} f$.

0.2 Abelian categories

From now on let $\mathcal{A}$ be an abelian category.

Remark. Let $f : X \to Y$ and $X \twoheadrightarrow \operatorname{coim} f \underset{u}{\xrightarrow{\cong}} \operatorname{im} f \hookrightarrow Y$ be the canonical factorization. Then either $X \twoheadrightarrow \operatorname{coim} f \hookrightarrow Y$ or $X \twoheadrightarrow \operatorname{im} f \hookrightarrow Y$ (or anything else isomorphic to them) is called "the" canonical factorization for f.

Proposition 0.17. *Let $X, Y \in \mathcal{A}$*

(a) $\operatorname{coker}(0 \to X) = X \xrightarrow{\mathrm{id}} X$ *and* $\ker(X \to 0) = X \xrightarrow{\mathrm{id}} X$.

(b) $f : X \to Y$ *is a monomorphism* $\iff$ $\ker f = 0$ $\iff$ *the canonical factorization of f is* $X \xrightarrow{\mathrm{id}} X \overset{f}{\hookrightarrow} Y$ $\iff$ *f is a kernel.*

(c) $g : C \to Y$ *is an epimorphism* $\iff$ $\operatorname{coker} g = 0$ $\iff$ *the canonical factorization of g is* $X \xrightarrow{g} Y \xrightarrow{\mathrm{id}} Y$ $\iff$ *g is a cokernel.*

(d) *u is an isomorphism* $\iff$ *u is a monomorphism and an epimorphism.*

Corollary 0.18. $X \xrightarrow{a} Z \xrightarrow{b} Y$ *is the canonical factorization for* $f = b \circ a$ $\iff$ *a ia a monomorphism and b an epimorphism.*

Definition 0.19. In an abelian category $f \in \mathcal{A}(X, Y)$ is called injective if $\ker f = 0$ and surjective if $\operatorname{coker} f = 0$.

Corollary 0.20. *Let $f \in \mathcal{A}(X, Y)$, then*

(a) *f is injective* $\iff$ *f is a monomorphism.*

(b) *f is surjective* $\iff$ *f is a epimorphism.*

(c) *f is injective and surjective* $\iff$ *f is a isomorphism.*

0.3 Exactness

Lemma 0.21. *Let (g, f) be a composable pair of morphisms in $\mathcal{A}$ such that $g \circ f = 0$, then $\exists$ a canonical injection* $\operatorname{im} f \hookrightarrow \ker g$.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ and write the epi-mono factorization of f:

$$X \xrightarrow{c} \operatorname{coim} f \xrightarrow{\cong} \operatorname{im} f \xhookrightarrow{d} Y \xrightarrow{g} Z$$

and call the isomorphism in the middle $u : \operatorname{coim} f \cong \operatorname{im} f$. Now $g \circ d \circ u \circ c = 0 \implies g \circ d \circ u = 0$ since c is an epimorphism and u isomorphism $\implies g \circ d = 0$. By the definition of the kernel d must factor through $\ker g$:

$$\operatorname{im} f \xrightarrow{e} \ker g \xhookrightarrow{\iota} Y \xrightarrow{g} Z$$

Since d is injective: if $e \circ \varphi = c \circ \varphi' \implies \iota \circ c \circ \varphi = \iota \circ c \circ \varphi' \implies d \circ \varphi = d \circ \varphi' \implies \varphi = \varphi' \implies e$ is a monomorphism $\implies \operatorname{im} f \hookrightarrow \ker g$. ☐

Definition 0.22. (a) A *complex* in $\mathcal{A}$ is a family of morphisms $(d_i : X_i \to X_{i+1})_{i \in J}$ and $\emptyset \neq J \subseteq \mathbb{Z}$ an interval such that $\forall i \in J^- : d_{i+1} \circ d_i = 0$ ($J^- := J \setminus \sup J$), in other notation:

$$\cdots \to X_{i-1} \xrightarrow{d_{i-1}} X_i \xrightarrow{d_i} X_{i+1} \to \cdots$$

such that $d_i \circ d_{i-1} = 0$.

(b) An exact sequence in $\mathcal{A}$ (or an acyclic complex) is a complex such that $\forall i \in J^-$ the canonical monomorphism from lemma 21 $\operatorname{im} d_i \hookrightarrow \ker d_{i+1}$ is an isomorphism

Proposition 0.23. *For any $f \in \mathcal{A}(X, Y)$ the sequence*

$$0 \to \ker f \xrightarrow{\iota} X \xrightarrow{f} Y \xrightarrow{\pi} \operatorname{coker} f \to 0$$

is exact, where ι is the kernel-morphism and π is the cokernel-morphism.

Proof.

- Exactness at $\ker f$: Kernels are monomorphisms and so

$$\ker \iota = 0 = \operatorname{im}(0 \to \ker f)$$

- Exactness at X:

$$\operatorname{im} \iota \underset{16}{=} \ker f \xrightarrow{\iota} X = \ker f$$

for Y and $\operatorname{coker} f$ pass to $\mathcal{A}^{\mathrm{op}}$.

☐

Lemma 0.24 (Decomposing and concatenating exact sequences)**.**

(a) If

$$\cdots \to A \xrightarrow{a} B \to 0 \quad \textit{and} \quad 0 \to B \xrightarrow{b} C \xrightarrow{c} D \to \cdots$$

are exact at B and C, then

$$\cdots \to A \xrightarrow{b \circ a} C \xrightarrow{c} D \to \cdots$$

is exact at C, i.e. $c \circ b \circ a = 0$ and $\ker c = \operatorname{im}(b \circ a)$.

(b) If

$$\cdots \to A \xrightarrow{a} B \xrightarrow{b} C \to 0 \quad \textit{and} \quad 0 \to C \xrightarrow{c} D \to \cdots$$

are exact at B and C, then

$$\cdots \to A \xrightarrow{a} B \xrightarrow{c \circ b} D \to \cdots$$

is exact at B.

(c) If

$$\cdots \to X \xrightarrow{f} Y \xrightarrow{g} Z \to \cdots$$

is exact at Y, then:

(i) $\cdots \to X \to \operatorname{im} f \to 0$ *and* $0 \to \ker g \to Y \xrightarrow{g} Z \to \cdots$ *are exact at* $\operatorname{im} f, \ker g, Y$.

(ii) $\cdots X \xrightarrow{f} Y \to \operatorname{coker} f \cong \operatorname{coim} g \to 0$ *and* $0 \to \operatorname{im} g \to Z \to \cdots$ *are exact at* $Y, \operatorname{coker} f, \operatorname{im} g$.

Example. Prop 23 + Lemma 24 gives for $f \in \mathcal{A}(X, Y)$:

$$0 \to \ker f \to X \to \operatorname{coim} f \to 0$$

and

$$0 \to \operatorname{im} f \to Y \to \operatorname{coker} f \to 0$$

are exact sequences.

Definition. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is also called a *short exact sequence* (s.e.s.)

Lemma 0.25. *Let $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ be a complex in $\mathcal{A}$, then:*

(a) The sequence

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, "b"] & C \arrow[r] \arrow[dr] & 0 \\ & & & X \end{tikzcd}$$

is (right) exact $\iff \forall X \in \mathcal{A}$

$$0 \to \operatorname{Hom}_{\mathcal{A}}(C, X) \xrightarrow{b^* = - \circ b} \operatorname{Hom}_{\mathcal{A}}(B, X) \xrightarrow{a^*} \operatorname{Hom}_{\mathcal{A}}(A, X)$$

is exact in Ab.

(b) The sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C$$

is (left) exact $\iff \forall X \in \mathcal{A}:$

$$0 \to \operatorname{Hom}_{\mathcal{A}}(X, A) \xrightarrow{a_*} \operatorname{Hom}_{\mathcal{A}}(X, B) \xrightarrow{b_*} \operatorname{Hom}_{\mathcal{A}}(X, C)$$

is exact in Ab.

Proof. Stacks Project. ∎

Proposition 0.26. *Let the following be a short exact sequence in $\mathcal{A}$*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "a"] & B \arrow[l, bend left, dashed, "s"] \arrow[r, "b"] & C \arrow[l, bend left, dashed, "t"] \arrow[r] & 0 \end{tikzcd}$$

Then the following are equivalent:

(i) $\exists s \in \mathcal{A}(B, A) : s \circ a = 1_A$

(ii) $\exists t \in \mathcal{A}(C, B) : b \circ t = 1_C$

(iii) $\exists s \in \mathcal{A}(B, A) \exists t \in \mathcal{A}(C, B) : s \circ a = 1_A, b \circ t = 1_C$ *and* $1_B = a \circ s + t \circ b$

(iv) $B \cong A \oplus C$ *with respect to the morphisms from (iii).*

Proof. Exercise. (See 1.28) ∎

Proposition 0.27. *Let $\mathcal{A}$ be an abelian category, then $\mathcal{A}$ possesses (all) finite limits and colimits, (finite meaning over finite index categories).*

Proof. Exercise (See 2.46) ∎

Proposition 0.28 (Special case of 27)**.** *If $\mathcal{A}$ is an abelian category, then $\mathcal{A}$ has pullbacks and pushouts. They are given by*

$$\text{p.b.} \left(\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \right) = \ker(X \oplus Y \xrightarrow{f-g} Z)$$

and

$$\text{p.o.} \left(\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd} \right) = \operatorname{coker}(X \xrightarrow{(f,-g)} Y \oplus Z)$$

Proof. Exercise. ∎

Definition 0.29. A commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd}$$

in $\mathcal{A}$ is called

(a) *cartesian* if

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \\ C & \end{tikzcd} \cong \text{p.b.} \left(\begin{tikzcd} & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd} \right)$$

(b) *cocartesian* if

$$\begin{tikzcd} & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd} \cong \text{p.o.} \left(\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \\ C & \end{tikzcd} \right)$$

Lemma 0.30. *Let $(*)$ be a commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd} \qquad (*)$$

Then:

(a) $(*)$ is cartesian $\iff 0 \to A \xrightarrow{(a,b)} B \oplus C \xrightarrow{c+d} D$ is exact.

(b) $(*)$ cocartesian $\iff A \to B \oplus C \to D \to 0$ is exact.

(c) If $(*)$ is cartesian, then $\ker a \cong \ker d$ via b.

(d) If $(*)$ is cocartesian, then $\operatorname{coker} a \cong \operatorname{coker} d$ via c.

(e) If $(*)$ is cartesian and d is surjective, then $(*)$ is cocartesian.

(f) If $(*)$ is cocartesian and a is injective, then $(*)$ is cartesian.

0.4 The homomorphism theorem and the isomorphism theorems

Definition 0.31. (a) On the set of monomorphisms to X define an equivalence relation $(f : U \hookrightarrow X) \sim (g : V \hookrightarrow X) \iff \exists$ commutative diagram

$$\begin{tikzcd} U \arrow[rr, dashed, "\alpha"] \arrow[dr, hook, "f"'] & & V \arrow[dl, hook', "g"] \\ & X & \end{tikzcd}$$

with α an isomorphism. Call the equivalence classes *subobjects* of X.

(b) On the set of epimorphisms from Y define an equivalence relation $(f : Y \twoheadrightarrow W) \sim (g : Y \twoheadrightarrow Z) \iff \exists$ commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, two heads, "f"'] \arrow[dr, two heads, "g"] & \\ W \arrow[rr, dashed, "\beta"'] & & Z \end{tikzcd}$$

with β an isomorphism. Call the equivalence classes *quotients* of Y.

Also: Call a monomorphism $f : U \to X$ (epimorphism $g : Y \to W$) a subobject (quotient) meaning its equivalence class.

Proposition 0.32. *On subobjects* $[U \xrightarrow{f} X], [V \xrightarrow{X}]$ *of* $X \in \mathcal{A}$ *define a relation* $[U \xrightarrow{f} X] \leq [V \xrightarrow{X}] \iff \exists$ *commutative diagram:*

$$\begin{tikzcd} U \arrow[rr, hook, dashed, "\alpha"] \arrow[dr, hook, "f"'] & & V \arrow[dl, hook', "g"] \\ & X & \end{tikzcd}$$

with α *a monomorphism. Then*

(a) $\leq$ *is a partial ordering on the class of subobjects.*

(b) The infimum of $[U \xrightarrow{f} X]$ *and* $[V \xrightarrow{g} X]$ *exists and is represented by the pullback*

$$\text{p.b.}\left(\begin{tikzcd} & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}\right)$$

together with the diagonal map $\text{p.b.}(\cdots) \to X$. *Write* $U \cap V$.

(c) The supremum of $[U \xrightarrow{f} X]$ *and* $[V \xrightarrow{g} X]$ *exists and is represented by the pushout*

$$\text{p.o.}\left(\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \\ V & \end{tikzcd}\right)$$

together with the canonical arrow $U \cap V \to \text{p.o.}(\cdots)$. *Write* $U + V$

Remark 0.33. Now we can formulate the homomorphism and isomorphism theorems:

(a) **Homomorphism theorem:** Given $f \in \mathcal{A}(X, Y) \exists$ canonical factorization:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "\text{can.}"'] & & Y \\ X \big/ \ker f := \operatorname{coim} f \arrow[rr, "\cong"'] & & \operatorname{im} f \arrow[u, "\text{can.}"'] \end{tikzcd}$$

(b) **First isomorphism theorem:** For subobjects $U, V \leq X$ we have

$$U + V \Big/ U \cong V \Big/ V \cap U$$

(c) **Second isomorphism theorem:** Given a monomorphism $\iota : U \to X, \exists$ bijection:

$$\{V \mid U \leq V \leq X\} \leftrightarrow \left\{V \leq X \Big/ U = \operatorname{coim} \iota\right\}$$

and for U, V subobjects with $\beta : U \to V$ and $\iota : V/U \hookrightarrow X/U$, we have an isomorphism

$$X \Big/ V \cong (X/U) \Big/ (V/U) = \operatorname{coim} \iota$$

Theorem 0.34 (Snake lemma)**.** *TODO.*