

Algebra 2 Lecture Notes
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Chapter 1

Modules

1.1 Modules

Let $(R, 0, 1, +, \cdot)$ or simply R be a ring.

Definition 1.1. (a) A left R -module $(M, 0, +, \cdot)$ or simply M is an abelian group $(M, 0, +)$, together with an operation $\cdot : R \times M \rightarrow M, (r, m) \mapsto r \cdot m = rm$, such that for all $a, b \in R, m, n \in M$

$$(M1) \quad a(m + n) = am + an \text{ and } (a + b)m = am + bm$$

$$(M2) \quad a(b \cdot m) = (ab) \cdot m$$

$$(M3) \quad 1 \cdot m = m$$

(b) Let M, N be left R -modules. A map $\varphi : M \rightarrow N$ is called R -linear or a left R -module homomorphism : $\iff \varphi : (M, 0, +) \rightarrow (N, 0, +)$ is a group homomorphism, and $\forall a \in R, m \in M : \varphi(am) = a\varphi(m)$. Define $\text{Hom}_R(M, N) = \{\varphi : M \rightarrow N \mid \varphi \text{ is } R\text{-linear}\}$.

Facts 1.2 (Excurs.). $\forall x \in M, a \in R : 0_R \cdot x = 0_M, a \cdot 0_M = 0_M, (-1) \cdot x = -x$

Remark 1.3 (Excurs.). (a) $\text{Hom}_R(M, N)$ is an abelian group with $0 =$ the map $M \rightarrow \{0_N\}$ and $\varphi + \psi : M \rightarrow N, m \mapsto \varphi(m) + \psi(m)$.

(b) If R is commutative, then $\text{Hom}_R(M, N)$ is an R -module via

$$r \cdot \varphi : M \rightarrow N, m \mapsto r \cdot \varphi(m)$$

(c) If an abelian group $(M, 0, +)$ carries an operation $\cdot : M \times R \rightarrow M, (m, r) \mapsto m \cdot r$ such that:

$$(M1') \quad (m + n) \cdot a = m \cdot a + n \cdot a, m \cdot (a + b) = ma + mb$$

$$(M2') \quad (m \cdot a) \cdot b = m \cdot (ab)$$

$$(M3') \quad m \cdot 1 = m$$

then $(M, 0, +, \cdot)$ is called a right R -module. Analogously we can define right R -module homomorphisms.

Convention 1.4. We shall use the term R -module for left R -module, since we will mainly work with these. In fact right R -modules are left R^{op} -modules.

Definition 1.5. The opposite ring (Gegenring) of $(R, 0, 1, +, \cdot)$ is $R^{\text{op}} = (R, 0, 1, +, \cdot^{\text{op}})$ with $a \cdot^{\text{op}} b = b \cdot a$

Facts 1.6 (Excercise). (a) R^{op} is a ring

(b) $\text{id}_R : R \rightarrow R$ is a ring homomorphism $\iff R$ is commutative.

(c) $\text{id}_R : R \rightarrow (R^{\text{op}})^{\text{op}}$ is an isomorphism.

In particular: If R is commutative, then left R -modules are right R -modules.

Remark 1.7 (Excercise). Let $(M, 0, +)$ be an abelian group.

(a) The abelian group $\text{End}_{\mathbb{Z}}(M) = \text{Hom}_{\mathbb{Z}}(M, M)$ is a ring with composition as multiplication.

(b) There is a bijection $\{\text{operations } * : R \times M \rightarrow M \mid (M, 0, +, *) \text{ is an } R\text{-module}\} \leftrightarrow \{\text{ring homomorphisms } \varphi : R \rightarrow \text{End}_{\mathbb{Z}}(M)\}$ via

$$* \mapsto \varphi_* : R \rightarrow \text{End}_{\mathbb{Z}}(M), r \mapsto (\varphi_*(r) : m \mapsto r \cdot m)$$

figure out an inverse.

(c) If M is an R -module, then $\text{End}_R(M) \subseteq \text{End}_{\mathbb{Z}}(M)$ is a subring

(d) The map $R^{\text{op}} \rightarrow \text{End}_R(R), r \mapsto \rho_r : a \mapsto a \cdot r$ is a ring isomorphism. The inverse is $\text{End}_R(R) \rightarrow R^{\text{op}}, \varphi \mapsto \varphi(1)$

Example 1.8. (a) Let K be a field, K -modules are K -vector spaces and vice versa.

(b) If $(M, 0, +)$ is an abelian group, it is in a unique way a \mathbb{Z} -module.

(c) Let K be a field, $R = M_{n \times n}(K), n > 1, V_n(K) = \text{column } Z_n(K) \text{ row vectors of length } n \text{ over } K$, then:

- $V_n(K)$ is a left R -module.
- $Z_n(K)$ is a right R -module.

(d) R is a left R -module and right R module with multiplication.

(e) If M_1 and M_2 are R -modules, we can define on $M_1 \times M_2$ a R -module structure via

$$r \cdot (m_1, m_2) := (rm_1, rm_2)$$

(group structure from Algebra 1)

(f) $\text{Hom}_R(R, M) \rightarrow M, \varphi \mapsto \varphi(1)$ is an isomorphism of abelian groups, and if R is commutative, then also an isomorphism of R -modules.

Definition 1.9. An R -linear map $\varphi : M \rightarrow M'$ is called a monomorphism/epimorphism/isomorphism $\iff \varphi$ is injective/surjective/bijective respectively. We say R -modules M, M' are isomorphic if there exists an isomorphism $M \rightarrow M'$.

Remark. φ is an R -linear isomorphism $\iff \varphi^{-1}$ is an R -linear isomorphism.

Definition 1.10. (a) Let M be an R -module. A subset $N \subseteq M$ is an R -submodule if it is a subgroup and $\forall a \in R, n \in N : a \cdot n \in N$ (i.e. $R \cdot N \subseteq N$)

(b) An R -submodule $I \subseteq R$ is called a left ideal.

(c) $I \subseteq R$ is called a two sided ideal iff it is a left ideal and $I \cdot R \subseteq I$

Example 1.11. (a) If $N' \subseteq N$ and $M' \subseteq M$ are R -submodules of R -modules M and N and if $\varphi : M \rightarrow N$ is an R -linear map, then:

$$\varphi(M') \subseteq N \text{ and } \varphi^{-1}(N') \subseteq M$$

are R -submodules. In particular $\ker(\varphi) \leq M$ and $\text{im}(\varphi) \leq N$ are submodules.

(b) If $(M_i)_{i \in I}$ is a family of submodules of M , then $\bigcap_{i \in I} M_i \subseteq M$ is the largest submodule of M contained in all M_i , and

$$\sum_{i \in I} M_i = \left\{ \sum_{i \in I} m_i \mid m_i \in M, \#\{i \mid m_i \neq 0\} < \infty \right\}$$

is the smallest submodule of M containing all M_i .

(c) 2-sided ideals of $M_{n \times n}(R)$ are of the form $M_{n \times n}(I)$ for $I \subseteq R$ a 2-sided ideal.

1.2 Quotient Modules

Definition 1.12. Let $N \subseteq M$ be a submodule. From linear algebra $(M/N, \bar{0}, \bar{+})$ is an abelian group. ($\bar{m} = m + N$ are the equivalence classes and $\bar{m} + \bar{m}' = \overline{m + m'}$). This is an R -module (exercise) via

$$\bar{\cdot} : R \times M/N \rightarrow M/N : (r, m + N) \mapsto rm + N$$

We call M/N (with $\bar{0}, \bar{+}, \bar{\cdot}$) the quotient module of M by N , and we write

$$\pi_{N \subseteq M} : M \twoheadrightarrow M/N, m \mapsto m + N$$

Definition 1.13. If $I \subseteq R$ is a 2-sided ideal of R , then

(a) $I \cdot M := \{\sum_{i \in I} a_i \cdot m_i \mid I \text{ finite, } a_i \in I, m_i \in M\}$ is an R -submodule of M (M an R -module)

(b) $(R/I, \bar{0}, \bar{1}, \bar{+}, \bar{\cdot})$ is a ring, and $M/I \cdot M$ is an R/I -module.

The following 3 results are proved as for groups:

Theorem 1.14 (Homomorphism theorem). Let $\varphi : M \rightarrow M'$ be an R -linear map, then

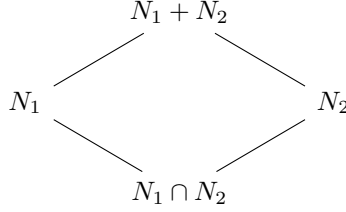
(a) \forall submodules $N \subseteq \ker(\varphi) : \exists ! R\text{-linear map } \bar{\varphi} : M/N \rightarrow M', m + N \mapsto \varphi(m)$ such that $\varphi = \bar{\varphi} \circ \pi_{N \subseteq M}$

(b) For $N = \ker(\varphi)$, the map $\bar{\varphi} : M/\ker(\varphi) \rightarrow \text{im}(\varphi)$ is an R -module isomorphism.

Theorem 1.15. (First isomorphism theorem) Let M be an R -module and $N_1, N_2 \leq M$ be R -submodules. Then the map

$$N_1 / N_1 \cap N_2 \rightarrow N_1 + N_2 / N_2, n_1 + N_1 \cap N_2 \mapsto n_1 + N_2$$

is a well-defined R -linear isomorphism.



Theorem 1.16 (Second isomorphism theorem). Let M be an R -module and $N \leq M$ an R -submodule. Then

(a) The following maps are bijective and mutually inverse to each other:

$$\{N' \subseteq M \text{ submodule} \mid N \subseteq N'\} \xrightleftharpoons[\psi]{\varphi} \{\bar{N} \subseteq M/N \text{ submodule}\}$$

$$\varphi : N' \mapsto N' / N \quad \pi_{N \subseteq M}^{-1}(\bar{N}) \mapsto \bar{N} : \psi$$

(b) For $N' \subseteq M$ a submodule with $N \subseteq N'$ we have the R -linear isomorphism:

$$(M/N) / (N'/N) \rightarrow M / N', \bar{m} + N' / M \mapsto m + N'$$

1.3 Direct sums and products

Let $(M_i)_{i \in I}$ be a family of R -modules.

Definition 1.17. (a) $\prod_{i \in I} M_i = \{(m_i)_{i \in I} \mid m_i \in M_i, \forall i \in I\}$ is an R -module with component-wise operations:

$$(m_i)_{i \in I} + (n_i)_{i \in I} = (m_i + n_i)_{i \in I}$$

$$r \cdot (m_i)_{i \in I} = (r \cdot m_i)_{i \in I}, \quad r \in R$$

is called the (direct) product of $(M_i)_{i \in I}$. One has the projection maps (R -module epimorphisms):

$$\pi_{i_0} : \prod_{i \in I} M_i \rightarrow M_{i_0}, (m_i) \mapsto m_{i_0}$$

(b) $\bigoplus_{i \in I} M_i = \{(m_i)_{i \in I} \in \prod_{i \in I} M_i \mid \{i \mid m_i \neq 0\} < \infty\}$ is an R -submodule of $\prod_{i \in I} M_i$. It is called the direct sum of $(M_i)_{i \in I}$. One has R -module monomorphisms

$$\iota_{i_0} : M_{i_0} \rightarrow \bigoplus_{i \in I} M_i, m_{i_0} \mapsto (\iota_{i_0}(m_{i_0}))$$

where the i -th component of $\iota_{i_0}(m_{i_0})$ is given by $\begin{cases} m_{i_0}, & i = i_0, \\ 0, & \text{otherwise} \end{cases}$

Theorem 1.18 (Universal property of the direct product/sum). (a) $\forall R$ -modules M , the map

$$\text{Hom}_R(M, \prod_{i \in I} M_i) \xrightarrow{\cong} \prod_{i \in I} \text{Hom}_R(M, M_i), \varphi \mapsto (\pi_i \circ \varphi)_{i \in I}$$

is well defined, bijective and a group isomorphism.

(b) $\forall R$ -modules M , the map

$$\text{Hom}_R(\bigoplus_{i \in I} M_i, M) \xrightarrow{\cong} \prod_{i \in I} \text{Hom}_R(M_i, M), \psi \mapsto (\psi \cdot \iota_i)_{i \in I}$$

is well defined, bijective and a group isomorphism.

Proof. (a) The inverse map is given by sending

$$\underline{\varphi} := (\varphi_i : M \rightarrow M_i)_{i \in I} \in \prod_{i \in I} \text{Hom}_R(M, M_i)$$

to

$$\pi_{\underline{\varphi}} : M \rightarrow \prod_{i \in I} M_i, m \mapsto (\varphi_i(m))_{i \in I}$$

now check: $\underline{\varphi} \mapsto \pi_{\underline{\varphi}}$ is inverse to the map in (a).

(b) The map is given by sending $\overline{\varphi} = (\varphi_i : M_i \rightarrow M)_{i \in I}$ to

$$\prod_{\overline{\varphi}} : \bigoplus_{i \in I} M_i \rightarrow M, (m_i)_{i \in I} \mapsto \sum_{i \in I} \varphi_i(m_i)$$

□

Corollary 1.19 (Important special case). Let I be finite, then:

(a) $M := \prod_{i \in I} M_i \stackrel{!}{=} \bigoplus_{i \in I} M_i$

(b) The maps $M_i \xrightleftharpoons[\pi_i]{\iota_i} M$ satisfy

$$\pi_i \circ \iota_j = \begin{cases} \text{id}_{M_i}, & i = j, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \sum_{i \in I} \iota_i \circ \pi_i = \text{id}_M$$

(c) If M' is a module with maps $M_i \xrightleftharpoons[\pi'_i]{\iota'_i} M'$ such that the formulas above hold, then $M \cong M'$

1.4 Generators and bases

From now onwards let R be a unitary ring and M, M', N be R -modules.

Notation. • For I a set we write $M^I := \prod_{i \in I} M$ and $M^{(I)} := \bigoplus_{i \in I} M$ (where $M_i = M, \forall i \in I$).

- For $r \in \mathbb{N}$ we will write $M^r := M^{\{1, \dots, r\}}$, so if I is finite then $M^I = M^{\#I} = M^{(I)}$

Definition 1.20. For $\underline{m} = (m_i)_{i \in I} \in M^{(I)}$ we define a map $\varphi_{\underline{m}} : R^{(I)} \rightarrow M, (r_i) \mapsto \sum_{i \in I} r_i \cdot m_i$ where r_i is non-zero only for finitely many i . We can also define $\varphi_{\underline{m}}$ via the universal property of $R^{(I)}$ using maps $R \rightarrow M, r \mapsto r \cdot m_i$ at component $i \in I$.

- (a) \underline{m} is a generating set of $M \iff \varphi_{\underline{m}}$ is surjective.
- (b) \underline{m} is a basis of $M \iff \varphi_{\underline{m}}$ is an isomorphism.
- (c) M is a free R -module $\iff M$ has a basis.
- (d) \underline{m} is finitely generated \iff it has a finite generating set.
- (e) \underline{m} is linearly independent $\iff \varphi_{\underline{m}}$ is injective.

Remark. Let $\iota_j : R \rightarrow R^{(I)}$ be the inclusion of the component $j \in I$ (1.18) and set $e_j := \iota_j(1)$. Then we call $(e_j)_{j \in I}$ the standard basis of $R^{(I)}$.

Example. (a) If K is a field, then any K -vector space has a basis.

(b) If $R = \mathbb{Z}$, then $M = \mathbb{Z}/(3)$ is finitely generated but not free (exercise).

Remark 1.21. Every R -module is a quotient of a free R -module.

Proof. Let $R^{(M)}$ be the free R -module over the index set M , then

$$\varphi_{\underline{m}} : R^{(M)} \rightarrow M, (r_m)_{m \in M} \mapsto \sum_{m \in M} r_m \cdot m$$

is surjective for $\underline{m} = (m)_{m \in M}$. □

Theorem 1.22. Let R be commutative, then for $n_1, n_2 \in \mathbb{N}_0$, then we have $R^{n_1} \cong R^{n_2} \iff n_1 = n_2$.

Proof. • “ \Leftarrow ”: (By induction to linear algebra.) Let $\mathfrak{m} \subseteq R$ be a maximal ideal. (Axiom of choice) Consider for $n \in \mathbb{N}$ the map $\varphi_n : R^n \rightarrow (R/\mathfrak{m})^n, (r_1, \dots, r_n) \mapsto (r_i \bmod \mathfrak{m})_{i \in \{1, \dots, n\}}$. Then φ_n is surjective with kernel $\mathfrak{m}^n \in R^n \implies R^n/\mathfrak{m}^n \cong (R/\mathfrak{m})^n$ by the homomorphism theorem. Now suppose $\psi : R^{n_1} \rightarrow R^{n_2}$ is an isomorphism. We show $n_1 \geq n_2$ (by symmetry of argument we get $n_1 = n_2$). Consider the map

$$\begin{array}{ccc} R^{n_1} & \xrightarrow{\cong} & R^{n_2} \twoheadrightarrow R^{n_2}/\mathfrak{m}^{n_2} \\ & \searrow \rho & \uparrow \end{array}$$

this map is surjective and contains \mathfrak{m}^{n_1} in its kernel (check this). By the homomorphism theorem we get a surjective homomorphism

$$\left(R/\mathfrak{m}\right)^{n_1} = R^{n_1}/\mathfrak{m}^{n_1} \rightarrow R^{n_2}/\mathfrak{m}^{n_2} = \left(R/\mathfrak{m}\right)^{n_2}$$

by linear algebra we conclude that $n_1 \geq n_2$. □

Definition 1.23. If M is free and finitely generated, then define $\text{rank}(M)$ (the rank of M) as the unique $n \in \mathbb{N}_0$ such that $M \cong R^n$.

Remark. If R is non-commutative, then the rank of the finitely generated R -modules is not well-defined. (Jantzen Schwermer Bsp VII.4.2: $R \cong M \cong R^2$ for $R = \text{End}_K(K[X])$)

1.5 Exact sequences

Definition 1.24. (a) A diagram of R -modules

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is called exact (at M) : $\iff \ker(g) = \text{im}(f)$

(b) An exact sequence of R -modules is a family $(f_j)_{j \in J}$ of R -module homomorphisms $f_j : M_j \rightarrow M_{j+1}$ index of an interval $J \subseteq \mathbb{Z}$, such that $\forall j \in J : j+1 \in J$, the sequence

$$M_j \xrightarrow{f_j} M_{j+1} \xrightarrow{f_{j+1}} M_{j+2}$$

is exact (at M_{j+1}). Other notation:

$$M_{j_0} \xrightarrow{f_{j_0}} M_{j_0+1} \xrightarrow{f_{j_0+1}} \cdots \rightarrow M_{j+2}$$

(c) An exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is called a short exact sequence (s.e.s.)

Remark. • $0 \rightarrow M' \xrightarrow{f} M$ is exact $\xLeftrightarrow{\text{Exercise}} f$ is injective.

• $M \xrightarrow{g} M'' \rightarrow 0$ is exact $\xLeftrightarrow{\text{Exercise}} g$ is surjective.
(0 stands for the 0-module $\{0\}$)

Example 1.25. Let $f : M \rightarrow N$ be an R -module homomorphism. Then one defines

$$\text{coker}(f) := N/\text{im}(f)$$

as the cokernel of f , it comes together with an R -module epimorphism $\pi : N \rightarrow \text{coker}(f)$. As an exercise: The sequence

$$0 \rightarrow \ker(f) \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{\pi} \text{coker}(f) \rightarrow 0$$

is exact. Subexamples:

- If f is injective, then $0 \rightarrow M \xrightarrow{f} N \rightarrow \text{coker}(f) \rightarrow 0$ is exact.
- If f is surjective, then $0 \rightarrow \ker(f) \rightarrow M \xrightarrow{f} N \rightarrow 0$ is exact.

Remark. For R -module homomorphisms $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with $\beta \circ \alpha = 0$, the following are equivalent:

- (i) $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$ is a s.e.s.
 - (ii) β is surjective and $\alpha : M' \rightarrow \ker(\beta)$ is an isomorphism.
 - (iii) α is injective and the homomorphism theorem induces an isomorphism $\text{coker}(\alpha) \cong M/\text{im}(\alpha) \rightarrow M''$
- $(\beta \circ \alpha = 0 \iff \text{im}(\alpha) \subseteq \ker(\beta))$

Proposition 1.26 (Exercise). (a) Let $0 \rightarrow M'_i \rightarrow M_i \rightarrow M''_i \rightarrow 0$ be short exact sequences $\forall i \in I$, then we get short exact sequences

$$0 \rightarrow \bigoplus_{i \in I} M'_i \rightarrow \bigoplus_{i \in I} M_i \rightarrow \bigoplus_{i \in I} M''_i \rightarrow 0$$

$$0 \rightarrow \prod_{i \in I} M'_i \rightarrow \prod_{i \in I} M_i \rightarrow \prod_{i \in I} M''_i \rightarrow 0$$

(b) Suppose $0 \rightarrow V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} V_n \rightarrow 0$ is an exact sequence of finite dimensional K -vector spaces, then:

$$\sum (-1)^i \dim_K(V_i) = 0.$$

Notation 1.27 (Commutativity of diagrams). A diagram of R -modules is a directed graph, where any vertex is an R -module and any arrow is an R -linear map from the module at its source to the module at its target. We call two arrows composable if the target of the first arrow is the source of the second; then the corresponding maps can be composed. So to any chain of composable arrows, the composition of maps defines a map from the source of the first to the target of the last arrow in the chain. A diagram is **commutative** if for any two chains of arrows with the same source and target, the resulting two maps agree.

Example. (a) To say that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ g \downarrow & & \downarrow g' \\ M_3 & \xrightarrow{f'} & M_4 \end{array}$$

commutes means that $g' \circ f = f' \circ g$.

(b) $M \xrightleftharpoons[g]{f} N$ commutes $\iff g = h$

Theorem-Definition 1.28. For a short exact sequence of R -modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \quad (*)$$

$\nwarrow \text{---} s \text{---}$ $\nwarrow \text{---} t \text{---}$

the following are equivalent:

(a) $\exists R$ -linear map $t : M'' \rightarrow M$ such that $g \circ t = \text{id}_{M''}$

(b) \exists submodule $N \subseteq M$ such that

$$\psi : \text{im}(f) \oplus N \rightarrow M, (b, n) \mapsto b + n$$

is an isomorphism.

(c) $\exists R$ -linear map $s : M \rightarrow M'$ such that $s \circ f = \text{id}_{M'}$.

In this case (if (a) - (c) hold), then the sequence (*) is called a *split exact sequence*. (simply (*) is split or splits), and t (or s) is called a *splitting of g* (or of f respectively).

Proof. • (a) \implies (b): Given t , define $N := \text{im}(t)$ and ψ as above, i.e. $\psi : \text{im}(f) \oplus N \rightarrow M, (b, n) \mapsto b + n$

- $\ker(\psi) = 0$: Let $(b, n) \in \ker(\psi)$, i.e. $n = t(m'')$, for some $m'' \in M''$ and $b = f(m')$ for some $m' \in M'$ and $n + b = 0$ ($\psi(b, n) = 0$).
- Apply $g : M \rightarrow M''$:

$$\underbrace{g(n + b)}_0 = \underbrace{g(t(m''))}_{g \circ t = \text{id}_{M''}} + \underbrace{g(f(m'))}_{g \circ f = 0} = m'' + 0$$

$$\implies m'' = 0 \implies n = t(m'') = 0 \xRightarrow{n+b=0} b = 0 \implies (b, n) = (0, 0)$$

- $\text{im}(\psi) = M$: Let $m \in M$, define $n = t(g(m))$ and $b = m - n$. So $n \in N = \text{im}(f)$. $b \in \text{im}(f)$?, to show $b \in \ker(g)$. For this $g(b) = g(m - n) = g(m) - \underbrace{g(t(g(m)))}_{g \circ t = \text{id}_{M''}} = g(m) - g(m) = 0$, so $(b, n) \in \text{im}(f) \oplus N$ and $\psi(b, n) = b + n = m$ by definition of b .

- (c) \implies (b) analogous. Define $N = \ker(s)$ ($M' \xrightleftharpoons[s]{f} M$). We want to show $\text{im}(f) \oplus N \rightarrow M, (b, n) \mapsto b + n$ is an isomorphism.

- $\ker(\psi) = 0$: Check.
- $\text{im}(\psi) = M$: For $m \in M$ observe that

$$\underbrace{f \circ s(m)}_{\in \text{im}(f)} + \underbrace{(m - f \circ s(m))}_{\in \ker(s) \text{ check.}} = m$$

- (b) \rightarrow (a) and (c): Consider the diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M' & \xrightarrow{f'} & \text{im}(f) \oplus N & \xrightarrow{g'} & M'' \longrightarrow 0 \\
& & \downarrow \text{id}_{M'} & & \downarrow \psi & & \downarrow \text{id}_{M''} \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \longrightarrow 0
\end{array}$$

$\xleftarrow{\quad ? \quad} \quad \xleftarrow{\quad ? \quad}$

The diagram commutes. $\psi \circ f' = f, g \circ \psi = g'$, e.g:

$$\psi \circ f'(m') = \psi(f(m'), 0) = f(m') + 0 = f(m')$$

and

$$g \circ \psi(b, n) = g(b + n) = \underbrace{g(b)}_{=0} = g(n) = g'(b, n)$$

($g(b) = 0$ is because $b \in \text{im}(f) = \ker(g)$).

- For s : $f : M' \rightarrow \text{im}(f)$ is an isomorphism (f is injective) $\implies f^{-1} : \text{im}(f) \rightarrow M'$ is an isomorphism. Check

$$s = (f^{-1}, 0) \circ \psi^{-1} : M \xrightarrow{\psi^{-1}} \text{im}(f) \oplus N \xrightarrow{(b,n) \mapsto f^{-1}(b)} M'$$

- For t : Check that $s : N \rightarrow M''$ is an isomorphism using (b). Set $t := i \circ g^{-1}$ for i the inclusion so

$$t : M'' \rightarrow N \hookrightarrow M$$

Check. □

Remark. $M' \xrightleftharpoons[s]{f} M$ and $M'' \xrightleftharpoons[g]{t} M$ satisfy the condition from corollary 1.19, namely:

- $s \circ f = \text{id}_{M'}$
- $g \circ t = \text{id}_{M''}$
- $t \circ g + f \circ s = \text{id}_M$

shows again: the sequence is split if $M \cong M' \oplus M''$ (for the “right maps”)

Remark 1.29. One also has short exact sequences for groups

$$1 \rightarrow \ker(\pi) \xrightarrow{s} G \xrightarrow[\pi]{t} \overline{G} \rightarrow 1$$

Here one has to be careful what splitting means. Having a t is not equivalent to having an s .

$$\exists t \iff G \cong \ker(\pi) \rtimes \overline{G}$$

$$\exists s \iff G \cong \ker(\pi) \times \overline{G}$$

1.6 Projective Modules

Definition 1.30. An R -module P is called projective \iff it has the following lifting property (LP; Hochhebungseigenschaft): In every diagram of R -modules

$$\begin{array}{ccc} & P & \\ \widehat{\varphi} \swarrow & \downarrow \varphi & \\ M & \xrightarrow{\pi} M' & \longrightarrow 0 \end{array}$$

with π surjective, there exists a lifting $\widehat{\varphi} : P \rightarrow M$ such that $\pi \circ \widehat{\varphi} = \varphi$.

Proposition 1.31. (a) Every free R -module is projective.

(b) For an R -module P TFAE:

- (i) P is projective
- (ii) every s.e.s. $0 \rightarrow M' \rightarrow M \rightarrow P \rightarrow 0$ of R -modules splits.
- (iii) P is a direct summand of a free module, i.e. $\exists R$ -module Q , such that $P \oplus Q$ is a free R -module.

Proof. (a) Let $P = R^{(I)}$ for a set I . Consider the diagram

$$\begin{array}{ccc} & R^{(I)} & \\ \widehat{\varphi} \swarrow & \downarrow \varphi & \\ M & \xrightarrow{\pi} M' & \longrightarrow 0 \end{array}$$

Denote by $(e_i)_{i \in I}$ the standard basis of $R^{(I)}$. φ is characterized by $m'_i := \varphi(e_i)$ for all $i \in I$. (by universal property of $R^{(I)} = \bigoplus_{i \in I} R$). Because π is surjective, we can choose a preimage $m_i \in M$ with $\pi(m_i) = m'_i$. Define $\widehat{\varphi} : R^{(I)} \rightarrow M$ as the unique R -module-homomorphism with $\widehat{\varphi}(e_i) = m_i$. Then $(\pi \circ \widehat{\varphi})(e_i) = \pi(m_i) = m'_i = \varphi(e_i) \implies \pi \circ \widehat{\varphi} = \varphi$.

(b) • (i) \implies (ii): Let P be projective, consider a s.e.s.

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \xrightarrow{\pi} & P \longrightarrow 0 \\ & & & & & \nwarrow \psi & \uparrow \text{id}_P \\ & & & & & & P \end{array}$$

By the lifting property $\exists \psi : P \rightarrow M$ such that $\pi \circ \psi = \text{id}_P$, i.e. ψ is a splitting \implies the s.e.s. splits.

- (ii) \implies (iii): From Remark 21 we have an R -module epimorphism $R^{(I)} \xrightarrow{\pi} P$ (for $I = P$). Take $Q := \ker \pi$ (\implies s.e.s. $0 \rightarrow Q \rightarrow R^{(I)} \rightarrow P \rightarrow 0$) By splitness $R^{(I)} = P \oplus Q$ (by Theorem 28).
- (iii) \implies (i): Start with a diagram

$$\begin{array}{ccc} & P & \\ & \downarrow \varphi & \\ M & \xrightarrow{\pi} M' & \longrightarrow 0 \end{array}$$

and assume $\exists R$ -module Q such that $P \oplus Q = R^{(I)}$ (really \cong). Extend φ to

$$P \oplus Q \xrightarrow{\tilde{\varphi}} M', (p, q) \mapsto \varphi(p) + 0$$

By (a) $\exists \hat{\varphi} : P \oplus Q \rightarrow M$ with $\pi \circ \hat{\varphi} = \tilde{\varphi}$.

$$\begin{array}{ccc} & & P \oplus Q \\ & \nwarrow \hat{\varphi} & \downarrow \tilde{\varphi} \\ M & \xrightarrow{\pi} & M' \end{array}$$

Set $\hat{\varphi} := \hat{\varphi}|_{P \oplus Q} : P \oplus Q \rightarrow M$, check $\pi \circ \hat{\varphi} = \varphi$. \square

Corollary 1.32. *Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a s.e.s. of R -modules.*

- (a) M'' projective $\implies (M \cong M' \oplus M'' \text{ and } M \text{ is projective} \iff M' \text{ projective})$.
- (b) If M' and M'' are free R -modules, then so is M .
- (c) If $M' \cong R^{(I)}$ and $M'' \cong R^{(I')}$, then $M \cong R^{(I \cup I')}$. In particular, $\text{rank}(M) = \text{rank}(M') + \text{rank}(M'')$ if $I \cup I'$ is finite.

Proof. (c) clear: $R^{(I)} \oplus R^{(I')} \cong R^{I \cup I'}$

(c) Follows from (a)

- (c) • First assertion in (a) ($M'' \implies M \cong M' \oplus M''$) from Proposition 31.
 - Second assertion: we know $M \cong M' \oplus M''$.
 - Suppose first: M is projective. Then by 31(b)(iii): $\exists Q$ an R -module such that $M \oplus Q$
- \square

Theorem 1.33 (Horse shoe lemma). *Given a diagra of R -modules with P', P'' projective, and the first row exact*

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \longrightarrow 0 \\ & & \alpha \uparrow & & & & \gamma \uparrow \\ & & P' & & & & P'' \end{array}$$

- (a) *The diagram can be completed by the dotted part to a commutative diagram, for a suitable $\beta : P' \oplus P'' \rightarrow M$, so that:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \longrightarrow 0 \\ & & \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow \\ 0 & \dashrightarrow & P' & \xrightarrow{f': p' \mapsto (p', 0)} & P' \oplus P'' & \xrightarrow{g': (p', p'') \mapsto p''} & P'' \dashrightarrow 0 \end{array}$$

and the second row is then also exact.

- (b) *If α and γ are surjective, then so is β .*

Proof. (a) Construction of β : Use the lifting property of P'' to complete

$$\begin{array}{ccccc} M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & \nwarrow \hat{\gamma} & \uparrow \gamma & & \\ & & P'' & & \end{array}$$

By the diagonal arrow $\hat{\gamma} : P'' \rightarrow M$ to a commutative diagram. Define

$$\beta : P' \oplus P'' \rightarrow M, (x', x'') \mapsto f \circ \alpha(x') + \hat{\gamma}(x'')$$

Check commutativities:

- $\beta \circ f' \stackrel{?}{=} f \circ \alpha$:

$$\beta \circ f'(x') = \beta(x', 0) = f \circ \alpha(x') + \hat{\gamma}(0)$$

- $\gamma \circ g' \stackrel{?}{=} g \circ \beta$:

$$g \circ \beta(x', x'') = g(f \circ \alpha(x') + \hat{\gamma}(x'')) = \text{TODO}$$

(b) Diagram chase:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \alpha \uparrow & & \beta \uparrow & & \uparrow \gamma & & \text{TODO} \\ 0 & \longrightarrow & P' & \xrightarrow{f'} & P & \xrightarrow{g'} & P'' & \longrightarrow & 0 \end{array}$$

To show: β is surjective. Let $m \in M$, γ surjective $\implies \exists x'' \in P'' : \gamma(x'') = g(m)$. g' surjective $\implies \exists x \in P : g'(x) = x'$.

Compare m with $\beta(x)$, consider $m - \beta(x)$. Observe: $g(m - \beta(x)) = g(m) - g \circ \beta(x) = \gamma(x'') - g(x'') = 0$ TODO

□

1.7 Finite generation, exact sequences and \oplus

Corollary 1.34. Let $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ be a s.e.s. of R -modules, then

(a) If M is finitely generated as R -module, then so is M''

(b) If M' and M'' are finitely generated (as R -modules), then so is M .

Proof. (a) M finitely generated R -module means \exists finite set I and R -module epimorphism $\pi : R^{(I)} \rightarrow M \implies R^{(I)} \rightarrow M''$ is given by $g \circ \pi$ is an epimorphism
 $\text{imp} M''$ is finitely generated as an R -module.

(b) Suppose we know R -module epimorphisms $\alpha : R^{(I')} \rightarrow M'$ and $\gamma : R^{(I'')} \rightarrow M''$, then Theorem 33 gives an R -module epimorphism

$$\beta : R^{(I')} \oplus R^{(I'')} \rightarrow M$$

$\implies M$ is finitely generated.

□

Remark. M is finitely generated as an R -module $\not\Rightarrow M' \leq M$ is finitely generated. Example: let $R = M = K[X_i \mid i \in \mathbb{N}]$ and consider

$$g : M \rightarrow K, X_i \mapsto 0, \forall i$$

The kernel is the ideal I of R generated by $\{X_i \mid i \in \mathbb{N}\}$. We can check: I is not a finitely generated R -module. If $I = (f_1, \dots, f_m)$ say $f_1, \dots, f_m \in K[X_1, \dots, X_n]$ TODO

Corollary 1.35 (exer). Let M_1, \dots, M_n be R -modules, then

- (a) $M = \bigoplus_{1 \leq i \leq n} M_i$ is finitely generated $\iff M_i$ is finitely generated $\forall i$
- (b) Suppose $M_0 \subseteq \dots \subseteq M_n$ with M_i/M_{i-1} finitely generated for all $i \in \{1, \dots, n\}$. Then M_n is finitely generated.

Theorem 1.36 (Snake lemma). Suppose we are given the following commutative diagram of R -modules with exact rows.

$$\begin{array}{ccccccc} M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' & \\ 0 & \longrightarrow & N' & \xrightarrow{f'} & N & \xrightarrow{g'} & N'' \end{array}$$

Then:

- (a) $\exists R$ -linear map δ (called the connecting homomorphism) from $\ker(\varphi'')$ to $\text{coker}(\varphi')$, such that the following sequence of R -modules is exact: TODO
- (b) If f is injective, then so is f' .
- (c) If g' is surjective, then so is g .

Proof. Construction of δ : Given $m'' \in \ker \varphi''$ map it to $m'' \in M''$ not $\varphi''(m'') = 0$

□

Theorem 1.37 (5-lemma). Suppose we are given the following commutative diagram of R -modules with exact rows

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{\alpha} & M_2 & \xrightarrow{\beta} & M_3 & \xrightarrow{\gamma} & M_4 & \xrightarrow{\delta} & M_5 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 & & \downarrow \varphi_5 \\ N_1 & \xrightarrow{\alpha'} & N_2 & \xrightarrow{\beta'} & N_3 & \xrightarrow{\gamma'} & N_4 & \xrightarrow{\delta'} & N_5 \end{array}$$

and suppose that φ_1 is surjective, φ_5 is injective, and φ_2 and φ_4 are isomorphisms, then φ_3 is also an isomorphism.

Proof. (in parts) Exercise.

1. version: diagram chase.
2. version: break up the diagram into 3 diagrams to which the snake lemma applies. □

1.8 Noetherian and Artinian modules and rings

Let R be a ring, M, M', M'', M_i R -modules. A sequence $(M_i)_{i \in \mathbb{N}}$ is said to *become stationary* if $\exists i_0 : \forall i \geq i_0 : M_i = M_{i_0}$.

Definition 1.38. M is called

- (a) noetherian: \iff each ascending chain of submodules

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots \subseteq M$$

becomes stationary (ACC for ascending chain condition)

- (b) artinian: \iff each descending chain of submodules

$$M \supseteq M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

becomes stationary (DCC for descending chain condition)

and R is called

- (c) left noetherian: \iff it is noetherian as a left R -module

- (d) left artinian: \iff it is artinian as a left R -module

analogously one defines right artinian/noetherian rings and modules.

Examples 1.39. (a) \mathbb{Z} is noetherian but not artinian.

- (b) Finite dimensional K -vector spaces are noetherian and artinian (use the dimension-function)

- (c) (Exersize) Let D be a skew field (division algebra), then any D -module is a free D -module. If a D -module is finitely generated, it is artinian and noetherian. *In the present case one has a well-defined dimension for finitely generated D -modules.*

- (d) Every field and every skew field is left and right artinian and noetherian. (D^{op} is a skew field if D is a skew field)

Definition 1.40. (a) The center of R is $Z(R) := \{r \in R \mid \forall r' \in R : r \cdot r' = r' \cdot r\}$, $Z(R)$ is a commutative subring (exersize)

- (b) Let S be any commutative ring and $\varphi : S \rightarrow R$ be a ring homomorphism such that $\varphi(S) \subseteq Z(R)$, then R is called an S -algebra (via φ).

Examples. (a) Every ring is a \mathbb{Z} -algebra (in a unique way)

- (b) $K[X]$ is a K -algebra.

- (c) If R is finite dimensional K -algebra, then R is left and right noetherian and artinian. (exercise) For instance, if M is a finite monoid (or a finite group), then the monoid ring $K[M]$ is left and right artinian and noetherian.

- (d) $S = \mathbb{Q}$ -subalgebra of 2×2 matrices over \mathbb{Q} generated by $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies S$ is commutative, but $MM_{2 \times 2}(\mathbb{Q})$ over S is not an S -algebra.

Facts 1.41 (Exercise; compare to linear algebra 2 (or Jantzen-Schwermer Ch. VIII.)).

- (a) For M the following are equivalent:
- (i) Each subset of submodules of M contains a maximal element.
 - (ii) Each submodule of M is finitely generated.
- (b) For M the following are equivalent:
- (i) M is artinian
 - (ii) Each subset of submodules of M contains a minimal element.

Lemma 1.42. For submodules $N, P_1, P_2 \subseteq M$ with

$$(i) \ P_1 \supseteq P_2$$

$$(ii) \ P_1 + N = P_2 + N$$

$$(iii) \ P_1 \cap N = P_2 \cap N$$

it follows that $P_1 = P_2$.

Proof. We need to show that $P_1 \subseteq P_2$. Take $m_1 \in P_1 \xRightarrow{(ii)} \exists m_2 \in P_2, n \in N$ such that $m_1 = m_2 + n$. $\implies n = m_1 - m_2 \underset{P_2 \subseteq P_1}{\in} P_1 \cap N \underset{(iii)}{=} P_2 \cap N$.
 $\implies m_1 = m_2 + \underbrace{n}_{\in P_2 \cap N} \in P_2$. \square

Theorem 1.43. Let $N \subseteq M$ be a submodule, then

- (a) M is noetherian (artinian) $\implies N$ and M/N are noetherian (artinian).
(b) N and M/N are noetherian $\iff M$ is noetherian (artinian).

Proof. (a) For N : use directly the characterization (ii) from Facts 41. For M/N : use the homomorphism theorem to identify submodules of M/N with those of M containing N and apply again (ii) from 41.

- (b) Proof only in the artinian case: assume that N and M/N are artinian. Let $M \supseteq M_0 \supseteq M_1 \supseteq \dots \supseteq M_n \supseteq \dots$ be a descending chain. Then by hypothesis:

$$M \cap N \supseteq M_0 \cap N \supseteq M_1 \cap N \supseteq \dots \supseteq M_n \cap N \supseteq \dots$$

becomes stationary, as does

$$M + N \supseteq M_0 + N \supseteq M_1 + N \supseteq \dots \supseteq M_n + N \supseteq \dots$$

$\implies \exists i_0 : \forall i \geq i_0 : M_i + N = M_{i_0} + N$ and $M_i \cap N = M_{i_0} \cap N$, we also have $M_i \subseteq M_{i_0}$, so by lemma 42 we have $M_i = M_{i_0} \implies (M_i)$ becomes stationary. \square

Corollary 1.44 (Exercise).

- (a) Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of R -modules. Then M is noetherian (artinian) $\iff M'$ and M'' are noetherian (artinian).

(b) If I is a finite set and $M := \bigoplus_{i \in I} M_i$, then M is noetherian (artinian) \iff all M_i are noetherian (artinian).

Note: R left-right noetherian (artinian) $\implies R^n$ is also left-right noetherian (artinian) R -module.

Corollary 1.45. Let R be left noetherian (artinian) and M a finitely generated R -module, then M is noetherian (artinian).

Proof. \exists an epimorphism $R^n \rightarrow M$. Now apply 44(a). \square

Corollary 1.46. Let R be left noetherian (artinian) and $I \subseteq R$ a two-sided ideal, then the ring R/I is also left noetherian (artinian).

Proof. R/I is a ring (because I is a two-sided ideal). R/I is left noetherian (artinian) as an R -module by 44(a) $\implies R/I$ is left noetherian (artinian) as an R/I -module. \square

Remark. R is noetherian and $S \subseteq R$ a subring $\nRightarrow S$ is noetherian because not every integral domain is noetherian, but its fraction field certainly is.

Proposition 1.47. Suppose we have $M \cong M \oplus N$ for some R -module $N \neq 0$. Then M is neither noetherian nor artinian.

proof sketch. 1. $M \neq 0$ because $N \neq 0$ is a direct summand of it.

2. $M \cong M \oplus N \cong (M \oplus N) \oplus N \cong ((M \oplus N) \oplus N) \oplus N \cong \dots$

• ∞ ascending chain:

$$0 \oplus N \subsetneq (0 \oplus N) \oplus N \subsetneq ((0 \oplus N) \oplus N) \oplus N \subsetneq \dots$$

$\implies M$ is not noetherian.

• ∞ descending chain:

$$M \supsetneq (M \oplus 0) \supsetneq (M \oplus 0) \oplus 0 \supsetneq ((M \oplus 0) \oplus 0) \oplus 0 \supsetneq \dots$$

$\implies M$ is not artinian. \square

Corollary 1.48 (Exercise from 42 and 45). Suppose $R \neq 0$ is left noetherian (artinian), then for $n_1, n_2 \in \mathbb{N}_0$: $R^{n_1} \cong R^{n_2} \implies n_1 = n_2$ (In particular a rank of free finitely generated R -modules is defined.)

Proof. Assume $\exists n_1, n_2 \in \mathbb{N}_0$ such that $R^{n_1} \cong R^{n_2}$, then $R^{n_1} \cong R^{n_1} \oplus R^{n_2+n_1} \implies R^{n_1}$ not left noetherian (artinian). But 45 implies that R^{n_1} is left noetherian (artinian) because R has these properties. \square

Theorem 1.49 (Hilbert's basis theorem). If R is left noetherian, then $R[X]$ is left noetherian (here X commutes with elements of R).

Proof. TODO \square

1.9 Simple modules

Let R be a ring, M, M', M'', M_i R -modules.

Definition 1.50. M is called simple (or irreducible) if $M \neq 0$ and 0 and M are the only R -submodules of M .

Examples 1.51. (a) Simple K -vector spaces are the 1-dimensional K -vectorspaces.

(b) Simple \mathbb{Z} -modules are $\mathbb{Z}/p\mathbb{Z}$ for p a prime.

(c) A simple $M_{n \times n}(K)$ -module is $V_n(K)$ (space of column vectors).

Definition 1.52. M is said to have a composition series $\iff \exists$ finite descending chain of submodules

$$M = M_n \supsetneq M_{n-1} \supsetneq \cdots \supsetneq M_1 \supsetneq M_0 = 0$$

such that $\forall i \in \{1, \dots, n\}$: the quotients M_i/M_{i-1} are simple. The index n is called the *length* of M and the quotients M_i/M_{i-1} are called the *factors* of M .

Proposition 1.53. M has a decomposition series $\iff M$ is artinian and noetherian.

Proof. • “ \Leftarrow ”: Construct an ascending chain of submodules of M as follows:

$$\begin{aligned} M_0 &= 0 \\ &\mid \cap \\ M_1 &= \text{a minimal submodule in } \{M' \leq M \mid 0 \subsetneq M'\} \\ &\mid \cap \\ M_2 &= \text{a minimal submodule in } \{M' \leq M \mid M_1 \subsetneq M'\} \\ &\mid \cap \\ &\vdots \end{aligned}$$

because M is artinian, the M_i exist (unless $M_{i+1} = M$), and because M is noetherian, we will find an n such that $M_n = M$. By the Homomorphism theorem: M_i/M_{i-1} is simple $\forall i \in \{1, \dots, n\}$.

• “ \Rightarrow ”: Suppose M has a decomposition series and do induction on the minimal length of the series:

- $n = 1$: M is simple.
- Induction step: Let $0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n+1} = M$, then $0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n$ is a decomposition series of M_n of minimal length n . (otherwise there would exist a shorter decomposition series of M which would be a contradiction.) Induction hypothesis implies M_n is artinian and noetherian, but also: $M/M_n = M_{n+1}/M_n$ is simple and hence artinian and noetherian. Consider $0 \rightarrow M_n \rightarrow M \rightarrow M/M_n \rightarrow 0 \implies M$ is artinian and noetherian. \square

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Theorem 1.54 (Jordan-Hölder). *Suppose M has a decomposition series, then*

- (a) *Any 2 decomposition series have the same length.*
- (b) *The list of factors of M (coming from any decomposition series) is unique up to permutation (and isomorphism).*

Definition. Consider chains of submodules of M (not necessarily decomposition series)

$$\underline{\mathcal{M}} := 0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

$$\underline{\mathcal{N}} := 0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = N$$

- (a) Call $\underline{\mathcal{N}}$ a *refinement* of $\underline{\mathcal{M}}$: $\iff \{N_1, \dots, N_\ell\} \supseteq \{M_1, \dots, M_n\}$.
- (b) Call $\underline{\mathcal{N}}$ and $\underline{\mathcal{M}}$ *equivalent* if $\ell = n$ and $\exists \sigma \in S_k$ such that $\forall i \in \{1, \dots, k\}$:

$$N_i / N_{i-1} \cong M_{\sigma(i)} / M_{\sigma(i)-1}$$

Lemma 1.55 (Schreier refinement lemma). *[Jacobson Basic Algebra II, 3.6] Any two finite length submodule chains possess equivalent refinements.*

Proof idea: Use $\underline{\mathcal{M}}$ to refine each step $N_{i-1} \subsetneq N_i$.

- (1) $N_{i-1} \subseteq N_{i-1} + (M_1 \cap N_i) \subseteq N_{i-1} + (M_2 \cap N_i) \subseteq \cdots \subseteq N_{i-1} + (M_k \cap N_i) = N_i$.
- (2) Similarly $M_{j-1} \subseteq M_{j-1} + (N_1 \cap M_j) \subseteq \cdots \subseteq M_{j-1} + (N_\ell \cap M_j) = M_j$.
Schreier verifies that the j -th subquotient of (1) and the i -th subquotient of (2) are isomorphic. (Butterfly lemma) \square

Proof of Jordan-Hölder using Schreier refinement. Suppose $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are decomposition series of M . Schreier refinement gives us refinements $\underline{\mathcal{M}}'$ of $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}'$ of $\underline{\mathcal{N}}$ such that $\underline{\mathcal{M}}'$ and $\underline{\mathcal{N}}'$ are equivalent, i.e. they have the same length and the same subfactors up to permutation and isomorphism. But $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ have no proper refinements $\implies \underline{\mathcal{M}}' = \underline{\mathcal{M}}$ and $\underline{\mathcal{N}}' = \underline{\mathcal{N}}$. \square

Definition 1.56. (a) We say M has finite length if M is artinian and noetherian.

- (b) If M has finite length, then its length $\text{len}(M)$ is the length of any decomposition series.

Proposition 1.57. *Let $0 \rightarrow M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \rightarrow 0$ be a short exact sequence of R -modules, then:*

- (a) M has finite length $\iff M'$ and M'' have finite length.
- (b) $\text{len}(M) = \text{len}(M') + \text{len}(M'')$.

Proof. (a) See Cor. 44.

- (b) Say $0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M'$ and $0 \subsetneq \overline{M}_1 \subsetneq \cdots \subsetneq \overline{M}_\ell = M''$ are decomposition series, then check (2nd isomorphism theorem)

$$0 \subsetneq \iota(M_1) \subsetneq \cdots \subsetneq \iota(M_k) = \iota(M') = \pi^{-1}(0) \subsetneq \pi^{-1}(\overline{M}_1) \subsetneq \cdots \subsetneq \pi^{-1}(\overline{M}_\ell) = M$$

is a decomposition series of M of length $\ell + k = \text{len}(M'') + \text{len}(M')$. \square

1.10 Indecomposable modules

Definition 1.58. An R -module M is indecomposable $\iff M \neq 0$ and there are no proper submodules $0 \subsetneq M_1, M_2 \subsetneq M$ such that $M_1 \oplus M_2 \xrightarrow{\cong} M, (m_1, m_2) \mapsto m_1 + m_2$.

Remark. (a) If M is simple, M is indecomposable.

(b) If every submodule of finite length is a direct sum of simple submodules, then M (indecomposable) of finite length $\implies M$ is simple.

E.g. if R is a field/skew field/ $R = \mathbb{Q}[G]$ finite group, then indecomposable \iff simple.

Examples. • Indecomposable \mathbb{Z} -modules are $\mathbb{Z}/p^n\mathbb{Z}$ for p prime and $n \in \mathbb{N}$.

- All non-zero \mathbb{Z} -submodules of \mathbb{Q} are indecomposable (e.g. \mathbb{Z}, \mathbb{Q}).
- Simple \mathbb{Z} -modules are $\mathbb{Z}/p\mathbb{Z}$ for p prime.
- Indecomposable $K[X]$ -modules are $K[X]/(f^n)$ for $f \in K[X]$ irreducible and $n \in \mathbb{N}$ and $K[X]$ -submodules of $K(X)$. (Any proper submodule of $K[X]/(f^n)$ is contained in $fK[X]/(f^n)$)

Theorem 1.59. Suppose M is noetherian or artinian, then $\exists n \in \mathbb{N}, \{M_1, \dots, M_n\} \subseteq M$ indecomposable submodules such that

$$M = \bigoplus_{1 \leq i \leq n} M_i.$$

Call this statement $(*)_M$ for M . (A generalization of the structure theorem of finitely generated modules over a principle ideal domain; existence part once indecomposable R -modules are understood).

Proof. (Assuming M is artinian; other case is an exercise) Assume the statement of the theorem $(*)_M$ does not hold for M . Define $X = \{M' \leq M \mid \neg (*)_{M'}\}$, then $X \neq \emptyset$ because $M \in X$ (by assumption). Let $M' \in X$ be a minimal element under \subseteq (this exists since M is artinian), then M' decomposable $\implies M' = M_1 \oplus M_2$ for proper submodules $0 \neq M_1, M_2 \subsetneq M' \implies M_1$ and $M_2 \notin X \implies (*)_{M_1}$ and $(*)_{M_2}$ hold, i.e.

$$M_1 = \bigoplus_{1 \leq i \leq t} N_i, \quad M_2 = \bigoplus_{1 \leq j \leq s} P_j$$

where N_i, P_j are indecomposable.

$$\implies M_1 \oplus M_2 = \bigoplus_{1 \leq i \leq t} N_i \oplus \bigoplus_{1 \leq j \leq s} P_j \implies (*)_{M_1 \oplus M_2}.$$

This is a contradiction to $M' \in X$ so X must be empty. \square

Remark. The decomposition in Theorem 59 is not unique

Exercise. Let M be a finitely generated module over a principle ideal domain, then M is indecomposable $\iff M \cong R$ or $\exists n \in \mathbb{N}, \exists$ prime element $p \in R$ such that $M \cong R/p^n$.

Proof. Using structure theorem for modules over PIDs. □

Remark (Exercise). Recall: $e \in S$ a ring is called an idempotent $\iff e^2 = e$
(nontrivial $\iff e \neq 0, 1$) M is indecomposable $\iff 0_M, \text{id}_M \in \text{End}_R(M)$ are
the only idempotents in $\text{End}_R(M)$ (or else $M = e \cdot M \oplus (1 - e) \cdot M$).

Chapter 2

Category Theory

2.1 Preliminary remarks on set theory

References. Literature for this chapter:

- Sophie Morel - Homological Algebra I.1,
- Daniel Murfet - Foundations for Category Theory,
- Saunders MacLane - Categories for the Working Mathematician I.6.

In this course we always assume a model of set theory that satisfies the Zermelo-Fraenkel axioms + the axiom of choice (ZFC).

Definition (Grothendieck universe; we assume ZFC). A *universe* \mathcal{U} is a set which has the following properties:

- (i) $\emptyset, \mathbb{N} \in \mathcal{U}$,
- (ii) $X \in \mathcal{U}$ and $y \in X \implies y \in \mathcal{U}$,
- (iii) $X \in \mathcal{U} \implies \{X\} \in \mathcal{U}$,
- (iv) $X \in \mathcal{U} \implies \mathcal{P}(X) \in \mathcal{U}$,
- (v) If $I \in \mathcal{U}$ and $\{X_i\}_{i \in I}$ is a family of members $X_i \in \mathcal{U}$, then $\bigcup_{i \in I} X_i \in \mathcal{U}$.

The existence of a universe is equivalent to the existence of a strongly inaccessible cardinal. (Thomas Jech - Set Theory)

Axiom (Axiom of universes (Grothendieck)). *Every set lies in a universe. (We will assume this)*

Definition. If \mathcal{U} is our chosen universe, then:

- A \mathcal{U} -set is an element in \mathcal{U} .
- A \mathcal{U} -class is a subset of \mathcal{U} .
- A \mathcal{U} -group is a group (G, e, \cdot) with $G \in \mathcal{U}$ and $\cdot : G \times G \rightarrow G \in \mathcal{U}$.
- A \mathcal{U} -ring is a ring $(R, 0, 1, +, \cdot)$ with $R \in \mathcal{U}$ and also $+, \cdot$
- etc.

Convention. We fix a \mathcal{U} and drop \mathcal{U} - in all terms.

2.2 Categories

Definition 2.1. (a) A *directed graph* (a diagram scheme) is a tuple $(O, A, \text{dom}, \text{cod})$ consisting of \mathcal{U} -classes O and A and maps $\text{dom}, \text{cod} : A \rightarrow O$. We call elements of O *objects* (or vertices) and elements of A *arrows* (or directed edges). For an arrow $f \in A$ call $\text{dom}(f)$ the *source* (or domain) of f and $\text{cod}(f)$ the target (or codomain) of f .

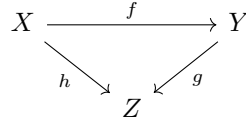
(b) For a graph as in (a) call $A \times_O A := \{(g, f) \in A \times A \mid \text{dom}(g) = \text{cod}(f)\}$ set of composable arrow pairs.

(c) A subgraph of $(O, A, \text{dom}, \text{cod})$ is a graph $(O', A', \text{dom}', \text{cod}')$ such that $O' \subseteq O, A' \subseteq A, \text{dom}' = \text{dom}|_{A'}$ and $\text{cod}' = \text{cod}|_{A'}$.

Example 2.2. Let $O = \{X, Y, Z\}, A = \{f, g, h\}, \text{dom}, \text{cod} : A \rightarrow O$ given by the table

	f	g	h
dom	X	Y	X
cod	Y	Z	Z

Illustration:



Definition 2.3. A *category* \mathcal{C} is a tuple $(\text{Ob } \mathcal{C}, \text{Mor } \mathcal{C}, \text{dom}, \text{cod}, \circ, 1)$ consisting of a graph $(\text{Ob } \mathcal{C}, \text{Mor } \mathcal{C}, \text{dom}, \text{cod})$ and maps

$$1 : \text{Ob } \mathcal{C} \rightarrow \text{Mor } \mathcal{C}, X \mapsto 1_X$$

and

$$\circ : \text{Mor } \mathcal{C} \times_{\text{Ob } \mathcal{C}} \text{Mor } \mathcal{C} \rightarrow \text{Mor } \mathcal{C}, (g, f) \mapsto g \circ f$$

such that:

(i) $\text{dom}(1_X) = \text{cod}(1_X) = X, \forall X \in \text{Ob } \mathcal{C},$

(ii) $\text{dom}(g \circ f) = \text{dom}(f)$ and $\text{cod}(g \circ f) = \text{cod}(g),$

(iii) $\forall f \in \text{Mor } \mathcal{C}$ with $X = \text{dom}(f), Y = \text{cod}(f)$

$$f \circ 1_X = 1_Y \circ f = f$$

(iv) \forall arrows $f, g, h \in \text{Mor } \mathcal{C}$ such that (h, g) and (g, f) are composable arrow pairs we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Call elements of $\text{Ob } \mathcal{C}$ the objects of \mathcal{C} and elements of $\text{Mor } \mathcal{C}$ the morphisms of \mathcal{C} .

Notation 2.4. For a category \mathcal{C} as in definition 3

(a) (often) write $X, Y \in \mathcal{C}$ to mean $X, Y \in \text{Ob } \mathcal{C}$

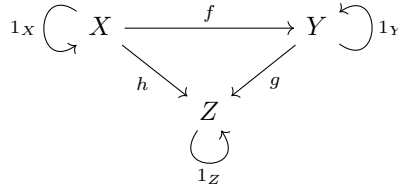
(b) For $X, Y \in \mathcal{C}$ write

$$\mathcal{C}(X, Y) := \text{Mor}_{\mathcal{C}}(X, Y) := \{f \in \mathcal{C} \mid \text{dom } f = X, \text{cod } f = Y\}$$

Definition 2.5. (a) Call a category \mathcal{C} locally small if $\mathcal{C}(X, Y)$ is a set $\forall X, Y \in \mathcal{C}$,

(b) Call \mathcal{C} small if $\text{Ob } \mathcal{C}, \text{Mor } \mathcal{C}$ are sets.

Remark 2.6 (Extension of example 2 to a category). Let $O = \{X, Y, Z\}$, $A = \{f, g, h\} \cup \{1_X, 1_Y, 1_Z\}$, cod, dom as before on $\{f, g, h\}$ and uniquely extended to $\{1_X, 1_Y, 1_Z\}$ by axiom (i) and \circ the only possible composition satisfying the axioms



composable arrow pairs:

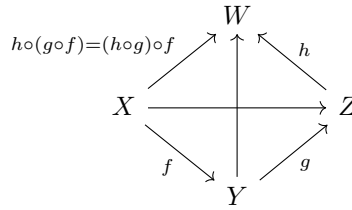
$$(1_X, 1_X), (f, 1_X), (1_Y, 1_Y), (1_Y, f), (g, 1_Y), (1_Z, 1_Z), (1_Z, g), (1_Z, h), (h, 1_X), (g, h)$$

Canonical universal extension would contain a second arrow $X \rightarrow Z$ since it would not want to impose the condition $g \circ f = h$.

Definition 2.7. (a) A diagram in \mathcal{C} is a subgraph Γ of $(\text{Ob } \mathcal{C}, \text{Mor } \mathcal{C}, \text{dom}, \text{cod})$.

(b) A diagram is commutative if for all objects X, Y of Γ and all chains of arrows from X to Y , their composition is the same (i.e. it only depends on X and Y).

Example (For associativity).



Examples (Examples of categories). • **Set** (category of \mathcal{U} -sets): where

- $\text{Ob Set} =$ class of all \mathcal{U} -sets,
- $\text{Mor Set} =$ class of all \mathcal{U} -maps between sets,
- dom, cod are the domain and codomain (range) of a map. (Think of a map as a triple $(X, Y, \text{graph map in } X \times Y)$)
- $\circ =$ composition of maps,
- $1_X = \text{id}_X$ the identity map.

- Grp (category of abelian groups)
- Ring
- CRing
- Top
- ${}_R\text{Mod}$
- Mod_R
- Vec_K
- $\text{Ab} =_{\mathbb{Z}} \text{Mod}$

Examples (Abstract examples). 1. $\text{Ob } \mathcal{C} = \text{Mor } \mathcal{C} = \emptyset$ (empty category)

2. $\text{Ob } \mathcal{C} = \{X\}, \text{Mor } \mathcal{C} = \{1_X\}$ (1 arrow category)

3. Let G be a group, define a category \underline{G} by $\text{Ob } \underline{G} = \{*\}$ (singleton set) and $\text{Mor } \underline{G} = G$, dom, cod the unique map $G \rightarrow \{*\}$, $1_* = e$ (unit element of G). \circ = composition in G :

$$\text{Mor } \underline{G} \times \text{Mor } \underline{G} = G \times G \rightarrow G = \text{Mor } \underline{G}$$

4. Let $\underline{A} = (M, \leq)$ be a partially ordered set. Define the associated category $\text{Ord } \underline{M}$ with $\text{Ob } \text{Ord } \underline{M} = \text{elements of } M$, morphisms are determined by

$$\text{Ord } \underline{M}(X, Y) = \begin{cases} \text{singleton set,} & X \leq Y, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Unit is clear. composition dictated by $\text{Mor}(\text{Ord } \underline{M})$ (i.e. by \leq)

Definition 2.8. For a category $\mathcal{C} = (\text{Ob } \mathcal{C}, \text{Mor } \mathcal{C}, \text{dom}, \text{cod}, \circ, 1)$ define the tuple $\mathcal{C}^{\text{op}} = (\text{Ob } \mathcal{C}, \text{Mor } \mathcal{C}, \text{cod}, \text{dom}, \circ^{\text{op}}, 1)$ with

$$\begin{aligned} \circ^{\text{op}} : \{(f, g) \in \text{Mor } \mathcal{C} \times \text{Mor } \mathcal{C} \mid \text{cod } f = \text{dom } g\} &\rightarrow \text{Mor } \mathcal{C} \\ (f, g) &\mapsto f \circ^{\text{op}} g := g \circ f \end{aligned}$$

(change the direction of arrows!)

Proposition 2.9 (Exercise). \mathcal{C}^{op} is a category, the opposite category to \mathcal{C} .

Example. $(\underline{G})^{\text{op}} = (\underline{G}^{\text{op}})$, $(G^{\text{op}} = (G, e, \circ^{\text{op}}))$ with $g \circ^{\text{op}} h = h \circ g$.

Warning 2.10. $\text{Vec}_K^{\text{op}}(V, W) \neq \text{not the set of maps } V \rightarrow W$, it is $\{f : W \rightarrow V \mid f \text{ is } K\text{-linear}\}$

Definition 2.11. A subcategory of $\mathcal{C} = (\text{Ob } \mathcal{C}, \text{Mor } \mathcal{C}, \text{dom}, \text{cod}, \circ, 1)$ is a category $\mathcal{C}' = (\text{Ob } \mathcal{C}', \text{Mor } \mathcal{C}', \text{dom}', \text{cod}', \circ', 1')$ such that $\text{Ob } \mathcal{C}' \subseteq \text{Ob } \mathcal{C}, \text{Mor } \mathcal{C}' \subseteq \text{Mor } \mathcal{C}, \text{dom}' = \text{dom}|_{\text{Mor } \mathcal{C}'}, \text{cod}' = \text{cod}|_{\text{Mor } \mathcal{C}'}, \circ' = \circ|_{\text{Mor } \mathcal{C}' \times_{\text{Ob } \mathcal{C}} \text{Mor } \mathcal{C}'}, 1' = 1|_{\text{Ob } \mathcal{C}'}$. We write $\mathcal{C}' \subseteq \mathcal{C}$.

Example. $\text{Ab} \subseteq \text{Grp}$ and $\text{CRing} \subseteq \text{Ring}$, etc.

Definition 2.12 (Product of categories). The product of two categories \mathcal{C} and \mathcal{C}' is the six-tuple:

$$(\text{Ob } \mathcal{C} \times \text{Ob } \mathcal{C}', \text{Mor } \mathcal{C} \times \text{Mor } \mathcal{C}', \text{dom} \times \text{dom}', \text{cod} \times \text{cod}', \circ, 1)$$

where \circ is componentwise composition $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$ and $1_{X \times X'} = (1_X, 1_{X'})$

Definition 2.13 (Concepts inside categories). Let $X, Y \in \mathcal{C}$, then call $f \in \mathcal{C}(X, Y)$

- (a) an *isomorphism* $\iff \exists g \in \mathcal{C}(Y, X)$ such that $g \circ f = 1_X, f \circ g = 1_Y$,
- (b) an *endomorphism* $\iff X = Y$,
- (c) an *automorphism* \iff it is an isomorphism and an endomorphism

Moreover \mathcal{C} is called a groupoid category \iff all morphisms are isomorphisms.

Example. Let G be a group, then \underline{G} is a groupoid category. \mathcal{C} a groupoid category $\implies \mathcal{C}(X, X)$ is a group (under $\circ, \forall X \in \text{Ob } \mathcal{C}$).

Definition 2.14. Let $X, Y \in \mathcal{C}$, then call $f \in \mathcal{C}(X, Y)$:

- (a) a *monomorphism* $\iff f$ is left cancellable $\iff \forall W \in \mathcal{C}$ the map $f_* : \mathcal{C}(W, X) \rightarrow \mathcal{C}(W, Y), g \mapsto f \circ g$ is injective.

$$W \begin{matrix} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{matrix} X \xrightarrow{f} Y : f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

- (b) an *epimorphism* $\iff f$ is right cancellable $\iff \forall Z \in \mathcal{C}$ the map $f^* : \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z), h \mapsto h \circ f$ is injective.

$$X \xrightarrow{f} Y \begin{matrix} \xrightarrow{h_1} \\ \xrightarrow{h_2} \end{matrix} Z : h_1 \circ f = h_2 \circ f \implies h_1 = h_2.$$

- (c) a *split monomorphism* $\iff \exists g \in \mathcal{C}(Y, X)$ such that $g \circ f = 1_X$

$$X \begin{matrix} \xrightarrow{\quad g \quad} \\ \xrightarrow{\quad f \quad} \end{matrix} Y$$

- (d) a *split epimorphism* $\iff \exists h \in \mathcal{C}(Y, X)$ such that $f \circ h = 1_Y$

$$X \begin{matrix} \xrightarrow{\quad h \quad} \\ \xrightarrow{\quad f \quad} \end{matrix} Y$$

Facts 2.15. (a) f split mono-/epimorphism $\implies f$ mono-/epimorphism.

(b) f (split) mono-/epimorphism in $\mathcal{C} \implies f$ (split) mono-/epimorphism in \mathcal{C}^{op} .

(c) (Exercise) For $f \in \mathcal{C}(X, Y), (X, Y) \in \mathcal{C}$ the following are equivalent:

- (i) f is an isomorphism

- (ii) $\forall W \in \mathcal{C} : f_* : \mathcal{C}(W, X) \rightarrow \mathcal{C}(W, Y), g \mapsto f \circ g$ is bijective.
 (iii) $\forall Z \in \mathcal{C} : f^* : \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z), h \mapsto h \circ f$ is bijective.

(b') f is an isomorphism in $\mathcal{C} \iff f$ is an isomorphism in \mathcal{C}^{op} .

Proof. (b), (b') are exercises. (c) “serious” exercise.

(a) For an epimorphism (check right cancellability) consider

$$X \xrightarrow{f} Y \xrightarrow[h_2]{h_1} Z : h_1 \circ f \stackrel{(1)}{=} h_2 \circ f$$

By f a split epimorphism, we have $h : Y \rightarrow X$ such that $f \circ h = 1_Y$ (2).
 Apply $- \circ h$ to (1):

$$\begin{array}{ccc} (h_1 \circ f) \circ h & = & (h_2 \circ f) \circ h \\ \parallel & & \parallel \\ h_1 & = & h_1 \circ 1_Y = h_1 \circ (f \circ h) = h_2 \circ (f \circ h) = h_2 \circ 1_Y = h_2 \end{array}$$

□

Examples. In **Set**, **Grp**, **Ring** the monomorphisms are the injective maps and in **Set**, **Grp** the epimorphisms are the surjective maps. But $\mathbb{Z} \rightarrow \mathbb{Q}$ (inclusion) is an epimorphism in **Ring**. If $K \subseteq E$ is purely inseparable, then it's an epimorphism in the category of fields.

Definition 2.16. (a) $X \in \mathcal{C}$ is called an *initial object* $\iff \forall Y \in \mathcal{C} : \#\mathcal{C}(X, Y) = 1$

(b) $X \in \mathcal{C}$ is called a *terminal object* $\iff \forall Z \in \mathcal{C} : \#\mathcal{C}(Z, X) = 1$

(c) $X \in \mathcal{C}$ is called a *null object* $\iff X$ is initial and terminal.

Example. • \emptyset is initial in **Set**, **Top**,

- $\{*\}$ is terminal in **Set**, **Top**
- $0 = \{0\}$ is a null object in ${}_R\text{Mod}$, **Ab**, Vec_K

2.3 Functors

Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories.

Definition 2.17. A *functor* F from \mathcal{C} to \mathcal{D} ($F : \mathcal{C} \rightarrow \mathcal{D}$) is a pair of maps

$$\begin{aligned} F : \text{Ob } \mathcal{C} &\rightarrow \text{Ob } \mathcal{D}, X \mapsto F(X), \\ F : \text{Mor } \mathcal{C} &\rightarrow \text{Mor } \mathcal{D}, f \mapsto F(f). \end{aligned}$$

that “preserve sources, targets, units and composition”, i.e.

- (i) $\forall f \in \text{Mor } \mathcal{C} : \text{dom}(Ff) = F(\text{dom } f)$ and $\text{cod}(Ff) = F(\text{cod } f)$
 (ii) $\forall X \in \text{Ob } \mathcal{C} : F(1_X) = 1_{FX}$

- (iii) \forall composable pairs (g, f) in $\text{Mor } \mathcal{C} \times_{\text{Ob } \mathcal{C}} \text{Mor } \mathcal{C} : F(g \circ f) = F(g) \circ F(f)$.
 (other notation $F(X \xrightarrow{f} Y) = FX \xrightarrow{Ff} FY$)

Examples. (a) Powerset:

$$\begin{aligned} \mathcal{P} : \text{Set} &\rightarrow \text{Set} \\ X &\mapsto \mathcal{P}(X) \\ f : X \rightarrow Y &\mapsto \mathcal{P}f : \mathcal{P}X \rightarrow \mathcal{P}Y, \\ (U \subseteq X) &\mapsto (f(U) \subseteq Y) \end{aligned}$$

(b) Forgetful functor (it forgets structure)

$$\begin{aligned} V : \text{Grp} &\rightarrow \text{Set}, (G, e, \circ) \mapsto G \\ V : \text{Top} &\rightarrow \text{Set}, (X, \mathcal{T}) \mapsto X \\ V : {}_R\text{Mod} &\rightarrow \text{Ab}, (M, 0, +, \cdot) \mapsto (M, 0, +) \end{aligned}$$

(c) ${}_R\text{Mod} \rightarrow \text{Mod}_{R^{\text{op}}}$, left R -modules \mapsto right R -modules.

Remark. Functors in definition 17 are also called covariant functors.

Definition 2.18. A *contravariant* functor from $\mathcal{C} \rightarrow \mathcal{D}$ is a functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$, i.e.

$$F(X \xrightarrow{f} Y) = (FX \xleftarrow{Ff} FY)$$

$= (Y \xrightarrow{f} X)$ in \mathcal{C}^{op} and

$$F((Y \xrightarrow{g} Z) \circ (X \xrightarrow{f} Y)) = F(X \xrightarrow{f} Y) \circ F(Y \xrightarrow{g} Z)$$

Visually:

$$\begin{array}{ccc} X & X & FX \\ \downarrow f & \uparrow f & \uparrow Ff \\ Y & Y & FY \\ \downarrow g & \uparrow g & \uparrow Fg \\ Z & Z & FZ \end{array}$$

in \mathcal{C} in \mathcal{C}^{op}

Remark (Exercise). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be any functor $\implies F$ maps isomorphisms to isomorphisms.

Examples (Contravariant functors). (a) Passage to the dual vector space

$$\begin{array}{ccc} D : \text{Vec}_K^{\text{op}} & \longrightarrow & \text{Vec}_K \\ V & \longmapsto & V^* = \text{Hom}_K(V, K) \\ f \downarrow & & \uparrow Df = f^* \\ W & \longmapsto & W^* \end{array}$$

linear algebra: $(f \circ g)^* = g^* \circ f^*$ for (f, g) a composable pair.

- (b) Let \mathbf{Poset} be the category of partially ordered sets, then we have a contravariant functor

$$\begin{array}{ccccc} \mathcal{O} : \mathbf{Top}^{\text{op}} & \longrightarrow & \mathbf{Poset} \\ (X, \mathcal{T}) & \longmapsto & (\mathcal{T}, \subseteq) \ni f^{-1}(V) \subseteq X \\ f \downarrow & & \uparrow \mathcal{O}(f) & & \uparrow \text{open} \\ (Y, \mathcal{T}') & \longmapsto & (\mathcal{T}, \subseteq) \ni V \subseteq Y \\ & & & & \uparrow \text{open} \end{array}$$

- (c) The contravariant powerset functor:

$$\begin{array}{ccc} \mathcal{P}^* : \mathbf{Set} & \longrightarrow & \mathbf{Set} \\ X & \longmapsto & \mathcal{P}^*(X) = \mathcal{P}(X) \\ f \downarrow & & \uparrow \mathcal{P}^* f \\ Y & \longmapsto & \mathcal{P}^*(Y) = \mathcal{P}(Y) \end{array}$$

Definition 2.19. Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories, a functor $\mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{D}$ is called a *bifunctor*.

Example 2.20 (Important example). Let \mathcal{C} be any category

$$\begin{array}{ccccc} \mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} & \longrightarrow & \mathbf{Set} \\ (X, Y) & \longmapsto & \mathcal{C}(X, Y) \ni g \\ f \uparrow \downarrow h & & \downarrow & & \downarrow \\ (W, Z) & \longmapsto & \mathcal{C}(W, Z) \ni h \circ g \circ f \end{array}$$

If we fix a first argument X , we get

$$h_X := \mathcal{C}(X, -) \rightarrow \mathbf{Set}$$

If we fix a second argument Y , we get

$$h^Y := \mathcal{C}(-, Y) \rightarrow \mathbf{Set}$$

Soon: we will also have another important bifunctor

$$- \otimes - : \mathbf{Mod}_R \times {}_R \mathbf{Mod} \rightarrow \mathbf{Ab}$$

Definition 2.21. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, F is called

- (a) *faithful* $\iff \forall X, Y \in \mathcal{C} : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$ is injective.
- (b) *full* $\iff \forall X, Y \in \mathcal{C} : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$ is surjective.
- (c) *fully faithful* F is full and faithful.
- (d) *essentially surjective* $\iff \forall Y \in \mathcal{D} \exists X \in \mathcal{C} \exists \text{ isomorphism } FX \xrightarrow{\cong} Y$.
- (e) *conservative* $\iff \forall f \in \text{Mor } \mathcal{C} : f \text{ is an isomorphism} \iff Ff \in \text{Mor } \mathcal{D} \text{ is an isomorphism. (} \implies \text{ always holds)}$

- (f) *an isomorphism* $\iff \exists G : \mathcal{D} \rightarrow \mathcal{C}$ functor such that $F \circ G = \text{id}_{\mathcal{D}}$ and $G \circ F = \text{id}_{\mathcal{C}}$.

Examples. (a) Forgetful functors are “often” faithful but not full

$$V : \text{Grp} \rightarrow \text{Set}, \text{Ab} \rightarrow \text{Set}, {}_R\text{Mod} \rightarrow \text{Set}, \text{Ring} \rightarrow \text{Set}$$

are conservative.

- (b) The forgetful functor $V : \text{Top} \rightarrow \text{Set}$ is not conservative and not full but essentially surjective.
- (c) The inclusion of a subcategory \mathcal{C}' into its ambient category \mathcal{C} is always faithful. Call \mathcal{C}' a *full subcategory* $\iff \forall X, Y \in \mathcal{C}' : \mathcal{C}'(X, Y) = \mathcal{C}(X, Y)$ ($\iff i : \mathcal{C}' \rightarrow \mathcal{C}$ is full)

2.4 Natural transformations

They are morphisms between functors.

Definition 2.22. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors.

- (a) A morphism from F to G (or a *natural transformation*) is a family $u = (u_X : FX \rightarrow GX)_{X \in \text{Ob } \mathcal{C}}$ of morphisms in \mathcal{D} , such that for all $f : X \rightarrow Y$ in \mathcal{C} we have the commutative diagram:

$$\begin{array}{ccc} FX & \xrightarrow{u_X} & GX \\ Ff \downarrow & & \downarrow Gf \\ FY & \xrightarrow{u_Y} & GY \end{array}$$

Notation:

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow u \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \end{array}$$

- (b) Composition: Let $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$ be functors and $u : F \Rightarrow G, v : G \Rightarrow H$ natural transformations. The composition $v \circ u : F \Rightarrow H$ is the natural transformation (check) $(v_X \circ u_X : FX \xrightarrow{u_X} GX \xrightarrow{v_X} HX)_{X \in \text{Ob } \mathcal{C}}$

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow u \\ \text{---} G \text{---} \\ \Downarrow v \\ \curvearrowleft \end{array} & \mathcal{D} \\ & H & \end{array}$$

- (c) The category $\mathcal{D}^{\mathcal{C}}$ (or $\text{Fun}(\mathcal{C}, \mathcal{D})$) whose objects are the functors $\mathcal{C} \rightarrow \mathcal{D}$ and whose morphisms are the natural transformations $(F : \mathcal{C} \rightarrow \mathcal{D}) \Rightarrow (G : \mathcal{C} \rightarrow \mathcal{D})$. The composition is from (b), and the unit natural transformation is

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow 1_F \\ \curvearrowleft \end{array} & \mathcal{D} \end{array}, 1_F = (FX \xrightarrow{1_{FX}} FX)_{X \in \text{Ob } \mathcal{C}}$$

(dom, cod are clear). Remark: One can also define 2-categories (and the category of categories is an example of such, objects: $\mathcal{C}, \mathcal{D}, \dots$ and morphisms are $F : \mathcal{C} \Rightarrow \mathcal{D}$ 2-morphisms = natural transformations)

- (d) A natural transformation $u : F \Rightarrow G$ is called a *natural isomorphism* $\iff \forall X \in \text{Ob } \mathcal{C} : u_X : FX \rightarrow GX$ is an isomorphism $\iff \exists$ natural transformation $v : G \Rightarrow F : v \circ u = \text{id}_F, u \circ v = \text{id}_G$. Exerc.

Example (Famous linear algebra example of a natural transformation). Let $(\cdot)^{**} : \text{Vec}_K \rightarrow \text{Vec}_K, V \mapsto V^{**}, f \mapsto f^{**}$ be the (covariant) bidual functor. $\text{id} : \text{Vec}_K \rightarrow \text{Vec}_K$ denotes the identity, we set $u_V : V \rightarrow V^{**}, v \mapsto (b_v : V^* \rightarrow K, \xi \mapsto \xi(v))$ then $u = (u_V)_{V \in \text{Vec}_K}$ is a natural transformation $u : \text{id} \Rightarrow (\cdot)^{**}$ and restricted to the full subcategory $\text{Vec}_K^{\text{f.d.}} \subseteq \text{Vec}_K$ on finite dimensional K -vector spaces, it gives a natural isomorphism $u : \text{id} \Rightarrow (\cdot)^{**}$

$$\begin{array}{ccc} & \text{id} & \\ \text{Vec}_K^{\text{f.d.}} & \begin{array}{c} \downarrow u \\ \downarrow \end{array} & \text{Vec}_K^{\text{f.d.}} \\ & (\cdot)^{**} & \end{array}$$

Definition 2.23 (important concept). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called an equivalence of categories $\iff \exists$ functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such that one has natural transformations $\text{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\text{id}_{\mathcal{D}} \Rightarrow F \circ G$. Call \mathcal{C} and \mathcal{D} *equivalent categories* $\iff \exists$ equivalence of categories $F : \mathcal{C} \rightarrow \mathcal{D}$.

Remark. The notion of “equivalence of categories” is far more important than the notion of isomorphism of categories.

Example (linear algebra). Let $\text{Vec}_K^{\text{std.}}$ be the full subcategory of the $\text{Vec}_K^{\text{f.d.}}$ on the object set $\{K^n \mid n \in \mathbb{N}_0\}$. Then the inclusion $\iota : \text{Vec}_K^{\text{std.}} \rightarrow \text{Vec}_K^{\text{f.d.}}$ is an equivalence of categories. For $G : \text{Vec}_K^{\text{std.}} \rightarrow \text{Vec}_K^{\text{std.}}$ take $V \mapsto K^{\dim_K V}$, choose a basis \underline{B}_V for any $V \in \text{Vec}_K^{\text{f.d.}}$ then we get an isomorphism $K^{\dim_K V} \xrightarrow{\alpha_V} V$. Define:

$$\begin{array}{ccc} V & \xrightarrow{\quad} & K^{\dim_K V} \\ f \downarrow & & \downarrow \alpha_W^{-1} \circ f \circ \alpha_V \\ W & \xrightarrow{\quad} & K^{\dim_K W} \end{array}$$

Find natural isomorphism $G \circ \iota \Leftarrow \text{id} \Rightarrow \iota \circ G$.

Remark. One also calls $\text{Vec}_K^{\text{std.}}$ a *skeleton* of $\text{Vec}_K^{\text{f.d.}}$.

Theorem 2.24. For a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ the following are equivalent:

- (i) F is an equivalence of categories.
- (ii) F is fully faithful and essentially surjective.

Proof. • (i) \implies (ii): Exercise.

- (ii) \implies (i): Standard textbook. □

Definition 2.25. The *essential image* of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ in \mathcal{D} is the full subcategory \mathcal{D}' of \mathcal{D} on objects isomorphic to FX for some $X \in \text{Ob } \mathcal{C}$.

Corollary 2.26 (of 24 and the definition). *Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ is fully faithful. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the essential image of F , then $F : \mathcal{C} \rightarrow \mathcal{D}'$ is an equivalence of categories.*

2.5 The Yoneda lemma and presheaves

Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories.

Definition 2.27. (a) A \mathcal{D} -valued presheaf on \mathcal{C} is a functor

$$\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$$

(b) The category of \mathcal{D} -valued presheaves on \mathcal{C} is $\text{PSh}(\mathcal{C}, \mathcal{D}) = \mathcal{D}^{\mathcal{C}^{\text{op}}}$

(c) If $\mathcal{D} = \text{Set}$, then we omit it from the notation, so $\text{PSh}(\mathcal{C}) = \text{Set}^{\mathcal{C}^{\text{op}}}$

Note that if \mathcal{D} is a small category, then $\mathcal{D}^{\mathcal{C}^{\text{op}}}$ is a category.

Remark (On the terminology). (Pre-)sheaves come from topology/geometry. Example: Let (X, \mathcal{T}) be a topological space (e.g. \mathbb{C} with the metric topology), For $U \subseteq X$ define $O_X(U) := \{f : U \rightarrow \mathbb{C} \text{ continuous}\}$ or $(O_{\mathbb{C}}(U) := \{f : U \rightarrow \mathbb{C} \text{ holomorphic}\})$. Check:

$$\begin{array}{ccc} O_X : \text{ord}(T, \subseteq) & \longrightarrow & \text{Set} \\ U & \longmapsto & O_X(U) \\ \uparrow \cup & & \downarrow \text{restriction} \\ V & \longmapsto & O_X(V) \end{array}$$

this is a presheaf.

Definition 2.28. The Yoneda embedding is the functor $h : \mathcal{C} \rightarrow \text{PSh}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set}), X \mapsto h_X := \mathcal{C}(-, X) : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

$$\begin{array}{ccc} h : \mathcal{C} & \longrightarrow & \text{PSh}(\mathcal{C}) \\ X & \longmapsto & h_X \quad \text{Hom}(-, X) \\ f \downarrow & & \downarrow h f \\ Y & \longmapsto & h_Y \quad \text{Hom}(-, Y) \end{array}$$

Lemma 2.29. For $X \in \mathcal{C}, F \in \text{PSh}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$

$$\begin{array}{ccc} \text{Hom}_{\text{PSh}(\mathcal{C})}(h_X, \mathcal{F}) & \xrightarrow{\Phi} & \mathcal{F}X \\ u := (u_Y : h_X Y \rightarrow \mathcal{F}Y)_{Y \in \text{Ob } \mathcal{C}} & \longmapsto & u_X 1_X \end{array}$$

is a bijection. ($\text{Hom}_{\text{PSh}(\mathcal{C})}(h_x, \mathcal{F})$ is a set.)

Proof. Reconstruct a natural transformation u^α from $\alpha \in \mathcal{F}X$, first consider what $u \in \text{Mor}_{\text{PSh}(\mathcal{C})}(h_x, \mathcal{F})$ gives us

$$\begin{array}{ccccc} X & h_X X = \text{Hom}_{\mathcal{C}}(X, X) & \xrightarrow{u_X} & \mathcal{F}X & 1_X \longmapsto u_X(1_X) = \alpha \\ \uparrow g & \downarrow - \circ g & & \downarrow \mathcal{F}g & \downarrow \\ Y & h_X Y = \text{Hom}_{\mathcal{C}}(Y, X) & \xrightarrow{u_Y} & \mathcal{F}Y & g \longmapsto u_Y(g) = \mathcal{F}g(\alpha) \end{array}$$

Define $\psi : \mathcal{F}(X) \rightarrow \text{Mor}_{\text{PSh}(\mathcal{C})}(h_X, \mathcal{F}), \alpha \mapsto (u_Y^\alpha)_{Y \in \text{Ob } \mathcal{C}}$ by setting $u_Y^\alpha : h_X(Y) \rightarrow \mathcal{F}Y, g \mapsto \mathcal{F}g(\alpha)$. Check u_Y^α is a natural transformation: For any $f : Z \rightarrow Y$ in \mathcal{C} we get \square

Corollary 2.30. *The functor $h : \mathcal{C} \rightarrow \text{PSh}(\mathcal{C})$ is fully faithful, i.e. $\mathcal{C}(X, Y) \leftrightarrow \text{Mor}_{\text{PSh}(\mathcal{C})}(h_X, h_Y)$.*

Proof. We need to show $\forall X, Y \in \text{Ob } \mathcal{C}$ the map $\mathcal{C}(X, Y) \rightarrow \text{Mor}_{\text{PSh}(\mathcal{C})}(h_X, h_Y), f \mapsto h(f) = f \circ -$ is bijective. Observe: Yoneda $\Phi : \text{Mor}_{\text{PSh}(\mathcal{C})}(h_X, h_Y) \rightarrow h_Y(X) = \mathcal{C}(X, Y), u \mapsto u_X(1_X)$ is a bijection, so it suffices to show $\Phi \circ h$ is a bijection. For this: $\Phi \circ h(f : X \rightarrow Y) = f \circ 1_X = f \implies \Phi \circ h = \text{id}$. \square

Definition 2.31. (a) Call $\mathcal{F} \in \text{PSh}(\mathcal{C})$ *representable* $\iff \exists X \in \mathcal{C}$ such that $h_X \cong \mathcal{F}$.

(b) A presentation of a (representable) $\mathcal{F} \in \text{PSh}(\mathcal{C})$ is a pair (X, α) with $X \in \text{Ob } \mathcal{C}, \alpha \in \mathcal{F}X$ such that $\Psi(\alpha) : h_X \Rightarrow \mathcal{F}$ from the proof of lemma 29 is a natural isomorphism.

Proposition 2.32. *Suppose (X, α) and (Y, β) are presentations of $\mathcal{F} \in \text{PSh}(\mathcal{C})$, then $\exists!$ isomorphism $f : X \rightarrow Y$ such that $\mathcal{F}(f)(\beta) = \alpha$*

Proof. Exercise. \square

2.6 Conatravariant Yoneda

Proposition 2.33. *The functor*

$$\begin{array}{ccc} h^{\text{op}} : \mathcal{C}^{\text{op}} & \longrightarrow & \text{Fun}(\mathcal{C}, \text{Set}) \\ X & \longmapsto & \mathcal{C}(X, -) \\ \downarrow f & & \uparrow h^{\text{op}}(f) \\ Y & \longmapsto & \mathcal{C}(Y, -) \end{array}$$

is fully faithful and for $X \in \text{Ob } \mathcal{C}$ and $\mathcal{F} : \mathcal{C} \rightarrow \text{Set}$ a functor, the map $\Phi' : \text{Mor}_{\text{Set}^{\mathcal{C}}}(h_X^{\text{op}}, \mathcal{F}) \rightarrow \mathcal{F}(X), u \mapsto u_X(1_X)$ is bijective.

Proof. (Exercise) Apply Yoneda to \mathcal{C}^{op} . \square

Definition 2.34. (a) A covariant functor $F : \mathcal{C} \rightarrow \text{Set}$ is *corepresentable* $\iff F \cong h_X^{\text{op}}$ for some $X \in \mathcal{C}$.

(b) A presentation of a (corepresentable) functor F is a pair (X, α) such that $(\Phi')^{-1}(\alpha)$ is an isomorphism $h_X^{\text{op}} \rightarrow F$

Proposition 2.35 (analog of 32). *If (X, α) and (Y, β) are 2 presentations of $F : \mathcal{C} \rightarrow \mathbf{Set}$, then $\exists!$ isomorphism $f : X \rightarrow Y$ such that $F(f)(\alpha) = \beta$.*

Remark. We mostly drop co- in corepresentable because the functor dictates if it is representable or corepresentable (if F co- or contravariant)

Remark. For $f : X \rightarrow Y$ we have

$$\begin{aligned} f \circ - : \mathcal{C}(Z, X) \rightarrow \mathcal{C}(X, Y) \text{ bij.} &\iff h(f) \cong \iff f \cong \iff h^{\text{op}}(f) \cong \\ &\iff - \circ f : \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z) \text{ bij.} \end{aligned}$$

because h is fully faithful.

2.7 Universal pairs

Definition 2.36. (a) Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $B \in \text{Ob } \mathcal{D}$. A pair (U, β) with $U \in \text{Ob } \mathcal{C}$ and $\beta : B \rightarrow F(U)$ (in \mathcal{D}) is *(co-)universal* for $(F, \beta) : \iff (U, \beta)$ (co-)represents $\mathcal{D}(B, F(-)) = h_B^{\text{op}} \circ F : \mathcal{C} \rightarrow \mathbf{Set}$.

(b) Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and $A \in \text{Ob } \mathcal{C}$. A pair (V, α) with $V \in \text{Ob } \mathcal{D}$ and $\alpha : G(V) \rightarrow A$ (in \mathcal{C}) is universal for $(G, A) \iff (V, \alpha)$ represents $(\mathcal{C}(G(-), A)) = h_A \circ G : \mathcal{D}^{\text{op}} \rightarrow \mathbf{Set}$.

TODO: interpretation

Examples 2.37. TODO

2.8 Limits and colimits

Let \mathcal{C} be a category.

Definition 2.38. A diagram in \mathcal{C} is a functor $F : J \rightarrow \mathcal{C}$ for J a small category (call J the *index category* of the diagram.)

Remark (Relation to previous notions of diagrams). Let $V : \mathbf{Cat} \rightarrow \mathbf{Diag}$, [MacLane II.7]: \exists functor TODO

Definition 2.39. A diagram $F : J \rightarrow \mathcal{C}$ *commutes* $\iff \forall i, j \in J$:

$$\underbrace{F(J(i, j))}_{\text{is a singleton}} \subseteq \mathcal{C}(Fi, Fj)$$

(naive diagram commutes $\iff F\varphi$ commutes.)

Definition 2.40. Let $F : J \rightarrow \mathcal{C}$ denote a diagram in \mathcal{C} .

(a) The *constant functor* from J to \mathcal{C} for $X \in \text{Ob } \mathcal{C}$ is $\Delta X : J \rightarrow \mathcal{C}$ with $\Delta X(i) = X, \forall i \in J$ and $\Delta X(h) = 1_X, \forall h \in \text{Mor } J$.

(b) The *diagonal* $\Delta : \mathcal{C} \rightarrow \mathcal{C}^J = \text{Fun}(J, \mathcal{C})$ with $\Delta(X) := \Delta X$ from (a) and $\Delta(f) :=$ the natural transformation $\Delta X \Rightarrow \Delta Y$ given for any $i \in J$ by $\Delta X(i) = X \xrightarrow{f} \Delta Y(i) = Y$.

- (c) A *cone* to $F : J \rightarrow \mathcal{C}$ (any fixed F) with apex $X \in \mathcal{C}$ is a natural transformation $\Delta X \Rightarrow F$. A *cocone* from F with vertex $X \in \text{Ob } \mathcal{C}$ is a natural transformation $F \Rightarrow \Delta X$.
- (d) Cones and cocones give rise to the following functors:
- $\text{Cone}(-, F) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ defined by $\text{Cone}(X, F) = \text{set of cones } \Delta X \Rightarrow F$, $\text{Cone}(f : X \rightarrow Y, F)$ maps $\text{Cone}(Y, F) \rightarrow \text{Cone}(X, F)$, $(\Delta Y \Rightarrow F) \mapsto (\Delta X \Rightarrow \Delta Y \Rightarrow F)$.
 - Similarly $\text{Cocone}(F, -) : \mathcal{C} \rightarrow \mathbf{Set}$ is the functor defined by $\text{Cocone}(F, X) = \text{set of cocones } F \Rightarrow \Delta X$ etc.

Observe:

$$\begin{aligned}\text{Cone}(-, F) &= \mathcal{C}^J(\Delta(-), F) \\ \text{Cocone}(F, -) &= \mathcal{C}^J(F, \Delta(-))\end{aligned}$$

Visualization: a natural transformation $u : \Delta X \Rightarrow F$ (where $X \in \mathcal{C}, F : J \rightarrow \mathcal{C}$) is for any $i \in J$ a morphism $X = \Delta X(i) \xrightarrow{u_i} F(i)$ such that $\forall h : i \rightarrow j$ in J the following diagram commutes:

$$\begin{array}{ccc}\Delta X(i) = X & \xrightarrow{1_X} & X = \Delta X(j) \\ \downarrow u_i & \searrow u_j & \downarrow u_j \\ Fi & \xrightarrow{Fh} & Fj\end{array}$$

for instance if TODO

Remark 2.41. (a) The cones to F form a full subcategory $F\text{-cones} \subseteq \mathcal{C}^J/F$ on objects $\Delta X \Rightarrow F$ ($X \in \mathcal{C}$).

(b) Similarly cocones from F form a full subcategory $F\text{-cocones} \subseteq F/\mathcal{C}^J$ on objects $F \Rightarrow \Delta X$.

Definition 2.42. (a) If $\text{Cone}(-, F) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ is representable, the representing object is called a *limit* over F . Notation: $\lim F$ or $\lim_J F$ for the representing universal object.

(b) If $\text{Cocone}(F, -) : \mathcal{C} \rightarrow \mathbf{Set}$ is representable, the representing object is called a *colimit* over F . Notation: $\text{colim } F$ or $\text{colim}_J F$

More explicitly: $\lim_J F$ is an object $L \in \mathcal{C}$ together with a (universal) cone $\Delta L \Rightarrow F$ such that \forall cones $\varphi : \Delta X \Rightarrow F$ in $\mathcal{C}^J \exists!$ morphism $\psi : X \rightarrow L$ such that the diagram commutes

$$\begin{array}{ccc}\Delta L & \xrightarrow{\psi} & \Delta X \\ & \searrow & \downarrow \varphi \\ \Delta L & \xrightarrow{\quad} & X\end{array}$$

Yoneda implies that $\lim F$ (if it exists) is unique up to unique isomorphism (similarly for $\text{colim } F$).

Exercise 2.43. (a) $\lim F$ exists \iff category of F -cones has a terminal object.

(b) $\text{colim } F$ exists \iff category of F -cocones has an initial object.

Proposition 2.44 (Exercise). Let $\Delta : \mathcal{C} \rightarrow \mathcal{C}^J$ be the diagonal from above and F any diagram in \mathcal{C}^J . Then

(a) $\lim F$ is a universal object for the pair (Δ, F) i.e. $\mathcal{C}^J(\Delta(-), F) \leftrightarrow \mathcal{C}(-, \lim_J F)$.

(b) $\text{colim } F$ is a couniversal object for the pair (Δ, F) i.e. $\mathcal{C}^J(F, \Delta(-)) = \mathcal{C}(\text{colim}_J F, -)$.

Examples 2.45. TODO

Definition 2.46. \mathcal{C} is (co-)complete $\iff \mathcal{C}$ contains all (co)-limits.

Theorem 2.47. (a) If \mathcal{C} contains all products and equalizers, then \mathcal{C} is complete.

(b) If \mathcal{C} contains all coproducts and coequalizers, then \mathcal{C} is cocomplete.

Corollary 2.48. Set and ${}_R\text{Mod}$ are complete and cocomplete.

2.9 Adjoint Functors

Definition 2.49. (a) Functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ form an *adjoint pair*, when one has a natural isomorphism α of bifunctors

$$\begin{array}{ccc} & \mathcal{D}(F(-), -) & \\ \mathcal{C}^{\text{op}} \times \mathcal{D} & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \\ \xrightarrow{\quad} \end{array} & \text{Set} \\ & \mathcal{C}(-, G(-)) & \end{array}$$

In this situation one says that F is a *left adjoint* for G and G is a *right adjoint* for F . We write $F \dashv G$ or

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xrightarrow{G} \end{array} \mathcal{D}.$$

(b) The tuple (F, G, α) is called an *adjunction*.

(c) We say $F : \mathcal{C} \rightarrow \mathcal{D}$ has a *right adjoint* if $\exists G : \mathcal{D} \rightarrow \mathcal{C} : F \dashv G$ (similarly $G : \mathcal{D} \rightarrow \mathcal{C}$ has left adjoint.)

Theorem 2.50. For functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ we have:

(a) F has a right adjoint $\iff \forall B \in \mathcal{D} \exists$ universal pair $(A, v : FA \rightarrow B)$ for (F, B) such that $\text{Hom}_{\mathcal{D}}(F(-), B) \cong \text{Hom}_{\mathcal{C}}(-, A)$.

(b) G has a left adjoint $\iff \forall A \in \mathcal{C} \exists$ universal pair $(B, u : A \rightarrow GB)$ for (G, A) such that $\text{Hom}_{\mathcal{C}}(A, G(-)) \cong \text{Hom}_{\mathcal{D}}(B, -)$.

Chapter 3

Tensor Products

Let R be a ring.

Definition 3.1. Let $M \in \text{Mod}_R$ and $N \in {}_R\text{Mod}$ and A an abelian group,

(a) A map $f : M \times N \rightarrow A$ is called *R -balanced* if

- its left \mathbb{Z} -linear, i.e. $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$.
- its right \mathbb{Z} -linear, i.e. $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$
- $\forall r \in R : f(mr, n) = f(m, rn)$.

(b) $\text{Bal}_{M,N}^R(A) = \{f : M \times N \rightarrow A \mid f \text{ is } R\text{-balanced}\}$ is an abelian group.

(c) $\text{Bal}_{M,N}^R(-) : \text{Ab} \rightarrow \text{Ab}$ is a functor via

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & A \\ & \searrow \text{Bal}_{M,N}^R(\varphi) & \downarrow \varphi \\ & & A' \end{array}$$

Idea: R -balanced (bilinear) maps appear naturally, but one needs to treat them separately (they don't live in Ab). To fix this we want to turn these R -balanced maps $M \times N \xrightarrow{f} A$ into a usual group homomorphism

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & A \\ - \otimes - \downarrow & \nearrow \in \text{Ab} & \\ M \otimes N & & \end{array}$$

Theorem 3.2. With the notation from definition 1, the functor $\text{Bal}_{M,N}^R : \text{Ab} \rightarrow \text{Ab}$ is representable, we denote the universal pair by

$$(M \otimes_R N, - \otimes - : M \times N \rightarrow M \otimes N)$$

More concretely, $- \otimes - : M \times N \rightarrow M \otimes N$ is an R -balanced map, such that

$$\begin{aligned} \text{Bal}_{M,N}^R(A) &\cong \text{Hom}_{\mathbb{Z}}(M \otimes_R N, A) \\ \varphi \circ (- \otimes -) &\leftarrow \varphi \end{aligned}$$

Definition 3.3. $M \otimes_R N$ is called the *tensor product* of M and N and elements $m \otimes n$ in the $\text{im}(- \otimes - : M \times N \rightarrow M \otimes_R N)$ are called *tensors*.

Remark. Its easy to see from the universal property that $m \otimes n$'s generate the group $M \otimes_R N$ (exercise)

$$\begin{array}{ccc} M \times N & \xrightarrow{- \otimes -} & M \otimes N \\ & \searrow & \downarrow q \\ & & M \otimes N / \langle \text{im}(- \otimes -) \rangle \end{array} \quad \Rightarrow \quad q = 0$$

we have

$$M \otimes_R N = \bigoplus_{(m,n) \in M \times N} \mathbb{Z}(m \otimes n) / \left\langle \begin{array}{l} (m_1 + m_2) - m_1 \otimes n - m_2 \otimes n \\ m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2 \\ mr \otimes n - m \otimes rn \end{array} \middle| \begin{array}{l} m, m_1, m_2 \in M \\ n, n_1, n_2 \in N \\ r \in R \end{array} \right\rangle$$

Proposition 3.4. $- \otimes_R - : \text{Mod}_R \times {}_R \text{Mod} \rightarrow \text{Ab}$ is a bifunctor. More explicitly, for $M, M', M'' \in \text{Mod}_R$ and $N, N', N'' \in {}_R \text{Mod}$, the following holds:

- (a) For any $\varphi \in \text{Hom}_{\text{Mod}_R}(M, M')$ and $\psi \in \text{Hom}_{{}_R \text{Mod}}(N, N') \exists!$ homomorphism $\varphi \otimes \psi \in \text{Hom}_{\text{Ab}}(M \otimes_R N, M' \otimes_R N')$ such that $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$
- (b) If additionally we have $\varphi' \in \text{Hom}_{\text{Mod}_R}(M', M'')$ and $\psi' \in \text{Hom}_{{}_R \text{Mod}}(N', N'')$, then $(\varphi' \circ \varphi) \otimes (\psi' \circ \psi) = (\varphi' \otimes \psi') \circ (\varphi \otimes \psi)$.
- (c) $\text{id}_M \otimes \text{id}_N = \text{id}_{M \otimes_R N}$.

Proof. TODO. □

Proposition 3.5. For $M \in \text{Mod}_R$ and $N \in {}_R \text{Mod}$

one has:

- (a) $M \rightarrow M \otimes_R R$ given by $m \mapsto m \otimes 1_R$ is an isomorphism in Ab .
- (b) $N \rightarrow R \otimes_R N$ given by $n \mapsto 1_R \otimes n$ is an isomorphism in Ab .

Proof. Only (a): This map is clearly \mathbb{Z} -linear, we construct the inverse map by TODO. □

Proposition 3.6. Let I be a set and $(M_i)_{i \in I}$ a family $M_i \in \text{Mod}_R$ and $N \in {}_R \text{Mod}$ (or the opposite), then $\exists!$ isomorphism

$$\psi : \left(\bigoplus_{i \in I} M_i \right) \otimes N \xrightarrow{\cong} \bigoplus_{i \in I} (M_i \otimes N), (m_i)_i \otimes n \mapsto (m_i \otimes n)_i.$$

Proof. TODO. □

Corollary 3.7. For sets I and J , $R^{(I)} \otimes_R R^{(J)} \cong R^{(I \times J)}$ given by $e_i \otimes f_j \mapsto e_{(i,j)}$ on the basis.

3.1 Tensor products over commutative rings

When R is commutative, $R \cong R^{\text{op}}$ and $\text{Mod}_R \cong {}_R\text{Mod}$, in this case $M \otimes_R N$ admits further structures:

Proposition 3.8. *Suppose that R is commutative and M, N are two R -modules, then*

(a) $M \otimes_R N$ is an R -module with scalar multiplication given by

$$r(m \otimes n) := rm \otimes n = m \otimes rn$$

on pure tensors. For $A \in {}_R\text{Mod}$ define

$$\text{Bil}_{M \times N}^R(A) := \left\{ f : M \times N \rightarrow A \left| \begin{array}{l} f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \\ f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \\ f(rm, n) = f(m, rn) = rf(m, n) \end{array} \right. \right\}$$

to be the set of R -bilinear maps from $M \times N$ to A .

(b) The functor $\text{Bil}_{M \times N}^R : {}_R\text{Mod} \rightarrow \text{Set}$ is representable by $(M \otimes_R N, - \otimes -)$.

(c) $- \otimes - : {}_R\text{Mod} \times {}_R\text{Mod} \rightarrow {}_R\text{Mod}$ is a bifunctor.

Proof. (a) Let $\ell_r : M \rightarrow M$ be given by $m \mapsto rm$, this gives $\ell_r \otimes \text{id}_N : M \otimes_R N \rightarrow M \otimes_R N$ by proposition 4. We define scalar multiplication by r on $M \otimes_R N$ to be the above $r \cdot - := \ell_r \otimes \text{id}$. Check that this gives $M \otimes_R N$ on ${}_R\text{Mod}$ structure.

(b) and (c) exercises. □

Remark. Note that we have less bilinear maps than balanced maps:

$$\text{Hom}_R(M \otimes_R N, A) \cong \text{Bil}_{M \times N}^R(A) \subseteq \text{Bal}_{M \times N}^R(A) \cong \text{Hom}_{\mathbb{Z}}(M \otimes_R N, A)$$

3.2 Tensor product of algebras

Let A be a commutative ring and R, R' two A -algebras via $\varphi : A \rightarrow R$ and $\varphi' : A \rightarrow R'$ (where $\varphi(A) \subseteq Z(R), \varphi'(A) \subseteq Z(R')$).

Proposition 3.9. (a) $\exists!$ A -bilinear multiplication $- \cdot - : (R \otimes_A R') \times (R \otimes_A R') \rightarrow R \otimes_A R'$ given by

$$(r \otimes r') \cdot (s \otimes s') := rs \otimes r's'$$

on pure tensors.

(b) $(R \otimes_A R', +, \cdot, 0_R \otimes 0_{R'}, 1_R \otimes 1_{R'})$ is a ring.

(c) $R \otimes_A R'$ is an A -algebra via $\varphi_{\otimes} : A \rightarrow R \otimes_A R', a \mapsto a \otimes 1 = 1 \otimes a = a(1 \otimes 1)$.

Proof. TODO. □

Examples. (a) If R is an A -algebra then $M_{n \times n}(A) \otimes_A R \cong M_{n \times n}(R)$.

$$(b) \ M_{n \times n}(A) \otimes_A M_{m \times m}(A) = M_{nm \times nm}(A).$$

$$(c) \ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}.$$

$$(d) \ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_{2 \times 2}(\mathbb{C}) \text{ where } \mathbb{H} \text{ is Hamilton's quaternion algebra.}$$

Definition 3.10. An (R, R') -bimodule is a tuple $(M, 0, +, \cdot, \cdot')$, where $(M, 0, +, \cdot) \in {}_R\text{Mod}$ and $(M, 0, +, \cdot') \in \text{Mod}_{R'}$ such that $\forall r \in R, r' \in R', m \in M$ we have

$$r \cdot (m \cdot' r') = (r \cdot m) \cdot' r'$$

We denote the category of bimodules by ${}_R\text{Mod}_{R'}$.

Remark. (a) If $M \in {}_R\text{Mod}_{R'}$ then one has ring homomorphisms

$$\begin{aligned} R &\rightarrow \text{End}_{\text{Mod}_{R'}}(M), r \mapsto r \cdot - \\ (R')^{\text{op}} &\rightarrow \text{End}_{\text{Mod}_R}(M), r' \mapsto - \cdot' r' \end{aligned}$$

(b) We have an equivalence of categories ${}_R\text{Mod}_{R'} \cong {}_{R \otimes_{\mathbb{Z}} (R')^{\text{op}}} \text{Mod}$. The $R \otimes_{\mathbb{Z}} (R')^{\text{op}}$ -module structure comes from

$$R \times (R')^{\text{op}} \rightarrow \text{End}_{\mathbb{Z}}(M), (r, r') \mapsto r \cdot - \cdot' r', r \otimes r' \in R \otimes_{\mathbb{Z}} (R')^{\text{op}}$$

notice that this is bilinear.

Proposition 3.11. The bifunctor $- \otimes -$ extends to a bifunctor

$$- \otimes_{R'} - : {}_R\text{Mod}_{R'} \times {}_{R'}\text{Mod}_{R''} \rightarrow {}_R\text{Mod}_{R''}$$

More explicitly for $M \in {}_R\text{Mod}_{R'}$ and $N \in {}_{R'}\text{Mod}_{R''}$ we define the (R, R'') -bimodule on $M \otimes_{R'} N$ by

$$r \cdot (m \otimes n) \cdot r'' := rm \otimes nr''$$

Proof. Exercise. □

Remark. Similarly one has $- \otimes - : {}_R\text{Mod}_{R'} \times {}_{R'}\text{Mod} \rightarrow {}_R\text{Mod}$ and $- \otimes - : \text{Mod}_{R'} \times {}_{R'}\text{Mod}_{R''} \rightarrow \text{Mod}_{R''}$.

Examples.

1. **Base change:** Let $\varphi : R \rightarrow S$ be a ring homomorphism, then S is an (S, R) -bimodule. The functors ${}_R\text{Mod} \rightarrow {}_S\text{Mod}, M \mapsto S \otimes_R M$ and $\text{Mod}_R \rightarrow \text{Mod}_S, N \mapsto N \otimes_R S$ are called *base change* or *base extension* from R to S . These are left adjoints to restriction of scalars, for example:

$$S \otimes_R R = S, \quad S \otimes_R R^{(I)} = S^{(I)}$$

2. **Associativity of \otimes :** There exists a natural isomorphism

$$(- \otimes_R -) \otimes_S - \cong - \otimes_R (- \otimes_S -) : {}_T\text{Mod}_R \times {}_R\text{Mod}_S \times {}_S\text{Mod}_Q \rightarrow {}_T\text{Mod}_Q$$

Remark. Let R be commutative and $M_1, \dots, M_n \in R\text{-Mod}$, then $\bigotimes_R^{1 \leq i \leq n} M_i = M_1 \otimes_R \dots \otimes_R M_n$ represents the functor $\text{Multi}_{M_1 \times \dots \times M_n}^R(-)$ of R -multilinear maps on $\times_{1 \leq i \leq n} M_i$.

Proposition 3.12. *Another functor on bimodules:*

- (a) *For $M \in {}_S\mathbf{Mod}_R$ and $N \in {}_T\mathbf{Mod}_R$ the abelian group $\mathrm{Hom}_R(M, N)$ carries a natural (T, S) -bimodule structure defined by*

$$(t \cdot f \cdot s)(x) = t \cdot f(sx)$$

for $f \in \mathrm{Hom}_R(M, N), s \in S, t \in T, x \in M$, this gives a bifunctor

$${}_S\mathbf{Mod}_R \times {}_T\mathbf{Mod}_R \rightarrow {}_T\mathbf{Mod}_S$$

- (b) *Similarly one has a bifunctor ${}_R\mathbf{Mod}_S \times {}_R\mathbf{Mod}_T \rightarrow {}_S\mathbf{Mod}_T$*

Proof. Exercise. □

Theorem 3.13 (Hom, \otimes adjunction, Jacobson Prop. 3.8). *Let R, S, T, U be rings and $M \in {}_R\mathbf{Mod}_S, N \in {}_S\mathbf{Mod}_T, P \in {}_U\mathbf{Mod}_T$. There exists a natural isomorphism:*

$$\begin{array}{ccc} & \curvearrowright & \\ {}_R\mathbf{Mod}_S^{\mathrm{op}} \times {}_U\mathbf{Mod}_T & \Downarrow & {}_U\mathbf{Mod}_R \text{ TODO} \\ & \curvearrowleft & \end{array}$$

Chapter 4

Abelian Categories

Definition 4.1. Let R be a commutative ring.

(a) A category \mathcal{A} is called R -linear if

- (i) $\forall X, Y \in \mathcal{A}, \mathcal{A}(X, Y)$ is an R -module $(\mathcal{A}(X, Y), 0_{X,Y}, +_{X,Y}, \cdot_{X,Y})$.
- (ii) $\forall X, Y, Z \in \mathcal{A}$ the composition map

$$\begin{aligned} \mathcal{A}(X, Y) \times \mathcal{A}(Y, Z) &\rightarrow \mathcal{A}(X, Z) \\ (\varphi, \psi) &\mapsto \psi \circ \varphi \end{aligned}$$

is R -bilinear. (in particular $r(\psi \circ \varphi) = (r \cdot \psi) \circ \varphi = \psi \circ (r\varphi)$).

- (b) A functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ between R -linear categories is called R -linear if $\forall X, Y \in \mathcal{A}$ the map $F : \mathcal{A}(X, Y) \rightarrow \mathcal{A}'(FX, FY)$ is R -linear.
- (c) For \mathbb{Z} -linear (preadditive) categories \mathcal{A} and \mathcal{A}' , the full subcategory $\text{Add}(\mathcal{A}, \mathcal{A}') \subseteq \text{Fun}(\mathcal{A}, \mathcal{A}')$ of \mathbb{Z} -linear (additive) functors between them is again a \mathbb{Z} -linear category, i.e. there is a natural addition on natural transformations.

Examples. (a) ${}_R\text{Mod}$ is \mathbb{Z} -linear and in fact (R is commutative) R -linear.

(b) \mathcal{A} is R -linear $\iff \mathcal{A}^{\text{op}}$ is also R -linear.

(c) If \mathcal{A} and \mathcal{A}' are R -linear, then $\mathcal{A} \times \mathcal{A}'$ is R -linear.

(d) Let S be a (not necessarily commutative) ring and \underline{S} the category

$$(\{*\}, S, \text{dom} = \{*\}, \text{cod} = \{*\}, \text{id}_* = \text{id}_S, \circ = \cdot_S)$$

associated to S , then \underline{S} is \mathbb{Z} -linear ($\text{Hom}_{\underline{S}}(*, *) = S$).

Lemma 4.2. If \mathcal{A} is \mathbb{Z} -linear, then for $X \in \mathcal{A}$ the following are equivalent:

- (i) X is initial.
- (ii) X is terminal.
- (iii) $\mathcal{A}(X, X) = 0$ (in particular $\text{id}_X = 0$)

Proof. Exercise. □

Definition 4.3. An $X \in \mathcal{A}$ satisfying conditions of Lemma 2 is called a *zero object*

Notation. The zero object of \mathcal{A} (if it exists) is unique up to unique isomorphism so we just write 0 or 0_A for this object.

Lemma 4.4. Let \mathcal{A} be \mathbb{Z} -linear with a 0-object and $X, Y \in \mathcal{A}$, then the zero map $0_{X,Y} \in \mathcal{A}(X, Y)$ (remember $\mathcal{A}(X, Y)$ is an abelian group) is equal to the composition $X \xrightarrow{\exists!} 0 \xrightarrow{\exists!} Y$

Proof. Exercise. □

Proposition 4.5. Let \mathcal{A} be \mathbb{Z} -linear, then for $X_1, X_2 \in \mathcal{A}$ the following are equivalent:

- (i) The product $X_1 \amalg X_2$ exists.
- (ii) The coproduct $X_1 \amalg X_2$ exists.
- (iii) $\exists Y \in \mathcal{A}, p_1, p_2 : Y \rightarrow X_i$ and $\iota_1, \iota_2 : X_i \rightarrow Y$ such that

$$p_i \circ \iota_j = \begin{cases} 1_{X_i}, & i = j \\ 0, & i \neq j \end{cases}$$

Proof. TODO. □

Remark. In (iii) (Y, ι_1, ι_2) is the coproduct and (Y, p_1, p_2) is the product.

Definition 4.6. In a \mathbb{Z} -linear category we denote $X \amalg Y = X \amalg Y$ by $X \oplus Y$ and call it the *direct sum* of X and Y .

Lemma 4.7. Let \mathcal{A} and \mathcal{A}' be \mathbb{Z} -linear and $F : \mathcal{A} \rightarrow \mathcal{A}'$ an additive functor, then

- (i) If 0_A exists then $0_{A'}$ exists and $F(0_A) \cong 0_{A'}$.
- (ii) If $X, Y \in \mathcal{A}$ and $X \oplus Y$ exists, then $FX \oplus FY$ exists and $FX \oplus FY \cong F(X \oplus Y)$.

Proof. Exercise. □

Definition 4.8. A category \mathcal{A} is called *additive* (or R -linear additive) if \mathcal{A} is \mathbb{Z} -linear (or R -linear), 0_A exists and $\forall X, Y \in \mathcal{A} : X \oplus Y$ exists.

Remark. If \mathcal{A} is additive then the addition on $\mathcal{A}(X, Y)$ is determined by the composition map! (Morel II 1.2.4)

4.1 Kernels and cokernels

Recall that the equalizer of two morphisms $f, g : X \rightarrow Y$ is the limit of the diagram $X \rightrightarrows Y$, so $Z \xrightarrow{u} X \rightrightarrows Y$ such that for every $W \xrightarrow{v} X \rightrightarrows Y$ with $f \circ v = g \circ v$, v factors through u :

$$\begin{array}{ccccc} Z & \xrightarrow{u} & X & \rightrightarrows & Y \\ & \nearrow v & & & \\ W & & & & \end{array}$$

similarly the coequalizer is the colimit of $X \rightrightarrows Y$.

In the category of R -modules, $\text{eq}(f, g) = \ker(f - g)$ and $\text{coeq}(f, g) = \text{coker}(f - g)$. In particular $\ker f = \text{eq}(f, 0)$.

Definition 4.9. Let \mathcal{A} be a \mathbb{Z} -linear category and $f \in \mathcal{A}(X, Y)$.

- (a) define $\ker f := \text{eq}(f, 0)$ and $\text{coker } f = \text{coeq}(f, 0)$ (if they exist)
- (b) If $\ker f$ exists, define the coimage as $X \rightarrow \text{coim } f = \text{coker}(\ker f \rightarrow X)$ (it might not exist).
- (c) If $\text{coker } f$ exists, define the image as $\text{im } f \rightarrow Y = \ker(Y \rightarrow \text{coker } f)$.

Example 4.10. Let $X_1, X_2 \in \mathcal{A}$ and $(X_1 \oplus X_2, \iota_1, \iota_2, p_1, p_2)$ the direct sum. Then $\iota_1 = \ker p_2, \iota_2 = \ker p_1, p_1 = \text{coker } \iota_2, p_2 = \text{coker } \iota_1$

Lemma 4.11. *Kernels are monomorphisms and cokernels are epimorphisms.*

Proof. We only prove the statement for kernels. Let $\ker f \xrightarrow{\iota} X \rightrightarrows Y$, now assume $\iota \circ \varphi = \iota \circ \varphi'$

$$Z \rightrightarrows \ker f \xrightarrow{\iota} X \rightrightarrows Y$$

By the definition of equalizer

$$\begin{aligned} \mathcal{A}(Z, \ker f) &\simeq \{\psi : Z \rightarrow \ker f \mid f \circ \psi = 0\} \\ \varphi &\mapsto \iota \circ \varphi \end{aligned}$$

is bijective, so $\varphi = \varphi'$. □

Theorem 4.12. *Let $f \in \mathcal{A}(X, Y)$ and assume $\ker f, \text{coker } f, \text{im } f, \text{coim } f$ exist. Then there exists a unique morphism $u : \text{coim } f \rightarrow \text{im } f$ such that f is equal to the composition*

$$X \xrightarrow{c} \text{coim } f \xrightarrow{\exists! u} \text{im } f \xrightarrow{d} Y$$

This is called the canonical factorization of f (or epi-mono factorization).

Remark. Note that by lemma 11, c is an epimorphism and d is a monomorphism.

Proof. TODO. □

Note that in the category of R -modules, u is an isomorphism.

Definition 4.13. A category \mathcal{A} is called *abelian* if

- (i) \mathcal{A} is additive (\mathbb{Z} -linear, 0 exists and $X \oplus Y$ exists)
- (ii) $\forall f \in \mathcal{A}(X, Y) : \ker f, \operatorname{coker} f$ exist.
- (iii) $\forall f \in \mathcal{A}(X, Y)$ with canonical factorization $f = d \circ u \circ c$, u is an isomorphism.

Example 4.14. • For any ring R the category ${}_R\mathbf{Mod}$ is abelian.

- For rings R, R' the category ${}_R\mathbf{Mod}_{R'}$ is abelian ($\cong {}_{R \otimes_{\mathbb{Z}}(R')}^{\operatorname{op}}\mathbf{Mod}$).

Example 4.15.

Let $\mathcal{A} \subseteq \mathbf{Ab}$ be the fullsubcategory of finitely generated free \mathbb{Z} -modules, then \mathcal{A} is additive.

For $f : X \rightarrow Y$ in \mathcal{A} the usual $\ker_{\mathbf{Ab}} f$ as abelian group is again a finitely generated free abelian group and is also the kernel in \mathcal{A} .

The cokernel however might have torsion. In fact

$$\operatorname{coker}_{\mathcal{A}} f = \operatorname{coker}_{\mathbf{Ab}} f / \operatorname{Tor}(\operatorname{coker}_{\mathbf{Ab}} f)$$

So kernels and cokernels exists in \mathcal{A} , now let $f : \mathbb{Z} \rightarrow \mathbb{Z}, n \mapsto 2n$, the canonical factorization is

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\operatorname{id}} & \mathbb{Z} & \xrightarrow{u} & \mathbb{Z} & \xrightarrow{\operatorname{id}} & \mathbb{Z} \\ & & n & \longmapsto & 2n & & \end{array}$$

since u is not an isomorphism, \mathcal{A} is not abelian.

Proposition 4.16. Let \mathcal{C} be a category, then

- (i) \mathcal{C} is \mathbb{Z} -linear $\iff \mathcal{C}^{\operatorname{op}}$ is \mathbb{Z} -linear.
- (ii) \mathcal{C} is additive $\iff \mathcal{C}^{\operatorname{op}}$ is additive.
- (iii) \mathcal{C} is abelian $\iff \mathcal{C}^{\operatorname{op}}$ is abelian.

Moreover (if they exist) $\ker_{\mathcal{C}} f = \operatorname{coker}_{\mathcal{C}^{\operatorname{op}}} f$ and $\operatorname{im}_{\mathcal{C}} f = \operatorname{coim}_{\mathcal{C}^{\operatorname{op}}} f$.

4.2 Abelian categories

From now on let \mathcal{A} be an abelian category.

Remark. Let $f : X \rightarrow Y$ and $X \twoheadrightarrow \operatorname{coim} f \xrightarrow[\cong]{u} \operatorname{im} f \hookrightarrow Y$ be the canonical factorization. Then either $X \twoheadrightarrow \operatorname{coim} f \hookrightarrow Y$ or $X \twoheadrightarrow \operatorname{im} f \hookrightarrow Y$ (or anything else isomorphic to them) is called “the” canonical factorization for f .

Proposition 4.17. *Let $X, Y \in \mathcal{A}$*

- (a) $\operatorname{coker}(0 \rightarrow X) = X \xrightarrow{\operatorname{id}} X$ and $\ker(X \rightarrow 0) = X \xrightarrow{\operatorname{id}} X$.
- (b) $f : X \rightarrow Y$ is a monomorphism $\iff \ker f = 0 \iff$ the canonical factorization of f is $X \xrightarrow{\operatorname{id}} X \xrightarrow{f} Y \iff f$ is a kernel.
- (c) $g : C \rightarrow Y$ is an epimorphism $\iff \operatorname{coker} g = 0 \iff$ the canonical factorization of g is $X \xrightarrow{g} Y \xrightarrow{\operatorname{id}} Y \iff g$ is a cokernel.
- (d) u is an isomorphism $\iff u$ is a monomorphism and an epimorphism.

Corollary 4.18. $X \xrightarrow{a} Z \xrightarrow{b} Y$ is the canonical factorization for $f = b \circ a \iff a$ is a monomorphism and b an epimorphism.

Definition 4.19. In an abelian category $f \in \mathcal{A}(X, Y)$ is called injective if $\ker f = 0$ and surjective if $\operatorname{coker} f = 0$.

Corollary 4.20. *Let $f \in \mathcal{A}(X, Y)$, then*

- (a) f is injective $\iff f$ is a monomorphism.
- (b) f is surjective $\iff f$ is an epimorphism.
- (c) f is injective and surjective $\iff f$ is an isomorphism.

4.3 Exactness

Lemma 4.21. *Let (g, f) be a composable pair of morphisms in \mathcal{A} such that $g \circ f = 0$, then \exists a canonical injection $\operatorname{im} f \hookrightarrow \ker g$.*

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ and write the epi-mono factorization of f :

$$X \xrightarrow{c} \operatorname{coim} f \xrightarrow[\cong]{} \operatorname{im} f \xrightarrow{d} Y \xrightarrow{g} Z$$

and call the isomorphism in the middle $u : \operatorname{coim} f \cong \operatorname{im} f$. Now $g \circ d \circ u \circ c = 0 \implies g \circ d \circ u = 0$ since c is an epimorphism and u isomorphism $\implies g \circ d = 0$. By the definition of the kernel d must factor through $\ker g$:

$$\operatorname{im} f \xrightarrow{e} \ker g \xrightarrow{\iota} Y \xrightarrow{g} Z$$

Since d is injective: if $e \circ \varphi = c \circ \varphi' \implies \iota \circ c \circ \varphi = \iota \circ c \circ \varphi' \implies d \circ \varphi = d \circ \varphi' \implies \varphi = \varphi' \implies e$ is a monomorphism $\implies \operatorname{im} f \hookrightarrow \ker g$. \square