Algebra 2 Lecture Notes
from Prof. Gebhard Böckle

Yousef Khell

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Contents

Chapter 1

Modules

1.1 Modules

Let $(R, 0, 1, +, \cdot)$ or simply R be a ring.

Definition 1.1. (a) A left R-module $(M, 0, +, \cdot)$ or simply M is an abelian group $(M, 0, +)$, together with an operation $\cdot : R \times M \to M, (r, m) \mapsto r \cdot m = rm$, such that for all $a, b \in R, m, n \in M$

(M1) $a(m + n) = am + an$ and $(a + b)m = am + bm$

(M2) $a(b \cdot m) = (ab) \cdot m$

(M3) $1 \cdot m = m$

(b) Let M, N be left R-modules. A map $\varphi : M \to N$ is called R-linear or a left R-module homomorphism $:\iff \varphi : (M, 0, +) \to (N, 0, +)$ is a group homomorphism, and $\forall a \in R, m \in M : \varphi(am) = a\varphi(m)$. Define $\mathrm{Hom}_R(M, N) = \{\varphi : M \to N \mid \varphi \text{ is } R\text{-linear}\}$.

Facts 1.2 (Excers.)**.** $\forall x \in M, a \in R : 0_R \cdot x = 0_M, a \cdot 0_M = 0_M, (-1) \cdot x = -x$

Remark 1.3 (Excers.)**.** (a) $\mathrm{Hom}_R(M, N)$ is an abelian group with $0 =$ the map $M \to \{0_N\}$ and $\varphi + \psi : M \to N, m \mapsto \varphi(m) + \psi(m)$.

(b) If R is commutative, then $\mathrm{Hom}_R(M, N)$ is an R-module via

$$r \cdot \varphi : M \to N, m \mapsto r \cdot \varphi(m)$$

(c) If an abelian group $(M, 0, +)$ carries an operation $\cdot : M \times R \to M, (m, r) \mapsto m \cdot r$ such that:

(M1') $(m + n) \cdot a = m \cdot a + n \cdot a, m \cdot (a + b) = ma + mb$

(M2') $(m \cdot a) \cdot b = m \cdot (ab)$

(M3') $m \cdot 1 = m$

then $(M, 0, +, \cdot)$ is called a right R-module. Analogously we can define right R-module homomorphisms.

Convention 1.4. We shall use the term R-module for left R-module, since we will mainly work with these. In fact right R-modules are left R^{op}-modules.

Definition 1.5. The opposite ring (Gegenring) of $(R, 0, 1, +, \cdot)$ is $R^{\text{op}} = (R, 0, 1, +, \cdot^{\text{op}})$ with $a \cdot^{\text{op}} b = b \cdot a$

Facts 1.6 (Excersize)**.** (a) R^{op} is a ring

(b) $\text{id}_R : R \to R$ is a ring homomorphism $\iff$ R is commutative.

(c) $\text{id}_R : R \to (R^{\text{op}})^{\text{op}}$ is an isomorphism.
In particular: If R is commutative, then left R-modules are right R-modules.

Remark 1.7 (Excersize)**.** Let $(M, 0, +)$ be an abelian group.

(a) The abelian group $\text{End}_{\mathbb{Z}}(M) = \text{Hom}_{\mathbb{Z}}(M, M)$ is a ring with composition as multiplication.

(b) There is a bijection $\{$operations $* : R \times M \to M \mid (M, 0, +, *)$ is an R-module$\} \leftrightarrow \{$ring homomorphisms $\varphi : R \to \text{End}_{\mathbb{Z}}(M)\}$ via

$$* \mapsto \varphi_* : R \to \text{End}_{\mathbb{Z}}(M), r \mapsto (\varphi_*(r) : m \mapsto r \cdot m)$$

figure out an inverse.

(c) If M is an R-module, then $\text{End}_R(M) \subseteq \text{End}_{\mathbb{Z}}(M)$ is a subring

(d) The map $R^{\text{op}} \to \text{End}_R(R), r \mapsto \rho_r : a \mapsto a \cdot r$ is a ring isomorphism. The inverse is $\text{End}_R(R) \to R^{\text{op}}, \varphi \mapsto \varphi(1)$

Example 1.8. (a) Let K be a field, K-modules are K-vector spaces and vice versa.

(b) If $(M, 0, +)$ is an abelian group, it is in a unique way a $\mathbb{Z}$-module.

(c) Let K be a field, $R = M_{n \times n}(K), n > 1, V_n(K) =$ column $Z_n(K)$ row vectors of length n over K, then:

- $V_n(K)$ is a left R-module.
- $Z_n(K)$ is a right R-module.

(d) R is a left R-module and right R module with multiplication.

(e) If M_1 and M_2 are R-modules, we can define on $M_1 \times M_2$ a R-module structure via

$$r \cdot (m_1, m_2) := (rm_1, rm_2)$$

(group structure from Algebra 1)

(f) $\text{Hom}_R(R, M) \to M, \varphi : \varphi(1)$ is an isomorphism of abelian groups, and if R is commutative, then also an isomorphism of R-modules.

Definition 1.9. An R-linear map $\varphi : M \to M'$ is called a monomorphism/epimorphism/isomorphism $\iff$ φ is injective/surjective/bijective respectively. We say R-modules M, M' are isomorphic if there exists an isomorphism $M \to M'$.

Remark. φ is an R-linear isomorphism $\iff$ φ^{-1} is an R-linear isomorphism.

Definition 1.10. (a) Let M be an R-module. A subset $N \subseteq M$ is an R-submodule if it is a subgroup and $\forall a \in R, n \in N : a \cdot n \in N$ (i.e. $R \cdot N \subseteq N$)

(b) An R-submodule $I \subseteq R$ is called a left ideal.

(c) $I \subseteq R$ is called a two sided ideal iff it is a left ideal and $I \cdot R \subseteq I$

Example 1.11. (a) If $N' \subseteq N$ and $M' \subseteq M$ are R-submodules of R-modules M and N and if $\varphi : M \to N$ is an R-linear map, then:

$$\varphi(M') \subseteq N \text{ and } \varphi^{-1}(N') \subseteq M$$

are R-submodules. In particular $\ker(\varphi) \leq M$ and $im(\varphi) \leq N$ are submodules.

(b) If $(M_i)_{i \in I}$ is a family of submodules of M, then $\bigcap_{i \in I} M_i \subseteq M$ is the largest submodule of M contained in all M_i, and

$$\sum_{i \in I} M_i = \{\sum_{i \in I} m_i \mid m_{i \in M)i}, \#\{i \mid m_{i \neq 0}\} < \infty\}$$

is the smallest submodule of M containing all M_i.

(c) 2-sided ideals of $M_{n \times n}(R)$ are of the form $M_{n \times n}(I)$ for $I \subseteq R$ a 2-sided ideal.

1.2 Quotient Modules

Definition 1.12. Let $N \subseteq N$ be a submodule. From linear algebra $(M/N, \overline{0}, \overline{+})$ is an abelian group. ($\overline{m} = m + N$ are the equivalence classes and $\overline{m} + \overline{m}' = \overline{m + m'}$). This is an R-module (exercise) via

$$\bar{\cdot} : R \times {}^{M}\!\Big/_{N} \to {}^{M}\!\Big/_{N} : (r, m + N) \mapsto rm + N$$

We call M/N (with $\overline{0}, \overline{+}, \bar{\cdot}$) the quotient module of M by N, and we write

$$\pi_{N \subseteq M} : M \twoheadrightarrow {}^{M}\!\Big/_{N}, m \mapsto m + N$$

Definition 1.13. If $I \subseteq R$ is a 2-sided ideal of R, then

(a) $I \cdot M := \{\sum_{i \in I} a_i \cdot m_i \mid I \text{ finite}, a_{i \in I, m_i \in M}\}$ is an R-submodule of M (M an R-module)

(b) $(R/I, \overline{0}, \overline{1}, \overline{+}, \bar{\cdot})$ is a ring, and $M/I \cdot M$ is an R/I-module.

The following 3 results are proved as for groups:

Theorem 1.14 (Homomorphism theorem)**.** *Let $\varphi : M \to M'$ be an R-linear map, then*

(a) $\forall$ submodules $N \subseteq \ker(\varphi) : \exists! R$-linear map $\overline{\varphi} : M/N \to M', m + N \mapsto \varphi(m)$ such that $\varphi = \overline{\varphi} \circ \pi_{N \subseteq M}$

(b) For $N = \ker(\varphi)$, the map $\overline{\varphi} : M/\ker(\varphi) \to im(\varphi)$ is an R-module isomorphism.

Theorem 1.15. *(First isomorphism theorem) Let M be an R-module and $N_1, N_2 \leq M$ be R-submodules. Then the map*

$$ {}^{N_1}\Big/_{N_1 \cap N_2} \to {}^{N_1 + N_2}\Big/_{N_2}, n_1 + N_1 \cap N_2 \mapsto n_1 + N_2$$

is a well-defined R-linear isomorphism.

$$\begin{tikzcd}
 & N_1 + N_2 \arrow[dl, dash] \arrow[dr, dash] & \\
N_1 \arrow[dr, dash] & & N_2 \arrow[dl, dash] \\
 & N_1 \cap N_2 &
\end{tikzcd}$$

Theorem 1.16 (Second isomorphism theorem)**.** *Let M be an R-module and $N \leq M$ an R-submodule. Then*

(a) The following maps are bijective and mutually inverse to each other:

$$\{N' \subseteq M \textit{ submodule} \mid N \subseteq N'\} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\overline{N} \subseteq {}^{M}\Big/_{N} \textit{ submodule}\}$$

$$\varphi : N \mapsto {}^{N'}\Big/_{N} \qquad \pi^{-1}_{N \subseteq M}(\overline{N}) \mapsfrom \overline{N} : \psi$$

(b) For $N' \subseteq M$ a submodule with $N \subseteq N'$ we have the R-linear isomorphism:

$$ {}^{(M/N)}\Big/_{(N'/N)} \to {}^{M}\Big/_{N'}, \overline{m} + {}^{N'}\Big/_{M} \mapsto m + N'$$

1.3 Direct sums and products

Let $(M_i)_{i \in I}$ be a family of R-modules.

Definition 1.17. (a) $\prod_{i \in I} M_i = \{(m_i)_{i \in I} \mid m_i \in M_i, \forall i \in I\}$ is an R-module with component-wise operations:

$$(m_i)_{i \in I} + (n_i)_{i \in I} = (m_i + n_i)_{i \in I}$$

$$r \cdot (m_i)_{i \in I} = (r \cdot m_i)_{i \in I}, \quad r \in R$$

is called the (direct) product of $(M_i)_{i \in I}$. One has the projection maps (R-module epimorphisms):

$$\pi_{i_0} : \prod_{i \in I} M_i \to M_{i_0}, (m_i) \mapsto m_{i_0}$$

(b) $\bigoplus_{i \in I} M_i = \{(m_i)_{i \in I} \in \prod_{i \in I} M_i \mid \{i \mid m_i \neq 0\} < \infty\}$ is an R-submodule of $\prod_{i \in I} M_i$. It is called the direct sum of $(M_i)_{i \in I}$. One has R-module monomorphisms

$$\iota_{i_0} : M_{i_0} \to \bigoplus_{i \in I} M_i, m_{i_0} \mapsto (\iota_{i_0}(m_{i_0}))$$

where the i-th component of $\iota_{i_0}(m_{i_0})$ is given by $\begin{cases} m_{i_0}, & i = i_0, \\ 0, & \text{otherwise} \end{cases}$

Theorem 1.18 (Universal property of the direct product/sum). *(a) $\forall R$-modules M, the map*

$$\operatorname{Hom}_R(M, \prod_{i \in I} M_i) \xrightarrow{\cong} \prod_{i \in I} \operatorname{Hom}_R(M, M_i), \varphi \mapsto (\pi_i \circ \varphi)_{i \in I}$$

is well defined, bijective and a group isomorphism.

(b) $\forall R$-modules M, the map

$$\operatorname{Hom}_R(\bigoplus_{i \in I} M_i, M) \xrightarrow{\cong} \prod_{i \in I} \operatorname{Hom}_R(M_i, M), \psi \mapsto (\psi \cdot \iota_i)_{i \in I}$$

is well defined, bijective and a group isomorphism.

Proof. (a) The inverse map is given by sending

$$\underline{\varphi} := (\varphi_i : M \to M_i)_{i \in I} \in \prod_{i \in I} \operatorname{Hom}_R(M, M_i)$$

to

$$\pi_{\underline{\varphi}} : M \to \prod_{i \in I} M_i, m \mapsto (\varphi_i(m))_{i \in I}$$

now check: $\underline{\varphi} \mapsto \pi_{\underline{\varphi}}$ is inverse to the map in (a).

(b) The map is given by sending $\overline{\varphi} = (\varphi_i : M_i \to M)_{i \in I}$ to

$$\coprod_{\overline{\varphi}} : \bigoplus_{i \in I} : M_i \to M, (m_i)_{i \in I} \mapsto \sum_{i \in I} \varphi_i(m_i)$$

□

Corollary 1.19 (Important special case). *Let I be finite, then:*

(a) $M := \prod_{i \in I} M_i \overset{!}{=} \bigoplus_{i \in I} M_i$

(b) The maps $M_i \underset{\pi_i}{\overset{\iota_i}{\rightleftarrows}} M$ satisfy

$$\pi_i \circ \iota_j = \begin{cases} \mathrm{id}_{M_i}, & i = j, \\ 0, & \textit{otherwise} \end{cases} \quad \textit{and} \quad \sum_{i \in I} \iota_i \circ \pi_i = \mathrm{id}_M$$

(c) If M' is a module with maps $M_i \underset{\pi_i'}{\overset{\iota_i'}{\rightleftarrows}} M'$ such that the formulas above hold, then $M \cong M'$

1.4 Generators and bases

From now onwards let R be a unitary ring and M, M', N be R-modules.

Notation.

- For I a set we write $M^I := \prod_{i\in I} M$ and $M^{(I)} := \bigoplus_{i\in I} M$ (where $M_i = M, \forall i \in I$).
- For $r \in \mathbb{N}$ we will write $M^r := M^{\{1,\dots,r\}}$, so if I is finite then $M^I = M^{\#I} = M^{(I)}$

Definition 1.20. For $\underline{m} = (m_i)_{i\in I} \in M^{(I)}$ we define a map $\varphi_{\underline{m}} : R^{(I)} \to M, (r_i) \mapsto \sum_{i\in I} r_i \cdot m_i$ where r_i is non-zero only for finitely many i. We can also define $\varphi_{\underline{m}}$ via the universal property of $R^{(I)}$ using maps $R \to M, r \mapsto r \cdot m_i$ at component $i \in I$.

(a) $\underline{m}$ is a generating set of $M \iff \varphi_{\underline{m}}$ is surjective.

(b) $\underline{m}$ is a basis of $M \iff \varphi_{\underline{m}}$ is an isomorphism.

(c) M is a free R-module $\iff M$ has a basis.

(d) $\underline{m}$ is finitely generated $\iff$ it has a finite generating set.

(e) $\underline{m}$ is linearly independent $\iff \varphi_{\underline{m}}$ is injective.

Remark. Let $\iota_j : R \to R^{(I)}$ be the inclusion of the component $j \in I$ (1.18) and set $e_j := \iota_j(1)$. Then we call $(e_j)_{j\in I}$ the standard basis of $R^{(I)}$.

Example. (a) If K is a field, then any K-vector space has a basis.

(b) If $R = \mathbb{Z}$, then $M = \mathbb{Z}/(3)$ is finitely generated but not free (exercise).

Remark 1.21. Every R-module is a quotient of a free R-module.

Proof. Let $R^{(M)}$ be the free R-module over the index set M, then

$$\varphi_{\underline{m}} : R^{(M)} \to M, (r_m)_{m\in M} \mapsto \sum_{m\in M} r_m \cdot m$$

is surjective for $\underline{m} = (m)_{m\in M}$. □

Theorem 1.22. *Let R be commutative, then for $n_1, n_2 \in \mathbb{N}_0$, then we have $R^{n_1} \cong R^{n_2} \iff n_1 = n_2$.*

Proof.

- “ $\Longleftarrow$ ”: (By induction to linear algebra.) Let $\mathfrak{m} \subseteq R$ be a maximal ideal. (Axiom of choice) Consider for $n \in \mathbb{N}$ the map $\varphi_n : R^n \to (R/\mathfrak{m})^n, (r_1, \dots r_n) \mapsto (r_i \mod n)_{i\in\{1,\dots,n\}}$. Then φ_n is surjective with kernel $\mathfrak{m}^n \in R^n \implies R^n/\mathfrak{m}^n \cong (R/\mathfrak{m})^n$ by the homomorphism theorem. Now suppose $\psi : R^{n_1} \to R^{n_2}$ is an isomorphism. We show $n_1 \geq n_2$ (by symmetry of argument we get $n_1 = n_2$). Consider the map

$$\begin{tikzcd} R^{n_1} \arrow[r, "\cong"] \arrow[rr, bend right, "\rho"'] & R^{n_2} \arrow[r, two heads] & R^{n_2}\big/\mathfrak{m}^{n_2} \end{tikzcd}$$

this map is surjective and contains $\mathfrak{m}^{n_1}$ in its kernel (check this). By the homomorphism theorem we get a surjective homomorphism

$$\left(R\Big/\mathfrak{m}\right)^{n_1} = R^{n_1}\Big/\mathfrak{m}^{n_1} \to R^{n_2}\Big/\mathfrak{m}^{n_2} = \left(R\Big/\mathfrak{m}\right)^{n_2}$$

by linear algebra we conclude that $n_1 \geq n_2$.

□

Definition 1.23. If M is free and finitely generated, then define rank(M) (the rank of M) as the unique $n \in \mathbb{N}_0$ such that $M \cong R^n$.

Remark. If R is non-commutative, then the rank of the finitely generated R-modules is not well-defined. (Jantzen Schwermer Bsp VII.4.2: $R \cong M \cong R^2$ for $R = \mathrm{End}_K(K[X])$)

1.5 Exact sequences

Definition 1.24. (a) A diagram of R-modules

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is called exact (at M) $:\Longleftrightarrow$ $\ker(g) = \mathrm{im}(f)$

(b) An exact sequence of R-modules is a family $(f_j)_{j\in J}$ of R-module homomorphisms $f_j : M_j \to M_{j+1}$ index of an interval $J \subseteq \mathbb{Z}$, such that $\forall j \in J : j+1 \in J$, the sequence

$$M_j \xrightarrow{f_j} M_{j+1} \xrightarrow{f_{j+1}} M_{j+2}$$

is exact (at M_{j+1}). Other notation:

$$M_{j_0} \xrightarrow{f_{j_0}} M_{j_0+1} \xrightarrow{f_{j_0+1}} \cdots \to M_{j+2}$$

(c) An exact sequence $0 \to M' \to M \to M'' \to 0$ is called a short exact sequence (s.e.s.)

Remark.
- $0 \to M' \xrightarrow{f} M$ is exact $\overset{\text{Exercise}}{\Longleftrightarrow}$ f is injective.
- $M \xrightarrow{g} M'' \to 0$ is exact $\overset{\text{Exercise}}{\Longleftrightarrow}$ g is surjective.
(0 stands for the 0-module $\{0\}$)

Example 1.25. Let $f : M \to N$ be an R-module homomorphism. Then one defines

$$\mathrm{coker}(f) := N\Big/\mathrm{im}(f)$$

as the cokernel of f , it comes toether with an R-module epimorphism $\pi : N \to \mathrm{coker}(f)$. As an exercise: The sequence

$$0 \to \ker(f) \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{\pi} \mathrm{coker}(f) \to 0$$

is exact. Subexamples:

- If f is injective, then $0 \to M \xrightarrow{f} N \to \operatorname{coker}(f) \to 0$ is exact.
- If f is surjective, then $0 \to \ker(f) \to M \xrightarrow{f} N \to 0$ is exact.

Remark. For R-module homomorphisms $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with $\beta \circ \alpha = 0$, the following are equivalent:

(i) $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is a s.e.s.

(ii) β is surjective and $\alpha : M' \to \ker(\beta)$ is an isomorphism.

(iii) α is injective and the homomorphism theorem induces an isomorphism $\operatorname{coker}(\alpha) \cong M/\operatorname{im}(\alpha) \to M''$

($\beta \circ \alpha = 0 \iff \operatorname{im}(\alpha) \subseteq \ker(\beta)$)

Proposition 1.26 (Exercise)**.** *(a) Let* $0 \to M_i' \to M_i \to M_i'' \to 0$ *be short exact sequences* $\forall i \in I$*, then we get short exact sequences*

$$0 \to \bigoplus_{i \in I} M_i' \to \bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} M_i'' \to 0$$

$$0 \to \prod_{i \in I} M_i' \to \prod_{i \in I} M_i \to \prod_{i \in I} M_i'' \to 0$$

(b) Suppose $0 \to V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} V_n \to 0$ *is an exact sequence of finite dimensional* K*-vector spaces, then:*

$$\sum (-1)^i \dim_K(V_i) = 0.$$

Notation 1.27 (Commutativity of diagrams)**.** A diagram of R-modules is a directed graph, where any vertex is an R-module and any arrow is an R-linear map from the module at its source to the module at its target. We call two arrows composable if the target of the first arrow is the source of the second; then the correspoding maps can be composed. So to any chain of composable arrows, the composition of maps defines a map from the source of the first to the target of the last arrow in the chain. A diagram is **commutative** if for any two chains of arrows with the same source and target, the resulting two maps agree.

Example. (a) To say that the diagram

$$\begin{tikzcd} .M_1 \arrow[r, "f"] \arrow[d, "g"'] & M_2 \arrow[d, "g'"] \\ M_3 \arrow[r, "f'"'] & M_4 \end{tikzcd}$$

commutes means that $g' \circ f = f' \circ g$.

(b) $M \underset{g}{\overset{f}{\rightleftarrows}} N$ commutes $\iff g = h$

Theorem-Definition 1.28. *For a short exact sequence of R-modules*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] \arrow[l, dashed, bend left, "s"] & M'' \arrow[r] \arrow[l, dashed, bend left, "t"] & 0 \end{tikzcd} \qquad (*)$$

the following are equivalent:

(a) $\exists R$*-linear map* $t : M'' \to M$ *such that* $g \circ t = \mathrm{id}_{M''}$

(b) $\exists$ *submodule* $N \subseteq M$ *such that*

$$\psi : \mathrm{im}(f) \oplus N \to M, (b, n) \mapsto b + n$$

is an isomorphism.

(c) $\exists R$*-linear map* $s : M \to M'$ *such that* $s \circ f = \mathrm{id}_{M'}$*.*

In this case (if (a) - (c) hold), then the sequence $(*)$ *is called a split exact sequence. (simply* $(*)$ *is split or splits), and* t *(or* s*) is called a splitting of* g *(or of* f *respectively).*

Proof.

- $(a) \implies (b)$: Given t, define $N := \mathrm{im}(t)$ and ψ as above, i.e. $\psi : \mathrm{im}(f) \oplus N \to M, (b, n) \mapsto b + n$
  - $\ker(\psi) = 0$: Let $(b, n) \in \ker(\psi)$, i.e. $n = t(m'')$, for some $m'' \in M''$ and $b = f(m')$ for some $m' \in M'$ and $n + b = 0$ ($\psi(b, n) = 0$).
  - Apply $g : M \to M''$:

$$\underbrace{g(n + b)}_{0} = \underbrace{g(t(m''))}_{g \circ t = \mathrm{id}_{M''}} + \underbrace{g(f(m'))}_{g \circ f = 0} = m'' + 0$$

$$\implies m'' = 0 \implies n = t(m'') = 0 \underset{n+b=0}{\implies} b = 0 \implies (b, n) = (0, 0)$$

  - $\mathrm{im}(\psi) = M$: Let $m \in M$, define $n = t(g(m))$ and $b = m - n$. So $n \in N = \mathrm{im}(f)$. $b \in \mathrm{im}(f)$?, to show $b \in \ker(g)$. For this $g(b) = g(m - n) = g(m) - \underbrace{g(t(g(m)))}_{g \circ t = \mathrm{id}_{M''}} = g(m) - g(m) = 0$, so $(b, n) \in \mathrm{im}(f) \oplus N$ and $\psi(b, n) = b + n = m$ by definition of b.

- $(c) \implies (b)$ analogous. Define $N = \ker(s)$ ($M' \underset{s}{\overset{f}{\rightleftarrows}} M$). We want to show $\mathrm{im}(f) \oplus N \to M, (b, n) \to b + n$ is an isomorphism.
  - $\ker(\psi) = 0$: Check.
  - $\mathrm{im}(\psi) = M$: For $m \in M$ observe that

$$\underbrace{f \circ s(m)}_{\in \mathrm{im}(f)} + \underbrace{(m - f \circ s(m))}_{\in \ker(s) \text{ check.}} = m$$

- $(b) \to (a)$ and (c): Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f'", "m'\mapsto(f(m'),0)"'] \arrow[d, "\mathrm{id}_{M'}"] & \mathrm{im}(f) \oplus N \arrow[r, "g'", "(b,n)\mapsto g(n)"'] \arrow[d, "\psi"] \arrow[l, dashed, bend right] & M'' \arrow[r] \arrow[d, "\mathrm{id}_{M''}"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] \arrow[l, dashed, bend left, "?"] & M'' \arrow[r] \arrow[l, dashed, bend left, "?"] & 0
\end{tikzcd}$$

  The diagram commutes. $\psi \circ f' = f, g \circ \psi = g'$, e.g:

$$\psi \circ f'(m') = \psi(f(m'), 0) = f(m') + 0 = f(m')$$

  and

$$g \circ \psi(b, n) = g(b + n) = \underbrace{g(b)}_{=0} = g(n) = g(n) = g'(b, n)$$

  ($g(b) = 0$ is because $b \in \mathrm{im}(f) = \ker(g)$).

- For s: $f : M' \to \mathrm{im}(f)$ is an isomorphism (f is injective) $\implies f^{-1} : \mathrm{im}(f) \to M'$ is an isomorphism. Check

$$s = (f^{-1}, 0) \circ \psi^{-1} : \ M \xrightarrow{\psi^{-1}} \mathrm{im}(f) \oplus N \xrightarrow{(b,n)\mapsto f^{-1}(b)} M'$$

- For t: Check that $s : N \to M''$ is an isomorphism using (b). Set $t := \imath \circ g^{-1}$ for $\imath$ the inclusion so

$$t : M'' \to N \hookrightarrow M$$

  Check. $\square$

Remark. $M' \underset{s}{\overset{f}{\rightleftarrows}} M$ and $M'' \underset{g}{\overset{t}{\rightleftarrows}} M$ satisfy the condition from corollary 1.19, namely:

- $s \circ f = \mathrm{id}_{M'}$
- $g \circ t = \mathrm{id}_{M''}$
- $t \circ g + f \circ s = \mathrm{id}_M$

shows again: the sequence is split if $M \cong M' \oplus M''$ (for the "right maps")

Remark 1.29. One also has short exact sequences for groups

$$1 \to \ker(\pi) \overset{s}{\leftrightarrows} G \underset{\pi}{\overset{t}{\leftrightarrows}} \overline{G} \to 1$$

Here one has to be careful what splitting means. Having a t is not equivalent to having an s.

$$\exists t \iff G \cong \ker(\pi) \rtimes \overline{G}$$

$$\exists s \iff G \cong \ker(\pi) \times \overline{G}$$

1.6 Projective Modules

Definition 1.30. An R-module P is called projective $\iff$ it has the following lifting property (LP; Hochhebungseigenschaft): In every diagram of R-modules

$$\begin{tikzcd}
 & P \arrow[d, "\varphi"] \arrow[dl, dashed, "\widehat{\varphi}"'] & \\
M \arrow[r, "\pi"] & M' \arrow[r] & 0
\end{tikzcd}$$

with π surjective, there exists a lifting $\widehat{\varphi} : P \to M$ such that $\pi \circ \widehat{\varphi} = \varphi$.

Proposition 1.31. *(a) Every free R-module is projective.*

(b) For an R-module P TFAE:

- *(i) P is projective*
- *(ii) every s.e.s. $0 \to M' \to M \to P \to 0$ of R-modules splits.*
- *(iii) P is a direct summand of a free module, i.e. $\exists R$-module Q, such that $P \oplus Q$ is a free R-module.*

Proof. (a) Let $P = R^{(I)}$ for a set I. Consider the diagram

$$\begin{tikzcd}
 & R^{(I)} \arrow[d, "\varphi"] \arrow[dl, dashed, "\widehat{\varphi}"'] & \\
M \arrow[r, "\pi"] & M' \arrow[r] & 0
\end{tikzcd}$$

Denote by $(e_i)_{i \in I}$ the standard basis of $R^{(I)}$. φ is characterized by $m'_i := \varphi(e_i)$ for all $i \in I$. (by universal property of $R^{(I)} = \bigoplus_{i \in I} R$). Because π is surjective, we can choose a preimage $m_i \in M$ with $\pi(m_i) = m'_i$. Define $\widehat{\varphi} : R^{(I)} \to M$ as the unique R-module-homomorphism with $\widehat{\varphi}(e_i) = m_i$. Then $(\pi \circ \widehat{\varphi})(e_i) = \pi(m_i) = m'_i = \varphi(e_i) \implies \pi \circ \widehat{\varphi} = \varphi$.

(b)

- (i) $\implies$ (ii): Let P be projective, consider a s.e.s.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M \arrow[r, "\pi"] & P \arrow[r] & 0 \\
 & & & P \arrow[u, "\mathrm{id}_P"'] \arrow[ul, dashed, "\psi"] &
\end{tikzcd}$$

  By the lifting property $\exists \psi : P \to M$ such that $\pi \circ \psi = \mathrm{id}_P$, i.e. ψ is a splitting $\implies$ the s.e.s. splits.

- (ii) $\implies$ (iii): From Remark 21 we have an R-module epimorphism $R^{(I)} \xrightarrow{\pi} P$ (for $I = P$). Take $Q := \ker \pi$ ($\implies$ s.e.s. $0 \to Q \to R^{(I)} \to P \to 0$) By splitness $R^{(I)} = P \oplus Q$ (by Theorem 28).

- (iii) $\implies$ (i): Start with a diagram

$$\begin{tikzcd}
 & P \arrow[d, "\varphi"] & \\
M \arrow[r, "\pi"] & M' \arrow[r] & 0
\end{tikzcd}$$

and assume $\exists R$-module Q such that $P \oplus Q = R^{(I)}$ (really $\cong$). Extend φ to

$$P \oplus Q \xrightarrow{\widetilde{\varphi}} M', (p,q) \mapsto \varphi(p) + 0$$

By (a) $\exists \widehat{\widetilde{\varphi}} : P \oplus Q \to M$ with $\pi \circ \widehat{\widetilde{\varphi}} = \widetilde{\varphi}$.

$$\begin{tikzcd}
 & P \oplus Q \arrow[dl, dashed, "\widehat{\widetilde{\varphi}}"'] \arrow[d, "\widetilde{\varphi}"] \\
M \arrow[r, "\pi"] & M'
\end{tikzcd}$$

Set $\widehat{\varphi} := \widehat{\widetilde{\varphi}}|_{P \oplus Q} : P \to M$, check $\pi \circ \widehat{\varphi} = \varphi$. ☐

Corollary 1.32. *Let $0 \to M' \to M \to M'' \to 0$ be a s.e.s. of R-modules.*

(a) M'' projective $\implies$ ($M \cong M' \oplus M''$ and M is projective $\iff$ M' projective).

(b) If M' and M'' are free R-modules, then so is M.

(c) If $M' \cong R^{(I)}$ and $M'' \cong R^{(I'')}$, then $M \cong R^{(I' \cup I'')}$. In particular, $rank(M) = rank(M') + rank(M'')$ if $I' \cup I''$ is finite.

Proof. (c) clear: $R^{(I')} \oplus R^{(I'')} \cong R^{I' \cup I''}$

(c) Follows from (a)

(c)
- First assertion in (a) ($M'' \implies M \cong M' \oplus M''$) from Proposition 31.
- Second assertion: we know $M \cong M' \oplus M''$.
- Suppose first: M is projective. Then by 31(b)(iii): $\exists Q$ an R-module such that $M \oplus Q$

☐

Theorem 1.33 (Horse shoe lemma)**.** *Given a diagra of R-modules with P', P'' projective, and the first row exact*

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0 \\
 & P' \arrow[u, "\alpha"] & & P'' \arrow[u, "\gamma"] &
\end{tikzcd}$$

(a) The diagram can be completed by the dotted part to a commutative diagram, for a suitable $\beta : P' \oplus P'' \to M$, so that:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0 \\
0 \arrow[r, dashed] & P' \arrow[u, "\alpha"] \arrow[r, dashed, "f' : p' \mapsto (p', 0)"'] & P' \oplus P'' \arrow[u, dashed, "\beta"] \arrow[r, dashed, "g' : (p', p'') \mapsto p''"'] & P'' \arrow[u, "\gamma"] \arrow[r, dashed] & 0
\end{tikzcd}$$

and the second row is then also exact.

(b) If α and γ are surjective, then so is β.

Proof. (a) Construction of β: Use the lifting property of P'' to complete

$$\begin{tikzcd}
M \arrow[r, "g"] & M'' \arrow[r] & 0 \\
& P'' \arrow[u, "\gamma"'] \arrow[ul, dashed, "\widehat{\gamma}"] &
\end{tikzcd}$$

By the diagonal arrow $\widehat{\gamma} : P'' \to M$ to a commutative diagram. Define

$$\beta : P' \oplus P'' \to M, (x', x'') \mapsto f \circ \alpha(x') + \widehat{\gamma}(x'')$$

Check commutativities:

- $\beta \circ f' \overset{?}{=} f \circ \alpha$:

$$\beta \circ f'(x') = \beta(x', 0) = f \circ \alpha(x') + \widehat{\gamma}(0)$$

- $\gamma \circ g' \overset{?}{=} g \circ \beta$:

$$g \circ \beta(x', x'') = g(f \circ \alpha(x') + \widehat{\gamma}(x'')) = TODO$$

(b) Diagram chase:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0 & \\
& & & & & TODO \\
0 \arrow[r] & P' \arrow[r, "f'"] \arrow[uu, "\alpha"] & P \arrow[r, "g'"] \arrow[uu, "\beta"] & P'' \arrow[r] \arrow[uu, "\gamma"'] & 0 &
\end{tikzcd}$$

To show: β is surjective. Let $m \in M$, γ surjective $\implies \exists x'' \in P'' : \gamma(x'') = g(m)$. g' surjective $\implies \exists x \in P : g'(x) = x'$.

Compare m with $\beta(x)$, consider $m - \beta(x)$. Observe: $g(m - \beta(x)) = g(m) - g \circ \beta(x) = \gamma(x'') - g(x'') = 0$ TODO

□

1.7 Finite generation, exact sequences and $\oplus$

Corollary 1.34. *Let* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *be a s.e.s. of* R*-modules, then*

(a) If M *is finitely generated as* R*-module, then so is* M''

(b) If M' *and* M'' *are finitely generated (as* R*-modules), then so is* M.

Proof. (a) M finitely generated R-module means $\exists$ finite set I and R-module epimorphism $\pi : R^{(I)} \to M \implies R^{(I)} \to M''$ is given by $g \circ \pi$ is an epimorphism
$impM''$ is finitely generated as an R-module.

(b) Suppose we know R-module epimorphisms $\alpha : R^{(I')} \to M'$ and $\gamma : R^{(I'')} \to M''$, then Theorem 33 gives an R-module epimorphism

$$\beta : R^{(I')} \oplus R^{(I'')} \to M$$

$\implies M$ is finitely generated. □

Remark. M is finitely generated as an R-module $\not\Longrightarrow$ $M' \leq M$ is finitely generated. Example: let $R = M = K[X_i \mid i \in \mathbb{N}]$ and consider

$$g : M \to K, X_i \mapsto 0, \forall i$$

The kernel is the ideal I of R generated by $\{X_i \mid i \in \mathbb{N}\}$. We can check: I is not a finitely generated R-module. If $I = (f_1, \dots, f_m)$ say $f_1, \dots, f_m \in K[X_1, \dots, X_n]$ TODO

Corollary 1.35 (exer). *Let $M_1, \dots, M_n$ be R-modules, then*

(a) $M = \bigoplus_{1 \leq i \leq n} M_i$ is finitely generated $\iff$ M_i is finitely generated $\forall i$

(b) Suppose $M_0 \subseteq \cdots \subseteq M_n$ with M_i / M_{i-1} finitely generated for all $i \in \{1, \dots, n\}$. Then M_n is finitely generated.

Theorem 1.36 (Snake lemma). *Suppose we are given the following commutative diagram of R-modules with exact rows.*

$$\begin{tikzcd}
 & M' \arrow[r, "f"] \arrow[d, "\varphi'"] & M \arrow[r, "g"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "f'"] & N \arrow[r, "g'"] & N'' &
\end{tikzcd}$$

Then:

(a) $\exists R$-linear map δ (called the connecting homomorphism) from $\ker(\varphi'')$ *to* $\operatorname{coker}(\varphi')$, *such that the following sequence of R-modules is exact: TODO*

(b) If f is injective, then so is f.

(c) If g' is surjective, then so is

Proof. Construction of δ: Given $m'' \in \ker \varphi''$ map it to $m'' \in M''$ not $\varphi''(m'') = 0$

□

Theorem 1.37 (5-lemma). *Suppose we are given the following commutative diagram of R-modules with exact rows*

$$\begin{tikzcd}
M_1 \arrow[r, "\alpha"] \arrow[d, two heads, "\varphi_1"] & M_2 \arrow[r, "\beta"] \arrow[d, "\varphi_2"] & M_3 \arrow[r, "\gamma"] \arrow[d, "\varphi_3"] & M_4 \arrow[r, "\delta"] \arrow[d, "\varphi_4"] & M_5 \arrow[d, hook, "\varphi_5"] \\
N_1 \arrow[r, "\alpha'"] & N_2 \arrow[r, "\beta'"] & N_3 \arrow[r, "\gamma'"] & N_4 \arrow[r, "\delta'"] & N_5
\end{tikzcd}$$

and suppose that φ_1 is surjective, φ_5 is injective, and φ_2 and φ_4 are isomorphisms, then φ_3 is also an isomorphism.

Proof. (in parts) Exercise.

1. version: diagram chase.
2. version: break up the diagram into 3 diagrams to which the snake lemma applies. □

1.8 Noetherian and Artinian modules and rings

Let R be a ring, M, M', M'', M_i R-modules. A sequence $(M_i)_{i\in\mathbb{N}}$ is said to *become stationary* if $\exists i_0 : \forall i \geq i_0 : M_i = M_{i_0}$.

Definition 1.38. M is called

(a) noetherian: $\iff$ each ascending chain of submodules

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots \subseteq M$$

becomes stationary (ACC for ascending chain condition)

(b) artinian: $\iff$ each descending chain of submodules

$$M \supseteq M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

becomes stationary (DCC for descending chain condition)

and R is called

(c) left noetherian: $\iff$ it is noetherian as a left R-module

(d) left artinian: $\iff$ it is artinian as a left R-module

analogously one defines right artinian/noetherian rings and modules.

Examples 1.39. (a) $\mathbb{Z}$ is noetherian but not artinian.

(b) Finite dimensional K-vector spaces are noetherian and artinian (use the dimension-function)

(c) (Exersize) Let D be a skew field (division algebra), then any D-module is a free D-module. If a D-module is finitely generated, it is artinian and noetherian.*In the present case one has a well-defined dimension for finitely generated D-modules.*

(d) Every field and every skew field is left and right artinian and noetherian. (D^{op} is a skew field if D is a skew field)

Definition 1.40. (a) The center of R is $Z(R) := \{r \in R \mid \forall r' \in R : r \cdot r' = r' \cdot r\}$, $Z(R)$ is a commutative subring (exersize)

(b) Let S be any commutative ring and $\varphi : S \to R$ be a ring homomorphism such that $\varphi(S) \subseteq Z(R)$, then R is called an S-algebra (via φ).

Examples. (a) Every ring is a $\mathbb{Z}$-algebra (in a unique way)

(b) $K[X]$ is a K-algebra.

(c) If R is finite dimensional K-algebra, then R is left and right noetherian and artinian. (exercise) For instance, if M is a finite monoid (or a finite group), then the monoid ring $K[M]$ is left and right artinian and noetherian.

(d) $S = \mathbb{Q}$-subalgebra of 2×2 matrices over $\mathbb{Q}$ generated by $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\implies$ S is commutative, but $MM_{2\times 2}(\mathbb{Q})$ over S is not an S-algebra.

Facts 1.41 (Exercise; compare to linear algebra 2 (or Jantzen-Schwermer Ch. VIII.))**.**

(a) For M the following are equivalent:

  (i) Each subset of submodules of M contains a maximal element.

  (ii) Each submodule of M is finitely generated.

(b) For M the following are equivalent:

  (i) M is artinian

  (ii) Each subset of submodules of M contains a minimal element.

Lemma 1.42. *For submodules $N, P_1, P_2 \subseteq M$ with*

(i) $P_1 \supseteq P_2$

(ii) $P_1 + N = P_2 + N$

(iii) $P_1 \cap N = P_2 \cap N$

it follows that $P_1 = P_2$.

Proof. We need to show that $P_1 \subseteq P_2$. Take $m_1 \in P_1 \underset{(ii)}{\Longrightarrow} \exists m_2 \in P_2, n \in N$ such that $m_1 = m_2 + n$. $\Longrightarrow$ $n = m_1 - m_2 \underset{P_2 \subseteq P_1}{\in} P_1 \cap N \underset{(iii)}{=} P_2 \cap N$. $\Longrightarrow m_1 = m_2 + \underbrace{n}_{\in P_2 \cap N} \in P_2$. ☐

Theorem 1.43. *Let $N \subseteq M$ be a submodule, then*

(a) M is noetherian (artinian) $\Longrightarrow$ N and M/N are noetherian (artinian).

(b) N and M/N are noetherian $\Longleftrightarrow$ M is noetherian (artinian).

Proof. (a) For N: use directly the characterization (ii) from Facts 41. For M/N: use the homomorphism theorem to identify submodules of M/N with those of M containing N and apply again (ii) from 41.

(b) Proof only in the artinian case: assume that N and M/N are artinian. Let $M \supseteq M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$ be a descending chain. Then by hypothesis:

$$M \cap N \supseteq M_0 \cap N \supseteq M_1 \cap N \supseteq \cdots \supseteq M_n \cap N \supseteq \cdots$$

becomes stationary, as does

$$M + N \supseteq M_0 + N \supseteq M_1 + N \supseteq \cdots \supseteq M_n + N \supseteq \cdots$$

$\Longrightarrow \exists i_0 : \forall i \geq i_0 : M_i + N = M_{i_0} + N$ and $M_i \cap N = M_{i_0} \cap N$, we also have $M_i \subseteq M_{i_0}$, so by lemma 42 we have $M_i = M_{i_0} \Longrightarrow (M_i)$ becomes stationary. ☐

Corollary 1.44 (Exercise)**.**

(a) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then M is noetherian (artinian) $\Longleftrightarrow$ M' and M'' are noetherian (artinian).

(b) If I is a finite set and $M := \bigoplus_{i\in I} M_i$, then M is noetherian (artinian) $\iff$ all M_i are noetherian (artinian).

Note: R left-right noetherian (artinian) $\implies$ R^n is also left-right noetherian (artinian) R-module.

Corollary 1.45. *Let R be left noetherian (artinian) and M a finitely generated R-module, then M is noetherian (artinian).*

Proof. $\exists$ an epimorphism $R^n \to M$. Now apply 44(a). ☐

Corollary 1.46. *Let R be left noetherian (artinian) and $I \subseteq R$ a two-sided ideal, then the ring R/I is also left noetherian (artinian).*

Proof. R/I is a ring (because I is a two-sided ideal). R/I is left noetherian (artinian) as an R-module by 44(a) $\implies$ R/I is left noetherian (artinian) as an R/I-module. ☐

Remark. R is noetherian and $S \subseteq R$ a subring $\not\Longrightarrow$ S is noetherian because not every integral domain is noetherian, but its fraction field certainly is.

Proposition 1.47. *Suppose we have $M \cong M \oplus N$ for some R-module $N \neq 0$. Then M is neither noetherian nor artinian.*

proof sketch. 1. $M \neq 0$ because $N \neq 0$ is a direct summand of it.

2. $M \cong M \oplus N \cong (M \oplus N) \oplus N \cong ((M \oplus N) \oplus N) \oplus N \cong \cdots$.

- ∞ ascending chain:

$$0 \oplus N \subsetneq (0 \oplus N) \oplus N \subsetneq ((0 \oplus N) \oplus N) \oplus N \subsetneq \cdots$$

  $\implies$ M is not noetherian.

- ∞ descending chain:

$$M \supsetneq (M \oplus 0) \supsetneq (M \oplus 0) \oplus 0 \supsetneq ((M \oplus 0) \oplus 0) \oplus 0 \supsetneq \cdots$$

  $\implies$ M is not artinian. ☐

Corollary 1.48 (Exercise from 42 and 45)**.** *Suppose $R \neq 0$ is left noetherian (artinian), then for $n_1, n_2 \in \mathbb{N}_0 : R^{n_1} \cong R^{n_2} \implies n_1 = n_2$ (In particular a rank of free finitely generated R-modules is defined.)*

Proof. Assume $\exists n_1, n_2 \in \mathbb{N}_0$ such that $R^{n_1} \cong R^{n_2}$, then $R^{n_1} \cong R^{n_1} \oplus R^{n_2+n_1} \implies R^{n_1}$ not left noetherian (artinian). But 45 implies that R^{n_1} is left noetherian (artinian) because R has these properties. ☐

Theorem 1.49 (Hilbert's basis theorem)**.** *If R is left noetherian, then $R[X]$ is left noetherian (here X commutes with elements of R).*

Proof. TODO ☐

1.9 Simple modules

Let R be a ring, M, M', M'', M_i R-modules.

Definition 1.50. *M is called simple (or irreducibile) if $M \neq 0$ and 0 and M are the only R-submodules of M.*

Examples 1.51. (a) Simple K-vector spaces are the 1-dimensional K-vectorspaces.

(b) Simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$ for p a prime.

(c) A simple $M_{n\times n}(K)$-module is $V_n(K)$ (space of column vectors).

Definition 1.52. M is said to have a composition series $\iff$ $\exists$ finite descending chain of submodules

$$M = M_n \supsetneq M_{n-1} \supsetneq \cdots \supsetneq M_1 \supsetneq M_0 = 0$$

such that $\forall i \in \{1, \ldots, n\}$: the quotients M_i/M_{i-1} are simple. The index n is called the *length* of M and the quotients M_i/M_{i-1} are called the *factors* of M.

Proposition 1.53. *M has a decomposition series $\iff$ M is artinian and noetherian.*

Proof.

- " $\Longleftarrow$ ": Construct an ascending chain of submodules of M as follows:

$$\begin{aligned} &M_0 = 0 \\ &\ \ \cap\!\!| \\ &M_1 = \text{a minimal submodule in } \{M' \leq M \mid 0 \subsetneq M'\} \\ &\ \ \cap\!\!| \\ &M_2 = \text{a minimal submodule in } \{M' \leq M \mid M_1 \subsetneq M'\} \\ &\ \ \cap\!\!| \\ &\ \ \vdots \end{aligned}$$

  because M is artinian, the M_i exist (unless $M_{i+1} = M$), and because M is noetherian, we will find an n such that $M_n = M$. By the Homomorphism theorem: M_i/M_{i-1} is simple $\forall i \in \{1, \ldots, n\}$.

- " $\Longrightarrow$ " : Suppose M has a decomposition series and do induction on the minimal length of the series:
  - $n = 1$: M is simple.
  - Induction step: Let $0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n+1} = M$, then $0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n$ is a decomposition series of M_n of minimal length n. (otherwise there would exist a shorter decomposition series of M which would be a contradiction.) Induction hypothesis implies M_n is artinian and noetherian, but also: $M/M_n = M_{n+1}/M_n$ is simple and hence artinian and noetherian. Consider $0 \to M_n \to M \to M/M_n \to 0 \underset{44}{\Longrightarrow} M$ is artinian and noetherian. □

Theorem 1.54 (Jordan-Hölder)**.** *Suppose M has a decomposition series, then*

(a) Any 2 decomposition series have the same length.

(b) The list of factors of M (coming from any decomposition series) is unique up to permutation (and isomorphism).

Definition. Consider chains of submodules of M (not necessarily decomposition series)

$$\underline{\mathcal{M}} := 0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

$$\underline{\mathcal{N}} := 0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = N$$

(a) Call $\underline{\mathcal{N}}$ a *refinement* of $\underline{\mathcal{M}} :\iff \{N_1, \ldots, N_\ell\} \supseteq \{M_1, \ldots, M_n\}$.

(b) Call $\underline{\mathcal{N}}$ and $\underline{\mathcal{M}}$ *equivalent* if $\ell = n$ and $\exists \sigma \in S_k$ such that $\forall i \in \{1, \ldots, k\}$:

$$N_i \Big/ N_{i-1} \cong M_{\sigma(i)} \Big/ M_{\sigma(i)-1}$$

Lemma 1.55 (Schreier refinement lemma)**.** *[Jacobson Basic Algebra II, 3.6]Any two finite length submodule chains possess equivalent refinements.*

Proof idea: Use $\underline{\mathcal{M}}$ to refine each step $N_{i-1} \subsetneq N_i$.

(1) $N_{i-1} \subseteq N_{i-1}+(M_1\cap N_i) \subseteq N_{i-1}+(M_2\cap N_i) \subseteq \cdots \subseteq N_{i-1}+(M_k\cap N_i) = N_i$.

(2) Similarly $M_{j-1} \subseteq M_{j-1} + (N_1 \cap M_j) \subseteq \cdots \subseteq M_{j-1} + (N_{ell} \cap M_j) = M_j$. Schreier verifies that the j-th subquotient of (1) and the i-th subquotient of (2) are isomorphic. (Butterfly lemma) □

Proof of Jordan-Hölder using Schreier refinement. Suppose $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are decomposition series of M. Schreier refinement gives us refinements $\underline{\mathcal{M}}'$ of $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ of $\underline{\mathcal{N}}'$ such that $\underline{\mathcal{M}}'$ and $\underline{\mathcal{N}}'$ are equivalent, i.e. they have the same length and the same subfactors up to permutation and isomorphism. But $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ have no proper refinements $\implies \underline{\mathcal{M}}' = \underline{\mathcal{M}}$ and $\underline{\mathcal{N}}' = \underline{\mathcal{N}}$. □

Definition 1.56. (a) We say M has finite length if M is artinian and noetherian.

(b) If M has finite length, then its length $\operatorname{len}(M)$ is the length of any decomposition series.

Proposition 1.57. *Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ be a short exact sequence of R-modules, then:*

(a) M has finite length $\iff$ M' and M'' have finite length.

(b) $\operatorname{len}(M) = \operatorname{len}(M') + \operatorname{len}(M'')$.

Proof. (a) See Cor. 44.

(b) Say $0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M'$ and $0 \subsetneq \overline{M_1} \subsetneq \cdots \subsetneq \overline{M_\ell} = M''$ are decomposition series, then check (2nd isomorphism theorem)

$$0 \subsetneq \iota(M_1) \subsetneq \cdots \subsetneq \iota(M_k) = \iota(M') = \pi^{-1}(0) \subsetneq \pi^{-1}(\overline{M}_1) \subsetneq \cdots \subsetneq \pi^{-1}(\overline{M}_\ell) = M$$

is a decomposition series of M of length $\ell + k = \operatorname{len}(M'') + \operatorname{len}(M')$. □

1.10 Indecomposable modules

Definition 1.58. An R-module M is indecomposable $\iff$ $M \neq 0$ and there are no proper submodules $0 \subsetneq M_1, M_2 \subsetneq M$ such that $M_1 \oplus M_2 \xrightarrow{\cong} M, (m_1, m_2) \mapsto m_1 + m_2$.

Remark. (a) If M is simple, M is indecomposable.

(b) If eveery submodule of finite length is a direct sum of simples submodules, then M (indecomposable) of finite length $\implies$ M is simple.

E.g. if R is a field/skew field/$R = \mathbb{Q}[G]$ finite group, then indecomposable $\iff$ simple.

Examples.

- Indecomposable $\mathbb{Z}$-modules are $\mathbb{Z}/p^n\mathbb{Z}$ for p prime and $n \in \mathbb{N}$.
- All non-zero $\mathbb{Z}$-submodules of $\mathbb{Q}$ are indecomposable (e.g. $\mathbb{Z}, \mathbb{Q}$).
- Simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$ for p prime.
- Indecomposable $K[X]$-modules are $K[X]/(f^n)$ for $f \in K[X]$ irreducible and $n \in \mathbb{N}$ and $K[X]$-submodules of $K(X)$. (Any propoer submodule of $K[X]/(f^n)$ is contained in $fK[X]/(f^n)$)

Theorem 1.59. *Suppose M is noetherian or artinian, then $\exists n \in \mathbb{N}, \{M_1, \ldots, M_n\} \subseteq M$ indecomposable submodules such that*

$$M = \bigoplus_{1 \leq i \leq n} M_i.$$

Call this statement $(*)_M$ for M. (A generalization of the structure theorem of finitely generated modules over a principle ideal domain; existence part onca indecomposable R-modules are understood).

Proof. (Assuming M is artinian; other case is an exercise) Assume the statement of the theorem $(*)_M$ does not hold for M. Define $X = \{M' \leq M \mid \neg(*)_{M'}\}$, then $X \neq \emptyset$ because $M \in X$ (by assumption). Let $M' \in X$ be a minimal element under $\subseteq$ (this existst since M is artinian) , then M' decomposable $\implies$ $M' = M_1 \oplus M_2$ for proper submodules $0 \neq M_1, M_2 \subsetneq M'$ $\implies$ M_1 and $M_2 \notin X$ $\implies$ $(*)_{M_1}$ and $(*)_{M_2}$ hold, i.e.

$$M_1 = \bigoplus_{1 \leq i \leq t} N_i, \quad M_2 = \bigoplus_{1 \leq j \leq s} P_j$$

where N_i, P_j are indecomponible.

$$\implies M_1 \oplus M_2 = \bigoplus_{1 \leq i \leq t} N_i \oplus \bigoplus_{1 \leq j \leq s} P_j \implies (*)_{M_1 \oplus M_2}.$$

This is a contradiction to $M' \in X$ so X must be empty. □

Remark. The decomposition in Theorem 59 is not unique

Exercise. Let M be a finitely generated module over a principle ideal domain, then M is indecomposable $\iff$ $M \cong R$ or $\exists n \in \mathbb{N}, \exists$ prime element $p \in R$ such that $M \cong R/p^n$.

Proof. Using structure theorem for modules over PIDs. $\square$

Remark (Exercise)**.** Recall: $e \in S$ a ring is called an idoempotent $\iff e^2 = 2$ (nontrivial $\iff e \neq 0, 1$) M is indecomposable $\iff 0_M, \mathrm{id}_M \in \mathrm{End}_R(M)$ are the only idempotents in $\mathrm{End}_R(M)$ (or else $M = e \cdot M \oplus (1 - e) \cdot M$).

Chapter 2

Category Theory

2.1 Preliminary remarks on set theory

References. Literature for this chapter:

- Sophie Morel - Homological Algebra I.1,
- Daniel Murfet - Foundations for Category Theory,
- Saunders MacLane - Categories for the Working Mathematician I.6.

In this course we always assume a model of set theory that satisfies the Zermelo-Fraenkel axioms + the axiom of choice (ZFC).

Definition (Grothendieck universe; we assume ZFC)**.** A *universe* $\mathcal{U}$ is a set which has the following properties:

(i) $\emptyset, \mathbb{N} \in \mathcal{U}$,

(ii) $X \in \mathcal{U}$ and $y \in X \implies y \in \mathcal{U}$,

(iii) $X \in \mathcal{U} \implies \{X\} \in \mathcal{U}$,

(iv) $X \in \mathcal{U} \implies \mathcal{P}(X) \in \mathcal{U}$,

(v) If $I \in \mathcal{U}$ and $\{X_i\}_{i \in I}$ is a family of members $X_i \in \mathcal{U}$, then $\bigcup_{i \in I} X_i \in \mathcal{U}$.

The existence of a universe is equivalent to the existence of a strongly inaccessible cardinal. (Thomas Jech - Set Theory)

Axiom (Axiom of universes (Grothendieck))**.** *Every set lies in a universe. (We will assume this)*

Definition. If $\mathcal{U}$ is our chosen universe, then:

- A $\mathcal{U}$-set is an element in $\mathcal{U}$.
- A $\mathcal{U}$-class is a subset of $\mathcal{U}$.
- A $\mathcal{U}$-group is a group $(G, e, \cdot)$ with $G \in \mathcal{U}$ and $\cdot : G \times G \to G \in \mathcal{U}$.
- A $\mathcal{U}$-ring is a ring $(R, 0, 1, +, \cdot)$ with $R \in \mathcal{U}$ and also $+, \cdot$
- etc.

Convention. We fix a $\mathcal{U}$ and drop $\mathcal{U}$- in all terms.

2.2 Categories

Definition 2.1. (a) A *directed graph* (a diagram scheme) is a tuple $(O, A, \mathrm{dom}, \mathrm{cod})$ consisting of $\mathcal{U}$-classes O and A and maps $\mathrm{dom}, \mathrm{cod} : A \to O$. We call elements of O *objects* (or vertices) and elements of A *arrows* (or directed edges). For an arrow $f \in A$ call $\mathrm{dom}(f)$ the *source* (or domain) of f and $\mathrm{cod}(f)$ the target (or codomain) of f.

(b) For a graph as in (a) call $A \times_O A := \{(g, f) \in A \times A \mid \mathrm{dom}(g) = \mathrm{cod}(f)\}$ set of composable arrow pairs.

(c) A subgraph of $(O, A, \mathrm{dom}, \mathrm{cod})$ is a graph $(O', A', \mathrm{dom}', \mathrm{cod}')$ such that $O' \subseteq O, A' \subseteq A, \mathrm{dom}' = \mathrm{dom}|_{A'}$ and $\mathrm{cod}' = \mathrm{cod}|_{A'}$.

Example 2.2. Let $O = \{X, Y, Z\}, A = \{f, g, h\}, \mathrm{dom}, \mathrm{cod} : A \to O$ given by the table

	f	g	h
dom	X	Y	X
cod	Y	Z	Z

Illustration:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

Definition 2.3. A *category* $\mathcal{C}$ is a tuple $(\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C}, \mathrm{dom}, \mathrm{cod}, \circ, 1)$ consisting of a graph $(\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C}, \mathrm{dom}, \mathrm{cod})$ and maps

$$1 : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}, X \mapsto 1_X$$

and

$$\circ : \mathrm{Mor}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}, (g, f) \mapsto g \circ f$$

such that:

(i) $\mathrm{dom}(1_X) = \mathrm{cod}(1_X) = X, \forall X \in \mathrm{Ob}\,\mathcal{C}$,

(ii) $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$,

(iii) $\forall f \in \mathrm{Mor}\,\mathcal{C}$ with $X = \mathrm{dom}(f), Y = \mathrm{cod}(f)$

$$f \circ 1_X = 1_Y \circ f = f$$

(iv) $\forall$ arrows $f, g, h \in \mathrm{Mor}\,\mathcal{C}$ such that (h, g) and (g, f) are composable aarrow pairs we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Call elements of $\mathrm{Ob}\,\mathcal{C}$ the objects of $\mathcal{C}$ and elements of $\mathrm{Mor}\,\mathcal{C}$ the morphisms of $\mathcal{C}$.

Notation 2.4. For a category $\mathcal{C}$ as in definition 3

(a) (often) write $X, Y \in \mathcal{C}$ to mean $X, Y \in \mathrm{Ob}\,\mathcal{C}$

(b) For $X, Y \in \mathcal{C}$ write

$$\mathcal{C}(X,Y) := \mathrm{Mor}_{\mathcal{C}}(X,Y) := \{f \in \mathcal{C} \mid \operatorname{dom} f = X, \operatorname{cod} f = Y\}$$

Definition 2.5. (a) Call a category $\mathcal{C}$ locally small if $\mathcal{C}(X,Y)$ is a set $\forall X, Y \in \mathcal{C}$,

(b) Call $\mathcal{C}$ small if $\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}$ are sets.

Remark 2.6 (Extension of example 2 to a category)**.** Let $O = \{X, Y, Z\}, A = \{f, g, h\} \cup \{1_X, 1_Y, 1_Z\}$, cod, dom as before on $\{f, g, h\}$ and uniquely extended to $\{1_X, 1_Y, 1_Z\}$ by axiom (i) and $\circ$ the only possible composition satisfying the axioms

$$\begin{tikzcd}
X \arrow[loop left, "1_X"] \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[loop right, "1_Y"] \arrow[dl, "g"] \\
& Z \arrow[loop below, "1_Z"] &
\end{tikzcd}$$

composable arrow pairs:

$(1_X, 1_X), (f, 1_X), (1_Y, 1_Y), (1_Y, f), (g, 1_Y), (1_Z, 1_Z), (1_Z, g), (1_Z, h), (h, 1_X), (g, h)$

Canonical universal extension would contain a second arrow $X \to Z$ since it would not want to impose the condition $g \circ f = h$.

Definition 2.7. (a) A diagram in $\mathcal{C}$ is a subgraph Γ of $(\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}, \operatorname{dom}, \operatorname{cod})$.

(b) A diagram is commutative if for all objects X, Y of Γ and all chains of arrows from X to Y, their composition is the same (i.e. it only depends on X and Y).

Example (For associativity)**.**

$$\begin{tikzcd}
& W & \\
X \arrow[ur, "h\circ(g\circ f)=(h\circ g)\circ f"] \arrow[rr] \arrow[dr, "f"'] & & Z \arrow[ul, "h"'] \\
& Y \arrow[uu] \arrow[ur, "g"'] &
\end{tikzcd}$$

Examples (Examples of categories)**.**

- Set (category of $\mathcal{U}$-sets): where
  - $\operatorname{Ob}\mathsf{Set}$ = class of all $\mathcal{U}$-sets,
  - $\operatorname{Mor}\mathsf{Set}$ = class of all $\mathcal{U}$-maps between sets,
  - dom, cod are the domain and codomain (range) of a map. (Think of a map as a triple $(X, Y,$ graph map in $X \times Y))$
  - $\circ$ = composition of maps,
  - $1_X = \mathrm{id}_X$ the identity map.

- Grp (category of abelian groups)
- Ring
- CRing
- Top
- ${}_R\mathsf{Mod}$
- Mod_R
- Vec_K
- $\mathsf{Ab} =_{\mathbb{Z}} \mathsf{Mod}$

Examples (Abstract examples).

1. $\operatorname{Ob}\mathcal{C} = \operatorname{Mor}\mathcal{C} = \emptyset$ (empty category)
2. $\operatorname{Ob}\mathcal{C} = \{X\}, \operatorname{Mor}\mathcal{C} = \{1_X\}$ (1 arrow category)
3. Let G be a group, define a category $\underline{G}$ by $\operatorname{Ob}\underline{G} = \{*\}$ (singleton set) and $\operatorname{Mor}\underline{G} = G$, dom, cod the unique map $G \to \{*\}$, $1_* = e$ (unit element of G). $\circ =$ composition in G:
$$\operatorname{Mor}\underline{G} \times \operatorname{Mor}\underline{G} = G \times G \to G = \operatorname{Mor}\underline{G}$$
4. Let $\underline{A} = (M, \leq)$ be a partially ordered set. Define the associated category $\operatorname{Ord}\underline{M}$ with $\operatorname{Ob}\operatorname{Ord}\underline{M} =$ elements of M, morphisms are determined by
$$\operatorname{Ord}\underline{M}(X,Y) = \begin{cases} \text{singleton set}, & X \leq Y, \\ \emptyset, & \text{otherwise.} \end{cases}$$
Unit is clear. composition dictated by $\operatorname{Mor}(\operatorname{Ord}\underline{M})$ (i.e. by $\leq$)

Definition 2.8. For a category $\mathcal{C} = (\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}, \operatorname{dom}, \operatorname{cod}, \circ, 1)$ define the tuple $\mathcal{C}^{\mathrm{op}} = (\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}, \operatorname{cod}, \operatorname{dom}, \circ^{\mathrm{op}}, 1)$ with

$$\begin{aligned} \circ^{\mathrm{op}} : \{(f,g) \in \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C} \mid \operatorname{cod} f = \operatorname{dom} g\} &\to \operatorname{Mor}\mathcal{C} \\ (f,g) &\mapsto f \circ^{\mathrm{op}} g := g \circ f \end{aligned}$$

(change the direction of arrows!)

Proposition 2.9 (Exercise). *$\mathcal{C}^{\mathrm{op}}$ is a category, the* opposite category *to $\mathcal{C}$.*

Example. $(\underline{G})^{\mathrm{op}} = \underline{(G^{\mathrm{op}})}$, $(G^{\mathrm{op}} = (G, e, \circ^{\mathrm{op}})$ with $g \circ^{\mathrm{op}} h = h \circ g)$.

Warning 2.10. $\mathsf{Vec}_K^{\mathrm{op}}(V,W) \neq$ not the set of maps $V \to W$, it is $\{f : W \to V \mid f \text{ is } K\text{-linear}\}$

Definition 2.11. A subcategory of $\mathcal{C} = (\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}, \operatorname{dom}, \operatorname{cod}, \circ, 1)$ is a category $\mathcal{C}' = (\operatorname{Ob}\mathcal{C}', \operatorname{Mor}\mathcal{C}', \operatorname{dom}', \operatorname{cod}', \circ', 1')$ such that $\operatorname{Ob}\mathcal{C}' \subseteq \operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C}' \subseteq \operatorname{Mor}\mathcal{C}, \operatorname{dom}' = \operatorname{dom}|_{\operatorname{Mor}\mathcal{C}'}, \operatorname{cod}' = \operatorname{cod}|_{\operatorname{Mor}\mathcal{C}'}, \circ' = \circ|_{\operatorname{Mor}\mathcal{C}' \times_{\operatorname{Ob}\mathcal{C}} \operatorname{Mor}\mathcal{C}'}, 1' = 1|_{\operatorname{Ob}\mathcal{C}'}$. We write $\mathcal{C}' \subseteq \mathcal{C}$.

Example. $\mathsf{Ab} \subseteq \mathsf{Grp}$ and $\mathsf{CRing} \subseteq \mathsf{Ring}$, etc.

Definition 2.12 (Product of categories)**.** The product of two categories $\mathcal{C}$ and $\mathcal{C}'$ is the six-tuple:

$$(\operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C}', \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C}', \operatorname{dom} \times \operatorname{dom}', \operatorname{cod} \times \operatorname{cod}', \circ, 1)$$

where $\circ$ is componentwise composition $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$ and $1_{X \times X'} = (1_X, 1_{X'})$

Definition 2.13 (Concepts inside categories)**.** Let $X, Y \in \mathcal{C}$, then call $f \in \mathcal{C}(X, Y)$

(a) an *isomorphism* $\iff$ $\exists g \in \mathcal{C}(Y, X)$ such that $g \circ f = 1_X, f \circ g = 1_Y$,

(b) an *endomorphism* $\iff$ $X = Y$,

(c) an *automorphism* $\iff$ it is an isomorphism and an endomorphism

Moreover $\mathcal{C}$ is called a groupoid category $\iff$ all morphisms are isomorphisms.

Example. Let G be a group, then $\underline{G}$ is a groupoid category. $\mathcal{C}$ a groupoid category $\implies$ $\mathcal{C}(X, X)$ is a group (under $\circ, \forall X \in \operatorname{Ob}\mathcal{C}$).

Definition 2.14. Let $X, Y \in \mathcal{C}$, then call $f \in \mathcal{C}(X, Y)$:

(a) a *monomorphism* $\iff$ f is left cancellable $\iff$ $\forall W \in \mathcal{C}$ the map $f_* : \mathcal{C}(W, X) \to \mathcal{C}(W, Y), g \mapsto f \circ g$ is injective.

$$\begin{tikzcd} W \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & X \arrow[r, "f"] & Y \end{tikzcd} : f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

(b) an *epimorphism* $\iff$ f is right cancellable $\iff$ $\forall Z \in \mathcal{C}$ the map $f^* : \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z), h \mapsto h \circ f$ is injective.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z \end{tikzcd} : h_1 \circ f = h_2 \circ f \implies h_1 = h_2.$$

(c) a *split monomorphism* $\iff$ $\exists g \in \mathcal{C}(Y, X)$ such that $g \circ f = 1_X$

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[l, dashed, bend right, "g"'] \end{tikzcd}$$

(d) a *split epimorphism* $\iff$ $\exists h \in \mathcal{C}(Y, X)$ such that $f \circ h = 1_Y$

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[l, dashed, bend right, "h"'] \end{tikzcd}$$

Facts 2.15. (a) f split mono-/epimorphism $\implies$ f mono-/epimorphism.

(b) f (split) mono-/epimorphism in $\mathcal{C}$ $\implies$ f (split) mono-/epimorphism in $\mathcal{C}^{\text{op}}$.

(c) (Exercise) For $f \in \mathcal{C}(X, Y), (X, Y) \in \mathcal{C}$ the following are equivalent:

  (i) f is an isomorphism

(ii) $\forall W \in \mathcal{C} : f_* : \mathcal{C}(W, X) \to \mathcal{C}(W, Y), g \mapsto f \circ g$ is bijective.

(iii) $\forall Z \in \mathcal{C} : f^* : \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z), h \mapsto h \circ f$ is bijective.

(b') f is an isomorphism in $\mathcal{C} \iff f$ is an isomorphism in $\mathcal{C}^{\mathrm{op}}$.

Proof. (b), (b') are exercises. (c) "serious" exercise.

(a) For an epimorphism (check right cancellability) consider

$$X \xrightarrow{f} Y \overset{h_1}{\underset{h_2}{\rightrightarrows}} Z : h_1 \circ f \overset{(1)}{=} h_2 \circ f$$

By f a split epimorphism, we have $h : Y \to X$ such that $f \circ h = 1_Y$ (2). Apply $- \circ h$ to (1):

$$\begin{tikzcd}
 & (h_1 \circ f) \circ h \arrow[d, phantom, "\parallel"] & = & (h_2 \circ f) \circ h \arrow[d, phantom, "\parallel"] & \\
h_1 = h_1 \circ 1_Y = & h_1 \circ (f \circ h) & = & h_2 \circ (f \circ h) & = h_2 \circ 1_Y = h_2
\end{tikzcd}$$

□

Examples. In Set, Grp, Ring the monomorphisms are the injective maps and in Set, Grp the epimorphisms are the surjective maps. But $\mathbb{Z} \to \mathbb{Q}$ (inclusion) is an epimorphism in Ring. If $K \subseteq E$ is purely inseperable, then it's an epimorphism in the category of fields.

Definition 2.16. (a) $X \in \mathcal{C}$ is called an *initial object* $\iff \forall Y \in \mathcal{C} : \#\mathcal{C}(X, Y) = 1$

(b) $X \in \mathcal{C}$ is called a *terminal object* $\iff \forall Z \in \mathcal{C} : \#\mathcal{C}(Z, X) = 1$

(c) $X \in \mathcal{C}$ is called a *null object* $\iff X$ is initial and terminal.

Example.

- $\emptyset$ is initial in Set, Top,
- $\{*\}$ is terminal in Set, Top
- $0 = \{0\}$ is a null object in ${}_R\mathsf{Mod}, \mathsf{Ab}, \mathsf{Vec}_K$

2.3 Functors

Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories.

Definition 2.17. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ ($F : \mathcal{C} \to \mathcal{D}$) is a pair of maps

$$F : \operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{D}, X \mapsto F(X),$$
$$F : \operatorname{Mor} \mathcal{C} \to \operatorname{Mor} \mathcal{D}, f \mapsto F(f).$$

that "preserve sources, targets, units and composition", i.e.

(i) $\forall f \in \operatorname{Mor} \mathcal{C} : \operatorname{dom}(Ff) = F(\operatorname{dom} f)$ and $\operatorname{cod}(Ff) = F(\operatorname{cod} f)$

(ii) $\forall X \in \operatorname{Ob} \mathcal{C} : F(1_X) = 1_{FX}$

(iii) $\forall$ composable pairs (g, f) in $\mathrm{Mor}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Mor}\,\mathcal{C} : F(g \circ f) = F(g) \circ F(f)$.
(other notation $F(X \xrightarrow{f} Y) = FX \xrightarrow{Ff} FY$)

Examples. (a) Powerset:

$$\begin{aligned} \mathcal{P} : \mathsf{Set} &\to \mathsf{Set} \\ X &\mapsto \mathcal{P}(X) \\ f : X \to Y &\mapsto \mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y, \\ &\quad (U \subseteq X) \mapsto (f(U) \subseteq Y) \end{aligned}$$

(b) Forgetful functor (it forgets structure)

$$\begin{gathered} V : \mathsf{Grp} \to \mathsf{Set}, (G, e, \circ) \mapsto G \\ V : \mathsf{Top} \to \mathsf{Set}, (X, \mathcal{T}) \mapsto X \\ V : {}_R\mathsf{Mod} \to \mathsf{Ab}, (M, 0, +, \cdot) \mapsto (M, 0, +) \end{gathered}$$

(c) ${}_R\mathsf{Mod} \to \mathsf{Mod}_{R^{\mathrm{op}}}$, left R-modules $\mapsto$ right R-modules.

Remark. Functors in definition 17 are also called covariant functors.

Definition 2.18. A *contravariant* functor from $\mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, i.e.

$$F(X \xrightarrow{f} Y) = (FX \xleftarrow{Ff} FY)$$

$= (Y \xrightarrow{f} X)$ in $\mathcal{C}^{\mathrm{op}}$ and

$$F((Y \xrightarrow{g} Z) \circ (X \xrightarrow{f} Y)) = F(X \xrightarrow{f} Y) \circ F(Y \xrightarrow{g} Z)$$

Visually:

$$\begin{tikzcd}
X \arrow[d, "f"] & X & FX \\
Y \arrow[d, "g"] & Y \arrow[u, "f"] & FY \arrow[u, "Ff"] \\
Z & Z \arrow[u, "g"] & FZ \arrow[u, "Fg"]
\end{tikzcd}$$

in $\mathcal{C}$ in $\mathcal{C}^{\mathrm{op}}$

Remark (Exercise)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be any functor $\implies$ F maps isomorphisms to isomorphisms.

Examples (Contravariant functors)**.** (a) Passage to the dual vector space

$$\begin{tikzcd}
D : \mathsf{Vec}_K^{\mathrm{op}} \arrow[r] & \mathsf{Vec}_K \\
V \arrow[r, mapsto] \arrow[d, "f"'] & V^* = \mathrm{Hom}_K(V, K) \\
W \arrow[r, mapsto] & W^* \arrow[u, "Df=f^*"']
\end{tikzcd}$$

linear algebar: $(f \circ g)^* = g^* \circ f^*$ for (f, g) a composable pair.

(b) Let Poset be the category of partially ordered sets, then we have a contravariant functor

$$\begin{tikzcd}
\mathcal{O} : \mathsf{Top}^{\mathrm{op}} \arrow[r] & \mathsf{Poset} & & \\
(X,\mathcal{T}) \arrow[r, mapsto] \arrow[d, "f"'] & (\mathcal{T},\subseteq) & \ni & f^{-1}(V) \underset{\text{open}}{\subseteq} X \\
(Y,\mathcal{T}') \arrow[r, mapsto] & (\mathcal{T},\subseteq) \arrow[u, "\mathcal{O}(f)"'] & \ni & V \underset{\text{open}}{\subseteq} Y \arrow[u, mapsto]
\end{tikzcd}$$

(c) The contravariant powerset functor:

$$\begin{tikzcd}
\mathcal{P}^* : \mathsf{Set} \arrow[r] & \mathsf{Set} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{P}^*(X) = \mathcal{P}(X) \\
Y \arrow[r, mapsto] & \mathcal{P}^*(Y) = \mathcal{P}(Y) \arrow[u, "\mathcal{P}^* f"']
\end{tikzcd}$$

Definition 2.19. Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories, a functor $\mathcal{C} \times \mathcal{C}' \to \mathcal{D}$ is called a *bifunctor*.

Example 2.20 (Important example)**.** Let $\mathcal{C}$ be any category

$$\begin{tikzcd}
\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r] & \mathsf{Set} & & \\
(X,Y) \arrow[r, mapsto] \arrow[d, shift left, "h"] & \mathcal{C}(X,Y) \arrow[d] & \ni & g \arrow[d, mapsto] \\
(W,Z) \arrow[r, mapsto] \arrow[u, shift left, "f"] & \mathcal{C}(W,Z) & \ni & h \circ g \circ f
\end{tikzcd}$$

If we fix a first argument X, we get

$$h_X := \mathcal{C}(X,-) \to \mathsf{Set}$$

If we fix a second argument Y, we get

$$h^Y := \mathcal{C}(-,Y) \to \mathsf{Set}$$

Soon: we will also have another important bifunctor

$$- \otimes - : \mathsf{Mod}_R \times {}_R\mathsf{Mod} \to \mathsf{Ab}$$

Definition 2.21. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, F is called

(a) *faithful* $\iff \forall X, Y \in \mathcal{C} : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$is injective.

(b) *full* $\iff \forall X, Y \in \mathcal{C} : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$is surjective.

(c) *fully faithful* F is full and faithful.

(d) *essentially surjective* $\iff \forall Y \in \mathcal{D} \exists X \in \mathcal{C} \exists$ isomorphism $FX \xrightarrow{\cong} Y$.

(e) *conservative* $\iff \forall f \in \operatorname{Mor}\mathcal{C} : f$ is an isomorphism $\iff Ff\imath \operatorname{Mor}\mathcal{D}$ is an isomorphism. ($\implies$ always holds)

(f) *an isomorphism* $\iff$ $\exists G : \mathcal{D} \to \mathcal{C}$ functor such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$.

Examples. (a) Forgetful functors are "often" faithful but not full

$$V : \mathsf{Grp} \to \mathsf{Set}, \mathsf{Ab} \to \mathsf{Set}, {}_R\mathsf{Mod} \to \mathsf{Set}, \mathsf{Ring} \to \mathsf{Set}$$

are conservative.

(b) The forgetful functor $V : \mathsf{Top} \to \mathsf{Set}$ is not conservative and not full but essentially surjective.

(c) The inclusion of a subcategory $\mathcal{C}'$ into its ambient category $\mathcal{C}$ is always faithful. Call $\mathcal{C}'$ a *full subcategory* $\iff$ $\forall X, Y \in \mathcal{C}' : \mathcal{C}'(X, Y) = \mathcal{C}(X, Y)$ ($\iff$ $i : \mathcal{C}' \to \mathcal{C}$ is full)

2.4 Natural transforations

They are morphisms between functors.

Definition 2.22. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors.

(a) A morphism from F to G (or a *natural transforation*) is a family $u = (u_X : FX \to GX)_{X \in \mathrm{Ob}\, \mathcal{C}}$ of morphisms in $\mathcal{D}$, such that forall $f : X \to Y$ in $\mathcal{C}$ we have the commutative diagram:

$$\begin{tikzcd}
FX \arrow[r, "u_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "u_Y"'] & GY
\end{tikzcd}$$

Notation:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=D, "u"]
\end{tikzcd}$$

(b) Composition: Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors and $u : F \Rightarrow G, v : G \Rightarrow H$ natural transformations. The composition $v \circ u : F \Rightarrow H$ is the natural transformation (check) $(v_X \circ u_X : FX \xrightarrow{u_X} GX \xrightarrow{v_X} HX)_{X \in \mathrm{Ob}\, \mathcal{C}}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"'{name=M}] \arrow[r, bend right=60, "H"'{name=D}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=M, "u"]
\arrow[Rightarrow, from=M, to=D, "v"]
\end{tikzcd}$$

(c) The category $\mathcal{D}^{\mathcal{C}}$ (or $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$) whose objects are the functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are the natural transformations $(F : \mathcal{C} \to \mathcal{D}) \Rightarrow (G : \mathcal{C} \to \mathcal{D})$. The composition is from (b), and the unit natural transforation is

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F"'{name=D}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=D, "1_F"]
\end{tikzcd} , 1_F = (FX \xrightarrow{1_{FX}} FX)_{X \in \mathrm{Ob}\, \mathcal{C}}$$

(dom, cod are clear). Remark: One can also define 2-categories (and the category of categories is an example of such, objects: $\mathcal{C}, \mathcal{D}, ...$ and morphisms are $F : \mathcal{C} \rightrightarrows \mathcal{D}$ 2-morphisms = natural transformations)

(d) A natural transformation $u : F \Rightarrow G$ is called a *natural isomorphism* $\iff \forall X \in \operatorname{Ob}\mathcal{C} : u_X : FX \to GX$ is an isomorphism $\underset{\text{Exerc.}}{\iff} \exists$ natural transformation $v : G \Rightarrow F : v \circ u = \mathrm{id}_F, u \circ v = \mathrm{id}_G$.

Example (Famous linear algebra example of a natural transformation)**.** Let $(\cdot)^{**} : \mathsf{Vec}_K \to \mathsf{Vec}_K, V \mapsto V^{**}, f \mapsto f^{**}$ be the (covariant) bidual functor. $\mathrm{id} : \mathsf{Vec}_K \to \mathsf{Vec}_K$ denotes the identity, we set $u_V : V \to V^{**}, v \mapsto (b_v : V^* \to K, \xi \mapsto \xi(v))$ then $u = (u_V)_{V \in \mathsf{Vec}_K}$ is a natural transformation $u : \mathrm{id} \Rightarrow (\cdot)^{**}$ and restricted to the full subcategory $\mathsf{Vec}_K^{\text{f.d.}} \subseteq \mathsf{Vec}_K$ on finite dimensional K-vector spaces, it gives a natural isomorphism $u : \mathrm{id} \Rightarrow (\cdot)^{**}$

$$\begin{tikzcd}
\mathsf{Vec}_K^{\text{f.d.}} \arrow[r, bend left=50, "\mathrm{id}"] \arrow[r, bend right=50, "(\cdot)^{**}"'] \arrow[r, phantom, "\Downarrow u"] & \mathsf{Vec}_K^{\text{f.d.}}
\end{tikzcd}$$

Definition 2.23 (important concept)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is called an equivalence of categories $\iff \exists$ functor $G : \mathcal{D} \to \mathcal{C}$ such that one has natural transforations $\mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\mathrm{id}_{\mathcal{D}} \Rightarrow F \circ G$. Call $\mathcal{C}$ and $\mathcal{D}$ *equivalent categories* $\iff \exists$ equivalence of categories $F : \mathcal{C} \to \mathcal{D}$.

Remark. The notion of "equivalence of categories" is far more important than the notion of isomorphism of categories.

Example (linear algebra)**.** Let $\mathsf{Vec}_K^{\text{std.}}$ be the full subcategory of the $\mathsf{Vec}_K^{\text{f.d.}}$ on the object set $\{K^n \mid n \in \mathbb{N}_0\}$. Then the inclusion $\iota : \mathsf{Vec}_K^{\text{std.}} \to \mathsf{Vec}_K^{\text{f.d.}}$ is an equivalence of categories. For $G : \mathsf{Vec}_K^{\text{std}} \to \mathsf{Vec}_K^{\text{std}}$ take $V \mapsto K^{\dim_K V}$, choose a basis $\underline{B}_V$ for any $V \in \mathsf{Vec}_K^{\text{f.d.}}$ then we get an isomorphism $K^{\dim_K V} \xrightarrow{\alpha_V} V$. Define:

$$\begin{tikzcd}
V \arrow[r, mapsto] \arrow[d, "f"'] & K^{\dim_K V} \arrow[d, "\alpha_W^{-1} \circ f \circ \alpha_V"] \\
W \arrow[r, mapsto] & K^{\dim_K W}
\end{tikzcd}$$

Find natural isomorphism $G \circ \iota \Leftarrow \mathrm{id} \Rightarrow \iota \circ G$.

Remark. One also calls $\mathsf{Vec}_K^{\text{std.}}$ a *skeleton* of $\mathsf{Vec}_K^{\text{f.d.}}$

Theorem 2.24. *For a functor $F : \mathcal{C} \to \mathcal{C}'$ the following are equivalent:*

(i) F is an equivalence of categories.

(ii) F is fully faithful and essentially surjective.

Proof.
- (i) $\Longrightarrow$ (ii): Exercise.
- (ii) $\Longrightarrow$ (i): Standard textbook. □

Definition 2.25. The *essential image* of a functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathcal{D}$ is the full subcategory $\mathcal{D}'$ of $\mathcal{D}$ on objects isomorphic to FX for some $X \in \operatorname{Ob} \mathcal{C}$.

Corollary 2.26 (of 24 and the definition)**.** *Suppose $F : \mathcal{C} \to \mathcal{D}$ is fully faithful. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the essential image of F, then $F : \mathcal{C} \to \mathcal{D}'$ is an equivalence of categories.*

2.5 The Yoneda lemma and presheaves

Let $\mathcal{C}, \mathcal{C}', \mathcal{D}$ be categories.

Definition 2.27. (a) A $\mathcal{D}$-valued presheaf on $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$$

(b) The category of $\mathcal{D}$-valued presheaves on $\mathcal{C}$ is $\mathrm{PSh}(\mathcal{C}, \mathcal{D}) = \mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$

(c) If $\mathcal{D} = \mathsf{Set}$, then we omit it from the notation, so $\mathrm{PSh}(\mathcal{C}) = \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$
Note that if $\mathcal{D}$ is a small category, then $\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$ is a category.

Remark (On the terminoloy)**.** (Pre-)sheaves come from topology/geometry. Example: Let $(X, \mathcal{T})$ be a topological space (e.g. $\mathbb{C}$ with the metric topology), For $U \subseteq X$ define $O_X(U) := \{f : U \to \mathbb{C} \text{ continuous}\}$ or $(O_{\mathbb{C}}(U) := \{f : U \to \mathbb{C} \text{ holomorphic }\})$. Check:

$$\begin{tikzcd}
O_X : \mathsf{ord}(T, \subseteq) \arrow[r] & \mathsf{Set} \\
U \arrow[r, mapsto] \arrow[d, phantom, "\cup|" left] & O_X(U) \arrow[d, "\text{restriction}"] \\
V \arrow[r, mapsto] \arrow[u] & O_X(V)
\end{tikzcd}$$

this is a presheaf.

Definition 2.28. The Yoneda embedding is the functor $h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}), X \mapsto h_X := \mathcal{C}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$

$$\begin{tikzcd}
h : \mathcal{C} \arrow[r] & \mathsf{PSh}(\mathcal{C}) & \\
X \arrow[r, mapsto] \arrow[d, "f"'] & h_X \arrow[d, Rightarrow, "hf"] & \mathrm{Hom}(-, X) \arrow[d, "f \circ -"] \\
Y \arrow[r, mapsto] & h_Y & \mathrm{Hom}(-, Y)
\end{tikzcd}$$

Lemma 2.29. *For $X \in \mathcal{C}, F \in \mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_X, \mathcal{F}) \arrow[r, "\Phi"] & \mathcal{F}X \\
u := (u_Y : h_X Y \to \mathcal{F}Y)_{Y \in \mathrm{Ob}\mathcal{C}} \arrow[r, mapsto] & u_X 1_X
\end{tikzcd}$$

is a bijection. ($\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_x, \mathcal{F})$ *is a set.)*

Proof. Reconstruct a natural transformation u^α from $\alpha \in \mathcal{F}X$, first consider what $u \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_x, \mathcal{F})$ gives us

$$\begin{tikzcd}
X & h_X X = \mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[r, "u_X"] \arrow[d, "-\circ g"] & \mathcal{F}X \arrow[d, "\mathcal{F}g"] & 1_X \arrow[r, mapsto] \arrow[d, mapsto] & u_X(1_X) = \alpha \arrow[d, mapsto] \\
Y \arrow[u, "g"] & h_X Y = \mathrm{Hom}_{\mathcal{C}}(Y,X) \arrow[r, "u_Y"] & \mathcal{F}Y & g \arrow[r, mapsto] & u_Y(g) = \mathcal{F}g(\alpha)
\end{tikzcd}$$

Define $\psi : \mathcal{F}(X) \to \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_X, \mathcal{F}), \alpha \mapsto (u^\alpha_Y)_{Y \in \mathrm{Ob}\,\mathcal{C}}$ by setting $u^\alpha_Y : h_X(Y) \to \mathcal{F}Y, g \mapsto \mathcal{F}g(\alpha)$. Check u^α_Y is a natural transforation: For any $f : Z \to Y$ in $\mathcal{C}$ we get TODO ☐

Corollary 2.30. *The functor $h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is fully faithful, i.e. $\mathcal{C}(X,Y) \leftrightarrow \mathrm{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y)$.*

Proof. We need to show $\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$ the map $\mathcal{C}(X,Y) \to \mathrm{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y), f \mapsto h(f) = f \circ -$ is bijective. Observe: Yoneda $\Phi : \mathrm{Mor}_{\mathsf{PSh}(\mathcal{C})}(h_X, h_Y) \to h_y(X) = \mathcal{C}(X,Y), u \mapsto u_X(1_X)$ is a bijection, so it suffices to show $\Phi \circ h$ is a bijection. For this: $\Phi \circ h(f : X \to Y) = f \circ 1_X = f \implies \Phi \circ h = \mathrm{id}$. ☐

Definition 2.31. (a) Call $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$ *representable* $\iff \exists X \in \mathcal{C}$ such that $h_X \cong \mathcal{F}$.

(b) A presentation of a (representable) $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$ is a pair (X, α) with $X \in \mathrm{Ob}\,\mathcal{C}, \alpha \in \mathcal{F}X$ such that $\Psi(\alpha) : h_X \Rightarrow \mathcal{F}$ from the proof of lemma 29 is a natural isomorphism.

Proposition 2.32. *Suppose (X, α) and (Y, β) are presentations of $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$, then $\exists!$ isomorphism $f : X \to Y$ such that $\mathcal{F}(f)(\beta) = \alpha$*

Proof. Exercise. ☐

2.6 Conatravariant Yoneda

Proposition 2.33. *The functor*

$$\begin{tikzcd}
h^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathsf{Set}) \\
X \arrow[r, mapsto] \arrow[d, "f"] & \mathcal{C}(X, -) \\
Y \arrow[r, mapsto] & \mathcal{C}(Y, -) \arrow[u, "h^{\mathrm{op}}(f)"]
\end{tikzcd}$$

is fully faithful and for $X \in \mathrm{Ob}\,\mathcal{C}$ and $\mathcal{F} : \mathcal{C} \to \mathsf{Set}$ a functor, the map $\Phi' : \mathrm{Mor}_{\mathsf{Set}^{\mathcal{C}}}(h^{\mathrm{op}}_X, \mathcal{F}) \to \mathcal{F}(X), u \mapsto u_X(1_X)$ is bijective.

Proof. (Exercise) Apply Yoneda to $\mathcal{C}^{\mathrm{op}}$. ☐

Definition 2.34. (a) A covariant functor $F : \mathcal{C} \to \mathsf{Set}$ is *corepresentable* $\iff F \cong h^{\mathrm{op}}_X$ for some $X \in \mathcal{C}$.

(b) A presentation of a (corepresentable) functor F is a pair (X, α) such that $(\Phi')^{-1}(\alpha)$ is an isomorphism $h^{\mathrm{op}}_X \to \mathcal{F}$

Proposition 2.35 (analog of 32)**.** *If* (X,α) *and* (Y,β) *are 2 presentations of* $F : \mathcal{C} \to \mathsf{Set}$*, then* $\exists!$ *isomorphism* $f : X \to Y$ *such that* $F(f)(\alpha) = \beta$*.*

Remark. We mostly drop co- in corepresentable because the functor dictates if it is representable or corepresentable (if F co- or contravariant)

Remark. For $f : X \to Y$ we have

$$\begin{aligned} f \circ - : \mathcal{C}(Z,X) \to \mathcal{C}(X,Y) \text{ bij.} &\iff h(f) \cong \iff f \cong \iff h^{\mathrm{op}}(f) \cong \\ &\iff - \circ f : \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z) \text{ bij.} \end{aligned}$$

because h is fully faithful.

2.7 Universal pairs

Definition 2.36. (a) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $B \in \operatorname{Ob}\mathcal{D}$. A pair (U,β) with $U \in \operatorname{Ob}\mathcal{C}$ and $\beta : B \to F(U)$ (in $\mathcal{D}$) is *(co-)universal* for $(F,\beta) :\iff (U,\beta)$ (co-)represents $\mathcal{D}(B, F(-)) = h_B^{\mathrm{op}} \circ F : \mathcal{C} \to \mathsf{Set}$.

(b) Let $G : \mathcal{D} \to \mathcal{C}$ be a functor and $A \in \operatorname{Ob}\mathcal{C}$. A pair (V,α) with $V \in \operatorname{Ob}\mathcal{D}$ and $\alpha : G(V) \to A$ (in $\mathcal{C}$) is universal for $(G,A) \iff (V,\alpha)$ represents $(\mathcal{C}(G(-),A)) = h_A \circ G : \mathcal{D}^{\mathrm{op}} \to \mathsf{Set}$.
TODO: interpretation

Examples 2.37. TODO

2.8 Limits and colimits

Let $\mathcal{C}$ be a category.

Definition 2.38. A diagram in $\mathcal{C}$ is a functor $F : J \to \mathcal{C}$ for J a small category (call J the *index category* of the diagram.)

Remark (Relation to previous notions of diagrams)**.** Let $V : \mathsf{Cat} \to \mathsf{Diag}$, [MacLane II.7]: $\exists$ functor TODO

Definition 2.39. A diagram $F : J \to \mathcal{C}$ *commutes* $\iff \forall i,j \in J$:

$$\underbrace{F(J(i,j))}_{\text{is a singleton}} \subseteq \mathcal{C}(Fi, Fj)$$

(naive diagram commutes $\iff F\varphi$ commutes.)

Definition 2.40. Let $F : J \to \mathcal{C}$ denote a diagram in $\mathcal{C}$.

(a) The *constant functor* from J to $\mathcal{C}$ for $X \in \operatorname{Ob}\mathcal{C}$ is $\Delta X : J \to \mathcal{C}$ with $\Delta X(i) = X, \forall i \in J$ and $\Delta X(h) = 1_X, \forall h \in \operatorname{Mor} J$.

(b) The *diagonal* $\Delta : \mathcal{C} \to \mathcal{C}^J = \operatorname{Fun}(J,\mathcal{C})$ with $\Delta(X) := \Delta X$ from (a) and $\Delta(f) :=$ the natural transformation $\Delta X \Rightarrow \Delta Y$ given for any $i \in J$ by $\Delta X(i) = X \xrightarrow{f} \Delta Y(i) = Y$.

(c) A *cone* to $F : J \to \mathcal{C}$ (any fixed F) with apex $X \in \mathcal{C}$ is a natural transforation $\Delta X \Rightarrow F$. A *cocone* from F with vertex $X \in \operatorname{Ob}\mathcal{C}$ is a natural transforation $F \Rightarrow \Delta X$.

(d) Cones and cocones give rise to the following functors:

- $\text{Cone}(-, F) : \mathcal{C}^{\text{op}} \to \mathsf{Set}$ defined by $\text{Cone}(X, F) =$ set of cones $\Delta X \Rightarrow F$, $\text{Cone}(f : X \to Y, F)$ maps $\text{Cone}(Y, F) \to \text{Cone}(X, F), (\Delta Y \Rightarrow F) \mapsto (\Delta X \Rightarrow \Delta Y \Rightarrow F)$.
- Similarly $\text{Cocone}(F, -) : \mathcal{C} \to \mathsf{Set}$ is the functor defined by $\text{Cocone}(F, X) =$ set of cocones $F \Rightarrow \Delta X$ etc.

Observe:

$$\text{Cone}(-, F) = \mathcal{C}^J(\Delta(-), F)$$
$$\text{Cocone}(F, -) = \mathcal{C}^J(F, \Delta(-))$$

Visualization: a natural transforation $u : \Delta X \Rightarrow F$ (where $X \in \mathcal{C}, F : J \to \mathcal{C}$) is for any $i \in J$ a morphism $X = \Delta X(i) \xrightarrow{u_i} F(i)$ such that $\forall h : i \to j$ in J the following diagram commutes:

$$\begin{tikzcd}
\Delta X(i) = X \arrow[r, "1_X"] \arrow[d, "u_i"] \arrow[dr, "u_j"] & X = \Delta X(j) \arrow[d, "u_j"] \\
Fi \arrow[r, "Fh"'] & Fj
\end{tikzcd}$$

for instance if TODO

Remark 2.41. (a) The cones to F form a full subcategory F-cones $\subseteq \mathcal{C}^J/F$ on objects $\Delta X \Rightarrow F$ $(X \in \mathcal{C})$.

(b) Similarly cocones from F form a full subcategory F-cocones $\subseteq F/\mathcal{C}^J$ on objects $F \Rightarrow \Delta X$.

Definition 2.42. (a) If $\text{Cone}(-, F) : \mathcal{C}^{\text{op}} \to \mathsf{Set}$ is representable, the representing object is called a *limit* over F. Notation: $\lim F$ or $\lim_J F$ for the representing universal object.

(b) If $\text{Cocone}(F, -) : \mathcal{C} \to \mathsf{Set}$ is representable, the representing object is called a *colimit* over F. Notation: $\operatorname{colim} F$ or $\operatorname{colim}_J F$

More explicitly: $\lim_J F$ is an object $L \in \mathcal{C}$ together with a (universal) ocne $\Delta L \Rightarrow F$ such that $\forall$ cones $\varphi : \Delta X \Rightarrow F$ in $\mathcal{C}^J \exists!$ morphism $\psi : X \to L$ such that the diagram commutes

$$\begin{tikzcd}
 & \Delta X \arrow[dl, dashed, "\psi"'] \arrow[d, Rightarrow, "\varphi"] \\
\Delta L \arrow[r, Rightarrow] & X
\end{tikzcd}$$

Yoneda implies that $\lim F$ (if it exists) is unique up to unique isomorphism (similarly for $\operatorname{colim} F$).

Exercise 2.43. (a) $\lim F$ exists $\iff$ category of F-cones has a terminal object.

(b) $\operatorname{colim} F$ exists $\iff$ category of F-cocones has an initial object.

Proposition 2.44 (Exercise)**.** *Let $\Delta : \mathcal{C} \to \mathcal{C}^J$ be the diagonal from above and F any diagram in $\mathcal{C}^J$. Then*

(a) $\lim F$ *is a universal object for the pair* (Δ, F) *i.e.* $\mathcal{C}^J(\Delta(-), F) \leftrightarrow \mathcal{C}(-, \lim_J F)$.

(b) $\operatorname{colim} F$ *is a couniveral object for the pair* (Δ, F) *i.e.* $\mathcal{C}^J(F, \Delta(-)) = \mathcal{C}(\operatorname{colim}_J F, -)$.

Examples 2.45. (a) $J :=$ the discrete category on the set $\{0, 1\}$, i.e. (Ob $= \{0, 1\}, \operatorname{Mor} J = \{1_0, 1_1\}, ...$). A functor $F : J \to \mathcal{C}$ is given by the datum of a pair $(F_0 = F(0), F_1 = F(1))$ of objects of $\mathcal{C}$,

cones:
$$\begin{tikzcd}
& X \arrow[dl, "u_0"'] \arrow[dr, "u_1"] & \\
F_0 & & F_1
\end{tikzcd}$$
cocones:
$$\begin{tikzcd}
F_0 & & F_1 \\
& X \arrow[ul, "v_0"] \arrow[ur, "v_0"'] &
\end{tikzcd}$$

any pair of morphisms $(u_0 : X \to F_0, u_1 : X \to F_1)$ resp. $(v_0 : F_0 \to X, v_1 : F_1 \to X)$ defines a cone $\lim F$ resp. $\operatorname{colim} F$ and satisfies

$$\mathcal{C}(Y, \lim F) = \mathcal{C}(Y, F_0) \times \mathcal{C}(Y, F_1)$$

resp.

$$\mathcal{C}(\operatorname{colim}(F, Z)) = \mathcal{C}(F_0, Z) \times \mathcal{C}(F_1, Z)$$

If $\lim F$ exists write formally $\lim F = F_0 \; \Pi \; F_1$ (product), and if $\operatorname{colim} F$ exists write $F_0 \amalg F_1$ (coproduct). Concretely

- $\mathcal{C} = \mathsf{Set}, F_0 \times F_1 = F_0 \sqcup F_1$
- $\mathcal{C} = {}_R\mathsf{Mod}, F_0 \times F_1 = F_0 \oplus F_1$
- $\mathcal{C} = \mathsf{Grp}, F_0 \times F_1 = F_0 * F_1$ (free product)

(b) We can generalize to arbitrary discrete (small) categories with unterlying set I.Names for universal objects

$$\prod_{i \in I} F_i, \qquad \coprod_{i \in I} F_i$$

defining property:

$$\mathcal{C}(\prod_{i \in I} F_i, Y) \overset{!}{=} \prod_{i \in I} \mathcal{C}(F_i, Y) \text{ and } \mathcal{C}(Z, \coprod_{i \in I} F_i) \overset{!}{=} \prod_{i \in I} \mathcal{C}(Z, F_i)$$

- $\mathsf{Set}, \prod_{i \in I} F_i = \bigsqcup_{i \in I} F_i$ (disjoint union)
- ${}_R\mathsf{Mod}, \prod_{i \in I} F_i = \bigoplus_{i \in I} F_i$ (direct sum)
- $\mathsf{Grp}, \prod_{i \in I} F_i = \mathop{\Large *}_{i \in I} F_i$ (free product)

(c) $J =$ category on 2 objects $0, 1$ with 2 morphisms $0 \rightrightarrows 1$ (besides $1_0, 1_1$). $F : J \to \mathcal{C}$ is determined by $F_0 \overset{a}{\underset{b}{\rightrightarrows}} F_1$

cones:
$$\begin{tikzcd}
& X \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\
F_0 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] & & F_1
\end{tikzcd}$$
cocones:
$$\begin{tikzcd}
F_0 \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] & & F_1 \\
& X \arrow[ul, "f_0"] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

- $\mathcal{C}(X, \lim F) = \{f_0 \in \mathcal{C}(X, F_0) \mid a \circ f_0 = b \circ f_0 = f_1\}$. $\lim F$ if it exists is called the *equalizer* $\mathrm{eq}(F_0 \overset{a}{\underset{b}{\rightrightarrows}} F_1)$.
- $\mathcal{C}(\operatorname{colim} F_1, X) = \{f_1 \in \mathcal{C}(F_1, X) \mid f_1 \circ a = f_1 \circ b\}$. $\operatorname{colim} F$ if it exists is called the *coequalizer* $\mathrm{coeq}(F_0 \overset{a}{\underset{b}{\rightrightarrows}} F_1)$. TODO

(d) Pullback and pushout:

Definition 2.46. $\mathcal{C}$ is *(co-)complete* $\iff$ $\mathcal{C}$ contains all (co)-limits.

Theorem 2.47. *(a) If $\mathcal{C}$ contains all products and equalizers, then $\mathcal{C}$ is complete.*

(b) If $\mathcal{C}$ contains all coproducts and coequalizers, then $\mathcal{C}$ is cocomplete.

Corollary 2.48. Set *and* $_R\mathsf{Mod}$ *are complete and cocomplete.*

2.9 Adjoint Functors

Definition 2.49. (a) Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ form an *adjoint pair*, when one has a natural isomorphism α of bifunctors

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\mathcal{D}(F(-){,}-)"] \arrow[r, bend right, "\mathcal{C}(-{,}G(-))"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{Set}
\end{tikzcd}$$

In this situation one says that F is a *left adjoint* for G and G is a *right adjoint* for F. We write $F \dashv G$ or $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \end{tikzcd}$.

(b) The tuple (F, G, α) is called an *adjunction.*

(c) We say $F : \mathcal{C} \to \mathcal{D}$ has a *right adjoint* if $\exists G : \mathcal{D} \to \mathcal{C} : F \dashv G$ (similarly $G : \mathcal{D} \to \mathcal{C}$ has left adjoint.)

Theorem 2.50. *For functors $F ; \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ we have:*

(a) F has a right adjoint $\iff$ $\forall B \in \mathcal{D} \exists$ universal pair $(A, v : FA \to B)$ for (F, B) such that $\mathrm{Hom}_{\mathcal{D}}(F(-), B) \cong \mathrm{Hom}_{\mathcal{C}}(-, A)$.

(b) G has a left adjoint $\iff$ $\forall A \in \mathcal{C} \exists$ universal pair $(B, u : A \to GB)$ for (G, A) such that $\mathrm{Hom}_{\mathcal{C}}(A, G(-)) \cong \mathrm{Hom}_{\mathcal{D}}(B, -)$.

Chapter 3

Tensor Products

Let R be a ring.

Definition 3.1. Let $M \in \mathsf{Mod}_R$ and $N \in {}_R\mathsf{Mod}$ and A an abelian group,

(a) A map $f : M \times N \to A$ is called R-*balanced* if

  - its left $\mathbb{Z}$-linear, i.e. $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$.
  - its right $\mathbb{Z}$-linear, i.e. $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$
  - $\forall r \in R : f(mr, n) = f(m, rn)$.

(b) $\mathrm{Bal}^R_{M,N}(A) = \{f : M \times N \to A \mid f \text{ is } R\text{-balanced}\}$ is an abelian group.

(c) $\mathrm{Bal}^R_{M,N}(-) : \mathsf{Ab} \to \mathsf{Ab}$ is a functor via

$$\begin{tikzcd}
M \times N \arrow[r, "f"] \arrow[dr, dashed, "\mathrm{Bal}^R_{M,N}(\varphi)"'] & A \arrow[d, "\varphi"] \\
& A'
\end{tikzcd}$$

Idea: R-balaned (bilinear) maps appear naturally, but one needs to treat them seperately (they don't live in Ab). To fix this we want to turn these R-balanced maps $M \times N \xrightarrow{f} A$ into a usual group homomorphism

$$\begin{tikzcd}
M \times N \arrow[r, "f"] \arrow[d, "-\otimes-"'] & A \\
M \otimes N \arrow[ur, dashed, "\in \mathsf{Ab}"'] &
\end{tikzcd}$$

Theorem 3.2. *With the notation from definition 1, the functor* $\mathrm{Bal}^R_{M,N} : \mathsf{Ab} \to \mathsf{Ab}$ *is representable, we denote the universal pair by*

$$(M \otimes_R N, - \otimes - : M \times N \to M \otimes N)$$

More concretely, $- \otimes - : M \times N \to M \otimes N$ *is an* R*-balanced map, such that*

$$\begin{aligned} \mathrm{Bal}^R_{M,N}(A) &\cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, A) \\ \varphi \circ (- \otimes -) &\leftarrow\!\shortmid \varphi \end{aligned}$$

Definition 3.3. $M \otimes_R N$ is called the *tensor product* of M and N and elements $m \otimes n$ in the $\mathrm{im}(- \otimes - : M \times N \to M \otimes_R N)$ are called *tensors.*

Remark. Its easy to see from the universal property that $m \otimes n$'s generate the group $M \otimes_R N$ (exercise)

$$\begin{tikzcd} M \times N \arrow[r, "-\otimes-"] \arrow[rd] & M \otimes N \arrow[d, shift right, "q"'] \arrow[d, shift left, "0"] \\ & M \otimes N \big/ \langle \mathrm{im}(- \otimes -) \rangle \end{tikzcd} \implies q = 0$$

we have

$$M \otimes_R N = \bigoplus_{(m,n) \in M \times N} \mathbb{Z}(m \otimes n) \Bigg/ \left\langle \begin{array}{l|l} (m_1 + m_2) - m_1 \otimes n - m_2 \otimes n & m, m_1, m_2 \in M \\ m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2 & n, n_1, n_2 \in N \\ mr \otimes n - m \otimes rn & r \in R \end{array} \right\rangle$$

Proposition 3.4. $- \otimes_R - : \mathsf{Mod}_R \times {}_R\mathsf{Mod} \to \mathsf{Ab}$ *is a bifunctor. More explicitly, for* $M, M', M'' \in \mathsf{Mod}_R$ *and* $N, N', N'' \in {}_R\mathsf{Mod}$*, the following holds:*

(a) For any $\varphi \in \mathrm{Hom}_{\mathsf{Mod}_R}(M, M')$ *and* $\psi \in \mathrm{Hom}_{RMod}(N, N')\exists!$ *homomorphism* $\varphi \otimes \psi \in \mathrm{Hom}_{\mathsf{Ab}}(M \otimes_R N, M' \otimes_R N')$ *such that* $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$

(b) If additionally we have $\varphi' \in \mathrm{Hom}_{ModR}(M', M'')$ *and* $\psi' \in \mathrm{Hom}_{{}_R\mathsf{Mod}}(N', N'')$*, then* $(\varphi' \circ \varphi) \otimes (\psi' \circ \psi) = (\varphi' \otimes \psi') \circ (\varphi \otimes \psi)$.

(c) $\mathrm{id}_M \otimes \mathrm{id}_N = \mathrm{id}_{M \otimes N}$.

Proof. TODO. □

Proposition 3.5. *For* $M \in \mathsf{Mod}_R$ *and* $N \in {}_R\mathsf{Mod}$

one has:

(a) $M \to M \otimes_R R$ given by $m \mapsto m \otimes 1_R$ is an isomorphism in Ab.

(b) $N \to R \otimes_R N$ given by $n \mapsto 1_R \otimes n$ is an isomorphism in Ab.

Proof. Only (a): This map is clearly $\mathbb{Z}$-linear, we construct the inverse map by TODO. □

Proposition 3.6. *Let* I *be a set and* $(M_i)_{i \in I}$ *a family* $M_i \in \mathsf{Mod}_R$ *and* $N \in {}_R\mathsf{Mod}$ *(or the opposite), then* $\exists!$ *isomorphism*

$$\psi : \left(\bigoplus_{i \in I} M_i \right) \otimes N \xrightarrow{\cong} \bigoplus_{i \in I} (M_i \otimes N), (m_i)_i \otimes n \mapsto (m_i \otimes n)_i.$$

Proof. TODO.

□

Corollary 3.7. *For sets* I *and* J, $R^{(I)} \otimes_R R^{(J)} \cong R^{(I \times J)}$ *given by* $e_i \otimes f_j \mapsto e_{(i,j)}$ *on the basis.*

3.1 Tensor products over commutative rings

When R is commutative, $R \cong R^{\mathrm{op}}$ and $\mathsf{Mod}_R \cong {}_R\mathsf{Mod}$, in this casse $M \otimes_R N$ admits further structures:

Proposition 3.8. *Suppose that R is commutative and M, N are two R-modules, then*

(a) $M \otimes_R N$ is an R-module with scalar multiplication given by

$$r(m \otimes n) := rm \otimes n = m \otimes rn$$

on pure tensors. For $A \in {}_R\mathsf{Mod}$ define

$$\mathrm{Bil}^R_{M\times N}(A) := \left\{ f : M \times N \to A \;\middle|\; \begin{array}{l} f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \\ f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \\ f(rm, n) = f(m, rn) = rf(m, n) \end{array} \right\}$$

to be the set of R-bilinear maps from $M \times N$ to A.

(b) The functor $\mathrm{Bil}^R_{M\times N} : {}_R\mathsf{Mod} \to \mathsf{Set}$ is representable by $(M \otimes_R N, - \otimes -)$.

(c) $- \otimes - : {}_R\mathsf{Mod} \times {}_R\mathsf{Mod} \to {}_R\mathsf{Mod}$ is a bifunctor.

Proof. (a) Let $\ell_r : M \to M$ be given by $m \mapsto rm$, this gives $\ell_r \otimes \mathrm{id}_N : M \otimes_R N \to M \otimes_R N$ by proposition 4. We define scalar multiplication by r on $M \otimes_R N$ to be the above $r \cdot - := \ell_r \otimes \mathrm{id}$. Check that this gives $M \otimes_R N$ on ${}_R\mathsf{Mod}$ structure.

(b) and (c) exercises.

□

Remark. Note that we have less bilinear maps than balanced maps:

$$\mathrm{Hom}_R(M \otimes_R N, A) \cong \mathrm{Bil}^R_{M\times N}(A) \subseteq \mathrm{Bal}^R_{M\times N}(A) \cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, A)$$

3.2 Tensor product of algebras

Let A be a commutative ring and R, R' two A-algebras via $\varphi : A \to R$ and $\varphi' A \to R'$ (where $\varphi(A) \subseteq Z(R), \varphi'(A) \subseteq Z(R')$).

Proposition 3.9. *(a) $\exists!A$-bilinear multiplication $- \cdot - : (R \otimes_A R') \times (R \otimes_A R') \to R \otimes_A R'$ given by*

$$(r \otimes r') \cdot (s \otimes s') := rs \otimes r's'$$

on pure tensors.

(b) $(R \otimes_A R', +, \cdot, 0_R \otimes 0_{R'}, 1_R \otimes 1_{R'})$ is a ring.

(c) $R \otimes_A R'$ is an A-algebra via $\varphi_\otimes : A \to R \otimes_A R', a \mapsto a \otimes 1 = 1 \otimes a = a(1 \otimes 1)$.

Proof. TODO. □

Examples. (a) If R is an A-algebra then $M_{n\times n}(A) \otimes_A R \cong M_{n\times n}(R)$.

(b) $M_{n\times n}(A)\otimes_A M_{m\times m}(A)=M_{nm\times nm}(A)$.

(c) $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}\times\mathbb{C}$.

(d) $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}\cong M_{2\times 2}(\mathcal{C})$ where $\mathbb{H}$ is Hamilton's quaternion algebra.

Definition 3.10. An (R,R')-*bimodule* is a tuple $(M,0,+,\cdot,\cdot')$, where $(M,0,+,\cdot)\in {}_R\mathsf{Mod}$ and $(M,0,+,\cdot)\in\mathsf{Mod}_{R'}$ such that $\forall r\in R, r'\in R', m\in M$ we have

$$r\cdot(m\cdot' r')=(r\cdot m)\cdot' r'$$

We denote the category of bimodules by ${}_R\mathsf{Mod}_{R'}$.

Remark. (a) If $M\in {}_R\mathsf{Mod}_{R'}$ then one has ring homomorphisms

$$R\to\mathrm{End}_{\mathsf{Mod}_{R'}}(M), r\mapsto r\cdot -$$
$$(R')^{\mathrm{op}}\to\mathrm{End}_{RMod}(M), r'\mapsto -\cdot r'$$

(b) We have an equivalence of categories ${}_R\mathsf{Mod}_{R'}\cong {}_{R\otimes_{\mathbb{Z}}(R')^{\mathrm{op}}}\mathsf{Mod}$. The $R\otimes_{\mathbb{Z}}(R')^{\mathrm{op}}$-module structure comes from

$$R\times(R')^{\mathrm{op}}\to\mathrm{End}_{\mathbb{Z}}(M), (r,r')\mapsto r\cdot-\cdot r', r\otimes r'\in R\otimes_{\mathbb{Z}}(R')^{\mathrm{op}}$$

notice that this is bilinear.

Proposition 3.11. *The bifunctor $-\otimes-$ extends to a bifunctor*

$$-\otimes_{R'}-:{}_R\mathsf{Mod}_{R'}\times{}_{R'}\mathsf{Mod}_{R''}\to{}_R\mathsf{Mod}_{R''}$$

More explicitly for $M\in{}_R\mathsf{Mod}_{R'}$ and $N\in{}_{R'}\mathsf{Mod}_{R''}$ we define the (R,R'')-bimodule on $M\otimes_{R'}N$ by

$$r\cdot(m\otimes n)\cdot r'':=rm\otimes nr''$$

Proof. Exercise. □

Remark. Similarly one has $-\otimes-:{}_R\mathsf{Mod}_{R'}\times{}_{R'}\mathsf{Mod}\to{}_R\mathsf{Mod}$ and $-\otimes-:\mathsf{Mod}_{R'}\times{}_{R'}\mathsf{Mod}_{R''}\to\mathsf{Mod}_{R''}$.

Examples.

1. **Base change:** Let $\varphi:R\to S$ be a ring homomorphism, then S is an (S,R)-bimodule. The functors ${}_R\mathsf{Mod}\to{}_S\mathsf{Mod}, M\mapsto S\otimes_R M$ and $\mathsf{Mod}_R\to\mathsf{Mod}_S, N\mapsto N\otimes_R S$ are called *base change* or base extension from R to S. These are left adjoints to restriction of scalars, for example:

$$S\otimes_R R=S,\quad S\otimes_R R^{(I)}=S^{(I)}$$

2. **Associativity of $\otimes$:** There exists a natural isomorphism

$$(-\otimes_R-)\otimes_S-\cong-\otimes_R(-\otimes_S-):{}_T\mathsf{Mod}_R\times{}_R\mathsf{Mod}_S\times{}_S\mathsf{Mod}_Q\to{}_T\mathsf{Mod}_Q$$

Remark. Let R be commutative and $M_1,\dots,M_n\in R-\mathsf{Mod}$, then $\bigotimes_R^{1\le i\le n}M_i=M_1\otimes_R\cdots\otimes_R M_n$ represents the functor $\mathrm{Multi}^R_{M_1\times\dots\times M_n}(-)$ of R-multilinear maps on $\bigtimes_{1\le i\le n}M_i$.

Proposition 3.12. *Another functor on bimodules:*

(a) For $M \in {}_S\mathsf{Mod}_R$ and $N \in {}_T\mathsf{Mod}_R$ the abelian group $\mathrm{Hom}_R(M, N)$ carries a natural (T, S)-bimodule structure defined by

$$(t \cdot f \cdot s)(x) = t \cdot f(sx)$$

for $f \in \mathrm{Hom}_R(M, N), s \in S, t \in T, x \in M$, this gives a bifunctor

$${}_S\mathsf{Mod}_R \times {}_T\mathsf{Mod}_R \to {}_T\mathsf{Mod}_S$$

(b) Similarly one has a bifunctor ${}_R\mathsf{Mod}_S \times {}_R\mathsf{Mod}_T \to {}_S\mathsf{Mod}_T$

Proof. Exercise. □

Theorem 3.13 (Hom, $\otimes$ adjunction, Jacobson Prop. 3.8)**.** *Let Let R, S, T, U be rings and $M \in {}_R\mathsf{Mod}_S, N \in {}_S\mathsf{Mod}_T, P \in {}_U\mathsf{Mod}_T$. There exists a natural isomorphism:*

$$\begin{tikzcd}
{}_R\mathsf{Mod}_S^{\mathrm{op}} \times {}_U\mathsf{Mod}_T \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & {}_U\mathsf{Mod}_R TODO
\end{tikzcd}$$

Chapter 4

Abelian Categories

Definition 4.1. Let R be a commutative ring.

(a) A category $\mathcal{A}$ is called R-linear if

  (i) $\forall X, Y \in \mathcal{A}, \mathcal{A}(X, Y)$ is an R-module $(\mathcal{A}(X, Y), 0_{X,Y}, +_{X,Y}, \cdot_{X,Y})$.

  (ii) $\forall X, Y, Z \in \mathcal{A}$ the composition map

$$\begin{aligned}\mathcal{A}(X, Y) \times \mathcal{A}(Y, Z) &\to \mathcal{A}(X, Z)\\ (\varphi, \psi) &\mapsto \psi \circ \varphi\end{aligned}$$

  is R-bilinear. (in particular $r(\psi \circ \varphi) = (r \cdot \psi) \circ \varphi = \psi \circ (r\varphi)$).

(b) A functor $F : \mathcal{A} \to \mathcal{A}'$ between R-linear categories is called R-linear if $\forall X, Y \in \mathcal{A}$ the map $F : \mathcal{A}(X, Y) \to \mathcal{A}'(FX, FY)$ is R-linear.

(c) For $\mathbb{Z}$-linear (preadditive) categories $\mathcal{A}$ and $\mathcal{A}'$, the full subcategory $\operatorname{Add}(\mathcal{A}, \mathcal{A}') \subseteq \operatorname{Fun}(\mathcal{A}, \mathcal{A}')$ of $\mathbb{Z}$-linear (additive) functors between them is again a $\mathbb{Z}$-linear category, i.e. there is a natural addition on natural transforations.

Examples. (a) ${}_R\mathsf{Mod}$ is $\mathbb{Z}$-linear and in fact (R is commutative) R-linear.

(b) $\mathcal{A}$ is R-linear $\iff$ $\mathcal{A}^{\mathrm{op}}$ is also R-linear.

(c) If $\mathcal{A}$ and $\mathcal{A}'$ are R-linear, then $\mathcal{A} \times \mathcal{A}'$ is R-linear.

(d) Let S be a (not necessarily commutative) ring and $\underline{S}$ the category

$$(\{*\}, S, \mathrm{dom} = \{*\}, \mathrm{cod} = \{*\}, \mathrm{id}_* = \mathrm{id}_S, \circ = \cdot_S)$$

associated to S, then $\underline{S}$ is $\mathbb{Z}$-linear ($\operatorname{Hom}_{\underline{S}}(*, *) = S$).

Lemma 4.2. *If $\mathcal{A}$ is $\mathbb{Z}$-linear, then for $X \in \mathcal{A}$ the following are equivalent:*

(i) X is initial.

(ii) X is terminal.

(iii) $\mathcal{A}(X, X) = 0$ (in particular $\mathrm{id}_X = 0$)

Proof. Exercise. □

Definition 4.3. An $X \in \mathcal{A}$ satisfying conditions of Lemma 2 is called a *zero object*

Notation. The zero object of $\mathcal{A}$ (if it exists) is unique up to unique isomorphism so we just write 0 or 0_A for this object.

Lemma 4.4. *Let $\mathcal{A}$ be $\mathbb{Z}$-linear with a 0-object and $X, Y \in \mathcal{A}$, then the zero map $0_{X,Y} \in \mathcal{A}(X,Y)$ (remember $\mathcal{A}(X,Y)$ is an abelian group) is equal to the composition $X \xrightarrow{\exists!} 0 \xrightarrow{\exists!} Y$*

Proof. Exercise. □

Proposition 4.5. *Let $\mathcal{A}$ be $\mathbb{Z}$-linear, then for $X_1, X_2 \in \mathcal{A}$ the following are equivalent:*

(i) The product $X_1 \amalg X_2$ exists.

(ii) The coproduct $X_1 \amalg X_2$ exists.

(iii) $\exists Y \in \mathcal{A}, p_1, p_2 : Y \to X_i$ and $\iota_1, \iota_2 : X_i \to Y$ such that

$$p_i \circ \iota_j = \begin{cases} 1_{X_i}, & i = j \\ 0, & i \neq j \end{cases}$$

Proof. TODO. □

Remark. In (iii) (Y, ι_1, ι_2) is the coproduct and (Y, p_1, p_2) is the product.

Definition 4.6. In a $\mathbb{Z}$-linear category we deonte $X \amalg Y = X \amalg Y$ by $X \oplus Y$ and call it the *direct sum* of X and Y.

Lemma 4.7. *Let $\mathcal{A}$ and $\mathcal{A}'$ be $\mathbb{Z}$-linear and $F : \mathcal{A} \to \mathcal{A}'$ an additive functor, then*

(i) If $0_{\mathcal{A}}$ exists then $0_{\mathcal{A}'}$ exists and $F(0_{\mathcal{A}}) \cong 0_{\mathcal{A}'}$.

(ii) If $X, Y \in \mathcal{A}$ and $X \oplus Y$ exists, then $FX \oplus FY$ exists and $FX \oplus FY \cong F(X \oplus Y)$.

Proof. Exercise. □

Definition 4.8. A category $\mathcal{A}$ is called *additive* (or R-linear additive) if $\mathcal{A}$ is $\mathbb{Z}$-linear (or R-linear), $0_{\mathcal{A}}$ exists and $\forall X, Y \in \mathcal{A} : X \oplus Y$ exists.

Remark. If $\mathcal{A}$ is additive then the addition on $\mathcal{A}(X,Y)$ is determined by the composition map! (Morel II 1.2.4)

4.1 Kernels and cokernels

Recall that the equalizer of two morphisms $f, g : X \to Y$ is the limit of the diagram $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, so $Z \xrightarrow{u} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ such that for every $W \xrightarrow{v} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ with $f \circ v = g \circ v$, v factors through u:

$$\begin{tikzcd}
Z \arrow[r, "u"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
W \arrow[u, dashed] \arrow[ur, "v"'] & &
\end{tikzcd}$$

similarly the coequalizer is the colimit of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$.
In the category of R-modules, $\mathrm{eq}(f, g) = \ker(f - g)$ and $\mathrm{coeq}(f, g) = \mathrm{coker}(f - g)$. In particular $\ker f = \mathrm{eq}(f, 0)$.

Definition 4.9. Let $\mathcal{A}$ be a $\mathbb{Z}$-linear category and $f \in \mathcal{A}(X, Y)$.

(a) define $\ker f := \mathrm{eq}(f, 0)$ and $\mathrm{coker}\, f = \mathrm{coeq}(f, 0)$ (if they exist)

(b) If $\ker f$ exists, define the coimage as $X \to \mathrm{coim}\, f = \mathrm{coker}(\ker f \to X)$ (it might not exist).

(c) If $\mathrm{coker}\, f$ exists, define the image as $\mathrm{im}\, f \to Y = \ker(Y \to \mathrm{coker}\, f)$.

Example 4.10. Let $X_1, X_2 \in \mathcal{A}$ and $(X_1 \oplus X_2, \iota_1, \iota_2, p_1, p_2)$ the direct sum. Then $\iota_1 = \ker p_2, \iota_2 = \ker p_1, p_1 = \mathrm{coker}\, \iota_2, p_2 = \mathrm{coker}\, \iota_1$

Lemma 4.11. *Kernels are monomorphisms and cokernels are epimorphisms.*

Proof. We only prove the statement for kernels. Let $\ker f \xrightarrow{\iota} X \overset{f}{\underset{0}{\rightrightarrows}} Y$, now assume $\iota \circ \varphi = \iota \circ \varphi'$

$$\begin{tikzcd}
Z \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\varphi'"'] & \ker f \arrow[r, "\iota"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

By the definition of equalizer

$$\begin{aligned} \mathcal{A}(Z, \ker f) &\simeq \{\psi : Z \to X \mid f \circ \psi = 0\} \\ \varphi &\mapsto \iota \circ \varphi \end{aligned}$$

is bijective, so $\varphi = \varphi'$. $\square$

Theorem 4.12. *Let* $f \in \mathcal{A}(X, Y)$ *and assume* $\ker f, \mathrm{coker}\, f, \mathrm{im}\, f, \mathrm{coim}\, f$ *exist. Then there exists a unique morphism* $u : \mathrm{coim}\, f \to \mathrm{im}\, f$ *such that* f *is equal to the composition*

$$\begin{tikzcd}
X \arrow[r, "c"] & \mathrm{coim}\, f \arrow[r, dashed, "\exists ! u"] & \mathrm{im}\, f \arrow[r, "d"] & Y
\end{tikzcd}$$

This is called the canonical factorization *of* f *(or epi-mono factorization).*

Remark. Note that by lemma 11, c is an epimorphism and d is a monomorphism.

Proof. TODO.

$\square$

Note that in the category of R-modules, u is an isomorphism.

Definition 4.13. A category $\mathcal{A}$ is called *abelian* if

(i) $\mathcal{A}$ is additive ($\mathbb{Z}$-linear, 0 exists and $X \oplus Y$ exists)

(ii) $\forall f \in \mathcal{A}(X, Y) : \ker f, \operatorname{coker} f$ exist.

(iii) $\forall f \in \mathcal{A}(X, Y)$ with canonical factorization $f = d \circ u \circ c$, u is an isomorphism.

Example 4.14.

- For any ring R the category ${}_R\mathsf{Mod}$ is abelian.
- For rings R, R' the category ${}_R\mathsf{Mod}_{R'}$ is abelian ($\cong {}_{R\otimes_{\mathbb{Z}}(R')^{\mathrm{op}}}\mathsf{Mod}$).

Example 4.15.

Let $\mathcal{A} \subseteq \mathsf{Ab}$ be the fullsubcategory of finitely generated free $\mathbb{Z}$-modules, then $\mathcal{A}$ is additive.

For $f : X \to Y$ in $\mathcal{A}$ the usual $\ker_{\mathsf{Ab}} f$ as abelian group is again a finitely generated free abelian group and is also the kernel in $\mathcal{A}$.

The cokernel however might have torsion. In fact

$$\operatorname{coker}_{\mathcal{A}} f = {}^{\operatorname{coker}_{\mathsf{Ab}} f}\Big/_{\operatorname{Tor}(\operatorname{coker}_{\mathsf{Ab}} f)}$$

So kernels and cokernels exists in $\mathcal{A}$, now let $f : \mathbb{Z} \to \mathbb{Z}, n \mapsto 2n$, the canonical factorization is

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "u"] & \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z} \\ & n \arrow[r, mapsto] & 2n & \end{tikzcd}$$

since u is not an isomorphism, $\mathcal{A}$ is not abelian.

Proposition 4.16. *Let $\mathcal{C}$ be a category, then*

(i) $\mathcal{C}$ is $\mathbb{Z}$-linear $\iff$ $\mathcal{C}^{\mathrm{op}}$ is $\mathbb{Z}$-linear.

(ii) $\mathcal{C}$ is additive $\iff$ $\mathcal{C}^{\mathrm{op}}$ is additive.

(iii) $\mathcal{C}$ is abelian $\iff$ $\mathcal{C}^{\mathrm{op}}$ is abelian.

Moreover (if they exist) $\ker_{\mathcal{C}} f = \operatorname{coker}_{\mathcal{C}^{\mathrm{op}}} f$ *and* $\operatorname{im}_{\mathcal{C}} f = \operatorname{coim}_{\mathcal{C}^{\mathrm{op}}} f$.

4.2 Abelian categories

From now on let $\mathcal{A}$ be an abelian category.

Remark. Let $f : X \to Y$ and $X \twoheadrightarrow \operatorname{coim} f \underset{u}{\overset{\cong}{\to}} \operatorname{im} f \hookrightarrow Y$ be the canonical factorization. Then either $X \twoheadrightarrow \operatorname{coim} f \hookrightarrow Y$ or $X \twoheadrightarrow \operatorname{im} f \hookrightarrow Y$ (or anything else isomorphic to them) is called "the" canonical factorization for f.

Proposition 4.17. *Let $X, Y \in \mathcal{A}$*

(a) $\operatorname{coker}(0 \to X) = X \xrightarrow{\text{id}} X$ *and* $\ker(X \to 0) = X \xrightarrow{\text{id}} X$.

(b) $f : X \to Y$ is a monomorphism $\iff$ $\ker f = 0$ $\iff$ the canonical factorization of f is $X \xrightarrow{\text{id}} X \overset{f}{\hookrightarrow} Y$ $\iff$ f is a kernel.

(c) $g : C \to Y$ is an epimorphism $\iff$ $\operatorname{coker} g = 0$ $\iff$ the canonical factorization of g is $X \overset{g}{\to} Y \xrightarrow{\text{id}} Y$ $\iff$ g is a cokernel.

(d) u is an isomorphism $\iff$ u is a monomorphism and an epimorphism.

Corollary 4.18. *$X \overset{a}{\to} Z \overset{b}{\to} Y$ is the canonical factorization for $f = b \circ a$ $\iff$ a ia a monomorphism and b an epimorphism.*

Definition 4.19. In an abelian category $f \in \mathcal{A}(X, Y)$ is called injective if $\ker f = 0$ and surjective if $\operatorname{coker} f = 0$.

Corollary 4.20. *Let $f \in \mathcal{A}(X, Y)$, then*

(a) f is injective $\iff$ f is a monomorphism.

(b) f is surjective $\iff$ f is a epimorphism.

(c) f is injective and surjective $\iff$ f is a isomorphism.

4.3 Exactness

Lemma 4.21. *Let (g, f) be a composable pair of morphisms in $\mathcal{A}$ such that $g \circ f = 0$, then $\exists$ a canonical injection* $\operatorname{im} f \hookrightarrow \ker g$.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ and write the epi-mono factorization of f:

$$\begin{tikzcd} X \arrow[r, two heads, "c"] & \operatorname{coim} f \arrow[r, "\cong"] & \operatorname{im} f \arrow[r, hook, "d"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

and call the isomorphism in the middle $u : \operatorname{coim} f \cong \operatorname{im} f$. Now $g \circ d \circ u \circ c = 0 \implies g \circ d \circ u = 0$ since c is an epimorphism and u isomorphism $\implies g \circ d = 0$. By the definition of the kernel d must factor through $\ker g$:

$$\begin{tikzcd} \operatorname{im} f \arrow[r, "e"] & \ker g \arrow[r, hook, "\iota"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Since d is injective: if $e \circ \varphi = c \circ \varphi' \implies \iota \circ c \circ \varphi = \iota \circ c \circ \varphi' \implies d \circ \varphi = d \circ \varphi' \implies \varphi = \varphi' \implies e$ is a monomorphism $\implies \operatorname{im} f \hookrightarrow \ker g$. $\square$

Definition 4.22. (a) A *complex* in $\mathcal{A}$ is a family of morphisms $(d_i : X_i \to X_{i+1})_{i\in J}$ and $\emptyset \neq J \subseteq \mathbb{Z}$ an interval such that $\forall i \in J^- : d_{i+1} \circ d_i = 0$ $(J^- := J \setminus \sup J)$, in other notation:

$$\cdots \to X_{i-1} \xrightarrow{d_{i-1}} X_i \xrightarrow{d_i} X_{i+1} \to \cdots$$

such that $d_i \circ d_{i-1} = 0$.

(b) An exact sequence in $\mathcal{A}$ (or an acyclic complex) is a complex such that $\forall i \in J^-$ the canonical monomorphism from lemma 21 $\operatorname{im} d_i \hookrightarrow \ker d_{i+1}$ is an isomorphism

Proposition 4.23. *For any $f \in \mathcal{A}(X, Y)$ the sequence*

$$0 \to \ker f \xrightarrow{\iota} X \xrightarrow{f} Y \xrightarrow{\pi} \operatorname{coker} f \to 0$$

is exact, where ι is the kernel-morphism and π is the cokernel-morphism.

Proof.

- Exactness at $\ker f$: Kernels are monomorphisms and so

$$\ker \iota = 0 = \operatorname{im}(0 \to \ker f)$$

- Exactness at X:

$$\operatorname{im} \iota \underset{16}{=} \ker f \xrightarrow{\iota} X = \ker f$$

for Y and $\operatorname{coker} f$ pass to $\mathcal{A}^{\mathrm{op}}$.

□

Lemma 4.24 (Decomposing and concatenating exact sequences)**.**

(a) If

$$\cdots \to A \xrightarrow{a} B \to 0 \quad \textit{and} \quad 0 \to B \xrightarrow{b} C \xrightarrow{c} D \to \cdots$$

are exact at B and C, then

$$\cdots \to A \xrightarrow{b\circ a} C \xrightarrow{c} D \to \cdots$$

is exact at C, i.e. $c \circ b \circ a = 0$ and $\ker c = \operatorname{im}(b \circ a)$.

(b) If

$$\cdots \to A \xrightarrow{a} B \xrightarrow{b} C \to 0 \quad \textit{and} \quad 0 \to C \xrightarrow{c} D \to \cdots$$

are exact at B and C, then

$$\cdots \to A \xrightarrow{a} B \xrightarrow{c\circ b} D \to \cdots$$

is exact at B.

(c) If

$$\cdots \to X \xrightarrow{f} Y \xrightarrow{g} Z \to \cdots$$

is exact at Y, then:

(i) $\cdots \to X \to \operatorname{im} f \to 0$ *and* $0 \to \ker g \to Y \xrightarrow{g} Z \to \cdots$ *are exact at* $\operatorname{im} f, \ker g, Y$.

(ii) $\cdots X \xrightarrow{f} Y \to \operatorname{coker} f \cong \operatorname{coim} g \to 0$ *and* $0 \to \operatorname{im} g \to Z \to \cdots$ *are exact at* $Y, \operatorname{coker} f, \operatorname{im} g$.

Example. Prop 23 + Lemma 24 gives for $f \in \mathcal{A}(X,Y)$:

$$0 \to \ker f \to X \to \operatorname{coim} f \to 0$$

and

$$0 \to \operatorname{im} f \to Y \to \operatorname{coker} f \to 0$$

are exact sequences.

Definition. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is also called a *short exact sequence* (s.e.s.)

Lemma 4.25. *Let* $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ *be a complex in* $\mathcal{A}$*, then:*

(a) The sequence

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, "b"] & C \arrow[r] \arrow[dr] & 0 \\ & & & X \end{tikzcd}$$

is (right) exact $\iff \forall X \in \mathcal{A}$

$$0 \to \operatorname{Hom}_{\mathcal{A}}(C,X) \xrightarrow{b^* = - \circ b} \operatorname{Hom}_{\mathcal{A}}(B,X) \xrightarrow{a^*} \operatorname{Hom}_{\mathcal{A}}(A,X)$$

is exact in Ab.

(b) The sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C$$

is (left) exact $\iff \forall X \in \mathcal{A}:$

$$0 \to \operatorname{Hom}_{\mathcal{A}}(X,A) \xrightarrow{a_*} \operatorname{Hom}_{\mathcal{A}}(X,B) \xrightarrow{b_*} \operatorname{Hom}_{\mathcal{A}}(X,C)$$

is exact in Ab.

Proof. Stacks Project. ☐

Proposition 4.26. *Let the following be a short exact sequence in* $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "a"] & B \arrow[r, "b"] \arrow[l, dashed, bend left, "s"] & C \arrow[r] \arrow[l, dashed, bend left, "t"] & 0 \end{tikzcd}$$

Then the following are equivalent:

(i) $\exists s \in \mathcal{A}(B,A) : s \circ a = 1_A$

(ii) $\exists t \in \mathcal{A}(C,B) : b \circ t = 1_C$

(iii) $\exists s \in \mathcal{A}(B,A) \exists t \in \mathcal{A}(C,B) : s \circ a = 1_A, b \circ t = 1_C$ *and* $1_B = a \circ s + t \circ b$

(iv) $B \cong A \oplus C$ with respect to the morphisms from (iii).

Proof. Exercise. (See 1.28) ☐

Proposition 4.27. *Let $\mathcal{A}$ be an abelian category, then $\mathcal{A}$ possesses (all) finite limits and colimits, (finite meaning over finite index categories).*

Proof. Exercise (See 2.46) ☐

Proposition 4.28 (Special case of 27)**.** *If $\mathcal{A}$ is an abelian category, then $\mathcal{A}$ has pullbacks and pushouts. They are given by*

$$\text{p.b.}\left(\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}\right) = \ker(X \oplus Y \xrightarrow{f-g} Z)$$

and

$$\text{p.o.}\left(\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd}\right) = \operatorname{coker}(X \xrightarrow{(f,-g)} Y \oplus Z)$$

Proof. Exercise. ☐

Definition 4.29. A commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd}$$

in $\mathcal{A}$ is called

(a) *cartesian* if

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \\ C & \end{tikzcd} \cong \text{p.b.}\left(\begin{tikzcd} & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd}\right)$$

(b) *cocartesian* if

$$\begin{tikzcd} & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd} \cong \text{p.o.}\left(\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \\ C & \end{tikzcd}\right)$$

Lemma 4.30. *Let $(*)$ be a commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\ C \arrow[r, "d"'] & D \end{tikzcd} \qquad (*)$$

Then:

(a) $(*)$ *is cartesian* $\iff 0 \to A \xrightarrow{(a,b)} B \oplus C \xrightarrow{c+d} D$ *is exact.*

(b) $(*)$ *cocartesian* $\iff A \to B \oplus C \to D \to 0$ *is exact.*

(c) If $(*)$ *is cartesian, then* $\ker a \cong \ker d$ *via* b*.*

(d) If $(*)$ *is cocartesian, then* $\operatorname{coker} a \cong \operatorname{coker} d$ *via* c*.*

(e) If $(*)$ *is cartesian and* d *is surjective, then* $(*)$ *is cocartesian.*

(f) If $(*)$ *is cocartesian and* a *is injective, then* $(*)$ *is cartesian.*

4.4 The homomorphism theorem and the isomorphism theorems

Definition 4.31. (a) On the set of monomorphisms to X define an equivalence relation $(f : U \hookrightarrow X) \sim (g : V \hookrightarrow X) \iff \exists$ commutative diagram

$$\begin{tikzcd}
U \arrow[rr, dashed, "\alpha"] \arrow[dr, hook, "f"'] & & V \arrow[dl, hook', "g"] \\
& X &
\end{tikzcd}$$

with α an isomorphism. Call the equivalence classes *subobjects* of X.

(b) On the set of epimorphisms from Y define an equivalence relation $(f : Y \twoheadrightarrow W) \sim (g : Y \twoheadrightarrow Z) \iff \exists$ commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, two heads, "f"'] \arrow[dr, two heads, "g"] & \\
W \arrow[rr, dashed, "\beta"'] & & Z
\end{tikzcd}$$

with β an isomorphism. Call the equivalence classes *quotients* of Y.

Also: Call a monomorphism $f : U \to X$ (epimorphism $g : Y \to W$) a subobject (quotient) meaning its equivalence class.

Proposition 4.32. *On subobjects* $[U \xrightarrow{f} X], [V \xrightarrow{X}]$ *of* $X \in \mathcal{A}$ *define a relation* $[U \xrightarrow{f} X] \leq [V \xrightarrow{X}] \iff \exists$ *commutative diagram:*

$$\begin{tikzcd}
U \arrow[rr, hook, dashed, "\alpha"] \arrow[dr, hook, "f"'] & & V \arrow[dl, hook', "g"] \\
& X &
\end{tikzcd}$$

with α *a monomorphism. Then*

(a) $\leq$ *is a partial ordering on the class of subobjects.*

(b) The infimum of $[U \xrightarrow{f} X]$ *and* $[V \xrightarrow{g} X]$ *exists and is represented by the pullback*

$$\text{p.b.}\left(\begin{tikzcd} & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}\right)$$

together with the diagonal map $\text{p.b.}(\cdots) \to X$. *Write* $U \cap V$.

(c) The supremum of $[U \xrightarrow{f} X]$ *and* $[V \xrightarrow{g} X]$ *exists and is represented by the pushout*

$$\text{p.o.}\left(\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \\ V & \end{tikzcd}\right)$$

together with the canonical arrow $U \cap V \to \text{p.o.}(\cdots)$. *Write* $U + V$

Remark 4.33. Now we can formulate the homomorphism and isomorphism theorems:

(a) **Homomorphism theorem:** Given $f \in \mathcal{A}(X, Y)\exists$ canonical factorization:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "\text{can.}"'] & & Y \\ X\big/_{\ker f} := \operatorname{coim} f \arrow[rr, "\cong"'] & & \operatorname{im} f \arrow[u, "\text{can.}"'] \end{tikzcd}$$

(b) **First isomorphism theorem:** For subobjects $U, V \leq X$ we have

$$U + V\big/_{U} \cong V\big/_{V \cap U}$$

(c) **Second isomorphism theorem:** Given a monomorphism $\iota : U \to X, \exists$ bijection:

$$\{V \mid U \leq V \leq X\} \leftrightarrow \left\{V \leq X\big/_{U} = \operatorname{coim} \iota\right\}$$

and for U, V subobjects with $\beta : U \to V$ and $\iota : V/U \hookrightarrow X/U$, we have an isomorphism

$$X\big/_{V} \cong (X/U)\big/_{(V/U)} = \operatorname{coim} \iota$$

Theorem 4.34 (Snake lemma)**.** *TODO.*

Chapter 5

Homological Algebra

5.1 Exact functors

Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor.

Definition 5.1. F is called

(a) *left exact* $\iff$ F commutes with finite limits.

(b) *right exact* $\iff$ F commutes with finite colimits.

(c) *exact* $\iff$ F is left and right exact.

Remark. Since $\mathcal{A}, \mathcal{A}'$ are additive categories, all finite limits and colimits exist in $\mathcal{A}$ and $\mathcal{A}'$. So if $D : J \to \mathcal{A}$ is a finite diagram, we have $\lim_J D$ exists in $\mathcal{A}$, $\lim_J F \circ D$ exists in $\mathcal{A}'$ and we have a natural morphism

$$F(\lim_J D) \to \lim_J F \circ D$$

in $\mathcal{A}'$. F is left exact if this morphism is an isomorphism $\forall D$.

Proposition 5.2. *Let $F : \mathcal{A} \to \mathcal{A}'$ be additive, then:*

(a) The following are equivalent:

(i) F is left exact.

(ii) F commutes with the formation of kernels, i.e. $\forall f : X \to Y$ in $\mathcal{A}$, the natural morphism

$$F(\ker f) \to \ker F(f)$$

is an isomorphism.

(iii) $\forall$ exact sequence $0 \to X' \to X \to X''$ in $\mathcal{A}$, the sequence $0 \to FX' \to FX \to FX''$ is exact in $\mathcal{A}'$.

(iv) $\forall$ exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, the sequence $0 \to FX' \to FX \to FX''$ is exact in $\mathcal{A}'$.

(b) The following are equivalent:

(i) F is right exact.

(ii) F commutes with the formation of cokernels.

(iii) $\forall$ exact sequence $X' \to X \to X'' \to 0$ in $\mathcal{A}$, the sequence $FX' \to FX \to FX'' \to 0$ is exact in $\mathcal{A}'$.

(iv) $\forall$ exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, the sequence $FX' \to FX \to FX'' \to 0$ is exact in $\mathcal{A}'$.

(c) F is exact $\iff$ $\forall$ exact sequences $X' \xrightarrow{f} X \xrightarrow{g} X''$, the sequence

$$FX' \xrightarrow{Ff} FX \xrightarrow{Fg} FX''$$

is exact.

Proof. TODO. ☐

Proposition 5.3. *$\forall X \in \mathcal{A}$ the co- and contravariant Hom functors* $\mathrm{Hom}_{\mathcal{A}}(X,-), \mathrm{Hom}_{\mathcal{A}}(-,X)$ *are left exact.*

Proof. TODO. ☐

Proposition 5.4. *Let $F : \mathcal{A} \to \mathcal{A}'$ and $G : \mathcal{A}' \to \mathcal{A}$ be additive functors with $F \dashv G$. Then F is right exact and G is left exact.*

Proof. TODO. ☐

Example 5.5. By Hom-$\otimes$ Adjunction (roughly $\mathrm{Hom}(M \otimes -, N) = \mathrm{Hom}(M, \mathrm{Hom}(-, N))$) $M \otimes -$ is left adjoint to $\mathrm{Hom}(-,)$ $\implies$ $M \otimes_R - : {}_R\mathsf{Mod} \to \mathsf{Ab}$ is right exact, as is $- \otimes_R M : \mathsf{Mod}_R \to \mathsf{Ab}$.

$$(- \otimes_R M : \mathsf{Mod}_R \to Ab) = (M \otimes_{R^{\mathrm{op}}} - : {}_{R^{\mathrm{op}}}\mathsf{Mod} \to \mathsf{Ab})$$

Remark. Next small goal: J any small index category. Are $\lim_J : \mathcal{A}^J \to \mathcal{A}, \mathrm{colim}_J\, \mathcal{A}^J \to \mathcal{A}$ exact?

Proposition 5.6. *$\mathcal{A}^J = \mathrm{Fun}(J, \mathcal{A})$ is an abelian category.*

Proposition 5.7. *Suppose $\mathcal{A}$ contains all limits (colimits) for a given small index category J, then $\lim_J : \mathcal{A}^J \to \mathcal{A}$ is left exact. ($\mathrm{colim}_J : \mathcal{A}^J \to \mathcal{A}$ is right exact.)*

Corollary 5.8. *Let I be a set and $\underline{I}$ the discrete category associated to I. Then:*

(a) $\prod_{i \in I} : A^I \to A, (A_i)_{i \in I} \mapsto \prod_{i \in I} A_i$

(b) $\bigoplus_{i \in I} : A^I \to A, (A_i)_{i \in I} \mapsto \bigoplus_{i \in I} A_i)$ assuming existance, are exact functors.

Definition 5.9. (a) A non-empty category J is called *filtered* iffy

(i) $\forall i, j \in J : \exists$ diagram $\begin{tikzcd} i \arrow[dr] & \\ j \arrow[r] & k \end{tikzcd} \in J.$

(ii) $\forall f, g : i \rightrightarrows j \exists h : j \to k$ such that

$$h \circ f = h \circ g : i \to k$$

(b) A directed poset $(I, \subseteq)$ is a poset such that $\forall i, j \in I \exists h \in I$ such that $h \geq i, h \geq j$ ($\implies$ $(I, \subseteq)$ is directed $\implies$ $\operatorname{ord}(I, \subseteq)$ is filtered.)

Definition 5.10. $\mathcal{A}$ has exact filtered colimits $\iff$ for each filtered index category J, the functor $\operatorname{colim} \mathcal{A}^J \to \mathcal{A}$ is exact (and defined).

Theorem 5.11. ${}_R\mathsf{Mod}$ *has exact filtered colimits.*

Fundamental question: how to investigate the non-exactness of functors (that are left or right exact). Answer of homological algebra: e.g. $F : A \to B$ left exact, define "higher right derived functors" $(R^iF)_{i\geq 0}$ from F such that $\forall$ s.e.s.

$$0 \to X' \to X \to X'' \to 0$$

in A

$$\begin{tikzcd}
0 \arrow[r] & FX' \arrow[r] & FX \arrow[r] & FX'' \arrow[dll] \\
 & R^1FX' \arrow[r] & R^1FX \arrow[r] & R^1FX'' \arrow[dll] \\
 & R^2FX' \arrow[r] & R^2FX \arrow[r] & R^2FX'' \arrow[r] & \cdots
\end{tikzcd}$$

and $R^0F = F$. Study R^iF to understand the nonexactness of F, or to gain insight into some invariants of $\mathcal{A}$. Some R^iF (typically $i \leq 3$) have concrete meanings.

5.1.1 To define R^iF (or L_iF)

One wants "enough" injectives (projectives) in $\mathcal{A}$.

Theorem-Definition 5.12. *For $I \in \mathcal{A}$ the following are equivalent:*

(i) $\forall$ diagrams with ι monomorphism $\exists$ extension $g : B \to I$ such that the following commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "f"'] & B \arrow[dl, dashed, "g"] \\
 & I &
\end{tikzcd}$$

(ii) The functor $\operatorname{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ *is exact.*

(iii) Every s.e.s. $0 \to I \xrightarrow{h} C \xrightarrow{k} D \to 0$ *in $\mathcal{A}$ is split.*

If any of these hold then I is called an injective object.

Theorem-Definition 5.13. *For $P \in \mathcal{A}$ the following are equivalent:*

(i) $\forall$ diagrams with π epimorphism $\exists$ lifting $g : P \to B$ such that $\pi \circ g = f$

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\
B \arrow[r, "\pi"'] & C \arrow[r] & 0
\end{tikzcd}$$

(ii) The functor $\mathrm{Hom}_{\mathcal{A}}(P,-) : \mathcal{A} \to \mathsf{Ab}$ *is exact.*

(iii) Every s.e.s. $0 \to A \to B \to P \to 0$ *is split.*

If these hold, then P *is called a projective object in* $\mathcal{A}$*.*

Remark. I injective in $\mathcal{A} \iff I$ projective in $\mathcal{A}^{\mathrm{op}}$.

Proposition 5.14. *(a) If* $(I_j)_{j\in J}$ *(*J *a set) is a family of injectives in* $\mathcal{A}$*, such that* $\prod_J I_j$ *exist, then* $\prod_J I_j$ *is injective.*

(b) If $(P_j)_{j\in J}$ *(*J *a set) is a family of projectives in* $\mathcal{A}$*, such that* $\bigoplus_{j\in J} P_j$ *exist, then* $\bigoplus_J P_j$ *is projective.*

Example 5.15. (a) R is a projective R-module (hence so is $R^{(I)}$)

(b) $\mathbb{Q}/\mathbb{Z}$ is an injective in ${}_{\mathbb{Z}}\mathsf{Mod}$.

Definition 5.16. (a) $\mathcal{A}$ has enough injective $\iff \forall X \in \mathcal{A} \exists$ monomorphism $X \to I$ with $I \in \mathcal{A}$ injective.

(b) $\mathcal{A}$ has enough projectives $\iff \forall X \in \mathcal{A} \exists$ epimorphism $P \to X$ with $P \in \mathcal{A}$ projective.

Definition 5.17. $Q \in \mathcal{A}$ is called a

(a) *generator* $\iff \mathrm{Hom}_{\mathcal{A}}(Q,-) : \mathcal{A} \to \mathsf{Ab}$ is faithful.

(b) *cogenerator* $\iff \mathrm{Hom}_{A}(-,Q) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ is faithful.

Remark 5.18. For $Q \in \mathcal{A}$ the following are equivalent:

1. Q is a generator.
2. $\forall X, Y \in \mathcal{A} \forall f, g \in \mathcal{A}(X,Y)$:

$$f \neq g \implies \exists h \in \mathcal{A}(Q,X) : f \circ h \neq g \circ h$$

3. $\forall X, Y \in \mathcal{A} \forall f \in \mathcal{A}(X,Y)$:

$$f \neq 0 \implies \exists h \in \mathcal{A}(Q,X) : f \circ h = 0$$

Examples 5.19. 1. R is a generator for ${}_R\mathsf{Mod}$: Suppose $f : M \to N$ in ${}_R\mathsf{Mod}$ is nonzero. Then $\exists m \in M : f(m) \neq 0$. Define $h : R \to M, r \mapsto rm \implies f \circ h : R \to N$ is nonzero.

2. $\mathbb{Q}/\mathbb{Z}$ is a cogenerator in $\mathsf{Ab} = {}_{\mathbb{Z}}\mathsf{Mod}$ let $f : M \to N$ be non-zero in Ab. Let $X \in \mathrm{im}(f) \setminus \{0\}$. TODO

Definition 5.20. $\mathcal{A}$ is called a *Grothendieck abelian category* (GAC) iffy

1. $\mathcal{A}$ is cocomplete.
2. For any filtered small category J : colim : $A^J \to A$ is exact.
3. $\mathcal{A}$ possesses a cogenerator.

Theorem 5.21. *For a GAC* $\mathcal{A}$ *the following hold:*

1. *The subobjects and quotient objects of any $A \in \mathcal{A}$ form a set.*
2. *$\mathcal{A}$ has enough injectives.*
3. *$\mathcal{A}$ has an injective cogenerator.*
4. *$\mathcal{A}$ is complete.*

Example 5.22. 1. ${}_R\mathsf{Mod}$ is a GAC by examples 19(a) Thm 11 Cor II48

2. If $\mathcal{A}$ is a GAC and J is a small category then $\mathsf{PSh}(J, A)$ is a GAC

Lemma 5.23. *Suppose $F : \mathcal{A} \to \mathcal{A}'$ and $G : \mathcal{A}' \to \mathcal{A}$ are functors such that $F \dashv G$, then:*

1. *F exact $\implies$ G maps injectives in $\mathcal{A}'$ to injectives in $\mathcal{A}$.*
2. *G exact $\implies$ F maps projectives in $\mathcal{A}$ to projectives in $\mathcal{A}'$.*
3. *F faithful $\implies$ G maps cogenerators in $\mathcal{A}'$ to cogenerators in $\mathcal{A}$.*
4. *G faithful $\implies$ F maps generators in $\mathcal{A}$ to generators in $\mathcal{A}'$.*

Theorem 5.24. *Let $\mathcal{A}$ be a GAC with generator Q, then:*

Corollary 5.25. $\mathbb{Q}/\mathbb{Z}$ *is injective in* Ab.

Corollary 5.26. *For any ring R,* $\mathrm{Hom}(R, \mathbb{Q}/\mathbb{Z})$ *is an injective R-module and a cogenerator.*

Proposition 5.27. *Let $Q, J \in \mathcal{A}$, then*

1. *Suppose $\mathcal{A}$ contains all coproducts over index sets. Then the following are equivalent:*
   - *(a) Q is a generator.*
   - *(b) $\forall X \in \mathcal{A} \exists$ set $I, \exists$ epimorphism $Q^{(I)} := \coprod_{i \in I} Q \to X$. Moreover, if Q is a rojective generator then $\mathcal{A}$ has enough projectievs of the form $Q^{(I)}$, where I is a set.*
2. *Suppose $\mathcal{A}$ contains all products over index sets then the following are equivalent:*
   - *(a) J is a cogenerator.*
   - *(b) $\forall Y \in \mathcal{A} \exists$ set I, monomorphism $Y \to J^I := \prod_{i \in I} J$. Moreover if J is an injective cogenerrator then $\mathcal{A}$ has enough injectives of the form J^I where I is a set.*

Corollary 5.28. ${}_R\mathsf{Mod}$ *has enough injectives of the form* $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})^I$*, I a set.*

Remark 5.29. TODO

5.2 (Co-)chain complexes, (co-)homology

Let $\mathcal{A}$ be an additive category.

Definition 5.30. (a) A *chain complex* (C_*, ∂_*) over $\mathcal{A}$ is a sequence $(\partial_i : C_i \to C_{i-1})_{i\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$ such that $\partial_i \circ \partial_{i+1} = 0, \forall i \in \mathbb{Z}$. The map ∂_i is called the i-th *differential* or i-th *boundary map* of the complex. A morphism $f_* : C_* \to D_*$ of chain complexes is a sequence of morphisms $f_* = (f_i : C_i \to D_i)_{i\in\mathbb{Z}}$ such that the diagram commutes $\forall i \in \mathbb{Z}$

$$\begin{tikzcd}
C_{i-1} \arrow[d, "f_{i-1}"'] & C_i \arrow[l, "\partial_i^C"'] \arrow[d, "f_i"] \\
D_{i-1} & D_i \arrow[l, "\partial_i^D"]
\end{tikzcd}$$

The *category of chain complexes* over $\mathcal{A}$ is denoted $\mathsf{Ch}_*(\mathcal{A})$.

(b) A *cochain complex* (C^*, δ^*) over $\mathcal{A}$ is a sequence $(\delta^i : C^i \to C^{i+1})_{i\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$, such that $\delta^{i+1} \circ \delta^i = 0, \forall i \in \mathbb{Z}$. A morphism $f^* : C^* \to D^*$ of cochain complexes is a sequence $f^* = (f^i : C^i \to D^i)_{i\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$ such that the following diagram commutes $\forall i \in \mathbb{Z}$

$$\begin{tikzcd}
D_i \arrow[r, "\delta_D^i"] & D_{i+1} \\
C_i \arrow[u, "f_i"] \arrow[r, "\delta_C^i"] & C_{i+1} \arrow[u, "f_{i+1}"']
\end{tikzcd}$$

The *category of cochain complexes* over $\mathcal{A}$ is denoted $\mathsf{Ch}^*(\mathcal{A})$

Exercise 5.31. TODO

In the following we do most things only for cochain complexes.

Definition 5.32. (a) The *support* of a cochain complex $C^* \in \mathsf{Ch}^*(\mathcal{A})$ is $\operatorname{supp} C = \{i \in \mathbb{Z} \mid C^i \neq 0\}$

(b) The full subcategory of $\mathsf{Ch}^*(\mathcal{A})$ on complexes supported on $\mathbb{N}_0$ (or on $-\mathbb{N}_0$) is denoted $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$ (or $\mathsf{Ch}^*_{\leq 0}(\mathcal{A})$).

Proposition 5.33. *If $\mathcal{A}$ is additive (or abelian), then so are* $\mathsf{Ch}^*(\mathcal{A}), \mathsf{Ch}^*_{\geq 0}(\mathcal{A}), \mathsf{Ch}^*_{\leq 0}(\mathcal{A})$.

Definition 5.34. For $i \in \mathbb{Z}$ define left shift by i by

$$\begin{tikzcd}
\mathsf{Ch}^*(\mathcal{A}) \arrow[r] & \mathsf{Ch}^*(\mathcal{A}) \\
C \arrow[r, mapsto] \arrow[d, "f"'] & C[i] \arrow[d, "f[i]"] \\
D \arrow[r, mapsto] & D[i]
\end{tikzcd}$$

where $C[i]^n = C^{n+i}, \delta^n_{C[i]} = \delta^{n+i}_C$ and $f[i]^n = f^{n+1}$.

Convention 5.35. We regard $\mathcal{A}$ as a subcategory of $\mathsf{Ch}^*(\mathcal{A})$, as the subcategory of complexes C^* with $\operatorname{supp} C^* \subseteq \{0\}$. Identify $X \in \mathcal{A}$ with complex

$$cdots \to 0 \to 0 \to X \to 0 \to 0 \to \cdots$$

Example 5.36. TODO

5.3 Double complexes

Can iterate the formation of $\mathcal{A} \to \mathsf{Ch}^*(\mathcal{A})$.

Definition 5.37. The category of double (cochain) complexes is $\mathsf{Ch}^{**}(\mathcal{A}) := \mathsf{Ch}^*(\mathsf{Ch}^*(\mathcal{A}))$, so objects of $\mathsf{Ch}^{**}(\mathcal{A})$ are complexes of complexes

Definition 5.38. $C \in \mathsf{Ch}^{**}(\mathcal{A})$ is called *bounded* $\iff \forall k \in \mathbb{Z} : \#\{(i,j) \in \mathbb{Z}^2 \mid i+j=k, C^{ij} \neq 0\} < \infty$. Write $\mathsf{Ch}^{**}_b(\mathcal{A}) \subseteq \mathsf{Ch}^{**}(\mathcal{A})$ for the full subcategory on bounded double complexes.

Exercise 5.39. If $\mathcal{A}$ is additive or abelian, then so is $\mathsf{Ch}^{**}_b(\mathcal{A})$.

Definition 5.40. The *total complex* $\mathrm{Tot}(C)$ of $C \in \mathsf{Ch}^{**}_b(\mathcal{A})$ is the complex $\overline{C} \in \mathsf{Ch}^*(\mathcal{A})$ defined as follows:

$$\overline{C}^k := \bigoplus_{\substack{(i,j)\in\mathbb{Z}^2 \\ i+j=k}} C^{ij}$$

$\delta^k_{\overline{C}}$ is constructed as follows:

$$\delta^{ij} : C^{ij} \xrightarrow{(\delta_1^{ij},(-1)^i\delta_2^{ij})} C^{i+1,j} \oplus C^{i,j+1} \hookrightarrow \bigoplus_{i'+j'=k+1} C^{i'j'} = \overline{C}^{k+1}$$

Use the universal property of the direct sum to define

$$\delta^k_{\overline{C}} = \bigoplus_{i+j=k} \delta^{ij} : \overline{C}^k \to \overline{C}^{k+1}$$

TODO

Exercise 5.41. TODO

Definition 5.42. For $C = (C^*, \delta^*) \in \mathsf{Ch}^*(\mathcal{A})$ define:

- $Z^i(C) := \ker(\delta^i)$ as the i-th cocycle object.
- $B^i(C) := \mathrm{im}(\delta^{i-1})$ as the i-th coboundary object.
- $u^i(C) : B^i(C) \to Z^i(C)$ the canonical monomorphism.
- $H^i(C) := \mathrm{coker}(u^i(C)) = Z^i(C)/B^i(C)$ as the i-th cohomology object.

(Co-)homology measures the non-exactness of the complex C.

Lemma 5.43 (Alternative description of cohomology object)**.** *For $C \in \mathsf{Ch}^*(\mathcal{A})$ consider TODO*

Lemma 5.44 (exer)**.** *TODO*

Theorem 5.45. *(a) Given a s.e.s.*

$$\mathcal{E} : 0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$$

one obtains a long exact sequence

$$\cdots \to H^i(C) \xrightarrow{H^i(f)} H^i(D) \xrightarrow{H^i(g)} H^i(E) \xrightarrow{d^i_{\mathcal{E}}} H^{i+1}(C) \xrightarrow{H^{i+1}(f)} \cdots$$

for $i \in \mathbb{Z}$, where the connecting homomorphism $d^i_{\mathcal{E}}$ is defined by the snake lemma and $\mathcal{E} \mapsto d^i_{\mathcal{E}}$ is "functorial".

(b) Given a morphism of short exact sequences $\varphi : \mathcal{E} \to \mathcal{E}'$ in $\mathsf{Ch}^*(\mathcal{A})$:

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] \arrow[d, "\varphi"'] & C \arrow[r, "f"] \arrow[d, "\alpha"] & D \arrow[r, "g"] \arrow[d, "\beta"] & E \arrow[r] \arrow[d, "\gamma"] & 0 \\
\mathcal{E}' : 0 \arrow[r] & C' \arrow[r, "f'"'] & D' \arrow[r, "g'"'] & E' \arrow[r] & 0
\end{tikzcd}$$

one obtains a commutative ladder of long exact sequences from (a)

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(C) \arrow[r, "H^i(f)"] \arrow[d, "H^i(\alpha)"] & H^i(D) \arrow[r, "H^i(g)"] \arrow[d, "H^i(\beta)"] & H^i(E) \arrow[r, "d^i_{\mathcal{E}}"] \arrow[d, "H^i(\gamma)"] & H^{i+1}(C) \arrow[r] \arrow[d, "H^{i+1}(\alpha)"] & \cdots \\
\cdots \arrow[r] & H^i(C') \arrow[r, "H^i(f')"'] & H^i(D') \arrow[r, "H^i(g')"'] & H^i(E') \arrow[r, "d^i_{\mathcal{E}'}"'] & H^{i+1}(C') \arrow[r] & \cdots
\end{tikzcd}$$

Definition 5.46. $C \in \mathsf{Ch}^*(\mathcal{A})$ is called *acyclic* if C is exact $\iff H^i(C) = 0, \forall i \in \mathbb{Z}$.

Corollary 5.47 (To theorem 45)**.** *Let $\mathcal{E} : 0 \to C' \to C \to C'' \to 0$ be a s.e.s. in $\mathsf{Ch}^*(\mathcal{A})$, then if any two of C', C, C'' are acyclic, then so is the third.*

Theorem 5.48 (Acyclicity criterion for total complexes)**.** *Let $C \in \mathsf{Ch}_b^{**}(\mathcal{A})$ such that*

(a) Each row $(C^{ij}, \delta_1^{ij})_{i\in\mathbb{Z}}$ is acyclic $\forall j \in \mathbb{Z}$, or

(b) Each column $(C^{ij}, \delta_1^{ij})_{j\in\mathbb{Z}}$ is acyclic $\forall i \in \mathbb{Z}$

then $\operatorname{Tot}(C)$ *is acyclic.*

Definition 5.49. An arrow $f : C \to D$ in $\mathsf{Ch}^*(\mathcal{A})$ is called a *quasi-isomorphism* (quism) $\iff \forall i \in \mathbb{Z} : H^i(f) : H^i(C) \to H^i(D)$ is an isomorphism.

Lemma 5.50. *TODO*

Corollary 5.51. *For $f : C \to D$ in $\mathsf{Ch}^*(\mathcal{A})$ the following are equivalent:*

(a) f is a quasi-isomorphism.

(b) $\operatorname{Cone} f$ *is acyclic.*

Definition 5.52. Let $f, g : C \to D$ in $\mathsf{Ch}^*(\mathcal{A})$.

(a) A *homotopy* from f to g is a sequence of morphisms $(s^i : C^i \to D^{i-1})_{i\in\mathbb{Z}}$ such that $\forall i \in \mathbb{Z} : f^i - g^i = \delta_D^{i-1} \circ s^i + s^{i+1} \circ \delta_C^i$, i.e.

$$\begin{tikzcd}
 & C^i \arrow[r, "\delta_C^i"] \arrow[dl, "s^i"'] \arrow[d, "f^i-g^i"] & C^{i+1} \arrow[dl, "s^{i+1}"] \\
D^{i-1} \arrow[r, "\delta_D^{i-1}"'] & D^i &
\end{tikzcd}$$

(b) f is called *homotopic* to g if $\exists$ homotopy from f to g. Write $f \sim g$.

(c) f is called *nullhomotopic* if $f \sim 0$.

Note: This definition only requires that $\mathcal{A}$ is additive.

Proposition 5.53. *(a) For $C, D \in \mathsf{Ch}^*(\mathcal{A})$, homotopy defines an equivalence relation on $\mathrm{Hom}_{\mathsf{Ch}^*(\mathcal{A})}(C, D)$.*

(b) For $f, f' : C \to D$ and $g, g' : D \to E$ in $\mathsf{Ch}^*\mathcal{A}$ one has:

$$f \sim f', g \sim g' \implies g \circ f \sim g' \circ f'$$

(c) Suppose $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, then one has a functor $F : \mathsf{Ch}^*\mathcal{A} \to \mathsf{Ch}^*\mathcal{B}$ by

$$F(\delta^n : C^n \to C^{n+1})_{n\in\mathbb{Z}} = (F\delta^n : FC^n \to FC^{n+1})_{n\in\mathbb{Z}}$$

This functor preserves homotopy, i.e. $f \sim g \implies Ff \sim Fg$.

Proposition 5.54. *Suppose $f, g : C \to D$ in $\mathsf{Ch}^*\mathcal{A}$ are homotopic, then*

$$H^i(f) = H^i(g) : H^i(C) \to H^i(D), \forall i \in \mathbb{Z}$$

Definition 5.55. A morphism $f : C \to D$ in $\mathsf{Ch}^*\mathcal{A}$ is called a *homotopy equivalence* from C to D if $\exists g : D \to C$ in $\mathsf{Ch}^*\mathcal{A}$ such that

$$g \circ f \sim 1_C, \quad f \circ g \sim 1_D$$

and in this case C and D are called homotopy equivalent.

Proposition 5.56. *Suppose $f : C \to D$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Example. TODO

5.4 Injective and projective resolutions

Notation. Inj or $\mathrm{Inj}_{\mathcal{A}}$ and Proj or $\mathrm{Proj}_{\mathcal{A}}$ are the full subcategories of $\mathcal{A}$ on injective or projective objects respectively. Note that $\mathrm{Inj}_{\mathcal{A}}^{\mathrm{op}} = \mathrm{Proj}_{\mathcal{A}^{\mathrm{op}}}$.

Definition 5.57. (a) An *injective resolution* of $A \in \mathcal{A}$ is a quism $f : \underline{A} \to I$ in $\mathsf{Ch}^*_{\geq 0}\mathcal{A}$ with $I \in \mathsf{Ch}^*_{\geq 0}(\mathrm{Inj}_{\mathcal{A}})$ i.e. $\mathrm{Cone}\, f$, which is

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

is acyclic and all the I^j are injective.

(b) A *projective resolution* of $A \in \mathcal{A}$ is a quism $g : P \to \underline{A}$ in $\mathsf{Ch}^*_{\leq 0}\mathcal{A}$ with $P \in \mathsf{Ch}^*_{\leq 0}(\mathrm{Proj}\,\mathcal{A})$.

Proposition 5.58. *Consider the functor*

$$\begin{tikzcd} \widehat{\cdot} : \mathsf{Ch}^*(\mathcal{A})^{\mathrm{op}} \arrow[r] & \mathsf{Ch}^*(\mathcal{A}^{\mathrm{op}}) \\ (C^n, \delta^n_C) \arrow[r, "\widehat{\cdot}"] \arrow[d, "f"'] & (\widehat{C}^n, \widehat{\delta}^n_C) \\ (D^n, \delta^n_D) \arrow[r, "\widehat{\cdot}"] & (\widehat{D}^n, \widehat{\delta}^n_D) \arrow[u, "\widehat{f}"'] \end{tikzcd}$$

where $\widehat{C}^n = C^{-n}$, $\widehat{\delta}^n_C = \delta^{-n-1} \in \mathcal{A}(C^{-n-1}, C^{-n}) = \mathcal{A}^{\mathrm{op}}(\widehat{C}^n, \widehat{C}^{n+1})$ and $\widehat{f}^n := f^{-n} \in \mathcal{A}(C^{-n}, D^{-n}) = \mathcal{A}^{\mathrm{op}}(\widehat{D}^n, \widehat{C}^n)$. Then *(a)* $\widehat{\cdot}$ *is well defined, satisfies* $\widehat{\cdot} \circ \widehat{\cdot} = \mathrm{id}$ *and* $\widehat{\cdot}$ *is an isomorphism of categories.*

(b) $\widehat{\ }(\mathsf{Ch}^*_{\geq 0/\leq 0}(\mathcal{A})^{\mathrm{op}}) = \mathsf{Ch}^*_{\leq 0/\geq 0}(\mathcal{A}^{\mathrm{op}})$.

(c) $\widehat{\ }(\mathsf{Ch}^*_{\geq 0/\leq 0}(\mathrm{Inj}_{\mathcal{A}} \,/\, \mathrm{Proj}_{\mathcal{A}}))^{\mathrm{op}} = \mathsf{Ch}^*_{\leq 0/\geq 0}(\mathrm{Proj}_{\mathcal{A}^{\mathrm{op}}} \,/\, \mathrm{Inj}_{\mathcal{A}^{\mathrm{op}}})$

(d) If $\underline{A} \to I$ *is an injective resolution in* $\mathsf{Ch}^* \geq 0(\mathcal{A})$*, then* $\widehat{I} \to \underline{A}$ *is a projective resolution in* $\mathsf{Ch}^*_{\leq 0}(\mathcal{A}^{\mathrm{op}})$

Theorem 5.59. *Let* $\mathcal{A}$ *be an abelian category with enough injectives, then*

(a) Each $A \in \mathcal{A}$ *possesses an injective resolution.*

(b) Let $h : A \to B$ *be a morphism, let* $f : \underline{B} \to I^\bullet$ *be an injective resolution and* $g : \underline{A} \to C^\bullet$ *be a quism in* $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$ *(*$C^\bullet$ *is a resolution of* A*), then there is a commutative diagram in* $\mathsf{Ch}^*_{\geq 0}\mathcal{A}$

$$\begin{tikzcd}
\underline{A} \arrow[r, "g"] \arrow[d, "h"'] & C^\bullet \arrow[d, dashed, "H"] \\
\underline{B} \arrow[r, "f"] & I^\bullet
\end{tikzcd}$$

(c) If diagram in (b) commutes with $H, H' : C^\bullet \to I^\bullet$*, then* $H' \sim H$*.*

Corollary 5.60. *Suppose* $\underline{A} \xrightarrow{g} I^\bullet$ *and* $\underline{A} \xrightarrow{g} J^\bullet$ *are inj resolutions of* A*. Then:*

(a) $\exists H : I^\bullet \to J^\bullet$ *such that*

$$\begin{tikzcd}
 & \underline{A} \arrow[dl, "g"'] \arrow[dr, "f"] & \\
J^\bullet & & I^\bullet \arrow[ll, dashed, "H"]
\end{tikzcd}$$

commutes.

(b) H *in (a) is always a homotopy equivalence.*

Lemma 5.61 (Horseshoe 2)**.** *Let* $0 \to A' \to A \to A'' \to 0$ *be a s.e.s. in* $\mathcal{A}$*, let* $\underline{A}' \xrightarrow{f'} I^\bullet$ *and* $\underline{A}'' \xrightarrow{f''} J^\bullet$ *be injective resolutions. Then* $\exists$ *commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \underline{A}' \arrow[r] \arrow[d, "f'"] & \underline{A} \arrow[r] \arrow[d, dashed, "f"] & \underline{A}'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & I^\bullet \arrow[r] & K^\bullet \arrow[r] & J^\bullet \arrow[r] & 0
\end{tikzcd}$$

in $\mathsf{Ch}^*_{\geq 0}\mathcal{A}$ *with exact rows and an injective resolution* $f : \underline{A} \to K^\bullet$*. Moreover*

$\forall n \geq 0$ the following commutative diagram has exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{im}(\delta_I^{n-1}) \arrow[r] \arrow[d] & \operatorname{im}(\delta_K^{n-1}) \arrow[r] \arrow[d] & \operatorname{im}(\delta_J^{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^n \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & J^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{im}(\delta_I^n) \arrow[r] \arrow[d] & \operatorname{im}(\delta_K^n) \arrow[r] \arrow[d] & \operatorname{im}(\delta_J^n) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

So $f' = \delta_I^{-1}, f = \delta_K^{-1}$ and $f'' = \delta_J^{-1}$.

Definition 5.62. Define $\mathsf{Ex}_{\mathcal{A}}$ as the category of s.e.s. in $\mathcal{A}$ with objects:

$$\mathcal{E}_A : 0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, and morphisms are commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$ with exact rows represented by $\underline{f} = (f', f, f'')$ in $\mathsf{Ex}_{\mathcal{A}}(\mathcal{E}_A, \mathcal{E}_B)$. Composition of arrows is componentwise. TODO

Proposition 5.63. *(a) $\mathsf{Ex}_{\mathcal{A}}$ is an additive category.*

(b) We have additive functors $\mathrm{pr}_i : \mathsf{Ex}_{\mathcal{A}} \to \mathcal{A}$, mapping $0 \to A_1 \to A_2 \to A_3 \to 0$ to A_i.

(c) $\mathsf{Ex}_{\mathcal{A}}$ is not abelian.

Definition 5.64. An arrow $\underline{f} = (f', f, f'')$ in $\mathsf{Ex}_{\mathcal{A}}$ is called

(a) a *strict monomorphism (strict epimorphism)* $\iff$ f', f, f'' are monics (epics) in $\mathcal{A}$.

(b) *strict* $\iff$

$$0 \to \ker f' \to \ker f \to \ker f'' \to 0$$

is exact which is (by the snake lemma) equivalent to exactness of

$$0 \to \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f'' \to 0$$

Proposition 5.65. *If $\underline{f}$ is strict in $\operatorname{Mor}(\mathsf{Ex}_{\mathcal{A}})$ then $\ker \underline{f}, \operatorname{coker} \underline{f}, \operatorname{im} \underline{f}, \operatorname{coim} \underline{f}$ exist in $\mathsf{Ex}_{\mathcal{A}}$ and the canonical map $\operatorname{coim} \underline{f} \to \operatorname{im} \underline{f}$ is an isomorphism.*

Remark. $\mathsf{Ex}_{\mathcal{A}}$ is an exact category.

Definition 5.66. (a) A complex $\mathcal{E}^\bullet = (\underline{\delta}^i :)$ TODO

Theorem 5.67. *TODO*

5.5 Derived Functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories.

Definition 5.68. (a) A homological (resp. cohomological) δ-functor $(T_n, \delta_n)_{n\geq 0}$ (resp $(T^n, \delta^n)_{n\geq 0}$) from $\mathcal{A}$ to $\mathcal{B}$ consists of

(i) a sequence of additive functors $(T_n : \mathcal{A} \to \mathcal{B})_{n\geq 0}$ resp. $(T^n : \mathcal{A} \to \mathcal{B})_{n\geq 0}$.

(ii) A sequence of natural transformations

$$\begin{tikzcd}
\mathsf{Ex}_A \arrow[r, bend left=50, "T_{n+1}\circ\mathrm{pr}_3", ""{name=U, below}] \arrow[r, bend right=50, "T_n\circ\mathrm{pr}_1"', ""{name=D}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "\delta_n"] & \text{resp.} & \mathsf{Ex}_A \arrow[r, bend left=50, "T^n\circ\mathrm{pr}_3", ""{name=U2, below}] \arrow[r, bend right=50, "T^{n+1}\circ\mathrm{pr}_1"', ""{name=D2}] & \mathcal{B} \arrow[Rightarrow, from=U2, to=D2, "\delta^n"]
\end{tikzcd}$$

such that this data assigns to each $\underline{\mathcal{E}} : 0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ in $\mathsf{Ex}_{\mathcal{A}}$ a long exact sequence in $\mathcal{B}$

$$\cdots \to T_{n+1} \xrightarrow{\delta_n^{\mathcal{E}}} T_nA_1 \xrightarrow{T_nf} T_nA_2 \xrightarrow{T_ng} T_nA_3 \xrightarrow{\delta_{n-1}^{\mathcal{E}}} T_{n-1}A_1 \to \cdots$$

resp.

$$\cdots \to T^{n-1}A_3 \xrightarrow{\delta_{\mathcal{E}}^{n-1}} T^nA_1 \xrightarrow{T^nf} T^nA_2 \xrightarrow{T^ng} T^nA_3 \xrightarrow{\delta_{\mathcal{E}}^{n}} T^{n+1}A_1 \to \cdots$$

Moreover the assignmennt $\mathcal{E} \to$ l.e.s. is functorial in $\mathsf{Ex}_{\mathcal{A}}$

(iii) TODO WTF IS THIS?

Example 5.69. TODO

Construction 5.70. Suppose $\mathcal{A}$ has enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor.

1. $\forall A \in \mathcal{A}$ choose an injective resolution $\iota_A : \underline{A} \to I_A^\bullet$ in $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$ and define $R^nF(A) := H^n(FI_A^\bullet)$ choice in $\mathcal{B}$ of n-th cohomology object.

2. $\forall f : A \to A'$ morphism in $\mathcal{A}$ choose an arrow ι_f such that

$$\begin{tikzcd}
\underline{A'} \arrow[r, "\iota_{A'}"] & I_{A'}^\bullet \\
\underline{A} \arrow[u, "f"] \arrow[r, "\iota_A"'] & I_A^\bullet \arrow[u, "\iota_f"']
\end{tikzcd}$$

commutes in $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$ which implies that $F(\iota_f) : FI_A^\bullet \to FI_{A'}^\bullet$ morphism in $\mathsf{Ch}^*_{\geq 0}(\mathcal{B})$. Define: $R^nF(f) := H^n(F(\iota_f)) : R^nF(A) \to R^nF(A')$ ...

Lemma 5.71. *(a) R^nF is a functor $\mathcal{A} \to \mathcal{B}$ (additive.)*

(b) If one makes other choices of injective resolutions $\widetilde{\iota}_A : A \to \widetilde{I}_A^\bullet$ and $\widetilde{\iota}_f : \widetilde{I}_A^\bullet \to \widetilde{I}_{A'}^\bullet$ then we get a natural isomorphism $R^nF \cong \widetilde{R}^nF$

3. Given $\mathcal{E} : 0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$ in $\mathsf{Ex}_{\mathcal{A}}$ and an injective resolution $\iota_{\mathcal{E}} : \mathcal{E} \to J^\bullet_{\mathcal{E}}$ in $\mathsf{Ch}^*_{\geq 0}(\mathsf{Ex}_{\mathcal{A}})$

Lemma 5.72. *(a) $FJ^\bullet_{\mathcal{E}}$ lies in $\mathsf{Ch}^*_{\geq 0}(\mathsf{Ex}_{\mathcal{B}})$, in particular, it is a s.e.s. of complexes in $\mathsf{Ch}^*_{\geq 0}(\mathcal{B})$.*

(b) Write $J^\bullet_{\mathcal{E}} = 0 \to I^\bullet_1 \to I^\bullet_2 \to I^\bullet_3 \to 0$ as a s.e.s. in $\mathsf{Ch}^*_{\geq 0}(\mathcal{A})$, then the I^n_i are injective and by (a) we have a s.e.s.

$$0 \to FI^\bullet_1 \to FI^\bullet_2 \to FI^\bullet_3 \to 0 \quad (*)$$

in $\mathsf{Ch}^*_{\geq 0}(\mathcal{B})$, then the following diagram commutes and the top row is a l.e.s. in $\mathcal{B}$, and the vertical maps are isomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(FI^\bullet_j) \arrow[r, "\delta^{n-1}_{FJ^\bullet_{\mathcal{E}}}"] & H^n(FI^\bullet_1) \arrow[r] & H^n(FI^\bullet_2) \arrow[r] & H^n(FI^\bullet_3) \arrow[r, "\delta^n_{FJ^\bullet_{\mathcal{E}}}"] & \cdots \\
\cdots \arrow[r] & R^{n-1}F(A_3) \arrow[r] \arrow[u, "u^{n-1}_{A_3}"] & R^nF(A_1) \arrow[r, "R^nF(f)"'] \arrow[u, "u^n_{A_1}"] & R^nF(A_2) \arrow[r, "R^nF(g)"'] \arrow[u, "u^n_{A_2}"] & R^nF(A_3) \arrow[r] \arrow[u, "u^n_{A_3}"] & \cdots
\end{tikzcd}$$

This is not even funny anymore. Define $\delta^n_{RF} : R^nF \circ \mathrm{pr}_3 \Rightarrow R^nF \circ \mathrm{pr}_1$ for $\mathcal{E}$ as $(u^{n+1}_{A_1})^{-1} \circ \delta^n_{FJ^\bullet_{\mathcal{E}}} \circ u^n_{A_3}$.

4. Show that δ^n_{RF} is well defined.

Lemma 5.73. *TODO*

Theorem 5.74. *Suppose $\mathcal{A}$ has enouguh injectives and F is an additive left exact functor, then $RF := (R^nF, \delta^n_{RF})_{n\geq 0}$ is a* universal cohomological δ-functor *and R^nF is called the n-*th right derived functor *of F. It satisfies:*

(a) $R^0F = F$.

(b) $I \in \mathrm{Inj}_{\mathcal{A}} \implies \forall n \geq 1, R^nF(I) = 0$.

Suppose $\mathcal{A}$ has enough projectives, let $G : \mathcal{A} \to \mathcal{B}$ be right exact, then

(a) $\exists$ homological δ-functor $LG = (L_iG, \delta_i)_{i\geq 0}$ such that $(*)$

  (i) $L_0G = G$

  (ii) $P \in \mathrm{Proj}_{\mathcal{A}} \implies \forall n \geq 1, L_nG(P) = 0$.

(b) LG with $(*)$ is universal.

(c) LG with $(*)$ is unique up to unique isomorphism.

(d) LG can be computed via projective resolutions, i.e. $\forall A \in \mathcal{A}$ with projective resolution $P^\bullet \to A$ in $\mathsf{Ch}^*_{\leq 0}(\mathcal{A})$ we have

$$L_iG(A) = H^{-i}(GP^\bullet)$$

Lemma. *If $T = (T_n, \delta_n)_{n\geq 0}$ is a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and if $\mathcal{A}$ has enough projectives and $T_iP = 0, \forall P \in \mathrm{Proj}_{\mathcal{A}}, i \geq 1$, then T is universal.*

5.6 The Ext Functor

Let $\mathcal{A}$ be abelian, $M, N \in \mathcal{A}$ and suppose $\mathcal{A}$ has enough projectives/injectives if needed.

Definition 5.75. Define

$$\operatorname{Ext}^i_{\mathcal{A}}(-,N) := R^i \operatorname{Hom}_{\mathcal{A}}(-,N)$$

with the natural transformations δ^i.

Example. To compute: if

$$\cdots \to P^{-2} \to P^1 \to P^0 \to M \quad (*)$$

is a projective resolution, then we have the complex $P^\bullet$, apply $\operatorname{Hom}_{\mathcal{A}}(-,N)$ and get

$$0 \to \operatorname{Hom}_{\mathcal{A}}(P^0,N) \to \operatorname{Hom}_{\mathcal{A}}(P^{-1},N) \to \operatorname{Hom}_{\mathcal{A}}(P^{-2},N) \to \cdots$$

and $\operatorname{Ext}^i_{\mathcal{A}}(M,N) = H^i(\operatorname{Hom}_{\mathcal{A}}(P^\bullet,N))$

Proposition 5.76. *(a) M projective $\implies \operatorname{Ext}^i_{\mathcal{A}}(M,N) = 0, \forall i \geq 1$.*

(b) N injective $\implies \operatorname{Ext}^i_{\mathcal{A}}(M,N) = 0, \forall i \geq 1$.

Definition 5.77. Define

$$\overline{\operatorname{Ext}}^i_{\mathcal{A}}(M,-) := R^i(\operatorname{Hom}_{\mathcal{A}}(M,-))$$

with the natural transformations δ^i.

Example. Compute $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(M,-)$ via injective resolutions of the 2nd argument:

$$0 \to N \to I^0 \to I^1 \to I^2 \to \cdots$$

and we get

$$\operatorname{Hom}_{\mathcal{A}}(M,I^0) \to \operatorname{Hom}_{\mathcal{A}}(M,I^1) \to \operatorname{Hom}_{\mathcal{A}}(M,I^2) \to \cdots$$

so $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(M,-) = H^i(\mathcal{A}(M,I^\bullet))$.

Proposition 5.78. *For M projective or N injective we have $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(M,N) = 0, \forall i \geq 1$.*

Remark. If $\mathcal{A}$ has enough projectives and injectives, $\operatorname{Ext}^i_{\mathcal{A}}(-,-)$ and $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(-,-)$ turn out to be isomorphic!

Remark. $\operatorname{Ext}^i_{\mathcal{A}}(-,-)$ and $\overline{\operatorname{Ext}}^i_{\mathcal{A}}(-,-)$ are bifunctors.

Theorem 5.79. *Suppose $\mathcal{A}$ has enough injectives and projectives, then $\exists$ natural isomorphisms as bifunctors*

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$$
$$u^i_{M,N} : \operatorname{Ext}^i_{\mathcal{A}}(M,N) \to \overline{\operatorname{Ext}}^i_{\mathcal{A}}(M,N)$$

5.6.1 Classical interpretation of Ext^1 as extension objects

Definition 5.80. (a) An extension of M by N is a s.e.s. in $\mathcal{A}$:

$$\mathcal{E} : 0 \to N \to E \to M \to 0$$

(b) Extensions $\mathcal{E}$ and

$$\mathcal{E}' : 0 \to N \to E' \to M \to 0$$

are called *equivalent* (write $\sim$), iff $\exists$ commutative diagram:

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}_N"] & E \arrow[r] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
\mathcal{E}' : 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

(c) $\operatorname{Ext}^1_{\mathcal{A}}(M, N)$ denotes the set of equivalence classes of extensions of M by N.

Remark. (a) By the snake lemma: $\mathcal{E} \sim \mathcal{E}' \implies E \cong E'$.

(b) $\sim$ is an equivalence relation.

Proposition 5.81. *(a)* $\operatorname{Ext}_{\mathcal{A}}(M, N)$ *is an abelian group for addition* $\mathcal{E} + \mathcal{E}'$ *defined by the* Baer sum*:*

$$\begin{tikzcd}
0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, "\Sigma"'] & E \oplus E' \arrow[r] \arrow[d] & M \oplus M \arrow[r] \arrow[d, "\mathrm{id}_{M\oplus M}"] & 0 \\
0 \arrow[r] & N \arrow[r] & E \oplus E' \amalg_{N \oplus N} N \arrow[r] & M \oplus M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u, "\mathrm{id}_N"] & E + E' \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd}$$

where $E \oplus E' \amalg_{N\oplus N} N$ *is the pushout by the sum map* $+ : N \oplus N \to N, (n, m) \mapsto n + m$ *and* $E + E'$ *is the pullback by the diagonal map* $\Delta : M \to M \oplus M, m \mapsto (m, m)$. *And the zero object is given by the split s.e.s*

$$0 \to N \to N \oplus M \to M \to 0$$

(b) One has a natural isomorphism as bifunctors

$$u : \operatorname{Ext}_{\mathcal{A}}(-, -) \Rightarrow \operatorname{Ext}^1_{\mathcal{A}}(-, -)$$

if $\mathcal{A}$ *has enough projectives (or to* $\overline{\operatorname{Ext}}^1_{\mathcal{A}}(-, -)$ *if* $\mathcal{A}$ *has enough injectives).*

Remark. In fact, Yoneda also considered higher Ext-groups (in the absence of innjectives/projectives), e.g. $\operatorname{Ext}^2(M, N)$ is the "group" of exact sequences

$$0 \to N \to E_1 \to E_2 \to M \to 0$$

modulo a suitable equivalence relation.

Notation. For ${}_R\mathsf{Mod}$ one usually abbreviates

$$\operatorname{Ext}^i_R(-, -) := \operatorname{Ext}^i_{{}_R\mathsf{Mod}}(-, -)$$

For $R = \mathbb{Z}[G]$ and a group G, one also considers the group cohomology for $M \in {}_{\mathbb{Z}[G]}\mathsf{Mod}$ as

Definition 5.82. Define

$$H^i(G, M) := \operatorname{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

such that $\mathbb{Z}$ is the trivial G-action.

Proposition. *In face $H^i(G, M)$ is the i-th right derived functor of*

$$\begin{aligned} M \to \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) &\overset{exer.}{=} M^G := \{m \in M \mid \forall g \in G : gm = m\} \\ \varphi_m : n \mapsto nm &\mapsfrom m \in M^G \end{aligned}$$

So far we had $\operatorname{Ext}^i_{\mathcal{A}}$ valued in Ab. But recall that $\operatorname{Hom}_R(-,-)$ is also a bifunctor

$$\begin{aligned} \operatorname{Hom}_R(-,-) : {}_R\mathsf{Mod}_{R'} \times {}_R\mathsf{Mod}_{R''} &\to {}_{R'}\mathsf{Mod}_{R''} \\ (M, N) &\mapsto \operatorname{Hom}_R(M, N) \end{aligned}$$

Fact: ${}_R\mathsf{Mod}_{R'}$ and ${}_R\mathsf{Mod}_{R''}$ have enough injectives and projectives because ${}_R\mathsf{Mod}_{R'}$ is isomorphic to ${}_{R\otimes(R')^{\text{op}}}\mathsf{Mod}$.

Proposition 5.83 (Exer). *This Bifunctor induces*

$$\operatorname{Ext}^i_R(-,-) : {}_R\mathsf{Mod}_{R'} \times {}_R\mathsf{Mod}_{R''} \to {}_{R'}\mathsf{Mod}_{R''}$$

5.7 The Tor Functor

Let R be a ring, $M \in \mathsf{Mod}_R$ and $N \in {}_R\mathsf{Mod}$. We had the bifunctor

$$- \otimes - : \mathsf{Mod}_R \times {}_R\mathsf{Mod} \to \mathsf{Ab}$$

and it is right exact, and also Mod_R and ${}_R\mathsf{Mod}$ have enough projectives.

Definition 5.84. For $i \geq 0$ define:

$$\begin{aligned} \operatorname{Tor}^R_i(M, -) &:= L_i(M \otimes_R -) \\ \overline{\operatorname{Tor}}^R_i(-, N) &:= L_i(- \otimes_R N) \end{aligned}$$

Proposition 5.85. *If $P^\bullet \to M$ (or $Q^\bullet \to N$) are projective resolutions, then one has:*

$$\begin{aligned} \overline{\operatorname{Tor}}^R_i(M, N) &= H^{-i}(P^\bullet \otimes_R N) \\ \operatorname{Tor}^R_i(M, N) &= H^{-i}(M \otimes_R Q^\bullet) \end{aligned}$$

Moreover, if M or N are projective, then

$$\operatorname{Tor}^R_i(M, N) \cong 0 \cong \overline{\operatorname{Tor}}^R_i(M, N)$$

Theorem 5.86. *One has natural isomorphism of bifunctors*

$$\operatorname{Tor}^R_i(-,-) \cong \overline{\operatorname{Tor}}^R_i(-,-)$$

Other ways to compute Tor_i, not using projective resolutions.

Definition 5.87. Let $T = (T_n, \delta_n)$ be a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$. Call $A \in \mathcal{A}$ T-acyclic if $T_i A = 0, \forall i \geq 1$ (with T_0 right exact) For example M projective $\implies$ M is $L(- \otimes_R N)$-acyclic

Facts 5.88. Let $0 \to A' \to A \to A'' \to 0$ be a s.e.s. in $\mathcal{A}$, then

(a) A'' is T-acyclic then

$$0 \to T_0 A' \to T_0 A \to T_0 A'' \to 0$$

is a s.e.s. (because $T_1 A'' = 0$ and l.e.s. from δ-functor).

(b) If A and A'' are T-acyclic, then so is A' (because of the l.e.s. from being a δ-functor).

$$\to T_2 A' \to \underbrace{T_2 A}_{=0} \to \underbrace{T_2 A''}_{=0} \to T_1 A' \to \underbrace{T_1 A}_{=0} \to \underbrace{T_1 A''}_{=0} \to T_0 A' \to T_0 A \to T_0 A''$$

(c) If $0 \to A_0 \to A_1 \to \cdots \to A_n \to 0$ is exact in $\mathcal{A}$ and if $A_1, \cdots, A_n$ are T-acyclic then A_0 is also T-acyclic. This follows from (b) by ind. using the exact sequences $0 \to A_0 \to A_1 \to X \to 0$ and

$$0 \to X \to A_2 \to \cdots \to A_n \to 0$$

Lemma 5.89. *If $C^\bullet \in \mathsf{Ch}^*_{\leq 0}(\mathcal{A})$ is acyclic with all C^i T-acyclic, then $T_0 C^\bullet$ is acyclic (we assume T_0 is right exact)*

Corollary 5.90. *Suppose $\mathcal{A}$ has enough projectives, $G : \mathcal{A} \to \mathcal{B}$ is right exact and $T = LG$. If $Q^\bullet \xrightarrow{g} \underline{A}$ is a resolution by T-acyclic objects. Then:*

$$L_i G(A) \cong H^{-i}(GQ^\bullet)$$

Definition 5.91. $M \in \mathsf{Mod}_R$ is called *flat* if $M \otimes_R - : {}_R\mathsf{Mod} \to \mathsf{Ab}$ is exact.

Proposition 5.92. *For $M \in \mathsf{Mod}_R$ the following are equivalent:*

(a) M is flat.

(b) $\mathrm{Tor}_1^R(M, -) = 0$.

(c) $\mathrm{Tor}_i^R(M, -) = 0 \forall i \geq 1$.

Theorem 5.93. *For $M \in \mathsf{Mod}_R$ the following are equivalent:*

(a) M is flat.

(b) $\forall N \in {}_R\mathsf{Mod} : M$ is $L(- \otimes N)$-acyclic.

And thus Tor_i^R can be computed via flat resolutions.

Chapter 6

Commutative Algebra