Talk 3: Ostrowski's theorem

0 Review

Remark 0.1. $|\cdot|$ be a non-archimedean absolute value over $\mathbb{Q}, n \in \mathbb{Z}$.
$\implies |n| \leq 1$

Remark 0.2 (Strong triangle inequality)**.** Let $|\cdot|$ be a non-archimedean absolute value over $\mathbb{F}$
$\implies \forall x, y \in \mathbb{F}, |x| \neq |y| : |x+y| = \max\{|x|, |y|\}$

Proofs for all following statements can be found in [Gou, Chapter 3.2]

1 Equivalence of absolute values

Definition 1.1. Two absolute values are considered equivalent if and only if they create the same topology on K.

Example 1.2. Example from [Con]:
Define an absolute value as $|\cdot|' : \mathbb{Q} \to \mathbb{R}_{>0} \quad x \mapsto \sqrt{|x|}$.
Then $|\cdot|'$ is equivalent to the regular absolute value on $\mathbb{Q}$

Lemma 1.3. *Let $(a_n) \subset \mathbb{F}$ be a sequence in $\mathbb{F}$*
$\lim_{n\to\infty}(a_n) = a \iff$ *Any open set containing a also contains almost all a_n.*

Proposition 1.4. *Let $|\cdot|_1$ and $|\cdot|_2$ be absolute values on a field $\mathbb{F}$. The following statements are equivalent:*

(i) $|\cdot|_1$ *and* $|\cdot|_2$ *are equivalent absolute values.*

(ii) A sequence $(a_n) \subset \mathbb{F}$ converges with respect to $|\cdot|_1$ if and only if it converges with respect to $|\cdot|_2$

(iii) $\forall x \in \mathbb{F} : |x|_1 < 1 \iff |x|_2 < 1$

(iv) $\exists \alpha \in \mathbb{R}_{>0} \forall x \in \mathbb{F} : |x|_1 = |x|_2^\alpha$

Example 1.5. Let $c \in \mathbb{R}_{>1}$ be a constant.
Recall the absolute value defined on $\mathbb{Q}$ as $\forall n \in \mathbb{Z} : |n| := c^{-v_p(n)}$.
$|\cdot|$ is equivalent to the p-adic absolute value $|\cdot|_p$.

2 Ostrowski's theorem and the product formula

Theorem 2.1 (Ostrowski)**.** *Any non-trivial absolute value on $\mathbb{Q}$ is equivalent either to the regular absolute value or a p-adic absolute value for some prime p.*

Proposition 2.2 (Product formula)**.**

$$\prod_{r \leq \infty} |x|_r = 1$$

References

[Gou] Fernando Q. Gouvêa: *p-adic Numbers.*

[Con] Keith Conrad: *Equivalence of absolute values*, available on the author's website. Last visited on 04/16/23