# 1 Revision

**Proposition 1.1.** *[Gou, Prop. 5.1.4] Let $b_{ij} \in \mathbb{Q}_p$ and suppose $\forall i : \lim_{j\to\infty} b_{ij} = 0$ and $\lim_{i\to\infty} b_{ij} = 0$ uniformly in $j$, then both series $\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} b_{ij}\right)$ and $\sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty} b_{ij}\right)$ converge and have equal sum.*

# 2 Strassman's Theorem

**Remark 2.1.** Let $(R, +, \cdot)$ be a ring, $x, y \in R$, then we have $x^n - y^n = (x-y)\sum_{j=0}^{n-1} x^j y^{n-1-j}, \forall n \in \mathbb{N}_0$

*Proof.* , we use induction on $n$, Base case: $n = 2$, it's easy to see that

$$(x-y)\sum_{j=0}^{n-1} x^j y^{n-1-j} = (x-y)(x+y) = x^2 - y^2$$

Induction hypothesis: we assume for an arbitrary $n \geq 2 : x^n - y^n = (x-y)\sum_{j=0}^{n-1} x^j y^{n-1-j}$, Induction step: consider

$$(x-y)\sum_{j=0}^{n} x^j y^{n-j} = (x-y)(y^n + y^{n-1}x + \cdots + x^{n-1}y + x^n)$$

$$= (x-y)(y(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}) + x^n) = y\underbrace{(x-y)\sum_{j=0}^{n-1} x^j y^{n-1-j}}_{=x^n - y^n} + x^n(x-y)$$

$$= y(x^n - y^n) + x^n(x-y) = yx^n - y^{n+1} + x^{n+1} - yx^n = x^{n+1} - y^{n+1}.$$

$\square$

**Lemma 2.2.** *Let $f(X) \in \mathbb{Q}_p[[X]]$ be a non-zero power series which converges $\forall x \in \mathbb{Z}_p$, then $\exists N \in \mathbb{N}_0$ such that $|a_N| = \max_{n\in\mathbb{N}_0} |a_n|$ and $|a_n| < |a_N|\ \forall n > N$*

*Proof.* Since $f(X)$ converges $\forall x \in \mathbb{Z}_p$, then we have

$$\forall x \in \mathbb{Z}_p : \lim_{n\to\infty} |a_n x^n| = 0 = \lim_{n\to\infty} |a_n| \cdot |x^n| \implies \lim_{n\to\infty} |a_n| = 0$$

$\square$

**Theorem 2.3** (Strassman)**.** *Let $f(X) \in \mathbb{Q}_p[[X]]$ and suppose we have $\lim_{n\to\infty} a_n = 0$, so that $f(x)$ converges $\forall x \in \mathbb{Z}_p$. Define $N \in \mathbb{N}_0$ like in Lemma 2.2 then the function $f$ has at most $N$ zeros.*

*Proof.* induction on $N$.

- Base case: if $N = 0$, then $|a_0| > |a_n|, \forall n \geq 1$, we want to show that there are no zeros: $f(x) \neq 0 \forall x \in \mathbb{Z}_p$, if we had $f(x) = 0$, then

  $$0 = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

  $$\implies |a_0| = |a_1 x + a_2 x^2 + \cdots| \leq \max_{n\geq 1} |a_n x^n| \leq \max_{n\geq 1} |a_n|$$

  But this contradicts the assumption that $|a_0| > |a_n|, \forall n \geq 1$, so there are no zeros in this case.

- Induction step: Suppose $N$ was defined like before, and $\exists \alpha \in \mathbb{Z}_p : f(\alpha) = 0$, then we have for any $x \in \mathbb{Z}_p$

  $$f(x) = f(x) - f(\alpha) = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n \alpha^n = \sum_{n=0}^{\infty} a_n (x^n - \alpha^n) \overset{2.1}{=} (x-\alpha)\sum_{n=0}^{\infty}\sum_{j=0}^{n-1} a_n x^j \alpha^{n-1-j}$$

  $$= (x-\alpha)\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} c_{nj}, \quad c_{nj} := \begin{cases} a_n x^j \alpha^{n-1-j} & j < n, \\ 0 & j \geq n. \end{cases}$$

  We can use prop 1.1 to change the order of the summation but first we have to show the conditions of the proposition:

1. $\forall n \in \mathbb{N}_0, \lim_{j\to\infty} c_{nj} = 0$: Clear, since we have $c_{nj} = 0, \forall j \geq n$.
2. $\lim_{n\to\infty} c_{nj} = 0$ uniformly in $j$: This is also easy to see, because we have $|a_n x^j \alpha^{n-1-j}| \leq |a_n| \to 0$ unrelated to $j$.

So we can switch the sums and then we have

$$(x-\alpha)\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} c_{nj} = (x-\alpha)\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} c_{nj}$$

since $\forall j \geq n : c_{nj} = 0$, we need to only consider when $n > j$ so its equal to

$$= (x-\alpha)\sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty} a_n x^j \alpha^{n-1-j} = (x-\alpha)\sum_{j=0}^{\infty} x^j \underbrace{\sum_{n=0}^{\infty} a_{n+j+1}\alpha^n}_{=:b_j}$$

$$= (x-\alpha)g(x), \quad g(x) := \sum_{j=0}^{\infty} b_j x^j$$

□

**Corollary 2.4.** *Let $f(X) = \sum a_n x^n$ be a non-zero power series which converges on $\mathbb{Z}_p$, and let $\alpha_1, ..., \alpha_m \in \mathbb{Z}_p$ be the roots of $f(X)$ in $\mathbb{Z}_p$, then there exists another power series $g(X)$ which also converges on $\mathbb{Z}_p$ but has no zeros in $\mathbb{Z}_p$, for which*

$$f(X) = \left(\prod_{i=1}^{m}(X-\alpha_i)\right) g(X)$$

*Proof.* □

**Corollary 2.5.** *Let $f(X) = \sum a_n x^n$ be a non-zero power series which converges on $p^m\mathbb{Z}_p$, for some $m \in \mathbb{Z}$. Then $f(X)$ has a finite number of roots in $p^m\mathbb{Z}_p$.*

*Proof.* □

**Corollary 2.6.** *Let $f(X) = \sum a_n x^n$ and $g(X) = \sum b_n X^n$ be two $p$-adic power series which converge in a disc $p^m\mathbb{Z}_p$. If there exist infinitely many numbers $\alpha \in p^m Z_p$ such that $f(\alpha) = g(\alpha)$, then $a_n = b_n, \forall n \geq 0$*

*Proof.* □

**Corollary 2.7.** *Let $f(X) = \sum a_n x^n$ be a $p$-adic power series which converges in some disc $p^m\mathbb{Z}_p$. If the function $p^m\mathbb{Z}_p \to \mathbb{Q}_p, x \mapsto f(x)$ is periodic, that is, $\exists \pi \in p^m\mathbb{Z}_p : f(x+\pi) = f(x), \forall \in p^m\mathbb{Z}_p$ then $f(X)$ is constant.*

*Proof.* □

**Corollary 2.8.** *Let $f(X) = \sum a_n x^n$ be a $p$-adic power series which is entire, that is, $f(x)$ converges $\forall x \in \mathbb{Q}_p$. Then $f(X)$ has at most countably many zeros. Furthermore, if the set of zeros is not finite then the zeros form a sequence $\alpha_n$ with $|\alpha_n| \to \infty$.*

*Proof.* □

# 3 The $p$-adic Logarithm Function

# 4 Roots of Unity

# References


[Gou] Fernando Q. Gouvêa: *p-adic Numbers.*