1 Revision

2 Strassman's Theorem

For the entirety of section §2 let

$$f : \mathbb{Z}_p \to \mathbb{Q}_p, f(X) = \sum_{n=0}^{\infty} a_n X^n = a_0 + a_1 X + a_2 X^2 + \cdots$$

be a non-zero (formal?) power series with coefficients in $\mathbb{Q}_p$

Lemma 2.1. *For $x, y \in \mathbb{Z}_p$ we have*

$$f(x) - f(y) = (x - y) \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} a_n x^j y^{n-1-j}$$

Proof. □

Lemma 2.2. *If $f(x)$ converges ($\lim_{n\to\infty} a_n = 0$) $\forall x \in \mathbb{Z}_p$, then $\exists N \in \mathbb{N}_0$:*

$$|a_N| = \max_{n\in\mathbb{N}_0} |a_n| \text{ and } |a_n| < |a_N| \forall n > N$$

Proof. □

Theorem 2.3 (Strassman)**.** *Suppose we have $\lim_{n\to\infty} a_n = 0$, so that $f(x)$ converges $\forall x \in \mathbb{Z}_p$. Define $N \in \mathbb{N}_0$ by the following condition:*

$$|a_N| = \max_{n\in\mathbb{N}_0} |a_n| \text{ and } |a_n| < |a_N| \forall n > N$$

then the function f has at most N zeros.

Proof. First step is to show that N exists. □

Corollary 2.4. *Let $f(X) = \sum a_n x^n$ be a non-zero power series which converges on $\mathbb{Z}_p$, and let $\alpha_1, ..., \alpha_m \in \mathbb{Z}_p$ be the roots of $f(X)$ in $\mathbb{Z}_p$, then there exists another power series $g(X)$ which also converges on $\mathbb{Z}_p$ but has no zeros in $\mathbb{Z}_p$, for which*

$$f(X) = \left(\prod_{i=1}^{m} (X - \alpha_i) \right) g(X)$$

Proof. □

Corollary 2.5. *Let $f(X) = \sum a_n x^n$ be a non-zero power series which converges on $p^m\mathbb{Z}_p$, for some $m \in \mathbb{Z}$. Then $f(X)$ has a finite number of roots in $p^m\mathbb{Z}_p$.*

Proof. □

Corollary 2.6. *Let $f(X) = \sum a_n x^n$ and $g(X) = \sum b_n X^n$ be two p-adic power series which converge in a disc $p^m\mathbb{Z}_p$. If there exist infinitely many numbers $\alpha \in p^m Z_p$ such that $f(\alpha) = g(\alpha)$, then $a_n = b_n, \forall n \geq 0$*

Proof. □

Corollary 2.7. *Let $f(X) = \sum a_n x^n$ be a p-adic power series which converges in some disc $p^m\mathbb{Z}_p$. If the function $p^m\mathbb{Z}_p \to \mathbb{Q}_p, x \mapsto f(x)$ is periodic, that is, $\exists \pi \in p^m\mathbb{Z}_p : f(x + \pi) = f(x), \forall \in p^m\mathbb{Z}_p$ then $f(X)$ is constant.*

Proof. □

Corollary 2.8. *Let $f(X) = \sum a_n x^n$ be a p-adic power series which is entire, that is, $f(x)$ converges $\forall x \in \mathbb{Q}_p$. Then $f(X)$ has at most countably many zeros. Furthermore, if the set of zeros is not finite then the zeros form a sequence α_n with $|\alpha_n| \to \infty$.*

Proof. □

3 The p-adic Logarithm Function

4 Roots of Unity