FIBERWISE POINCARÉ–HOPF THEORY AND EXOTIC SMOOTH STRUCTURES ON FIBER BUNDLES

YAJIT JAIN

Abstract. We prove the Rigidity Conjecture of Goette and Igusa, which states that, after rationalizing, there are no stable exotic smoothings of manifold bundles with closed even dimensional fibers. The key ingredients of the proof are fiberwise Poincaré–Hopf theorems generalizing earlier such results about the Becker–Gottlieb transfer. These theorems show how to compute the smooth structure class, an invariant of smooth structures on fiber bundles, using the data of a fiberwise generalized Morse function. We use this result to prove a duality theorem for the smooth structure class, from which the conjecture directly follows. This duality theorem generalizes Milnor's duality theorems for Reidemeister and Whitehead torsion, as well as similar results for higher Franz–Reidemeister torsion due to Igusa.

Contents

1. Introduction

An *exotic smooth structure* on a smooth manifold bundle $p : M \to B$ is another smooth manifold bundle $p' : M' \to B$ together with a fiberwise tangential homeomorphism $f : M \to M'$. A fiberwise tangential homeomorphism is a fiberwise homeomorphism $f : M \to M'$ covered by an isomorphism of vertical tangent bundles $T^\vee M \cong T^\vee M'$. When the fibers of p are closed manifolds, a fiberwise tangential homeomorphism $f : M \to M'$ is equivalent to a fiberwise diffeomorphism by smoothing theory [SK77]. However, if M and M' have boundary, for instance if M is stabilized by multiplying with a trivial disk bundle, then this is not the case. Thus, when considering bundles with closed fibers, we will be interested in the existence of *stable exotic smooth structures*. We postpone precise definitions and further details to Section 3.

In [GIW14], Goette, Igusa, and Williams define the *smooth structure class*, an element of the real homology of the total space that distinguishes exotic smooth structures on manifold bundles after the fiber dimension is stabilized. When comparing fiberwise tangentially homeomorphic bundles M and M', this invariant is denoted $\Theta(M, M')$ and is closely related to the relative higher Franz–Reidemeister torsion. In a subsequent paper [GI14], Goette and Igusa give a procedure to construct stable exotic smooth structures on a bundle with closed odd dimensional fibers. In other words, they construct examples for which the smooth structure class is nonzero. Furthermore, for fixed M, they show that their procedure for constructing M' generates all possible values of $\Theta(M, M')$, up to linear combinations of rational multiples. They go on to conjecture that when the fibers are closed even dimensional manifolds, there are no stable exotic smooth structures, i.e. the smooth structure class always vanishes. In this paper we prove the Rigidity Conjecture of Goette and Igusa [GI14].

Main Theorem (Theorem F)**.** *Let M be a smooth manifold bundle with closed $2n$-dimensional fibers, $n > 2$. There does not exist a bundle M' with closed $2n$-dimensional fibers and a positive integer k for which $M' \times I^k$ is a rationally nontrivial stable exotic smoothing of $M \times I^k$. In particular, $\Theta(M' \times I^k, M \times I^k)$ vanishes for any such $M' \times I^k$.*

The main theorem stated above has several antecedents for related invariants. These include vanishing theorems for the Euler characteristic, Reidemeister torsion, the Becker–Gottlieb transfer, and the higher Franz–Reidemeister torsion. Vanishing theorems for each of these invariants are proven in stylistically equivalent ways, though the technical ingredients vary widely. They generally follow by applying a Poincaré–Hopf type theorem in combination with Poincaré duality. In order to motivate the contents of this paper, we will briefly describe these theorems and their proofs. This discussion is summarized in the table below. At the end of this section we summarize the proof of our main theorem and give an outline of the contents of this paper.

	Euler characteristic	Reidemeister torsion	Becker–Gottlieb transfer	higher torsion τ^{IK}	smooth structure class Θ
Poincaré–Hopf theorem	$\chi(M) = \sum_{z \in Z} (-1)^{\mathsf{Ind}_X(z)}$	Compute torsion from explicit choice of triangulation	Fiberwise Poincaré–Hopf theorem	Framing Principle	Theorem D
Poincaré duality	$\mathsf{Ind}_X(z) = -\mathsf{Ind}_{-X}(z)$ for odd dimensional manifolds	Recompute for dual cell complex	Recompute fiberwise index map after negating Morse function and compare	Apply the framing principle for a fiberwise GMF f and compare with results for $-f$	Theorem E
⇒Vanishing result	$\chi(M) = 0$ when $\dim M$ is odd	Torsion is trivial on even dimensional manifolds	transfer map on real cohomology vanishes for odd dimensional fibers	higher torsion of even dimensional manifold bundles is MMM class⇒relative torsion of fiberwise tangentially homeomorphic manifold bundles vanishes	Rigidity Conjecture: smooth structure class vanishes

Recall the classical proof that the Euler characteristic of an odd dimensional manifold is zero: The Poincare–Hopf theorem states that the Euler characteristic of a closed manifold M is equal to the sum of the indices of isolated critical points of a Morse function f on M:

$$\chi(M) = \sum_{z \in Z} (-1)^{\mathsf{Ind}_{\nabla f}(z)}$$

On an odd dimensional manifold, $\mathsf{Ind}_{\nabla(-f)}(z)$ and $\mathsf{Ind}_{\nabla f}(z)$ have opposite parity. It follows that $\chi(M) = 0$.

We adopt a stylized view of this proof: the vanishing result for the Euler characteristic is proven by applying the Poincaré–Hopf theorem in combination with Poincaré duality.

The Reidemeister torsion is a K-theoretic generalization of the Euler characteristic that admits an analogous vanishing theorem: the Reidemeister torsion of an even dimensional manifold is zero. To prove this, recall that the Reidemeister torsion of a manifold is computed in terms of a triangulation of the manifold that can be obtained from a Morse function. We can compare the torsion of one cell decomposition to the torsion of the dual cell decomposition obtained by inverting the Morse function. The specific computation is due to Milnor [Mil62], and when the dimension is even it follows that the torsion must be zero. Stylistically this proof is the same as the proof of the vanishing result for the Euler characteristic: the formula for torsion in terms of the data of a triangulation is an instance of a Poincaré–Hopf theorem. The comparison to the dual cell complex is an instance of Poincaré duality. A nearly identical argument is also used to prove a duality theorem for Whitehead torsion [Mil66].

The Becker–Gottlieb transfer is a generalization of the Euler characteristic to families. For a smooth bundle of smooth manifolds $p : E \to B$, one can associate a wrong way map of spectra $\Sigma^\infty B_+ \to \Sigma^\infty E_+$. If the base is a point, this is equivalently a map of infinite loop spaces from S^0 to $\Omega^\infty \Sigma^\infty E_+$. On components, if E is connected, we have a map $S^0 \to \mathbb{Z}$, and the non-basepoint element maps to $\chi(E) \in \mathbb{Z}$.

One can easily prove a vanishing result for the Becker–Gottlieb transfer using a small amount of technology. Fiberwise Poincaré–Hopf theorems for the Becker–Gottlieb transfer have been proven by [BM76, Dou06]. Briefly, assume that X is a smooth nondegenerate vertical vector field on the total space of a smooth bundle $p : E \to B$. This vector field might be obtained by computing the gradient of a fiberwise Morse function, so long as such a function exists. Let Z be the vanishing locus of the vector field, which forms a covering space π over B. Then the fiberwise Poincaré–Hopf theorem is expressed in terms of the following homotopy commutative diagram of spectra:

$$\begin{tikzcd}
\Sigma^\infty B_+ \arrow[r, "\mathsf{tr}_p"] \arrow[d, "\mathsf{tr}_\pi"'] & \Sigma^\infty M_+ \\
\Sigma^\infty Z_+ \arrow[r, "\mathsf{Ind}_X"] & \Sigma^\infty Z_+ \arrow[u, "+"']
\end{tikzcd}$$

In the diagram above, tr_π and tr_p denote the transfers associated to π and p. The map Ind_X denotes a fiberwise index map associated to the vertical vector field X. On real cohomology one can easily prove from the definitions that $(\mathsf{Ind}_X)^* = (-1)^d (\mathsf{Ind}_{-X})^*$, where d denotes the fiber dimension. It follows that $(\mathsf{tr}_p)^* = (-1)^d (\mathsf{tr}_p)^*$. Thus the transfer map on cohomology vanishes when the fiber dimension is odd. In this case the classical Poincaré–Hopf theorem was replaced by a parametrized version, and Poincaré duality arose in the comparison of the vector field and its negative.

A common generalization of the Euler characteristic to both the K-theoretic and parametrized settings is the higher Franz–Reidemeister torsion. This invariant is a characteristic class in the cohomology of the base of a smooth fiber bundle. The primary tool that enables computations of this invariant is Igusa's framing principle [Igu05]. The framing principle describes the higher Franz–Reidemeister torsion as the sum of an 'exotic' class and a 'tangential' term

A consequence of the framing principle is that for smooth manifold bundles with closed even dimensional fibers, the torsion class is congruent to a Miller–Morita–Mumford class. To prove this, we write down the framing principle for a fiberwise generalized Morse function f and compare it to the analogous formula for $-f$. By studying the canonical involution on the Whitehead space, one can prove that the exotic term is two-torsion. Thus we are left only with the tangential term which agrees with a Miller–Morita–Mumford class.

Once again, this proof is analogous to those from above: the framing principle resembles a Poincaré–Hopf theorem, and the comparison of the formulas for f and $-f$ resembles an application of Poincaré duality. However, the proof of this vanishing theorem requires significantly more technology than those which came previously. In particular, to define the exotic term in the framing principle, Igusa uses a Waldhausen category model for the Whitehead space which encodes the combinatorics of colliding critical points of fiberwise generalized Morse functions. The proof of the framing principle requires an understanding of the deformation properties of the critical loci of fiberwise generalized Morse functions.

We postpone giving a precise definition of the smooth structure class until Section 3.5, however we point out that the smooth structure class is closely related to the invariants discussed above. One explanation for this is that the higher Franz–Reidemeister torsion and the smooth structure class can both be defined in terms of nullhomotopies of maps that factor through the Becker–Gottlieb transfer. An explicit relationship between these invariants at the level of homology groups of the base is proven in [GI14]: the pushdown class $p_*\Theta(M, M')$ is congruent to $D\tau^{IK}(M, M')$, the Poincaré dual of the relative higher torsion. Thus one should expect that a proof of the main theorem above should follow from a sufficiently general version of a Poincaré–Hopf theorem along with an application of Poincaré duality. This paper provides such a proof, which is summarized in the next section.

1.1. **Proof Summary.** In this paper we prove a vanishing result for the smooth structure class, an invariant of smooth structures on fiber bundles introduced by Goette, Igusa, and Williams in [GIW14], after work of Dwyer, Weiss, and Williams [DWW03]. In analogy with the examples above, the proof is an application of a Poincaré–Hopf type theorem in combination with Poincaré duality. In this section we give a precise outline of the proof, as well as a summary of the individual sections in this paper.

By a fiberwise Poincaré–Hopf theorem we broadly mean a computation of a fiberwise characteristic, e.g. the Becker–Gottlieb transfer, the excisive A-theory Euler characteristics, etc., in terms of the critical locus of a fiberwise generalized Morse function. Examples of such theorems can be found in [BM76, CJ98, Dou06]. These theorems generalize the classical Poincaré–Hopf theorem, which computes the Euler characteristic in terms of local data at the isolated critical points of a Morse function.

The following list is an itemized outline of the proof of the Main Theorem.

(0) The following recollections are necessary for this outline:

- The smooth structure class $\Theta(M, M')$ is an element of $\pi_0\Gamma_B\mathcal{H}^{\%}_B(M)\otimes\mathbb{Q}$. This is the space of sections of the fiberwise homology bundle obtained by taking fiberwise smash products with the stable h-cobordism space of a point. See Section 3.5 for a precise definition.

- By the stable parametrized h-cobordism theorem, $\Gamma_B \mathcal{H}_B^{\%}(M)$ is the homotopy fiber of the map $\Gamma_B Q_B(M) \to \Gamma_B A_B^{\%}(M)$, which is induced by the unit map from the sphere spectrum to $A(*)$. The spectrum $A(*)$ is the Waldhausen K-theory of spaces functor, otherwise known as A-theory, evaluated at a point. The functor $A^{\%}$ is the excisive approximation to A-theory.
- All smooth bundles admit fiberwise generalized Morse functions by [Igu84, **?**, EM12]. In stark contrast, smooth bundles rarely admit fiberwise Morse functions.

(1) We prove a fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer, an element of $\Gamma_B Q_B(M)$. This result, Theorem A, expresses the transfer in terms of a tubular neighborhood of the critical locus of a fiberwise generalized Morse function on M.

(2) We further generalize the fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer to the excisive A-theory characteristic, an element of $\Gamma_B A_B^{\%}(M)$. Furthermore, this factorization is compatible with the factorization of the Becker–Gottlieb transfer in the previous theorem. This is Theorem B.

(3) We generalize Theorem B for any *stratified deformation* of the critical locus of a fiberwise generalized Morse function. This is Theorem C.

(4) If two smooth bundles $M \times I^k$ and $M' \times I^k$ are fiberwise tangentially homeomorphic, then for large k, we can embed $M \times I^k$ in $M' \times I^k$ so that the complement of the image of $M \times I^k$ is an h-cobordism N between $\partial_0 N := \partial(M \times I^k)$ and $\partial_1 N := \partial(M' \times I^k)$. With this setup, $\chi^{\%}(N, \partial_0 N)$ and $\chi^{\%}(N, \partial_1 N)$ are nullhomotopic, so we can define the smooth structure characteristics $\theta^{\%}(N, \partial_0 N)$ and $\theta^{\%}(N, \partial_1 N)$.

(5) Theorem D gives a fiberwise Poincaré–Hopf theorem for the smooth structure characteristic. This theorem relies on Theorem C, in that the smooth structure class is expressed in terms of a particular stratified deformation of the critical locus of a fiberwise generalized Morse function. The stratified deformation that we use encodes a parametrized cancellation argument used by Hatcher in [Hat75] and Igusa in [Igu84, **?**, Igu02, Igu05]

(6) The fiberwise Poincaré–Hopf theorem can be used to prove a duality theorem for the smooth structure class, Theorem E, by inverting the Morse function. The Rigidity Conjecture follows from this duality theorem.

To complete the analogy in the exposition from the previous section, we will indicate how this proof can be thought of as an application of a Poincaré–Hopf type theorem and Poincaré duality. The Poincaré–Hopf type theorem that we ultimately apply is Theorem D, and as the outline indicates, this is a generalization of other Poincaré–Hopf theorems that we prove along the way. Poincaré duality appears in the proof of the duality theorem for the smooth structure class, Theorem E.

1.2. **Outline.** We now give a detailed outline of the contents of this paper.

Section 2 contains descriptions of related work pertaining to fiberwise Poincaré–Hopf theorems, the Hatcher construction, and exotic smoothings.

Section 3 contains precise definitions of the key invariants in this paper: higher torsion invariants (3.1) and the smooth structure class (3.4). Of crucial importance is the homotopy theoretic definition of the smooth structure class appearing in (3.5), which expresses the smooth structure class as the data of a lift of the excisive A-theory characteristic to the Becker–Gottlieb transfer.

Section 4 recalls definitions and key geometric results pertaining to fiberwise generalized Morse functions.

Section 5 contains the main technical results of this paper. These are the fiberwise Poincaré–Hopf theorems for the Becker–Gottlieb transfer, the excisive A-theory characteristic, and the smooth structure class. The first theorem, Theorem A in (5.1), is a computation of the Becker–Gottlieb transfer in terms of the critical locus of a fiberwise generalized Morse function. The second theorem, Theorem B in (5.2), generalizes Theorem A to compute the excisive A-theory Euler characteristic in terms of a fiberwise generalized Morse function. The third theorem, Theorem C in (5.3) generalizes Theorem B to stratified deformations of the critical locus of a generalized Morse functions.

Section 6 contains the main results for the smooth structure class. Each subsection relies on the statements proven in the preceding subsections. Section 6.1 gives the detailed geometric setup for the proof of the Rigidity Conjecture. Section 6.2 explains the stratified deformation used in subsequent sections. Section 6.3 contains the statement and proof of the fiberwise Poincaré–Hopf theorem for the smooth structure class. Section 6.4 contains the statement and proof of the duality theorem for the smooth structure class, Theorem E. Section 6.5 uses the duality theorem for the smooth structure class to prove the Rigidity Conjecture.

2. Related Work

Two main ideas arising in this paper, fiberwise Poincaré–Hopf theorems and the Hatcher construction, are of much independent interest in the literature. In this section we give a brief survey of the literature and recent progress pertaining to both keywords. We omit a discussion of the most immediate literature pertaining to the Rigidity Conjecture, including [DWW03, GIW14, GI14], as detailed descriptions of these works appear elsewhere in this paper.

Hatcher's construction associates to an element of the kernel of the J-homomorphism a disk bundle which is fiber homotopy trivial but not smoothly trivial. The construction should be interpreted as a stable map from G/O to $\Omega \mathrm{Wh}^{\mathrm{Diff}}(*)$. Waldhausen gave a different formulation of the same map in [Wal82], and in [Bök84] Bokstedt proved this map to be a rational homotopy equivalence. Later Igusa gave another proof using parametrized Morse theory. Exciting new developments by Kragh [Kra18] have identified the homotopy fiber of the Hatcher–Waldhausen map as a certain functional space $\mathcal{M}_\infty$ considered by Eliashberg and Gromov in [EG98], establishing a connection between the study of Lagrangians to algebraic K-theory of spaces. Kragh associates to every exact Lagrangian an element of $\pi_*(\mathcal{M}_\infty)$. If any of these examples were proven to be nontrivial, they would be counterexamples to the nearby Lagrangian conjecture in symplectic topology. In essence, counter examples to the nearby Lagrangian conjecture might be found in the kernel of Hatcher's construction. Recent work by Igusa and Alvarez-Gavela elaborates further on these developments [AGI19].

Goodwillie, Igusa, and Ohrt have developed an equivariant version of Hatcher's construction [GIO15]. Ordinarily, the space G/O classifies vector bundles whose spherical fibrations are fiber homotopy trivial. In the equivariant version, G/O is replaced by the space G_n/U, which is the colimit of spaces $G_n(N)/U(N)$, classifying rank N complex vector bundles together with a C_n-equivariant fiber homotopy trivialization of the associated sphere bundle. The equivariant Hatcher construction is then a map $G_n/U \to \mathcal{H}^s(BC_n)$, where the target is the stable h-cobordism space of the classifying space of C_n. The geometric outcome of the construction is no longer a disk bundle, but instead a bundle of h-cobordisms of the product of a disk with lens spaces.

Bunke and Gepner have reformulated the Becker–Gottlieb transfer in the context of derived algebraic K-theory [BG13]. Their work leads to the Transfer Index Conjecture, essentially a derived version of the parametrized index theorem of Dwyer, Weiss, and Williams. This conjecture suggests as a corollary the existence of certain classes in algebraic K-theory of a ring of integers in a number field. The authors prove that the Hatcher construction produces nontrivial representatives for these classes in special cases.

The Farell–Hsiang [FH78] results on diffeomorphism groups of disks relative to their boundary prove that in the pseudoisotopy stable range,

$$\pi_i \mathrm{BDiff}_\partial(D^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 3 \mod 4 \text{ and } n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

According to [Igu02], the nontrivial generator for odd dimensional disks can be obtained from Hatcher's construction. There has been much recent progress outside of the stable range by a number of authors, including distinct but related work by Kupers, Randal-Williams, Watanabe, and Weiss. In [Wei15] Weiss identifies nontrivial Pontryagin classes $p_{n+k} \in H^{4n+4k}(\mathrm{BTop}(2n); \mathbb{Q})$. These are shown to evaluate nontrivially on $\pi_{4n+4k}(\mathrm{BTop}(2n); \mathbb{Q})$. It then follows by the Morlet equivalence that there must be nonzero rational homotopy in $\pi_*(\mathrm{BDiff}_\partial(D^{2n}))$ outside of the stable range. One might reasonably suspect that these examples could produce unstable exotic smoothings of manifold bundles. However, stably these classes are trivial, so they do not provide counterexamples to the Rigidity Conjecture.

Fiberwise Poincaré–Hopf theorems first appeared in work by Brumfiel and Madsen [BM76], and have proven to be a useful computational tool with many applications. For instance, in [MT01] the authors then apply the theorem towards early progress on the Mumford conjecture. Similar theorems are used in [RW08] to compute the $\mod 2$ homology of the stable nonorientable mapping class group, as well as in [Rei19] to establish the existence of 'non-kinetic' smooth bundles over the classifying space $BSU(2)$. Douglas [Dou06] gave alternative proofs to [BM76] using Dold's Euclidean neighborhood rectracts. This paper establishes fiberwise Poincaré–Hopf theorems for smooth manifold bundles admitting fiberwise Morse functions, and provided motivation for the present work. Our results provide fiberwise Poincaré–Hopf theorems for bundles that admit fiberwise generalized Morse functions. This is a significant strengthening, as all bundles admit fiberwise generalized Morse functions, but bundles rarely admit fiberwise Morse functions.

This paper is concerned with a characteristic in the homotopy fiber of the fibration

$$\Gamma_B \mathcal{H}^{\%}_B(M) \to \Gamma_B Q_B(M) \to \Gamma_B A^{\%}_B(M)$$

that arises from a nullhomotopy of the excisive A-theory characteristic. One could naturally ask about characteristics in the homotopy fiber of the fibration

$$\Gamma_B \Omega \mathrm{Wh}^{\mathrm{PL}}_B(M) \to \Gamma_B A^{\%}_B(M) \to \Gamma_B A_B(M)$$

given by a nullhomotopy of the ordinary A-theory characteristic. This is the premise of Steimle's PhD thesis [Ste10], in which the author studies the 'parametrized excisive characteristic'. One of the main technical results of their work is an additivity theorem for the parametrized excisive characteristic, which parallels the Poincaré–Hopf theorems in the present work, if they were restricted to bundles admitting fiberwise Morse decompositions. For comparison, the smooth structure class appearing in $\Gamma_B \mathcal{H}^{\%}_B(M)$ concerns the existence of stable exotic smoothings of fiber bundles, whereas the parametrized excisive characteristic appearing in $\Gamma_B \Omega \mathrm{Wh}^{\mathrm{PL}}_B(M)$ concerns the existence of topological manifold bundles whose projection maps are homotopic to stabilizations of an arbitrary map of compact topological manifolds.

3. Key Invariants and Constructions

In this section we summarize the invariants and constructions that appear in this work. We begin by recalling two equivalent definitions of higher torsion resulting from work by Igusa and Klein, as well as Dwyer, Weiss, and Williams. Next we give the formal statement of the Rigidity conjecture using the definition of the smooth structure class from [GI14]. Then we state and prove an alternative homotopy theoretic definition of the smooth structure class.

3.1. **Higher Torsion Invariants.**

In this subsection we outline the main theories of higher torsion, emphasizing the role that generalized Morse functions play in computations of these invariants.

Igusa–Klein torsion is obtained from a smooth manifold bundle using Morse theory. Recall Waldhausen's theorem [Wal87] that $A(X)$ splits as the product $Q(X_+)\times \mathrm{Wh}^{\mathrm{Diff}}(X)$. Igusa–Klein torsion relies on a rational model for the loopspace of the smooth Whitehead space. This model, referred to as the Whitehead category, is constructed so that under sufficient conditions one can use the data of a fiberwise generalized Morse function to construct a map from the base to the Whitehead category [Igu02, Igu08, Ohr19]. Pulling back a particular rational characteristic class in the cohomology of the Whitehead space along this map gives the higher Franz–Reidemeister torsion of the bundle as a cohomology class in the base.

The model for the smooth Whitehead space is given in detail in [Igu02,Igu05]. We merely point out that the zero simplices are given by the fiberwise Morse complexes over points in the base, and the one simplices capture the combinatorics of the Morse complex as it passes through birth-death critical points. This construction relies heavily on the geometry of fiberwise generalized Morse functions, which we discuss at length in Section 3.

The smooth torsion of Dwyer, Weiss, and Williams is defined in purely homotopy theoretic terms, yet it surprisingly captures information about the smooth structure of the bundle. The starting point is the commutative diagram below, which was proven to commute for smooth manifold bundles in [DWW03]. In the diagram below, η is the unit map, α is the assembly map, and λ is the linearization functor.

$$\begin{tikzcd}
B \arrow[r, "\mathsf{tr}(p)"] \arrow[dr, "\chi^{\%}(p)"] \arrow[ddr, "\chi^{h}(p)"] \arrow[dddr] & Q(M_+) \arrow[d, "\eta"] \\
& A^{\%}(M) \arrow[d, "\alpha"] \\
& A(M) \arrow[d, "\lambda"] \\
& K(\mathbb{R})
\end{tikzcd}$$

For a smooth fiber bundle with acyclic fibers one can construct a nullhomotopy of the composition $\lambda\circ\alpha\circ\eta\circ\mathsf{tr}(p)$. This nullhomotopy provides a map to the homotopy fiber of the composition $Q(M_+)\to K(\mathbb{R})$, which is $\Omega\mathrm{Wh}^{\mathrm{Diff}}(M)$ by Waldhausen's theorem. Thus we have a lift

$$\begin{tikzcd}
 & \Omega \mathrm{Wh}^{\mathrm{Diff}}(M) \arrow[d] \\
B \arrow[ur, "\tau^s(p)"] \arrow[r, "\mathsf{tr}(p)"] \arrow[dr] & Q(M_+) \arrow[d, "\lambda\circ\alpha\circ\eta"] \\
 & K(\mathbb{R})
\end{tikzcd}$$

The top diagonal map is the smooth torsion. We produce a characteristic class in B by pulling back a cohomology class in $\Omega\mathrm{Wh}^{\mathrm{Diff}}(M)$ known as the universal Franz–Reidemeister torsion invariant. This class is constructed from the Kamber–Tondeur form in Volodin K-theory [Igu02].

In [BDW09] the authors explicitly construct smooth torsion for bundles with acyclic fibers as well as unipotent bundles by using 'partitions' from Waldhausen's manifold approach. The manifold approach gives an explicit model for $Q(X_+)$ and $A(X)$ and the authors carefully study linearization maps out of this model to algebraic K-theory, explicitly constructing the nullhomotopies in K-theory for bundles satisiying the assumptions. In [BDKW11] the authors go on to prove that the smooth torsion invariants satisfy Igusa's axioms for higher torsion [Igu08].

There is a third construction of higher torsion due to Bismut and Lott [BL95] using a purely analytic approach. The axiomatic approach by Igusa was the first attempt to unify the three definitions. Recently, Ohrt in [Ohr19] proved that the Igusa Klein torsion and the smooth torsion of Dwyer, Weiss, Williams, Badzioch, and Dorabiala agree. The main thrust of the proof is to use unpublished expansion category models of Igusa and Waldhausen to provide a unifying model in which one can compare the definitions of the two torsion invariants. The expansion categories are Waldhausen categories that model the smooth Whitehead space, $Q(X_+)$, and $A(X)$ and again rely heavily on combinatorial data obtained from generalized Morse functions.

As mentioned before, Igusa–Klein torsion is more computable than DWW torsion (of course, since these invariants are equivalent, computations of one translate to computations of the other). The key formula that is used to compute Igusa Klein torsion is the Framing Principle, Theorem 4.11 in [Igu08]. Loosely, it states that the higher torsion invariants can be computed in terms of fiberwise generalized Morse functions. In particular, it decomposes the higher Franz Reidemeister torsion class into two terms, the first is an algebraic torsion of a parametrized chain complex associated to the generalized Morse function, and the second is a Chern character of the complexification of the negative eigenspace bundle on the critical locus of the generalized Morse function. The framing principle can be used to prove that on even dimensional manifold bundles higher torsion is congruent to a Miller–Morita–Mumford class. The method of proof is to multiply the fiberwise generalized Morse function by -1. This has the effect of negating the algebraic term and leaving the Chern character the same. Adding together the formulas given by the framing principle for f and $-f$ yields the result. The idea behind the framing principle, that higher torsion invariants can be computed in terms of the data of the critical locus of a generalized Morse function, and the consequences for even dimensional manifold bundles are the main motivations for the techniques developed in this work.

3.2. **Stable Exotic Smooth Structures on Fiber Bundles.** In this section we define exotic smoothings and stable exotic smoothings of smooth manifold bundles. We also give examples of how such bundles can be obtained.

Definition 3.1. A *fiberwise tangential homeomorphism* between two smooth bundles $p : M \to B$ and $p' : M \to B$ is a fiberwise homeomorphism between M and M' that is covered by an isomorphism of the vertical tangent bundles $T^\vee M$ and $T^\vee M'$.

Remark 3.2. If M and M' have closed fibers, then M and M' are fiberwise diffeomorphic by smoothing theory. However, if the fibers of M and M' have boundary, then this is not the case.

Definition 3.3. Suppose that the fibers of M have boundary. Then an exotic smooth structure on M is another smooth bundle $p : M' \to B$ that is fiberwise tangentially homeomorphic to M.

The previous definition excludes the case when the fibers of M are closed because in this case exotic smooth structures on M do not exist. However, if M has closed fibers, then we can still consider exotic smoothings of stabilizations of M, such as $M \times I$. We call these stable exotic smoothings of M.

Definition 3.4. A stable exotic smoothing of a smooth bundle M is a bundle $W \to B$ that is fiberwise tangentially homeomorphic to $M \times I^{k+1}$ for some $k \geq 0$.

The following example can be used to produce a fiberwise tangential homeomorphism between two stabilized manifolds that are the boundaries of a parametrized h-cobordisms that is topologically trivial.

Example 3.5 (Example iii.a in 1.3.3. of [GIW14])**.** Let $\pi : W \to B$ satisfy the following:

- $\pi : W \to B$ is a smooth manifold bundle.
- $\psi : W \to B \times I$ is a topological manifold bundle.
- $\psi|_{B\times 0} : M_0 \to B$ is a smooth manifold bundle.
- $\psi|_{B\times 1} : M_1 \to B$ is a smooth manifold bundle.

From this data we will obtain a fiberwise tangential homeomorphism between $M_0 \times I \xrightarrow{\psi_0 \times \mathrm{id}_I} B$ and $M_1 \times I \xrightarrow{\psi_1 \times \mathrm{id}_I} B$, or equivalently, a stable exotic smoothing of M_0.

(1) Let M_t denote $\psi^{-1}(b, 0)$. Consider the fiberwise homeomorphism $W \to M_0 \times I$ over $B \times I$ given by the homeomorphism $f_t : M_0 \to M_t$.
(2) We will obtain a linearization of the once stabilized vertical topological tangent microbundle of M_t. This linearization is given by the restriction of the vertical tangent bundle of $\pi : W \to B$ to M_t. In the notation of [GIW14], this is a microbundle map
$$\mu_t : T^\vee W|_{M_t} \to E^\vee M_t \oplus \epsilon^1$$
(3) This provides a one parameter family of fiberwise linearized bundles. Explicitly the family is $M_t \times I$, and μ_t gives the linearization of each $M_t \times I$. Furthermore it is smooth when $t = 0, 1$. We do not have a linearization of the family M_t.
(4) Proposition 1.3.4 in [GIW14] proves that this determines a fiberwise tangential homeomorphism between $M_0 \times I$ and $M_1 \times I$ up to a contractible choice.

3.3. **Moduli Spaces of Exotic Smoothings on Fiber Bundles.** First, we will give an explicit construction of the unstable and stable spaces of exotic smoothings, $\tilde{\mathcal{S}}_B(M)$ and $\tilde{\mathcal{S}}^s_B(M)$ from [GIW14]. Then we will introduce an auxilliary space to which the stable space of exotic smoothings is homotopy equivalent. Finally, we will state the result from [DWW03] that this auxilliary space is homotopy equivalent to $\Gamma_B \mathcal{H}^{\%}_B(M)$. Thus we will recover the main smoothing theorem of [GIW14], which states that $\tilde{\mathcal{S}}^s_B(M) \simeq \Gamma_B \mathcal{H}^{\%}_B(M)$.

Let $\mathcal{S}^t_\bullet(n)$ denote the simplicial set of n-dimensional compact topological manifolds. We can identify the geometric realization of this simplicial set as a more familiar space $|\mathcal{S}^t_\bullet(n)| \simeq \coprod_M \mathrm{BHomeo}(M)$, where the disjoint union varies over all homeomorphism classes of compact n-dimensional manifolds M. Likewise, define the simplicial set $\mathcal{S}^d_\bullet(n)$ of n-dimensional compact smooth manifolds. Again we have an identification with a more familiar space: $|\mathcal{S}^d_\bullet(n)| \simeq \coprod_M \mathrm{BDiff}(M)$ where the disjoint union varies over all diffeomorphism classes of n-dimensional compact smooth manifolds. We also have an intermediate moduli space $\tilde{\mathcal{S}}^t_\bullet(n)$ which consists of those compact topological n manifolds with a vector bundle structure on their topological tangent microbundle. The spaces $|\mathcal{S}^d_\bullet(n)|$, $|\widetilde{\mathcal{S}}^t_\bullet(n)|$, and $|\mathcal{S}^t_\bullet(n)|$ respectively classify: smooth manifold bundles with compact fibers of dimension n, fiber bundles of compact topological manifolds of dimension n along with fiberwise linearizations of the vertical tangent microbundles, and fiber bundles of compact topological manifolds of dimension n. We can consider the space of maps from a compact base B into each classifying space, which are related by forgetful functors:

$$|\mathcal{S}^d_\bullet(n)|^B \to |\widetilde{\mathcal{S}}^t_\bullet(n)|^B \to |\mathcal{S}^t_\bullet(n)|^B$$

Then for a smooth manifold bundle $p : M \to B$ we define $\widetilde{\mathcal{S}}_B(M)$ to be the homotopy fiber of $|\mathcal{S}^d_\bullet(n)|^B \to |\widetilde{\mathcal{S}}^t_\bullet(n)|^B$ over the point $M \in |\widetilde{\mathcal{S}}^t_\bullet(n)|^B$. By definition, a point in $\widetilde{\mathcal{S}}_B(M)$ is a one parameter family of fiberwise linearized topological manifold bundles that is smooth at both endpoints. By Proposition 1.3.4 in [GIW14], this data is equivalent to a fiberwise tangential homeomorphism between the smooth endpoints. Thus, $\widetilde{\mathcal{S}}_B(M)$ contains those smooth manifold bundles $M' \to B$ that are fiberwise tangentially homeomorphic to M.

Remark 3.6. When the fibers of M are closed, the unstable space $\widetilde{\mathcal{S}}_B(M)$ is contractible by smoothing theory. However, when the fibers have boundary this space is not contractible. For instance, $\widetilde{\mathcal{S}}_B(M \times I)$ is not contractible.

Definition 3.7. The stable space $\widetilde{\mathcal{S}}^s_B(M)$ is the direct limit of $\widetilde{\mathcal{S}}_B(M)$ with respect to linear disk bundles.

Remark 3.8. If M is a smooth bundle with closed fibers, then stable exotic smoothings of M are by definition elements of $\widetilde{\mathcal{S}}_B(M \times I^{k+1})$ for some $k \geq 0$, and are thus elements of $\widetilde{\mathcal{S}}^s_B(M)$.

Example 3.9. In 3.5 we obtained a fiberwise tangential homeomorphism between $M_0 \times I$ and $M_1 \times I$, both bundles with closed fibers. Thus, $M_1 \times I$ is a point in $\widetilde{\mathcal{S}}_B(M \times I)$, and therefore in $\widetilde{\mathcal{S}}^s_B(M \times I)$. However, M_1 is not a point in the unstable space $\widetilde{\mathcal{S}}_B(M)$, which is contractible.

Next we will identify the homotopy type of $\widetilde{\mathcal{S}}^s_B(M)$. We first define an auxilliary space. Consider again the smooth manifold bundle p, and denote the vertical tangent bundle by γ. Let $\mathcal{S}^d(p, \gamma)$ denote the subspace of $|\mathcal{S}^d_\bullet(n)|^B$ consisting of those smooth manifold bundles that are fiberwise tangentially homeomorphic to (p, γ). Let $\tilde{\mathcal{S}}^t(p, \gamma)$ denote the subspace of $|\tilde{\mathcal{S}}^t_\bullet(n)|^B$ consisting of those topological manifold bundles with fiberwise linearizations of their topological tangent microbundles that are fiberwise tangentially homeomorphic to (p, γ). Then the following diagram commutes:

$$\begin{tikzcd}
{|\mathcal{S}^d_\bullet(n)|^B} \arrow[r] & {|\tilde{\mathcal{S}}^t_\bullet(n)|^B} \arrow[r] & {|\mathcal{S}^t_\bullet(n)|^B} \\
\mathcal{S}^d_n(p,\gamma) \arrow[r] \arrow[u, "inc."] & \tilde{\mathcal{S}}^t_n(p,\gamma) \arrow[r] \arrow[u, "inc."] & \star \arrow[u]
\end{tikzcd}$$

It is clear that the homotopy fiber of the map $\mathcal{S}^d_n(p,\gamma) \xrightarrow{inc.} \tilde{\mathcal{S}}^t_n(p,\gamma)$ over the point p is homotopy equivalent to the homotopy fiber of $|\mathcal{S}^d_\bullet(n)|^B \to |\tilde{\mathcal{S}}^t_\bullet(n)|^B$ over the point p, thus the stabilizations of the two spaces must agree. Denote the stabilization of the map $\mathcal{S}^d_n(p,\gamma) \to \mathcal{S}^t_n(p,\gamma)$ by $\mathcal{S}^d(p,\gamma) \to \mathcal{S}^t(p,\gamma)$, and denote the stable homotopy fiber at the point p by $\tilde{\mathcal{S}}(p,\gamma)$. Then we are left with a homotopy equivalence

$$\tilde{\mathcal{S}}(p,\gamma) \xrightarrow{\simeq} \tilde{\mathcal{S}}^s_B(M)$$

Now we explain why $\tilde{\mathcal{S}}(p,\gamma)$ is homotopy equivalent to $\Gamma_B \mathcal{H}^{\%}_B(M)$. The vertical arrows in the diagram below compute characteristics on each stable moduli space. On smooth bundles one computes the Becker–Gottlieb transfer. On fiberwise linearized topological bundles one computes the excisive A-theory characteristic. Both of these characteristics are refinements of the fiberwise A-theory characteristic of p, $\chi^h(p) \in \Gamma_B A_B(E)$, so the diagram below commutes by the main theorems of [DWW03]

$$\begin{tikzcd}
\mathcal{S}^d(p,\gamma) \arrow[r] \arrow[d, "\mathsf{tr}(-)"] & \tilde{\mathcal{S}}^t(p,\gamma) \arrow[r] \arrow[d, "\chi^{\%}(-)"] & \star \arrow[d, "\chi^h(p)"] \\
\Gamma_B Q_B(E_+) \arrow[r] & \Gamma_B A^{\%}_B(E) \arrow[r] & \Gamma_B A_B(E)
\end{tikzcd}$$

The following theorem is a stronger statement.

Theorem (Theorem 12.8 in [DWW03])**.** *The two squares and thus the outer rectangle in diagram (6) are homotopy cartesian.*

The homotopy fiber of the top row of the left square is $\tilde{\mathcal{S}}(p,\gamma)$ and the homotopy fiber of the bottom row is $\Gamma_B \mathcal{H}^{\%}_B(M)$. The map $\tilde{\mathcal{S}}(p,\gamma) \to \Gamma_B \mathcal{H}^{\%}_B(M)$ is the induced map on homotopy fibers, and it is an equivalence because the left square is homotopy cartesian. Combining this with the homotopy equivalence $\tilde{\mathcal{S}}(p,\gamma) \xrightarrow{\simeq} \tilde{\mathcal{S}}^s_B(M)$ yields a homotopy equivalence $\tilde{\mathcal{S}}^s_B(M) \simeq \Gamma_B \mathcal{H}^{\%}_B(M)$. We have proven the following theorem.

Theorem (Theorem 1.5.14 in [GIW14])**.** *For M a smooth manifold bundle with compact fibers there is a homotopy equivalence*

$$\tilde{\mathcal{S}}^s_B(M) \simeq \Gamma_B \mathcal{H}^{\%}_B(M) \tag{2}$$

3.4. **Statement of the Rigidity Conjecture.** In this section we summarize the results of [GIW14, GI14], and state the Rigidity Conjecture, lending special attention to the crucial role played by the higher Franz–Reidemeister torsion invariants of Igusa and Klein and the Hatcher construction. We begin by stating precisely what we mean by 'exotic smoothing'.

Definition 3.10. Suppose that $p : M \to B$ is a smooth manifold bundle with fiber F, where F, M, and B are compact smooth manifolds. Let $T^\vee M$ be the vertical tangent bundle of M. An *exotic smooth structure* on M is another smooth bundle $M' \to B$ with a fiberwise tangential homeomorphism $h : M \to M'$. This means that h is a fiberwise homeomorphism covered by an isomorphism of vector bundles $T^\vee M \cong T^\vee M'$.

Let $\tilde{\mathcal{S}}_B(M)$ denote the space of exotic smoothings of M. Let $\tilde{\mathcal{S}}_B^s(M)$ be the stabilization obtained by taking a direct limit with respect to all linear disk bundles of M (see Section 3.5 or [GIW14] for details). Recall the stable space of h-cobordisms on X, denoted $\mathcal{H}(X)$. Finally, let $\Gamma_B \mathcal{H}_B^\%(M)$ denote the space of sections of the fiberwise homology bundle obtained by applying fiberwise smash with $\mathcal{H}(*)$. Then there is a weak equivalence which follows from the results of [DWW03] and is proven independently in [GIW14]. They go on to compute $\pi_0 \Gamma_B \mathcal{H}_B^\%(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{\dim B - 4k}(M;\mathbb{R})$ using the fact that the rational homotopy type of $\mathcal{H}(*)$ is a wedge of Eilenberg-Maclane spaces.

Definition 3.11. Let M be fiberwise tangentially homeomorphic to M' so that $[M'] \in \tilde{\mathcal{S}}_B(M)$. Then let the smooth structure class $\Theta(M, M')$ be the image of $[M']$ under the composition

$$\pi_0 \tilde{\mathcal{S}}_B(M) \to \pi_0 \tilde{\mathcal{S}}_B^s(M) \otimes \mathbb{R} \cong \pi_0 \Gamma_B \mathcal{H}_B^\%(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{\dim B - 4k}(M;\mathbb{R})$$

This definition makes use of the equivalence in (2), but does not explicitly make use of the map. In Section 3.5 we will give an alternative proof of (2). One outcome of this proof will be a detailed description of the map. This will be used to provide a homotopy theoretic refinement of the smooth structure class.

The smooth structure class is proven in [GI14] to satisfy the following duality relationship with the relative higher Franz–Reidemeister torsion class of Igusa and Klein, which is a characteristic class in the cohomology of B.

$$p_* \Theta(M', M) = D\tau^{IK}(M', M). \tag{3}$$

The relation above allows one to compute the relative smooth structure class in situations where torsion is computable and p_* is trivial. One such instance is the Hatcher construction, which produces exotic smoothings of odd dimensional disk bundles that have nontrivial higher torsion invariants [BDKW11, BG13, Igu02, Igu05]. By gluing the exotic disk bundles from Hatcher's example into smooth manifold bundles, one can produce exotic smoothings that vary the relative torsion class, and thus vary the relative structure class. This construction is called the immersed Hatcher construction, and it leads to the following theorem, which is a reformulation of Theorem 3.1.1 in [GI14].

Theorem (Main Theorem of [GI14])**.** *Rationally and stably the immersed Hatcher construction gives all possible exotic smooth structures on smooth manifold bundles with closed odd dimensional fibers.*

When the smooth bundle has closed even dimensional fibers, the immersed Hatcher construction does not produce exotic smooth structures. If the vertical boundary is nonempty, then the most that can be done is to produce an exotic smooth structure on the vertical boundary. Furthermore, on an even dimensional manifold bundle, the higher torsion classes are known to be tangential, in particular they are congruent to constant multiples of Miller–Morita–Mumford classes of the vertical tangent bundle [Igu05]. This implies that the relative torsion class $\tau^{IK}(M', M)$ is zero since M' is fiberwise tangentially homeomorphic to M. Thus, $p_*\Theta(M', M)$ is also trivial. This is evidence for the following Rigidity Conjecture.

Conjecture 3.12 (Rigidity Conjecture of Goette and Igusa – 0.3.3 in [GI14])**.** *The stable smooth structure class vanishes when the fiber is a closed oriented even dimensional manifold:*

$$\Theta(M', M) = 0$$

In other words, rationally and stably, there are no exotic smooth structures on manifold bundles with closed oriented even dimensional fibers.

Our main theorem, Theorem F, asserts that this conjecture is true.

3.5. **Homotopical Definition of the Smooth Structure Class.** In this section we give a homotopy theoretic definition of the smooth structure class which refines the homology class $\Theta(M, M')$ from Definition 3.11.

Definition 3.13. Suppose that $p : M \to B$ and $p' : M' \to B$ are fiberwise tangentially homeomorphic smooth manifold bundles with compact fibers so that there is a nullhomotopy of the difference $\chi^{\%}(p) - \chi^{\%}(p')$. This nullhomotopy gives a lift as in the following diagram:

$$\begin{tikzcd}
 & \mathcal{H}^{\%}(M) \arrow[d] \\
 & Q(M_+) \arrow[d, "\eta"] \\
B \arrow[ruu, "\theta(M,M')"] \arrow[ru, "\mathsf{tr}(p)-\mathsf{tr}(p')"'] \arrow[r, "\chi^{\%}(p)-\chi^{\%}(p')"'] & A^{\%}(M)
\end{tikzcd}$$

This lift we denote by $\theta(M, M')$, and refer to as the homotopical smooth structure class.

Proposition 3.14. The definitions of the smooth structure class given in Definition 3.11 and Definition 3.13 agree. That is, the image of $\theta(M, M')$ under the map $\Gamma_B\mathcal{H}^{\%}_B(M) \to \pi_0\Gamma_B\mathcal{H}^{\%}_B(M)\otimes\mathbb{R}$ is congruent to $\Theta(M, M')$.

The proof of this proposition will be given at the end of this section. First, we will give an explicit construction of the unstable and stable spaces of exotic smoothings, $\tilde{\mathcal{S}}_B(M)$ and $\tilde{\mathcal{S}}^s_B(M)$ from [GIW14]. Then we will observe that the stable space of exotic smoothings is homotopy equivalent to an auxilliary space. Finally, we will state the result from [DWW03] that this auxilliary space is homotopy equivalent to $\Gamma_B\mathcal{H}^{\%}_B(M)$. There are two consequences of this result: a proof of (2) from Section 3.4, and a proof of Proposition 3.14.

Let $\mathcal{S}^t_\bullet(n)$ denote the simplicial set of n-dimensional compact topological manifolds. We can identify the geometric realization of this simplicial set as a more familiar space $|\mathcal{S}^t_\bullet(n)| \simeq \coprod_M \mathrm{BHomeo}(M)$, where the disjoint union varies over all homeomorphism classes of compact n-dimensional manifolds M. Likewise, define the simplicial set $\mathcal{S}^d_\bullet(n)$ of n-dimensional compact smooth manifolds. Again we have an identification with a more familiar space: $|\mathcal{S}^d_\bullet(n)| \simeq \coprod_M \mathrm{BDiff}(M)$ where the disjoint union varies over all diffeomorphism classes of n-dimensional compact smooth manifolds. We also have an intermediate moduli space $\tilde{\mathcal{S}}^t_\bullet(n)$ which consists of those compact topological n manifolds with a vector bundle structure on their topological tangent microbundle. The spaces $|\mathcal{S}^d_\bullet(n)|$, $|\tilde{\mathcal{S}}^t_\bullet(n)|$, and $|\mathcal{S}^t_\bullet(n)|$ respectively classify: smooth manifold bundles with compact fibers of dimension n, fiber bundles of compact topological manifolds of dimension n along with fiberwise linearizations of the vertical tangent microbundles, and fiber bundles of compact topological manifolds of dimension n. We can consider the space of maps from a compact base B into each classifying space, which are related by forgetful functors:

$$|\mathcal{S}^d_\bullet(n)|^B \to |\tilde{\mathcal{S}}^t_\bullet(n)|^B \to |\mathcal{S}^t_\bullet(n)|^B$$

Then for a smooth manifold bundle $p : M \to B$ we define $\tilde{\mathcal{S}}_B(M)$ to be the homotopy fiber of $|\mathcal{S}^d_\bullet(n)|^B \to |\tilde{\mathcal{S}}^t_\bullet(n)|^B$ over the point $M \in |\tilde{\mathcal{S}}^t_\bullet(n)|^B$. The stable space $\tilde{\mathcal{S}}^s_B(M)$ is the direct limit of $\tilde{\mathcal{S}}_B(M)$ with respect to linear disk bundles.

Now we define an auxilliary space. Consider again the smooth manifold bundle p, and denote the vertical tangent bundle by γ. Let $\mathcal{S}^d(p,\gamma)$ denote the subspace of $|\mathcal{S}^d_\bullet(n)|^B$ consisting of those smooth manifold bundles that are fiberwise tangentially homeomorphic to (p,γ). Let $\tilde{\mathcal{S}}^t(p,\gamma)$ denote the subspace of $|\tilde{\mathcal{S}}^t_\bullet(n)|^B$ consisting of those topological manifold bundles with fiberwise linearizations of their topological tangent microbundles that are fiberwise tangentially homeomorphic to (p,γ). Then the following diagram commutes:

$$\begin{tikzcd}
|\mathcal{S}^d_\bullet(n)|^B \arrow[r] & |\tilde{\mathcal{S}}^t_\bullet(n)|^B \arrow[r] & |\mathcal{S}^t_\bullet(n)|^B \\
\mathcal{S}^d_n(p,\gamma) \arrow[r] \arrow[u, "inc."] & \tilde{\mathcal{S}}^t_n(p,\gamma) \arrow[r] \arrow[u, "inc."] & \star \arrow[u]
\end{tikzcd}$$

It is clear that the homotopy fiber of the map $\mathcal{S}^d_n(p,\gamma) \xrightarrow{inc.} \tilde{\mathcal{S}}^t_n(p,\gamma)$ over the point p is homotopy equivalent to the homotopy fiber of $|\mathcal{S}^d_\bullet(n)|^B \to |\tilde{\mathcal{S}}^t_\bullet(n)|^B$ over the point p, thus the stabilizations of the two spaces must agree. Denote the stabilization of the map $\mathcal{S}^d_n(p,\gamma) \to \mathcal{S}^t_n(p,\gamma)$ by $\mathcal{S}^d(p,\gamma) \to \mathcal{S}^t(p,\gamma)$, and denote the stable homotopy fiber at the point p by $\tilde{\mathcal{S}}(p,\gamma)$. Then we are left with a homotopy equivalence

$$\tilde{\mathcal{S}}(p,\gamma) \xrightarrow{\simeq} \tilde{\mathcal{S}}^s_B(M) \tag{4}$$

Now we explain why $\tilde{\mathcal{S}}(p,\gamma)$ is homotopy equivalent to $\Gamma_B\mathcal{H}^{\%}_B(M)$. The vertical arrows in the diagram below compute characteristics on each stable moduli space. On smooth bundles one computes the Becker–Gottlieb transfer. On fiberwise linearized topological bundles one computes the excisive A-theory characteristic. Both of these characteristics are refinements of the fiberwise A-theory characteristic of p, $\chi^h(p) \in \Gamma_B A_B(E)$, so the diagram below commutes by the main theorems of [DWW03]

$$\begin{tikzcd}
\mathcal{S}^d(p,\gamma) \arrow[r] \arrow[d, "\mathsf{tr}(-)"] & \tilde{\mathcal{S}}^t(p,\gamma) \arrow[r] \arrow[d, "\chi^{\%}(-)"] & \star \arrow[d, "\chi^h(p)"] \\
\Gamma_B Q_B(E_+) \arrow[r] & \Gamma_B A^{\%}_B(E) \arrow[r] & \Gamma_B A_B(E)
\end{tikzcd} \tag{5}$$

However, the following theorem is a stronger statement.

Theorem (Theorem 12.8 in [DWW03]). *The two squares and thus the outer rectangle in diagram (6) are homotopy cartesian.*

The homotopy fiber of the top row of the left square is $\tilde{\mathcal{S}}(p,\gamma)$ and the homotopy fiber of the bottom row is $\Gamma_B\mathcal{H}^{\%}_B(M)$. The map $\tilde{\mathcal{S}}(p,\gamma) \to \Gamma_B\mathcal{H}^{\%}_B(M)$ is the induced map on homotopy fibers, and it is an equivalence because the left square is homotopy cartesian. Combining this with the homotopy equivalence $\tilde{\mathcal{S}}(p,\gamma) \xrightarrow{\simeq} \tilde{\mathcal{S}}^s_B(M)$ yields the homotopy equivalence $\tilde{\mathcal{S}}^s_B(M) \simeq \Gamma_B\mathcal{H}^{\%}_B(M)$ as in (2).

Proof of Proposition 3.14. From the discussion above we have a description of the map $\tilde{\mathcal{S}}^s_B(M) \xrightarrow{\simeq} \Gamma_B \mathcal{H}^{\%}_B(M)$ as the induced map on the homotopy fibers in following diagram.

(6)
$$\begin{tikzcd}
\tilde{\mathcal{S}}^s_B(M) \arrow[r] \arrow[d, "\simeq"] & \mathcal{S}^d(p,\gamma) \arrow[r] \arrow[d, "\mathsf{tr}(-)"] & \tilde{\mathcal{S}}^t(p,\gamma) \arrow[d, "\chi^{\%}(-)"] & p' \arrow[l, hook'] \arrow[d, mapsto] \\
\Gamma_B \mathcal{H}^{\%}_B(E) \arrow[r] & \Gamma_B Q_B(E_+) \arrow[r] & \Gamma_B A^{\%}_B(E) & \chi^{\%}(p') \arrow[l, hook']
\end{tikzcd}$$

From this it is clear that the component of $\Theta(M, M') \in \pi_0 \Gamma_B \mathcal{H}^{\%}_B(M)$ is the data of a nullhomotopy of $\chi^{\%}(p) - \chi^{\%}(p')$, as in Definition 3.13. $\square$

4. Review of Generalized Morse Functions

In this section we will prove that the critical loci of fiberwise generalized Morse functions have an essential transversality property. More specifically, the gradient vector field is transverse to the zero section of the vertical tangent bundle. While this condition is well known and frequently used for fiberwise Morse functions, it leads to surprising and substantial results for fiberwise generalized Morse functions. In particular, it implies that the critical set of a fiberwise generalized Morse function is a smooth manifold.

Let $f : (E, \partial_0 E) \to (I, 0)$ be a fiberwise generalized Morse function. Such functions always exist on smooth fiber bundles when the dimension of the fiber is at least the dimension of the base [Igu90]. Let $\Sigma(f)$ be the singular set of f. We must describe the local behavior of this function, i.e. we need generalizations of the Morse lemma. We begin by giving the local behavior in a neighborhood of a compact subset of the collection of Morse singularities as well as in a neighborhood of a compact subset of the collection of birth-death singularities.

In the unparametrized setting, we consider a function f on a single fiber. A Morse singularity of the function f can be written in the form

$$f(x) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

with respect to local coordinates $(x_1, \cdots, x_n) \in \mathbb{R}^i \times \mathbb{R}^{n-i}$. At a birth-death point,

$$f(x) = -x_1^2 - \cdots - x_{i-1}^2 + x_i^3 + x_{i+1}^2 + \cdots + x_n^2$$

In the parametrized setting, we have the following proposition/summary from [Igu08]:

Proposition 4.1. In a generic p-parameter family of generalized Morse functions, birth-death points occur on a codimension one subspace of the parameter space. The family of functions f_t has the form

$$f_t(x) = -x_1^2 - \cdots - x_{i-1}^2 + x_i^3 + t_0 x_i + x_{i+1}^2 + \cdots + x_n^2$$

with respect to parameter coordinates $t_0, \cdots t_{p-1}$ and t-dependent local coordinates $(x_1, \cdots, x_n)$ for M.

Let $p : E \to B^q$ be a smooth fiber bundle with compact base and fiber F^n. Let $f : E \to \mathbb{R}$ be a fiberwise generalized Morse function. Then by the proposition above, in local coordinates at a birth-death singularity we have

$$f_t(x) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^3 + t_0 x_{i+1} + x_{i+2}^2 + \cdots + x_n^2.$$

where t_0 is the unfolding direction. The gradient of this function is a section $E \to TE$, and if we take the gradient with respect to fiber coordinates, we get a section of the vertical tangent bundle, $E \to T^\vee E$ (which could have also been obtained by projecting off of TE). We can explicitly compute the map $\nabla f : E \to T^\vee E$ as

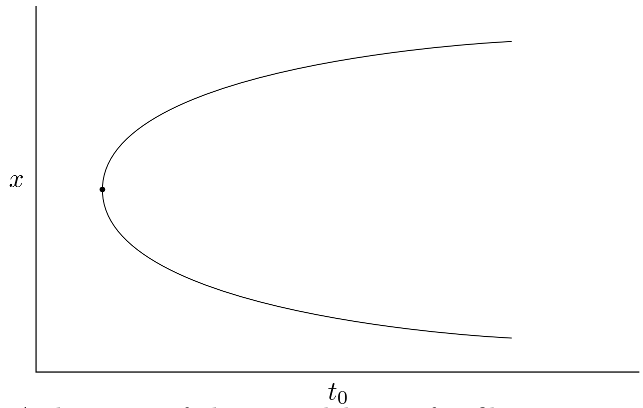


FIGURE 1. A depiction of the critical locus of a fiberwise generalized Morse function in the local neighborhood of a birth-death singularity.

$$(x,t) \mapsto \langle -2x_1, \cdots, -2x_i, 3x_{i+1}^2 + t_0, 2x_{i+2}, \cdots, 2x_n \rangle$$

Now the derivative of ∇f is a map on tangent spaces which takes the form of a rectangular matrix of size $(2n+q) \times (n+q)$. We write this matrix below.

$$T_{(x,t)}E \mapsto T_{\nabla f(x,t)}(T^{\vee}E) \cong T^{\vee}E \oplus T^{\vee}E \to T^{\vee}E$$

The last map in this composition is the projection off of the nonidentity component. The tangent space in the domain is labeled using coordinates $t_0, \cdots, t_{q-1}$ in the base, and $x_1, \cdots, x_n$ in the fiber. In the target we add labels $\frac{\partial}{\partial x_i}$ for $i, \cdots, n$ for the coordinates in the vertical tangent direction. Keep in mind that in the neighborhood of a birth-death singularity, t_0 is always identified with the 'unfolding' direction.

$$
\begin{array}{cc}
 & \overbrace{\hphantom{1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1}}^{x}\qquad \overbrace{\hphantom{1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1}}^{t} \\
\begin{array}{c}
x\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right. \\
t\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right. \\
\frac{\partial}{\partial x}\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right.
\end{array}
&
\begin{pmatrix}
1 & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & \\
 & & & & & 1 & & & & & & & & \\
 & & & & & & 1 & & & & & & & \\
 & & & & & & & 1 & & & & & & \\
 & & & & & & & & 1 & & & & & \\
 & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & 1 \\
-2 & & & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & & & \\
 & & -2 & & & & & & & & & & & \\
 & & & 6x_{i+1} & & & & 1 & & & & & & \\
 & & & & 2 & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & \\
 & & & & & & 2 & & & & & & &
\end{pmatrix} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\underbrace{\hphantom{1}}_{t_0}
\end{array}
$$

We have a smooth map $\nabla f : E \to T^\vee E$, and we can check to see whether the image of this map is transverse to the inclusion of the zero section of $T^\vee E$, $i_0 : E \to T^\vee E$. If p is in $\nabla f(E) \cap i_0 E$, then $\nabla f(E)$ is transverse to $i_0 E$ if, for all a, b, p so that $\nabla f(a) = i_0(b) = p$,

$$\operatorname{Im}(D(\nabla f)(a)) \oplus \operatorname{Im}(D(i_0)(b)) \twoheadrightarrow T_p(T^\vee E).$$

It is clear that the intersection $p \in \nabla f(E) \cap i_0 E$ is the set of critical points of f, and as these points are either Morse critical points, or birth-death singularities, we handle each of these cases independently. In the event that p is a Morse critical point, it is a standard exercise that the map above is surjective. The case of a birth-death singularity is identical, except in the row labelled by $\frac{\partial}{\partial x_{i+1}}$. At the birth-death singularity, the entry $6x_{i+1}$ vanishes, and if it were not for the 1 in the t_0 entry of the row, there would be no image in the 1-dimensional subspace spanned by $\frac{\partial}{\partial x_{i+1}}$ of the matrix above. So we do have surjectivity and thus transversality, but only because of the derivative in the unfolding direction t_0. So transversality is a direct consequence of the unfolding behavior of a parametrized family of generalized Morse functions. We summarize this discussion in the following lemma:

Lemma 4.2. For $f : E \to \mathbb{R}$ a fiberwise generalized Morse function on a smooth fiber bundle $E \to B$ with compact base, the section $\nabla f : E \to T^\vee E$ is transverse to the zero section of the vertical tangent bundle of E.

Corollary 4.3. *The normal bundle $\nu(\Sigma(f)) \xrightarrow{\pi} \Sigma(f)$ to the embedding $\Sigma(f) \hookrightarrow E$ is isomorphic to the restriction of the vertical tangent bundle of E to $\Sigma(f)$, $T^\vee E|_{\Sigma(f)}$.*

4.0.1. *Ghost sets.* The image of the birth-death set in B, $p(\Sigma_1^n(f))$, is known as the bifurcation set. The bifurcation set is a codimension zero submanifold of B. In the neighborhood of a birth-death singularity, the points given by $x_i = 0$ and $-\epsilon < t_0 < 0$ are inflection points on which the second derivative of f in the vertical direction vanishes. We call these points ghost points, and they allow us to define a ghost set, $\Sigma_g^n(f)$, which is formally a lift of a one-sided collar neighborhood of the bifurcation set. The ghost set is transverse to $\Sigma(f)$, as in the following figure.

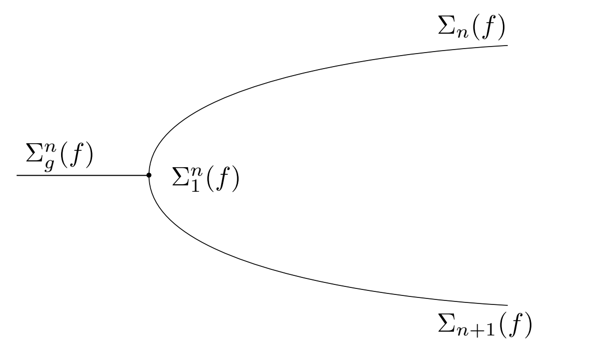


Figure 2. The ghost set is transversally attached to the critical locus at the birth-death set.

The ghost set is used to locally perturb $\Sigma(f)$ so that the critical points of the generalized Morse function do not cancel. In particular, we consider the manifolds with corners $\Sigma^n(f) \cup \Sigma_g^n(f)$ and $\Sigma^{n+1}(f) \cup \Sigma_g^n(f)$. Smoothing both of these manifolds gives us a smooth critical locus over a small simplex that intersects the bifurcation set. This construction of the ghost set is used to prove the framing principle [Igu08]. In this work the ghost set is used to conclude that locally in the neighborhood of a birth-death singularity, certain index maps of the gradient vector field are trivial.

5. Fiberwise Poinaré–Hopf Theorems

In this section we prove three fiberwise Poincaré–Hopf theorems. Each theorem expresses a characteristic in terms of the critical locus of a fiberwise generalized Morse function and an index map defined using the gradient vector field of the function. First we prove a fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer. In the second section we extend this theorem to a fiberwise Poincaré–Hopf theorem for the excisive A-theory characteristic. This is done by defining an index map for $A^{\%}$-theory that is compatible with the index map defined using the smooth vector field. By the smooth parametrized h-cobordism theorem [WJR13], we have the homotopy equivalence $\Omega \mathrm{Wh}^{\mathrm{Diff}}(*) \simeq \mathcal{H}(*)$, where the latter space is the stable h-cobordism space of a point. So the homotopy fiber of the map $\Gamma_B \Sigma_B^\infty(M_+) \to \Gamma_B A_B^{\%}(M)$ is homotopy equivalent to $\Gamma_B \mathcal{H}_B^{\%}(M)$. Thus a Poincaré–Hopf theorem for the smooth structure class in $\Gamma_B \mathcal{H}_B^{\%}(M)$ can be obtained from the previous theorems. This is done precisely in the third section.

For the remainder of this section, we assume that the bundle $p : M \to B$ admits a fiberwise generalized Morse function. Such a function can always be obtained by [Igu90, Lur09, EM12].

5.1. **Fiberwise Poincaré–Hopf Theorem for the Becker–Gottlieb transfer.** There are several preliminaries before the statement and proof of Theorem A at the end of this section. We begin by reviewing the notation we will use for the fiberwise generalized Morse function and the associated data. This allows us to display the structure of our fiberwise Poincaré–Hopf theorems in diagram (7) below.

Let $p : M^k \to B^q$ be a smooth manifold bundle with closed base and compact fiber F^{k-q} possibly with boundary. In the constructions that follows we will assume M is embedded in $B \times \mathbb{R}^n$ with normal bundle ν^{n+q-k} and vertical tangent bundle τ^{k-q}. We will also assume B is embedded in $\mathbb{R}^\ell$.

The function $f : M \to \mathbb{R}$ is a fiberwise generalized Morse function and we let the vector field X be the vertical gradient of f. The critical locus $\Sigma(f)$ is a smooth submanifold of M that is a union of closed subsets

$$\Sigma(f) = \bigcup \overline{\Sigma}^n(f)$$

which intersect along the birth-death sets

$$\Sigma_1^n(f) = \overline{\Sigma}^n(f) \cap \overline{\Sigma}^{n+1}(f).$$

We denote by $\mathscr{S}(f)$ the set of strata of $\Sigma(f)$. Elements in this set are q-manifolds $\Sigma^n(f)$ with boundary $\Sigma_1^n(f)$ consisting of the index n critical points and birth-death singularities, respectively. We let Z represent a generic stratum of $\mathscr{S}(f)$, and we let $\mathring{Z}$ denote the interior of Z, $\Sigma^n(f) - \Sigma_1^n(f)$. Then the tubular neighborhood of $\mathring{Z}$ in M is a bundle $\pi : D\mathring{Z} \to \mathring{Z}$ where DZ is an N-disk bundle on $\mathring{Z}$ with compact fibers.

Our goal is to prove that the following diagram commutes. The precise formulation of this diagram is given in the statement of Theorem A.

(7)
$$\begin{tikzcd}
\mathbb{S} \arrow[rr, "\overline{\mathrm{tr}}(p)"] \arrow[d, "\overline{\mathrm{tr}}(\pi)"'] & & \Gamma_B \Sigma^\infty_B (M_+) \\
\bigvee_{Z \in \mathscr{S}(f)} \Gamma_Z \Sigma^\infty_Z (DZ)_+ \arrow[rr, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{Ind}}_Z"] & & \bigvee_{Z \in \mathscr{S}(f)} \Gamma_Z \Sigma^\infty_Z (DZ)_+ \arrow[u, "+"']
\end{tikzcd}$$

To prove that diagram (7) commutes we must introduce several new definitions, which we enumerate below.

(1) We first present the section spaces appearing in the diagram as homotopy limits in the base of homotopy colimits in the fiber. Definitions 5.1 and 5.2 indicate the diagram categories over which these limits and colimits are indexed. The categories we use are similar to the indexing categories for factorization homology and cohomology as in [AF15, AF19], though our categories contain only disks of cardinality one.
(2) We use this presentation to give a refined version of the Becker–Gottlieb transfer in Definition 5.4.
(3) Next we define our index map using the gradient vector field, and use it to define a refined version of the vector field transfer appearing in [BG91]. These are Definitions 5.6 and 5.7.
(4) Finally, the first step in the proof of Theorem A is to prove that the Becker–Gottlieb transfer is homotopic to the vector field transfer when there is no vertical boundary, or when the vector field points outwards along the vertical boundary. This is proven in Proposition 5.9 after introducing an auxilliary term for the vector field transfer on the boundary in Definition 5.8.
(5) The remainder of the proof of Theorem A produces a factorization of the vector field transfer in terms of the index map.

In our diagram categories we choose to work with one point compactifications of open embeddings of q-disks in B and codimension-zero k-disks in M properly embedded along the fibers.

Definition 5.1. The objects of the category $\mathsf{Disk}_q^{B/}$ are one point compactifications of q-disks with one component embedded in B. The morphisms are one point compactifications of open embeddings.

Definition 5.2. An object in $\mathsf{Disk}_{k/p^{-1}U}$ is an embedding of $\mathbb{R}^q \times D^{k-q}$ into $p^{-1}U$ so that p embeds $\mathbb{R}^q \times \{0\}$ into B, and for each point x in $\mathbb{R}^q \times 0$, $\{x\} \times D^{k-q}$ is a disk with boundary embedded in the fiber $M_{p(x)}$. The morphisms in this category are inclusions.

Proposition 5.3. There are weak equivalences

$$\Gamma_B \Sigma^\infty_B M \simeq \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \Sigma^\infty (p^{-1}(U)^+) \simeq \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+$$

Proof. The section space on the left would ordinarily be defined using the category of simplices in B, $\mathsf{Sing}(B)$. The first equivalence follows from the fact that the functor $\mathrm{ev}_0 : \mathsf{Disk}_q^{B/} \to \mathsf{Sing}(B)$, which sends a disk to the zero simplex given by the point at the origin, is initial. This in turn is true because the functor $\mathsf{Disk}_q^{B/} \to \mathscr{D}\mathsf{isk}_q^{B/}$ is a localization, and the functor $\mathrm{ev}_0 : \mathscr{D}\mathsf{isk}_q^{B/} \to \mathsf{Sing}(B)$

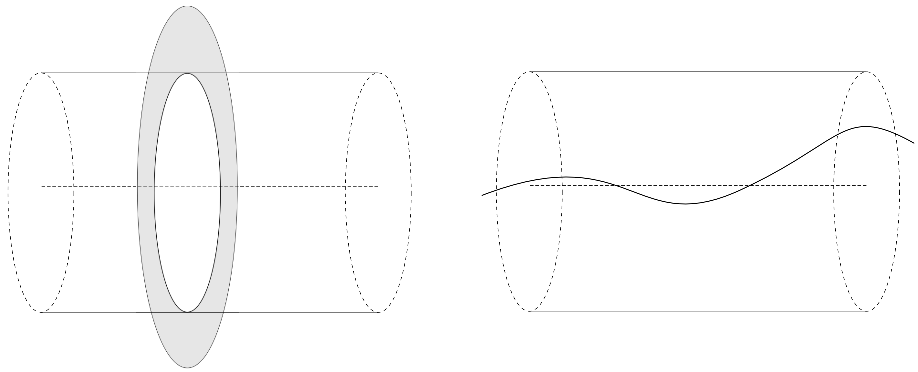


FIGURE 3. The disks in $\mathsf{Disk}_{k/p^{-1}U}$ are embedded so that the image in a particular fiber is a properly embedded disk, but the projection onto B is open. On the left is depicted an object in the category with the annulus represents a local neighborhood in a particular fiber. On the right is depicted a possible configuration of the disk relative to a q-dimensional submanifold of M on which the projection to B is an embedding. In practics this submanifold may belong to the set of critical points of a fiberwise Morse function.

is an equivalence of ∞-categories. Thus the composition is a localization, and is therefore both initial and final.

The second equivalence requires us to prove that

$$\operatorname*{\mathsf{hocolim}}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \simeq \Sigma^\infty(p^{-1}(U)^+)$$

For this we use a hypercover argument as in [DI04]. It suffices to show that for each point $x \in p^{-1}(U)$, the full subcategory of $\mathsf{Disk}_{k/p^{-1}U}$ whose objects contain x has a contractible classifying space. This subcategory is cofiltered, so it is contractible.

□

We will now construct a characteristic

$$\mathbb{S} \to \operatorname*{\mathsf{holim}}_{U \in \mathsf{Disk}_q^{B/}} \Sigma^\infty(p^{-1}(U)^+) \tag{8}$$

Temporarily let F denote the functor

$$U \mapsto \Sigma^\infty(p^{-1}(U)^+) \simeq \operatorname*{\mathsf{hocolim}}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+$$

The characteristic map is to the outer homotopy limit

$$\mathbb{S} \to \operatorname*{\mathsf{holim}}_{U \in \mathsf{Disk}_q^{B/}} F(U)$$

This map is assembled from local characteristic morphisms $\mathbb{S} \to F(U)$.

We construct the maps $\mathbb{S} \to F(U)$ explicitly using the Pontryagin–Thom collapse map. Then the map (8) is obtained by extending the functor F to the left cone $\mathsf{Disk}_q^{B/\triangleleft}$. To do this we must construct maps $S \to F(U)$ for all U that are compatible with the morphisms in $\mathsf{Disk}_q^{B/}$.

Let M_U denote $p^{-1}(U)$. M_U is open, but each fiber M_x over $x \in U$ is compact, possibly with boundary $\partial^\vee M_U$. The bundles ν and τ are understood to be restricted along the open embedding $M_U \hookrightarrow M$. Consider the composition of the Pontryagin-Thom collapse with the inclusion below.

$$(U \times R^n)^+ \to M_U^\nu/\partial^\vee M_U^\nu \to M_U^{\nu\oplus\tau} \simeq M_U^+ \wedge S^n \tag{9}$$

Let U' contain U. Then there is an open embedding $M_U \hookrightarrow M_{U'}$. We get compatibility of the associated compositions as in the following diagram, with vertical maps obtained by one-point compactifying open embeddings.

$$\begin{tikzcd}
(U' \times R^n)^+ \arrow[r] \arrow[d] & M_{U'}^\nu/\partial^\vee M_U^\nu \arrow[r] \arrow[d] & M_{U'}^{\nu\oplus\tau} \arrow[r, "\simeq"] \arrow[d] & M_{U'}^+ \wedge S^n \arrow[d] \\
(U \times R^n)^+ \arrow[r] & M_U^\nu/\partial^\vee M_U^\nu \arrow[r] & M_U^{\nu\oplus\tau} \arrow[r, "\simeq"] & M_U^+ \wedge S^n
\end{tikzcd} \tag{10}$$

Stabilize (9) to obtain a map $\Sigma^\infty U^+ \to \Sigma^\infty M_U^+$. Stabilize (10) to get a stable diagram

$$\begin{tikzcd}
\Sigma^\infty U'^+ \arrow[r] \arrow[d] & \Sigma^\infty M_{U'}^+ \arrow[d] \\
\Sigma^\infty U^+ \arrow[r] & \Sigma^\infty M_U^+
\end{tikzcd}$$

Recall that we have also chosen an embedding $B^q \hookrightarrow \mathbb{R}^\ell$ which induces an open embedding $U \times \mathbb{R}^n \hookrightarrow B^q \times \mathbb{R}^n \hookrightarrow \mathbb{R}^\ell \times \mathbb{R}^n$. One point compactifying gives a map $S^{\ell+n} \to (U \times \mathbb{R}^n)^+$. Doing the same for U' and stabilizing, we can extend our stable compatibility diagram to the left.

$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d, "="] & \Sigma^\infty U'^+ \arrow[r] \arrow[d] & \Sigma^\infty M_{U'}^+ \arrow[d] \\
\mathbb{S} \arrow[r] & \Sigma^\infty U^+ \arrow[r] & \Sigma^\infty M_U^+
\end{tikzcd}$$

With the construction above we have produced an extension of the functor F on the category $\mathsf{Disk}_q^{B/}$ to a functor $\widetilde{F}$ on the category $\mathsf{Disk}_q^{B/\triangleleft}$ so that $\widetilde{F}(*) = \mathbb{S}$. Thus we have the morphism $\mathbb{S} \to \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} F(U)$ as desired.

Definition 5.4. The morphism (8), as constructed in the discussion above, is the refined transfer denoted by $\overline{\mathsf{tr}}(p)$. The morphism is constructed by extending the functor $U \mapsto \Sigma^\infty(p^{-1}(U)^+)$ to the left cone $\mathsf{Disk}_q^{B/\triangleleft}$ via the characteristic maps given in (9) and (10).

Proposition 5.5. The Becker–Gottlieb transfer $\mathsf{tr}(p)$ is equivalent to $\overline{\mathsf{tr}}(p)$. In particular the following diagram is homotopy commutative.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\mathsf{tr}(p)"] \arrow[dr, "\overline{\mathsf{tr}}(p)"'] & \Gamma_B \Sigma_B^\infty(M)_+ \arrow[d, "\simeq"] \\
& \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \Sigma^\infty(p^{-1}(U)^+)
\end{tikzcd}$$

Our next task is to define an index map and use it to define a vector field transfer, $\overline{\mathsf{tr}}_X(p)$. Fix $U \in \mathsf{Disk}_q^{B/}$ and consider the equivalence $\Sigma^\infty(p^{-1}(U)^+) \simeq \underset{V \in \mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}} \Sigma^\infty(-)_+$. For each $V \in \mathsf{Disk}_{k/p^{-1}U}$, and point (a_y, b_y) in $\nu \oplus \tau|_{y \in V}$ we have a well defined map $\mathsf{Ind}_X(V)$.

$$S^n \simeq \mathrm{Th}(\nu \oplus \tau|_V) \to \mathrm{Th}(\nu \oplus \tau|_V) \simeq S^n \tag{11}$$

which sends (a_y, b_y) to $(a_y, b_y + X(y))$.

Since it is defined pointwise this map is compatible with the embeddings $V \hookrightarrow V'$ comprising the morphisms in $\mathsf{Disk}_{k/p^{-1}U}$. Stably this gives us a natural transformation from the functor $\Sigma^\infty(-)_+$ to itself. This natural transformation induces a map

$$\underset{V \in \mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}} \Sigma^\infty(-)_+ \to \underset{V \in \mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}} \Sigma^\infty(-)_+ \tag{12}$$

which in turn induces a map

$$\underset{U \in \mathsf{Disk}_q^{B/}}{\mathsf{holim}} \underset{V \in \mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}} \Sigma^\infty(-)_+ \xrightarrow{\mathsf{Ind}_X} \underset{U \in \mathsf{Disk}_q^{B/}}{\mathsf{holim}} \underset{V \in \mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}} \Sigma^\infty(-)_+ \tag{13}$$

Definition 5.6. We denote by $\mathsf{Ind}_X(M)$ the index map (13) associated to the vertical vector field X on the bundle M.

Definition 5.7. Let the vector field transfer, denoted $\overline{\mathsf{tr}}_X(p)$, be the composition of (8) and (13):

$$\mathbb{S} \to \underset{U \in \mathsf{Disk}_q^{B/}}{\mathsf{holim}} \Sigma^\infty(f^{-1}(U)^+) \simeq \underset{U \in \mathsf{Disk}_q^{B/}}{\mathsf{holim}} \underset{V \in \mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}} \Sigma^\infty(-)_+ \xrightarrow{\mathsf{Ind}_X(M)} \underset{U \in \mathsf{Disk}_q^{B/}}{\mathsf{holim}} \underset{V \in \mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}} \Sigma^\infty(-)_+$$

We compare our refined vector field transfer $\overline{\mathsf{tr}}_X(p)$ to the vector field transfer of Becker and Gottlieb [BG91], obtained by stabilizing the composition

$$B_+ \wedge S^n \to \mathrm{Th}(\nu|_M)/\mathrm{Th}(\nu|_{\partial^\vee M}) \to \mathrm{Th}(\nu \oplus \tau|_M) \simeq M_+ \wedge S^n$$

wherein the intermediate morphism $\mathrm{Th}(\nu|_M)/\mathrm{Th}(\nu|_{\partial^\vee M}) \to \mathrm{Th}(\nu \oplus \tau|_M)$ sends a_y to $a \oplus X(y)$. Note that for this to be well defined, X must be nonzero and of length 1 on $\partial^\vee M$. It is evident that our refined vector field transfer $\overline{\mathsf{tr}}_X(p)$ is homotopic to that of Becker and Gottlieb. In particular, the difference between the standard transfer of Becker and Gottlieb and their vector field transfer is equivalent to the difference in our refined versions. This difference is an error term computed on the boundary $\partial^\vee M$. For the following definition let $\dot{p}$ denote the bundle formed from the vertical boundary of M over B.

Definition 5.8. Let X_∂ be vertical vector field defined on an open set of $\partial^\vee M$ formed by taking those vectors pointing inside M and projecting them onto $\tau|_{\partial^\vee M}$. Let W be a manifold with boundary containing all zeros of X_∂. Define $\mathsf{tr}_{X_\partial}(\dot{p})$ to be the stabilization of the composition

$$B_+ \wedge S^n \to \mathrm{Th}(\nu|_W)/\mathrm{Th}(\nu|_{\partial^\vee W}) \to \mathrm{Th}(\nu \oplus \tau|_W) \simeq W^+ \wedge S^n$$

where $\mathrm{Th}(\nu|_W)/\mathrm{Th}(\nu|_{\partial^\vee W}) \to \mathrm{Th}(\nu \oplus \tau|_W)$ sends a_y to $a \oplus X_\partial(y)$.

Theorem 7.1 in [BG91] states that $\mathsf{tr}(p) = \mathsf{tr}_X(p) + \mathsf{tr}_{X_\partial}(\dot{p})$ and thus implies the following proposition.

Proposition 5.9. The difference $\overline{\mathsf{tr}}(p) - \overline{\mathsf{tr}}_X(p)$ is homotopic to $\mathsf{tr}_{X_\partial}(\dot{p})$. In particular, if fibers of p are closed, or if X points outwards on all of $\partial^\vee M$, then $\overline{\mathsf{tr}}(p) \sim \overline{\mathsf{tr}}_X(p)$.

We are now ready to state and prove a fiberwise Poincaré–Hopf theorem.

Theorem A. *The following diagram of spectra is homotopy commutative.*

$$
\begin{tikzcd}
\mathbb{S} \arrow[rr, "\overline{\mathsf{tr}}(p)"] \arrow[d, "\overline{\mathsf{tr}}(\pi)"'] & & \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \\
\bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-)_+ \arrow[r, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{Ind}}_X(D\mathring{Z})"] & \bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-)_+ \arrow[ur, "+"'] &
\end{tikzcd}
$$

Proof of Theorem A. Our proof of Theorem A follows by proving that the three subdiagrams of the diagram below commute. The top portion of the diagram with the curved arrow commutes by Proposition 5.9. We will prove the commutativity of the bottom triangle first and the middle triangle second.

$$
\begin{tikzcd}
\mathbb{S} \arrow[rr, bend left, "\overline{\mathsf{tr}}(p)"] \arrow[rr, "\overline{\mathsf{tr}}_X(p)"] \arrow[drr, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{tr}}_X(\pi_{\mathring{Z}})"] \arrow[d, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{tr}}(\pi_{\mathring{Z}})"'] & & \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \\
\bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-)_+ \arrow[rr, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{Ind}}_X(D\mathring{Z})"] & & \bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-)_+ \arrow[u, "inc."']
\end{tikzcd}
\tag{14}
$$

The bottom triangle commutes essentially by the definition of the vector field transfer. We explain the notation. Recall that $\mathring{Z}$ is the interior of the stratum Z. There is a bundle $\pi : D\mathring{Z} \to \mathring{Z}$. The vertical vector field X on M restricts to the vertical vector field X on $D\mathring{Z}$. Thus on each stratum Z we apply the definition of the vector field transfer to see that $\overline{\mathsf{tr}}_X(\pi_{\mathring{Z}})$ factors as $\overline{\mathsf{Ind}}_X(D\mathring{Z}) \circ \overline{\mathsf{tr}}(\pi_{\mathring{Z}})$. This identity applies to each wedge summand, and thus the bottom triangle commutes.

The main idea of the proof of the middle triangle is that the vector field transfer on p factors through the vector field transfer on the tubular neighborhood of the critical locus.

For $Z \in \mathscr{S}(f)$, and $U \in \mathsf{Disk}_q^{\mathring{Z}/}$, let F_Z denote the functor $U \mapsto \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-)_+$. Consider the functor $\rho : \mathsf{Disk}_q^{\mathring{Z}/} \to \mathsf{Disk}_q^{B/}$ given by the projection p, which is an embedding on Z. We can right Kan extend to get a functor $\mathsf{RKan}_\rho(F_Z)$ on $\mathsf{Disk}_q^{B/}$ so that

$$
\operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-)_+ \simeq \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} F_Z(U) \simeq \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \mathsf{RKan}_\rho(F_Z)(U) \tag{15}
$$

Fix $U \in \mathsf{Disk}_q^{B/}$ and consider the map obtained by precomposing (12) with the transfer on U:

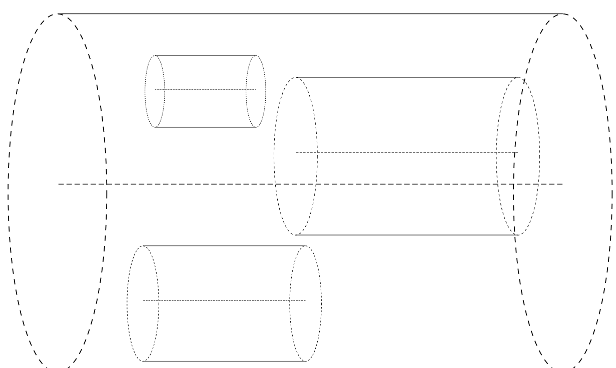
$$\Sigma^\infty U^+ \to \Sigma^\infty p^{-1}(U)^+ \simeq \operatorname*{\mathsf{hocolim}}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \to \operatorname*{\mathsf{hocolim}}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+$$

We will prove that this map factors as in the diagram below.

$$\begin{tikzcd}
\Sigma^\infty U^+ \arrow[r] \arrow[drr] & \Sigma^\infty p^{-1}(U)^+ \arrow[r, "\simeq"] & \operatorname*{\mathsf{hocolim}}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \arrow[r] & \operatorname*{\mathsf{hocolim}}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \\
 & & \bigvee_{Z \in \mathscr{S}(f)} \mathsf{RKan}_\rho(F_Z)(U) \arrow[ur, "inc."] &
\end{tikzcd} \tag{16}$$

The proof of the middle triangle in diagram (14) follows from the commutativity of the triangle above.

If $U \subset p(Z)$ then since p is an embedding when restricted to Z, $p|_Z^{-1}U$ is an open disk in Z, and $\pi^{-1}(p|_Z^{-1}U)$ is the tubular neighborhood on Z restricted to U. Then we identify $\mathsf{RKan}_\rho(F_Z)(U) \simeq \Sigma^\infty(\pi^{-1}(p|_Z^{-1}U))^+$. Using disk categories, this is expressed the homotopy colimit over the full subcategory of $\mathsf{Disk}_{k/p^{-1}U}$ containing exactly those disks which are properly embedded within the restriction of the tubular neighborhood of Z to $(p|_Z)^{-1}(U)$. We will not introduce new notation for this subcategory, but instead direct the reader to the picture below, which indicates how the objects in the subcategory are situated in the tubular neighborhood of Z.



Having identified $\mathsf{RKan}_\rho(F_Z)(U)$, we will now factor the composition along the top of diagram (16). Consider the local index $\mathsf{Ind}_X(V)$ on a properly embedded disk V as in (11). We will prove two assertions about the local index on V.

(1) If V does not intersect the critical locus $\Sigma(f)$ then $\mathsf{Ind}_X(V)$ is nullhomotopic.
(2) If V intersects the birth-death locus $\Sigma_1(f)$ then $\mathsf{Ind}_X(V)$ is nullhomotopic.

For now we proceed under the assumption that both statements above are true. It follows that the composition along the top of diagram (16) factors through a colimit over the subcategory of disks that intersect $\Sigma(f)$ and do not intersect $\Sigma^1(f)$. This is identical to our description of the right Kan extension appearing in the diagram. Now taking homotopy limits over $\mathsf{Disk}_q^{B/}$, extending to the left cone as before, and finally applying (15) gives us the middle triangle in diagram (14).

It remains only to prove assertions (1) and (2) above. Assertion (1) is obvious, as the vector field X is nonvanishing on the interior of V, and thus the index map is nullhomotopic. Assertion

(2) is a consequence of the existence of 'ghost sets' from Section 4.3 in [Igu08]. Recall the discussion in Section 4.0.1. In a neighborhood of the birth-death set the critical locus can be extended transversally to the birth death set in the unfolding direction, forming a 'ghost'. Thus instead of cancelling the critical points at the birth-death submanifold, we have two critical points on each fiber within a neighborhood of the birth-death submanifold. These critical points have opposite index, thus on this local disk the index is nullhomotopic.

□

Remark 5.10. We wish to point out that a simpler proof of Theorem A can be carried out using pure differential topology by understanding the local behavior in the neighborhood of birth-death singularities and then proceeding as in [BM76], [Dou06], or [CJ98]. However, we choose to give a proof using limits and colimits over disk categories because this facillitates further generalizations such as Theorem B.

5.2. **Fiberwise Poincaré–Hopf for the excisive A-theory characteristic.** In this section we extend the factorization of the Becker–Gottlieb transfer in Theorem A to a factorization of the excisive A-theory Euler characteristic. This requires more definitions analogous to those of the previous section. We enumerate them here.

(1) We first define a refined excisive A-theory characteristic using the definition of $\overline{\mathsf{tr}}_X(p)$ from the previous section as well as the unit map $\mathbb{S} \to A(*)$. This is Definition 5.11. Proposition 5.12 asserts that this characteristic is equivalent to the excisive A-theory characteristic of [DWW03].
(2) Next we define a version of the index map for $A^{\%}$-theory. This is defined using the smooth index map from before, as well as the unit map $\eta : \mathbb{S} \to A(*)$, and its section trace : $A(*) \to \mathbb{S}$. Proposition 5.14 proves that the $A^{\%}$-theory index map is compatible with the smooth index map.
(3) The proof of Theorem B follows from these two propositions and formal arguments.

Recall the excisive A-theory characteristic [DWW03] associated to a topological manifold bundle $p : M \to B$:

$$\mathbb{S} \xrightarrow{\chi^{\%}(p)} \Gamma_B A^{\%}_B(M)$$

If the bundle $p : M \to B$ is smooth or at least homotopy equivalent to a smooth fiber bundle, they prove that $\chi^{\%}(p)$ factors through the Becker–Gottlieb transfer $\mathsf{tr}(p)$ (in fact, they prove an if and only if). In particular, understanding η_* to be induced by the unit map $\eta : \mathbb{S} \to A(*)$, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\mathsf{tr}(p)"] \arrow[dr, "\chi^{\%}(p)"'] & \Gamma_B \Sigma^{\infty}_B(M) \arrow[d, "\eta_*"] \\
& \Gamma_B A^{\%}_B(M)
\end{tikzcd}$$

Since we are working with smooth manifold bundles, our refinement of the Becker–Gottlieb transfer leads to a refinement of the excisive A-theory characteristic.

Definition 5.11. We define the refined excisive A-theory characteristic $\overline{\chi}^{\%}(p)$ to be the composition

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\overline{\mathsf{tr}}(p)"] \arrow[dr, "\overline{\chi}^{\%}(p)"'] & \underset{U\in\mathsf{Disk}_q^{B/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}}\ \Sigma^\infty(-)_+ \arrow[d, "\eta_*"] \\
& \underset{U\in\mathsf{Disk}_q^{B/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_{k/p^{-1}U}}{\mathsf{hocolim}}\ A^{\%}(-)
\end{tikzcd}$$

Proposition 5.12. The refined excisive A-theory characteristic is homotopic to the excisive A-theory characteristic $\chi^{\%}(p)$.

We use the index map (13) to define an index map for $A^{\%}$. Recall the unstable local index map $\mathsf{Ind}_X(V)$ given in (11). We define a local index map on $V_+ \wedge A(*)$ as

$$V_+ \wedge A(*) \xrightarrow{\mathsf{Ind}_X(V)\wedge \mathrm{id}} V_+ \wedge A(*)$$

This induces a global index map for $A^{\%}$ on $D\overset{\circ}{Z}$:

$$\underset{U\in\mathsf{Disk}_q^{\overset{\circ}{Z}/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_{k/\pi^{-1}U}}{\mathsf{hocolim}}\ A^{\%}(-) \xrightarrow{\mathsf{Ind}^h_X(D\overset{\circ}{Z})} \underset{U\in\mathsf{Disk}_q^{\overset{\circ}{Z}/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_{k/\pi^{-1}U}}{\mathsf{hocolim}}\ A^{\%}(-) \tag{17}$$

Definition 5.13. We denote by $\mathsf{Ind}^h_X(D\overset{\circ}{Z})$ the index map (17) associated to the vertical vector field X on the bundle $D\overset{\circ}{Z}$.

Proposition 5.14. The index maps $\mathsf{Ind}_X(D\overset{\circ}{Z})$ and $\mathsf{Ind}^h_X(D\overset{\circ}{Z})$ are compatible as in the diagram below

$$\begin{tikzcd}
\underset{U\in\mathsf{Disk}_q^{\overset{\circ}{Z}/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_{k/\pi^{-1}U}}{\mathsf{hocolim}}\ \Sigma^\infty(-) \arrow[r, "\mathsf{Ind}_X(D\overset{\circ}{Z})"] \arrow[d, "\eta_*"] & \underset{U\in\mathsf{Disk}_q^{\overset{\circ}{Z}/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_{k/\pi^{-1}U}}{\mathsf{hocolim}}\ \Sigma^\infty(-) \arrow[d, "\eta_*"] \\
\underset{U\in\mathsf{Disk}_q^{\overset{\circ}{Z}/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_{k/\pi^{-1}U}}{\mathsf{hocolim}}\ A^{\%}(-) \arrow[r, "\mathsf{Ind}^h_X(D\overset{\circ}{Z})"] & \underset{U\in\mathsf{Disk}_q^{\overset{\circ}{Z}/}}{\mathsf{holim}}\ \underset{V\in\mathsf{Disk}_{k/\pi^{-1}U}}{\mathsf{hocolim}}\ A^{\%}(-)
\end{tikzcd}$$

Proof. It suffices to prove that the two index maps are compatible on a single disk V. This is easy to verify since the index map for $A^{\%}$ is given by the identity on the $A(*)$ component of the smash product. □

Theorem B. *The following diagram of spectra is homotopy commutative.*

$$\begin{tikzcd}[column sep=small]
& \mathbb{S} \arrow[rr, "\overline{\mathsf{tr}}(p)"] \arrow[dl, "\overline{\mathsf{tr}}(\pi)"'] \arrow[dd] & & \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \arrow[dd] \\
\bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-) \arrow[rr, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{Ind}}_X(D\mathring{Z})"'] \arrow[dd] & & \bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-) \arrow[ur, "+"] \arrow[dd] & \\
& \mathbb{S} \arrow[rr, "\overline{\chi^{\%}}(p)"] \arrow[dl, "\overline{\chi^{\%}}(\pi)"'] & & \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/p^{-1}U}} A^{\%}(-) \\
\bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} A^{\%}(-) \arrow[rr, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{Ind}}^h_X(D\mathring{Z})"'] & & \bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} A^{\%}(-) \arrow[ur, "+"] &
\end{tikzcd}$$

Proof of Theorem B. The top square commutes by Theorem A. The left vertical side and the back vertical side commute by Proposition 5.12. The right vertical side is self-evident. The front vertical side commutes by Proposition 5.14. Thus the five sides excluding the base commute. This is enough information to prove the homotopy commutativity of the bottom square.

We have shown that the boundary of the cube commutes. It remains to prove that the entire cube commutes. To prove this, we break the cube into two pieces which we handle separately. To describe the first piece, we must define an auxilliary vector field transfer for $A^{\%}$-theory, analogous to the vector field transfer $\overline{\mathsf{tr}}_X(p)$ which appeared in the proof of A. This definition is chosen so that the following diagram commutes. In particular, $\overline{\chi^{\%}}_X(p) := \eta \circ \overline{\mathsf{tr}}_X(p)$.

$$\begin{tikzcd}[column sep=small]
& \mathbb{S} \arrow[rr, dashed, "\overline{\mathsf{tr}}_X(p)"] \arrow[dl, "\overline{\mathsf{tr}}(\pi)"'] \arrow[dd] & & \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \arrow[dd] \\
\bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-) \arrow[rr, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{Ind}}_X(D\mathring{Z})"'] \arrow[dd] & & \bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} \Sigma^\infty(-) \arrow[ur, "+"] \arrow[dd] & \\
& \mathbb{S} \arrow[rr, dashed, "\overline{\chi^{\%}}_X(p)"] \arrow[dl, "\overline{\chi^{\%}}(\pi)"'] & & \operatorname*{holim}_{U \in \mathsf{Disk}_q^{B/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/p^{-1}U}} A^{\%}(-) \\
\bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} A^{\%}(-) \arrow[rr, "\bigvee_{Z \in \mathscr{S}(f)} \overline{\mathsf{Ind}}^h_X(D\mathring{Z})"'] & & \bigvee_{Z \in \mathscr{S}(f)} \operatorname*{holim}_{U \in \mathsf{Disk}_q^{\mathring{Z}/}} \operatorname*{hocolim}_{V \in \mathsf{Disk}_{k/\pi^{-1}U}} A^{\%}(-) \arrow[ur, "+"] &
\end{tikzcd}$$

To obtain the cube above, it suffices to see that for a pair (U, V), the following diagram of characteristics and vector field transfers commutes.

$$\begin{tikzcd}
\mathbb{S} \arrow[rr, "\mathsf{tr}_X(V)"] \arrow[dr, "\mathsf{tr}(V)"] \arrow[dd] & & \Sigma^\infty(V)_+ \arrow[dd, "\eta"] \\
& \Sigma^\infty(V)_+ \arrow[ur, "\mathsf{Ind}_X(V)"] \arrow[dd, "\eta"] & \\
\mathbb{S} \arrow[rr, "\chi^{\%}_X(V)"] \arrow[dr, "\chi^{\%}(V)"] & & A^{\%}(V) \\
& A^{\%}(V) \arrow[ur, "\mathsf{Ind}^h_X(V)"] &
\end{tikzcd}$$

The top face commutes as in the proof of Theorem A. The vertical left side commutes by [DWW03]. The vertical back side commutes by the definition of $\chi^{\%}_X(p)$. The right vertical side commutes by the definition of the index map Ind^h. It follows that the triangular prism commutes, and thus the cube above commutes.

Now it remains to relate the vector field transfers to the ordinary transfers, as in Proposition 5.9. This is expressed precisely in the diagram below.

$$\begin{tikzcd}
& & \operatorname*{holim}\limits_{U\in\mathsf{Disk}_q^{B/}} \operatorname*{hocolim}\limits_{V\in\mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \arrow[dd, "\eta"] \\
\mathbb{S} \arrow[urr, "\overline{\mathsf{tr}}(p)"] \arrow[r, dashed, "\overline{\mathsf{tr}}_X(p)"] \arrow[dd, equal] & \operatorname*{holim}\limits_{U\in\mathsf{Disk}_q^{B/}} \operatorname*{hocolim}\limits_{V\in\mathsf{Disk}_{k/p^{-1}U}} \Sigma^\infty(-)_+ \arrow[ur, equal] \arrow[dd, "\eta"] & \\
& & \operatorname*{holim}\limits_{U\in\mathsf{Disk}_q^{B/}} \operatorname*{hocolim}\limits_{V\in\mathsf{Disk}_{k/p^{-1}U}} A^{\%}(-) \\
\mathbb{S} \arrow[urr, "\overline{\chi^{\%}}(p)"] \arrow[r, dashed, "\overline{\chi^{\%}}_X(p)"] & \operatorname*{holim}\limits_{U\in\mathsf{Disk}_q^{B/}} \operatorname*{hocolim}\limits_{V\in\mathsf{Disk}_{k/p^{-1}U}} A^{\%}(-) \arrow[ur, equal] &
\end{tikzcd}$$

Each face of the diagram above commutes, except for the bottom. Our goal is to fill in the bottom face with a homotopy so that the triangular prism commutes. This argument is entirely formal. Identifying the two copies of $\mathbb{S}$ in the diagram above gives us a pyramid, which can be broken into two tetrahedra. For brevity we replace the objects with numbered vertices, but the morphisms are labeled in the same way as before. The dotted arrow from vertex 1 to vertex 3 is an auxilliary morphism which we use to decompose the pyramid. There are two 3-simplexes in the resulting diagram. The simplex containing the morphism $\overline{\mathsf{tr}}(p)$ we will refer to as simplex A. The other simplex will be referred to as simplex B.

$$\begin{tikzcd}
 & & 2 \arrow[dd, "\eta"] \\
0 \arrow[urr, "\overline{\mathsf{tr}}(p)"] \arrow[r, dashed, "\overline{\mathsf{tr}}_X(p)"] \arrow[drr, "\overline{\chi^{\%}}(p)"] \arrow[ddr, dashed, "\overline{\chi^{\%}}_X(p)"'] & 1 \arrow[ur, equal] \arrow[dd, "\eta"] \arrow[dr, dotted] & \\
 & & 3 \\
 & 2 \arrow[ur, equal] &
\end{tikzcd}$$

In simplex A, we use the inner horn filling property to choose the face '013' so that simplex A commutes. In simplex B we use the inner horn filling property to choose the face '023' so that simplex B commutes. It now follows that the triangular prism commutes.

□

We conclude this section with the commutative diagram below, which replaces index category notation used in this section with the homotopy equivalent section spaces. In the applications to come we will use the shorthand, keeping in mind that the actual definitions of spectra and morphisms are slightly more involved.

$$\begin{tikzcd}
 & \mathbb{S} \arrow[rr, "\mathsf{tr}(p)"] \arrow[dl, "\mathsf{tr}(\pi)"'] \arrow[dd] & & \Gamma_B \Sigma^\infty_B(M_+) \arrow[dd] \\
\bigvee_{Z \in \mathscr{S}(f)} \Gamma_Z \Sigma^\infty_Z(DZ_+) \arrow[rr, "\bigvee_{Z \in \mathscr{S}(f)} \mathsf{Ind}_X"'] \arrow[dd] & & \bigvee_{Z \in \mathscr{S}(f)} \Gamma_Z \Sigma^\infty_Z(DZ_+) \arrow[ur, "+"] \arrow[dd] & \\
 & \mathbb{S} \arrow[rr, "\chi^{\%}(p)"] \arrow[dl, "\chi^{\%}(\pi)"'] & & \Gamma_B A^{\%}_B(M) \\
\bigvee_{Z \in \mathscr{S}(f)} \Gamma_Z A^{\%}_Z(DZ) \arrow[rr, "\bigvee_{Z \in \mathscr{S}(f)} \mathsf{Ind}^h_X"'] & & \bigvee_{Z \in \mathscr{S}(f)} \Gamma_Z A^{\%}_Z(DZ) \arrow[ur, "+"] &
\end{tikzcd}$$

5.3. **Fiberwise Poincaré–Hopf Theorem for Stratified Deformations.** We now give a further generalization of Theorems A and B for stratified deformations of the critical locus of a generalized Morse function.

Definition 5.15. A stratified subset of a smooth bundle M over B^q is a pair (Σ, ψ) so that Σ is a compact smooth q-dimensional manifold together with a smooth mapping $\pi : \Sigma \to B$ and a continuous mapping $\psi : \Sigma \to X$. The map π may have singularities, however these singularities are required to be fold singularities: locally they can be given by $\pi(x_1, \cdots, x_q) = (x_1^2, x_2, \cdots, x_q)$. The singularities form a $q-1$ submanifold of Σ.

Example 5.16. For our purposes the pair (Σ, ψ) will be a stratified subset corresponding to the critical locus of a fiberwise generalized Morse function on a smooth manifold bundle. The map ψ will be a map $\Sigma \to BO$ classifying the restriction of the vertical tangent bundle to Σ.

Definition 5.17 (p. 67 in [Igu05])**.** A stratified deformation between stratified subsets (Σ, ψ) and (Σ', ψ') of M over B with coefficients in X is a stratified subset (S, Ψ) of $M \times I$ over $B \times I$ with coefficients in X such that the restrictions of (S, Ψ) to the $B \times 0$ and $B \times 1$ are (Σ, ψ) and (Σ', ψ'). In the event that (Σ, ψ) and (Σ', ψ') are related by a stratified deformation we say that they belong to the same stratified deformation class and use the notation $(\Sigma, \psi) \sim (\Sigma', \psi')$.

Note that when we consider a stratified deformation of a critical locus of a fiberwise generalized Morse function, the end result of the deformation may not necessarily be realized by a Morse function. Thus, Theorem C below does not immediately follow from Theorem B. When referencing the set of strata of a stratified subset (Σ, ψ), we use the notation $\mathscr{S}(\Sigma)$, since we cannot make reference to a function as was done previously.

Theorem C. *Suppose that (Σ', ψ') is a stratified deformation of (Σ, ψ), where Σ is the critical locus of a fiberwise generalized Morse function, and ψ is the classifiying map for the restriction of the vertical tangent bundle $T^\vee M$ to Σ. Let π' denote the disk bundle associated to the tubular neighborhood of Σ' in M. Then the following diagram commutes:*

$$\begin{tikzcd}
& \mathbb{S} \arrow[rr, "\mathsf{tr}(p)"] \arrow[dl, "\mathsf{tr}(\pi')"'] \arrow[dd] & & \Gamma_B \Sigma_B^\infty(M_+) \arrow[dd] \\
\bigvee\limits_{Z \in \mathscr{S}(\Sigma')} \Gamma_Z \Sigma_Z^\infty(DZ_+) \arrow[rr, "\bigvee\limits_{Z \in \mathscr{S}(\Sigma')} \mathsf{Ind}_X^d"] \arrow[dd] & & \bigvee\limits_{Z \in \mathscr{S}(\Sigma')} \Gamma_Z \Sigma_Z^\infty(DZ_+) \arrow[ur, "+"] \arrow[dd] & \\
& \mathbb{S} \arrow[rr, "\chi^\%(p)"] \arrow[dl, "\chi^\%(\pi')"'] & & \Gamma_B A_B^\%(M) \\
\bigvee\limits_{Z \in \mathscr{S}(\Sigma')} \Gamma_Z A_Z^\%(DZ) \arrow[rr, "\bigvee\limits_{Z \in \mathscr{S}(\Sigma')} \mathsf{Ind}_X^h"'] & & \bigvee\limits_{Z \in \mathscr{S}(\Sigma')} \Gamma_Z A_Z^\%(DZ) \arrow[ur, "+"] &
\end{tikzcd}$$

Proof. The data of a stratified deformation provides the data of a homotopy between the compositions along the top squares:

$$(+)\circ\left(\bigvee_{Z\in\mathscr{S}(\Sigma')}\mathsf{Ind}_X\right)\circ\mathsf{tr}(\pi')\sim(+)\circ\left(\bigvee_{Z\in\mathscr{S}(\Sigma)}\mathsf{Ind}_X\right)\circ\mathsf{tr}(\pi)$$

The same is true of the bottom squares: there is a homotopy between the two compositions

$$(+)\circ\left(\bigvee_{Z\in\mathscr{S}(\Sigma')}\mathsf{Ind}^h_X\right)\circ\chi^{\%}(\pi')\sim(+)\circ\left(\bigvee_{Z\in\mathscr{S}(\Sigma)}\mathsf{Ind}^h_X\right)\circ\chi^{\%}(\pi)$$

Now the result follows from Theorem B. □

6. Consequences for the Smooth Structure Class

In this section we will prove Conjecture 3.12. We will begin in Section 6.1 by reformulating the setup of the Rigidity Conjecture so that it is amenable to applying the results from Section 5. In Section 6.2 we will construct a specific stratified deformation motivated by parametrized handle cancellation arguments of Hatcher [Hat75] and Igusa [Igu84, Igu88, Igu02, Igu05]. In Section 6.3 we will use this stratified deformation to prove a fiberwise Poincaré–Hopf theorem for the smooth structure class, Theorem D. In Section 6.4 we will use Theorem D to prove a duality theorem, Theorem E, for the smooth structure class. Finally, in 6.5 we will apply Theorem E in the case where the fiber dimension is even and prove the Rigidity Conjecture.

6.1. **Setup for the Proof of the Rigidity Conjecture.** Suppose that $p: M\to B$ and $p': M'\to B$ are smooth manifold bundles with closed fibers of dimension n so that there exists a fiberwise tangential homeomorphism h from $M\times I^k$ to $M'\times I^k$ for some $k>0$. By fiberwise immersion theory, h is isotopic to a fiberwise immersion, which can be further be approximated by an embedding $\widetilde{h}$ for k large enough. The closure of the complement of the image of $\widetilde{h}: M\times I^k\hookrightarrow M'\times I^k$ under this embedding is a parametrized h-cobordism N over B with boundaries $\partial_0 N := \partial(M\times I^k)$ and $\partial_1 N := \partial(M'\times I^k)$.

Proposition 6.1. There are fiberwise diffeomorphisms $M_0\times I^k\cup_{\partial_0 N} N\cong M_1\times I^k$ and $N\cup_{\partial_1 N} M_1\times I^k\cong M_0\times I^k$.

Proof. The first assertion follows by construction. If we glue on $M_1\times I^k$, we have the following diffeomorphism:

$$M_0\times I^k\cup_{\partial_0 N} N\cup_{\partial_1 N} M_1\times I^k\cong M_1\times I^k\cup_{\mathrm{id}} M_1\times I^k$$

The fiberwise doubles $M_0\times I^k\cup_{\mathrm{id}} M_0\times I^k$ and $M_1\times I^k\cup_{\mathrm{id}} M_1\times I^k$ are fiberwise tangentially homeomorphic via the map h on each component. Since the fibers of each of these bundles are closed, it follows that these bundles are fiberwise diffeomorphic by smoothing theory. Therefore we have a diffeomorphism

$$M_0\times I^k\cup_{\partial_0 N} N\cup_{\partial_1 N} M_1\times I^k\cong M_0\times I^k\cup_{\mathrm{id}} M_0\times I^k$$

and so $N\cup_{\partial_1 N} M_1\times I^k\cong M_0\times I^k$. □

We will now use the diffeormorphisms in Proposition 6.1 to define smooth structure characteristics $\theta(N,\partial_0 N)$ and $\theta(N,\partial_1 N)$

The additivity results for the excisive A-theory Euler characteristic from [BD05] can be applied to $M_0\times I^k\cup_{\partial_0 N} N\cong M_1\times I^k$ to obtain a homotopy

$$\chi^{\%}(M_1\times I^k)-\chi^{\%}(M_0\times I^k)\sim\chi^{\%}(N)-\chi^{\%}(\partial_0 N). \tag{18}$$

Likewise, from the diffeomorphism $N \cup_{\partial_1 N} M_1 \times I^k \cong M_0 \times I^k$ we obtain a homotopy

$$\chi^{\%}(M_0 \times I^k) - \chi^{\%}(M_1 \times I^k) \sim \chi^{\%}(N) - \chi^{\%}(\partial_1 N). \tag{19}$$

Since $M_1 \times I^k$ is fiberwise tangentially homeomorphic to $M_0 \times I^k$, $\chi^{\%}(M_1 \times I^k) - \chi^{\%}(M_0 \times I^k)$ is nullhomotopic, and therefore $\chi^{\%}(N) - \chi^{\%}(\partial_0 N)$ is nullhomotopic. The smooth structure characteristic $\theta(N, \partial_0 N)$ is then defined as the lift obtained from this nullhomotopy.

Similarly, since $M_0 \times I^k$ is fiberwise tangentially homeomorphic to $M_1 \times I^k$, $\chi^{\%}(M_0 \times I^k) - \chi^{\%}(M_1 \times I^k)$ is nullhomotopic, and therefore $\chi^{\%}(N) - \chi^{\%}(\partial_1 N)$ is nullhomotopic. The smooth structure characteristic $\theta(N, \partial_1 N)$ is then be defined as the lift obtained from this nullhomotopy.

Proposition 6.2. There is a homotopy between smooth structure characteristics as follows:

$$\theta(N, \partial_0 N) \sim -\theta(N, \partial_1 N).$$

Proof. This follows immediately from comparing homotopies (18) and (19). The nullhomotopy of the left hand side of (18) is negative of the nullhomotopy of the left hand side of (19). □

6.2. **Constructing a Stratified Deformation.** We continue in the setup of the previous section. Let $f : N \to [0, 1]$ be a fiberwise generalized Morse function so that $f(\partial_0 N) = 0$ and $f(\partial_1 N) = 1$. In this section we will construct a stratified deformation of the critical locus of f which will be used in 6.3 to prove a fiberwise Poincaré–Hopf theorem for the smooth structure characteristic.

Recall the definitions and examples of stratified subsets and stratified deformations appearing in Section 5.3. We will begin with a definition of a specific type of stratified subset called an *immersed lens*.

Definition 6.3 (p.70 in [Igu05])**.** Let V be a compact connected q manifold with boundary so that V is immersed in B^q. Let $\psi_1, \psi_2 : V \to X$ be continuous maps which are trivial on ∂V. Then the *immersed lens* $L_i(V, \psi_1, \psi_2)$ is defined to be the stratified subset (L, ψ_L) where L is the double of V in indices i and $i + 1$, and ψ_L is ψ_1 on the lower stratum and ψ_2 on the upper stratum.

For the purposes of this section, the space X will be $BO \times BO$. The stratified subset corresponding the the critical locus of f is (Σ, ψ), where $\psi : \Sigma \to BO \times BO$ classifies the stable negative and positive eigenspace bundles. The classifying map of the vertical tangent bundle can be obtained from ψ by taking the direct sum of the two vector bundles.

Lemma 6.4. There exists a stratified deformation with boundaries (Σ, ψ) and (Σ', ψ'), so that the strata of (Σ', ψ') are concentrated in two indices. Furthermore, each component of the lower stratum of Σ' lies in a contractible subset of Σ'.

Remark 6.5. Lemma 6.4 is an excerpt from the proof of the transfer theorem in [Igu05]. In particular, the statement is identical to Step (c) on p.70, the proof of which appears on pages 71-73.

Briefly, the strategy of the proof is to first add and delete twisted lenses, immersed lenses for which ψ_1 is the same as ψ_2 after composition with the fold map, to concentrate the stratified subset into two degrees. The stratified deformations obtained by adding and deleting the twisted lenses reduces the number of components in kth stratum by one. An inductive argument starting in the minimal stratum will then concentrate all strata in two degrees.

The next task is to prove that Σ'_{-} lies in a contractible subset of Σ'. To do this, we choose a triangulation of Σ', and do a deformation on each simplex in the stratification. On the zero simplices the idea is to add a lens above a zero simplex, give a stratified deformation that cancels the lower stratum of this lens to obtain a 'mushroom', and then observe that the mushroom has the desired property: the '$-$' stratum on top of the mushroom (as well as it's boundary) lies in a contractible subset. We give a pictorial version of this stratified deformation in the figure below. This construction, as well as the inductive constructions for higher simplices, also appears with pictures in the proof of Lemma 3.2.1 in [GI14].

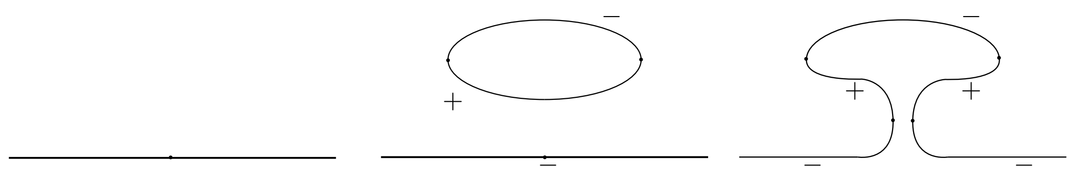


FIGURE 4. The stratified deformation introduces a lens above a designated point in Σ_- and then cancels the + and - strata to obtain a 'mushroom'.

Lemma 6.6 (Lemma 5.7 in [Igu05])**.** The stratified subset (Σ', ψ') can be deformed into a stratified subset (Σ'', ψ'') presented as a disjoint union of immersed lenses and components on which ψ'' is trivial. Furthermore, ψ'' is also trivial on the lower stratum of each of the immersed lenses.

Proposition 6.7. The stratified subset (Σ'', ψ'') of N from Lemma 6.6 has the following property: on each stratum Z of Σ'', the restriction of the vector bundle ψ'' is either trivial or fiber homotopy trivial.

Proof. We begin by applying Theorem C to the stratified subset (Σ'', ψ'') of N to obtain

$$\chi^{\%}(N, \partial_0 N) \sim \sum_{Z'' \in \mathscr{S}(\Sigma'')} \mathsf{Ind}^h_X(DZ'')\chi^{\%}(DZ'') \tag{20}$$

which we know to be nullhomotopic.

Let us partition $\mathscr{S}(\Sigma'')$ into three sets: V_+, V_-, W. The sets V_+ and V_- respectively contain the upper and lower strata of the immersed lenses. From Lemma 6.6, ψ'' is trivial on each element of V_-. The set W contains those strata which do not belong to immersed lenses, on which ψ'' is trivial. For each $Z \in W$, since $\psi''|_Z$ is trivial, $\chi^{\%}(DZ'')$ is nullhomotopic.

We are guaranteed a bijection between V_+ and V_-, since these sets correspond to the top strata and bottom strata of immersed lenses. There are finitely many immersed lenses, which we denote by (L^j, ψ_{L^j}) with $1 \leq j \leq \alpha$ for the purposes of this proof. Suppose that $Z^j_- \cup_\partial Z^j_+$ forms an immersed lens (L^j, ψ_{L^j}). Then each immersed lens (L^j, ψ_{L^j}) corresponds to two summands of the formula above:

$$\chi^{\%}(L^j, \psi_{L^j}) = \mathsf{Ind}^h_X(DZ^j_-)\chi^{\%}(DZ^j_-) + \mathsf{Ind}^h_X(DZ^j_+)\chi^{\%}(DZ^j_+)$$

Because $\chi^{\%}(N, \partial_0 N)$ is nullhomotopic and each immersed lens corresponds to a pair of cancelling handles in the same two degrees, $\chi^{\%}(L^j, \psi_{L^j})$ is nullhomotopic for each j. Since ψ'' is trivial on all lower strata, $\chi^{\%}(DZ^j_-)$ is nullhomotopic. Thus, $\chi^{\%}(DZ^j_+)$ is nullhomotopic for each j.

Next, observe that the composition $Z^j_+ \to A(*)$ in the following diagram factors through the section $\chi^{\%}(DZ^j_+)$, and is therefore nullhomotopic.

$$\begin{tikzcd}
Z^j_+ \arrow[r, "\psi''|_{Z_+}"] \arrow[dr] & BO \arrow[r] \arrow[d] & QS_0 \arrow[d] \\
& BG \arrow[r] & A(*)
\end{tikzcd}$$

This implies that $\psi''|_{Z^j_+}$ must be fiber homotopy trivial for each j.

$\square$

6.3. **A fiberwise Poincaré-Hopf Theorem for the Smooth Structure Class.** We now use the setup of Section 6.1 and the results of Section 6.2 to prove a fiberwise Poincaré–Hopf theorem for the smooth structure characteristic of the parametrized h-cobordism bundle N.

Theorem D. *Recall from the setup in Section 6.1 that N is a parametrized h-cobordism bundle with boundaries $\partial_0 N$ and $\partial_1 N$ so that $\chi^{\%}(N, \partial_0 N)$ and $\chi^{\%}(N, \partial_1 N)$ are nullhomotopic. Let f be a fiberwise generalized Morse function on N so that $f(\partial_0 N) = 0$ and $f(\partial_1 N) = 1$. Let (Σ, ψ) be the stratified subset corresponding to the tubular neighborhood of the critical locus of f. Let (Σ_{SD}, ψ_{SD}) denote the stratified subset produced in Lemmas 6.4 and 6.6 by constructing stratified deformations of (Σ, ψ). Then the following diagram commutes up to homotopy.*

$$\begin{tikzcd}
& \mathbb{S} \arrow[rr, "\theta(N,\partial_0 N)"] \arrow[dl, "\theta(DZ,SZ)"'] & & \Gamma_B \mathcal{H}^{\%}_B(M) \\
\bigvee_{Z\in\mathscr{S}(\Sigma_{SD})} \Gamma_Z \mathcal{H}^{\%}_Z(DZ) \arrow[rr, "\bigvee_{Z\in\mathscr{S}(\Sigma')} \mathsf{Ind}^{h/d}_{\nabla^\vee f}(Z)"'] & & \bigvee_{Z\in\mathscr{S}(\Sigma_{SD})} \Gamma_Z \mathcal{H}^{\%}_Z(DZ) \arrow[ur, "+"] &
\end{tikzcd}$$

In this diagram, the map $\mathsf{Ind}^{h/d}_{\nabla^\vee f}(Z)$ is the map on homotopy fibers induced by $\mathsf{Ind}^{d}_{\nabla^\vee f}(Z)$ and $\mathsf{Ind}^{h}_{\nabla^\vee f}(Z)$

Proof. Theorem B gives a homotopy

$$\chi^{\%}(N, \partial_0 N) \sim (+) \circ \left(\bigvee_{Z\in\mathscr{S}(\Sigma)} \mathsf{Ind}^h_X \right) \circ \chi^{\%}(\pi)$$

Theorem C gives another homotopy

$$\chi^{\%}(N, \partial_0 N) \sim (+) \circ \left(\bigvee_{Z\in\mathscr{S}(\Sigma_{SD})} \mathsf{Ind}^h_X \right) \circ \chi^{\%}(\pi)$$

which is constructed as a composition of homotopies

$$\chi^{\%}(N, \partial_0 N) \sim (+) \circ \left(\bigvee_{Z\in\mathscr{S}(\Sigma)} \mathsf{Ind}^h_X \right) \circ \chi^{\%}(\pi) \sim (+) \circ \left(\bigvee_{Z\in\mathscr{S}(\Sigma_{SD})} \mathsf{Ind}^h_X \right) \circ \chi^{\%}(\pi)$$

The second homotopy in the composition above is from the stratified deformation used to construct (Σ_{SD}, ψ_{SD}).

Now the nullhomotopy of $\chi^{\%}(N, \partial_0 N)$ used to define $\Theta(N, \partial_0 N)$ induces a nullhomotopy of the factorization

$$(+) \circ \left(\bigvee_{Z\in\mathscr{S}(\Sigma_{SD})} \mathsf{Ind}^h_X \right) \circ \chi^{\%}(\pi).$$

By Proposition 6.7, $\chi^{\%}(\pi)$ is nullhomotopic, therefore the nullhomotopy of $\chi^{\%}(N, \partial_0 N)$ decomposes as a sum of nullhomotopies of $\mathsf{Ind}^h_X \circ \chi^{\%}(\pi)$ over $Z \in \mathscr{S}(\Sigma_{SD})$. This gives us a lift of the cube diagram in Theorem B, which is the square diagram in statement of the theorem. □

Recall that the index map $\mathsf{Ind}^d_{\nabla^\vee f}(Z)$ is defined as the smash product

$$\mathsf{Ind}_{\nabla^\vee f}(Z) \wedge \mathrm{id} : \mathrm{Th}(T^\vee M|_Z) \wedge \mathbb{S} \to \mathrm{Th}(T^\vee M|_Z) \wedge \mathbb{S}$$

where $\mathsf{Ind}_{\nabla^\vee f}(Z)$ is the map on Thom spaces induced by the vertical gradient vector field of f. Likewise, the index map $\mathsf{Ind}^h_{\nabla^\vee f}(Z)$ is defined as the smash product

$$\mathsf{Ind}_{\nabla^\vee f}(Z) \wedge \mathrm{id} : \mathrm{Th}(T^\vee M|_Z) \wedge A(*) \to \mathrm{Th}(T^\vee M|_Z) \wedge A(*)$$

Together, these maps induce the index map $\mathsf{Ind}^{h/d}_{\nabla^\vee f}(Z)$ on the homotopy fiber

$$\mathsf{Ind}_{\nabla^\vee f}(Z) \wedge \mathrm{id} : \mathrm{Th}(T^\vee M|_Z) \wedge \mathcal{H}(*) \to \mathrm{Th}(T^\vee M|_Z) \wedge \mathcal{H}(*)$$

Rationally, the map $\mathsf{Ind}_{\nabla^\vee f}(Z)$ is multiplication by -1 raised to the dimension of the negative eigenspace bundle on Z. The following corollary for the rational smooth structure class will be used in the proof of the Rigidity Conjecture.

Corollary 6.8. *In the same setup as Theorem D, we have the following formula for the smooth structure class:*

$$\Theta(N, \partial_0 N) = \sum_{Z \in \mathscr{S}(\Sigma_{SD})} (-1)^{\mathsf{Ind}_f(Z)} \Theta(DZ, SZ) \tag{21}$$

6.4. **A Duality Theorem for the Smooth Structure Class.** In this section we prove a duality theorem for the smooth structure class using Theorem D and Corollary 6.8. Recall that in Section 6.3, we made use of a fiberwise generalized Morse function $f : N \to \mathbb{R}$ for which $f(\partial_0 N) = 0$ and $f(\partial_1 N) = 1$. In this section we will construct a fiberwise generalized Morse function $\overline{f} : N \to \mathbb{R}$ so that $\overline{f}(\partial_0 N) = 1$ and $\overline{f}(\partial_1 N) = 0$, apply Theorem D, and obtain a formula for $\Theta(N, \partial_1 N)$. We will compare this formula with (21) above to prove the duality theorem for the smooth structure class, Theorem E.

The function $-f$ is a fiberwise generalized Morse function on N which sends $\partial_0 N$ to 1 and $\partial_1 N$ to 0. The vertical gradient vector field $\nabla(-f)$ is a nondegenerate vector field given as a section of the vertical tangent bundle $T^\vee N$. Now N is a submanifold of $M' \times I^k$, and the tangent bundle of $M' \times I^k$ is $T^\vee M \oplus \epsilon^{k-1} \oplus \epsilon^1$. The trivial line bundle ϵ^1 is the normal bundle to $\partial_1 N = \partial(M' \times I^k)$. As N is a submanifold of $M' \times I^k$, we know that $T^\vee N$ splits off k trivial line bundles, one of which is distinguished as the normal direction to $\partial_1 N$. Let X be the gradient vector field obtained by multiplying $\nabla(-f)$ by -1 in each component corresponding to one of the $k-1$ trivial line bundles complementary to the pullback of ϵ^1. Then X is a nondegenerate vector field on N, which we can integrate to a fiberwise generalized Morse function $\overline{f} : N \to \mathbb{R}$. The critical locus of $\overline{f}$ is identical to the critical locus of f, however the positive and negative eigenspace data has changed, $\overline{f}(\partial_0 N) = 1$, and $\overline{f}(\partial_1 N) = 0$. Let $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$ be the stratified subset obtained by applying Lemmas 6.4 and 6.6 to the critical locus of $\overline{f}$.

Then from Theorem D and Corollary 6.8, we have the following formula for $\Theta(N, \partial_1 N)$:

$$\Theta(N, \partial_1 N) = \sum_{Z \in \mathscr{S}(\overline{\Sigma_{SD}})} (-1)^{\mathsf{Ind}_{\overline{f}}(Z)} \Theta(DZ, SZ) \tag{22}$$

Now we will prove the duality theorem below by comparing formulas (21) and (22).

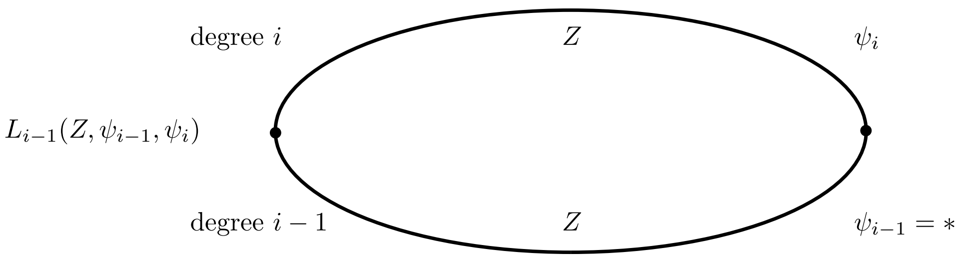
Theorem E. *For N as in Theorem D, the following formula holds for the smooth structure class:*

$$\Theta(N, \partial_0 N) = (-1)^n \Theta(N, \partial_1 N)$$

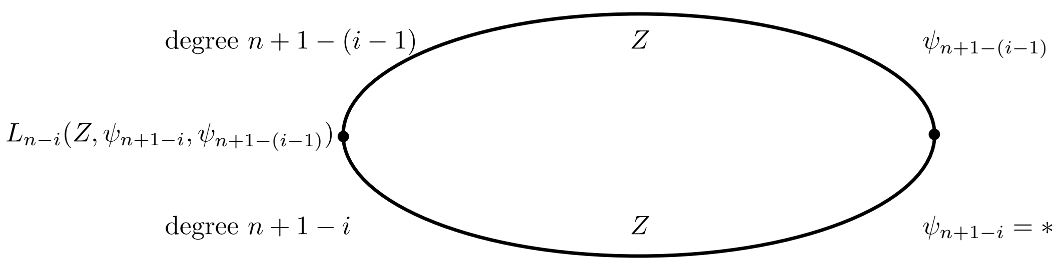
Remark 6.9. This formula is strikingly similar to Milnor's duality theorems for Reidemeister and Whitehead torsion [Mil62,Mil66], as well as Igusa's involution property for higher Franz–Reidemeister torsion [Igu02]. We view Theorem E as a generalization of these results. The proof below is comparable to Milnor's and Igusa's proofs.

Proof of Theorem E. The critical loci of f and $-f$, Σ_f and $\Sigma_{\overline{f}}$ are identical. Consider a stratum Z in $\mathscr{S}(f) = \mathscr{S}(-f)$. We can consider two different vector bundles on Z. Let γ_-^i be the negative eigenspace bundle of f on Z. Let $\overline{\gamma}_-^j$ be the negative eigenspace bundle of $\overline{f}$ on Z. By the construction of $\overline{f}$, $j = |n+1-i|$. Thus, if a stratum in $\mathscr{S}(f)$ is in degree i, the corresponding stratum in $\mathscr{S}(\overline{f})$ is in degree $n+1-i$. Similarly, a stratum of (Σ_{SD}, ψ_{SD}) in degree i corresponds to a stratum of $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$ in degree $n+1-i$.

Recall that the stratified subsets (Σ_{SD}, ψ_{SD}) and $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$ are each disjoint unions of immersed lenses concentrated in two degrees. Because the critical loci of f and $\overline{f}$ are identical, the submanifolds Σ_{SD} and $\overline{\Sigma_{SD}}$ constructed from the same stratified deformation are identical. In particular, there is a one-to-one correspondence between immersed lenses comprising (Σ_{SD}, ψ_{SD}) and $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$. It suffices to consider one such pair, and observe how the bundle data ψ_{SD} and $\overline{\psi_{SD}}$ has changed. Below we depict $L_{i-1}(Z, \psi_{i-1}, \psi_i)$, an immersed lens belonging to (Σ_{SD}, ψ_{SD}).



The immersed lens $L_{i-1}(Z, \psi_{i-1}, \psi_i)$ above corresponds to the immersed lens $L_{n+1-i}(Z, \psi_{n+1-i}, \psi_{(n+1)-(i-1)})$ belonging to $(\overline{\Sigma_{SD}}, \overline{\psi_{SD}})$, depicted below.



Recall that after the stratified deformation of Lemma 6.4, the bundle data on the lower stratum is trivial, so ψ_{i-1} and ψ_{n+1-i} are trivial as in the diagrams above. On the upper strata, ψ_{SD} is the map $Z \to BO \times BO$ classifying the stable negative and positive eigenspace bundles, γ_f and γ_{-f}, respectively. These stable bundles γ_f and γ_{-f} have the property that $\gamma_f \oplus \gamma_{-f} \cong T^\vee M|_Z \oplus \epsilon^n$. The smooth structure characteristic $\theta(DZ, SZ)$ is defined in terms of the restriction of the vertical tangent bundle $T^\vee M|_Z$, and therefore remains unchanged when comparing formulas (21) and (22). However, the coefficients $\mathsf{Ind}_f(Z)$ and $\mathsf{Ind}_{\overline{f}(Z)}$ change according to the formula $\mathsf{Ind}_{\overline{f}(Z)} - \mathsf{Ind}_f(Z) = n+1-(i-1)-i = n+2-2i$. In particular,

$$(-1)^{\mathsf{Ind}_{\overline{f}}(Z)} = (-1)^n \cdot (-1)^{\mathsf{Ind}_f(Z)}$$

The theorem statement above is a direct consequence of this computation. $\square$

6.5. **Proof of the Rigidity Conjecture.** We can now prove Conjecture 3.12.

Theorem F. *Let M be a smooth manifold bundle with closed $2n$-dimensional fibers, $n > 2$. There does not exist a bundle M' with closed $2n$-dimensional fibers and a positive integer k for which $M' \times I^k$ is a rationally nontrivial stable exotic smoothing of $M \times I^k$. In particular, $\Theta(M' \times I^k, M \times I^k)$ vanishes for any such $M' \times I^k$.*

Proof. If $p : M \to B$ and $p' : M' \to B$ are smooth fiber bundles with closed fibers of dimension n, and $M \times I^k$ is fiberwise tangentially homeomorphic to $M' \times I^k$ for some $k > 0$, then the smooth structure classes $\Theta(N, \partial_0 N)$ and $\Theta(N, \partial_1 N)$ are defined as in Section 6.1. Furthermore, if n is even as in the statement of Conjecture 3.12, then by Theorem E,

$$\Theta(N, \partial_0 N) = \Theta(N, \partial_1 N).$$

However, by Proposition 6.2,

$$\Theta(N, \partial_0 N) = -\Theta(N, \partial_1 N).$$

It must follow then that $\Theta(N, \partial_0 N)$ and $\Theta(N, \partial_1 N)$ are both rationally trivial. This means that the nullhomotopy of $\chi^{\%}(N) - \chi^{\%}(\partial_0 N)$ used to define $\theta(N, \partial_0 N)$ is rationally trivial. This nullhomotopy is homotopic to the nullhomotopy of $\chi^{\%}(M_1 \times I^k) - \chi^{\%}(M_0 \times I^k)$, which must then also be rationally trivial. By Definition 3.13, this means that the smooth structure characteristic $\theta(M \times I^k, M' \times I^k)$ is rationally trivial, and therefore the smooth structure class $\Theta(M \times I^k, M' \times I^k)$ is trivial. $\square$

References

[AF15] David Ayala and John Francis. Factorization homology of topological manifolds. *Journal of Topology*, 8(4):1045–1084, 2015.

[AF19] David Ayala and John Francis. Zero-pointed manifolds. *Journal of the Institute of Mathematics of Jussieu*, pages 1–74, 2019.

[AGI19] Daniel Alvarez-Gavela and Kiyoshi Igusa. Turaev torsion of ribbon Legendrians. *arXiv preprint arXiv:1911.01393*, 2019.

[BD05] Bernard Badzioch and Wojciech Dorabiala. Additivity for parametrized topological euler characteristic and reidemeister torsion. *arXiv preprint math/0506258*, 2005.

[BDKW11] Bernard Badzioch, Wojciech Dorabiała, John R Klein, and Bruce Williams. Equivalence of higher torsion invariants. *Advances in Mathematics*, 226(3):2192–2232, 2011.

[BDW09] Bernard Badzioch, Wojciech Dorabiała, and Bruce Williams. Smooth parametrized torsion: A manifold approach. *Advances in Mathematics*, 221:660–680, 2009.

[BG75] James C Becker and Daniel H Gottlieb. The transfer map and fiber bundles. *Topology*, 14(1):1–12, 1975.

[BG91] James C Becker and Daniel Henry Gottlieb. Vector fields and transfers. *Manuscripta Mathematica*, 72(1):111–130, 1991.

[BG13] Ulrich Bunke and David Gepner. Differential function spectra, the differential Becker–Gottlieb transfer, and applications to differential algebraic K-theory, 2013.

[BL95] Jean-Michel Bismut and John Lott. Flat vector bundles, direct images and higher real analytic torsion. *Journal of the American Mathematical Society*, 8(2):291–363, 1995.

[BM76] G Brumfiel and Ib Madsen. Evaluation of the transfer and the universal surgery classes. *Inventiones mathematicae*, 32(2):133–169, 1976.

[Bök84] Marcel Bökstedt. The rational homotopy type of $\Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$. In *Algebraic Topology Aarhus 1982*, pages 25–37. Springer, 1984.

[CJ98] Michael Charles Crabb and Ioan Mackenzie James. *Fibrewise homotopy theory*. Springer, 1998.

[DI04] Daniel Dugger and Daniel C Isaksen. Topological hypercovers and 1-realizations. *Mathematische Zeitschrift*, 246(4):667–689, 2004.

[Dou05] Christopher L Douglas. Trace and transfer maps in the algebraic K-theory of spaces. *K-Theory*, 36(1-2), 2005.

[Dou06] Christopher L Douglas. On the fibrewise Poincaré–Hopf theorem. *Contemporary Mathematics*, 407:101, 2006.

[DWW95] Wiliam Dwyer, Michael Weiss, and Bruce Williams. A parametrized index theorem for the algebraic K-theory Euler class. *Acta mathematica*, 190(1):1–104, 1995.

[DWW03] Wiliam Dwyer, Michael Weiss, and Bruce Williams. A parametrized index theorem for the algebraic K-theory Euler class. *Acta mathematica*, 190(1):1–104, 2003.

[EG98] Yasha Eliashberg and Misha Gromov. Lagrangian intersection theory: finite-dimensional approach. *Translations of the American Mathematical Society-Series 2*, 186:27–118, 1998.

[EM12] Yakov M Eliashberg and Nikolai M Mishachev. The space of framed functions is contractible. In *Essays in Mathematics and its Applications*, pages 81–109. Springer, 2012.

[FH78] F Thomas Farrell and Wu-Chung Hsiang. On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part*, volume 1, pages 325–337, 1978.

[FLS10] FT Farrell, Wolfgang Lück, and Wolfgang Steimle. Obstructions to fibering a manifold. *Geometriae Dedicata*, 148(1):35–69, 2010.

[GE71] Mikhail Leonidovich Gromov and Ja M Eliašberg. Removal of singularities of smooth mappings. *Mathematics of the USSR-Izvestiya*, 5(3):615–639, 1971.

[GI14] Sebastian Goette and Kiyoshi Igusa. Exotic smooth structures on topological fiber bundles ii. *Transactions of the American Mathematical Society*, 366(2):791–832, 2014.

[GIO15] Thomas Goodwillie, Kiyoshi Igusa, and Christopher Ohrt. An equivariant version of Hatcher's G/O construction. *Journal of Topology*, 8(3):675–690, 2015.

[GIW14] Sebastian Goette, Kiyoshi Igusa, and Bruce Williams. Exotic smooth structures on topological fiber bundles i. *Transactions of the American Mathematical Society*, 366(2):749–790, 2014.

[Goe01] Sebastian Goette. Morse theory and higher torsion invariants i. *arXiv preprint math/0111222*, 2001.

[Hat75] Allen E Hatcher. Higher simple homotopy theory. *Annals of Mathematics*, pages 101–137, 1975.

[Hir59] Morris W Hirsch. Immersions of manifolds. *Transactions of the American Mathematical Society*, 93(2):242–276, 1959.

[Igu84] Kiyoshi Igusa. On the homotopy type of the space of generalized Morse functions. *Topology*, 23(2):245–256, 1984.

[Igu88] Kiyoshi Igusa. The stability theorem for smooth pseudoisotopies. *K-theory*, 2(1-2), 1988.

[Igu90] Kiyoshi Igusa. Parametrized Morse theory and its applications. In *Proceedings of the International Congress of Mathematicians*, volume 1, pages 643–651, 1990.

[Igu02] Kiyoshi Igusa. *Higher Franz–Reidemeister torsion*. Number 31. American Mathematical Soc., 2002.

[Igu05] Kiyoshi Igusa. *Higher complex torsion and the framing principle*, volume 835. American Mathematical Soc., 2005.

[Igu08] Kiyoshi Igusa. Axioms for higher torsion invariants of smooth bundles. *Journal of Topology*, 1(1):159–186, 2008.

[Kra18] Thomas Kragh. Generating families for Lagrangians in $\mathbb{R}^{2n}$ and the Hatcher-Waldhausen map. *arXiv preprint arXiv:1804.02557*, 2018.

[Lur09] Jacob Lurie. On the classification of topological field theories. *Current developments in mathematics*, 2008:129–280, 2009.

[Mil62] John Milnor. A duality theorem for Reidemeister torsion. *Annals of Mathematics*, pages 137–147, 1962.

[Mil66] John Milnor. Whitehead torsion. *Bulletin of the American Mathematical Society*, 72(3):358–426, 1966.

[MP89] Stephen A Mitchell and Stewart B Priddy. A double coset formula for levi subgroups and splitting BGL_n. In *Algebraic topology*, pages 325–334. Springer, 1989.

[MT01] Ib Madsen and Ulrike Tillmann. The stable mapping class group and $Q\mathbb{CP}^\infty$. *Inventiones mathematicae*, 145(3):509–544, 2001.

[Ohr19] Christopher Ohrt. Comparison of higher smooth torsion, 2019.

[Rei19] Jens Reinhold. Tautological classes and smooth bundles over $BSU(2)$. *Proceedings of the American Mathematical Society*, 147(2):885–895, 2019.

[RW08] Oscar Randal-Williams. The homology of the stable nonorientable mapping class group. *Algebraic & Geometric Topology*, 8(3):1811–1832, 2008.

[SK77] Laurence C Siebenmann and Robion C Kirby. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*. Citeseer, 1977.

[Ste10] Wolfgang Steimle. *Obstructions to stably fibering manifolds*. 2010.

[Ste12] Wolfgang Steimle. Higher Whitehead torsion and the geometric assembly map. *Journal of Topology*, 5(3):529–574, 2012.

[Wal82] Friedhelm Waldhausen. Algebraic K-theory of spaces: a manifold approach. In *Current trends in algebraic topology*, volume 1. 1982.

[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology*, pages 318–419. Springer, 1985.

[Wal87] Friedhelm Waldhausen. Algebraic K-theory of spaces, concordance, and stable homotopy theory. In *Algebraic topology and algebraic K-theory: proceedings of a Conference October 24-28, 1983 (Entitled´ Algebraic Topology and Algebraic K-Theory´) at Princton University, dedicated to John C. Moore on his 60th birthday*, 1987.

[Wei15] Michael S Weiss. Dalian notes on rational Pontryagin classes. *arXiv preprint arXiv:1507.00153*, 2015.

[WJR13] Friedhelm Waldhausen, Bjørn Jahren, and John Rognes. *Spaces of PL manifolds and categories of simple maps*. Number 186. Princeton University Press, 2013.