

# FIBERWISE POINCARÉ–HOPF THEORY AND EXOTIC SMOOTH STRUCTURES ON FIBER BUNDLES

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ABSTRACT. We prove the Rigidity Conjecture of Goette and Igusa, which states that, after stabilizing and then rationalizing, there are no exotic smoothings of manifold bundles with closed even dimensional fibers. The key ingredients of the proof are fiberwise Poincaré–Hopf theorems generalizing earlier such results about the Becker–Gottlieb transfer. These theorems show how to compute the smooth structure class, an invariant of smooth structures on fiber bundles, using the data of a fiberwise generalized Morse function. The benefit of this approach is two-fold. It explains earlier work by Goette and Igusa in the case where the fiber dimension is odd, and, after applying Igusa’s extensive work on higher Franz–Reidemeister torsion invariants, it supplies a proof of the aforementioned conjecture.

## CONTENTS

1. Introduction	2
1.1. Proof Summary	4
1.2. Outline	5
2. Related Work	6
3. Key Invariants and Constructions	7
3.1. Higher Torsion Invariants	8
3.2. Statement of the Rigidity Conjecture	9
3.3. Homotopical Definition of the Smooth Structure Class	11
3.4. Hatcher Construction	13
4. Review of Generalized Morse Functions	16
5. Fiberwise Poincaré–Hopf Theorems	18
5.1. Fiberwise Poincaré–Hopf Theorem for the Becker–Gottlieb transfer	19
5.2. Fiberwise Poincaré–Hopf for the excisive A-theory characteristic	25
5.3. Fiberwise Poincaré–Hopf Theorem for Stratified Deformations	30
5.4. Fiberwise Poincaré–Hopf Theorem for the Smooth Structure Class	31
6. Application to exotic smoothings of fiber bundles	34
6.1. Pairs of compatible Morse functions.	35
6.2. Computations for the Immersed Hatcher construction	35
6.3. Proof of the Rigidity Conjecture	37
References	38

## 1. INTRODUCTION

The present work proves a conjecture about exotic smoothings of manifold bundles due to Goette and Igusa. An *exotic smooth structure* on a smooth manifold bundle  $p : M \rightarrow B$  is another smooth manifold bundle  $p' : M' \rightarrow B$  together with a fiberwise tangential homeomorphism  $f : M \rightarrow M'$ . A fiberwise tangential homeomorphism is a fiberwise homeomorphism  $f : M \rightarrow M'$  covered by an isomorphism of vertical tangent bundles  $T^\vee M \cong T^\vee M'$ .

In [GIW14], Goette, Igusa, and Williams define the *smooth structure class*, an element of the real homology of the total space that distinguishes exotic smooth structures on manifold bundles after the fiber dimension is stabilized. When comparing bundles  $M$  and  $M'$ , this invariant is denoted  $\Theta(M, M')$  and is closely related to the relative higher Franz–Reidemeister torsion. In a subsequent paper [GI14], Goette and Igusa give a procedure to construct stable exotic smooth structures on a bundle with closed odd dimensional fibers. In other words, they construct examples for which the smooth structure class is nonzero. Furthermore, for fixed  $M$ , they show that their procedure for constructing  $M'$  generates all possible values of  $\Theta(M, M')$ , up to linear combinations of rational multiples. They go on to conjecture that when the fibers are closed even dimensional manifolds, there are no stable exotic smooth structures, i.e. the smooth structure class always vanishes.

**Main Theorem** (Rigidity Conjecture of Goette and Igusa, Theorem F). *The Rigidity Conjecture of Goette and Igusa is true: there are no stable exotic smoothings of even dimensional manifold bundles with closed fibers. In particular, if  $M$  and  $M'$  are fiberwise tangentially homeomorphic smooth manifold bundles with closed even dimensional fibers then  $\Theta(M, M') = 0$ .*

The main theorem stated above has several antecedents for related invariants. These include vanishing theorems for the Euler characteristic, Reidemeister torsion, the Becker–Gottlieb transfer, and the higher Franz–Reidemeister torsion. Vanishing theorems for each of these invariants are proven in stylistically equivalent ways, though the technical ingredients vary widely. They generally follow by applying a Poincaré–Hopf type theorem in combination with Poincaré duality. In order to motivate the contents of this paper, we will briefly describe these theorems and their proofs. This discussion is summarized in the table below. At the end of this section we summarize the proof of our main theorem and give an outline of the contents of this paper.

	Euler characteristic	Reidemeister torsion	Becker–Gottlieb transfer	higher torsion $\tau^{IK}$	smooth structure class $\Theta$
Poincaré–Hopf theorem	$\chi(M) = \sum_{z \in Z} (-1)^{\text{Ind}_X(z)}$	Compute torsion from explicit choice of triangulation	Fiberwise Poincaré–Hopf theorem	Framing Principle	Theorem D
Poincaré duality	$\text{Ind}_X(z) = -\text{Ind}_{-X}(z)$ for odd dimensional manifolds	Recompute for dual cell complex	Recompute fiberwise index map after negating Morse function and compare	Apply the framing principle for a fiberwise GMF $f$ and compare with results for $-f$	The involution on pseudoisotopy space is used to prove $\pi_1(\text{BDiff}_\partial(D^{2n}) \otimes Q) = 0$ . See [FH78, Igu05].
$\Rightarrow$ Vanishing result	$\chi(M) = 0$ when $\dim M$ is odd	Torsion is trivial on even dimensional manifolds	transfer map on real cohomology vanishes for odd dimensional fibers	higher torsion of even dimensional manifold bundles is MMM class $\Rightarrow$ relative torsion of fiberwise tangentially homeomorphic manifold bundles vanishes	Rigidity Conjecture: smooth structure class vanishes

Recall the classical proof that the Euler characteristic of an odd dimensional manifold is zero: The Poincaré–Hopf theorem states that the Euler characteristic of a closed manifold  $M$  is equal to the sum of the indices of the zeros of a nondegenerate vector field  $X$  on  $M$ :

$$\chi(M) = \sum_{z \in Z} (-1)^{\text{Ind}_X(z)}$$

On an odd dimensional manifold,  $\text{Ind}_{-X}(z)$  and  $\text{Ind}_X(z)$  have opposite parity. It follows that  $\chi(M) = 0$ .

We adopt a stylized view of this proof: the vanishing result for the Euler characteristic is proven by applying the Poincaré–Hopf theorem in combination with Poincaré duality.

The Reidemeister torsion is a K-theoretic generalization of the Euler characteristic that admits an analogous vanishing theorem: the Reidemeister torsion of an even dimensional manifold is zero. To prove this, recall that the Reidemeister torsion of a manifold is computed in terms of a triangulation of the manifold. Thus, we can compare the torsion of one cell decomposition to the torsion of its dual. The specific computation is due to Milnor [Mil62], and when the dimension is even it follows that the torsion must be zero. Stylistically this proof is the same as the proof of the vanishing result for the Euler characteristic: the formula for torsion in terms of the data of a triangulation is an instance of a Poincaré–Hopf theorem. The comparison to the dual cell complex is an instance of Poincaré duality.

The Becker–Gottlieb transfer is a generalization of the Euler characteristic to families. For a smooth bundle of smooth manifolds  $p : E \rightarrow B$ , one can associate a wrong way map of spectra  $\Sigma^\infty B_+ \rightarrow \Sigma^\infty E_+$ . If the base is a point, this is a map from  $S^0$  to  $\Omega^\infty \Sigma^\infty E_+$ . On components, if  $E$  is connected, we have a map  $S^0 \rightarrow \mathbb{Z}$ , and the non-basepoint element maps to  $\chi(E) \in \mathbb{Z}$ .

One can easily prove a vanishing result for the Becker–Gottlieb transfer using a small amount of technology. Fiberwise Poincaré–Hopf theorems for the Becker–Gottlieb transfer have been proven by [BM76, Dou06]. Briefly, assume that  $X$  is a smooth nondegenerate vertical vector field on the total space of a smooth bundle  $p : E \rightarrow B$ . This vector field might be obtained by computing the gradient of a fiberwise Morse function, so long as such a function exists. Let  $Z$  be the vanishing locus of the vector field, which forms a covering space  $\pi$  over  $B$ . Then the fiberwise Poincaré–Hopf theorem is expressed in terms of the following homotopy commutative diagram of spectra:

$$\begin{array}{ccc} \Sigma^\infty B_+ & \xrightarrow{\text{tr}_p} & \Sigma^\infty M_+ \\ \text{tr}_\pi \downarrow & & \uparrow + \\ \Sigma^\infty Z_+ & \xrightarrow{\text{Ind}_X} & \Sigma^\infty Z_+ \end{array}$$

In the diagram above,  $\text{tr}_\pi$  and  $\text{tr}_p$  denote the transfers associated to  $\pi$  and  $p$ . The map  $\text{Ind}_X$  denotes a fiberwise index map associated to the vertical vector field  $X$ . On real cohomology one can easily prove from the definitions that  $(\text{Ind}_X)^* = (-1)^d (\text{Ind}_{-X})^*$ , where  $d$  denotes the fiber dimension. It follows that  $(\text{tr}_p)^* = (-1)^d (\text{tr}_\pi)^*$ . Thus the transfer map on cohomology vanishes when the fiber dimension is odd. In this case the classical Poincaré–Hopf theorem was replaced by a parametrized version, and Poincaré duality arose in the comparison of the vector field and its negative.

A common generalization of the Euler characteristic to both the K-theoretic and parametrized settings is the higher Franz–Reidemeister torsion. This invariant is a characteristic class in the cohomology of the base of a smooth fiber bundle. The primary tool that enables computations of this invariant is the Framing Principle due to Igusa [Igu05]. The framing principle describes the higher Franz–Reidemeister torsion in as the sum of an ‘exotic’ class and a ‘tangential’ term.

A consequence of the framing principle is that for smooth manifold bundles with closed even dimensional fibers, the torsion class is congruent to a Miller–Morita–Mumford class. To prove this, we write down the framing principle for a fiberwise generalized Morse function  $f$  and compare it to the analogous formula for  $-f$ . By studying the canonical involution on the Whitehead space,

one can prove that the exotic term is two-torsion. Thus we are left only with the tangential term which agrees with a Miller–Morita–Mumford class.

Once again, this proof is analogous to those from above: the framing principle resembles a Poincaré–Hopf theorem, and the comparison of the formulas for  $f$  and  $-f$  resembles an application of Poincaré duality. However, the proof of this vanishing theorem requires significantly more technology than those which came previously. In particular, to define the exotic term in the framing principle, Igusa uses a Waldhausen category model for the Whitehead space which encodes the combinatorics of colliding critical points of fiberwise generalized Morse functions. The proof of the framing principle requires an understanding of the deformation properties of the critical loci of fiberwise generalized Morse functions.

We postpone giving a precise definition of the smooth structure class until Section 3.3, however the smooth structure class is closely related to the invariants discussed above. One explanation for this is that the higher Franz–Reidemeister torsion and the smooth structure class can both be defined in terms of nullhomotopies of maps that factor through the Becker–Gottlieb transfer. An explicit relationship between these invariants at the level of homology groups of the base is proven in [GI14]: the pushdown class  $p_*\Theta(M, M')$  is congruent to  $D\tau^{IK}(M, M')$ , the Poincaré dual of the relative higher torsion. Thus one should expect that a proof of the main theorem above should follow from a sufficiently general version of a Poincaré–Hopf theorem along with an application of Poincaré duality. The following section provides a summary of such a proof.

**1.1. Proof Summary.** In this paper we prove a vanishing result for the smooth structure class, an invariant of smooth structures on fiber bundles introduced by Goette, Igusa, and Williams in [GIW14], after work of Dwyer, Weiss, and Williams [DWW03]. In analogy with the examples above, the proof is an application of a Poincaré–Hopf type theorem in combination with Poincaré duality. In this section we give a precise outline of the proof, as well as a summary of the individual sections in this paper.

By a fiberwise Poincaré–Hopf theorem we broadly mean a computation of a fiberwise characteristic, e.g. the Becker–Gottlieb transfer, the excisive A-theory Euler characteristics, etc., in terms of the critical locus of a fiberwise generalized Morse function. Examples of such theorems can be found in [BM76, CJ98, Dou06]. These theorems generalize the classical Poincaré–Hopf theorem, which computes the Euler characteristic in terms of local data at the isolated critical points of a Morse function.

The following list is an itemized outline of the proof of the Main Theorem.

(0) The following recollections are necessary for this outline:

- The smooth structure class  $\Theta(M, M')$  is an element of  $\pi_0\Gamma_B\mathcal{H}_B^\%(M) \otimes \mathbb{Q}$ . This is the space of sections of the fiberwise homology bundle obtained by taking fiberwise smash products with the stable h-cobordism space of a point. See Section 3.3 for a precise definition.
- By the stable parametrized h-cobordism theorem,  $\Gamma_B\mathcal{H}_B^\%(M)$  is the homotopy fiber of the map  $\Gamma_B Q_B(M) \rightarrow \Gamma_B A_B^\%(M)$ , which is induced by the unit map from the sphere spectrum to  $A(*)$ . The spectrum  $A(*)$  is the Waldhausen K-theory of spaces functor, otherwise known as A-theory, evaluated at a point. The functor  $A^\%$  is the excisive approximation to A-theory.
- A celebrated theorem by Igusa indicates that all smooth bundles for which the fiber dimension exceeds the dimension of the base admit fiberwise generalized Morse functions. In stark contrast, smooth bundles rarely admit fiberwise Morse functions.

- (1) We prove a fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer, an element of  $\Gamma_B Q_B(M)$ . This result, Theorem A, expresses the transfer in terms of a tubular neighborhood of the critical locus of a fiberwise generalized Morse function on  $M$ .
- (2) We further generalize the fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer to the excisive A-theory characteristic, an element of  $\Gamma_B A_B^{\%}(M)$ . Furthermore, this factorization is compatible with the factorization of the Becker–Gottlieb transfer in the previous theorem. This is Theorem B.
- (3) We generalize Theorem B for any *stratified deformation* of the critical locus of a fiberwise generalized Morse function. This is Theorem C.
- (4) Theorem D gives a fiberwise Poincaré–Hopf theorem for the smooth structure class in the setting where a smooth bundle has trivial excisive A-theory Euler characteristic. This theorem relies on Theorem C, in that the smooth structure class is expressed in terms of a particular stratified deformation of the critical locus of a fiberwise generalized Morse function. The stratified deformation that we use is a geometric construction due to Igusa that 1) concentrates the strata of critical locus into two indices and 2) vastly simplifies the tangential data on the submanifold.
- (5) If two smooth bundles  $M$  and  $M'$  are fiberwise tangentially homeomorphic, then for large  $k$ , we can embed  $M \times I^{k+1}$  in  $M' \times I^{k+1}$  so that the complement of the image of  $M \times I^{k+1}$  is an h-cobordism  $N$  on  $\partial(M \times I^{k+1})$ . After picking compatible Morse functions on  $M \times I^{k+1}$  and  $M' \times I^{k+1}$  that allow us to prove that  $\chi^{\%}(N) = 0$ , we apply Theorem D to  $N$ .
- (6) The fiberwise Poincaré–Hopf theorem for the smooth structure class expresses  $\Theta(M, M')$  in terms of  $\Theta(DZ)$ , where  $DZ$  is the tubular neighborhood of an embedded submanifold of  $M'$  with the same dimension as  $B$ . The rational class  $\Theta(DZ)$  depends only on a disk bundle classified by a map to  $\text{BDiff}_{\partial}(D^{2n}) \otimes \mathbb{Q}$ , which was computed to be trivial in the pseudoisotopy stable range by [FH78]. Since there are no exotic smoothings of even dimensional disk bundles,  $\Theta(DZ)$  must be trivial, so  $\Theta(M, M') = 0$ .

To complete the analogy in the exposition from the previous section, we will indicate how this proof can be thought of as an application of a Poincaré–Hopf type theorem and Poincaré duality. The Poincaré–Hopf type theorem that we ultimately apply is Theorem D, and as the outline indicates, this is a generalization of other Poincaré–Hopf theorems that we prove along the way. The instance of Poincaré duality that we make use of is the computation of  $\text{BDiff}_{\partial}(D^{2n}) \otimes \mathbb{Q}$  due to [FH78]. To explain why this can reasonably be thought of as Poincaré duality, we recall that their proof makes use of the involution on space. In rational models of the Whitehead space due to [Igu02], the analogous involution is modeled as the map obtained by negating a fiberwise generalized Morse function.

**1.2. Outline.** We now give a detailed outline of the contents of this paper.

Section 2 contains descriptions of related work pertaining to fiberwise Poincaré–Hopf theorems, the Hatcher construction, and exotic smoothings.

Section 3 contains precise definitions of the key invariants in this paper: higher torsion invariants (3.1) and the smooth structure class (3.2). Of crucial importance is the homotopy theoretic definition of the smooth structure class appearing in (3.3), which expresses the smooth structure class as the data of a lift of the excisive A-theory characteristic to the Becker–Gottlieb transfer. This section also includes details of the important constructions: the Hatcher construction (3.4), and the immersed Hatcher construction (3.4.1).

Section 4 recalls definitions and key geometric results pertaining to fiberwise generalized Morse functions.

Section 5 contains the main technical results of this paper. These are the fiberwise Poincaré–Hopf theorems for the Becker–Gottlieb transfer, the excisive A-theory characteristic, and the smooth structure class. The first theorem, Theorem A in (5.1), is a computation of the Becker–Gottlieb transfer in terms of the critical locus of a fiberwise generalized Morse function. The second theorem, Theorem B in (5.2), generalizes Theorem A to compute the excisive A-theory Euler characteristic in terms of a fiberwise generalized Morse function. The third theorem, Theorem C in (5.3) generalizes Theorem B to stratified deformations of the critical locus of a generalized Morse functions. The fourth theorem, Theorem D in (5.4) uses Theorem C and a constuction of a particular stratified deformation due to Igusa to give a fiberwise Poincaré–Hopf theorem for the smooth structure class.

Section 6 contains computations of the smooth structure class in the odd and even dimensional settings. In (6.1) we show how to arrange two fiberwise tangentially homeomorphic manifold bundles so that Theorem D can be applied. Next, in (6.2) we compute the structure class for smooth bundles with odd dimensional fibers, thereby reproving a main theorem in [GI14], Theorem E. Finally, in (6.3) prove the Rigidity Conjecture, Theorem F.

## 2. RELATED WORK

Two main ideas arising in this paper, fiberwise Poincaré–Hopf theorems and the Hatcher construction, are of much independent interest in the literature. In this section we give a brief survey of the literature and recent progress pertaining to both keywords. We omit a discussion of the most immediate literature pertaining to the Rigidity Conjecture, including [DWW03, GIW14, GI14], as detailed descriptions of these works appear elsewhere in this paper.

Hatcher’s construction associates to an element of the kernel of the J-homomorphism a disk bundle which is fiber homotopy trivial but not smoothly trivial. The construction should be interpreted as a stable map from  $G/O$  to  $\Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$ . Waldhausen gave a different formulation of the same map in [Wal82], and in [Bök84] Bokstedt proved this map to be a rational homotopy equivalence. Later Igusa gave another proof using parametrized Morse theory. Exciting new developments by Kragh [Kra18] have identified the homotopy fiber of the Hatcher–Waldhausen map as a certain functional space  $\mathcal{M}_\infty$  considered by Eliashberg and Gromov in [EG98], establishing a connection between the study of Lagrangians to algebraic K-theory of spaces. Kragh associates to every exact Lagrangian an element of  $\pi_*(\mathcal{M}_\infty)$ . If any of these examples were proven to be nontrivial, they would be counterexamples to the nearby Lagrangian conjecture in symplectic topology. In essence, counter examples to the nearby Lagrangian conjecture might be found in the kernel of Hatcher’s construction. Recent work by Igusa and Alvarez-Gavela elaborates further on these developments [AGI19].

Goodwillie, Igusa, and Ohrt have developed an equivariant version of Hatcher’s construction [GIO15]. Ordinarily, the space  $G/O$  classifies vector bundles whose spherical fibrations are fiber homotopy trivial. In the equivariant version,  $G/O$  is replaced by the space  $G_n/U$ , which is the colimit of spaces  $G_n(N)/U(N)$ , classifying rank  $N$  complex vector bundles together with a  $C_n$ -equivariant fiber homotopy trivialization of the associated sphere bundle. The equivariant Hatcher construction is then a map  $G_n/U \rightarrow \mathcal{H}^s(BC_n)$ , where the target is the stable h-cobordism space of the classifying space of  $C_n$ . The geometric outcome of the construction is no longer a disk bundle, but instead a bundle of h-cobordisms of the product of a disk with lens spaces.

Bunke and Gepner have reformulated the Becker–Gottlieb transfer in the context of derived algebraic K-theory [BG13]. Their work leads to the Transfer Index Conjecture, essentially a derived version of the parametrized index theorem of Dwyer, Weiss, and Williams. This conjecture suggests as a corollary the existence of certain classes in algebraic K-theory of a ring of integers in a number field. The authors prove that the Hatcher construction produces nontrivial representatives for these classes in special cases.

The Farrell–Hsiang [FH78] results on diffeomorphism groups of disks relative to their boundary prove that in the pseudoisotopy stable range,

$$(1) \quad \pi_i \mathrm{BDiff}_\partial(D^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 3 \pmod{4} \text{ and } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

According to [Igu02], the nontrivial generator for odd dimensional disks can be obtained from Hatcher’s construction. There has been much recent progress outside of the stable range by a number of authors, including distinct but related work by Kupers, Randal-Williams, Watanabe, and Weiss. In [Wei15] Weiss identifies nontrivial Pontryagin classes  $p_{n+k} \in H^{4n+4k}(\mathrm{BTop}(2n); \mathbb{Q})$ . These are shown to evaluate nontrivially on  $\pi_{4n+4k}(\mathrm{BTop}(2n); \mathbb{Q})$ . It then follows by the Morlet equivalence that there must be nonzero rational homotopy in  $\pi_*(\mathrm{BDiff}_\partial(D^{2n}))$  outside of the stable range. One might reasonably suspect that these examples could produce unstable exotic smoothings of manifold bundles. However, stably these classes are trivial, so they do not provide counterexamples to the Rigidity Conjecture.

Fiberwise Poincaré–Hopf theorems first appeared in work by Brumfiel and Madsen [BM76], and have proven to be a useful computational tool with many applications. For instance, in [MT01] the authors then apply the theorem towards early progress on the Mumford conjecture. Similar theorems are used in [RW08] to compute the  $\pmod{2}$  homology of the stable nonorientable mapping class group, as well as in [Rei19] to establish the existence of ‘non-kinetic’ smooth bundles over the classifying space  $BSU(2)$ . Douglas [Dou06] gave alternative proofs to [BM76] using Dold’s Euclidean neighborhood retracts. This paper establishes fiberwise Poincaré–Hopf theorems for smooth manifold bundles admitting fiberwise Morse functions, and provided motivation for the present work. Our results provide fiberwise Poincaré–Hopf theorems for bundles that admit fiberwise generalized Morse functions. This is a significant strengthening, as all bundles admit fiberwise generalized Morse functions, but bundles rarely admit fiberwise Morse functions.

This paper is concerned with a characteristic in the homotopy fiber of the fibration

$$\Gamma_B \mathcal{H}_B^\%(M) \rightarrow \Gamma_B Q_B(M) \rightarrow \Gamma_B A_B^\%(M)$$

that arises from a nullhomotopy of the excisive A-theory characteristic. One could naturally ask about characteristics in the homotopy fiber of the fibration

$$\Gamma_B \Omega \mathrm{Wh}_B^{\mathrm{PL}}(M) \rightarrow \Gamma_B A_B^\%(M) \rightarrow \Gamma_B A_B(M)$$

given by a nullhomotopy of the ordinary A-theory characteristic. This is the premise of Steimle’s PhD thesis [Ste10], in which the author studies the ‘parametrized excisive characteristic’. One of the main technical results of their work is an additivity theorem for the parametrized excisive characteristic, which parallels the Poincaré–Hopf theorems in the present work, if they were restricted to bundles admitting fiberwise Morse decompositions. For comparison, the smooth structure class appearing in  $\Gamma_B \mathcal{H}_B^\%(M)$  concerns the existence of stable exotic smoothings of fiber bundles, whereas the parametrized excisive characteristic appearing in  $\Gamma_B \Omega \mathrm{Wh}_B^{\mathrm{PL}}(M)$  concerns the existence of topological manifold bundles whose projection maps are homotopic to stabilizations of an arbitrary map of compact topological manifolds.

### 3. KEY INVARIANTS AND CONSTRUCTIONS

In this section we summarize the invariants and constructions that appear in this work. We begin by recalling two equivalent definitions of higher torsion resulting from work by Igusa and Klein, as well as Dwyer, Weiss, and Williams. Next we give the formal statement of the Rigidity conjecture using the definition of the smooth structure class from [GI14]. Then we state and prove an alternative homotopy theoretic definition of the smooth structure class. Finally, we reproduce

Hatcher's example of an odd dimensional disk bundle that is an exotic smoothing of a trivial bundle, and show how Goette and Igusa use a variation, the immersed Hatcher construction, to produce exotic smoothings of bundles with odd dimensional fibers.

**3.1. Higher Torsion Invariants.** In this subsection we outline the main theories of higher torsion, emphasizing the role that generalized Morse functions play in computations of these invariants.

Igusa–Klein torsion is obtained from a smooth manifold bundle using Morse theory. Recall Waldhausen's theorem [Wal87] that  $A(X)$  splits as the product  $Q(X_+) \times \text{Wh}^{\text{Diff}}(X)$ . Igusa–Klein torsion relies on a rational model for the loop space of the smooth whitehead space. This model, referred to as the Whitehead category, is constructed so that under sufficient conditions one can use the data of a fiberwise generalized Morse function to construct a map from the base to the Whitehead category [Igu02, Igu08, Ohr19]. Pulling back a particular rational characteristic class in the cohomology of the Whitehead space along this map gives the higher Franz–Reidemeister torsion of the bundle as a cohomology class in the base.

The model for the smooth Whitehead space is given in detail in [Igu02, Igu05]. We merely point out that the zero simplices are given by the fiberwise Morse complexes over points in the base, and the one simplices capture the combinatorics of the Morse complex as it passes through birth-death critical points. This construction relies heavily on the geometry of fiberwise generalized Morse functions, which we discuss at length in Section 3.

The smooth torsion of Dwyer, Weiss, and Williams is defined in purely homotopy theoretic terms, yet it surprisingly captures information about the smooth structure of the bundle. The starting point is the commutative diagram below, which was proven to commute for smooth manifold bundles in [DWW03]. In the diagram below,  $\eta$  is the unit map,  $\alpha$  is the assembly map, and  $\lambda$  is the linearization functor.

$$\begin{array}{ccc}
 B & \xrightarrow{\text{tr}(p)} & Q(M_+) \\
 & \searrow \chi^\%(p) & \downarrow \eta \\
 & & A^\%(M) \\
 & \searrow \chi^h(p) & \downarrow \alpha \\
 & & A(M) \\
 & \searrow & \downarrow \lambda \\
 & & K(\mathbb{R})
 \end{array}$$

For a smooth fiber bundle with acyclic fibers one can construct a nullhomotopy of the composition  $\lambda \circ \alpha \circ \eta \circ \text{tr}(p)$ . This nullhomotopy provides a map to the homotopy fiber of the composition  $Q(M_+) \rightarrow K(\mathbb{R})$ , which is  $\Omega \text{Wh}^{\text{Diff}}(M)$  by Waldhausen's theorem. Thus we have a lift

$$\begin{array}{ccc}
 & & \Omega \text{Wh}^{\text{Diff}}(M) \\
 & \nearrow \tau^s(p) & \downarrow \\
 B & \xrightarrow{\text{tr}(p)} & Q(M_+) \\
 & \searrow & \downarrow \lambda \circ \alpha \circ \eta \\
 & & K(\mathbb{R})
 \end{array}$$



The top diagonal map is the smooth torsion. We produce a characteristic class in  $B$  by pulling back a cohomology class in  $\Omega\mathrm{Wh}^{\mathrm{Diff}}(M)$  known as the universal Franz–Reidemeister torsion invariant. This class is constructed from the Kamber–Tondeur form in Volodin K-theory [Igu02].

In [BDW09] the authors explicitly construct smooth torsion for bundles with acyclic fibers as well as unipotent bundles by using ‘partitions’ from Waldhausen’s manifold approach. The manifold approach gives an explicit model for  $Q(X_+)$  and  $A(X)$  and the authors carefully study linearization maps out of this model to algebraic K-theory, explicitly constructing the nullhomotopies in K-theory for bundles satisfying the assumptions. In [BDKW11] the authors go on to prove that the smooth torsion invariants satisfy Igusa’s axioms for higher torsion [Igu08].

There is a third construction of higher torsion due to Bismut and Lott [BL95] using a purely analytic approach. The axiomatic approach by Igusa was the first attempt to unify the three definitions. Recently, Ohrt in [Ohr19] proved that the Igusa Klein torsion and the smooth torsion of Dwyer, Weiss, Williams, Badzioch, and Dorabiala agree. The main thrust of the proof is to use unpublished expansion category models of Igusa and Waldhausen to provide a unifying model in which one can compare the definitions of the two torsion invariants. The expansion categories are Waldhausen categories that model the smooth Whitehead space,  $Q(X_+)$ , and  $A(X)$  and again rely heavily on combinatorial data obtained from generalized Morse functions.

As mentioned before, Igusa Klein torsion is more computable than DWW torsion (of course, since these invariants are equivalent, computations of one translate to computations of the other). The key formula that is used to compute Igusa Klein torsion is the Framing Principle, Theorem 4.11 in [Igu08]. Loosely, it states that the higher torsion invariants can be computed in terms of fiberwise generalized Morse functions. In particular, it decomposes the higher Franz–Reidemeister torsion class into two terms, the first is an algebraic torsion of a parametrized chain complex associated to the generalized Morse function, and the second is a Chern character of the complexification of the negative eigenspace bundle on the critical locus of the generalized Morse function. The framing principle can be used to prove that on even dimensional manifold bundles higher torsion is congruent to a Miller–Morita–Mumford class. The method of proof is to multiply the fiberwise generalized Morse function by  $-1$ . This has the effect of negating the algebraic term and leaving the Chern character the same. Adding together the formulas given by the framing principle for  $f$  and  $-f$  yields the result. The idea behind the framing principle, that higher torsion invariants can be computed in terms of the data of the critical locus of a generalized Morse function, and the consequences for even dimensional manifold bundles are the main motivations for the techniques developed in this work.

**3.2. Statement of the Rigidity Conjecture.** In this section we summarize the results of [GIW14, GI14], and explicitly state the Rigidity Conjecture, lending special attention to the crucial role played by the higher Franz–Reidemeister torsion invariants of Igusa and Klein and the Hatcher construction. We begin by stating precisely what we mean by ‘exotic smoothing’.

**Definition 3.1.** Suppose that  $p : M \rightarrow B$  is a smooth manifold bundle with fiber  $F$ , where  $F$ ,  $M$ , and  $B$  are compact smooth manifolds. Let  $T^\vee M$  be the vertical tangent bundle of  $M$ . An *exotic smooth structure* on  $M$  is another smooth bundle  $M' \rightarrow B$  with a fiberwise tangential homeomorphism  $h : M \rightarrow M'$ . This means that  $h$  is a fiberwise homeomorphism covered by an isomorphism of vector bundles  $T^\vee M \cong T^\vee M'$ .

Let  $\tilde{\mathcal{S}}_B(M)$  denote the space of exotic smoothings of  $M$ . Let  $\tilde{\mathcal{S}}_B^s(M)$  be the stabilization obtained by taking a direct limit with respect to all linear disk bundles of  $M$  (see Section 3.3 or [GIW14] for details). Recall the stable space of h-cobordisms on  $X$ , denoted  $\mathcal{H}(X)$ . Finally, let  $\Gamma_B \mathcal{H}_B^\%(M)$  denote the space of sections of the fiberwise homology bundle obtained by applying fiberwise smash with  $\mathcal{H}(\ast)$ . Then there is a weak equivalence

$$(2) \quad \tilde{\mathcal{S}}_B^s(M) \simeq \Gamma_B \mathcal{H}_B^\%(M)$$

which follows from the results of [DWW03] and is proven independently in [GIW14]. They go on to compute  $\pi_0 \Gamma_B \mathcal{H}_B^\%(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{\dim B - 4k}(M; \mathbb{R})$  using the fact that the rational homotopy type of  $\mathcal{H}(\ast)$  is a wedge of Eilenberg-MacLane spaces.

**Definition 3.2.** Let  $M$  be fiberwise tangentially homeomorphic to  $M'$  so that  $[M'] \in \tilde{\mathcal{S}}_B(M)$ . Then let the smooth structure class  $\Theta(M, M')$  be the image of  $[M']$  under the composition

$$\pi_0 \tilde{\mathcal{S}}_B(M) \rightarrow \pi_0 \tilde{\mathcal{S}}_B^s(M) \otimes \mathbb{R} \cong \pi_0 \Gamma_B \mathcal{H}_B^\%(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{\dim B - 4k}(M; \mathbb{R})$$

This definition makes use of the equivalence in (2), but does not explicitly make use of the map. In Section 3.3 we will give an alternative proof of (2). One outcome of this proof will be a detailed description of the map. This will be used to provide a homotopy theoretic refinement of the smooth structure class.

The smooth structure class is proven in [GI14] to satisfy the following duality relationship with the relative higher Franz–Reidemeister torsion class of Igusa and Klein, which is a characteristic class in the cohomology of  $B$ .

$$(3) \quad p_* \Theta(M', M) = D\tau^{IK}(M', M).$$

The relation above allows one to compute the relative smooth structure class in situations where torsion is computable and  $p_*$  is trivial. One such instance is the Hatcher construction, which produces exotic smoothings of odd dimensional disk bundles that have nontrivial higher torsion invariants [BDKW11, BG13, Igu02, Igu05]. By gluing the exotic disk bundles from Hatcher’s example into smooth manifold bundles, one can produce exotic smoothings that vary the relative torsion class, and thus vary the relative structure class. This construction is called the immersed Hatcher construction, and it leads to the following theorem, which is a reformulation of Theorem 3.1.1 in [GI14].

**Theorem** (Main Theorem of [GI14]). *Rationally and stably the immersed Hatcher construction gives all possible exotic smooth structures on smooth manifold bundles with closed odd dimensional fibers.*

When the smooth bundle has closed even dimensional fibers, the immersed Hatcher construction does not produce exotic smooth structures. If the vertical boundary is nonempty, then the most that can be done is to produce an exotic smooth structure on the vertical boundary. Furthermore, on an even dimensional manifold bundle, the higher torsion classes are known to be tangential, in particular they are congruent to constant multiples of Miller–Morita–Mumford classes of the vertical tangent bundle [Igu05]. This implies that the relative torsion class  $\tau^{IK}(M', M)$  is zero since  $M'$  is fiberwise tangentially homeomorphic to  $M$ . Thus,  $p_* \Theta(M', M)$  is also trivial. This is evidence for the following Rigidity Conjecture.

**Conjecture 3.3** (Rigidity Conjecture of Goette and Igusa – 0.3.3 in [GI14]). *The stable smooth structure class vanishes when the fiber is a closed oriented even dimensional manifold:*

$$\Theta(M', M) = 0$$

*In other words, rationally and stably, there are no exotic smooth structures on manifold bundles with closed oriented even dimensional fibers.*

Our main theorem, Theorem F, asserts that this conjecture is true. The proof of this theorem relies on Theorem D and Proposition 6.1.

**3.3. Homotopical Definition of the Smooth Structure Class.** In this section we give a homotopy theoretic definition of the smooth structure class which refines the homology class  $\Theta(M, M')$  from Definition 3.2.

**Definition 3.4.** Suppose that  $p : M \rightarrow B$  and  $p' : M' \rightarrow B$  are fiberwise tangentially homeomorphic smooth manifold bundles with compact fibers so that there is a nullhomotopy of the difference  $\chi^\%(p) - \chi^\%(p')$ . This nullhomotopy gives a lift as in the following diagram:

$$\begin{array}{ccc}
 & & \mathcal{H}^\%(M) \\
 & \nearrow \theta(M, M') & \downarrow \\
 & & Q(M_+) \\
 & \nearrow \text{tr}(p) - \text{tr}(p') & \downarrow \eta \\
 B & \xrightarrow{\chi^\%(p) - \chi^\%(p')} & A^\%(M)
 \end{array}$$

This lift we denote by  $\theta(M, M')$ , and refer to as the homotopical smooth structure class.

**Proposition 3.5.** The definitions of the smooth structure class given in Definition 3.2 and Definition 3.4 agree. That is, the image of  $\theta(M, M')$  under the map  $\Gamma_B \mathcal{H}_B^\%(M) \rightarrow \pi_0 \Gamma_B \mathcal{H}_B^\%(M) \otimes \mathbb{R}$  is congruent to  $\Theta(M, M')$ .

The proof of this proposition will be given at the end of this section. First, we will give an explicit construction of the unstable and stable spaces of exotic smoothings,  $\tilde{\mathcal{S}}_B(M)$  and  $\tilde{\mathcal{S}}_B^s(M)$  from [GIW14]. Then we will observe that the stable space of exotic smoothings is homotopy equivalent to an auxilliary space. Finally, we will state the result from [DWW03] that this auxilliary space is homotopy equivalent to  $\Gamma_B \mathcal{H}_B^\%(M)$ . There are two consequences of this result: a proof of (2) from Section 3.2, and a proof of Proposition 3.5.

Let  $\mathcal{S}_\bullet^t(n)$  denote the simplicial set of  $n$ -dimensional compact topological manifolds. We can identify the geometric realization of this simplicial set as a more familiar space  $|\mathcal{S}_\bullet^t(n)| \simeq \coprod_M \text{BHomeo}(M)$ , where the disjoint union varies over all homeomorphism classes of compact  $n$ -dimensional manifolds  $M$ . Likewise, define the simplicial set  $\mathcal{S}_\bullet^d(n)$  of  $n$ -dimensional compact smooth manifolds. Again we have an identification with a more familiar space:  $|\mathcal{S}_\bullet^d(n)| \simeq \coprod_M \text{BDiff}(M)$  where the disjoint union varies over all diffeomorphism classes of  $n$ -dimensional compact smooth manifolds. We also have an intermediate moduli space  $\tilde{\mathcal{S}}_\bullet^t(n)$  which consists of those compact topological  $n$  manifolds with a vector bundle structure on their topological tangent microbundle. The spaces  $|\mathcal{S}_\bullet^d(n)|$ ,  $|\tilde{\mathcal{S}}_\bullet^t(n)|$ , and  $|\mathcal{S}_\bullet^t(n)|$  respectively classify: smooth manifold bundles with compact fibers of dimension  $n$ , fiber bundles of compact topological manifolds of dimension  $n$  along with fiberwise linearizations of the vertical tangent microbundles, and fiber bundles of compact topological manifolds of dimension  $n$ . We can consider the space of maps from a compact base  $B$  into each classifying space, which are related by forgetful functors:

$$|\mathcal{S}_\bullet^d(n)|^B \rightarrow |\tilde{\mathcal{S}}_\bullet^t(n)|^B \rightarrow |\mathcal{S}_\bullet^t(n)|^B$$

Then for a smooth manifold bundle  $p : M \rightarrow B$  we define  $\tilde{\mathcal{S}}_B(M)$  to be the homotopy fiber of  $|\mathcal{S}_\bullet^d(n)|^B \rightarrow |\mathcal{S}_\bullet^t(n)|^B$  over the point  $M \in |\mathcal{S}_\bullet^t(n)|^B$ . The stable space  $\tilde{\mathcal{S}}_B^s(M)$  is the direct limit of  $\tilde{\mathcal{S}}_B(M)$  with respect to linear disk bundles.

Now we define an auxilliary space. Consider again the smooth manifold bundle  $p$ , and denote the vertical tangent bundle by  $\gamma$ . Let  $\mathcal{S}^d(p, \gamma)$  denote the subspace of  $|\mathcal{S}_\bullet^d(n)|^B$  consisting

of those smooth manifold bundles that are fiberwise tangentially homeomorphic to  $(p, \gamma)$ . Let  $\tilde{\mathcal{S}}^t(p, \gamma)$  denote the subspace of  $|\tilde{\mathcal{S}}_\bullet^t(n)|^B$  consisting of those topological manifold bundles with fiberwise linearizations of their topological tangent microbundles that are fiberwise tangentially homeomorphic to  $(p, \gamma)$ . Then the following diagram commutes:

$$\begin{array}{ccccc} |\mathcal{S}_\bullet^d(n)|^B & \longrightarrow & |\tilde{\mathcal{S}}_\bullet^t(n)|^B & \longrightarrow & |\mathcal{S}_\bullet^t(n)|^B \\ \text{inc.} \uparrow & & \text{inc.} \uparrow & & \uparrow \\ \mathcal{S}_n^d(p, \gamma) & \longrightarrow & \tilde{\mathcal{S}}_n^t(p, \gamma) & \longrightarrow & \star \end{array}$$

It is clear that the homotopy fiber of the map  $\mathcal{S}_n^d(p, \gamma) \xrightarrow{\text{inc.}} \tilde{\mathcal{S}}_n^t(p, \gamma)$  over the point  $p$  is homotopy equivalent to the homotopy fiber of  $|\mathcal{S}_\bullet^d(n)|^B \rightarrow |\tilde{\mathcal{S}}_\bullet^t(n)|^B$  over the point  $p$ , thus the stabilizations of the two spaces must agree. Denote the stabilization of the map  $\mathcal{S}_n^d(p, \gamma) \rightarrow \tilde{\mathcal{S}}_n^t(p, \gamma)$  by  $\mathcal{S}^d(p, \gamma) \rightarrow \mathcal{S}^t(p, \gamma)$ , and denote the stable homotopy fiber at the point  $p$  by  $\tilde{\mathcal{S}}(p, \gamma)$ . Then we are left with a homotopy equivalence

$$(4) \quad \tilde{\mathcal{S}}(p, \gamma) \xrightarrow{\sim} \tilde{\mathcal{S}}_B^s(M)$$

Now we explain why  $\tilde{\mathcal{S}}(p, \gamma)$  is homotopy equivalent to  $\Gamma_B \mathcal{H}_B^\%(M)$ . The vertical arrows in the diagram below compute characteristics on each stable moduli space. On smooth bundles one computes the Becker–Gottlieb transfer. On fiberwise linearized topological bundles one computes the excisive A-theory characteristic. Both of these characteristics are refinements of the fiberwise A-theory characteristic of  $p$ ,  $\chi^h(p) \in \Gamma_B A_B(E)$ , so the diagram below commutes by the main theorems of [DWW03]

$$(5) \quad \begin{array}{ccccc} \mathcal{S}^d(p, \gamma) & \longrightarrow & \mathcal{S}^t(p, \gamma) & \longrightarrow & \star \\ \downarrow \text{tr}(-) & & \downarrow \chi^\%( - ) & & \downarrow \chi^h(p) \\ \Gamma_B Q_B(E_+) & \longrightarrow & \Gamma_B A_B^\%(E) & \longrightarrow & \Gamma_B A_B(E) \end{array}$$

However, the following theorem is a stronger statement.

**Theorem** (Theorem 12.8 in [DWW03]). *The two squares and thus the outer rectangle in diagram (6) are homotopy cartesian.*

The homotopy fiber of the top row of the left square is  $\tilde{\mathcal{S}}(p, \gamma)$  and the homotopy fiber of the bottom row is  $\Gamma_B \mathcal{H}_B^\%(M)$ . The map  $\tilde{\mathcal{S}}(p, \gamma) \rightarrow \Gamma_B \mathcal{H}_B^\%(M)$  is the induced map on homotopy fibers, and it is an equivalence because the left square is homotopy cartesian. Combining this with the homotopy equivalence  $\tilde{\mathcal{S}}(p, \gamma) \xrightarrow{\sim} \tilde{\mathcal{S}}_B^s(M)$  yields the homotopy equivalence  $\tilde{\mathcal{S}}_B^s(M) \simeq \Gamma_B \mathcal{H}_B^\%(M)$  as in (2).

*Proof of Proposition 3.5.* From the discussion above we have a description of the map  $\tilde{\mathcal{S}}_B^s(M) \xrightarrow{\sim} \Gamma_B \mathcal{H}_B^\%(M)$  as the induced map on the homotopy fibers in following diagram.

$$(6) \quad \begin{array}{ccccccc} \tilde{\mathcal{S}}_B^s(M) & \longrightarrow & \mathcal{S}^d(p, \gamma) & \longrightarrow & \mathcal{S}^t(p, \gamma) & \longleftarrow & p' \\ \downarrow \simeq & & \downarrow \text{tr}(-) & & \downarrow \chi^\%( - ) & & \downarrow \\ \Gamma_B \mathcal{H}_B^\%(E) & \longrightarrow & \Gamma_B Q_B(E_+) & \longrightarrow & \Gamma_B A_B^\%(E) & \longleftarrow & \chi^\%(p') \end{array}$$

From this it is clear that the component of  $\Theta(M, M') \in \pi_0 \Gamma_B \mathcal{H}_B^\%(M)$  is the data of a nullhomotopy of  $\chi^\%(p) - \chi^\%(p')$ , as in Definition 3.4.  $\square$

**3.4. Hatcher Construction.** Hatcher's example produces a smooth disk bundle over a compact base that is homeomorphic but not diffeomorphic to a trivial disk bundle. The construction proceeds in five steps below. Briefly, the idea is to use a vector bundle in the kernel of the J-homomorphism to obtain a disk bundle for which the corresponding sphere bundle is fiberwise homotopy equivalent to a trivial sphere bundle. After stabilizing so that this homotopy equivalence produces an embedding, we glue this disk bundle into the center of a trivial sphere bundle on which the fiber is a solid torus. The result is a disk bundle with the desired property. There are several descriptions of this construction in the literature, the construction used here is identical to the one used in [GI14], and when the base is a  $4k$ -dimensional sphere a similar construction can be found in [Goe01]. We proceed step by step, starting only with a compact manifold  $B$  of dimension  $q$ , and a continuous map  $B^q \rightarrow G/O$ .

- (1) We extract from the map  $B^q \rightarrow G/O$  an  $n$ -dimensional vector bundle whose associated sphere bundle,  $S^{n-1}(\xi) \rightarrow B$ , is fiber homotopy trivial. Here  $n \geq q + 1$ .
- (2) Let  $\eta$  be an  $m$ -dimensional vector bundle on  $B$  complementary to  $\xi$ . Construct the bundle  $D^n(\xi) \oplus D^m(\eta)$  over  $B$ . This bundle contains a subbundle  $S^{n-1}(\xi) \oplus D^m(\eta)$ . This will make up the boundary of the disk bundle that we later glue inside of the bundle of solid tori.
- (3) Construct the trivial sphere bundle  $S^{n-1} \times (I \times D^m) \times B \rightarrow B$ . This is the bundle of solid tori. We group the  $(I \times D^m)$  term because it is helpful to think of this as a  $D^{m+1}$  disk.
- (4) We construct a fiberwise smooth embedding  $S(j) : S^{n-1}(\xi) \oplus D^m(\eta) \hookrightarrow S^{n-1} \times D^m \times B$
- (5) Now we attach the disk bundle to the solid torus to obtain a smooth disk bundle  $E^{n,m}(\xi)$ .

$$E^{n,m}(\xi) := D^n(\xi) \oplus D^m(\eta) \cup_{S(j)} S^{n-1} \times D^m \times B$$

The bundle  $E^{n,m}(\xi)$  is Hatcher's example.

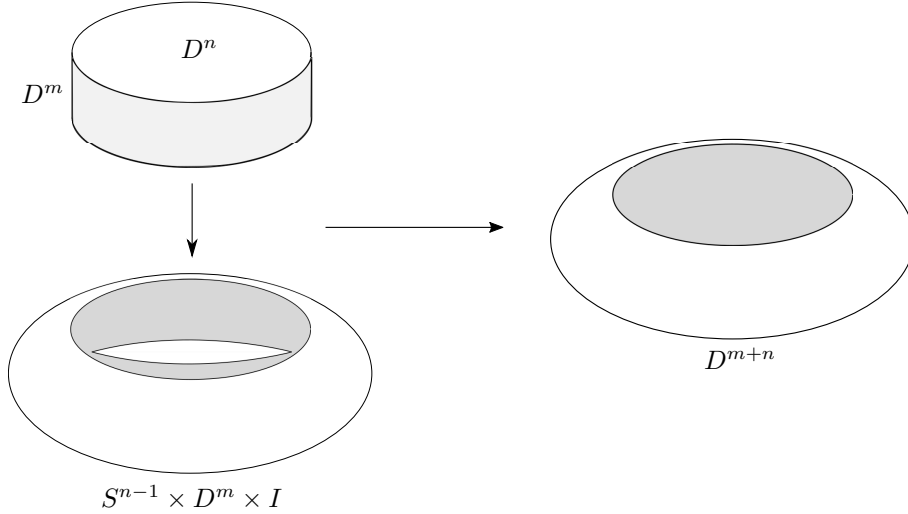


FIGURE 1. Fiberwise surgery in Hatcher's example

Steps (1) and (4) above require justification. The map  $B \rightarrow G/O$  classifies a stable vector bundle whose spherical fibration is trivial. Since the map  $BO(n) \rightarrow BO$  is  $q + 1$  connected when  $n \geq q + 1$ , this stable vector bundle is given by a unique  $n$ -plane bundle  $\xi$ . See [GI14] for a proof that  $S^{n-1}(\xi) \rightarrow B$  is fiber homotopy trivial (one must check that the composition

$B \xrightarrow{\xi} BO_n \rightarrow BG_n$  is nullhomotopic). The construction of the embedding  $S(j)$  follows from the following crucial lemma.

**Lemma 3.6** (Lemma 1.2.1 in [GI14]). If  $m > n > q$  then there is a smooth fiberwise embedding of pairs:

$$j : (D^n(\xi), S^{n-1}(\xi)) \rightarrow (D^n, S^{n-1}) \times D^m \times B$$

over  $B$  which is the standard embedding over  $\partial_0 B$  and which is transverse to  $S^{n-1} \times D^m$ . Furthermore, if  $m \geq q + 3$  then this fiberwise embedding will be unique up to fiberwise isotopy.

Let  $\eta$  be the normal bundle to the embedding  $j$ . Then we have a codimension 0 embedding

$$D(j) : D^n(\xi) \oplus D^m(\eta) \hookrightarrow D^n \times D^m \times B$$

from which we obtain  $S(j)$  by restricting to the boundary  $\partial D^n(\xi) \oplus D^m(\eta)$ .

**Remark 3.7.** The constraint  $m > n > q$  implies that  $m+n \geq 2q+3$ . Thus Hatcher's construction can only be used to produce disk bundles where the dimension of the fiber is  $2 \dim B + 3$ . Our intent is to perform surgery on manifold bundles by gluing in Hatcher's construction, so these bundles must also have the property that the dimension of the fiber is at least  $2 \dim B + 3$ . Thus this constraint appears in the statement of Theorem E.

**3.4.1. Immersed Hatcher Construction.** The goal of the immersed Hatcher construction is to use Hatcher's disk bundle to produce exotic smoothings of smooth manifold bundles that are supported on embedded disk bundles. Roughly, we attach thickenings of Hatcher's disk bundle along the top  $M \times 1$  boundary of  $M \times I$  at surgery sites which are tubular neighborhoods of the critical locus of a fiberwise generalized Morse function. Since Hatcher's construction only produces disk bundles whose fibers have dimension at least  $2 \dim B + 3$ , the same is true of the immersed Hatcher construction. The construction proceeds in four steps.

- (1) First, given a fiberwise generalized Morse function  $f : M \rightarrow \mathbb{R}$ , let  $L$  denote the submanifold of the critical locus consisting of those critical points of even index.  $L$  will likely have multiple components and has nonempty boundary consisting of the birth-death critical points. There is an embedding  $L \xrightarrow{\lambda} M$  and an immersion  $\lambda : L \rightarrow B$  given by restricting the projection  $p : M \rightarrow B$ . The reason  $\lambda$  is an immersion and not an embedding is because fibers may have multiple even index critical points. Let  $T$  be a tubular neighborhood of  $L$ , so that the following diagram commutes

$$\begin{array}{ccc} T & \xrightarrow{D(\tilde{\lambda})} & M \\ \downarrow \pi & & \downarrow p \\ L & \xrightarrow{\lambda} & B \end{array}$$

- (2) Construct a Hatcher handle  $B^{n,m}(\xi, \eta)$  by attaching the top of  $T \times I$  to Hatcher's construction  $E^{n,m}(\xi)$ .
- (3) Attach the Hatcher handle to the top of  $T \times I$ .

$$E_L^{n,m+1}(\xi, \eta) := T \times I \cup B^{n,m}(\xi, \eta)$$

- (4) Now we excise  $T \times I$  from  $M \times I$  and glue in  $E_L^{n,m+1}(\xi, \eta)$ . The result is

$$E_+^{n,m}(M, \tilde{\lambda}, \xi) = (M - T) \times I \cup E_L^{n,m+1}(\xi, \eta)$$

The top of  $E_+^{n,m}(M, \tilde{\lambda}, \xi)$ , i.e.  $M \times I$  plus the attached Hatcher handles, forms a smooth manifold bundle fiberwise tangentially homeomorphic to  $M$ . In other words, we obtain an element

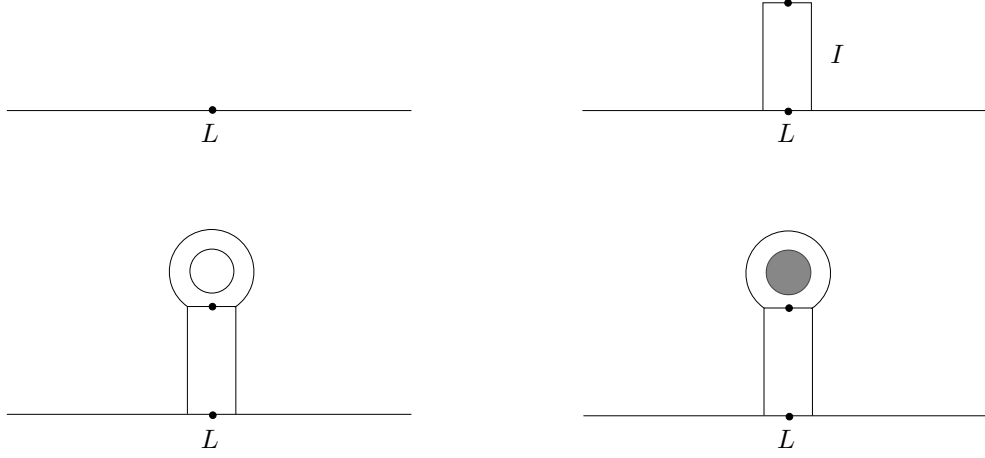


FIGURE 2. This figure depicts steps 1, 2, and 3 of the immersed Hatcher construction on a single fiber, in which case  $L$  is a point.

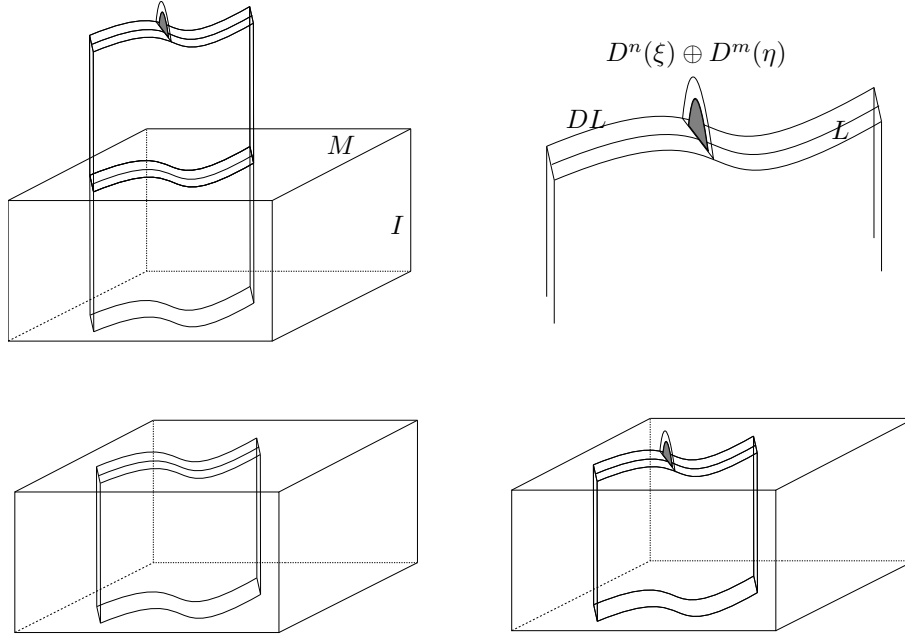


FIGURE 3. This figure depicts steps 3 and 4 of the immersed Hatcher construction when  $L$  is one dimensional.

$M' \in \tilde{\mathcal{S}}_B^s(M)$ .  $M'$  has a property that is essential for computing  $\Theta(M, M')$ , which we state in the proposition below.

**Proposition 3.8.**  $M'$  admits a fiberwise generalized Morse function  $f' : M' \rightarrow \mathbb{R}$ , the critical locus of which is homeomorphic to  $\Sigma(f) \subset M$ .

*Proof.* We construct this function by first constructing a fiberwise Morse function on  $E_L^{n,m+1}(\xi, \eta)$  using the two handle attachments in the Hatcher construction. After cancelling the handles, this gives a fiberwise generalized Morse function on  $E_L^{n,m+1}(\xi, \eta)$ . Using this function, we get a fiberwise generalized Morse function on  $E_+^{n,m}(M, \tilde{\lambda}, \xi)$  which is given by projection to  $I$  on

everything outside of  $E_L$ , and is the generalized Morse function inside  $E_L$ . Finally we restrict these functions to the top of  $E_+^{n,m}(M, \lambda, \xi)$ . The critical locus of the resulting function is exactly  $L \cup_{\partial L} L$ . Since  $L$  was obtained by restricting to only those critical points of even index, this proves the statement.  $\square$

#### 4. REVIEW OF GENERALIZED MORSE FUNCTIONS

In this section we will prove that the critical loci of fiberwise generalized Morse functions have an essential transversality property. More specifically, the gradient vector field is transverse to the zero section of the vertical tangent bundle. While this condition is well known and frequently used for fiberwise Morse functions, it leads to surprising and substantial results for fiberwise generalized Morse functions. In particular, it implies that the critical set of a fiberwise generalized Morse function is a smooth manifold.

Let  $f : (E, \partial_0 E) \rightarrow (I, 0)$  be a fiberwise generalized Morse function. Such functions always exist on smooth fiber bundles when the dimension of the fiber is at least the dimension of the base [Igu90]. Let  $\Sigma(f)$  be the singular set of  $f$ . We must describe the local behavior of this function, i.e. we need generalizations of the Morse lemma. We begin by giving the local behavior in a neighborhood of a compact subset of the collection of Morse singularities as well as in a neighborhood of a compact subset of the collection of birth-death singularities.

In the unparametrized setting, we consider a function  $f$  on a single fiber. A Morse singularity of the function  $f$  can be written in the form

$$f(x) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

with respect to local coordinates  $(x_1, \dots, x_n) \in \mathbb{R}^i \times \mathbb{R}^{n-i}$ . At a birth-death point,

$$f(x) = -x_1^2 - \cdots - x_{i-1}^2 + x_i^3 + x_{i+1}^2 + \cdots + x_n^2$$

In the parametrized setting, we have the following proposition/summary from [Igu08]:

**Proposition 4.1.** In a generic  $p$ -parameter family of generalized Morse functions, birth-death points occur on a codimension one subspace of the parameter space. The family of functions  $f_t$  has the form

$$f_t(x) = -x_1^2 - \cdots - x_{i-1}^2 + x_i^3 + t_0 x_i + x_{i+1}^2 + \cdots + x_n^2$$

with respect to parameter coordinates  $t_0, \dots, t_{p-1}$  and  $t$ -dependent local coordinates  $(x_1, \dots, x_n)$  for  $M$ .

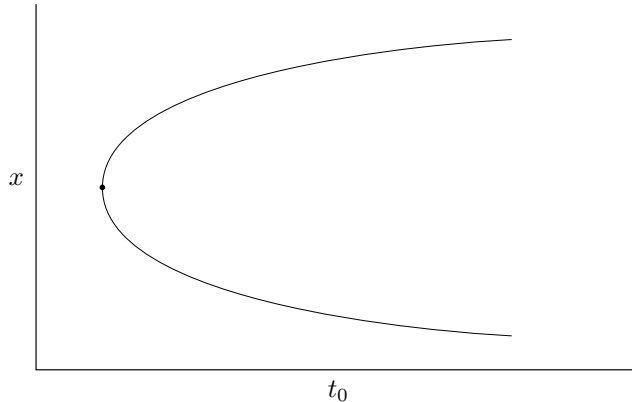


FIGURE 4. A depiction of the critical locus of a fiberwise generalized Morse function in the local neighborhood of a birth-death singularity.



Let  $p : E \rightarrow B^q$  be a smooth fiber bundle with compact base and fiber  $F^n$ . Let  $f : E \rightarrow \mathbb{R}$  be a fiberwise generalized Morse function. Then by the proposition above, in local coordinates at a birth-death singularity we have

$$f_t(x) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^3 + t_0 x_{i+1} + x_{i+2}^2 + \cdots + x_n^2.$$

where  $t_0$  is the unfolding direction. The gradient of this function is a section  $E \rightarrow TE$ , and if we take the gradient with respect to fiber coordinates, we get a section of the vertical tangent bundle,  $E \rightarrow T^\vee E$  (which could have also been obtained by projecting off of  $TE$ ). We can explicitly compute the map  $\nabla f : E \rightarrow T^\vee E$  as

$$(x, t) \mapsto \langle -2x_1, \dots, -2x_i, 3x_{i+1}^2 + t_0, 2x_{i+2}, \dots, 2x_n \rangle$$

Now the derivative of  $\nabla f$  is a map on tangent spaces which takes the form of a rectangular matrix of size  $(2n + q) \times (n + q)$ . We write this matrix below.

$$T_{(x,t)}E \mapsto T_{\nabla f(x,t)}(T^\vee E) \cong T^\vee E \oplus T^\vee E \rightarrow T^\vee E$$

The last map in this composition is the projection off of the nonidentity component. The tangent space in the domain is labeled using coordinates  $t_0, \dots, t_{q-1}$  in the base, and  $x_1, \dots, x_n$  in the fiber. In the target we add labels  $\frac{\partial}{\partial x_i}$  for  $i, \dots, n$  for the coordinates in the vertical tangent direction. Keep in mind that in the neighborhood of a birth-death singularity,  $t_0$  is always identified with the ‘unfolding’ direction.

$$\left( \begin{array}{c} x \\ t \\ \frac{\partial}{\partial x} \end{array} \right) \left( \begin{array}{cccccccccccccccc} 1 & & & & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & & & & \\ & & 1 & & & & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & & & & \\ & & & & 1 & & & & & & & & & & & & \\ & & & & & 1 & & & & & & & & & & & \\ & & & & & & 1 & & & & & & & & & & \\ & & & & & & & 1 & & & & & & & & & \\ & & & & & & & & 1 & & & & & & & & \\ & & & & & & & & & 1 & & & & & & & \\ & & & & & & & & & & 1 & & & & & & \\ & & & & & & & & & & & 1 & & & & & \\ & & & & & & & & & & & & 1 & & & & \\ & & & & & & & & & & & & & 1 & & & \\ & & & & & & & & & & & & & & 1 & & \\ & & & & & & & & & & & & & & & 1 & \\ -2 & & & & & & & & & & & & & & & & \\ & \ddots & & & & & & & & & & & & & & & \\ & & -2 & & & & & & & & & & & & & & \\ & & & 6x_{i+1} & & & & & & & & & & & & & \\ & & & & 2 & & & & & & & & & & & & \\ & & & & & \ddots & & & & & & & & & & & \\ & & & & & & 2 & & & & & & & & & & \\ & & & & & & & & & & & & & & & & \end{array} \right)$$

$\underbrace{\hspace{10em}}_{t_0}$

We have a smooth map  $\nabla f : E \rightarrow T^\vee E$ , and we can check to see whether the image of this map is transverse to the inclusion of the zero section of  $T^\vee E$ ,  $i_0 : E \rightarrow T^\vee E$ . If  $p$  is in  $\nabla f(E) \cap i_0 E$ , then  $\nabla f(E)$  is transverse to  $i_0 E$  if, for all  $a, b, p$  so that  $\nabla f(a) = i_0(b) = p$ ,

$$\text{Im}(D(\nabla f)(a)) \oplus \text{Im}(D(i_0)(b)) \rightarrow T_p(T^\vee E).$$

It is clear that the intersection  $p \in \nabla f(E) \cap i_0 E$  is the set of critical points of  $f$ , and as these points are either Morse critical points, or birth-death singularities, we handle each of these cases independently. In the event that  $p$  is a Morse critical point, it is a standard exercise that the map

above is surjective. The case of a birth-death singularity is identical, except in the row labelled by  $\frac{\partial}{\partial x_{i+1}}$ . At the birth-death singularity, the entry  $6x_{i+1}$  vanishes, and if it were not for the 1 in the  $t_0$  entry of the row, there would be no image in the 1-dimensional subspace spanned by  $\frac{\partial}{\partial x_{i+1}}$  of the matrix above. So we do have surjectivity and thus transversality, but only because of the derivative in the unfolding direction  $t_0$ . So transversality is a direct consequence of the unfolding behavior of a parametrized family of generalized Morse functions. We summarize this discussion in the following lemma:

**Lemma 4.2.** For  $f : E \rightarrow \mathbb{R}$  a fiberwise generalized Morse function on a smooth fiber bundle  $E \rightarrow B$  with compact base, the section  $\nabla f : E \rightarrow T^\vee E$  is transverse to the zero section of the vertical tangent bundle of  $E$ .

**Corollary 4.3.** The normal bundle  $\nu(\Sigma(f)) \xrightarrow{\pi} \Sigma(f)$  to the embedding  $\Sigma(f) \hookrightarrow E$  is isomorphic to the restriction of the vertical tangent bundle of  $E$  to  $\Sigma(f)$ ,  $T^\vee E|_{\Sigma(f)}$ .

4.0.1. *Ghost sets.* The image of the birth-death set in  $B$ ,  $p(\Sigma_1^n(f))$ , is known as the bifurcation set. The bifurcation set is a codimension zero submanifold of  $B$ . In the neighborhood of a birth-death singularity, the points given by  $x_i = 0$  and  $-\epsilon < t_0 < 0$  are inflection points on which the second derivative of  $f$  in the vertical direction vanishes. We call these points ghost points, and they allow us to define a ghost set,  $\Sigma_g^n(f)$ , which is formally a lift of a one-sided collar neighborhood of the bifurcation set. The ghost set is transverse to  $\Sigma(f)$ , as in the following figure.

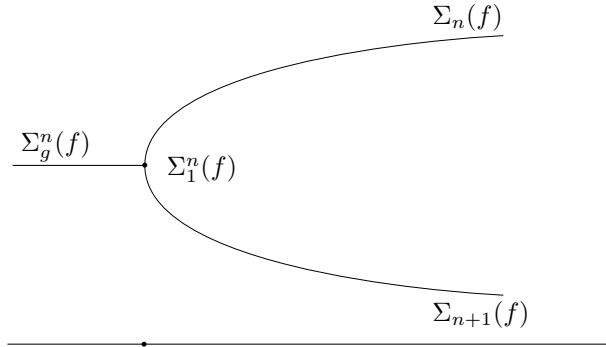


FIGURE 5. The ghost set is transversally attached to the critical locus at the birth-death set.

The ghost set is used to locally perturb  $\Sigma(f)$  so that the critical points of the generalized Morse function do not cancel. In particular, we consider the manifolds with corners  $\Sigma^n(f) \cup \Sigma_g^n(f)$  and  $\Sigma^{n+1}(f) \cup \Sigma_g^n(f)$ . Smoothing both of these manifolds gives us a smooth critical locus over a small simplex that intersects the bifurcation set. This construction of the ghost set is used to prove the framing principle [Igu08]. In this work the ghost set is used to conclude that locally in the neighborhood of a birth-death singularity, certain index maps of the gradient vector field are trivial.

## 5. FIBERWISE POINCARÉ–HOPF THEOREMS

In this section we prove three fiberwise Poincaré–Hopf theorems. Each theorem expresses a characteristic in terms of the critical locus of a fiberwise generalized Morse function and an index map defined using the gradient vector field of the function. First we prove a fiberwise Poincaré–Hopf theorem for the Becker–Gottlieb transfer. In the second section we extend this

theorem to a fiberwise Poincaré–Hopf theorem for the excisive A-theory characteristic. This is done by defining an index map for  $A^\%$ -theory that is compatible with the index map defined using the smooth vector field. By the smooth parametrized h-cobordism theorem [WJR13], we have the homotopy equivalence  $\Omega\mathrm{Wh}^{\mathrm{Diff}}(*) \simeq \mathcal{H}(*)$ , where the latter space is the stable h-cobordism space of a point. So the homotopy fiber of the map  $\Gamma_B \Sigma_B^\infty(M_+) \rightarrow \Gamma_B A_B^\%(M)$  is homotopy equivalent to  $\Gamma_B \mathcal{H}_B^\%(M)$ . Thus a Poincaré–Hopf theorem for the smooth structure class in  $\Gamma_B \mathcal{H}_B^\%(M)$  can be obtained from the previous theorems. This is done precisely in the third section.

For the remainder of this section, we assume that the bundle  $p : M \rightarrow B$  admits a fiberwise generalized Morse function. Such a function can always be obtained by [Igu90, Lur09, EM12].

**5.1. Fiberwise Poincaré–Hopf Theorem for the Becker–Gottlieb transfer.** There are several preliminaries before the statement and proof of Theorem A at the end of this section. We begin by reviewing the notation we will use for the fiberwise generalized Morse function and the associated data. This allows us to display the structure of our fiberwise Poincaré–Hopf theorems in diagram (7) below.

Let  $p : M^k \rightarrow B^q$  be a smooth manifold bundle with closed base and compact fiber  $F^{k-q}$  possibly with boundary. In the constructions that follows we will assume  $M$  is embedded in  $B \times \mathbb{R}^n$  with normal bundle  $\nu^{n+q-k}$  and vertical tangent bundle  $\tau^{k-q}$ . We will also assume  $B$  is embedded in  $\mathbb{R}^\ell$ .

The function  $f : M \rightarrow \mathbb{R}$  is a fiberwise generalized Morse function and we let the vector field  $X$  be the vertical gradient of  $f$ . The critical locus  $\Sigma(f)$  is a smooth submanifold of  $M$  that is a union of closed subsets

$$\Sigma(f) = \bigcup \bar{\Sigma}^n(f)$$

which intersect along the birth-death sets

$$\Sigma_1^n(f) = \bar{\Sigma}^n(f) \cap \bar{\Sigma}^{n+1}(f).$$

We denote by  $\mathcal{S}(f)$  the set of strata of  $\Sigma(f)$ . Elements in this set are  $q$ -manifolds  $\Sigma^n(f)$  with boundary  $\Sigma_1^n(f)$  consisting of the index  $n$  critical points and birth-death singularities, respectively. We let  $Z$  represent a generic stratum of  $\mathcal{S}(f)$ , and we let  $\overset{\circ}{Z}$  denote the interior of  $Z$ ,  $\Sigma^n(f) - \Sigma_1^n(f)$ . Then the tubular neighborhood of  $\overset{\circ}{Z}$  in  $M$  is a bundle  $\pi : D\overset{\circ}{Z} \rightarrow \overset{\circ}{Z}$  where  $DZ$  is an  $N$ -disk bundle on  $\overset{\circ}{Z}$  with compact fibers.

Our goal is to prove that the following diagram commutes. The precise formulation of this diagram is given in the statement of Theorem A.

$$(7) \quad \begin{array}{ccc} \mathbb{S} & \xrightarrow{\bar{\mathrm{tr}}(p)} & \Gamma_B \Sigma_B^\infty(M_+) \\ \bar{\mathrm{tr}}(\pi) \downarrow & & \uparrow + \\ \bigvee_{Z \in \mathcal{S}(f)} \Gamma_Z \Sigma_Z^\infty(DZ)_+ & \xrightarrow{\bigvee_{Z \in \mathcal{S}(f)} \overline{\mathrm{Ind}}_Z} & \bigvee_{Z \in \mathcal{S}(f)} \Gamma_Z \Sigma_Z^\infty(DZ)_+ \end{array}$$

To prove that diagram (7) commutes we must introduce several new definitions, which we enumerate below.

- (1) We first present the section spaces appearing in the diagram as homotopy limits in the base of homotopy colimits in the fiber. Definitions 5.1 and 5.2 indicate the diagram categories over which these limits and colimits are indexed. The categories we use are similar to the indexing categories for factorization homology and cohomology as in [AF15, AF19], though our categories contain only disks of cardinality one.
- (2) We use this presentation to give a refined version of the Becker–Gottlieb transfer in Definition 5.4.
- (3) Next we define our index map using the gradient vector field, and use it to define a refined version of the vector field transfer appearing in [BG91]. These are Definitions 5.6 and 5.7.
- (4) Finally, the first step in the proof of Theorem A is to prove that the Becker–Gottlieb transfer is homotopic to the vector field transfer when there is no vertical boundary, or when the vector field points outwards along the vertical boundary. This is proven in Proposition 5.9 after introducing an auxilliary term for the vector field transfer on the boundary in Definition 5.8.
- (5) The remainder of the proof of Theorem A produces a factorization of the vector field transfer in terms of the index map.

In our diagram categories we choose to work with one point compactifications of open embeddings of  $q$ -disks in  $B$  and codimension-zero  $k$ -disks in  $M$  properly embedded along the fibers.

**Definition 5.1.** The objects of the category  $\text{Disk}_q^{B/}$  are one point compactifications of  $q$ -disks with one component embedded in  $B$ . The morphisms are one point compactifications of open embeddings.

**Definition 5.2.** An object in  $\text{Disk}_{k/p^{-1}U}$  is an embedding of  $\mathbb{R}^q \times D^{k-q}$  into  $p^{-1}U$  so that  $p$  embeds  $\mathbb{R}^q \times \{0\}$  into  $B$ , and for each point  $x$  in  $\mathbb{R}^q \times 0$ ,  $\{x\} \times D^{k-q}$  is a disk with boundary embedded in the fiber  $M_{p(x)}$ . The morphisms in this category are inclusions.

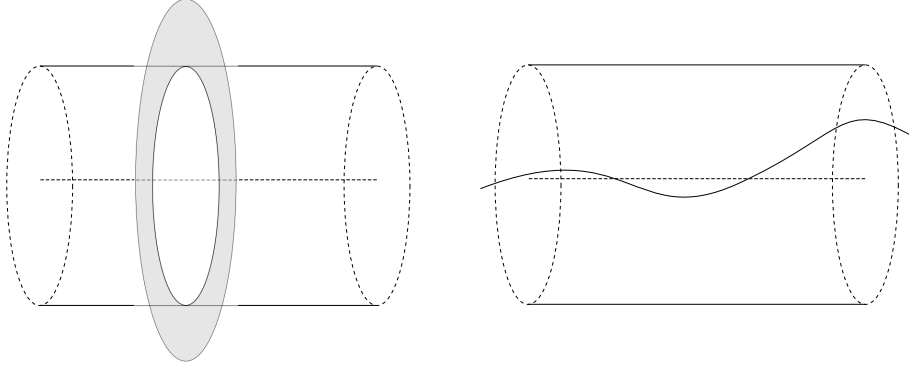


FIGURE 6. The disks in  $\text{Disk}_{k/p^{-1}U}$  are embedded so that the image in a particular fiber is a properly embedded disk, but the projection onto  $B$  is open. On the left is depicted an object in the category with the annulus represents a local neighborhood in a particular fiber. On the right is depicted a possible configuration of the disk relative to a  $q$ -dimensional submanifold of  $M$  on which the projection to  $B$  is an embedding. In practice this submanifold may belong to the set of critical points of a fiberwise Morse function.

**Proposition 5.3.** There are weak equivalences

$$\Gamma_B \Sigma_B^\infty M \simeq \operatorname{holim}_{U \in \operatorname{Disk}_q^{B/}} \Sigma^\infty(p^{-1}(U)^+) \simeq \operatorname{holim}_{U \in \operatorname{Disk}_q^{B/}} \operatorname{hocolim}_{V \in \operatorname{Disk}_{k/p-1} U} \Sigma^\infty(-)_+$$

*Proof.* The section space on the left would ordinarily be defined using the category of simplices in  $B$ ,  $\operatorname{Sing}(B)$ . The first equivalence follows from the fact that the functor  $\operatorname{ev}_0 : \operatorname{Disk}_q^{B/} \rightarrow \operatorname{Sing}(B)$ , which sends a disk to the zero simplex given by the point at the origin, is initial. This in turn is true because the functor  $\operatorname{Disk}_q^{B/} \rightarrow \mathcal{D}\operatorname{isk}_q^{B/}$  is a localization, and the functor  $\operatorname{ev}_0 : \mathcal{D}\operatorname{isk}_q^{B/} \rightarrow \operatorname{Sing}(B)$  is an equivalence of  $\infty$ -categories. Thus the composition is a localization, and is therefore both initial and final.

The second equivalence requires us to prove that

$$\operatorname{hocolim}_{V \in \operatorname{Disk}_{k/p-1} U} \Sigma^\infty(-)_+ \simeq \Sigma^\infty(p^{-1}(U)^+)$$

For this we use a hypercover argument as in [DI04]. It suffices to show that for each point  $x \in p^{-1}(U)$ , the full subcategory of  $\operatorname{Disk}_{k/p-1} U$  whose objects contain  $x$  has a contractible classifying space. This subcategory is cofiltered, so it is contractible.  $\square$

We will now construct a characteristic

$$(8) \quad \mathbb{S} \rightarrow \operatorname{holim}_{U \in \operatorname{Disk}_q^{B/}} \Sigma^\infty(p^{-1}(U)^+)$$

Temporarily let  $F$  denote the functor

$$U \mapsto \Sigma^\infty(p^{-1}(U)^+) \simeq \operatorname{hocolim}_{V \in \operatorname{Disk}_{k/p-1} U} \Sigma^\infty(-)_+$$

The characteristic map is to the outer homotopy limit

$$\mathbb{S} \rightarrow \operatorname{holim}_{U \in \operatorname{Disk}_q^{B/}} F(U)$$

This map is assembled from local characteristic morphisms  $\mathbb{S} \rightarrow F(U)$ .

We construct the maps  $\mathbb{S} \rightarrow F(U)$  explicitly using the Pontryagin–Thom collapse map. Then the map (8) is obtained by extending the functor  $F$  to the left cone  $\operatorname{Disk}_q^{B/\triangleleft}$ . To do this we must construct maps  $S \rightarrow F(U)$  for all  $U$  that are compatible with the morphisms in  $\operatorname{Disk}_q^{B/}$ .

Let  $M_U$  denote  $p^{-1}(U)$ .  $M_U$  is open, but each fiber  $M_x$  over  $x \in U$  is compact, possibly with boundary  $\partial^\vee M_U$ . The bundles  $\nu$  and  $\tau$  are understood to be restricted along the open embedding  $M_U \hookrightarrow M$ . Consider the composition of the Pontryagin–Thom collapse with the inclusion below.

$$(9) \quad (U \times R^n)^+ \rightarrow M_U^\nu / \partial^\vee M_U^\nu \rightarrow M_U^{\nu \oplus \tau} \simeq M_U^+ \wedge S^n$$

Let  $U'$  contain  $U$ . Then there is an open embedding  $M_U \hookrightarrow M_{U'}$ . We get compatibility of the associated compositions as in the following diagram, with vertical maps obtained by one-point compactifying open embeddings.

$$(10) \quad \begin{array}{ccccccc} (U' \times R^n)^+ & \longrightarrow & M_{U'}^\nu / \partial^\vee M_{U'}^\nu & \longrightarrow & M_{U'}^{\nu \oplus \tau} & \xrightarrow{\simeq} & M_{U'}^+ \wedge S^n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (U \times R^n)^+ & \longrightarrow & M_U^\nu / \partial^\vee M_U^\nu & \longrightarrow & M_U^{\nu \oplus \tau} & \xrightarrow{\simeq} & M_U^+ \wedge S^n \end{array}$$

Stabilize (9) to obtain a map  $\Sigma^\infty U^+ \rightarrow \Sigma^\infty M_U^+$ . Stabilize (10) to get a stable diagram

$$\begin{array}{ccc}
\Sigma^\infty U'^+ & \longrightarrow & \Sigma^\infty M_{U'}^+ \\
\downarrow & & \downarrow \\
\Sigma^\infty U^+ & \longrightarrow & \Sigma^\infty M_U^+
\end{array}$$

Recall that we have also chosen an embedding  $B^q \hookrightarrow \mathbb{R}^\ell$  which induces an open embedding  $U \times \mathbb{R}^n \hookrightarrow B^q \times \mathbb{R}^n \hookrightarrow \mathbb{R}^\ell \times \mathbb{R}^n$ . One point compactifying gives a map  $S^{\ell+n} \rightarrow (U \times \mathbb{R}^n)^+$ . Doing the same for  $U'$  and stabilizing, we can extend our stable compatibility diagram to the left.

$$\begin{array}{ccccc}
\mathbb{S} & \longrightarrow & \Sigma^\infty U'^+ & \longrightarrow & \Sigma^\infty M_{U'}^+ \\
\downarrow = & & \downarrow & & \downarrow \\
\mathbb{S} & \longrightarrow & \Sigma^\infty U^+ & \longrightarrow & \Sigma^\infty M_U^+
\end{array}$$

With the construction above we have produced an extension of the functor  $F$  on the category  $\mathbf{Disk}_q^{B/}$  to a functor  $\tilde{F}$  on the category  $\mathbf{Disk}_q^{B/\triangleleft}$  so that  $\tilde{F}(*) = \mathbb{S}$ . Thus we have the morphism  $\mathbb{S} \rightarrow \operatorname{holim}_{U \in \mathbf{Disk}_q^{B/}} F(U)$  as desired.

**Definition 5.4.** The morphism (8), as constructed in the discussion above, is the refined transfer denoted by  $\bar{\operatorname{tr}}(p)$ . The morphism is constructed by extending the functor  $U \mapsto \Sigma^\infty(p^{-1}(U)^+)$  to the left cone  $\mathbf{Disk}_q^{B/\triangleleft}$  via the characteristic maps given in (9) and (10).

**Proposition 5.5.** The Becker–Gottlieb transfer  $\operatorname{tr}(p)$  is equivalent to  $\bar{\operatorname{tr}}(p)$ . In particular the following diagram is homotopy commutative.

$$\begin{array}{ccc}
\mathbb{S} & \xrightarrow{\operatorname{tr}(p)} & \Gamma_B \Sigma_B^\infty(M)_+ \\
& \searrow \bar{\operatorname{tr}}(p) & \downarrow \simeq \\
& & \operatorname{holim}_{U \in \mathbf{Disk}_q^{B/}} \Sigma^\infty(p^{-1}(U)^+)
\end{array}$$

Our next task is to define an index map and use it to define a vector field transfer,  $\bar{\operatorname{tr}}_X(p)$ . Fix  $U \in \mathbf{Disk}_q^{B/}$  and consider the equivalence  $\Sigma^\infty(p^{-1}(U)^+) \simeq \operatorname{hocolim}_{V \in \mathbf{Disk}_{k/p-1} U} \Sigma^\infty(-)_+$ . For each

$V \in \mathbf{Disk}_{k/p-1} U$ , and point  $(a_y, b_y)$  in  $\nu \oplus \tau|_{y \in V}$  we have a well defined map  $\operatorname{Ind}_X(V)$ .

$$(11) \quad S^n \simeq \operatorname{Th}(\nu \oplus \tau|_V) \rightarrow \operatorname{Th}(\nu \oplus \tau|_V) \simeq S^n$$

which sends  $(a_y, b_y)$  to  $(a_y, b_y + X(y))$ .

Since it is defined pointwise this map is compatible with the embeddings  $V \hookrightarrow V'$  comprising the morphisms in  $\mathbf{Disk}_{k/p-1} U$ . Stably this gives us a natural transformation from the functor  $\Sigma^\infty(-)_+$  to itself. This natural transformation induces a map

$$(12) \quad \operatorname{hocolim}_{V \in \mathbf{Disk}_{k/p-1} U} \Sigma^\infty(-)_+ \rightarrow \operatorname{hocolim}_{V \in \mathbf{Disk}_{k/p-1} U} \Sigma^\infty(-)_+$$

which in turn induces a map

$$(13) \quad \text{holim}_{U \in \text{Disk}_q^B / V \in \text{Disk}_{k/p-1}U} \text{hocolim} \Sigma^\infty(-)_+ \xrightarrow{\text{Ind}_X} \text{holim}_{U \in \text{Disk}_q^B / V \in \text{Disk}_{k/p-1}U} \text{hocolim} \Sigma^\infty(-)_+$$

**Definition 5.6.** We denote by  $\text{Ind}_X(M)$  the index map (13) associated to the vertical vector field  $X$  on the bundle  $M$ .

**Definition 5.7.** Let the vector field transfer, denoted  $\overline{\text{tr}}_X(p)$ , be the composition of (8) and (13):

$$\mathbb{S} \rightarrow \text{holim}_{U \in \text{Disk}_q^B / V \in \text{Disk}_{k/p-1}U} \Sigma^\infty(f^{-1}(U)^+) \simeq \text{holim}_{U \in \text{Disk}_q^B / V \in \text{Disk}_{k/p-1}U} \text{hocolim} \Sigma^\infty(-)_+ \xrightarrow{\text{Ind}_X(M)} \text{holim}_{U \in \text{Disk}_q^B / V \in \text{Disk}_{k/p-1}U} \text{hocolim} \Sigma^\infty(-)_+$$

We compare our refined vector field transfer  $\overline{\text{tr}}_X(p)$  to the vector field transfer of Becker and Gottlieb [BG91], obtained by stabilizing the composition

$$B_+ \wedge S^n \rightarrow \text{Th}(\nu|_M)/\text{Th}(\nu|_{\partial^\vee M}) \rightarrow \text{Th}(\nu \oplus \tau|_M) \simeq M_+ \wedge S^n$$

wherein the intermediate morphism  $\text{Th}(\nu|_M)/\text{Th}(\nu|_{\partial^\vee M}) \rightarrow \text{Th}(\nu \oplus \tau|_M)$  sends  $a_y$  to  $a \oplus X(y)$ . Note that for this to be well defined,  $X$  must be nonzero and of length 1 on  $\partial^\vee M$ . It is evident that our refined vector field transfer  $\overline{\text{tr}}_X(p)$  is homotopic to that of Becker and Gottlieb. In particular, the difference between the standard transfer of Becker and Gottlieb and their vector field transfer is equivalent to the difference in our refined versions. This difference is an error term computed on the boundary  $\partial^\vee M$ . For the following definition let  $\dot{p}$  denote the bundle formed from the vertical boundary of  $M$  over  $B$ .

**Definition 5.8.** Let  $X_\partial$  be vertical vector field defined on an open set of  $\partial^\vee M$  formed by taking those vectors pointing inside  $M$  and projecting them onto  $\tau|_{\partial^\vee M}$ . Let  $W$  be a manifold with boundary containing all zeros of  $X_\partial$ . Define  $\text{tr}_{X_\partial}(\dot{p})$  to be the stabilization of the composition

$$B_+ \wedge S^n \rightarrow \text{Th}(\nu|_W)/\text{Th}(\nu|_{\partial^\vee W}) \rightarrow \text{Th}(\nu \oplus \tau|_W) \simeq W^+ \wedge S^n$$

where  $\text{Th}(\nu|_W)/\text{Th}(\nu|_{\partial^\vee W}) \rightarrow \text{Th}(\nu \oplus \tau|_W)$  sends  $a_y$  to  $a \oplus X_\partial(y)$ .

Theorem 7.1 in [BG91] states that  $\text{tr}(p) = \text{tr}_X(p) + \text{tr}_{X_\partial}(\dot{p})$  and thus implies the following proposition.

**Proposition 5.9.** The difference  $\overline{\text{tr}}(p) - \overline{\text{tr}}_X(p)$  is homotopic to  $\text{tr}_{X_\partial}(\dot{p})$ . In particular, if fibers of  $p$  are closed, or if  $X$  points outwards on all of  $\partial^\vee M$ , then  $\overline{\text{tr}}(p) \sim \overline{\text{tr}}_X(p)$ .

We are now ready to state and prove a fiberwise Poincaré–Hopf theorem.

**Theorem A.** *The following diagram of spectra is homotopy commutative.*

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\overline{\text{tr}}(p)} & \text{holim}_{U \in \text{Disk}_q^B / V \in \text{Disk}_{k/p-1}U} \text{hocolim} \Sigma^\infty(-)_+ \\ \downarrow \overline{\text{tr}}(\pi) & & \uparrow + \\ \bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{\mathbb{Z}} / V \in \text{Disk}_{k/\pi-1}U} \text{hocolim} \Sigma^\infty(-)_+ & \xrightarrow{\bigvee_{Z \in \mathcal{S}(f)} \overline{\text{Ind}}_X(D\mathring{Z})} & \bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{\mathbb{Z}} / V \in \text{Disk}_{k/\pi-1}U} \text{hocolim} \Sigma^\infty(-)_+ \end{array}$$

*Proof of Theorem A.* Our proof of Theorem A follows by proving that the three subdiagrams of the diagram below commute. The top portion of the diagram with the curved arrow commutes by Proposition 5.9. We will prove the commutativity of the bottom triangle first and the middle triangle second.

$$\begin{array}{ccc}
& \xrightarrow{\quad \overline{\mathrm{tr}}(p) \quad} & \\
\mathrm{S} & \xrightarrow{\quad \overline{\mathrm{tr}}_X(p) \quad} \mathrm{holim}_{U \in \mathrm{Disk}_q^{B/}} \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/p-1}U} \Sigma^\infty(-)_+ & (14) \\
& \searrow \bigvee_{Z \in \mathcal{S}(f)} \overline{\mathrm{tr}}_X(\pi_Z^\circ) & \\
\bigvee_{Z \in \mathcal{S}(f)} \mathrm{holim}_{U \in \mathrm{Disk}_q^{Z/}} \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/\pi-1}U} \Sigma^\infty(-)_+ & \xrightarrow{\bigvee_{Z \in \mathcal{S}(f)} \overline{\mathrm{ind}}_X(D\mathring{Z})} & \bigvee_{Z \in \mathcal{S}(f)} \mathrm{holim}_{U \in \mathrm{Disk}_q^{Z/}} \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/\pi-1}U} \Sigma^\infty(-)_+ \\
& \nearrow \mathrm{inc.} &
\end{array}$$

The bottom triangle commutes essentially by the definition of the vector field transfer. We explain the notation. Recall that  $\mathring{Z}$  is the interior of the stratum  $Z$ . There is a bundle  $\pi : D\mathring{Z} \rightarrow \mathring{Z}$ . The vertical vector field  $X$  on  $M$  restricts to the vertical vector field  $X$  on  $D\mathring{Z}$ . Thus on each stratum  $Z$  we apply the definition of the vector field transfer to see that  $\overline{\mathrm{tr}}_X(\pi_Z^\circ)$  factors as  $\overline{\mathrm{ind}}_X(D\mathring{Z}) \circ \overline{\mathrm{tr}}(\pi_Z^\circ)$ . This identity applies to each wedge summand, and thus the bottom triangle commutes.

The main idea of the proof of the middle triangle is that the vector field transfer on  $p$  factors through the vector field transfer on the tubular neighborhood of the critical locus.

For  $Z \in \mathcal{S}(f)$ , and  $U \in \mathrm{Disk}_q^{Z/}$ , let  $F_Z$  denote the functor  $U \mapsto \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/\pi-1}U} \Sigma^\infty(-)_+$ . Consider

the functor  $\rho : \mathrm{Disk}_q^{Z/} \rightarrow \mathrm{Disk}_q^{B/}$  given by the projection  $p$ , which is an embedding on  $Z$ . We can right Kan extend to get a functor  $\mathrm{RKan}_\rho(F_Z)$  on  $\mathrm{Disk}_q^{B/}$  so that

$$(15) \quad \mathrm{holim}_{U \in \mathrm{Disk}_q^{Z/}} \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/\pi-1}U} \Sigma^\infty(-)_+ \simeq \mathrm{holim}_{U \in \mathrm{Disk}_q^{Z/}} F_Z(U) \simeq \mathrm{holim}_{U \in \mathrm{Disk}_q^{B/}} \mathrm{RKan}_\rho(F_Z)(U)$$

Fix  $U \in \mathrm{Disk}_q^{B/}$  and consider the map obtained by precomposing (12) with the transfer on  $U$ :

$$\Sigma^\infty U^+ \rightarrow \Sigma^\infty p^{-1}(U)^+ \simeq \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/p-1}U} \Sigma^\infty(-)_+ \rightarrow \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/p-1}U} \Sigma^\infty(-)_+$$

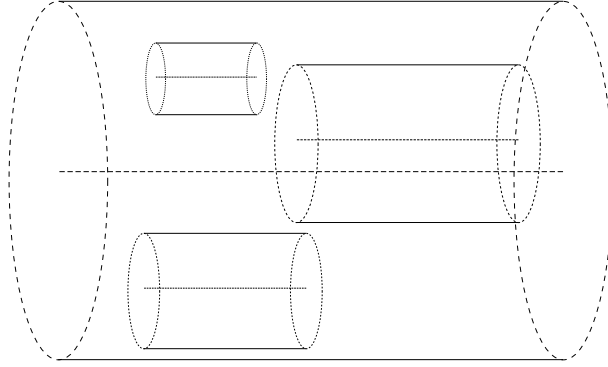
We will prove that this map factors as in the diagram below.

$$\begin{array}{ccc}
\Sigma^\infty U^+ & \longrightarrow & \Sigma^\infty p^{-1}(U)^+ \xrightarrow{\simeq} \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/p-1}U} \Sigma^\infty(-)_+ \longrightarrow \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/p-1}U} \Sigma^\infty(-)_+ \\
& \searrow & \nearrow \mathrm{inc.} \\
& \bigvee_{Z \in \mathcal{S}(f)} \mathrm{RKan}_\rho(F_Z)(U) &
\end{array} \quad (16)$$



The proof of the middle triangle in diagram (14) follows from the commutativity of the triangle above.

If  $U \subset p(Z)$  then since  $p$  is an embedding when restricted to  $Z$ ,  $p|_Z^{-1}U$  is an open disk in  $Z$ , and  $\pi^{-1}(p|_Z^{-1}U)$  is the tubular neighborhood on  $Z$  restricted to  $U$ . Then we identify  $\mathbf{R}\mathbf{Kan}_\rho(F_Z)(U) \simeq \Sigma^\infty(\pi^{-1}(p|_Z^{-1}U))^+$ . Using disk categories, this is expressed the homotopy colimit over the full subcategory of  $\mathbf{Disk}_{k/p^{-1}U}$  containing exactly those disks which are properly embedded within the restriction of the tubular neighborhood of  $Z$  to  $(p|_Z)^{-1}(U)$ . We will not introduce new notation for this subcategory, but instead direct the reader to the picture below, which indicates how the objects in the subcategory are situated in the tubular neighborhood of  $Z$ .



Having identified  $\mathbf{R}\mathbf{Kan}_\rho(F_Z)(U)$ , we will now factor the composition along the top of diagram (16). Consider the local index  $\mathbf{Ind}_X(V)$  on a properly embedded disk  $V$  as in (11). We will prove two assertions about the local index on  $V$ .

- (1) If  $V$  does not intersect the critical locus  $\Sigma(f)$  then  $\mathbf{Ind}_X(V)$  is nullhomotopic.
- (2) If  $V$  intersects the birth-death locus  $\Sigma_1(f)$  then  $\mathbf{Ind}_X(V)$  is nullhomotopic.

For now we proceed under the assumption that both statements above are true. It follows that the composition along the top of diagram (16) factors through a colimit over the subcategory of disks that intersect  $\Sigma(f)$  and do not intersect  $\Sigma^1(f)$ . This is identical to our description of the right Kan extension appearing in the diagram. Now taking homotopy limits over  $\mathbf{Disk}_q^{B/}$ , extending to the left cone as before, and finally applying (15) gives us the middle triangle in diagram (14).

It remains only to prove assertions (1) and (2) above. Assertion (1) is obvious, as the vector field  $X$  is nonvanishing on the interior of  $V$ , and thus the index map is nullhomotopic. Assertion (2) is a consequence of the existence of ‘ghost sets’ from Section 4.3 in [Igu08]. Recall the discussion in Section 4.0.1. In a neighborhood of the birth-death set the critical locus can be extended transversally to the birth death set in the unfolding direction, forming a ‘ghost’. Thus instead of cancelling the critical points at the birth-death submanifold, we have two critical points on each fiber within a neighborhood of the birth-death submanifold. These critical points have opposite index, thus on this local disk the index is nullhomotopic. □

**Remark 5.10.** We wish to point out that a simpler proof of Theorem A can be carried out using pure differential topology as in [BM76], [Dou06], or [CJ98]. However, we choose to give a proof using limits and colimits over disk categories because this facilitates the proof of Theorem B.

**5.2. Fiberwise Poincaré–Hopf for the excisive A-theory characteristic.** In this section we extend the factorization of the Becker–Gottlieb transfer in Theorem A to a factorization of

the excisive A-theory Euler characteristic. This requires more definitions analogous to those of the previous section. We enumerate them here.

- (1) We first define a refined excisive A-theory characteristic using the definition of  $\bar{\mathrm{tr}}_X(p)$  from the previous section as well as the unit map  $\mathbb{S} \rightarrow A(*)$ . This is Definition 5.11. Proposition 5.12 asserts that this characteristic is equivalent to the excisive A-theory characteristic of [DWW03].
- (2) Next we define a version of the index map for  $A^\%$ -theory. This is defined using the smooth index map from before, as well as the unit map  $\eta : \mathbb{S} \rightarrow A(*)$ , and its section trace  $: A(*) \rightarrow \mathbb{S}$ . Proposition 5.14 proves that the  $A^\%$ -theory index map is compatible with the smooth index map.
- (3) The proof of Theorem B follows from these two propositions and formal arguments.

Recall the excisive A-theory characteristic [DWW03] associated to a topological manifold bundle  $p : M \rightarrow B$ :

$$\mathbb{S} \xrightarrow{\chi^\%(p)} \Gamma_B A_B^\%(M)$$

If the bundle  $p : M \rightarrow B$  is smooth or at least homotopy equivalent to a smooth fiber bundle, they prove that  $\chi^\%(p)$  factors through the Becker–Gottlieb transfer  $\mathrm{tr}(p)$  (in fact, they prove an if and only if). In particular, understanding  $\eta_*$  to be induced by the unit map  $\eta : \mathbb{S} \rightarrow A(*)$ , the following diagram commutes.

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\mathrm{tr}(p)} & \Gamma_B \Sigma_B^\infty(M) \\ & \searrow \chi^\%(p) & \downarrow \eta_* \\ & & \Gamma_B A_B^\%(M) \end{array}$$

Since we are working with smooth manifold bundles, our refinement of the Becker–Gottlieb transfer leads to a refinement of the excisive A-theory characteristic.

**Definition 5.11.** We define the refined excisive A-theory characteristic  $\bar{\chi}^\%(p)$  to be the composition

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\bar{\mathrm{tr}}(p)} & \mathrm{holim}_{U \in \mathrm{Disk}_q^{B/}} \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/p-1} U} \Sigma^\infty(-)_+ \\ & \searrow \bar{\chi}^\%(p) & \downarrow \eta_* \\ & & \mathrm{holim}_{U \in \mathrm{Disk}_q^{B/}} \mathrm{hocolim}_{V \in \mathrm{Disk}_{k/p-1} U} A^\%(-) \end{array}$$

**Proposition 5.12.** The refined excisive A-theory characteristic is homotopic to the excisive A-theory characteristic  $\chi^\%(p)$ .

We use the index map (13) to define an index map for  $A^\%$ . Recall that (11) induced a stable map  $\Sigma^\infty V_+ \rightarrow \Sigma^\infty V_+$  for  $V \in \mathrm{Disk}_{k/\pi-1} U$ . Also recall the trace map  $A(*) \rightarrow \mathbb{S}$  that is well known to be a section of  $\eta$ . We define a local index map on  $V_+ \wedge A(*)$  as the composition

$$V_+ \wedge A(*) \xrightarrow{id \wedge \mathrm{trace}} V_+ \wedge \mathbb{S} \xrightarrow{\mathrm{Ind}} V_+ \wedge \mathbb{S} \xrightarrow{id \wedge \eta} V_+ \wedge A(*)$$

This induces a global index map for  $A^\%$  on  $D\mathring{Z}$ :

$$(17) \quad \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} A^\%(-) \xrightarrow{\text{Ind}_X^h(D\mathring{Z})} \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} A^\%(-)$$

**Definition 5.13.** We denote by  $\text{Ind}_X^h(D\mathring{Z})$  the index map (17) associated to the vertical vector field  $X$  on the bundle  $D\mathring{Z}$ .

**Proposition 5.14.** The index maps  $\text{Ind}_X(D\mathring{Z})$  and  $\text{Ind}_X^h(D\mathring{Z})$  are compatible as in the diagram below

$$\begin{array}{ccc} \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} \Sigma^\infty(-) & \xrightarrow{\text{Ind}_X(D\mathring{Z})} & \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} \Sigma^\infty(-) \\ \downarrow \eta_* & & \downarrow \eta_* \\ \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} A^\%(-) & \xrightarrow{\text{Ind}_X^h(D\mathring{Z})} & \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} A^\%(-) \end{array}$$

*Proof.* It suffices to prove that the two index maps are compatible on a single disk  $V$ . This is easy to verify given that trace is a section of  $\eta$ .  $\square$

**Theorem B.** The following diagram of spectra is homotopy commutative.

$$\begin{array}{ccccc} & & \mathbb{S} & \xrightarrow{\overline{\text{tr}}(p)} & \text{holim}_{U \in \text{Disk}_q^B} \text{hocolim}_{V \in \text{Disk}_{k/p-1} U} \Sigma^\infty(-)_+ \\ & \swarrow \overline{\text{tr}}(\pi) & \downarrow & & \uparrow + \\ \bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} \Sigma^\infty(-) & \xrightarrow{\bigvee_{Z \in \mathcal{S}(f)} \text{Ind}_X(D\mathring{Z})} & \bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} \Sigma^\infty(-) & & \\ & \downarrow & \downarrow & & \downarrow \\ & \mathbb{S} & \xrightarrow{\overline{\chi^\%}(p)} & \text{holim}_{U \in \text{Disk}_q^B} \text{hocolim}_{V \in \text{Disk}_{k/p-1} U} A^\%(-) \\ & \swarrow \overline{\chi^\%}(\pi) & \downarrow & & \uparrow + \\ \bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} A^\%(-) & \xrightarrow{\bigvee_{Z \in \mathcal{S}(f)} \text{Ind}_X^h(D\mathring{Z})} & \bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{\mathring{Z}/}} \text{hocolim}_{V \in \text{Disk}_{k/\pi-1} U} A^\%(-) & & \end{array}$$

*Proof of Theorem B.* The top square commutes by Theorem A. The left vertical side and the back vertical side commute by Proposition 5.12. The right vertical side is self-evident. The front vertical side commutes by Proposition 5.14. Thus the five sides excluding the base commute. This is enough information to prove the homotopy commutativity of the bottom square.

We have shown that the boundary of the cube commutes. It remains to prove that the entire cube commutes. To prove this, we break the cube into two pieces which we handle separately. To describe the first piece, we must define an auxilliary vector field transfer for  $A^\%$ -theory, analogous

to the vector field transfer  $\overline{\text{tr}}_X(p)$  which appeared in the proof of A. This definition is chosen so that the following diagram commutes. In particular,  $\overline{\chi}_X^{\%}(p) := \eta \circ \overline{\text{tr}}_X(p)$ .

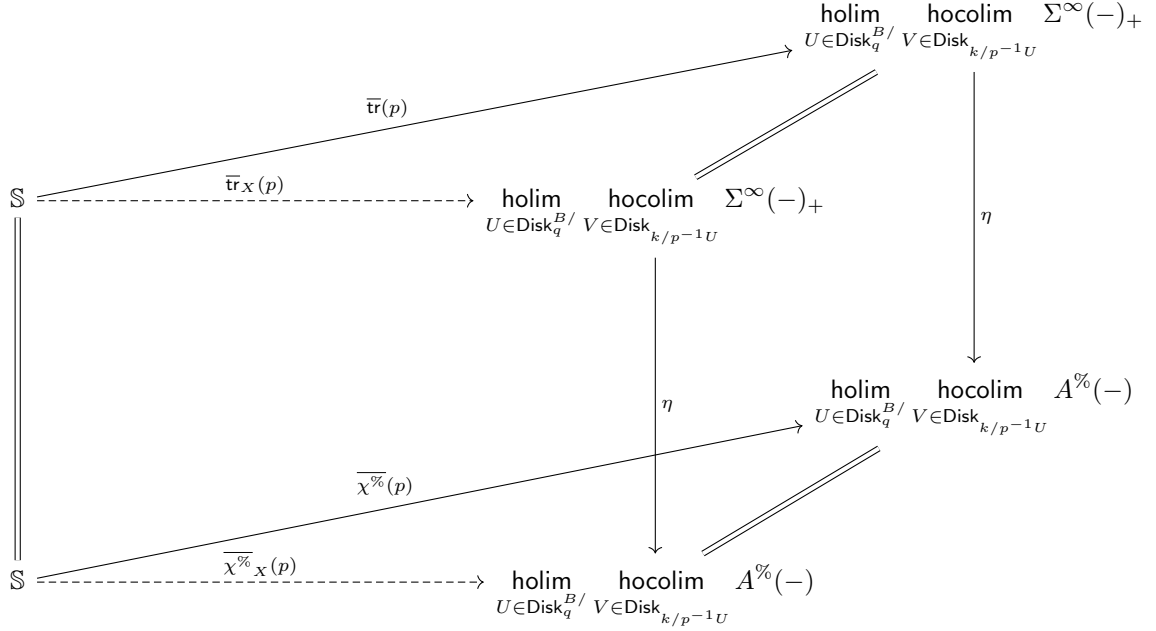
$$\begin{array}{ccccc}
& \mathbb{S} & \xrightarrow{\overline{\text{tr}}_X(p)} & \text{holim}_{U \in \text{Disk}_q^{B/}} & \text{hocolim}_{V \in \text{Disk}_{k/p-1}U} \Sigma^\infty(-)_+ \\
& \nwarrow \overline{\text{tr}}(\pi) & \downarrow & \nearrow + & \downarrow \\
\bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{B/}} & \text{hocolim}_{V \in \text{Disk}_{k/\pi-1}U} \Sigma^\infty(-) & \xrightarrow{\text{Ind}_X(D\check{Z})} & \bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{B/}} & \text{hocolim}_{V \in \text{Disk}_{k/\pi-1}U} \Sigma^\infty(-) \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
& \mathbb{S} & \xrightarrow{\overline{\chi}_X^{\%}(p)} & \text{holim}_{U \in \text{Disk}_q^{B/}} & \text{hocolim}_{V \in \text{Disk}_{k/p-1}U} A^\%( - ) \\
& \nwarrow \overline{\chi}^{\%}(\pi) & \downarrow & \nearrow + & \downarrow \\
\bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{B/}} & \text{hocolim}_{V \in \text{Disk}_{k/\pi-1}U} A^\%( - ) & \xrightarrow{\text{Ind}_X^h(D\check{Z})} & \bigvee_{Z \in \mathcal{S}(f)} \text{holim}_{U \in \text{Disk}_q^{B/}} & \text{hocolim}_{V \in \text{Disk}_{k/\pi-1}U} A^\%( - )
\end{array}$$

To obtain the cube above, it suffices to see that for a pair  $(U, V)$ , the following diagram of characteristics and vector field transfers commutes.

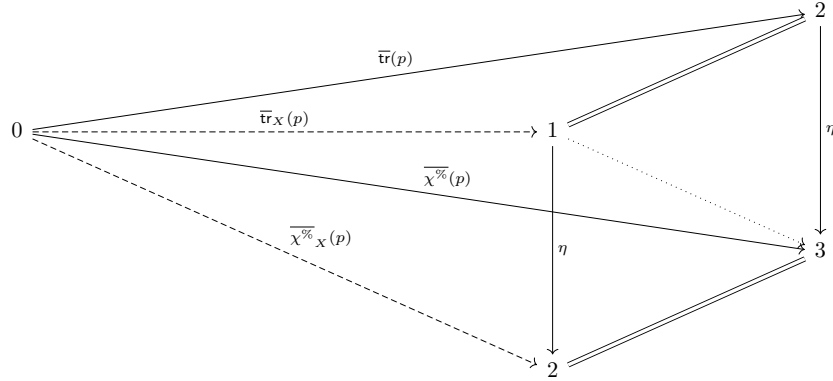
$$\begin{array}{ccc}
\mathbb{S} & \xrightarrow{\text{tr}_X(V)} & \Sigma^\infty(V)_+ \\
& \searrow \text{tr}(V) & \nearrow \text{Ind}_X(V) \\
& & \Sigma^\infty(V)_+ \\
& \downarrow & \downarrow \eta \\
\mathbb{S} & \xrightarrow{\chi_X^{\%}(V)} & A^\%(V) \\
& \searrow \chi^{\%}(V) & \nearrow \text{Ind}_X^h(V) \\
& & A^\%(V)
\end{array}$$

The top face commutes as in the proof of Theorem A. The vertical left side commutes by [DWW03]. The vertical back side commutes by the definition of  $\chi_X^{\%}(p)$ . The right vertical side commutes by the definition of the index map  $\text{Ind}^h$ . It follows that the triangular prism commutes, and thus the cube above commutes.

Now it remains to relate the vector field transfers to the ordinary transfers, as in Proposition 5.9. This is expressed precisely in the diagram below.



Each face of the diagram above commutes, except for the bottom. Our goal is to fill in the bottom face with a homotopy so that the triangular prism commutes. This argument is entirely formal. Identifying the two copies of  $\mathbb{S}$  in the diagram above gives us a pyramid, which can be broken into two tetrahedra. For brevity we replace the objects with numbered vertices, but the morphisms are labeled in the same way as before. The dotted arrow from vertex 1 to vertex 3 is an auxiliary morphism which we use to decompose the pyramid. There are two 3-simplices in the resulting diagram. The simplex containing the morphism  $\overline{\text{tr}}(p)$  we will refer to as simplex A. The other simplex will be referred to as simplex B.



In simplex A, we use the inner horn filling property to choose the face ‘013’ so that simplex A commutes. In simplex B we use the inner horn filling property to choose the face ‘023’ so that simplex B commutes. It now follows that the triangular prism commutes.

□

We conclude this section with the commutative diagram below, which replaces index category notation used in this section with the homotopy equivalent section spaces. In the applications to come we will use the shorthand, keeping in mind that the actual definitions of spectra and morphisms are slightly more involved.

$$\begin{array}{ccccc}
 & & \mathbb{S} & \xrightarrow{\text{tr}(p)} & \Gamma_B \Sigma_B^\infty(M_+) \\
 & \swarrow \text{tr}(\pi) & \downarrow & \nearrow + & \downarrow \\
 \bigvee_{Z \in \mathcal{S}(f)} \Gamma_Z \Sigma_Z^\infty(DZ_+) & \xrightarrow{\quad \bigvee_{Z \in \mathcal{S}(f)} \text{Ind}_X \quad} & \bigvee_{Z \in \mathcal{S}(f)} \Gamma_Z \Sigma_Z^\infty(DZ_+) & & \\
 \downarrow & & \downarrow & & \\
 & \swarrow \chi^\infty(\pi) & \mathbb{S} & \xrightarrow{\chi^\infty(p)} & \Gamma_B A_B^\infty(M) \\
 \bigvee_{Z \in \mathcal{S}(f)} \Gamma_Z A_Z^\infty(DZ) & \xrightarrow{\quad \bigvee_{Z \in \mathcal{S}(f)} \text{Ind}_X^h \quad} & \bigvee_{Z \in \mathcal{S}(f)} \Gamma_Z A_Z^\infty(DZ) & & \\
 & \nearrow + & & & \\
 & & & & \Gamma_B A_B^\infty(M)
 \end{array}$$

**5.3. Fiberwise Poincaré–Hopf Theorem for Stratified Deformations.** We now give a further generalization of Theorems A and B for stratified deformations of the critical locus of a generalized Morse function.

**Definition 5.15.** A stratified subset of a smooth bundle  $M$  over  $B^q$  is a pair  $(\Sigma, \psi)$  so that  $\Sigma$  is a compact smooth  $q$ -dimensional manifold together with a smooth mapping  $\pi : \Sigma \rightarrow B$  and a continuous mapping  $\psi : \Sigma \rightarrow X$ . The map  $\pi$  may have singularities, however these singularities are required to be fold singularities: locally they can be given by  $\pi(x_1, \dots, x_q) = (x_1^2, x_2, \dots, x_q)$ . The singularities form a  $q - 1$  submanifold of  $\Sigma$ .

**Example 5.16.** For our purposes the pair  $(\Sigma, \psi)$  will be a stratified subset corresponding to the critical locus of a fiberwise generalized Morse function on a smooth manifold bundle. The map  $\psi$  will be a map  $\Sigma \rightarrow BO$  classifying the restriction of the vertical tangent bundle to  $\Sigma$ .

**Definition 5.17** (p. 67 in [Igu05]). A stratified deformation between stratified subsets  $(\Sigma, \psi)$  and  $(\Sigma', \psi')$  of  $M$  over  $B$  with coefficients in  $X$  is a stratified subset  $(S, \Psi)$  of  $M \times I$  over  $B \times I$  with coefficients in  $X$  such that the restrictions of  $(S, \Psi)$  to the  $B \times 0$  and  $B \times 1$  are  $(\Sigma, \psi)$  and  $(\Sigma', \psi')$ . In the event that  $(\Sigma, \psi)$  and  $(\Sigma', \psi')$  are related by a stratified deformation we say that they belong to the same stratified deformation class and use the notation  $(\Sigma, \psi) \sim (\Sigma', \psi')$ .

Note that when we consider a stratified deformation of a critical locus of a fiberwise generalized Morse function, the end result of the deformation may not necessarily be realized by a Morse function. Thus, Theorem C below does not immediately follow from Theorem B. When referencing the set of strata of a stratified subset  $(\Sigma, \psi)$ , we use the notation  $\mathcal{S}(\Sigma)$ , since we cannot make reference to a function as was done previously.

**Theorem C.** Suppose that  $(\Sigma', \psi')$  is a stratified deformation of  $(\Sigma, \psi)$ , where  $\Sigma$  is the critical locus of a fiberwise generalized Morse function, and  $\psi$  is the classifying map for the restriction

of the vertical tangent bundle  $T^\vee M$  to  $\Sigma$ . Let  $\pi'$  denote the disk bundle associated to the tubular neighborhood of  $\Sigma'$  in  $M$ . Then the following diagram commutes:

$$\begin{array}{ccccc}
& \mathbb{S} & \xrightarrow{\text{tr}(p)} & \Gamma_B \Sigma_B^\infty(M_+) & \\
\swarrow \text{tr}(\pi') & \downarrow & & \nearrow + & \downarrow \\
\bigvee_{Z \in \mathcal{S}(\Sigma')} \Gamma_Z \Sigma_Z^\infty(DZ_+) & \xrightarrow{\bigvee_{Z \in \mathcal{S}(\Sigma')} \text{Ind}_X} & \bigvee_{Z \in \mathcal{S}(\Sigma')} \Gamma_Z \Sigma_Z^\infty(DZ_+) & & \\
\downarrow & & \downarrow & & \\
& \mathbb{S} & \xrightarrow{\chi^\%(p)} & \Gamma_B A_B^\%(M) & \\
\swarrow \chi^\%(\pi') & \downarrow & & \nearrow + & \downarrow \\
\bigvee_{Z \in \mathcal{S}(\Sigma')} \Gamma_Z A_Z^\%(DZ) & \xrightarrow{\bigvee_{Z \in \mathcal{S}(\Sigma')} \text{Ind}_X^h} & \bigvee_{Z \in \mathcal{S}(\Sigma')} \Gamma_Z A_Z^\%(DZ) & & 
\end{array}$$

*Proof.* The data of a stratified deformation provides the data of a homotopy between the compositions along the top squares:

$$(+) \circ \left( \bigvee_{Z \in \mathcal{S}(\Sigma')} \text{Ind}_X \right) \circ \text{tr}(\pi') \sim (+) \circ \left( \bigvee_{Z \in \mathcal{S}(\Sigma)} \text{Ind}_X \right) \circ \text{tr}(\pi)$$

The same is true of the bottom squares: there is a homotopy between the two compositions

$$(+) \circ \left( \bigvee_{Z \in \mathcal{S}(\Sigma')} \text{Ind}_X^h \right) \circ \chi^\%(\pi') \sim (+) \circ \left( \bigvee_{Z \in \mathcal{S}(\Sigma)} \text{Ind}_X^h \right) \circ \chi^\%(\pi)$$

Now the result follows from Theorem B.  $\square$

**5.4. Fiberwise Poincaré–Hopf Theorem for the Smooth Structure Class.** Our next task is to prove a fiberwise Poincaré–Hopf theorem for the homotopical smooth structure class. For our purposes in Section 6 it suffices to prove the rational formula for the smooth structure class:

$$\Theta(M, M') = \sum_{Z \in \mathcal{S}(f)} \Theta(DZ, DZ').$$

We instead prove a stronger decomposition for the homotopical smooth structure class. For the purposes of this section we assume that the bundle  $p : M \rightarrow B$  satisfies  $\chi^\%(p) = 0$  and work with the single manifold bundle  $M$ , instead of comparing two fiberwise tangentially homeomorphic manifolds  $M$  and  $M'$ . In other words we are assuming that  $M$  is fiberwise tangentially homeomorphic to a trivial bundle.

**Theorem D.** *Suppose that  $p : M \rightarrow B$  is a smooth manifold bundle for which  $\chi^\%(p) = 0$ . Let  $f$  be a fiberwise generalized Morse function on  $M$ . Let  $(\Sigma, \psi)$  be the stratified subset corresponding to the tubular neighborhood of the critical locus of  $f$ . Then there exists a stratified deformation of*

$(\Sigma, \psi)$  to a stratified subset  $(\Sigma_{SD}, \psi_{SD})$  for which the following diagram is homotopy commutative.

$$\begin{array}{ccc}
 & \mathbb{S} & \xrightarrow{\theta(M)} \Gamma_B \mathcal{H}_B^{\%}(M) \\
 \theta(DZ) \swarrow & & \nearrow + \\
 \bigvee_{Z \in \mathcal{S}(\Sigma_{SD})} \Gamma_Z \mathcal{H}_Z^{\%}(DZ) & \xrightarrow{\text{id}} & \bigvee_{Z \in \mathcal{S}(\Sigma_{SD})} \Gamma_Z \mathcal{H}_Z^{\%}(DZ)
 \end{array}$$

This diagram can be reformulated more concisely as:

$$(18) \quad \theta(M) \sim \sum_{Z \in \mathcal{S}(\Sigma_{SD})} \theta(DZ)$$

Our strategy is to first prove that  $\theta(M)$  is a *stratified deformation invariant*, i.e. an invariant of the stratified deformation class of  $(\Sigma, \psi)$ . Then we construct a stratified deformation of  $(\Sigma, \psi)$  for which Theorem D holds.

**Lemma 5.18.** The homotopical smooth structure class  $\theta(M)$  is an invariant of the stratified deformation class of the stratified subset corresponding to a fiberwise generalized Morse function on  $M$ .

*Proof.* As in the statement of Theorem D, let  $\Sigma(f)$  be the critical locus of a fiberwise generalized Morse function  $f$ , and let  $\psi : \Sigma(f) \rightarrow BO$  be the restriction of  $T^\vee M$  to  $\Sigma(f)$ . Let  $(\Sigma', \psi')$  be a deformation of  $(\Sigma, \psi)$ . Theorem C indicates that the fiberwise Poincaré–Hopf theorems apply to both stratified subsets.

Recall that  $\Theta(M)$  is equivalent to the data of a nullhomotopy of  $\chi^{\%}(M)$ . By Theorem C, this is equivalent to a nullhomotopy of  $(+) \circ \left( \bigvee_{Z \in \mathcal{S}(\Sigma)} \text{Ind}_X^h \right) \circ \chi^{\%}(\pi)$ . But by Theorem C,  $\Theta(M)$  is also equivalent to a nullhomotopy of  $(+) \circ \left( \bigvee_{Z \in \mathcal{S}(\Sigma')} \text{Ind}_X^h \right) \circ \chi^{\%}(\pi')$ . Since these two index sums are homotopic via the data of the stratified deformation,  $\Theta(M)$  is invariant under the stratified deformation. This concludes the proof of this stratified deformation lemma.  $\square$

**Remark 5.19.** The previous lemma and its proof can be interpreted as saying that  $\Theta(M)$  depends only on the homotopy type of  $\chi^{\%}(M)$ , which is itself a stratified deformation invariant by Theorem C.

The stratified deformation  $(\Sigma', \psi')$  for which Theorem D holds is a disjoint union of *immersed lenses* along with components on which  $\psi'$  is trivial. We define immersed lenses now.

**Definition 5.20** (p.70 in [Igu05]). Let  $V$  be a compact connected  $q$  manifold with boundary so that  $V$  is immersed in  $B^q$ . Let  $\psi_1, \psi_2 : V \rightarrow X$  be continuous maps which are trivial on  $\partial V$ . Then the *immersed lens*  $L_i(V, \psi_1, \psi_2)$  is defined to be the stratified subset  $(L, \psi_L)$  where  $L$  is the double of  $V$  in indices  $i$  and  $i+1$ , and  $\psi_L$  is  $\psi_1$  on the lower stratum and  $\psi_2$  on the upper stratum.

**Lemma 5.21.** There exists a stratified deformation with boundaries  $(\Sigma, \psi)$  and  $(\Sigma', \psi')$ , so that the strata of  $(\Sigma', \psi')$  are concentrated in two indices. Furthermore, each component of the lower stratum of  $\Sigma'$  lies in a contractible subset of  $\Sigma'$ .

**Remark 5.22.** Lemma 5.21 is an excerpt from the proof of the transfer theorem in [Igu05]. In particular, the statement is identical to Step (c) on p.70, the proof of which appears on pages 71–73.



Briefly, the strategy of the proof is to first add and delete twisted lenses, immersed lenses for which  $\psi_1$  is the same as  $\psi_2$  after composition with the fold map, to concentrate the stratified subset into two degrees. The stratified deformations obtained by adding and deleting the twisted lenses reduces the number of components in  $k$ th stratum by one. An inductive argument starting in the minimal stratum will then concentrate all strata in two degrees.

The next task is to prove that  $\Sigma'_-$  lies in a contractible subset of  $\Sigma'$ . To do this, we choose a triangulation of  $\Sigma'$ , and do a deformation on each simplex in the stratification. On the zero simplices the idea is to add a lens above a zero simplex, give a stratified deformation that cancels the lower stratum of this lens to obtain a ‘mushroom’, and then observe that the mushroom has the desired property: the ‘-’ stratum on top of the mushroom (as well as it’s boundary) lies in a contractible subset. We give a pictorial version of this stratified deformation in the figure below. This construction, as well as the inductive constructions for higher simplices, also appears with pictures in the proof of Lemma 3.2.1 in [GI14].

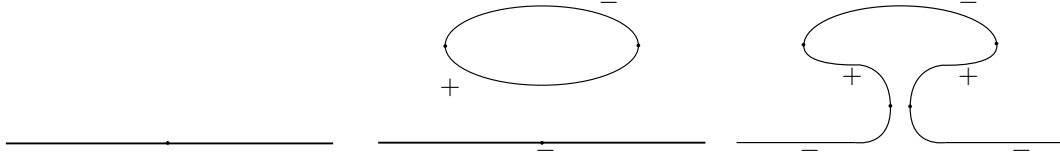


FIGURE 7. The stratified deformation introduces a lens above a designated point in  $\Sigma_-$  and then cancels the + and - strata to obtain a ‘mushroom’.

**Lemma 5.23** (Lemma 5.7 in [Igu05]). The stratified subset  $(\Sigma', \psi')$  can be deformed into a stratified subset  $(\Sigma'', \psi'')$  presented as a disjoint union of immersed lenses and components on which  $\psi'$  is trivial. Furthermore,  $\psi''$  is trivial on the lower stratum of each of the immersed lenses.

**Proposition 5.24.** Recall the requirement from Theorem D that  $\chi^\%(M) = 0$ . In this setting, the stratified subset  $(\Sigma'', \psi'')$  of  $M$  from Lemma 5.23 has the property that on each stratum  $Z$  of  $\Sigma''$ , the restriction of the vector bundle  $\psi''$  is either trivial or fiber homotopy trivial. In particular,  $\chi^\%(DZ)$  vanishes on each stratum.

*Proof.* Begin by applying Theorem C to  $\Sigma''$  to obtain

$$\chi^\%(M) = \sum_{Z'' \in \mathcal{S}(\Sigma'')} \text{Ind}_X^h(DZ) \chi^\%(DZ'') = 0$$

Let us partition  $\mathcal{S}(\Sigma'')$  into three sets:  $V_+, V_-, W$ . The sets  $V_+$  and  $V_-$  respectively contain the upper and lower strata of the immersed lens. The set  $W$  contains those strata which do not belong to immersed lenses, on which  $\psi''$  is trivial. For each  $Z \in W$ , since  $\psi''|_Z$  is trivial,  $\chi^\%(DZ'')$  is trivial.

We are guaranteed a bijection between  $V_+$  and  $V_-$  by definition, so suppose that  $Z_- \cup_{\partial} Z_+$  forms an immersed lens. Since each of the  $Z_+$  have the same index, the characteristics  $\chi^\%(DZ_+)$  varying over  $Z_+ \in V_+$  cannot satisfy a relation in  $\Gamma_B A_B^\%(N)$ . It follows that each pair  $Z_+$  and  $Z_-$  must be a cancelling pair, and therefore  $\chi^\%(DZ_-) + \chi^\%(DZ_+)$  is trivial. On each stratum in  $Z_- \in V_-$ ,  $\psi''$  is trivial, so  $\chi^\%(DZ_-)$  is trivial. It follows that  $\chi^\%(DZ_+)$  must be trivial as well. However,  $\psi''|_{Z_+}$  is not necessarily a trivial vector bundle.

Observe that the characteristic map  $\chi^\%(DZ_+)$  factors through the following diagram:

$$\begin{array}{ccccc}
Z_+ & \xrightarrow{\psi''|_{Z_+}} & BO & \longrightarrow & QS_0 \\
& \searrow & \downarrow & & \downarrow \\
& & BG & \longrightarrow & A(*)
\end{array}$$

View  $\chi^\%(DZ_+)$  as the composition from  $Z_+$  to  $A(*)$ . For this map to be nullhomotopic, it suffices for the composition  $Z_+ \rightarrow BO \rightarrow BG$  to be nullhomotopic. Thus  $\psi''|_{Z_+}$  must be fiber homotopy trivial.  $\square$

*Proof of Theorem D.* Apply the deformation in Lemma 5.21 to  $(\Sigma, \psi)$  to obtain  $(\Sigma', \psi')$ , and then apply the deformation in Lemma 5.23 to obtain  $(\Sigma'', \psi'')$ . The stratified subset  $(\Sigma'', \psi'')$  has three properties. First, the strata are concentrated in two indices,  $i$ , and  $i + 1$ . Second, the stratified subset is a disjoint union of components which are either immersed lenses, or are stratified subsets on which  $\psi'$  is trivial. The third property is that on the lower stratum (and birth-death points, by definition) of each of the immersed lenses,  $\psi''$  is trivial. These properties are used in Proposition 5.24 to conclude that  $\chi^\%(DZ)$  is trivial on each stratum  $Z$  in  $\mathcal{S}(\Sigma'')$ .

We now apply Theorem C to  $(\Sigma'', \psi'')$ . Since we have assumed that  $\chi^\%(p) = 0$ , the immediate consequence of this theorem is that

$$\sum_{Z'' \in \mathcal{S}(\Sigma'')} \chi^\%(DZ'') = 0$$

By Proposition 5.24,  $\chi^\%(\pi'')$  in the statement of Theorem C is nullhomotopic, so  $\theta(M)$  factors through the corresponding lift,  $\mathbb{S} \rightarrow \bigvee_{Z \in \mathcal{S}(\Sigma'')} \Gamma_Z \mathcal{H}_Z^\%(DZ)$ . Thus we have proven the following identity for the stratified subset  $(\Sigma'', \psi'')$ :

$$\theta(M) \sim \sum_{Z'' \in \mathcal{S}(\Sigma'')} \theta(DZ'')$$

$\square$

**Remark 5.25.** The proof of Theorem D parallels the proofs of the the transfer theorem and the Framing Principle in [Igu05]. Those theorems are rational formulas for higher Franz–Reidemeister torsion classes. They are proven by first proving ‘stratified deformation lemmas’ showing that particular classes from the formulas are stratified deformation invariants. The second step is to prove that the formula holds for a particular stratified deformation of the critical locus of a generalized Morse function. The stratified deformation used here is virtually identical to the stratified deformation used by Igusa.

## 6. APPLICATION TO EXOTIC SMOOTHINGS OF FIBER BUNDLES

In this section of the paper we give the main computations of our smooth structure class. We will begin by first explaining how we can apply the fiberwise Poincaré–Hopf theorems from the previous section to a pair of fiberwise tangentially homeomorphic manifold bundles. Then we will compute the smooth structure class of the immersed Hatcher construction, and finally, we prove the Rigidity Conjecture.

**6.1. Pairs of compatible Morse functions.** The goal of the next two sections is to use Theorem D to compute the smooth structure class for the immersed Hatcher construction and to prove Conjecture 3.3. In this section we start with two fiberwise tangentially homeomorphic manifolds  $M$  and  $M'$  and construct fiberwise generalized Morse functions  $f$  and  $f'$  on stabilizations of  $M$  and  $M'$  with some compatibility. We will use this pair of Morse functions to apply Theorem D.

Once again, begin with  $M'$  fiberwise tangentially homeomorphic to  $M$ , both smooth manifold bundles over  $B^q$  with closed  $d$ -dimensional fibers. Consider the induced fiberwise tangential homeomorphism  $M \times I^{k+1} \rightarrow M' \times I^{k+1}$ . By fiberwise immersion theory this is the same thing as a fiberwise immersion. This can further be approximated by an embedding for large  $k$ . The closure of the complement of the image of  $M \times I^{k+1}$  in  $M' \times I^{k+1}$  is an h-cobordism  $N$  over  $B$  on the image of  $\partial(M \times I^{k+1})$ .

We now choose a fiberwise generalized Morse function  $g$  on  $M \times I^{k+1}$  that is constant on  $\partial(M \times I^{k+1})$ . Next we choose a fiberwise generalized Morse function  $g'$  on the h-cobordism  $N$  over  $B$  that is constant on the ingoing and outgoing boundaries. Let  $f : M' \times I^{k+1} \rightarrow \mathbb{R}$  be the GMF obtained by applying  $g$  on the image of  $M \times I^{k+1}$  in  $M' \times I^{k+1}$  and applying  $g'$  on  $N$ . The function  $f$  is constant on  $\partial(M' \times I^{k+1})$  as well as on the image of  $\partial(M \times I^{k+1})$ .

**Proposition 6.1.** The characteristic  $\chi^\%(N)$  is trivial.

*Proof.* We apply Theorem B to the function  $f$  on  $M' \times I^{k+1}$  as well as to the function  $g$  on  $M \times I^{k+1}$ . Note that  $\mathcal{S}(f) = \mathcal{S}(g) \cup \mathcal{S}(g')$  by construction.

$$\begin{aligned} \chi^\%(M' \times I^{k+1}) &= \sum_{Z \in \mathcal{S}(f)} \text{Ind}_X(DZ) \circ \chi^\%(DZ) \\ &= \sum_{Z \in \mathcal{S}(g)} \text{Ind}_X(DZ) \circ \chi^\%(DZ) + \sum_{Z \in \mathcal{S}(g')} \text{Ind}_X(DZ) \circ \chi^\%(DZ) \\ &= \chi^\%(M \times I^{k+1}) + \sum_{Z \in \mathcal{S}(g')} \text{Ind}_X(DZ) \circ \chi^\%(DZ) \end{aligned}$$

The first line above is an application of Theorem B. The second line is a simple partition of  $\mathcal{S}(f)$ . The third line identifies the first summation as an application of Theorem B to  $M \times I^{k+1}$ . Because  $M$  is fiberwise tangentially homeomorphic to  $M'$ ,  $\chi^\%(M' \times I^{k+1}) = \chi^\%(M \times I^{k+1})$ . This means that

$$\sum_{Z \in \mathcal{S}(g')} \text{Ind}_X(DZ) \circ \chi^\%(DZ) = \chi^\%(N) = 0$$

□

**Corollary 6.2.** For bundles  $M$  and  $M'$  satisfying the conditions of Theorem D, there exists a stratified subset  $(\Sigma_{SD}, \psi_{SD})$  for which the rational smooth structure class  $\Theta(M, M')$  satisfies the following equation:

$$\Theta(M, M') = \Theta(N) = \sum_{Z \in \mathcal{S}(\Sigma_{SD})} \Theta(DZ, DZ').$$

*Proof.* This follows immediately by applying Theorem D to  $N$  and then applying Proposition 3.5 to obtain the rational statement. □

**6.2. Computations for the Immersed Hatcher construction.** In this section we compute the smooth structure class for the immersed Hatcher construction using Theorem D. Recall the notation from Sections 3.4 and 3.4.1.  $M$  is a smooth manifold bundle,  $\xi^n$  is a vector bundle in the kernel of the J-homomorphism,  $\eta^m$  is a vector bundle complementary to  $\xi$ , and  $\tilde{\lambda}$  is the inclusion of the critical locus of  $L \subset M$  a fiberwise generalized Morse function. Then the immersed Hatcher construction produces a smooth manifold bundle  $E_+^{n,m}(M, \tilde{\lambda}, \xi)$  that is fiberwise tangentially

homeomorphic to  $M$ . This is done by taking  $M \times I$ , identifying a disk bundle  $D^n \times D^m(\eta)$  embedded in  $M \times \{1\}$ , and attaching an exotic disk bundle  $E_L^{n,m}(\xi, \eta)$  to  $D^n \times D^m(\eta) \times I$ . For the remainder of this subsection, let us refer to the outcome of this construction as  $M'$ . Then  $M \times I$  embeds into  $M'$ , and the complement of  $M \times I$  is an h-cobordism on  $M \times \{1\}$ . This h-cobordism is comprised of a single disk bundle that is fiberwise tangentially homeomorphic to a trivial bundle.

The setup above is sufficient to apply Theorem D, in particular Corollary 6.2 to conclude that

$$\Theta(E_+^{n,m}(M, \tilde{\lambda}, \xi), M) = \tilde{\lambda}_* \Theta(E_L^{n,m}(\xi, \eta), D^n \times D^m(\eta) \times I)$$

The following is a computation from [GI14].

**Proposition 6.3.** The relative torsion  $\tau^{IK}(E_L^{n,m}(\xi, \eta), D^n \times D^m(\eta) \times I)$  is equal to  $(-1)^n 2\tilde{\text{ch}}(\xi)$ .

Since the Hatcher construction is a disk bundle, equation (3) indicates that

$$\Theta(E_L^{n,m}(\xi, \eta), D^n \times D^m(\eta) \times I) = D\tau^{IK}(E_L^{n,m}(\xi, \eta), D^n \times D^m(\eta) \times I)$$

Thus we have the following proposition:

**Proposition 6.4.** The exotic smooth structure class of the immersed Hatcher construction is as follows:

$$\Theta((E_+^{n,m}(M, \tilde{\lambda}, \xi)), M) = (-1)^n \tilde{\lambda}_* D(2\tilde{\text{ch}}(\xi))$$

This proposition is equivalent to Theorem 3.0.5 in [GI14], though our proof differs because it uses the fiberwise Poincaré–Hopf results from earlier in this paper. The main theorem of [GI14] is that the immersed Hatcher construction generates all exotic smoothings of manifold bundles with closed odd dimensional fibers. We restate this theorem and provide a different proof below.

**Theorem E.** *Rationally and stably, the immersed Hatcher construction produces all exotic smoothings of bundles with closed odd dimensional fibers when the dimension of the fiber is at least  $2 \dim B + 3$ .*

*Proof.* Recall that  $M$  and  $M'$  are fiberwise tangentially homeomorphic manifold bundles with even dimensional closed fibers. Embed  $M \times I^{k+1}$  in  $M' \times I^{k+1}$  with  $k$  even as in Section 6.1. Let  $N$  be an h-cobordism bundle identified as the complement of the closure of  $M \times I^{k+1}$  in  $M' \times I^{k+1}$ , and equipped with a fiberwise generalized Morse function  $g'$ . By Proposition 6.1,  $\chi^{\%}(N) = 0$ .

Let  $\Sigma(g')$  be the critical locus of  $g'$  and let  $\psi_{g'}$  be the restriction of  $T^{\vee}M$  to  $\Sigma(g')$ . By Theorem D and Corollary 6.2 there exists a stratified deformation of  $(\Sigma(g'), \psi_{g'})$  to a stratified subset  $(\Sigma_{SD}, \psi_{SD})$  for which

$$\Theta(M \times I^{k+1}, M' \times I^{k+1}) = \sum_{Z \in \mathcal{S}(\Sigma_{SD})} \Theta(DZ).$$

In order for  $\Theta(M, M')$  to be nonzero, at least one summand of  $\Theta(DZ)$  must be nonzero, and  $\Theta(DZ)$  only depends on the smooth structure on the interior of the tubular neighborhood of  $Z$ , which is the disk bundle of an odd dimensional vector bundle that has been stabilized  $k+1$  times by adding trivial line bundles. Thus  $\Theta(DZ)$  depends only on a classifying map from  $Z$  to  $\text{BDiff}(D^{2n+1} \text{ rel } \partial)$ . By [FH78], the rational homotopy of this space is nontrivial. These exotic odd dimensional disk bundles can all be obtained by Hatcher's construction according to [Igu02] or Corollary 2.2.1 in [GI14]. Thus  $M'$  must be the result of applying the immersed Hatcher construction to  $M$ . The dimension constraint appears because Hatcher's construction only produces disk bundles whose fibers are  $2 \dim B + 3$  (see Lemma 3.6 and Remark 3.7).  $\square$

**6.3. Proof of the Rigidity Conjecture.** We now give a proof of the Rigidity Conjecture.

**Theorem F.** *The Rigidity Conjecture, Conjecture 3.3 in the present work, is true.*

*Proof of Conjecture 3.3.* Recall that  $M$  and  $M'$  are fiberwise tangentially homeomorphic manifold bundles with even dimensional closed fibers. Embed  $M \times I^{k+1}$  in  $M' \times I^{k+1}$  with  $k$  even as in Section 6.1. Let  $N$  be an h-cobordism bundle identified as the complement of the closure of  $M \times I^{k+1}$  in  $M' \times I^{k+1}$ , and equipped with a fiberwise generalized Morse function  $g'$ . By Proposition 6.1,  $\chi^{\%}(N) = 0$ .

Let  $\Sigma(g')$  be the critical locus of  $g'$  and let  $\psi_{g'}$  be the restriction of  $T^{\vee}M$  to  $\Sigma(g')$ . By Theorem D and Corollary 6.2 there exists a stratified deformation of  $(\Sigma(g'), \psi_{g'})$  to a stratified subset  $(\Sigma_{SD}, \psi_{SD})$  for which

$$\Theta(M \times I^{k+1}, M' \times I^{k+1}) = \sum_{Z \in \mathcal{S}(\Sigma_{SD})} \Theta(DZ).$$

In order for  $\Theta(M, M')$  to be nonzero, at least one summand of  $\Theta(DZ)$  must be nonzero, and  $\Theta(DZ)$  only depends on the smooth structure on the interior of the tubular neighborhood of  $Z$ , which is a disk bundle of an even dimensional vector bundle that has been stabilized by adding  $k+1$  trivial line bundles. Thus  $\Theta(DZ)$  depends only on a classifying map from  $Z$  to  $\text{BDiff}(D^{2n} \text{ rel } \partial)$ . By [FH78], the rational homotopy of this space is trivial in the pseudoisotopy stable range: there are no exotic even dimensional disk bundles in the stable range. Thus  $\Theta(DZ)$  must always be zero. We must conclude then that  $\Theta(M, M') = \Theta(M \times I^{k+1}, M' \times I^{k+1}) = 0$ .  $\square$

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