H24

2

(2) 続き

(ii)

$$\mathrm{div}\left(\frac{1}{2}\,\mathrm{grad}\,\phi^2\right) = x^2+y^2+1$$

ガウスの発散定理より

$$\iint_S \phi\,\mathrm{grad}\,\phi\cdot n\,dS = \iiint_V \mathrm{div}(\phi\,\mathrm{grad}\,\phi)\,dV$$

$$= \iiint_V \mathrm{div}\left(\frac{1}{2}\,\mathrm{grad}\,\phi^2\right)dV$$

$$= \iiint_V (x^2+y^2+1)\,dV$$

$$= \int_0^2 dz \int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2+1)\,dx\,dy$$

$$= 2\int_0^1 \left[\frac{1}{3}x^3+(y^2+1)x\right]_0^{\sqrt{1-y^2}} dy$$

$$= 2\int_0^1 \left\{\frac{1}{3}(1-y^2)^{\frac{3}{2}}+(y^2+1)\sqrt{1-y^2}\right\}dy$$

$$= 2\int_0^1 \left\{\frac{1}{3}(1-y^2)^{\frac{3}{2}}+y^2(1-y^2)^{\frac{1}{2}}+(1-y^2)^{\frac{1}{2}}\right\}dy$$

$$y = \sin\theta \rightarrow dy = \cos\theta\,d\theta$$

$$= 2\int_0^1 \left\{\frac{1}{3}\cos^3\theta+\sin^2\theta\cos\theta+\cos\theta\right\}\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left\{\frac{1}{3}\left(\frac{1}{2}+\frac{\cos 2\theta}{2}+\frac{\cos 4\theta}{4}\right)+\frac{\sin 2\theta}{2}+\frac{1+\cos 2\theta}{2}\right\}d\theta$$

$$= \left[\frac{1}{3}\left(\theta+\frac{1}{2}\sin 2\theta+\frac{1}{8}\sin 4\theta\right)-\frac{1}{2}\cos 2\theta+\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{6}+\frac{1}{2}+\frac{\pi}{2}+\frac{1}{2}$$

$$= 1+\frac{2}{3}\pi$$