H20

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(1) $$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

特異点は $z = \zeta$

$\zeta = z$ において $f(\zeta)$ は正則

$$\mathrm{Res}[z] = \lim_{\zeta \to z} (\zeta - z) \frac{f(\zeta)}{\zeta - z} = f(z)$$

よって留数定理より

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z)$$

(2) $$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \int_C \frac{f(\zeta)}{\zeta} d\zeta + \int_C \frac{z f(\zeta)}{\zeta(\zeta-1)} d\zeta + \int_C \frac{z(z-1) f(\zeta)}{\zeta(\zeta-1)(\zeta-2)} d\zeta + \int_C \frac{z(z-1)(z-2) f(\zeta)}{\zeta(\zeta-1)(\zeta-2)(\zeta-z)} d\zeta$$

$$2\pi i f(z) = 2\pi i f(0) + 2\pi i (f(0) + f(1)) z + 2\pi i (f(0) + f(1) + f(2)) z(z-1) + \int_C \frac{z(z-1)(z-2) f(\zeta)}{\zeta(\zeta-1)(\zeta-2)(\zeta-z)} d\zeta$$

$$f(z) = f(0) + (f(0) + f(1)) z + (f(0) + f(1) + f(2)) z(z-1) + \frac{1}{2\pi i} \int_C \frac{z(z-1)(z-2) f(\zeta)}{\zeta(\zeta-1)(\zeta-2)(\zeta-z)} d\zeta$$

$$a_1 = f(0) + f(1)$$

$$a_2 = f(0) + f(1) + f(2)$$

(3) $$\int_{C_n} \frac{f(\zeta)}{\zeta - z} = \int_{C_n} \frac{f(\zeta)}{\zeta} d\zeta + \int_{C_n} \frac{z f(\zeta)}{\zeta(\zeta-1)} d\zeta + \int_C \frac{z(z-1) f(\zeta)}{\zeta(\zeta-1)(\zeta-2)} d\zeta + \cdots + \int_C \frac{z(z-1)\cdots(z-n) f(\zeta)}{\zeta(\zeta-1)\cdots(\zeta-n)(\zeta-z)} d\zeta$$

$$f(z) = f(0) + (f(0) + f(1)) z + (f(0) + f(1) + f(2)) z(z-1) + \cdots + \frac{1}{2\pi i} \int_C \frac{z(z-1)\cdots(z-n) f(\zeta)}{\zeta(\zeta-1)\cdots(\zeta-n)(\zeta-z)} d\zeta$$

$f(\zeta) = \sin \pi\zeta$ とすると $f(z) = \sin \pi z$

$f(n) = \sin n\pi = 0$

代入

$$\sin \pi z = \frac{1}{2\pi i} \int_C \frac{z(z-1)\cdots(z-n) \sin \pi\zeta}{\zeta(\zeta-1)\cdots(\zeta-n)(\zeta-z)} d\zeta$$