H27

$\boxed{4}$

(1)

$$\mathcal{L}[f''(t)](s) = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f'(t)\,dt$$

$$= 0 - f'(0) + \left[s e^{-st} f(t)\right]_0^\infty + \int_0^\infty s^2 e^{-st} f(t)\,dt$$

$$= -f'(0) + 0 - s f(0) + s^2 \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathcal{L}[f(t)](s)}$$

$$= s^2 \mathcal{L}[f(t)](s) - s f(0) - f'(0)$$

(2)

$$\mathcal{L}[t](s) = \int_0^\infty t e^{-st}\,dt$$

$$= \left[-\frac{1}{s} t e^{-st}\right]_0^\infty + \int_0^\infty \frac{1}{s} e^{-st}\,dt \qquad \lim_{t\to\infty} f(t) e^{-st} = 0$$

$$= 0 - \left[\frac{1}{s^2} e^{-st}\right]_0^\infty = \frac{1}{s^2}$$

$$\mathcal{L}[\sinh t](s) = \int_0^\infty \sinh t\, e^{-st}\,dt$$

$$= \frac{1}{2}\int_0^\infty \left\{ e^{(a-s)t} - e^{-(a+s)t} \right\} dt$$

$$= \frac{1}{2}\left[\frac{1}{a-s} e^{(a-s)t} + \frac{1}{a+s} e^{-(a+s)t}\right]_0^\infty$$

$$= \frac{1}{2}\left(\frac{1}{a-s} + \frac{1}{a+s}\right) = \frac{2a}{2(a^2-s^2)} = \frac{a}{a^2-s^2}$$

(3)

$$x''(t) - a^2 x(t) = t$$

$$\mathcal{L}[x''(t)](s) - a^2 \mathcal{L}[x(t)](s) = \mathcal{L}[t](s)$$

(1), (2) より

$$s^2 \mathcal{L}[x(t)](s) - s f(0) - f'(0) - a^2 \mathcal{L}[x(t)](s) = \frac{1}{s^2}$$

$$\mathcal{L}[x(t)](s) = \frac{1}{s^2-a^2}\cdot\frac{1}{s^2} = \frac{1}{a^2}\Big($$

$$= \frac{1}{a^2}\left(\frac{1}{s^2} - \frac{1}{s^2-a^2}\right)$$

$$= \frac{1}{a^3}\left(\frac{a}{s^2} + \frac{a}{a^2-s^2}\right).$$

(2) より

$$x(t) = \frac{1}{a^3}\left(at + \sinh t\right)$$

$$= \frac{t}{a^2} + \frac{1}{a^3}\sinh t$$