H23

[3]

(1) $f(z) = \dfrac{z^2+1}{z(z-a)(az-1)}$

特異点は $z = 0 . a . \frac{1}{a}$

$z=0$ aとき $\dfrac{z^2+1}{(z-a)(az-1)}$ は正則

$z=a$ aとき $\dfrac{z^2+1}{z(az-1)}$ は正則

$z=\frac{1}{a}$ aとき $\dfrac{z^2+1}{az(z-a)}$ は正則

$$Res[0] = \lim_{z\to 0} \frac{z^2+1}{(z-a)(az-1)} = \frac{1}{a}$$

$$Res[a] = \lim_{z\to a} \frac{z^2+1}{z(az-1)} = \frac{a^2+1}{a^3-a}$$

$$Res[\tfrac{1}{a}] = \lim_{z\to \frac{1}{a}} \frac{z^2+1}{az(z-a)} = \frac{1+a^2}{a-a^3}$$

(2) $z = e^{i\theta}$

$$\frac{dz}{d\theta} = iz$$

$$\cos\theta = \frac{1}{2}(z+z^{-1})$$

$$I = \int_{|z|=1} -\frac{1}{2i} \frac{z+z^{-1}}{az^2-(a^2+1)z+a} dz$$

$$= \int_{|z|=1} \frac{i}{2} \frac{z^2+1}{z(z-a)(az-1)} dz = \frac{i}{2}\int_{|z|=1} f(z)\, dz$$

(3) $a>1$ より $|z|=1$ 内の $f(z)$ の特異点は

$z = 0 . \frac{1}{a}$

よって

$$\int_{|z|=1} f(z)\, dz = 2\pi i \left(Res[0] + Res[\tfrac{1}{a}]\right)$$

$$= 2\pi i\left(\frac{1}{a} + \frac{1+a^2}{a-a^3}\right) = 2\pi i \frac{2}{a(1-a^2)}$$

したがって

$$I = \frac{i}{2}\cdot 2\pi i \frac{2}{a(1-a^2)} = \frac{2\pi}{a(a^2-1)}$$