2016 数学

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(1) $\dfrac{dy}{dx} - \dfrac{2}{x}y = -\dfrac{\sin 2x}{x}y^2$

$\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{2}{xy} = -\dfrac{\sin 2x}{x}$

$u = \dfrac{1}{y}$ とおき,

$-\dfrac{du}{dy}\cdot\dfrac{dy}{dx} - \dfrac{2}{x}u = -\dfrac{\sin 2x}{x}$

$\dfrac{du}{dx} + \dfrac{2}{x}u = \dfrac{\sin 2x}{x}$

ここで,

$e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^2 x$

から,

$(e^2 x u)' = e^2 \sin 2x$

$xu = -\dfrac{1}{2}\cos 2x + C$

$u = -\dfrac{1}{2x}\cos 2x + \dfrac{C}{x}$

よって,

$y = \dfrac{1}{-\frac{1}{2x}\cos 2x + \frac{C}{x}}$

$= \dfrac{2x}{2C - \cos 2x}$

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(2) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \dfrac{x}{2}$

特性方程式 $f(s)=0$ は,

$f(s) = s^2 - 6s + 9 = (s-3)^2 = 0$

$s = 3$ (重解)

だから,余関数は,

$y = e^{3x}(C_1 + C_2 x)$

また,

$y = ax + b$ とおくと,

$y' = a$, $y'' = 0$ から,

$-6a + 9ax + 9b = \dfrac{x}{2}$

$\cancel{3a + 9b = \frac{x}{2}}$

$9a = \dfrac{1}{2}$, $-6a + 9b = 0$

より,

$a = \dfrac{1}{18}$, $b = \dfrac{1}{27}$

よって,一般解は,

$y = e^{3x}(C_1 + C_2 x) + \dfrac{x}{18} + \dfrac{1}{27}$

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(3) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y = \sin x$

$f(s) = s^2 + 2s + 10 = 0$

$s = -1 \pm \sqrt{1-10}$

$= -1 \pm 3i$

よって,余関数は,

$y = e^{-x}(\cos 3x + \sin 3x)$

また,

$y = a\cos x + b\sin x$ とおくと,

$y' = -a\sin x + b\cos x$.

$y'' = -a\cos x - b\sin x$ から,

$(-a + 2b + 10a)\cos x$

$+ (-b - 2a + 10b)\sin x = \sin x$

$\begin{cases} 9a + 2b = 0. \\ -2a + 9b = 1 \end{cases}$

$b = -\dfrac{9}{2}a$

$-2a - \dfrac{81}{2}a = 1$

$-\dfrac{85}{2}a = 1$

$a = -\dfrac{2}{85}$, $b = \dfrac{9}{85}$

よって,一般解は,

$y = e^{-x}(\cos 3x + \sin 3x)$

$\quad - \dfrac{2}{85}\cos x + \dfrac{9}{85}\sin x$

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(1) $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+1)}\right]$

$= \int_0^t \sin\tau \cdot u(\tau - t)\,d\tau$

$= -[\cos\tau]_0^t$

$= 1 - \cos t$

また,

$\mathcal{L}^{-1}\left[\dfrac{1}{s^2+s+1}\right]$ は,

$\dfrac{1}{s^2+s+1} = \dfrac{\frac{2}{\sqrt3}\cdot\frac{\sqrt3}{2}}{(s+\frac12)^2 + \frac34}$

より,

$\mathcal{L}^{-1}\left[\dfrac{1}{s^2+s+1}\right]$

$= \dfrac{2}{\sqrt3}e^{-\frac{t}{2}}\sin\dfrac{\sqrt3}{2}t$

(2) $\mathcal{L}[x(t) * e^{-t}] = X(s)\cdot\dfrac{1}{s+1}$

$(\mathcal{L}[x(t)] = X(s))$

より,

$\begin{cases} sX(s) - x(0) - \dfrac{1}{s+1}X(s) = \dfrac{1}{s} \\ x(0) = 1 \end{cases}$

$\left(s - \dfrac{1}{s+1}\right)X(s) = \dfrac{1}{s} + 1$

$X(s) = \dfrac{\frac{1}{s} + 1}{s - \frac{1}{s+1}}$

$= \dfrac{1}{s}\cdot\dfrac{1+s}{s - \frac{1}{s+1}}$

$= \dfrac{1}{s}\cdot\dfrac{s^2+2s+1}{s^2+s-1}$

$= \dfrac{1}{s}\cdot\dfrac{1}{(s+\frac12)^2 - \frac54} + \dfrac{s+2}{(s+\frac12)^2 - \frac54}$

$= \dfrac{1}{s}\cdot\dfrac{2}{\sqrt5}\cdot\dfrac{\frac{\sqrt5}{2}}{(s+\frac12)^2 - \frac54} + \dfrac{s+\frac12}{(s+\frac12)^2 - \frac54}$

$\quad + \dfrac{\frac32\cdot\frac{\sqrt5}{\sqrt5}}{(s+\frac12)^2 - \frac54}$

ここで,

$\mathcal{L}^{-1}\left[\dfrac{1}{s}\cdot\dfrac{\frac{\sqrt5}{2}}{(s+\frac12)^2 - \frac54}\right]$

$= \int_0^t e^{-\frac{\tau}{2}}\sinh\dfrac{\sqrt5}{2}\tau\,d\tau$

$= \int_0^t e^{-\frac{\tau}{2}}\cdot\dfrac{e^{\frac{\sqrt5}{2}\tau} - e^{-\frac{\sqrt5}{2}\tau}}{2}d\tau$

$= \dfrac12\int_0^t \left(e^{\frac{\tau}{2}(\sqrt5-1)} - e^{-\frac{\tau}{2}(\sqrt5+1)}\right)d\tau$

$= \dfrac12\left[\dfrac{2}{\sqrt5-1}e^{\frac{\sqrt5-1}{2}\tau} + \dfrac{2}{\sqrt5+1}e^{-\frac{\sqrt5+1}{2}\tau}\right]_0^t$

$= \dfrac12\left(\dfrac{2}{\sqrt5-1}e^{\frac{\sqrt5-1}{2}t} + \dfrac{2}{\sqrt5+1}e^{-\frac{\sqrt5+1}{2}t} - \dfrac{2}{\sqrt5-1} + \dfrac{2}{\sqrt5+1}\right)$

$= \dfrac{1}{\sqrt5-1}e^{\frac{\sqrt5-1}{2}t} + \dfrac{1}{\sqrt5+1}e^{-\frac{\sqrt5+1}{2}t} - \dfrac{2\sqrt5}{5-1}$

$= \dfrac{1}{\sqrt5-1}e^{\frac{\sqrt5-1}{2}t} + \dfrac{1}{\sqrt5+1}e^{-\frac{\sqrt5+1}{2}t} - \dfrac{\sqrt5}{2}$

よって,

$x(t) = \mathcal{L}^{-1}[X(s)]$

$= e^{-\frac{t}{2}}\cosh\dfrac{\sqrt5}{2}t + \dfrac{3}{\sqrt5}e^{-\frac{t}{2}}\sinh\dfrac{\sqrt5}{2}t$

$\quad + \dfrac{2}{5-\sqrt5}e^{\frac{\sqrt5-1}{2}t} + \dfrac{2}{5+\sqrt5}e^{-\frac{\sqrt5+1}{2}t} - 1$