H19

$\boxed{3}$

(1)

$$C_1 = \{ z = x \mid 0 \leqq x \leqq R \}$$

$$dz = dx$$

$$I_1 = \int_{C_1} \frac{1}{x^3+1}\, dx$$

$$C_2 = \{ z = Re^{i\theta} \mid 0 \leqq \theta \leqq \tfrac{2}{3}\pi \}$$

$$\frac{dz}{d\theta} = iRe^{i\theta}$$

$$I_2 = \int_{C_2} \frac{iRe^{i\theta}}{R^3 e^{i3\theta}+1}\, d\theta$$

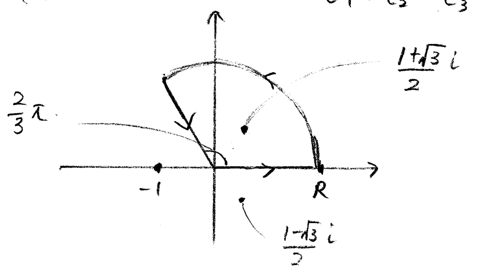
$$C_3 = \{ z = xe^{\frac{2}{3}\pi i} \mid 0 \leqq x \leqq R \}$$

$$\frac{dz}{dx} = e^{\frac{2}{3}\pi i}$$

$$I_3 = \int_{C_3} \frac{e^{\frac{2}{3}\pi i}}{x^3 e^{2\pi i}+1}\, dx = \int_{C_3} \frac{e^{\frac{2}{3}\pi i}}{x^3+1}\, dx$$

(2)

$C_1 + C_2 - C_3$



$$I = \int_C \frac{1}{z^3+1}\, dz = \int_C \frac{1}{(z+1)(z^2-z+1)}\, dz$$

特異点は $z = -1, \frac{1\pm\sqrt{3}i}{2}$

C内にあるのは $z = \frac{1+\sqrt{3}i}{2}$ のみ

$z = \frac{1+\sqrt{3}i}{2}$ のとき $\left(z - \frac{1+\sqrt{3}i}{2}\right)\frac{1}{z^3+1}$ は正則

$$I = 2\pi i \,\mathrm{Res}\left[\frac{1+\sqrt{3}i}{2}\right]$$

$$= 2\pi i \lim_{z \to \frac{1+\sqrt{3}i}{2}} \frac{1}{(z+1)\left(z - \frac{1-\sqrt{3}i}{2}\right)} = \frac{2\pi i}{3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{2\pi i}{3}\left(\cos\tfrac{2}{3}\pi - i\sin\tfrac{2}{3}\pi\right) = \frac{2\pi i}{3} e^{-\frac{2}{3}\pi i}$$