H21

2

(1)

$$\mathrm{rot}\, u = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z},\ \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x},\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right)$$

$$= (x - 2xz,\ 3z^3 - y - 1,\ z^2)$$

$$\mathrm{div}(\mathrm{rot}\, u) = 1 - 2z - 1 + 2z$$

$$= 0$$

(2)

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = r$$

$$u = (r^3 - r,\ r^3\cos\theta,\ r^2\sin\theta\cos\theta)$$

$$\mathrm{rot}\, u = (r\cos\theta - 2r^2\cos\theta,\ 3r^2 - r\sin\theta - 1,\ r^2)$$

$S_1 : x^2 + y^2 = z^2$ より

$$z = \sqrt{x^2+y^2}$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}},\quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$n = \frac{\left(-\frac{\partial z}{\partial x},\ -\frac{\partial z}{\partial y},\ -1\right)}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

$$= \frac{1}{\sqrt{2}}\left(-\frac{x}{\sqrt{x^2+y^2}},\ -\frac{y}{\sqrt{x^2+y^2}},\ 1 \right)$$

$$= -\frac{1}{\sqrt{2}}(\cos\theta,\ \sin\theta,\ 1)$$

$$\iint_{S_1} (\mathrm{rot}\, u)\cdot n\, dS = \int_0^{2\pi}\int_0^1 -\frac{1}{\sqrt{2}}(\cos\theta,\ \sin\theta,\ 1)\cdot(r\cos\theta - 2r^2\cos\theta,\ 3r^2 - 1 - r\sin\theta,\ r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -\frac{1}{\sqrt{2}}\left\{ r^2(3\sin\theta - 2\cos^2\theta + 1) + r(\cos^2\theta - \sin^2\theta) - \sin\theta \right\} dr\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{\sqrt{2}}\left[\frac{1}{3}r^3(3\sin\theta - 2\cos^2\theta + 1) + \frac{1}{2}r(\cos^2\theta - \sin^2\theta) - r\sin\theta \right]_0^1 d\theta$$

$$= -\frac{1}{\sqrt{2}}\int_0^{2\pi}\left(\sin\theta - \frac{2}{3}\cos^2\theta + \frac{1}{3} + \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta - \sin\theta \right) d\theta$$

$$\frac{1+\cos 2\theta}{2}$$

$$= -\frac{1}{\sqrt{2}}\int_0^{2\pi}\left(\frac{1}{3}\cos^2\theta - \frac{1}{6} \right) d\theta = -\frac{1}{\sqrt{2}}\left[\frac{1}{3}\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = -\frac{\sqrt{2}}{3}\pi$$