H18

3

(1) $f(z) = \dfrac{\pi \cos \pi z}{\sin \pi z}$

$\cos \pi z$, $\sin \pi z$ をマクローリン展開して

$\cos \pi z = 1 - \dfrac{(\pi z)^2}{2!} + \dfrac{(\pi z)^4}{4!} - \cdots$

$\sin \pi z = \pi z - \dfrac{(\pi z)^3}{3!} + \dfrac{(\pi z)^5}{5!} - \cdots$

$g(z) = \pi z - \dfrac{(\pi z)^3}{3!} + \dfrac{(\pi z)^5}{5!} - \cdots$

$g'(z) = \pi \left(1 - \dfrac{(\pi z)^2}{2!} + \dfrac{(\pi z)^4}{4!} - \cdots\right)$

ここで $f(z) = \dfrac{a_{-1}}{z} + a_0 + a_1 z + \cdots$

$f(z) = \dfrac{g'(z)}{g(z)}$

$f(z)\, g(z) = g'(z)$

$\left(\dfrac{a_{-1}}{z} + a_0 + a_1 z + \cdots\right)\left(\pi z - \dfrac{(\pi z)^3}{3!} + \dfrac{(\pi z)^5}{5!} - \cdots\right) = \pi \left(1 - \dfrac{(\pi z)^2}{2!} + \dfrac{(\pi z)^4}{4!} - \cdots\right)$

$a_{-1}\pi \left(1 - \dfrac{(\pi z)^2}{3!} + \dfrac{(\pi z)^4}{5!} - \cdots\right) + a_0 \left(\pi z - \dfrac{(\pi z)^3}{3!} + \dfrac{(\pi z)^5}{5!}\right) + a_1 \dfrac{1}{\pi}\left((\pi z)^2 - \dfrac{(\pi z)^4}{3!} + \dfrac{(\pi z)^6}{5!}\right) + \cdots = \pi \left(1 - \dfrac{(\pi z)^2}{2!} + \dfrac{(\pi z)^4}{4!} + \cdots\right)$

係数比較より

$\begin{cases} a_{-1}\pi = \pi \quad \rightarrow a_{-1} = 1 \\ -a_{-1}\pi \dfrac{(\pi z)^2}{3!} + a_1 \dfrac{1}{\pi}(\pi z)^2 = -\dfrac{\pi}{2!}(\pi z)^2 \\ a_0 = 0 \end{cases}$

$-\dfrac{\pi}{3!} + a_1 \dfrac{1}{\pi} = -\dfrac{\pi}{2!}$

$a_1 = \pi^2 \left(\dfrac{1}{2!} + \dfrac{1}{3!}\right) = -\dfrac{\pi^2}{3}$