2013 複素解析

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$f(z) = \dfrac{(z^2-1)^2}{z^2(z^2-6z+1)}$

(1) 特異点と留数

$f(z) = \dfrac{(z^2-1)^2}{z^2(z-(3+2\sqrt{2}))(z-(3-2\sqrt{2}))}$ なので

特異点 $z=0$（2重解），$3\pm2\sqrt{2}$（1重解）

$z = \dfrac{6\pm\sqrt{36-4}}{2}$

$= 3\pm2\sqrt{2}$

2位の極 $z=0$ の留数は

$$\mathrm{Res}[0] = \frac{1}{1!}\lim_{z\to0}\left(\frac{d}{dz}\right) z^2 f(z)$$

$$= \lim_{z\to0}\left(\frac{d}{dz}\right)\frac{(z^2-1)^2}{z^2-6z+1}$$

$$= \lim_{z\to0}\left(\frac{2\cdot2z(z^2-1)}{z^2-6z+1} - \frac{(z^2-1)^2(2z-6)}{(z^2-6z+1)^2}\right)$$

$$= -\frac{-6}{1} = \underline{6}$$

1位の極 $z = 3\pm2\sqrt{2}$ の留数は

$$\mathrm{Res}[3+2\sqrt{2}] = \lim_{z\to3+2\sqrt{2}}(z-(3+2\sqrt{2}))f(z)$$

$$= \lim_{z\to3+2\sqrt{2}}\frac{(z^2-1)^2}{z^2(z-(3-2\sqrt{2}))} = \frac{(16+12\sqrt{2})^2}{(17+12\sqrt{2})(4\sqrt{2})}$$

$$= \frac{4(34+12\sqrt{2})}{\sqrt{2}(17+12\sqrt{2})} = \underline{4\sqrt{2}}$$

$$\mathrm{Res}[3-2\sqrt{2}] = \lim_{z\to3-2\sqrt{2}}(z-(3-2\sqrt{2}))f(z)$$

$$= \lim_{z\to3-2\sqrt{2}}\frac{(z^2-1)^2}{z^2(z-(3+2\sqrt{2}))} = \frac{(16-12\sqrt{2})^2}{(17-12\sqrt{2})(-4\sqrt{2})}$$

$$= \frac{4}{-\sqrt{2}}\frac{34-24\sqrt{2}}{(17-12\sqrt{2})} = \underline{-4\sqrt{2}}$$

(2) $I_1 = \int_0^{2\pi}\frac{\sin^2\theta}{3-\cos\theta}\,d\theta$ を $|z|=1$ にうつし $I_2 = \int_{|z|=1} f(z)\,dz$ で表す

$|z|=1$ において $z = e^{i\theta}$ とおくと $dz = iz\,d\theta$

$$I_1 = \int_0^{2\pi}\frac{\sin^2\theta}{3-\cos\theta}\,d\theta = \int_{|z|=1}\frac{\left(\frac{z-z^{-1}}{2i}\right)^2}{3-\frac{z+z^{-1}}{2}}\frac{1}{iz}\,dz$$

$$= \frac{-1}{2i}\int_{|z|=1}\frac{(z+z^{-1})^2}{6z-(z^2+1)}\,dz$$

$$= \frac{1}{2i}\int_{|z|=1}\frac{(z^2+1)^2}{z^2(z^2-6z+1)}\,dz$$

$$= \frac{1}{2i}\int_{|z|=1} f(z)\,dz = \underline{\frac{1}{2i}I_2}$$

(3) I_1 の値

(2) より $I_1 = \dfrac{1}{2i}I_2$

$I_2 = \int_{|z|=1} f(z)\,dz$ において $f(z)$ の特異点で $|z|=1$ の中にあるのは，$z = 0,\ 3-2\sqrt{2}$ なので留数定理より

$$I_2 = 2\pi i\,(\mathrm{Res}[0] + \mathrm{Res}[3-2\sqrt{2}])$$

$$= 2\pi i\,(6-4\sqrt{2})$$

$$I_1 = \frac{1}{2i}\times2\pi i(6-4\sqrt{2}) = \underline{(6-4\sqrt{2})\pi}$$