H26

[3]

(1) $f(z) = \dfrac{\cos z}{\sin z} - \dfrac{1}{z}$

$= \dfrac{z\cos z - \sin z}{z \sin z}$　　特異点は $0,\ n\pi$ (nは整数)

(2)

マクローリン展開を用いて

$\sin z = 0 + z - \dfrac{0}{2!} - \dfrac{z^3}{3!} + \dfrac{0}{4!} + \dfrac{z^5}{5!} - \cdots$

$= z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots$

$f(z) = \dfrac{z\cos z - \sin z}{z\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\right)}$

$= \dfrac{z\cos z - \sin z}{z^2\left(1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots\right)}$

$z = 0$ のとき $\dfrac{z\cos z - \sin z}{\left(1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots\right)}$ は正則

$z = 0$ は 2位の極

$\mathrm{Res}[0] = \lim_{z\to 0} \dfrac{d}{dz} z^2 f(z) = \lim_{z\to 0} \dfrac{d}{dz}\left(\dfrac{z^2\cos z}{\sin z} - z\right)$

$= \lim_{z\to 0}\left\{\dfrac{(2z\cos z - z^2\sin z)\sin z - \cos z\,(z^2\cos z)}{\sin^2 z} - 1\right\}$

$= \lim_{z\to 0}\left\{\dfrac{(2\cos z - z\sin z)\dfrac{\sin z}{z} - \cos^2 z}{\left(\dfrac{\sin z}{z}\right)^2} - 1\right\}$　　$\leftarrow \lim_{z\to 0} \dfrac{\sin z}{z} = 1$

$= \dfrac{2 - 0 - 1}{1} - 1 = 0$

$h(z) = z\cos z - \sin z$

$g(z) = z\sin z$　　とすると

$g'(z) = z\cos z + \sin z$

$g(n\pi) = 0,\quad g'(n\pi) = n\pi \neq 0$

よって

$\mathrm{Res}[n\pi] = \dfrac{h(n\pi)}{g'(n\pi)} = \dfrac{n\pi}{n\pi} = 1.$