14 Sums and Asymptotics

Sums and products arise regularly in the analysis of algorithms, financial applications, physical problems, and probabilistic systems. For example, according to Theorem 2.2.1,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \tag{14.1}$$

Of course, the left-hand sum could be expressed concisely as a subscripted summation

$$\sum_{i=1}^{n} i$$

but the right-hand expression $n(n+1)/2$ is not only concise but also easier to evaluate. Furthermore, it more clearly reveals properties such as the growth rate of the sum. Expressions like $n(n+1)/2$ that do not make use of subscripted summations or products—or those handy but sometimes troublesome sequences of three dots—are called *closed forms*.

Another example is the closed form for a *geometric sum*

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \tag{14.2}$$

given in Problem 5.4. The sum as described on the left-hand side of (14.2) involves n additions and $1 + 2 + \cdots + (n-1) = (n-1)n/2$ multiplications, but its closed form on the right-hand side can be evaluated using fast exponentiation with at most $2 \log n$ multiplications, a division, and a couple of subtractions. Also, the closed form makes the growth and limiting behavior of the sum much more apparent.

Equations (14.1) and (14.2) were easy to verify by induction, but, as is often the case, the proofs by induction gave no hint about how these formulas were found in the first place. Finding them is part math and part art, which we'll start examining in this chapter.

Our first motivating example will be the value of a financial instrument known as an annuity. This value will be a large and nasty-looking sum. We will then describe several methods for finding closed forms for several sorts of sums, including those for annuities. In some cases, a closed form for a sum may not exist, and so we will provide a general method for finding closed forms for good upper and lower bounds on the sum.

The methods we develop for sums will also work for products, since any product can be converted into a sum by taking its logarithm. For instance, later in the chapter we will use this approach to find a good closed-form approximation to the *factorial function*

$$n! ::= 1 \cdot 2 \cdot 3 \cdots n.$$

We conclude the chapter with a discussion of asymptotic notation, especially “Big Oh” notation. Asymptotic notation is often used to bound the error terms when there is no exact closed form expression for a sum or product. It also provides a convenient way to express the growth rate or order of magnitude of a sum or product.

14.1 The Value of an Annuity

Would you prefer a million dollars today or \$50,000 a year for the rest of your life? On the one hand, instant gratification is nice. On the other hand, the *total dollars* received at \$50K per year is much larger if you live long enough.

Formally, this is a question about the value of an annuity. An *annuity* is a financial instrument that pays out a fixed amount of money at the beginning of every year for some specified number of years. In particular, an n-year, m-payment annuity pays m dollars at the start of each year for n years. In some cases, n is finite, but not always. Examples include lottery payouts, student loans, and home mortgages. There are even firms on Wall Street that specialize in trading annuities.[1]

A key question is, “What is an annuity worth?” For example, lotteries often pay out jackpots over many years. Intuitively, \$50,000 a year for 20 years ought to be worth less than a million dollars right now. If you had all the cash right away, you could invest it and begin collecting interest. But what if the choice were between \$50,000 a year for 20 years and a *half* million dollars today? Suddenly, it’s not clear which option is better.

14.1.1 The Future Value of Money

In order to answer such questions, we need to know what a dollar paid out in the future is worth today. To model this, let’s assume that money can be invested at a fixed annual interest rate p. We’ll assume an 8% rate[2] for the rest of the discussion, so $p = 0.08$.

[1] Such trading ultimately led to the subprime mortgage disaster in 2008–2009. We’ll talk more about that in a later chapter.

[2] U.S. interest rates have dropped steadily for several years, and ordinary bank deposits now earn around 1.0%. But just a few years ago the rate was 8%; this rate makes some of our examples a little more dramatic. The rate has been as high as 17% in the past thirty years.

Here is why the interest rate p matters. Ten dollars invested today at interest rate p will become $(1+p)\cdot 10 = 10.80$ dollars in a year, $(1+p)^2 \cdot 10 \approx 11.66$ dollars in two years, and so forth. Looked at another way, ten dollars paid out a year from now is only really worth $1/(1+p)\cdot 10 \approx 9.26$ dollars today, because if we had the \$9.26 today, we could invest it and would have \$10.00 in a year anyway. Therefore, p determines the value of money paid out in the future.

So for an n-year, m-payment annuity, the first payment of m dollars is truly worth m dollars. But the second payment a year later is worth only $m/(1+p)$ dollars. Similarly, the third payment is worth $m/(1+p)^2$, and the n-th payment is worth only $m/(1+p)^{n-1}$. The total value V of the annuity is equal to the sum of the payment values. This gives:

$$\begin{aligned} V &= \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}} \\ &= m \cdot \sum_{j=0}^{n-1} \left(\frac{1}{1+p}\right)^j && \text{(substitute } j = i-1\text{)} \\ &= m \cdot \sum_{j=0}^{n-1} x^j && \text{(substitute } x = 1/(1+p)\text{).} \qquad (14.3) \end{aligned}$$

The goal of the preceding substitutions was to get the summation into the form of a simple geometric sum. This leads us to an explanation of a way you could have discovered the closed form (14.2) in the first place using the *Perturbation Method*.

14.1.2 The Perturbation Method

Given a sum that has a nice structure, it is often useful to "perturb" the sum so that we can somehow combine the sum with the perturbation to get something much simpler. For example, suppose

$$S = 1 + x + x^2 + \cdots + x^n.$$

An example of a perturbation would be

$$xS = x + x^2 + \cdots + x^{n+1}.$$

The difference between S and xS is not so great, and so if we were to subtract xS from S, there would be massive cancellation:

$$\begin{aligned} S &= 1 + x + x^2 + x^3 + \cdots + x^n \\ -xS &= \quad\; - x - x^2 - x^3 - \cdots - x^n - x^{n+1}. \end{aligned}$$

The result of the subtraction is

$$S - xS = 1 - x^{n+1}.$$

Solving for S gives the desired closed-form expression in equation 14.2, namely,

$$S = \frac{1 - x^{n+1}}{1 - x}.$$

We'll see more examples of this method when we introduce *generating functions* in Chapter 16.

14.1.3 A Closed Form for the Annuity Value

Using equation 14.2, we can derive a simple formula for V, the value of an annuity that pays m dollars at the start of each year for n years.

$$\begin{aligned} V &= m\left(\frac{1 - x^n}{1 - x}\right) && \text{(by equations 14.3 and 14.2)} \quad (14.4) \\ &= m\left(\frac{1 + p - (1/(1+p))^{n-1}}{p}\right) && \text{(substituting } x = 1/(1+p)\text{)}. \quad (14.5) \end{aligned}$$

Equation 14.5 is much easier to use than a summation with dozens of terms. For example, what is the real value of a winning lottery ticket that pays \$50,000 per year for 20 years? Plugging in $m = \$50{,}000$, $n = 20$ and $p = 0.08$ gives $V \approx \$530{,}180$. So because payments are deferred, the million dollar lottery is really only worth about a half million dollars! This is a good trick for the lottery advertisers.

14.1.4 Infinite Geometric Series

We began this chapter by asking whether you would prefer a million dollars today or \$50,000 a year for the rest of your life. Of course, this depends on how long you live, so optimistically assume that the second option is to receive \$50,000 a year *forever*. This sounds like infinite money! But we can compute the value of an annuity with an infinite number of payments by taking the limit of our geometric sum in equation 14.2 as n tends to infinity.

Theorem 14.1.1. *If $|x| < 1$, then*

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}.$$

Proof.

$$
\begin{aligned}
\sum_{i=0}^{\infty} x^i &::= \lim_{n\to\infty} \sum_{i=0}^{n} x^i \\
&= \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} && \text{(by equation 14.2)} \\
&= \frac{1}{1 - x}.
\end{aligned}
$$

The final line follows from the fact that $\lim_{n\to\infty} x^{n+1} = 0$ when $|x| < 1$. ■

In our annuity problem $x = 1/(1 + p) < 1$, so Theorem 14.1.1 applies, and we get

$$
\begin{aligned}
V &= m \cdot \sum_{j=0}^{\infty} x^j && \text{(by equation 14.3)} \\
&= m \cdot \frac{1}{1 - x} && \text{(by Theorem 14.1.1)} \\
&= m \cdot \frac{1 + p}{p} && (x = 1/(1 + p)).
\end{aligned}
$$

Plugging in $m = \$50{,}000$ and $p = 0.08$, we see that the value V is only \$675,000. It seems amazing that a million dollars today is worth much more than \$50,000 paid every year for eternity! But on closer inspection, if we had a million dollars today in the bank earning 8% interest, we could take out and spend \$80,000 a year, *forever*. So as it turns out, this answer really isn't so amazing after all.

14.1.5 Examples

Equation 14.2 and Theorem 14.1.1 are incredibly useful in computer science.

Here are some other common sums that can be put into closed form using equation 14.2 and Theorem 14.1.1:

$$1 + 1/2 + 1/4 + \cdots = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{1-(1/2)} = 2 \tag{14.6}$$

$$0.99999\cdots = 0.9 \sum_{i=0}^{\infty} \left(\frac{1}{10}\right)^i = 0.9 \left(\frac{1}{1-1/10}\right) = 0.9 \left(\frac{10}{9}\right) = 1 \tag{14.7}$$

$$1 - 1/2 + 1/4 - \cdots = \sum_{i=0}^{\infty} \left(\frac{-1}{2}\right)^i = \frac{1}{1-(-1/2)} = \frac{2}{3} \tag{14.8}$$

$$1 + 2 + 4 + \cdots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i = \frac{1-2^n}{1-2} = 2^n - 1 \tag{14.9}$$

$$1 + 3 + 9 + \cdots + 3^{n-1} = \sum_{i=0}^{n-1} 3^i = \frac{1-3^n}{1-3} = \frac{3^n-1}{2} \tag{14.10}$$

If the terms in a geometric sum grow smaller, as in equation 14.6, then the sum is said to be *geometrically decreasing*. If the terms in a geometric sum grow progressively larger, as in equations 14.9 and 14.10, then the sum is said to be *geometrically increasing*. In either case, the sum is usually approximately equal to the term in the sum with the greatest absolute value. For example, in equations 14.6 and 14.8, the largest term is equal to 1 and the sums are 2 and 2/3, both relatively close to 1. In equation 14.9, the sum is about twice the largest term. In equation 14.10, the largest term is 3^{n-1} and the sum is $(3^n - 1)/2$, which is only about a factor of 1.5 greater. You can see why this rule of thumb works by looking carefully at equation 14.2 and Theorem 14.1.1.

14.1.6 Variations of Geometric Sums

We now know all about geometric sums—if you have one, life is easy. But in practice one often encounters sums that cannot be transformed by simple variable substitutions to the form $\sum x^i$.

A non-obvious but useful way to obtain new summation formulas from old ones is by differentiating or integrating with respect to x. As an example, consider the following sum:

$$\sum_{i=1}^{n} i x^i = x + 2x^2 + 3x^3 + \cdots + n x^n$$

This is not a geometric sum. The ratio between successive terms is not fixed, and so our formula for the sum of a geometric sum cannot be directly applied. But differentiating equation 14.2 leads to:

$$\frac{d}{dx}\left(\sum_{i=0}^{n} x^i\right) = \frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x}\right). \tag{14.11}$$

The left hand side of equation 14.11 is simply

$$\sum_{i=0}^{n} ix^{i-1}.$$

The right hand side of 14.11 is

$$\frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

Now multiplying both sides of 14.11 by x, we get

$$\sum_{i=0}^{n} ix^i == \frac{x(1-(n+1)x^n + nx^{n+1})}{(1-x)^2} = \frac{x-(n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} \tag{14.12}$$

So we have the desired closed-form expression for our sum. It seems a little complicated, but it's easier to work with than the sum itself. Incidentally, Problem 14.2 shows how the perturbation method could also be applied to derive an equivalent expression

$$\sum_{i=0}^{n} ix^i = \frac{1-x^{n+1}}{(1-x)^2} - \frac{1+nx^{n+1}}{1-x} \tag{14.13}$$

Notice that if $|x| < 1$, then this series converges to a finite value even if there are infinitely many terms. Taking the limit of equation 14.12 as n tends to infinity gives the following theorem:

Theorem 14.1.2. *If* $|x| < 1$*, then*

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}. \tag{14.14}$$

As a consequence, suppose that there is an annuity that pays im dollars at the end of each year i *forever*. For example, if m = \$50,000, then the payouts are \$50,000 and then \$100,000 and then \$150,000 and so on. It is hard to believe that the value of this annuity is finite! But we can use Theorem 14.1.2 to compute the value:

$$\begin{aligned} V &= \sum_{i=1}^{\infty} \frac{im}{(1+p)^i} \\ &= m \cdot \frac{1/(1+p)}{(1-\frac{1}{1+p})^2} \\ &= m \cdot \frac{1+p}{p^2}. \end{aligned}$$

The second line follows by an application of Theorem 14.1.2. The third line is obtained by multiplying the numerator and denominator by $(1+p)^2$.

For example, if $m = \$50{,}000$, and $p = 0.08$ as usual, then the value of the annuity is $V = \$8{,}437{,}500$. Even though the payments increase every year, the increase is only additive with time; by contrast, dollars paid out in the future decrease in value exponentially with time. The geometric decrease swamps out the additive increase. Payments in the distant future are almost worthless, so the value of the annuity is finite.

The trick of taking the derivative (or integral) of a summation formula is a good trick to remember, even though it may require computing some nasty derivatives.

14.2 Sums of Powers

In Chapter 5, we verified the formula (14.1), but the source of this formula is still a mystery. Sure, we can prove that it's true by using well ordering or induction, but where did the expression on the right come from in the first place? Even more inexplicable is the closed form expression for the sum of consecutive squares:

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}. \tag{14.15}$$

It turns out that there is a way to derive these expressions, but before we explain it, we thought it would be fun—OK, our definition of "fun" may be different than yours—to show you how Gauss is supposed to have proved equation 14.1 when he was a young boy.

Gauss's idea is related to the perturbation method we used in Section 14.1.2. Let

$$S = \sum_{i=1}^{n} i.$$

Then we can write the sum in two orders:

$$\begin{aligned} S &= 1 + \quad 2 \quad + \ldots + (n-1) + n, \\ S &= n + (n-1) + \ldots + \quad 2 \quad + 1. \end{aligned}$$

Adding these two equations gives

$$\begin{aligned} 2S &= (n+1) + (n+1) + \cdots + (n+1) + (n+1) \\ &= n(n+1). \end{aligned}$$

Hence,

$$S = \frac{n(n+1)}{2}.$$

Not bad for a young child—Gauss showed some potential. . . .

Unfortunately, the same trick does not work for summing consecutive squares. However, we can observe that the result might be a third-degree polynomial in n, since the sum contains n terms that average out to a value that grows quadratically in n. So we might guess that

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d.$$

If our guess is correct, then we can determine the parameters a, b, c and d by plugging in a few values for n. Each such value gives a linear equation in a, b, c and d. If we plug in enough values, we may get a linear system with a unique solution. Applying this method to our example gives:

$$\begin{aligned} n = 0 &\quad \text{implies} \quad 0 = d \\ n = 1 &\quad \text{implies} \quad 1 = a + b + c + d \\ n = 2 &\quad \text{implies} \quad 5 = 8a + 4b + 2c + d \\ n = 3 &\quad \text{implies} \quad 14 = 27a + 9b + 3c + d. \end{aligned}$$

Solving this system gives the solution $a = 1/3$, $b = 1/2$, $c = 1/6$, $d = 0$. Therefore, *if* our initial guess at the form of the solution was correct, then the summation is equal to $n^3/3 + n^2/2 + n/6$, which matches equation 14.15.

The point is that if the desired formula turns out to be a polynomial, then once you get an estimate of the *degree* of the polynomial, all the coefficients of the polynomial can be found automatically.

Be careful! This method lets you discover formulas, but it doesn't guarantee they are right! After obtaining a formula by this method, it's important to go back and *prove* it by induction or some other method. If the initial guess at the solution was not of the right form, then the resulting formula will be completely wrong! A later chapter will describe a method based on generating functions that does not require any guessing at all.

14.3 Approximating Sums

Unfortunately, it is not always possible to find a closed-form expression for a sum. For example, no closed form is known for

$$S = \sum_{i=1}^{n} \sqrt{i}.$$

In such cases, we need to resort to approximations for S if we want to have a closed form. The good news is that there is a general method to find closed-form upper and lower bounds that works well for many sums. Even better, the method is simple and easy to remember. It works by replacing the sum by an integral and then adding either the first or last term in the sum.

Definition 14.3.1. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *strictly increasing* when

$$x < y \text{ IMPLIES } f(x) < f(y),$$

and it is *weakly increasing*[3] when

$$x < y \text{ IMPLIES } f(x) \leq f(y).$$

Similarly, f is *strictly decreasing* when

$$x < y \text{ IMPLIES } f(x) > f(y),$$

and it is *weakly decreasing*[4] when

$$x < y \text{ IMPLIES } f(x) \geq f(y).$$

For example, 2^x and $\sqrt{x}$ are strictly increasing functions, while $\max\{x, 2\}$ and $\lceil x \rceil$ are weakly increasing functions. The functions $1/x$ and 2^{-x} are strictly decreasing, while $\min\{1/x, 1/2\}$ and $\lfloor 1/x \rfloor$ are weakly decreasing.

Theorem 14.3.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a weakly increasing function. Define*

$$S ::= \sum_{i=1}^{n} f(i) \tag{14.16}$$

[3] Weakly increasing functions are usually called *nondecreasing* functions. We will avoid this terminology to prevent confusion between being a nondecreasing function and the much weaker property of *not* being a decreasing function.

[4] Weakly decreasing functions are usually called *nonincreasing*.

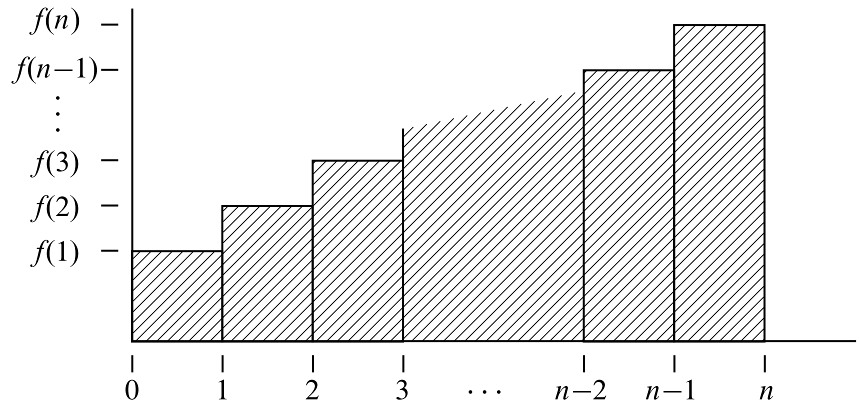


Figure 14.1 The area of the ith rectangle is $f(i)$. The shaded region has area $\sum_{i=1}^{n} f(i)$.

and

$$I ::= \int_1^n f(x)\,dx.$$

Then

$$I + f(1) \le S \le I + f(n). \tag{14.17}$$

Similarly, if f is weakly decreasing, then

$$I + f(n) \le S \le I + f(1).$$

Proof. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is weakly increasing. The value of the sum S in (14.16) is the sum of the areas of n unit-width rectangles of heights $f(1), f(2), \ldots, f(n)$. This area of these rectangles is shown shaded in Figure 14.1.

The value of

$$I = \int_1^n f(x)\,dx$$

is the shaded area under the curve of $f(x)$ from 1 to n shown in Figure 14.2.

Comparing the shaded regions in Figures 14.1 and 14.2 shows that S is at least I plus the area of the leftmost rectangle. Hence,

$$S \ge I + f(1) \tag{14.18}$$

This is the lower bound for S given in (14.17).

To derive the upper bound for S given in (14.17), we shift the curve of $f(x)$ from 1 to n one unit to the left as shown in Figure 14.3.

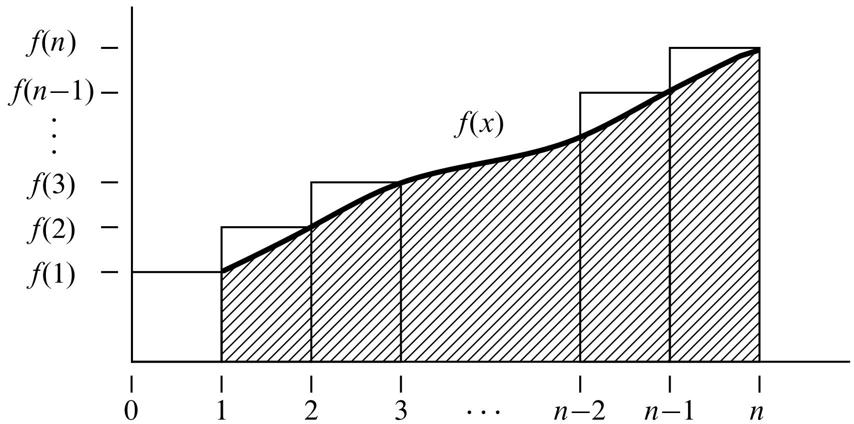


Figure 14.2 The shaded area under the curve of $f(x)$ from 1 to n (shown in bold) is $I = \int_1^n f(x)\,dx$.

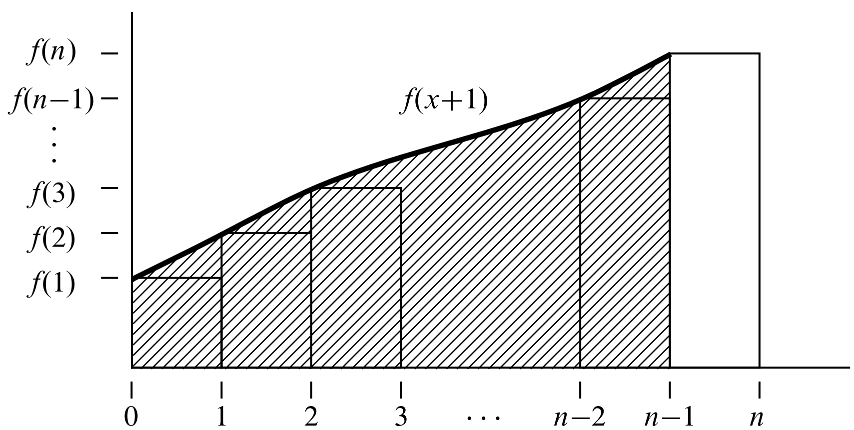


Figure 14.3 This curve is the same as the curve in Figure 14.2 shifted left by 1.

Comparing the shaded regions in Figures 14.1 and 14.3 shows that S is at most I plus the area of the rightmost rectangle. That is,

$$S \leq I + f(n),$$

which is the upper bound for S given in (14.17).

The very similar argument for the weakly decreasing case is left to Problem 14.10. ■

Theorem 14.3.2 provides good bounds for most sums. At worst, the bounds will be off by the largest term in the sum. For example, we can use Theorem 14.3.2 to bound the sum

$$S = \sum_{i=1}^{n} \sqrt{i}$$

as follows.

We begin by computing

$$\begin{aligned} I &= \int_1^n \sqrt{x}\,dx \\ &= \left.\frac{x^{3/2}}{3/2}\right|_1^n \\ &= \frac{2}{3}(n^{3/2} - 1). \end{aligned}$$

We then apply Theorem 14.3.2 to conclude that

$$\frac{2}{3}(n^{3/2} - 1) + 1 \;\leq\; S \;\leq\; \frac{2}{3}(n^{3/2} - 1) + \sqrt{n}$$

and thus that

$$\frac{2}{3}n^{3/2} + \frac{1}{3} \;\leq\; S \;\leq\; \frac{2}{3}n^{3/2} + \sqrt{n} - \frac{2}{3}.$$

In other words, the sum is very close to $\frac{2}{3}n^{3/2}$. We'll define several ways that one thing can be "very close to" something else at the end of this chapter.

As a first application of Theorem 14.3.2, we explain in the next section how it helps in resolving a classic paradox in structural engineering.

14.4 Hanging Out Over the Edge

Suppose you have a bunch of books and you want to stack them up, one on top of another in some off-center way, so the top book sticks out past books below it without falling over. If you moved the stack to the edge of a table, how far past the edge of the table do you think you could get the top book to go? Could the top book stick out completely beyond the edge of table? You're not supposed to use glue or any other support to hold the stack in place.

Most people's first response to the Book Stacking Problem—sometimes also their second and third responses—is "No, the top book will never get completely past the edge of the table." But in fact, you can get the top book to stick out as far as you want: one booklength, two booklengths, any number of booklengths!

14.4.1 Formalizing the Problem

We'll approach this problem recursively. How far past the end of the table can we get one book to stick out? It won't tip as long as its center of mass is over the table, so we can get it to stick out half its length, as shown in Figure 14.4.

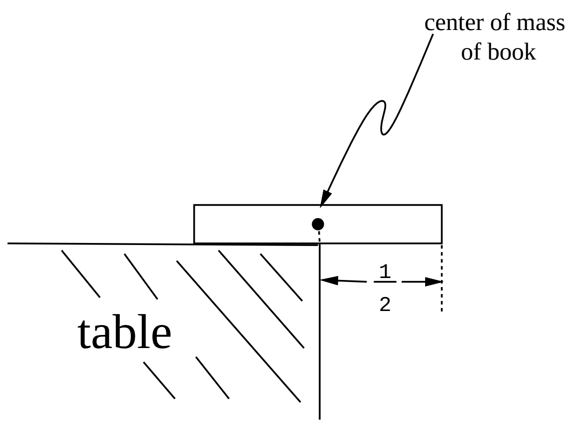


Figure 14.4 One book can overhang half a book length.

Now suppose we have a stack of books that will not tip over if the bottom book rests on the table—call that a *stable stack*. Let's define the *overhang* of a stable stack to be the horizontal distance from the center of mass of the stack to the furthest edge of the top book. So the overhang is purely a property of the stack, regardless of its placement on the table. If we place the center of mass of the stable stack at the edge of the table as in Figure 14.5, the overhang is how far we can get the top

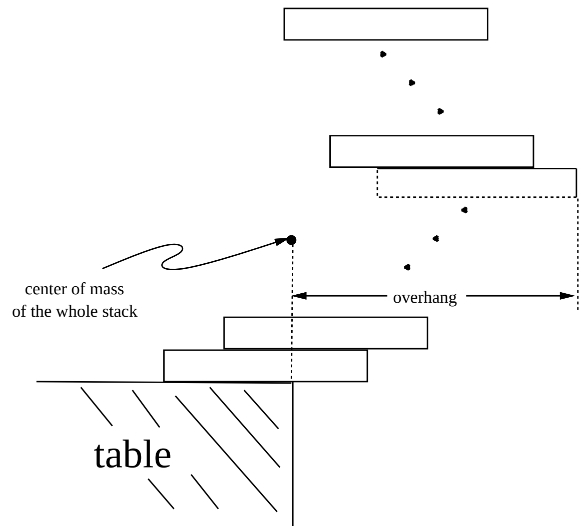


Figure 14.5 Overhanging the edge of the table.

book in the stack to stick out past the edge.

In general, a stack of n books will be stable if and only if the center of mass of the top i books sits over the $(i+1)$st book for $i = 1, 2, \ldots, n-1$.

So we want a formula for the maximum possible overhang B_n achievable with a stable stack of n books.

We've already observed that the overhang of one book is 1/2 a book length. That is,

$$B_1 = \frac{1}{2}.$$

Now suppose we have a stable stack of $n+1$ books with maximum overhang. If the overhang of the n books on top of the bottom book was not maximum, we could get a book to stick out further by replacing the top stack with a stack of n books with larger overhang. So the maximum overhang B_{n+1} of a stack of $n+1$ books is obtained by placing a maximum overhang stable stack of n books on top of the bottom book. And we get the biggest overhang for the stack of $n+1$ books by placing the center of mass of the n books right over the edge of the bottom book as in Figure 14.6.

So we know where to place the $n+1$st book to get maximum overhang. In fact, the reasoning above actually shows that this way of stacking $n+1$ books is the *unique* way to build a stable stack where the top book extends as far as possible. All we have to do is calculate what this extension is.

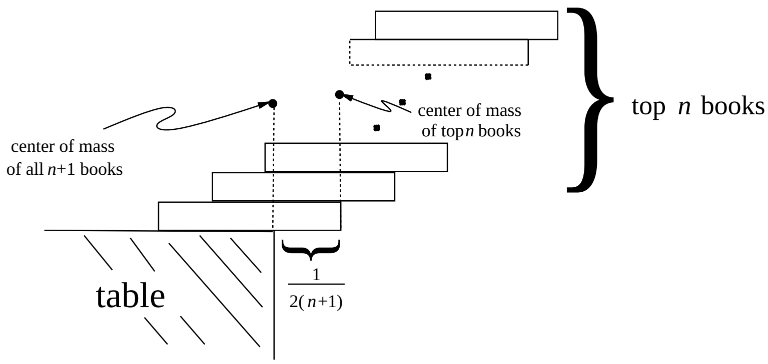


Figure 14.6 Additional overhang with $n + 1$ books.

The simplest way to do that is to let the center of mass of the top n books be the origin. That way the horizontal coordinate of the center of mass of the whole stack of $n + 1$ books will equal the increase in the overhang. But now the center of mass of the bottom book has horizontal coordinate $1/2$, so the horizontal coordinate of center of mass of the whole stack of $n + 1$ books is

$$\frac{0 \cdot n + (1/2) \cdot 1}{n + 1} = \frac{1}{2(n + 1)}.$$

In other words,

$$B_{n+1} = B_n + \frac{1}{2(n + 1)}, \tag{14.19}$$

as shown in Figure 14.6.

Expanding equation (14.19), we have

$$\begin{aligned} B_{n+1} &= B_{n-1} + \frac{1}{2n} + \frac{1}{2(n + 1)} \\ &= B_1 + \frac{1}{2 \cdot 2} + \cdots + \frac{1}{2n} + \frac{1}{2(n + 1)} \\ &= \frac{1}{2} \sum_{i=1}^{n+1} \frac{1}{i}. \end{aligned} \tag{14.20}$$

So our next task is to examine the behavior of B_n as n grows.

14.4.2 Harmonic Numbers

Definition 14.4.1. The nth *harmonic number* H_n is

$$H_n ::= \sum_{i=1}^{n} \frac{1}{i}.$$

So (14.20) means that

$$B_n = \frac{H_n}{2}.$$

The first few harmonic numbers are easy to compute. For example, $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} > 2$. The fact that H_4 is greater than 2 has special significance: it implies that the total extension of a 4-book stack is greater than one full book! This is the situation shown in Figure 14.7.

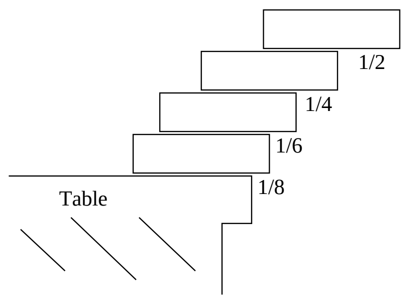


Figure 14.7 Stack of four books with maximum overhang.

There is good news and bad news about harmonic numbers. The bad news is that there is no known closed-form expression for the harmonic numbers. The good news is that we can use Theorem 14.3.2 to get close upper and lower bounds on H_n. In particular, since

$$\int_1^n \frac{1}{x}\,dx = \ln(x)\Big|_1^n = \ln(n),$$

Theorem 14.3.2 means that

$$\ln(n) + \frac{1}{n} \le H_n \le \ln(n) + 1. \tag{14.21}$$

In other words, the nth harmonic number is very close to $\ln(n)$.

Because the harmonic numbers frequently arise in practice, mathematicians have worked hard to get even better approximations for them. In fact, it is now known that

$$H_n = \ln(n) + \gamma + \frac{1}{2n} + \frac{1}{12n^2} + \frac{\epsilon(n)}{120n^4} \tag{14.22}$$

Here γ is a value $0.577215664\ldots$ called *Euler's constant*, and $\epsilon(n)$ is between 0 and 1 for all n. We will not prove this formula.

We are now finally done with our analysis of the book stacking problem. Plugging the value of H_n into (14.20), we find that the maximum overhang for n books is very close to $\ln(n)/2$. Since $\ln(n)$ grows to infinity as n increases, this means that if we are given enough books we can get a book to hang out arbitrarily far over the edge of the table. Of course, the number of books we need will grow as an exponential function of the overhang; it will take 227 books just to achieve an overhang of 3, never mind an overhang of 100.

Extending Further Past the End of the Table

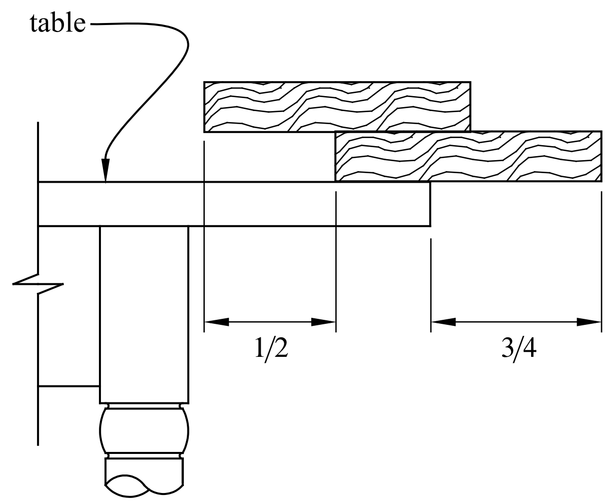
The overhang we analyzed above was the furthest out the *top* book could extend past the table. This leaves open the question of if there is some better way to build a stable stack where some book other than the top stuck out furthest. For example, Figure 14.8 shows a stable stack of two books where the bottom book extends further out than the top book. Moreover, the bottom book extends 3/4 of a book length past the end of the table, which is the same as the maximum overhang for the top book in a two book stack.

Since the two book arrangement in Figure 14.8(a) ties the maximum overhang stack in Figure 14.8(b), we could take the unique stable stack of n books where the top book extends furthest, and switch the top two books to look like Figure 14.8(a). This would give a stable stack of n books where the second from the top book extends the same maximum overhang distance. So for $n > 1$, there are at least two ways of building a stable stack of n books which both extend the maximum overhang distance—one way where the top book is furthest out, and another way where the second from the top book is furthest out.

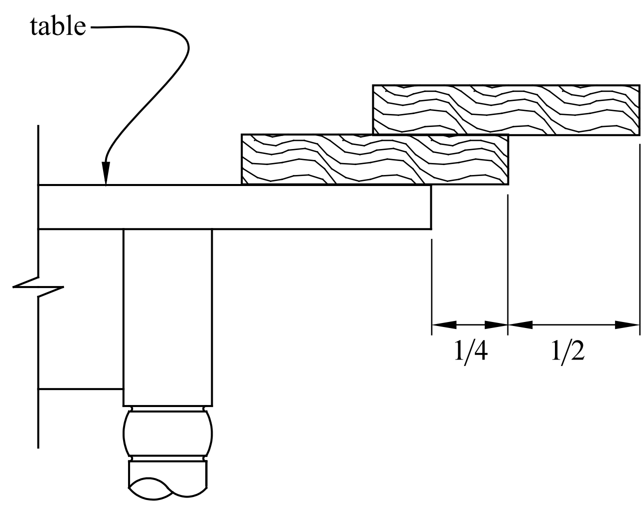
It is not too hard to prove that these are the *only* two ways to get a stable stack of books that achieves maximum overhang, providing we stick to stacking only *one* book on top of another. But there is more to the story. Building book piles with more than one book resting on another—think of an inverted pyramid—it is possible to get a stack of n books to extend proportional to $\sqrt[3]{n}$—much more than $\ln n$—book lengths without falling over. See [16], *Maximum Overhang*.

14.4.3 Asymptotic Equality

For cases like equation 14.22 where we understand the growth of a function like H_n up to some (unimportant) error terms, we use a special notation, $\sim$, to denote the leading term of the function. For example, we say that $H_n \sim \ln(n)$ to indicate that the leading term of H_n is $\ln(n)$. More precisely:



(a)



(b)

Figure 14.8 Figure (a) shows a stable stack of two books where the bottom book extends the same amount past the end of the table as the maximum overhang two-book stack shown in Figure (b).

Definition 14.4.2. For functions $f, g : \mathbb{R} \to \mathbb{R}$, we say f is *asymptotically equal* to g, in symbols,

$$f(x) \sim g(x)$$

iff

$$\lim_{x \to \infty} f(x)/g(x) = 1.$$

Although it is tempting to write $H_n \sim \ln(n) + \gamma$ to indicate the two leading terms, this is not really right. According to Definition 14.4.2, $H_n \sim \ln(n) + c$ where c is *any constant*. The correct way to indicate that γ is the second-largest term is $H_n - \ln(n) \sim \gamma$.

The reason that the $\sim$ notation is useful is that often we do not care about lower order terms. For example, if $n = 100$, then we can compute $H(n)$ to great precision using only the two leading terms:

$$|H_n - \ln(n) - \gamma| \leq \left| \frac{1}{200} - \frac{1}{120000} + \frac{1}{120 \cdot 100^4} \right| < \frac{1}{200}.$$

We will spend a lot more time talking about asymptotic notation at the end of the chapter. But for now, let's get back to using sums.

14.5 Products

We've covered several techniques for finding closed forms for sums but no methods for dealing with products. Fortunately, we do not need to develop an entirely new set of tools when we encounter a product such as

$$n! ::= \prod_{i=1}^{n} i. \tag{14.23}$$

That's because we can convert any product into a sum by taking a logarithm. For example, if

$$P = \prod_{i=1}^{n} f(i),$$

then

$$\ln(P) = \sum_{i=1}^{n} \ln(f(i)).$$

We can then apply our summing tools to find a closed form (or approximate closed form) for $\ln(P)$ and then exponentiate at the end to undo the logarithm.

For example, let's see how this works for the factorial function $n!$. We start by taking the logarithm:

$$\begin{aligned}\ln(n!) &= \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n) \\ &= \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n-1) + \ln(n) \\ &= \sum_{i=1}^{n} \ln(i).\end{aligned}$$

Unfortunately, no closed form for this sum is known. However, we can apply Theorem 14.3.2 to find good closed-form bounds on the sum. To do this, we first compute

$$\begin{aligned}\int_1^n \ln(x)\,dx &= x\ln(x) - x\Big|_1^n \\ &= n\ln(n) - n + 1.\end{aligned}$$

Plugging into Theorem 14.3.2, this means that

$$n\ln(n) - n + 1 \;\le\; \sum_{i=1}^{n} \ln(i) \;\le\; n\ln(n) - n + 1 + \ln(n).$$

Exponentiating then gives

$$\frac{n^n}{e^{n-1}} \;\le\; n! \;\le\; \frac{n^{n+1}}{e^{n-1}}. \tag{14.24}$$

This means that $n!$ is within a factor of n of n^n/e^{n-1}.

14.5.1 Stirling's Formula

The most commonly used product in discrete mathematics is probably $n!$, and mathematicians have workedto find tight closed-form bounds on its value. The most useful bounds are given in Theorem 14.5.1.

Theorem 14.5.1 (*Stirling's Formula*)**.** *For all $n \ge 1$,*

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\epsilon(n)}$$

where

$$\frac{1}{12n+1} \le \epsilon(n) \le \frac{1}{12n}.$$

Theorem 14.5.1 can be proved by induction (with some pain), and there are lots of proofs using elementary calculus, but we won't go into them.

There are several important things to notice about Stirling's Formula. First, $\epsilon(n)$ is always positive. This means that

$$n! > \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{14.25}$$

for all $n \in \mathbb{N}^+$.

Second, $\epsilon(n)$ tends to zero as n gets large. This means that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{14.26}$$

which is impressive. After all, who would expect both π and e to show up in a closed-form expression that is asymptotically equal to $n!$?

Third, $\epsilon(n)$ is small even for small values of n. This means that Stirling's Formula provides good approximations for $n!$ for most all values of n. For example, if we use

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

as the approximation for $n!$, as many people do, we are guaranteed to be within a factor of

$$e^{\epsilon(n)} \leq e^{\frac{1}{12n}}$$

of the correct value. For $n \geq 10$, this means we will be within 1% of the correct value. For $n \geq 100$, the error will be less than 0.1%.

If we need an even closer approximation for $n!$, then we could use either

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}$$

or

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n+1)}$$

depending on whether we want an upper, or a lower, bound. By Theorem 14.5.1, we know that both bounds will be within a factor of

$$e^{\frac{1}{12n}-\frac{1}{12n+1}} = e^{\frac{1}{144n^2+12n}}$$

of the correct value. For $n \geq 10$, this means that either bound will be within 0.01% of the correct value. For $n \geq 100$, the error will be less than 0.0001%.

For quick future reference, these facts are summarized in Corollary 14.5.2 and Table 14.1.

Approximation	$n \geq 1$	$n \geq 10$	$n \geq 100$	$n \geq 1000$
$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$	$< 10\%$	$< 1\%$	$< 0.1\%$	$< 0.01\%$
$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}$	$< 1\%$	$< 0.01\%$	$< 0.0001\%$	$< 0.000001\%$

Table 14.1 Error bounds on common approximations for $n!$ from Theorem 14.5.1. For example, if $n \geq 100$, then $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ approximates $n!$ to within 0.1%.

Corollary 14.5.2.

$$n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot \begin{cases} 1.09 & \textit{for } n \geq 1, \\ 1.009 & \textit{for } n \geq 10, \\ 1.0009 & \textit{for } n \geq 100. \end{cases}$$

14.6 Double Trouble

Sometimes we have to evaluate sums of sums, otherwise known as *double summations*. This sounds hairy, and sometimes it is. But usually, it is straightforward—you just evaluate the inner sum, replace it with a closed form, and then evaluate the outer sum (which no longer has a summation inside it). For example,[5]

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(y^n \sum_{i=0}^{n} x^i\right) &= \sum_{n=0}^{\infty}\left(y^n \frac{1-x^{n+1}}{1-x}\right) && \text{equation 14.2} \\
&= \left(\frac{1}{1-x}\right)\sum_{n=0}^{\infty} y^n - \left(\frac{1}{1-x}\right)\sum_{n=0}^{\infty} y^n x^{n+1} \\
&= \frac{1}{(1-x)(1-y)} - \left(\frac{x}{1-x}\right)\sum_{n=0}^{\infty}(xy)^n && \text{Theorem 14.1.1} \\
&= \frac{1}{(1-x)(1-y)} - \frac{x}{(1-x)(1-xy)} && \text{Theorem 14.1.1} \\
&= \frac{(1-xy) - x(1-y)}{(1-x)(1-y)(1-xy)} \\
&= \frac{1-x}{(1-x)(1-y)(1-xy)} \\
&= \frac{1}{(1-y)(1-xy)}.
\end{aligned}
$$

When there's no obvious closed form for the inner sum, a special trick that is often useful is to try *exchanging the order of summation.* For example, suppose we want to compute the sum of the first n harmonic numbers

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j} \tag{14.27}$$

For intuition about this sum, we can apply Theorem 14.3.2 to equation 14.21 to conclude that the sum is close to

$$\int_1^n \ln(x)\,dx = x\ln(x) - x \Big|_1^n = n\ln(n) - n + 1.$$

Now let's look for an exact answer. If we think about the pairs (k, j) over which

[5] OK, so maybe this one is a little hairy, but it is also fairly straightforward. Wait till you see the next one!

we are summing, they form a triangle:

$$
\begin{array}{cc|ccccccc}
 & & j & & & & & & \\
 & & 1 & 2 & 3 & 4 & 5 & \dots & n \\
\hline
k & 1 & 1 & & & & & & \\
 & 2 & 1 & 1/2 & & & & & \\
 & 3 & 1 & 1/2 & 1/3 & & & & \\
 & 4 & 1 & 1/2 & 1/3 & 1/4 & & & \\
 & & \dots & & & & & & \\
 & n & 1 & 1/2 & & \dots & & & 1/n
\end{array}
$$

The summation in equation 14.27 is summing each row and then adding the row sums. Instead, we can sum the columns and then add the column sums. Inspecting the table we see that this double sum can be written as

$$
\begin{aligned}
\sum_{k=1}^{n} H_k &= \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{j} \\
&= \sum_{j=1}^{n} \sum_{k=j}^{n} \frac{1}{j} \\
&= \sum_{j=1}^{n} \frac{1}{j} \sum_{k=j}^{n} 1 \\
&= \sum_{j=1}^{n} \frac{1}{j} (n - j + 1) \\
&= \sum_{j=1}^{n} \frac{n+1}{j} - \sum_{j=1}^{n} \frac{j}{j} \\
&= (n+1) \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} 1 \\
&= (n+1) H_n - n. \qquad (14.28)
\end{aligned}
$$

14.7 Asymptotic Notation

Asymptotic notation is a shorthand used to give a quick measure of the behavior of a function $f(n)$ as n grows large. For example, the asymptotic notation $\sim$ of Definition 14.4.2 is a binary relation indicating that two functions grow at the *same* rate. There is also a binary relation “little oh” indicating that one function grows at a significantly *slower* rate than another and “Big Oh” indicating that one function grows not much more rapidly than another.

14.7.1 Little Oh

Definition 14.7.1. For functions $f, g : \mathbb{R} \to \mathbb{R}$, with g nonnegative, we say f is *asymptotically smaller* than g, in symbols,

$$f(x) = o(g(x)),$$

iff

$$\lim_{x \to \infty} f(x)/g(x) = 0.$$

For example, $1000x^{1.9} = o(x^2)$ because

$$\lim_{x \to \infty} \frac{1000x^{1.9}}{x^2} = 1000 \lim_{x \to \infty} \frac{1}{x^{0.1}} = 1000 \cdot 0 = 0.$$

This argument generalizes directly to yield

Lemma 14.7.2.

$$f = o(g) \text{ IMPLIES } c \cdot f = o(g) \qquad \textit{for any constant } c. \tag{14.29}$$

$$x^a = o(x^b) \qquad \textit{for constants } 0 \leq a < b. \tag{14.30}$$

Also, log’s grow more slowly than roots:

Lemma 14.7.3. $\log_a x = o(x^{\epsilon})$ *for all* $a > 1, \epsilon > 0$.

Proof. For $y \geq 1$, we have $1/y \leq y$. Taking integrals from 1 to z, we conclude

$$\ln z \leq \frac{z^2}{2} \tag{14.31}$$

for $z \geq 1$. Choose $\epsilon > \delta > 0$ and let $z = \sqrt{x^{\delta}}$ in (14.31). Then

$$\frac{\delta \ln x}{2} \leq \frac{x^{\delta}}{2},$$

$$\ln x \leq \frac{x^{\delta}}{\delta} = o(x^{\epsilon}) \qquad \text{by Lemma 14.7.2.} \tag{14.32}$$

Finally, for any real number $a > 1$,

$$\log_a x = \frac{\ln x}{\ln a} = o(x^{\epsilon})$$

by (14.32) and (14.29). ∎

Corollary 14.7.4. *$x^b = o(a^x)$ for any $a, b \in \mathbb{R}$ with $a > 1$.*

Lemma 14.7.3 and Corollary 14.7.4 can also be proved using l'Hôpital's Rule or the Maclaurin Series for $\log x$ and e^x. Proofs can be found in most calculus texts.

14.7.2 Big Oh

"Big Oh" is the most frequently used asymptotic notation. It is used to give an upper bound on the growth of a function, such as the running time of an algorithm. There is a standard definition of Big Oh given below in 14.7.9, but we'll begin with an alternative definition that makes several basic properties of Big Oh more apparent.

Definition 14.7.5. Given functions $f, g : \mathbb{R} \to \mathbb{R}$ with g nonnegative, we say that

$$f = O(g)$$

iff

$$\limsup_{x \to \infty} |f(x)| / g(x) < \infty.$$

Here we're using the technical notion of *limit superior*[6] instead of just limit. But because limits and lim sup's are the same when limits exist, this formulation makes it easy to check basic properties of Big Oh. We'll take the following Lemma for granted.

Lemma 14.7.6. *If a function $f : \mathbb{R} \to \mathbb{R}$ has a finite or infinite limit as its argument approaches infinity, then its limit and limit superior are the same.*

Now Definition 14.7.5 immediately implies:

Lemma 14.7.7. *If $f = o(g)$ or $f \sim g$, then $f = O(g)$.*

[6] The precise definition of lim sup is

$$\limsup_{x \to \infty} h(x) ::= \lim_{x \to \infty} \text{lub}_{y \geq x} h(y),$$

where "lub" abbreviates "least upper bound."

Proof. $\lim f/g = 0$ or $\lim f/g = 1$ implies $\lim f/g < \infty$, so by Lemma 14.7.6, $\limsup f/g < \infty$. ∎

Note that the converse of Lemma 14.7.7 is not true. For example, $2x = O(x)$, but $2x \not\sim x$ and $2x \neq o(x)$.

We also have:

Lemma 14.7.8. *If $f = o(g)$, then it is* not *true that $g = O(f)$.*

Proof.

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \frac{1}{\lim_{x\to\infty} f(x)/g(x)} = \frac{1}{0} = \infty,$$

so by Lemma 14.7.6, $g \neq O(f)$. ∎

We need lim sup's in Definition 14.7.5 to cover cases when limits don't exist. For example, if $f(x)/g(x)$ oscillates between 3 and 5 as x grows, then $\lim_{x\to\infty} f(x)/g(x)$ does not exist, but $f = O(g)$ because $\limsup_{x\to\infty} f(x)/g(x) = 5$.

An equivalent, more usual formulation of big O does not mention lim sup's:

Definition 14.7.9. Given functions $f, g : \mathbb{R} \to \mathbb{R}$ with g nonnegative, we say

$$f = O(g)$$

iff there exists a constant $c \geq 0$ and an x_0 such that for all $x \geq x_0$, $|f(x)| \leq cg(x)$.

This definition is rather complicated, but the idea is simple: $f(x) = O(g(x))$ means $f(x)$ is less than or equal to $g(x)$, except that we're willing to ignore a constant factor, namely c, and to allow exceptions for small x, namely, for $x < x_0$. So in the case that $f(x)/g(x)$ oscillates between 3 and 5, $f = O(g)$ according to Definition 14.7.9 because $f \leq 5g$.

Proposition 14.7.10. $100x^2 = O(x^2)$.

Proof. Choose $c = 100$ and $x_0 = 1$. Then the proposition holds, since for all $x \geq 1$, $\left|100x^2\right| \leq 100x^2$. ∎

Proposition 14.7.11. $x^2 + 100x + 10 = O(x^2)$.

Proof. $(x^2+100x+10)/x^2 = 1+100/x+10/x^2$ and so its limit as x approaches infinity is $1+0+0 = 1$. So in fact, $x^2+100x+10 \sim x^2$, and therefore $x^2+100x+10 = O(x^2)$. Indeed, it's conversely true that $x^2 = O(x^2 + 100x + 10)$. ∎

Proposition 14.7.11 generalizes to an arbitrary polynomial:

Proposition 14.7.12. $a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 = O(x^k)$.

We'll omit the routine proof.

Big Oh notation is especially useful when describing the running time of an algorithm. For example, the usual algorithm for multiplying $n \times n$ matrices uses a number of operations proportional to n^3 in the worst case. This fact can be expressed concisely by saying that the running time is $O(n^3)$. The asymptotic notation highlights the behavior of the algorithm at a high level that abstracts away implementations details that may be programming language or machine-specific.

It turns out that there is another matrix multiplication procedure that uses $O(n^{2.55})$ operations.[7] This is an asymptotically faster procedure, and it will definitely be much faster on large enough matrices. But being asymptotically faster does not mean that it is a better choice. In fact, the $O(n^{2.55})$-operation multiplication procedure is almost never used in practice because it only becomes relatively efficient on matrices of impractical size.

However, the fact that the $O(n^{2.55})$ procedure is asymptotically faster indicates that it involves *new ideas* that go beyond optimized implementations of the $O(n^3)$ method. We can expect that these new ideas can lead to practical matrix multiplication procedures that are significantly faster than the usual ones.

14.7.3 Theta

Sometimes we want to specify that a running time $T(n)$ is precisely quadratic up to constant factors (both upper bound *and* lower bound). We could do this by saying that $T(n) = O(n^2)$ and $n^2 = O(T(n))$, but rather than say both, mathematicians have devised yet another symbol Θ to do the job.

Definition 14.7.13.

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \text{ and } g = O(f).$$

The statement $f = \Theta(g)$ can be paraphrased intuitively as "f and g are equal to within a constant factor."

The Theta notation allows us to highlight growth rates and suppress distracting factors and low-order terms. For example, if the running time of an algorithm is

$$T(n) = 10n^3 - 20n^2 + 1,$$

then we can more simply write

$$T(n) = \Theta(n^3).$$

[7] It is conceivable that there is an $O(n^2)$ matrix multiplication procedure, but none is known.

In this case, we would say that T *is of order* n^3 or that $T(n)$ *grows cubically*, which is often the main thing we really want to know. Another such example is

$$\pi^2 3^{x-7} + \frac{(2.7x^{113} + x^9 - 86)^4}{\sqrt{x}} - 1.08^{3x} = \Theta(3^x).$$

Just knowing that the running time of an algorithm is $\Theta(n^3)$, for example, is useful, because if n doubles we can predict that the running time will *by and large*[8] increase by a factor of at most 8 for large n. In this way, Theta notation preserves information about the scalability of an algorithm or system. Scalability is, of course, a big issue in the design of algorithms and systems.

14.7.4 Pitfalls with Asymptotic Notation

There is a long list of ways to make mistakes with asymptotic notation. This section presents some of the ways that big O notation can lead to trouble. With minimal effort, you can cause just as much chaos with the other symbols.

The Exponential Fiasco

Sometimes relationships involving big O are not so obvious. For example, one might guess that $4^x = O(2^x)$ since 4 is only a constant factor larger than 2. This reasoning is incorrect, however; 4^x actually grows as the square of 2^x.

Constant Confusion

Every constant is $O(1)$. For example, $17 = O(1)$. This is true because if we let $f(x) = 17$ and $g(x) = 1$, then there exists a $c > 0$ and an x_0 such that $|f(x)| \leq cg(x)$. In particular, we could choose $c = 17$ and $x_0 = 1$, since $|17| \leq 17 \cdot 1$ for all $x \geq 1$. We can construct a false theorem that exploits this fact.

False Theorem 14.7.14.

$$\sum_{i=1}^{n} i = O(n)$$

Bogus proof. Define $f(n) = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$. Since we have shown that every constant i is $O(1)$, $f(n) = O(1) + O(1) + \cdots + O(1) = O(n)$. ■

Of course in reality $\sum_{i=1}^{n} i = n(n+1)/2 \neq O(n)$.

---

[8] Since $\Theta(n^3)$ only implies that the running time $T(n)$ is between cn^3 and dn^3 for constants $0 < c < d$, the time $T(2n)$ could regularly exceed $T(n)$ by a factor as large as $8d/c$. The factor is sure to be close to 8 for all large n only if $T(n) \sim n^3$.

The error stems from confusion over what is meant in the statement $i = O(1)$. For any *constant* $i \in \mathbb{N}$ it is true that $i = O(1)$. More precisely, if f is any constant function, then $f = O(1)$. But in this False Theorem, i is not constant—it ranges over a set of values $0, 1, \ldots, n$ that depends on n.

And anyway, we should not be adding $O(1)$'s as though they were numbers. We never even defined what $O(g)$ means by itself; it should only be used in the context “$f = O(g)$” to describe a relation between functions f and g.

Equality Blunder

The notation $f = O(g)$ is too firmly entrenched to avoid, but the use of “=” is regrettable. For example, if $f = O(g)$, it seems quite reasonable to write $O(g) = f$. But doing so might tempt us to the following blunder: because $2n = O(n)$, we can say $O(n) = 2n$. But $n = O(n)$, so we conclude that $n = O(n) = 2n$, and therefore $n = 2n$. To avoid such nonsense, we will never write “$O(f) = g$.”

Similarly, you will often see statements like

$$H_n = \ln(n) + \gamma + O\left(\frac{1}{n}\right)$$

or

$$n! = (1 + o(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

In such cases, the true meaning is

$$H_n = \ln(n) + \gamma + f(n)$$

for some $f(n)$ where $f(n) = O(1/n)$, and

$$n! = (1 + g(n))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

where $g(n) = o(1)$. These last transgressions are OK as long as you (and your reader) know what you mean.

Operator Application Blunder

It's tempting to assume that familiar operations preserve asymptotic relations, but it ain't necessarily so. For example, $f \sim g$ in general does not even imply that $3^f = \Theta(3^g)$. On the other hand, some operations preserve and even strengthen asymptotic relations, for example,

$$f = \Theta(g) \text{ IMPLIES } \ln f \sim \ln g.$$

See Problem 14.27.

14.7.5 Omega (Optional)

Sometimes people incorrectly use Big Oh in the context of a lower bound. For example, they might say, “The running time $T(n)$ is at least $O(n^2)$.” This is another blunder! Big Oh can only be used for *upper* bounds. The proper way to express the lower bound would be

$$n^2 = O(T(n)).$$

The lower bound can also be described with another special notation “big Omega.”

Definition 14.7.15. Given functions $f, g : \mathbb{R} \to \mathbb{R}$ with f nonnegative, define

$$f = \Omega(g)$$

to mean

$$g = O(f).$$

For example, $x^2 = \Omega(x)$, $2^x = \Omega(x^2)$ and $x/100 = \Omega(100x + \sqrt{x})$.

So if the running time of your algorithm on inputs of size n is $T(n)$, and you want to say it is at least quadratic, say

$$T(n) = \Omega(n^2).$$

There is a similar “little omega” notation for lower bounds corresponding to little o:

Definition 14.7.16. For functions $f, g : \mathbb{R} \to \mathbb{R}$ with f nonnegative, define

$$f = \omega(g)$$

to mean

$$g = o(f).$$

For example, $x^{1.5} = \omega(x)$ and $\sqrt{x} = \omega(\ln^2(x))$.

The little omega symbol is not as widely used as the other asymptotic symbols we defined.

Problems for Section 14.1

Class Problems

Problem 14.1.
We begin with two large glasses. The first glass contains a pint of water, and the

second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

(a) Describe a closed-form formula for the amount of wine in the first glass after n back-and-forth pourings.

(b) What is the limit of the amount of wine in each glass as n approaches infinity?

Problem 14.2.
You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \cdots + z^n \\ zS &= z + z^2 + \cdots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \text{ (where } z \neq 1\text{)} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \cdots + nz^n$$

Problem 14.3.
Sammy the Shark is a financial service provider who offers loans on the following terms.

- Sammy loans a client m dollars in the morning. This puts the client m dollars in debt to Sammy.
- Each evening, Sammy first charges a service fee which increases the client's debt by f dollars, and then Sammy charges interest, which multiplies the debt by a factor of p. For example, Sammy might charge a "modest" ten cent service fee and 1% interest rate per day, and then f would be 0.1 and p would be 1.01.

(a) What is the client's debt at the end of the first day?

(b) What is the client's debt at the end of the second day?

(c) Write a formula for the client's debt after d days and find an equivalent closed form.

(d) If you borrowed \$10 from Sammy for a year, how much would you owe him?

Homework Problems

Problem 14.4.
Is a Harvard degree really worth more than an MIT degree? Let us say that a new Harvard graduate starts with a monthly salary of \$10K and gets a \$1K raise every month. So after one month, the Harvard grad gets paid \$10K, at the end of the second month they get paid \$11K, at the end of the third month \$12K, ...

On the other hand, a new MIT graduate starts with a monthly salary of \$6,000, but gets a 2% raise every month. So after one month, the MIT grad gets paid \$6K, at the end of the second month they get paid $1.02 \cdot \$6K = \$6,120$, at the end of the third month $1.02 \cdot \$6120 = \$6242.40, \ldots$. Assume inflation is a fixed 0.3% per month. That is, \$1.003 a month from now is worth \$1.00 today.

(a) Write a closed form formula for the total number of thousands of dollars a Harvard grad will earn during the n years after graduation. What is the total when $n = 25$?

(b) Write a closed form formula for the value in today's dollars of a Harvard degree, assuming the Harvard grad will work for n years following graduation. What is the value when $n = 25$?

(c) Likewise for an MIT degree.

Problem 14.5.
Suppose you deposit \$100 into your MIT Credit Union account today, then\$99 at the end of the first month from now, \$98 at tne end of the second months from now, and so on. Given that the interest rate is constantly 0.3% per month, how long will it take to save \$5,000?

Problems for Section 14.2

Class Problems

Problem 14.6.
Find a closed form for each of the following sums:

(a)

$$\sum_{i=1}^{n} \left(\frac{1}{i+2012} - \frac{1}{i+2013} \right).$$

(b) Assuming the following sum equals a polynomial in n, find the polynomial. Then verify by induction that the sum equals the polynomial you find.

$$\sum_{i=1}^{n} i^3$$

Problems for Section 14.3

Practice Problems

Problem 14.7.
Let

$$S ::= \sum_{n=1}^{5} \sqrt{3}n .$$

Using the Integral Method of Section 14.3, we can find integers a, b, c, d and a real number e such that

$$\int_a^b x^e \, dx \leq S \leq \int_c^d x^e \, dx$$

What are appropriate values for $a, \ldots, e$?

Class Problems

Problem 14.8.
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, weakly increasing function. Say that f *grows slowly* when

$$f(n) = o\left(\int_1^n f(x) \, dx \right).$$

(a) Prove that the function $f_a(n) ::= n^a$ grows slowly for any $a > 0$.

(b) Prove that the function e^n does not grow slowly.

(c) Prove that if f grows slowly, then

$$\int_1^n f(x) \, dx \sim \sum_{i=1}^{n} f(i).$$

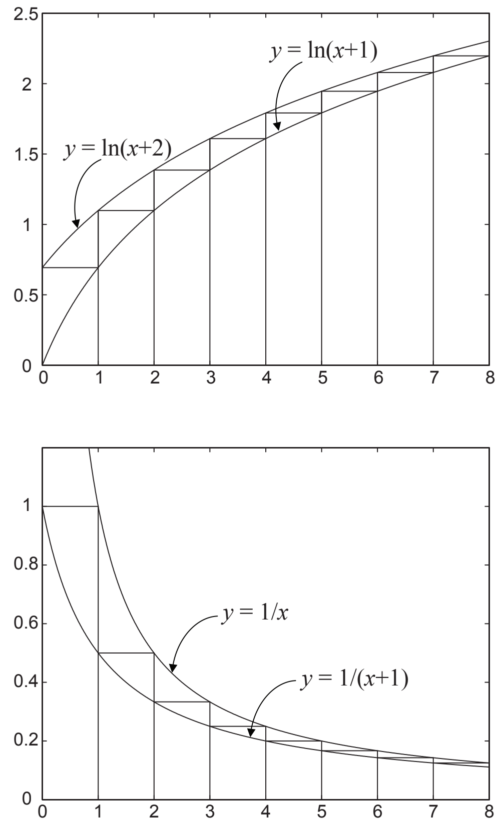


Figure 14.9 Integral bounds for two sums

Exam Problems

Problem 14.9.
Assume n is an integer larger than 1. Circle all the correct inequalities below.

Explanations are not required, but partial credit for wrong answers will not be given without them. *Hint:* You may find the graphs in Figure 14.9 helpful.

- $\sum_{i=1}^{n} \ln(i+1) \leq \ln 2 + \int_1^n \ln(x+1)dx$
- $\sum_{i=1}^{n} \ln(i+1) \leq \int_0^n \ln(x+2)dx$

- $\displaystyle\sum_{i=1}^{n} \frac{1}{i} \geq \int_0^n \frac{1}{x+1} dx$

Homework Problems

Problem 14.10.
Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a weakly decreasing function. Define

$$S ::= \sum_{i=1}^{n} f(i)$$

and

$$I ::= \int_1^n f(x)\, dx.$$

Prove that

$$I + f(n) \leq S \leq I + f(1).$$

(Proof by very clear picture is OK.)

Problem 14.11.
Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

Problems for Section 14.4

Class Problems

Problem 14.12.
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and then return to the oasis, with no water precached in the desert, if she takes a total of only 1 gallon from the oasis?

(b) What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

(c) The explorer will travel using a recursive strategy to go far into the desert and back, drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the nth Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

(d) Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Problem 14.13.
There is a number a such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of a? Prove it.

Hint: Find a value for a you think that works, then apply the integral bound.

Problem 14.14.
An infinite sum of nonnegative terms will converge to the same value—or diverge—no matter the order in which the terms are summed. This may not be true when there are an infinite number of both nonnegative and negative terms. An extreme example is

$$\sum_{i=0}^{\infty}(-1)^i = 1 + (-1) + 1 + (-1) + \cdots$$

because by regrouping the terms we can deduce:

$$\begin{aligned}[1 + (-1)] + [1 + (-1)] + \cdots &= 0 + 0 + \cdots = 0,\\ 1 + [(-1) + 1] + [(-1) + 1] + \cdots &= 1 + 0 + 0 + \cdots = 1.\end{aligned}$$

The problem here with this infinite sum is that the sum of the first n terms oscillates between 0 and 1, so the sum does not approach any limit.

But even for convergent sums, rearranging terms can cause big changes when the sum contains positive and negative terms. To illustrate the problem, we look at the Alternating Harmonic Series:

$$1 - 1/2 + 1/3 - 1/4 + \cdots \pm .$$

A standard result of elementary calculus, [2], p.403, is that this series converges to $\ln 2$, but things change if we reorder the terms in the series.

Explain for example how to reorder terms in the Alternating Harmonic Series so that the reordered series converges to 7. Then explain how to reorder so it diverges.

Homework Problems

Problem 14.15.
There is a bug on the edge of a 1-meter rug. The bug wants to cross to the other side of the rug. It crawls at 1 cm per second. However, at the end of each second, a malicious first-grader named Mildred Anderson *stretches* the rug by 1 meter. Assume that her action is instantaneous and the rug stretches uniformly. Thus, here's what happens in the first few seconds:

- The bug walks 1 cm in the first second, so 99 cm remain ahead.
- Mildred stretches the rug by 1 meter, which doubles its length. So now there are 2 cm behind the bug and 198 cm ahead.
- The bug walks another 1 cm in the next second, leaving 3 cm behind and 197 cm ahead.

- Then Mildred strikes, stretching the rug from 2 meters to 3 meters. So there are now $3 \cdot (3/2) = 4.5$ cm behind the bug and $197 \cdot (3/2) = 295.5$ cm ahead.
- The bug walks another 1 cm in the third second, and so on.

Your job is to determine this poor bug's fate.

(a) During second i, what *fraction* of the rug does the bug cross?

(b) Over the first n seconds, what fraction of the rug does the bug cross altogether? Express your answer in terms of the Harmonic number H_n.

(c) The known universe is thought to be about $3 \cdot 10^{10}$ light years in diameter. How many universe diameters must the bug travel to get to the end of the rug? (This distance is NOT the inflated distance caused by the stretching but only the actual walking done by the bug).

Problem 14.16.
Prove that the Alternating Harmonic Series

$$1 - 1/2 + 1/3 - 1/4 + \cdots \pm$$

converges.

Exam Problems

Problem 14.17.
Show that

$$\sum_{i=1}^{\infty} i^p$$

converges to a finite value iff $p < -1$.

Problems for Section 14.7

Practice Problems

Problem 14.18.
Find the least nonnegative integer n such that $f(x)$ is $O(x^n)$ when f is defined by each of the expressions below.

(a) $2x^3 + (\log x)x^2$

(b) $2x^2 + (\log x)x^3$

(c) $(1.1)^x$

(d) $(0.1)^x$

(e) $(x^4 + x^2 + 1)/(x^3 + 1)$

(f) $(x^4 + 5\log x)/(x^4 + 1)$

(g) $2^{(3\log_2 x^2)}$

Problem 14.19.
Let $f(n) = n^3$. For each function $g(n)$ in the table below, indicate which of the indicated asymptotic relations hold.

$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$
$6 - 5n - 4n^2 + 3n^3$				
$n^3 \log n$				
$(\sin(\pi n/2) + 2)\, n^3$				
$n^{\sin(\pi n/2)+2}$				
$\log n!$				
$e^{0.2n} - 100n^3$				

Problem 14.20.
Circle each of the true statements below.

Explanations are not required, but partial credit for wrong answers will not be given without them.

- $n^2 \sim n^2 + n$
- $3^n = O(2^n)$
- $n^{\sin(n\pi/2)+1} = o(n^2)$
- $n = \Theta\left(\dfrac{3n^3}{(n+1)(n-1)}\right)$

Problem 14.21.
Show that
$$\ln(n^2!) = \Theta(n^2 \ln n)$$

Hint: Stirling's formula for $(n^2)!$.

Problem 14.22.
The quantity
$$\frac{(2n)!}{2^{2n}(n!)^2} \tag{14.33}$$
will come up later in the course (it is the probability that in 2^{2n} flips of a fair coin, exactly n will be Heads). Show that it is asymptotically equal to $\dfrac{1}{\sqrt{\pi n}}$.

Problem 14.23.
Suppose let f and g be real-valued functions.

(a) Give an example of f, g such that
$$\limsup fg < \limsup f \cdot \limsup g,$$
and all the lim sup's are finite.

(b) Give an example of f, g such that
$$\limsup fg > \limsup f \cdot \limsup g.$$
and all the lim sup's are finite.

Homework Problems

Problem 14.24. **(a)** Prove that the relation R on positive functions such that $f \; R \; g$ iff $g = o(f)$ is a strict partial order.

(b) If g is a positive function, prove than $f \sim g$ iff $f = g + h$ for some function $h = o(g)$.

Problem 14.25.
Indicate which of the following holds for each pair of functions $(f(n), g(n))$ in the table below. Assume $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants. Pick the four table entries you consider to be the most challenging or interesting and justify your answers to these.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$	$f = \Theta(g)$	$f \sim g$
2^n	$2^{n/2}$						
$\sqrt{n}$	$n^{\sin(n\pi/2)}$						
$\log(n!)$	$\log(n^n)$						
n^k	c^n						
$\log^k n$	n^ϵ						

Problem 14.26.
Arrange the following functions in a sequence $f_1, f_2, \dots f_{24}$ so that $f_i = O(f_{i+1})$. Additionally, if $f_i = \Theta(f_{i+1})$, indicate that too:

1. $n \log n$
2. $2^{100} n$
3. n^{-1}
4. $n^{-1/2}$
5. $(\log n)/n$
6. $\binom{n}{64}$
7. $n!$
8. $2^{2^{100}}$
9. 2^{2^n}
10. 2^n
11. 3^n
12. $n 2^n$
13. 2^{n+1}

14. $2n$
15. $3n$
16. $\log(n!)$
17. $\log_2 n$
18. $\log_{10} n$
19. $2.1^{\sqrt{n}}$
20. 2^{2n}
21. 4^n
22. n^{64}
23. n^{65}
24. n^n

Problem 14.27.
Let f, g be nonnegative real-valued functions such that $\lim_{x\to\infty} f(x) = \infty$ and $f \sim g$.

(a) Give an example of f, g such that $\text{NOT}(2^f \sim 2^g)$.

(b) Prove that $\log f \sim \log g$.

(c) Use Stirling's formula to prove that in fact

$$\log(n!) \sim n \log n$$

Problem 14.28.
Determine which of these choices

$$\Theta(n), \quad \Theta(n^2 \log n), \quad \Theta(n^2), \quad \Theta(1), \quad \Theta(2^n), \quad \Theta(2^{n \ln n}), \quad \text{none of these}$$

describes each function's asymptotic behavior. Full proofs are not required, but briefly explain your answers.

(a)

$$n + \ln n + (\ln n)^2$$

(b)

$$\frac{n^2 + 2n - 3}{n^2 - 7}$$

(c)

$$\sum_{i=0}^{n} 2^{2i+1}$$

(d)

$$\ln(n^2!)$$

(e)

$$\sum_{k=1}^{n} k\left(1 - \frac{1}{2^k}\right)$$

Problem 14.29. **(a)** Either prove or disprove each of the following statements.

- $n! = O((n+1)!)$
- $(n+1)! = O(n!)$
- $n! = \Theta((n+1)!)$
- $n! = o((n+1)!)$
- $(n+1)! = o(n!)$

(b) Show that $\left(\frac{n}{3}\right)^{n+e} = o(n!)$.

Problem 14.30.
Prove that

$$\sum_{k=1}^{n} k^6 = \Theta(n^7).$$

Class Problems

Problem 14.31.
Give an elementary proof (without appealing to Stirling's formula) that $\log(n!) = \Theta(n \log n)$.

Problem 14.32.
Suppose $f, g : \mathbb{N}^+ \to \mathbb{N}^+$ and $f \sim g$.

(a) Prove that $2f \sim 2g$.

(b) Prove that $f^2 \sim g^2$.

(c) Give examples of f and g such that $2^f \not\sim 2^g$. Briefly explain your answer.

Problem 14.33.
Recall that for functions f, g on $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|. \tag{14.34}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is $O()$ of the other, indicate the *smallest nonnegative integer* c and for that smallest c, the *smallest corresponding nonnegative integer* n_0 ensuring that condition (14.34) applies.

(a) $f(n) = n^2, g(n) = 3n$.

$f = O(g)$	YES	NO	If YES, $c =$ _______, $n_0 =$ _______
$g = O(f)$	YES	NO	If YES, $c =$ _______, $n_0 =$ _______

(b) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

$f = O(g)$	YES	NO	If YES, $c =$ _______, $n_0 =$ _______
$g = O(f)$	YES	NO	If YES, $c =$ _______, $n_0 =$ _______

(c) $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

$f = O(g)$	YES	NO	If yes, $c =$ _______ $n_0 =$ _______
$g = O(f)$	YES	NO	If yes, $c =$ _______ $n_0 =$ _______

Problem 14.34.

False Claim.

$$2^n = O(1). \tag{14.35}$$

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

Bogus proof. The proof is by induction on n where the induction hypothesis $P(n)$ is the assertion (14.35).

base case: $P(0)$ holds trivially.

inductive step: We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n+1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all n. That is, the exponential function is bounded by a constant.

■

Problem 14.35. **(a)** Prove that the relation R on functions such that $f \; R \; g$ iff $f = o(g)$ is a strict partial order.

(b) Describe two functions f, g that are incomparable under big Oh:

$$f \neq O(g) \text{ AND } g \neq O(f).$$

Conclude that R is not a linear order. How about three such functions?

Exam Problems

Problem 14.36.
Give an example of a pair of strictly increasing total functions, $f : \mathbb{N}^+ \to \mathbb{N}^+$ and $g : \mathbb{N}^+ \to \mathbb{N}^+$, that satisfy $f \sim g$ but **not** $3^f = O(3^g)$.

Problem 14.37.
Let f, g be real-valued functions such that $f = \Theta(g)$ and $\lim_{x \to \infty} f(x) = \infty$. Prove that

$$\ln f \sim \ln g.$$

Problem 14.38.

(a) Define a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = \Theta(n^2)$ and $\text{NOT}(f(n) \sim n^2)$. Explain why your function satisfies these conditions.

$$f(n) =$$

(b) Define a function $g : \mathbb{N} \to \mathbb{R}$ such that $g(n) = O(n^2)$, $g(n) \neq \Theta(n^2)$, $g(n) \neq o(n^2)$, and $n = O(g(n))$, and explain why your example works.

$$g(n) =$$

Problem 14.39. **(a)** Show that

$$(an)^{b/n} \sim 1.$$

where a, b are positive constants and $\sim$ denotes asymptotic equality. *Hint:* $an = a2^{\log_2 n}$.

(b) Show that

$$\sqrt[n]{n!} = \Theta(n).$$

Problem 14.40.

(a) Indicate which of the following asymptotic relations below on the set of non-negative real-valued functions are equivalence relations (**E**), strict partial orders (**S**), weak partial orders (**W**), or *none* of the above (**N**).

(i) $f = o(g)$, the “little Oh” relation.

(ii) $f = O(g)$, the “big Oh” relation.

(iii) $f \sim g$, the “asymptotically equal” relation.

(iv) $f = \Theta(g)$, the “Theta” relation.

(v) $f = O(g)$ AND NOT$(g = O(f))$.

(b) Indicate the implications among the items (i)–(v) in part (a). For example,

- item (i) IMPLIES item (ii).

Briefly explain your answers.

Problem 14.41.
Recall that if f and g are nonnegative real-valued functions on $\mathbb{Z}^+$, then $f = O(g)$ iff there exist $c, n_0 \in \mathbb{Z}^+$ such that

$$\forall n \geq n_0.\ f(n) \leq cg(n).$$

For each pair of functions f and g below, indicate the **smallest** $c \in \mathbb{Z}^+$, and for that smallest c, the **smallest corresponding** $n_0 \in \mathbb{Z}^+$, that would establish $f = O(g)$ by the definition given above. If there is no such c, write ∞.

(a) $f(n) = \frac{1}{2}\ln n^2, g(n) = n.$ $c =$ _______, $n_0 =$ _______

(b) $f(n) = n, g(n) = n \ln n.$ $c =$ _______, $n_0 =$ _______

(c) $f(n) = 2^n, g(n) = n^4 \ln n$ $c =$ _______, $n_0 =$ _______

(d) $f(n) = 3\sin\left(\frac{\pi(n-1)}{100}\right) + 2, g(n) = 0.2.$ $c =$ _______, $n_0 =$ _______

Problem 14.42.
Let f, g be positive real-valued functions on finite, *connected*, simple graphs. We will extend the $O()$ notation to such graph functions as follows: $f = O(g)$ iff there is a constant $c > 0$ such that

$$f(G) \leq c \cdot g(G) \text{ for all connected simple graphs } G \text{ with more than one vertex.}$$

For each of the following assertions, state whether it is **True** or **False** and briefly explain your answer. You are **not** expected to offer a careful proof or detailed counterexample.

Reminder: $V(G)$ is the set of vertices and $E(G)$ is the set of edges of G, and G is connected.

(a) $|V(G)| = O(|E(G)|)$.

(b) $|E(G)| = O(|V(G)|)$.

(c) $|V(G)| = O(\chi(G))$, where $\chi(G)$ is the chromatic number of G.

(d) $\chi(G) = O(|V(G)|)$.