13 Planar Graphs

13.1 Drawing Graphs in the Plane

Suppose there are three dog houses and three human houses, as shown in Figure 13.1. Can you find a route from each dog house to each human house such that no route crosses any other route?

A similar question comes up about a little-known animal called a *quadrapus* that looks like an octopus with four stretchy arms instead of eight. If five quadrapi are resting on the sea floor, as shown in Figure 13.2, can each quadrapus simultaneously shake hands with every other in such a way that no arms cross?

Both these puzzles can be understood as asking about drawing graphs in the plane. Replacing dogs and houses by nodes, the dog house puzzle can be rephrased as asking whether there is a planar drawing of the graph with six nodes and edges between each of the first three nodes and each of the second three nodes. This graph is called the *complete bipartite graph* $K_{3,3}$ and is shown in Figure 13.3.(a). The quadrapi puzzle asks whether there is a planar drawing of the complete graph K_5 shown in Figure 13.3.(b).

In each case, the answer is, "No—but almost!" In fact, if you remove an edge from either of these graphs, then the resulting graph *can* be redrawn in the plane so that no edges cross, as shown in Figure 13.4.

Planar drawings have applications in circuit layout and are helpful in displaying graphical data such as program flow charts, organizational charts and scheduling conflicts. For these applications, the goal is to draw the graph in the plane with as few edge crossings as possible. (See the box on the following page for one such example.)

13.2 Definitions of Planar Graphs

We took the idea of a planar drawing for granted in the previous section, but if we're going to *prove* things about planar graphs, we better have precise definitions.

Definition 13.2.1. A *drawing* of a graph assigns to each node a distinct point in the plane and assigns to each edge a smooth curve in the plane whose endpoints correspond to the nodes incident to the edge. The drawing is *planar* if none of the

Figure 13.1 Three dog houses and and three human houses. Is there a route from each dog house to each human house so that no pair of routes cross each other?

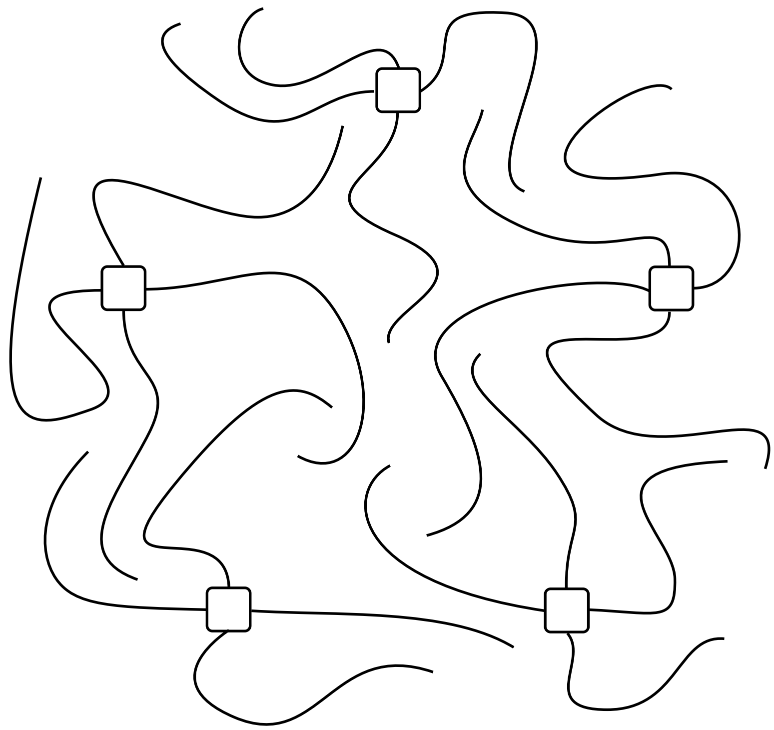
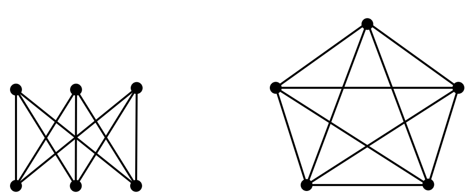


Figure 13.2 Five quadrapi (4-armed creatures).



(a) (b)

Figure 13.3 $K_{3,3}$ (a) and K_5 (b). Can you redraw these graphs so that no pairs of edges cross?

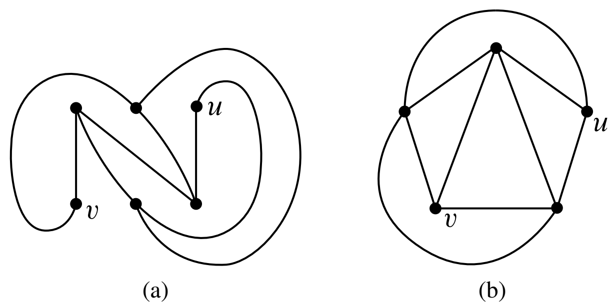


Figure 13.4 Planar drawings of (a) $K_{3,3}$ without $\langle u—v \rangle$, and (b) K_5 without $\langle u—v \rangle$.

Steve Wozniak and a Planar Circuit Design

When wires are arranged on a surface, like a circuit board or microchip, crossings require troublesome three-dimensional structures. When Steve Wozniak designed the disk drive for the early Apple II computer, he struggled mightily to achieve a nearly planar design according to the following excerpt from `apple2history.org` which in turn quotes *Fire in the Valley* by Freiberger and Swaine:

> For two weeks, he worked late each night to make a satisfactory design. When he was finished, he found that if he moved a connector he could cut down on feedthroughs, making the board more reliable. To make that move, however, he had to start over in his design. This time it only took twenty hours. He then saw another feedthrough that could be eliminated, and again started over on his design. "The final design was generally recognized by computer engineers as brilliant and was by engineering aesthetics beautiful. Woz later said, 'It's something you can only do if you're the engineer and the PC board layout person yourself. That was an artistic layout. The board has virtually no feedthroughs.'

curves cross themselves or other curves, namely, the only points that appear more than once on any of the curves are the node points. A graph is *planar* when it has a planar drawing.

Definition 13.2.1 is precise but depends on further concepts: "smooth planar curves" and "points appearing more than once" on them. We haven't defined these concepts—we just showed the simple picture in Figure 13.4 and hoped you would get the idea.

Pictures can be a great way to get a new idea across, but it is generally not a good idea to use a picture to replace precise mathematics. Relying solely on pictures can sometimes lead to disaster—or to bogus proofs, anyway. There is a long history of bogus proofs about planar graphs based on misleading pictures.

The bad news is that to prove things about planar graphs using the planar drawings of Definition 13.2.1, we'd have to take a chapter-long excursion into continuous mathematics just to develop the needed concepts from plane geometry and point-set topology. The good news is that there is another way to define planar graphs that uses only discrete mathematics. In particular, we can define planar graphs as a recursive data type. In order to understand how it works, we first need to understand the concept of a *face* in a planar drawing.

13.2.1 Faces

The curves in a planar drawing divide up the plane into connected regions called the *continuous faces*[1] of the drawing. For example, the drawing in Figure 13.5 has four continuous faces. Face IV, which extends off to infinity in all directions, is called the *outside face*.

The vertices along the boundary of each continuous face in Figure 13.5 form a cycle. For example, labeling the vertices as in Figure 13.6, the cycles for each of the face boundaries can be described by the vertex sequences

$$abca \qquad abda \qquad bcdb \qquad acda. \tag{13.1}$$

These four cycles correspond nicely to the four continuous faces in Figure 13.6—so nicely, in fact, that we can identify each of the faces in Figure 13.6 by its cycle. For example, the cycle $abca$ identifies face III. The cycles in list 13.1 are called the *discrete faces* of the graph in Figure 13.6. We use the term "discrete" since cycles in a graph are a discrete data type—as opposed to a region in the plane, which is a continuous data type.

[1]Most texts drop the adjective *continuous* from the definition of a face as a connected region. We need the adjective to distinguish continuous faces from the *discrete* faces we're about to define.

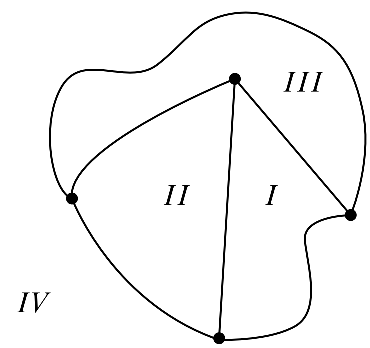


Figure 13.5 A planar drawing with four continuous faces.

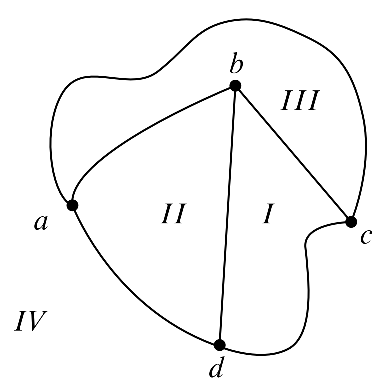


Figure 13.6 The drawing with labeled vertices.

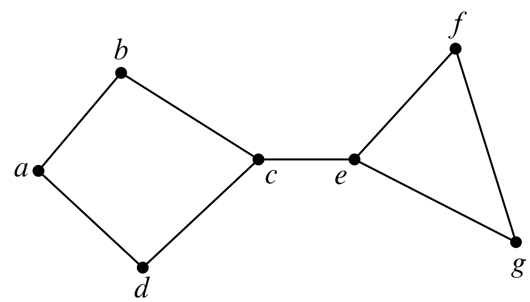


Figure 13.7 A planar drawing with a *bridge*.

Unfortunately, continuous faces in planar drawings are not always bounded by cycles in the graph—things can get a little more complicated. For example, the planar drawing in Figure 13.7 has what we will call a *bridge*, namely, a cut edge $\langle c—e \rangle$. The sequence of vertices along the boundary of the outer region of the drawing is

$$abcefgecda.$$

This sequence defines a closed walk, but does not define a cycle since the walk has two occurrences of the bridge $\langle c—e \rangle$ and each of its endpoints.

The planar drawing in Figure 13.8 illustrates another complication. This drawing has what we will call a *dongle*, namely, the nodes v, x, y and w, and the edges incident to them. The sequence of vertices along the boundary of the inner region is

$$rstvxyxvwvtur.$$

This sequence defines a closed walk, but once again does not define a cycle because it has two occurrences of *every* edge of the dongle—once "coming" and once "going."

It turns out that bridges and dongles are the only complications, at least for connected graphs. In particular, every continuous face in a planar drawing corresponds to a closed walk in the graph. These closed walks will be called the *discrete faces* of the drawing, and we'll define them next.

13.2.2 A Recursive Definition for Planar Embeddings

The association between the continuous faces of a planar drawing and closed walks provides the discrete data type we can use instead of continuous drawings. We'll define a *planar embedding* of *connected* graph to be the set of closed walks that are its face boundaries. Since all we care about in a graph are the connections between

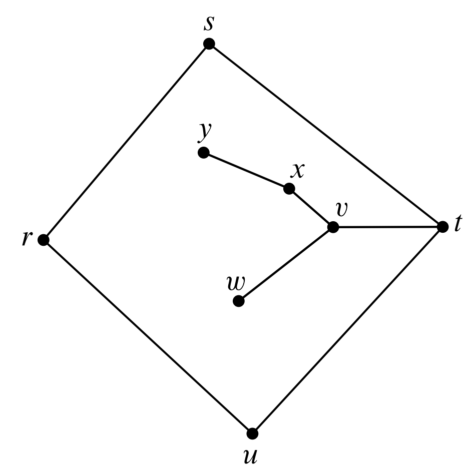


Figure 13.8 A planar drawing with a *dongle*.

vertices—not what a drawing of the graph actually looks like—planar embeddings are exactly what we need.

The question is how to define planar embeddings without appealing to continuous drawings. There is a simple way to do this based on the idea that any continuous drawing can drawn step by step:

- either draw a new point somewhere in the plane to represent a vertex,
- or draw a curve between two vertex points that have already been laid down, making sure the new curve doesn't cross any of the previously drawn curves.

A new curve won't cross any other curves precisely when it stays within one of the continuous faces. Alternatively, a new curve won't have to cross any other curves if it can go between the outer faces of two different drawings. So to be sure it's ok to draw a new curve, we just need to check that its endpoints are on the boundary of the same face, or that its endpoints are on the outer faces of different drawings. Of course drawing the new curve changes the faces slightly, so the face boundaries will have to be updated once the new curve is drawn. This is the idea behind the following recursive definition.

Definition 13.2.2. A *planar embedding* of a *connected* graph consists of a nonempty set of closed walks of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

Base case: If G is a graph consisting of a single vertex v then a planar embedding of G has one discrete face, namely, the length zero closed walk v.

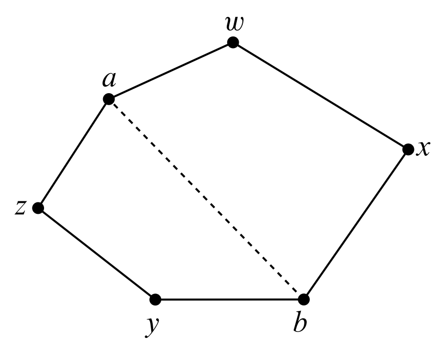


Figure 13.9 The “split a face” case: $awxbyza$ splits into $awxba$ and $abyza$.

Constructor case (split a face): Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that occur in some discrete face γ of the planar embedding. That is, γ is a closed walk of the form

$$\gamma = \alpha \widehat{\ } \beta$$

where α is a walk from a to b and β is a walk from b to a. Then the graph obtained by adding the edge $\langle a—b \rangle$ to the edges of G has a planar embedding with the same discrete faces as G, except that face γ is replaced by the two discrete faces[2]

$$\alpha \widehat{\ } \langle b—a \rangle \quad \text{and} \quad \langle a—b \rangle \widehat{\ } \beta \tag{13.2}$$

as illustrated in Figure 13.9.[3]

Constructor case (add a bridge): Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Let γ be a discrete face of the embedding of G and suppose that γ begins and ends at vertex a.

Similarly, let δ be a discrete face of the embedding of H that begins and ends at vertex b.

[2] There is a minor exception to this definition of embedding in the special case when G is a line graph beginning with a and ending with b. In this case the cycles into which γ splits are actually the same. That’s because adding edge $\langle a—b \rangle$ creates a cycle that divides the plane into “inner” and “outer” continuous faces that are both bordered by this cycle. In order to maintain the correspondence between continuous faces and discrete faces in this case, we define the two discrete faces of the embedding to be two “copies” of this same cycle.

[3] Formally, merge is an operation on walks, not a walk and an edge, so in (13.2), we should have used a walk $(a\ \langle a—b \rangle\ b)$ instead of an edge $\langle a—b \rangle$ and written

$$\alpha \widehat{\ } (b\ \langle b—a \rangle\ a) \quad \text{and} \quad (a\ \langle a—b \rangle\ b) \widehat{\ } \beta$$

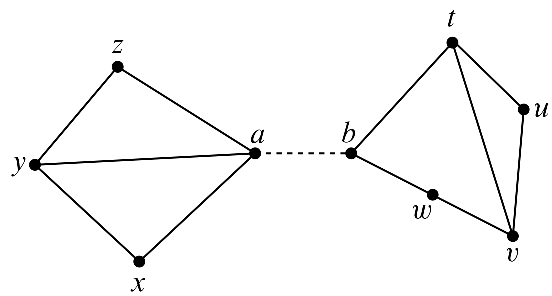


Figure 13.10 The "add a bridge" case.

Then the graph obtained by connecting G and H with a new edge $\langle a—b \rangle$ has a planar embedding whose discrete faces are the union of the discrete faces of G and H, except that faces γ and δ are replaced by one new face

$$\gamma \,\hat{}\, \langle a—b \rangle \,\hat{}\, \delta \,\hat{}\, \langle b—a \rangle .$$

This is illustrated in Figure 13.10, where the vertex sequences of the faces of G and H are:

$$G : \{axyza,\ axya,\ ayza\} \qquad H : \{btuvwb,\ btvwb,\ tuvt\},$$

and after adding the bridge $\langle a—b \rangle$, there is a single connected graph whose faces have the vertex sequences

$$\{axyzabtuvwba,\ axya,\ ayza,\ btvwb,\ tuvt\}.$$

A bridge is simply a cut edge, but in the context of planar embeddings, the bridges are precisely the edges that occur *twice on the same discrete face*—as opposed to once on each of two faces. Dongles are trees made of bridges; we only use dongles in illustrations, so there's no need to define them more precisely.

13.2.3 Does It Work?

Yes! In general, a graph is planar because it has a planar drawing according to Definition 13.2.1 if and only if each of its connected components has a planar embedding as specified in Definition 13.2.2. Of course we can't prove this without an excursion into exactly the kind of continuous math that we're trying to avoid. But now that the recursive definition of planar graphs is in place, we won't ever need to fall back on the continuous stuff. That's the good news.

The bad news is that Definition 13.2.2 is a lot more technical than the intuitively simple notion of a drawing whose edges don't cross. In many cases it's easier to

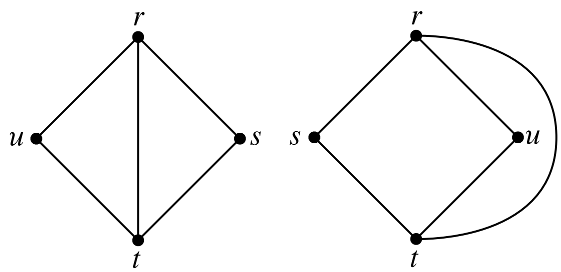


Figure 13.11 Two illustrations of the same embedding.

stick to the idea of planar drawings and give proofs in those terms. For example, erasing edges from a planar drawing will surely leave a planar drawing. On the other hand, it's not so obvious, though of course it is true, that you can delete an edge from a planar embedding and still get a planar embedding (see Problem 13.9).

In the hands of experts, and perhaps in your hands too with a little more experience, proofs about planar graphs by appeal to drawings can be convincing and reliable. But given the long history of mistakes in such proofs, it's safer to work from the precise definition of planar embedding. More generally, it's also important to see how the abstract properties of curved drawings in the plane can be modelled successfully using a discrete data type.

13.2.4 Where Did the Outer Face Go?

Every planar drawing has an immediately-recognizable outer face —it's the one that goes to infinity in all directions. But where is the outer face in a planar embedding?

There isn't one! That's because there really isn't any need to distinguish one face from another. In fact, a planar embedding could be drawn with any given face on the outside. An intuitive explanation of this is to think of drawing the embedding on a *sphere* instead of the plane. Then any face can be made the outside face by "puncturing" that face of the sphere, stretching the puncture hole to a circle around the rest of the faces, and flattening the circular drawing onto the plane.

So pictures that show different "outside" boundaries may actually be illustrations of the same planar embedding. For example, the two embeddings shown in Figure 13.11 are really the same—check it: they have the same boundary cycles.

This is what justifies the "add bridge" case in Definition 13.2.2: whatever face is chosen in the embeddings of each of the disjoint planar graphs, we can draw a bridge between them without needing to cross any other edges in the drawing, because we can assume the bridge connects two "outer" faces.

13.3 Euler’s Formula

The value of the recursive definition is that it provides a powerful technique for proving properties of planar graphs, namely, structural induction. For example, we will now use Definition 13.2.2 and structural induction to establish one of the most basic properties of a connected planar graph, namely, that the number of vertices and edges completely determines the number of faces in every possible planar embedding of the graph.

Theorem 13.3.1 (Euler’s Formula)**.** *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges and f is the number of faces.

For example, in Figure 13.5, $v = 4$, $e = 6$ and $f = 4$. Sure enough, $4-6+4 = 2$, as Euler’s Formula claims.

Proof. The proof is by structural induction on the definition of planar embeddings. Let $P(\mathcal{E})$ be the proposition that $v - e + f = 2$ for an embedding $\mathcal{E}$.

Base case ($\mathcal{E}$ is the one-vertex planar embedding): By definition, $v = 1$, $e = 0$ and $f = 1$, and $1 - 0 + 1 = 2$, so $P(\mathcal{E})$ indeed holds.

Constructor case (split a face): Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face $\gamma = a \ldots b \cdots a$ of the planar embedding.

Then the graph obtained by adding the edge $\langle a — b \rangle$ to the edges of G has a planar embedding with one more face and one more edge than G. So the quantity $v - e + f$ will remain the same for both graphs, and since by structural induction this quantity is 2 for G’s embedding, it’s also 2 for the embedding of G with the added edge. So P holds for the constructed embedding.

Constructor case (add bridge): Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Then connecting these two graphs with a bridge merges the two bridged faces into a single face, and leaves all other faces unchanged. So the bridge operation yields a planar embedding of a connected graph with $v_G + v_H$ vertices, $e_G + e_H + 1$ edges, and $f_G + f_H - 1$ faces. Since

$$\begin{aligned}
&(v_G + v_H) - (e_G + e_H + 1) + (f_G + f_H - 1) \\
&\quad = (v_G - e_G + f_G) + (v_H - e_H + f_H) - 2 \\
&\quad = (2) + (2) - 2 \qquad \text{(by structural induction hypothesis)} \\
&\quad = 2,
\end{aligned}$$

$v - e + f$ remains equal to 2 for the constructed embedding. That is, $P(\mathcal{E})$ also holds in this case.

This completes the proof of the constructor cases, and the theorem follows by structural induction. ■

13.4 Bounding the Number of Edges in a Planar Graph

Like Euler's formula, the following lemmas follow by structural induction directly from Definition 13.2.2.

Lemma 13.4.1. *In a planar embedding of a connected graph, each edge occurs once in each of two different faces, or occurs exactly twice in one face.*

Lemma 13.4.2. *In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.*

Combining Lemmas 13.4.1 and 13.4.2 with Euler's Formula, we can now prove that planar graphs have a limited number of edges:

Theorem 13.4.3. *Suppose a connected planar graph has $v \geq 3$ vertices and e edges. Then*

$$e \leq 3v - 6. \tag{13.3}$$

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Lemma 13.4.1, every edge has exactly two occurrences in the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Lemma 13.4.2, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{13.4}$$

But $f = e - v + 2$ by Euler’s formula, and substituting into (13.4) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$ ■

13.5 Returning to K_5 and $K_{3,3}$

Finally we have a simple way to answer the quadrapi question at the beginning of this chapter: the five quadrapi can’t all shake hands without crossing. The reason is that we know the quadrupi question is the same as asking whether a complete graph K_5 is planar, and Theorem 13.4.3 has the immediate:

Corollary 13.5.1. *K_5 is not planar.*

Proof. K_5 is connected and has 5 vertices and 10 edges. But since $10 > 3 \cdot 5 - 6$, K_5 does not satisfy the inequality (13.3) that holds in all planar graphs. ■

We can also use Euler’s Formula to show that $K_{3,3}$ is not planar. The proof is similar to that of Theorem 13.3 except that we use the additional fact that $K_{3,3}$ is a bipartite graph.

Lemma 13.5.2. *In a planar embedding of a connected bipartite graph with at least 3 vertices, each face has length at least 4.*

Proof. By Lemma 13.4.2, every face of a planar embedding of the graph has length at least 3. But by Lemma 12.6.2 and Theorem 12.8.3.3, a bipartite graph can’t have odd length closed walks. Since the faces of a planar embedding are closed walks, there can’t be any faces of length 3 in a bipartite embedding. So every face must have length at least 4. ■

Theorem 13.5.3. *Suppose a connected bipartite graph with $v \geq 3$ vertices and e edges is planar. Then*

$$e \leq 2v - 4. \tag{13.5}$$

Proof. Lemma 13.5.2 implies that all the faces of an embedding of the graph have length at least 4. Now arguing as in the proof of Theorem 13.4.3, we find that the sum of the lengths of the face boundaries is exactly $2e$ and at least $4f$. Hence,

$$4f \leq 2e \tag{13.6}$$

for any embedding of a planar bipartite graph. By Euler's theorem, $f = 2 - v + e$. Substituting $2 - v + e$ for f in (13.6), we have

$$4(2 - v + e) \leq 2e,$$

which simplies to (13.5). ■

Corollary 13.5.4. *$K_{3,3}$ is not planar.*

Proof. $K_{3,3}$ is connected, bipartite and has 6 vertices and 9 edges. But since $9 > 2 \cdot 6 - 4$, $K_{3,3}$ does not satisfy the inequality (13.3) that holds in all bipartite planar graphs. ■

13.6 Coloring Planar Graphs

We've covered a lot of ground with planar graphs, but not nearly enough to prove the famous 4-color theorem. But we can get awfully close. Indeed, we have done almost enough work to prove that every planar graph can be colored using only 5 colors.

There are two familiar facts about planarity that we will need.

Lemma 13.6.1. *Any subgraph of a planar graph is planar.*

Lemma 13.6.2. *Merging two adjacent vertices of a planar graph leaves another planar graph.*

Merging two adjacent vertices, n_1 and n_2 of a graph means deleting the two vertices and then replacing them by a new "merged" vertex m adjacent to all the vertices that were adjacent to either of n_1 or n_2, as illustrated in Figure 13.12.

Many authors take Lemmas 13.6.1 and 13.6.2 for granted for continuous drawings of planar graphs described by Definition 13.2.1. With the recursive Definition 13.2.2 both Lemmas can actually be proved using structural induction (see Problem 13.9).

We need only one more lemma:

Lemma 13.6.3. *Every planar graph has a vertex of degree at most five.*

Proof. Assuming to the contrary that every vertex of some planar graph had degree at least 6, then the sum of the vertex degrees is at least $6v$. But the sum of the vertex degrees equals $2e$ by the Handshake Lemma 12.2.1, so we have $e \geq 3v$ contradicting the fact that $e \leq 3v - 6 < 3v$ by Theorem 13.4.3. ■

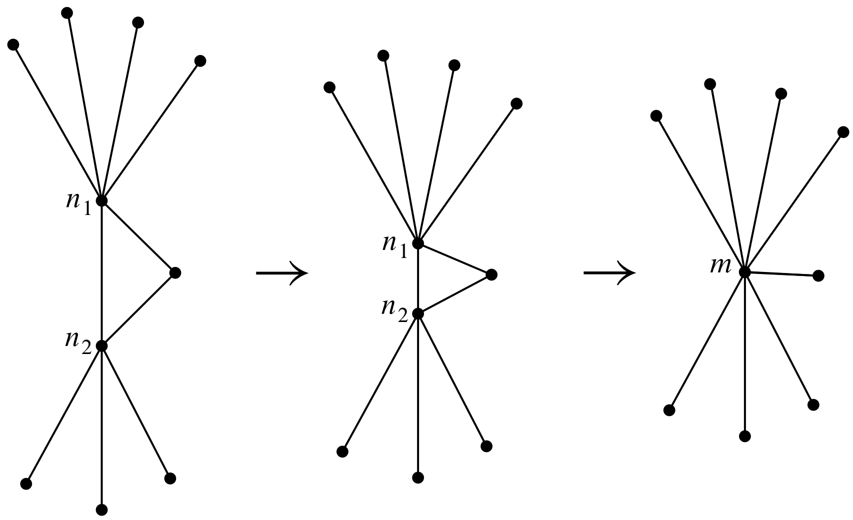


Figure 13.12 Merging adjacent vertices n_1 and n_2 into new vertex m.

Theorem 13.6.4. *Every planar graph is five-colorable.*

Proof. The proof will be by strong induction on the number v of vertices, with induction hypothesis:

> Every planar graph with v vertices is five-colorable.

Base cases ($v \leq 5$): immediate.

Inductive case: Suppose G is a planar graph with $v + 1$ vertices. We will describe a five-coloring of G.

First, choose a vertex g of G with degree at most 5; Lemma 13.6.3 guarantees there will be such a vertex.

Case 1: ($\deg(g) < 5$): Deleting g from G leaves a graph H that is planar by Lemma 13.6.1, and since H has v vertices, it is five-colorable by induction hypothesis. Now define a five coloring of G as follows: use the five-coloring of H for all the vertices besides g, and assign one of the five colors to g that is not the same as the color assigned to any of its neighbors. Since there are fewer than 5 neighbors, there will always be such a color available for g.

Case 2: ($\deg(g) = 5$): If the five neighbors of g in G were all adjacent to each other, then these five vertices would form a nonplanar subgraph isomorphic to K_5, contradicting Lemma 13.6.1 (since K_5 is not planar). So there must be two neighbors, n_1 and n_2, of g that are not adjacent. Now merge n_1 and g into a new vertex, m. In this new graph, n_2 is adjacent to m, and the graph is planar by Lemma 13.6.2. So we can then merge m and n_2 into a another new vertex m', resulting in a new graph G' which by Lemma 13.6.2 is also planar. Since G' has $v - 1$ vertices, it is five-colorable by the induction hypothesis.

Now define a five coloring of G as follows: use the five-coloring of G' for all the vertices besides g, n_1 and n_2. Next assign the color of m' in G' to be the color of the neighbors n_1 and n_2. Since n_1 and n_2 are not adjacent in G, this defines a proper five-coloring of G except for vertex g. But since these two neighbors of g have the same color, the neighbors of g have been colored using fewer than five colors altogether. So complete the five-coloring of G by assigning one of the five colors to g that is not the same as any of the colors assigned to its neighbors.

■

13.7 Classifying Polyhedra

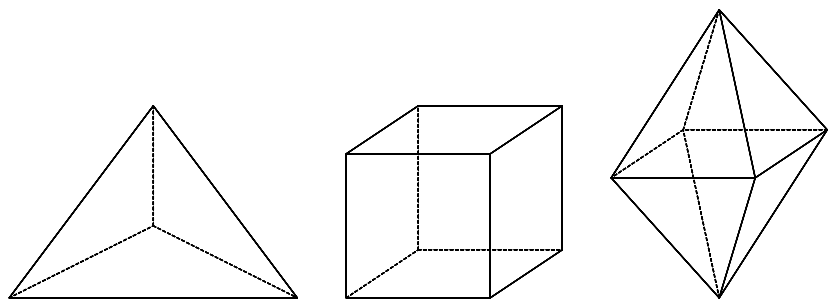
The Pythagoreans had two great mathematical secrets, the irrationality of $\sqrt{2}$ and a geometric construct that we're about to rediscover!

A *polyhedron* is a convex, three-dimensional region bounded by a finite number of polygonal faces. If the faces are identical regular polygons and an equal number of polygons meet at each corner, then the polyhedron is *regular*. Three examples of regular polyhedra are shown in Figure 13.13: the tetrahedron, the cube, and the octahedron.

We can determine how many more regular polyhedra there are by thinking about planarity. Suppose we took *any* polyhedron and placed a sphere inside it. Then we could project the polyhedron face boundaries onto the sphere, which would give an image that was a planar graph embedded on the sphere, with the images of the corners of the polyhedron corresponding to vertices of the graph. We've already observed that embeddings on a sphere are the same as embeddings on the plane, so Euler's formula for planar graphs can help guide our search for regular polyhedra.

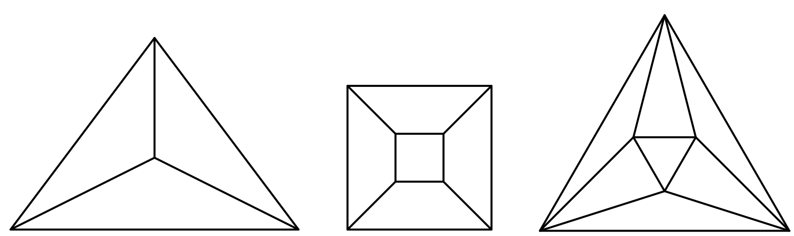
For example, planar embeddings of the three polyhedra in Figure 13.1 are shown in Figure 13.14.

Let m be the number of faces that meet at each corner of a polyhedron, and let n be the number of edges on each face. In the corresponding planar graph, there are m edges incident to each of the v vertices. By the Handshake Lemma 12.2.1, we



(a) (b) (c)

Figure 13.13 The tetrahedron (a), cube (b), and octahedron (c).
v



(a) (b) (c)

Figure 13.14 Planar embeddings of the tetrahedron (a), cube (b), and octahedron (c).

n	m	v	e	f	polyhedron
3	3	4	6	4	tetrahedron
4	3	8	12	6	cube
3	4	6	12	8	octahedron
3	5	12	30	20	icosahedron
5	3	20	30	12	dodecahedron

Figure 13.15 The only possible regular polyhedra.

know:

$$mv = 2e.$$

Also, each face is bounded by n edges. Since each edge is on the boundary of two faces, we have:

$$nf = 2e$$

Solving for v and f in these equations and then substituting into Euler's formula gives:

$$\frac{2e}{m} - e + \frac{2e}{n} = 2$$

which simplifies to

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2} \tag{13.7}$$

Equation 13.7 places strong restrictions on the structure of a polyhedron. Every nondegenerate polygon has at least 3 sides, so $n \geq 3$. And at least 3 polygons must meet to form a corner, so $m \geq 3$. On the other hand, if either n or m were 6 or more, then the left side of the equation could be at most $1/3 + 1/6 = 1/2$, which is less than the right side. Checking the finitely-many cases that remain turns up only five solutions, as shown in Figure 13.15. For each valid combination of n and m, we can compute the associated number of vertices v, edges e, and faces f. And polyhedra with these properties do actually exist. The largest polyhedron, the dodecahedron, was the other great mathematical secret of the Pythagorean sect.

The 5 polyhedra in Figure 13.15 are the only possible regular polyhedra. So if you want to put more than 20 geocentric satellites in orbit so that they *uniformly* blanket the globe—tough luck!

13.8 Another Characterization for Planar Graphs

We did not pick K_5 and $K_{3,3}$ as examples because of their application to dog houses or quadrapi shaking hands. We really picked them because they provide another, famous, discrete characterizarion of planar graphs:

Theorem 13.8.1 (Kuratowski)**.** *A graph is not planar if and only if it contains K_5 or $K_{3,3}$ as a minor.*

Definition 13.8.2. A *minor* of a graph G is a graph that can be obtained by repeatedly[4] deleting vertices, deleting edges, and merging *adjacent* vertices of G.

For example, Figure 13.16 illustrates why C_3 is a minor of the graph in Figure 13.16(a). In fact C_3 is a minor of a connected graph G if and only if G is not a tree.

The known proofs of Kuratowski's Theorem 13.8.1 are a little too long to include in an introductory text, so we won't give one.

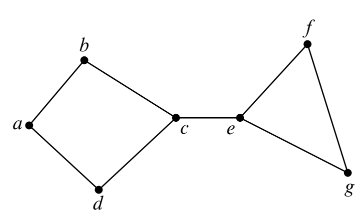
Problems for Section 13.2

Practice Problems

Problem 13.1.
What are the discrete faces of the following two graphs?

Write each cycle as a sequence of letters without spaces, starting with the alphabetically earliest letter in the clockwise direction, for example "`adbfa`." Separate the sequences with spaces.

(a)



(b)

[4] The three operations can each be performed any number of times in any order.

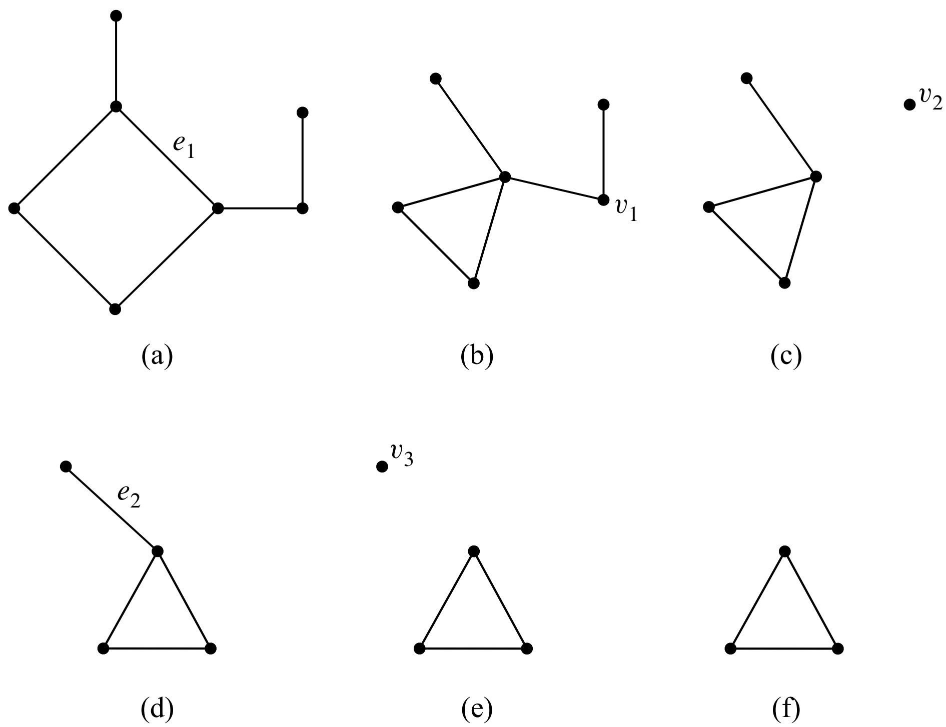
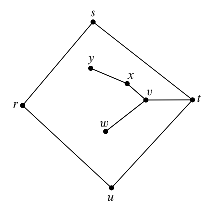


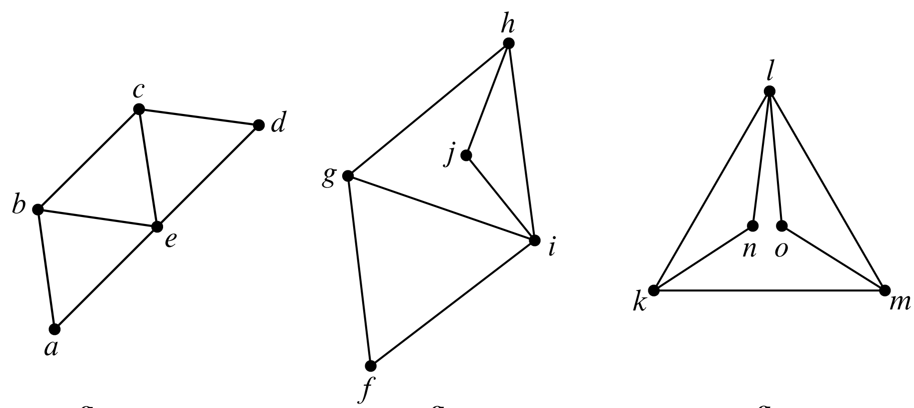
Figure 13.16 One method by which the graph in (a) can be reduced to C_3 (f), thereby showing that C_3 is a minor of the graph. The steps are: merging the nodes incident to e_1 (b), deleting v_1 and all edges incident to it (c), deleting v_2 (d), deleting e_2, and deleting v_3 (f).



Problems for Section 13.8

Exam Problems

Problem 13.2.



g_1 g_2 g_3

(a) Describe an isomorphism between graphs G_1 and G_2, and another isomorphism between G_2 and G_3.

(b) Why does part (a) imply that there is an isomorphism between graphs G_1 and G_3?

Let G and H be planar graphs. An embedding E_G of G is isomorphic to an embedding E_H of H iff there is an isomorphism from G to H that also maps each face of E_G to a face of E_H.

(c) One of the embeddings pictured above is not isomorphic to either of the others. Which one? Briefly explain why.

(d) Explain why all embeddings of two isomorphic planar graphs must have the same number of faces.

Problem 13.3. **(a)** Give an example of a planar graph with two planar embeddings, where the first embedding has a face whose length is not equal to the length of any face in the secoind embedding. Draw the two embeddings to demonstrate this.

(b) Define the length of a planar embedding $\mathcal{E}$ to be the sum of the lengths of the faces of $\mathcal{E}$. Prove that all embeddings of the same planar graph have the same length.

Problem 13.4.
Definition 13.2.2 of planar graph embeddings applied only to connected planar graphs. The definition can be extended to planar graphs that are not necessarily connected by adding the following additional constructor case to the definition:

- **Constructor Case:** (collect disjoint graphs) Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are planar embeddings with no vertices in common. Then $\mathcal{E}_1 \cup \mathcal{E}_2$ is a planar embedding.

Euler's Planar Graph Theorem now generalizes to unconnected graphs as follows: if a planar embedding $\mathcal{E}$ has v vertices, e edges, f faces and c connected components, then

$$v - e + f - 2c = 0. \tag{13.8}$$

This can be proved by structural induction on the definition of planar embedding.

(a) State and prove the base case of the structural induction.

(b) Let v_i, e_i, f_i, and c_i be the number of vertices, edges, faces, and connected components in embedding $\mathcal{E}_i$ and let v, e, f, c be the numbers for the embedding from the (collect disjoint graphs) constructor case. Express v, e, f, c in terms of v_i, e_i, f_i, c_i.

(c) Prove the (collect disjoint graphs) case of the structural induction.

Problem 13.5. **(a)** A simple graph has 8 vertices and 24 edges. What is the average degree per vertex?

(b) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

(c) A connected simple graph has one more vertex than it has edges. Explain why it is a planar graph.

(d) How many faces does a planar graph from part c have?

(e) How many distinct isomorphisms are there between the graph given in Figure 13.17 and itself? (Include the identity isomorphism.)

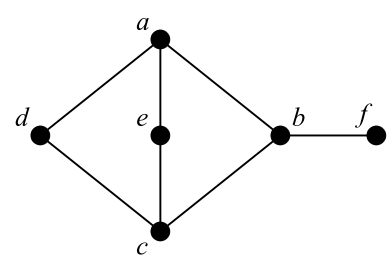


Figure 13.17

Class Problems

x

Problem 13.6.
Figure 13.18 shows four different pictures of planar graphs.

(a) For each picture, describe its discrete faces (closed walks that define the region borders).

(b) Which of the pictured graphs are isomorphic? Which pictures represent the same planar embedding?—that is, they have the same discrete faces.

(c) Describe a way to construct the embedding in Figure 4 according to the recursive Definition 13.2.2 of planar embedding. For each application of a constructor rule, be sure to indicate the faces (cycles) to which the rule was applied and the cycles which result from the application.

Problem 13.7.
Prove the following assertions by structural induction on the definition of planar embedding.

(a) In a planar embedding of a graph, each edge occurs exactly twice in the faces of the embedding.

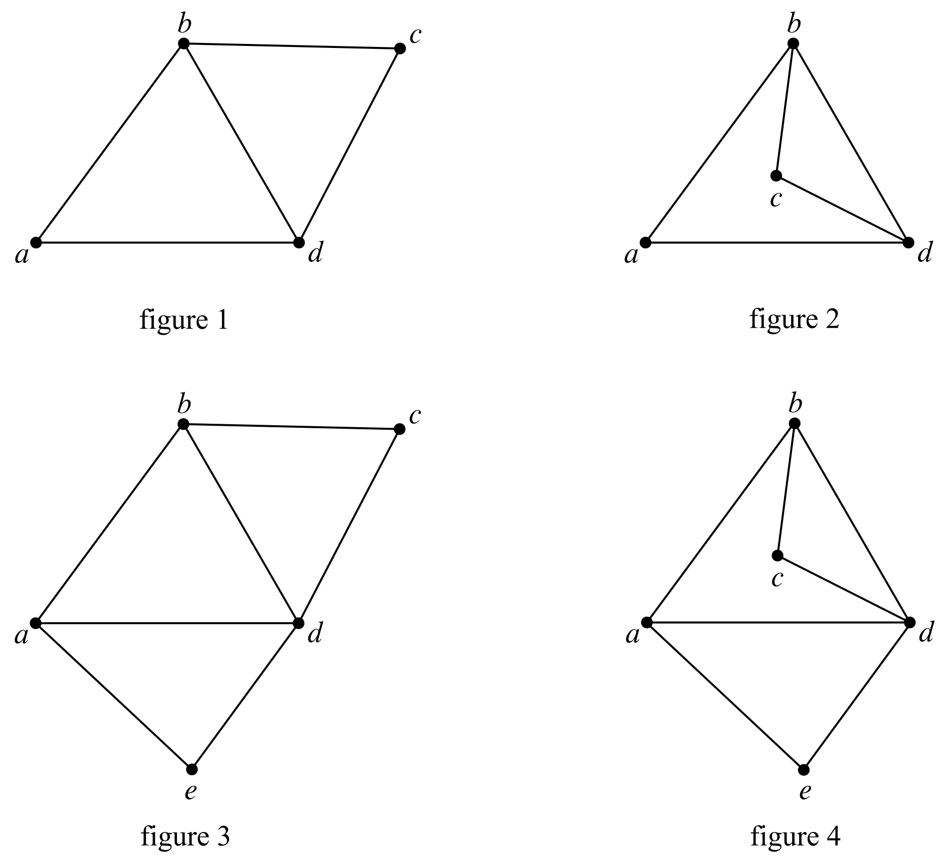


Figure 13.18

(b) In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.

Homework Problems

Problem 13.8.

A simple graph is *triangle-free* when it has no cycle of length three.

(a) Prove for any connected triangle-free planar graph with $v > 2$ vertices and e edges,

$$e \leq 2v - 4. \tag{13.9}$$

(b) Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

(c) Prove that any connected triangle-free planar graph is 4-colorable.

Problem 13.9. **(a)** Prove

Lemma (Switch Edges)**.** *Suppose that, starting from some embeddings of planar graphs with disjoint sets of vertices, it is possible by two successive applications of constructor operations to add edges e and then f to obtain a planar embedding $\mathcal{F}$. Then starting from the same embeddings, it is also possible to obtain $\mathcal{F}$ by adding f and then e with two successive applications of constructor operations.*

Hint: There are four cases to analyze, depending on which two constructor operations are applied to add e and then f. Structural induction is not needed.

(b) Prove

Corollary (Permute Edges)**.** *Suppose that, starting from some embeddings of planar graphs with disjoint sets of vertices, it is possible to add a sequence of edges $e_0, e_1, \ldots, e_n$ by successive applications of constructor operations to obtain a planar embedding $\mathcal{F}$. Then starting from the same embeddings, it is also possible to obtain $\mathcal{F}$ by applications of constructor operations that successively add any permutation[5] of the edges $e_0, e_1, \ldots, e_n$.*

Hint: By induction on the number of switches of adjacent elements needed to convert the sequence 0,1,...,n into a permutation $\pi(0), \pi(1), \ldots, \pi(n)$.

(c) Prove

Corollary (Delete Edge)**.** *Deleting an edge from a planar graph leaves a planar graph.*

(d) Conclude that any subgraph of a planar graph is planar.

[5] If $\pi : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ is a bijection, then the sequence $e_{\pi(0)}, e_{\pi(1)}, \ldots, e_{\pi(n)}$ is called a *permutation* of the sequence $e_0, e_1, \ldots, e_n$.