9 Number Theory

Number theory is the study of the integers. *Why* anyone would want to study the integers may not be obvious. First of all, what's to know? There's 0, there's 1, 2, 3, and so on, and, oh yeah, -1, -2, ... Which one don't you understand? What practical value is there in it?

The mathematician G. H. Hardy delighted at its impracticality. He wrote:

> [Number theorists] may be justified in rejoicing that there is one science, at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean.

Hardy was especially concerned that number theory not be used in warfare; he was a pacifist. You may applaud his sentiments, but he got it wrong: number theory underlies modern cryptography, which is what makes secure online communication possible. Secure communication is of course crucial in war—leaving poor Hardy spinning in his grave. It's also central to online commerce. Every time you buy a book from Amazon, use a certificate to access a web page, or use a PayPal account, you are relying on number theoretic algorithms.

Number theory also provides an excellent environment for us to practice and apply the proof techniques that we developed in previous chapters. We'll work out properties of greatest common divisors (gcd's) and use them to prove that integers factor uniquely into primes. Then we'll introduce modular arithmetic and work out enough of its properties to explain the RSA public key crypto-system.

Since we'll be focusing on properties of the integers, we'll adopt the default convention in this chapter that *variables range over the set $\mathbb{Z}$ of integers*.

9.1 Divisibility

The nature of number theory emerges as soon as we consider the *divides* relation.

Definition 9.1.1. *a divides b* (notation $a \mid b$) iff there is an integer k such that

$$ak = b.$$

The divides relation comes up so frequently that multiple synonyms for it are used all the time. The following phrases all say the same thing:

- $a \mid b$,
- a divides b,
- a is a *divisor* of b,
- a is a *factor* of b,
- b is *divisible* by a,
- b is a *multiple* of a.

Some immediate consequences of Definition 9.1.1 are that for all n

$$n \mid 0, \qquad n \mid n, \text{ and } \qquad \pm 1 \mid n.$$

Also,

$$0 \mid n \text{ IMPLIES } n = 0.$$

Dividing seems simple enough, but let's play with this definition. The Pythagoreans, an ancient sect of mathematical mystics, said that a number is *perfect* if it equals the sum of its positive integral divisors, excluding itself. For example, $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are perfect numbers. On the other hand, 10 is not perfect because $1 + 2 + 5 = 8$, and 12 is not perfect because $1 + 2 + 3 + 4 + 6 = 16$. Euclid characterized all the *even* perfect numbers around 300 BC (Problem 9.2). But is there an *odd* perfect number? More than two thousand years later, we still don't know! All numbers up to about 10^{300} have been ruled out, but no one has proved that there isn't an odd perfect number waiting just over the horizon.

So a half-page into number theory, we've strayed past the outer limits of human knowledge. This is pretty typical; number theory is full of questions that are easy to pose, but incredibly difficult to answer. We'll mention a few more such questions in later sections.[1]

9.1.1 Facts about Divisibility

The following lemma collects some basic facts about divisibility.

Lemma 9.1.2.

1. *If $a \mid b$ and $b \mid c$, then $a \mid c$.*

[1] *Don't Panic*—we're going to stick to some relatively benign parts of number theory. These super-hard unsolved problems rarely get put on problem sets.

2. *If $a \mid b$ and $a \mid c$, then $a \mid sb + tc$ for all s and t.*

3. *For all $c \neq 0$, $a \mid b$ if and only if $ca \mid cb$.*

Proof. These facts all follow directly from Definition 9.1.1. To illustrate this, we'll prove just part 2:

Given that $a \mid b$, there is some $k_1 \in \mathbb{Z}$ such that $ak_1 = b$. Likewise, $ak_2 = c$, so

$$sb + tc = s(k_1 a) + t(k_2 a) = (sk_1 + tk_2)a.$$

Therefore $sb + tc = k_3 a$ where $k_3 ::= (sk_1 + tk_2)$, which means that

$$a \mid sb + tc.$$

■

A number of the form $sb + tc$ is called an *integer linear combination* of b and c, or, since in this chapter we're only talking about integers, just a *linear combination*. So Lemma 9.1.2.2 can be rephrased as

If a divides b and c, then a divides every linear combination of b and c.

We'll be making good use of linear combinations, so let's get the general definition on record:

Definition 9.1.3. An integer n is a *linear combination* of numbers $b_0, \ldots, b_k$ iff

$$n = s_0 b_0 + s_1 b_1 + \cdots + s_k b_k$$

for some integers $s_0, \ldots, s_k$.

9.1.2 When Divisibility Goes Bad

As you learned in elementary school, if one number does *not* evenly divide another, you get a "quotient" and a "remainder" left over. More precisely:

Theorem 9.1.4. *[Division Theorem][2] Let n and d be integers such that $d \neq 0$. Then there exists a unique pair of integers q and r, such that*

$$n = q \cdot d + r \text{ AND } 0 \leq r < |d|. \tag{9.1}$$

[2]This theorem is often called the "Division Algorithm," but we prefer to call it a theorem since it does not actually describe a division procedure for computing the quotient and remainder.

The number q is called the *quotient* and the number r is called the *remainder* of n divided by d. We use the notation $\text{qcnt}(n, d)$ for the quotient and $\text{rem}(n, d)$ for the remainder.

The absolute value notation $|d|$ used above is probably familiar from introductory calculus, but for the record, let's define it.

Definition 9.1.5. For any real number r, the *absolute value* $|r|$ of r is:[3]

$$|r| ::= \begin{cases} r & \text{if } r \geq 0, \\ -r & \text{if } r < 0. \end{cases}$$

So by definition, the *remainder* $\text{rem}(n, d)$ *is nonnegative* regardless of the sign of n and d. For example, $\text{rem}(-11, 7) = 3$, since $-11 = (-2) \cdot 7 + 3$.

"Remainder" operations built into many programming languages can be a source of confusion. For example, the expression "32 % 5" will be familiar to programmers in Java, C, and C++; it evaluates to $\text{rem}(32, 5) = 2$ in all three languages. On the other hand, these and other languages are inconsistent in how they treat remainders like "32 % -5" or "-32 % 5" that involve negative numbers. So don't be distracted by your familiar programming language's behavior on remainders, and stick to the mathematical convention that *remainders are nonnegative*.

The remainder on division by d by definition is a number in the (integer) *interval* from 0 to $|d| - 1$. Such integer intervals come up so often that it is useful to have a simple notation for them. For $k \leq n \in \mathbb{Z}$,

$$\begin{aligned} [k..n] &::= \{i \mid k \leq i \leq n\}, \\ (k..n] &::= [k..n] - \{k\}, \\ [k..n) &::= [k..n] - \{n\}, \\ (k..n) &::= [k..n] - \{k, n\}. \end{aligned}$$

9.1.3 Die Hard

Die Hard 3 is just a B-grade action movie, but we think it has an inner message: everyone should learn at least a little number theory. In Section 6.2.3, we formalized a state machine for the Die Hard jug-filling problem using 3 and 5 gallon jugs,

[3] The absolute value of r could be defined as $\sqrt{r^2}$, which works because of the convention that square root notation always refers to the *nonnegative* square root (see Problem 1.2). Absolute value generalizes to complex numbers where it is called the *norm*. For $a, b \in \mathbb{R}$,

$$|a + bi| ::= \sqrt{a^2 + b^2}.$$

and also with 3 and 9 gallon jugs, and came to different conclusions about bomb explosions. What's going on in general? For example, how about getting 4 gallons from 12- and 18-gallon jugs, getting 32 gallons with 899- and 1147-gallon jugs, or getting 3 gallons into a jug using just 21- and 26-gallon jugs?

It would be nice if we could solve all these silly water jug questions at once. This is where number theory comes in handy.

A Water Jug Invariant

Suppose that we have water jugs with capacities a and b with $b \geq a$. Let's carry out some sample operations of the state machine and see what happens, assuming the b-jug is big enough:

$$\begin{aligned}
(0,0) &\to (a,0) && \text{fill first jug}\\
&\to (0,a) && \text{pour first into second}\\
&\to (a,a) && \text{fill first jug}\\
&\to (2a-b,b) && \text{pour first into second (assuming } 2a \geq b)\\
&\to (2a-b,0) && \text{empty second jug}\\
&\to (0,2a-b) && \text{pour first into second}\\
&\to (a,2a-b) && \text{fill first}\\
&\to (3a-2b,b) && \text{pour first into second (assuming } 3a \geq 2b)
\end{aligned}$$

What leaps out is that at every step, the amount of water in each jug is a linear combination of a and b. This is easy to prove by induction on the number of transitions:

Lemma 9.1.6 (Water Jugs)**.** *In the* Die Hard *state machine of Section 6.2.3 with jugs of sizes a and b, the amount of water in each jug is always a linear combination of a and b.*

Proof. The induction hypothesis $P(n)$ is the proposition that after n transitions, the amount of water in each jug is a linear combination of a and b.

Base case ($n = 0$): $P(0)$ is true, because both jugs are initially empty, and $0 \cdot a + 0 \cdot b = 0$.

Inductive step: Suppose the machine is in state (x, y) after n steps, that is, the little jug contains x gallons and the big one contains y gallons. There are two cases:

- If we fill a jug from the fountain or empty a jug into the fountain, then that jug is empty or full. The amount in the other jug remains a linear combination of a and b. So $P(n+1)$ holds.

- Otherwise, we pour water from one jug to another until one is empty or the other is full. By our assumption, the amount x and y in each jug is a linear combination of a and b before we begin pouring. After pouring, one jug is either empty (contains 0 gallons) or full (contains a or b gallons). Thus, the other jug contains either $x + y$, $x + y - a$ or $x + y - b$ gallons, all of which are linear combinations of a and b since x and y are. So $P(n + 1)$ holds in this case as well.

Since $P(n + 1)$ holds in any case, this proves the inductive step, completing the proof by induction. ■

So we have established that the jug problem has a preserved invariant, namely, the amount of water in every jug is a linear combination of the capacities of the jugs. Lemma 9.1.6 has an important corollary:

Corollary. *In trying to get 4 gallons from 12- and 18-gallon jugs, and likewise to get 32 gallons from 899- and 1147-gallon jugs,*

Bruce will die!

Proof. By the Water Jugs Lemma 9.1.6, with 12- and 18-gallon jugs, the amount in any jug is a linear combination of 12 and 18. This is always a multiple of 6 by Lemma 9.1.2.2, so Bruce can't get 4 gallons. Likewise, the amount in any jug using 899- and 1147-gallon jugs is a multiple of 31, so he can't get 32 either. ■

But the Water Jugs Lemma doesn't tell the complete story. For example, it leaves open the question of getting 3 gallons into a jug using just 21- and 26-gallon jugs: the only positive factor of both 21 and 26 is 1, and of course 1 divides 3, so the Lemma neither rules out nor confirms the possibility of getting 3 gallons.

A bigger issue is that we've just managed to recast a pretty understandable question about water jugs into a technical question about linear combinations. This might not seem like a lot of progress. Fortunately, linear combinations are closely related to something more familiar, greatest common divisors, and will help us solve the general water jug problem.

9.2 The Greatest Common Divisor

A *common divisor* of a and b is a number that divides them both. The *greatest common divisor* of a and b is written $\gcd(a, b)$. For example, $\gcd(18, 24) = 6$.

As long as a and b are not both 0, they will have a gcd. The gcd turns out to be very valuable for reasoning about the relationship between a and b and for reasoning about integers in general. We'll be making lots of use of gcd's in what follows.

Some immediate consequences of the definition of gcd are that

$$\begin{aligned} \gcd(n,1) &= 1 \\ \gcd(n,n) = \gcd(n,0) &= |n| \qquad\qquad for n \neq 0, \end{aligned}$$

where the last equality follows from the fact that everything is a divisor of 0.

9.2.1 Euclid's Algorithm

The first thing to figure out is how to find gcd's. A good way called *Euclid's algorithm* has been known for several thousand years. It is based on the following elementary observation.

Lemma 9.2.1. *For $b \neq 0$,*

$$\gcd(a,b) = \gcd(b, \operatorname{rem}(a,\ b)).$$

Proof. By the Division Theorem 9.1.4,

$$a = qb + r \tag{9.2}$$

where $r = \operatorname{rem}(a,\ b)$. So a is a linear combination of b and r, which implies that any divisor of b and r is a divisor of a by Lemma 9.1.2.2. Likewise, r is a linear combination $a - qb$ of a and b, so any divisor of a and b is a divisor of r. This means that a and b have the same common divisors as b and r, and so they have the same *greatest* common divisor. ■

Lemma 9.2.1 is useful for quickly computing the greatest common divisor of two numbers. For example, we could compute the greatest common divisor of 1147 and 899 by repeatedly applying it:

$$\begin{aligned} \gcd(1147, 899) &= \gcd(899, \underbrace{\operatorname{rem}(1147,\ 899)}_{=248}) \\ &= \gcd\,(248, \operatorname{rem}(899,\ 248) = 155) \\ &= \gcd\,(155, \operatorname{rem}(248,\ 155) = 93) \\ &= \gcd\,(93, \operatorname{rem}(155,\ 93) = 62) \\ &= \gcd\,(62, \operatorname{rem}(93,\ 62) = 31) \\ &= \gcd\,(31, \operatorname{rem}(62,\ 31) = 0) \\ &= 31 \end{aligned}$$

This calculation that $\gcd(1147, 899) = 31$ was how we figured out that with water jugs of sizes 1147 and 899, Bruce dies trying to get 32 gallons.

On the other hand, applying Euclid's algorithm to 26 and 21 gives

$$\gcd(26, 21) = \gcd(21, 5) = \gcd(5, 1) = 1,$$

so we can't use the reasoning above to rule out Bruce getting 3 gallons into the big jug. As a matter of fact, because the gcd here is 1, Bruce *will* be able to get any number of gallons into the big jug up to its capacity. To explain this, we will need a little more number theory.

Euclid's Algorithm as a State Machine

Euclid's algorithm can easily be formalized as a state machine. The set of states is $\mathbb{N}^2$ and there is one transition rule:

$$(x, y) \longrightarrow (y, \text{rem}(x,\ y)), \tag{9.3}$$

for $y > 0$. By Lemma 9.2.1, the gcd stays the same from one state to the next. That means the predicate

$$\gcd(x, y) = \gcd(a, b)$$

is a preserved invariant on the states (x, y). This preserved invariant is, of course, true in the start state (a, b). So by the Invariant Principle, if y ever becomes 0, the invariant will be true and so

$$x = \gcd(x, 0) = \gcd(a, b).$$

Namely, the value of x will be the desired gcd.

What's more x and therefore also y, gets to be 0 pretty fast. To see why, note that starting from (x, y), two transitions leads to a state whose the first coordinate is $\text{rem}(x,\ y)$, which is at most half the size of x.[4] Since x starts off equal to a and gets halved or smaller every two steps, it will reach its minimum value—which is $\gcd(a, b)$—after at most $2 \log a$ transitions. After that, the algorithm takes at most one more transition to terminate. In other words, Euclid's algorithm terminates after at most $1 + 2 \log a$ transitions.[5]

[4] In other words,

$$\text{rem}(x,\ y) \leq x/2 \qquad \text{for } 0 < y \leq x. \tag{9.4}$$

This is immediate if $y \leq x/2$, since the remainder of x divided by y is less than y by definition. On the other hand, if $y > x/2$, then $\text{rem}(x,\ y) = x - y < x/2$.

[5] A tighter analysis shows that at most $\log_\varphi(a)$ transitions are possible where φ is the golden ratio $(1 + \sqrt{5})/2$, see Problem 9.14.

9.2.2 The Pulverizer

We will get a lot of mileage out of the following key fact:

Theorem 9.2.2. *The greatest common divisor of a and b is a linear combination of a and b. That is,*

$$\gcd(a, b) = sa + tb,$$

for some integers s and t.[6]

We already know from Lemma 9.1.2.2 that every linear combination of a and b is divisible by any common factor of a and b, so it is certainly divisible by the greatest of these common divisors. Since any constant multiple of a linear combination is also a linear combination, Theorem 9.2.2 implies that any multiple of the gcd is a linear combination, giving:

Corollary 9.2.3. *An integer is a linear combination of a and b iff it is a multiple of* $\gcd(a, b)$.

We'll prove Theorem 9.2.2 directly by explaining how to find s and t. This job is tackled by a mathematical tool that dates back to sixth-century India, where it was called *kuttaka*, which means "the Pulverizer." Today, the Pulverizer is more commonly known as the "Extended Euclidean Gcd Algorithm," because it is so close to Euclid's algorithm.

For example, following Euclid's algorithm, we can compute the gcd of 259 and 70 as follows:

$$\begin{aligned}
\gcd(259, 70) &= \gcd(70, 49) && \text{since } \operatorname{rem}(259,\ 70) = 49\\
&= \gcd(49, 21) && \text{since } \operatorname{rem}(70,\ 49) = 21\\
&= \gcd(21, 7) && \text{since } \operatorname{rem}(49,\ 21) = 7\\
&= \gcd(7, 0) && \text{since } \operatorname{rem}(21,\ 7) = 0\\
&= 7.
\end{aligned}$$

The Pulverizer goes through the same steps, but requires some extra bookkeeping along the way: as we compute $\gcd(a, b)$, we keep track of how to write each of the remainders (49, 21, and 7, in the example) as a linear combination of a and b. This is worthwhile, because our objective is to write the last nonzero remainder,

[6] This result is often referred to as *Bezout's lemma*, which is a misattribution since it was first published in the West 150 years earlier by someone else, and was described a thousand years before that by Indian mathematicians Aryabhata and Bhaskara.

which is the gcd, as such a linear combination. For our example, here is this extra bookkeeping:

x	y	$(\text{rem}(x,\ y))$	$=$	$x - q \cdot y$
259	70	49	$=$	$a - 3 \cdot b$
70	49	21	$=$	$b - 1 \cdot 49$
			$=$	$b - 1 \cdot (a - 3 \cdot b)$
			$=$	$-1 \cdot a + 4 \cdot b$
49	21	7	$=$	$49 - 2 \cdot 21$
			$=$	$(a - 3 \cdot b) - 2 \cdot (-1 \cdot a + 4 \cdot b)$
			$=$	$\boxed{3 \cdot a - 11 \cdot b}$
21	7	0		

We began by initializing two variables, $x = a$ and $y = b$. In the first two columns above, we carried out Euclid's algorithm. At each step, we computed $\text{rem}(x,\ y)$ which equals $x - \text{qcnt}(x, y) \cdot y$. Then, in this linear combination of x and y, we replaced x and y by equivalent linear combinations of a and b, which we already had computed. After simplifying, we were left with a linear combination of a and b equal to $\text{rem}(x,\ y)$, as desired. The final solution is boxed.

This should make it pretty clear how and why the Pulverizer works. If you have doubts, you may work through Problem 9.13, where the Pulverizer is formalized as a state machine and then verified using an invariant that is an extension of the one used for Euclid's algorithm.

Since the Pulverizer requires only a little more computation than Euclid's algorithm, you can "pulverize" very large numbers very quickly by using this algorithm. As we will soon see, its speed makes the Pulverizer a very useful tool in the field of cryptography.

Now we can restate the Water Jugs Lemma 9.1.6 in terms of the greatest common divisor:

Corollary 9.2.4. *Suppose that we have water jugs with capacities a and b. Then the amount of water in each jug is always a multiple of* $\gcd(a, b)$.

For example, there is no way to form 4 gallons using 3- and 6-gallon jugs, because 4 is not a multiple of $\gcd(3, 6) = 3$.

9.2.3 One Solution for All Water Jug Problems

Corollary 9.2.3 says that 3 can be written as a linear combination of 21 and 26, since 3 is a multiple of $\gcd(21, 26) = 1$. So the Pulverizer will give us integers s and t such that

$$3 = s \cdot 21 + t \cdot 26 \tag{9.5}$$

The coefficient s could be either positive or negative. However, we can readily transform this linear combination into an equivalent linear combination

$$3 = s' \cdot 21 + t' \cdot 26 \tag{9.6}$$

where the coefficient s' is positive. The trick is to notice that if in equation (9.5) we increase s by 26 and decrease t by 21, then the value of the expression $s \cdot 21 + t \cdot 26$ is unchanged overall. Thus, by repeatedly increasing the value of s (by 26 at a time) and decreasing the value of t (by 21 at a time), we get a linear combination $s' \cdot 21 + t' \cdot 26 = 3$ where the coefficient s' is positive. (Of course t' must then be negative; otherwise, this expression would be much greater than 3.)

Now we can form 3 gallons using jugs with capacities 21 and 26: We simply repeat the following steps s' times:

1. Fill the 21-gallon jug.

2. Pour all the water in the 21-gallon jug into the 26-gallon jug. If at any time the 26-gallon jug becomes full, empty it out, and continue pouring the 21-gallon jug into the 26-gallon jug.

At the end of this process, we must have emptied the 26-gallon jug exactly $-t'$ times. Here's why: we've taken $s' \cdot 21$ gallons of water from the fountain, and we've poured out some multiple of 26 gallons. If we emptied fewer than $-t'$ times, then by (9.6), the big jug would be left with at least $3 + 26$ gallons, which is more than it can hold; if we emptied it more times, the big jug would be left containing at most $3 - 26$ gallons, which is nonsense. But once we have emptied the 26-gallon jug exactly $-t'$ times, equation (9.6) implies that there are exactly 3 gallons left.

Remarkably, we don't even need to know the coefficients s' and t' in order to use this strategy! Instead of repeating the outer loop s' times, we could just repeat *until we obtain 3 gallons*, since that must happen eventually. Of course, we have to keep track of the amounts in the two jugs so we know when we're done. Here's the

solution using this approach starting with empty jugs, that is, at $(0, 0)$:

$$
\begin{array}{lllllll}
\xrightarrow{\text{fill 21}} & (21,0) & \xrightarrow{\text{pour 21 into 26}} & & & & (0,21) \\
\xrightarrow{\text{fill 21}} & (21,21) & \xrightarrow{\text{pour 21 to 26}} & (16,26) & \xrightarrow{\text{empty 26}} & (16,0) & \xrightarrow{\text{pour 21 to 26}} (0,16) \\
\xrightarrow{\text{fill 21}} & (21,16) & \xrightarrow{\text{pour 21 to 26}} & (11,26) & \xrightarrow{\text{empty 26}} & (11,0) & \xrightarrow{\text{pour 21 to 26}} (0,11) \\
\xrightarrow{\text{fill 21}} & (21,11) & \xrightarrow{\text{pour 21 to 26}} & (6,26) & \xrightarrow{\text{empty 26}} & (6,0) & \xrightarrow{\text{pour 21 to 26}} (0,6) \\
\xrightarrow{\text{fill 21}} & (21,6) & \xrightarrow{\text{pour 21 to 26}} & (1,26) & \xrightarrow{\text{empty 26}} & (1,0) & \xrightarrow{\text{pour 21 to 26}} (0,1) \\
\xrightarrow{\text{fill 21}} & (21,1) & \xrightarrow{\text{pour 21 to 26}} & & & & (0,22) \\
\xrightarrow{\text{fill 21}} & (21,22) & \xrightarrow{\text{pour 21 to 26}} & (17,26) & \xrightarrow{\text{empty 26}} & (17,0) & \xrightarrow{\text{pour 21 to 26}} (0,17) \\
\xrightarrow{\text{fill 21}} & (21,17) & \xrightarrow{\text{pour 21 to 26}} & (12,26) & \xrightarrow{\text{empty 26}} & (12,0) & \xrightarrow{\text{pour 21 to 26}} (0,12) \\
\xrightarrow{\text{fill 21}} & (21,12) & \xrightarrow{\text{pour 21 to 26}} & (7,26) & \xrightarrow{\text{empty 26}} & (7,0) & \xrightarrow{\text{pour 21 to 26}} (0,7) \\
\xrightarrow{\text{fill 21}} & (21,7) & \xrightarrow{\text{pour 21 to 26}} & (2,26) & \xrightarrow{\text{empty 26}} & (2,0) & \xrightarrow{\text{pour 21 to 26}} (0,2) \\
\xrightarrow{\text{fill 21}} & (21,2) & \xrightarrow{\text{pour 21 to 26}} & & & & (0,23) \\
\xrightarrow{\text{fill 21}} & (21,23) & \xrightarrow{\text{pour 21 to 26}} & (18,26) & \xrightarrow{\text{empty 26}} & (18,0) & \xrightarrow{\text{pour 21 to 26}} (0,18) \\
\xrightarrow{\text{fill 21}} & (21,18) & \xrightarrow{\text{pour 21 to 26}} & (13,26) & \xrightarrow{\text{empty 26}} & (13,0) & \xrightarrow{\text{pour 21 to 26}} (0,13) \\
\xrightarrow{\text{fill 21}} & (21,13) & \xrightarrow{\text{pour 21 to 26}} & (8,26) & \xrightarrow{\text{empty 26}} & (8,0) & \xrightarrow{\text{pour 21 to 26}} (0,8) \\
\xrightarrow{\text{fill 21}} & (21,8) & \xrightarrow{\text{pour 21 to 26}} & (3,26) & \xrightarrow{\text{empty 26}} & (3,0) & \xrightarrow{\text{pour 21 to 26}} (0,3)
\end{array}
$$

The same approach works regardless of the jug capacities and even regardless of the amount we're trying to produce! Simply repeat these two steps until the desired amount of water is obtained:

1. Fill the smaller jug.
2. Pour all the water in the smaller jug into the larger jug. If at any time the larger jug becomes full, empty it out, and continue pouring the smaller jug into the larger jug.

By the same reasoning as before, this method eventually generates every multiple—up to the size of the larger jug—of the greatest common divisor of the jug capacities, all the quantities we can possibly produce. No ingenuity is needed at all!

So now we have the complete water jug story:

Theorem 9.2.5. *Suppose that we have water jugs with capacities a and b. For any $c \in [0..a]$, it is possible to get c gallons in the size a jug iff c is a multiple of* $\gcd(a, b)$.

9.2.4 Properties of the Greatest Common Divisor

It can help to have some basic gcd facts on hand:

Lemma 9.2.6.

a) $\gcd(ka, kb) = k \cdot \gcd(a, b)$ *for all* $k > 0$.

b) $(d \mid a \text{ AND } d \mid b)$ IFF $d \mid \gcd(a, b)$.

c) If $\gcd(a, b) = 1$ *and* $\gcd(a, c) = 1$, *then* $\gcd(a, bc) = 1$.

d) If $a \mid bc$ *and* $\gcd(a, b) = 1$, *then* $a \mid c$.

Showing how all these facts follow from Theorem 9.2.2 that gcd is a linear combination is a good exercise (Problem 9.11).

These properties are also simple consequences of the fact that integers factor into primes in a unique way (Theorem 9.4.1). But we'll need some of these facts to prove unique factorization in Section 9.4, so proving them by appeal to unique factorization would be circular.

9.3 Prime Mysteries

Some of the greatest mysteries and insights in number theory concern properties of prime numbers:

Definition 9.3.1. A *prime* is a number greater than 1 that is divisible only by itself and 1. A number other than 0, 1, and -1 that is not a prime is called *composite*.[7]

Here are three famous mysteries:

Twin Prime Conjecture There are infinitely many primes p such that $p + 2$ is also a prime.

In 1966, Chen showed that there are infinitely many primes p such that $p + 2$ is the product of at most two primes. So the conjecture is known to be *almost* true!

Conjectured Inefficiency of Factoring Given the product of two large primes $n = pq$, there is no efficient procedure to recover the primes p and q. That is, no *polynomial time* procedure (see Section 3.5) is guaranteed to find p and

[7] So 0, 1, and -1 are the only integers that are neither prime nor composite.

q in a number of steps bounded by a polynomial in the length of the binary representation of n (not n itself). The length of the binary representation at most $1 + \log_2 n$.

The best algorithm known is the "number field sieve," which runs in time proportional to:

$$e^{1.9(\ln n)^{1/3}(\ln \ln n)^{2/3}}.$$

This number grows more rapidly than any polynomial in $\log n$ and is infeasible when n has 300 digits or more.

Efficient factoring is a mystery of particular importance in computer science, as we'll explain later in this chapter.

Goldbach's Conjecture We've already mentioned Goldbach's Conjecture 1.1.6 several times: every even integer greater than two is equal to the sum of two primes. For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, etc.

In 1939, Schnirelman proved that every even number can be written as the sum of not more than 300,000 primes, which was a start. Today, we know that every even number is the sum of at most 6 primes.

Primes show up erratically in the sequence of integers. In fact, their distribution seems almost random:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots.$$

One of the great insights about primes is that their density among the integers has a precise limit. Namely, let $\pi(n)$ denote the number of primes up to n:

Definition 9.3.2.

$$\pi(n) ::= |\{p \in [2..n] \mid p \text{ is prime}\}|.$$

For example, $\pi(1) = 0$, $\pi(2) = 1$ and $\pi(10) = 4$, because 2, 3, 5, and 7 are the primes less than or equal to 10. Step by step, π grows erratically according to the erratic spacing between successive primes, but its overall growth rate is known to smooth out to be the same as the growth of the function $n/\ln n$:

Theorem 9.3.3 (Prime Number Theorem)**.**

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln n} = 1.$$

Thus, primes gradually taper off. As a rule of thumb, about 1 integer out of every $\ln n$ in the vicinity of n is a prime.

The Prime Number Theorem was conjectured by Legendre in 1798 and proved a century later by de la Vallée Poussin and Hadamard in 1896. However, after his death, a notebook of Gauss was found to contain the same conjecture, which he apparently made in 1791 at age 15. (You have to feel sorry for all the otherwise "great" mathematicians who had the misfortune of being contemporaries of Gauss.)

A proof of the Prime Number Theorem is beyond the scope of this text, but there is a manageable proof (see Problem 9.22) of a related result that is sufficient for our applications:

Theorem 9.3.4 (Chebyshev's Theorem on Prime Density)**.** *For $n > 1$,*

$$\pi(n) > \frac{n}{3\ln n}.$$

A Prime for Google

In late 2004 a billboard appeared in various locations around the country:

$$\left\{\begin{array}{l}\text{first 10-digit prime found}\\ \text{in consecutive digits of } e\end{array}\right\}\textbf{.com}$$

Substituting the correct number for the expression in curly-braces produced the URL for a Google employment page. The idea was that Google was interested in hiring the sort of people that could and would solve such a problem.

How hard is this problem? Would you have to look through thousands or millions or billions of digits of e to find a 10-digit prime? The rule of thumb derived from the Prime Number Theorem says that among 10-digit numbers, about 1 in

$$\ln 10^{10} \approx 23$$

is prime. This suggests that the problem isn't really so hard! Sure enough, the first 10-digit prime in consecutive digits of e appears quite early:

$$\begin{aligned} e =& 2.7182818284590452353602874713526624977572470936999595749669\\ & 6762772407663035354759457138217852516642\mathbf{7427466391}9320030\\ & 5992181741359662904357290033429526059563073813232862794 34\ldots \end{aligned}$$

9.4 The Fundamental Theorem of Arithmetic

There is an important fact about primes that you probably already know: every positive integer number has a *unique* prime factorization. So every positive integer can be built up from primes in *exactly one way*. These quirky prime numbers are the building blocks for the integers.

Since the value of a product of numbers is the same if the numbers appear in a different order, there usually isn't a unique way to express a number as a product of primes. For example, there are three ways to write 12 as a product of primes:

$$12 = 2 \cdot 2 \cdot 3 = 2 \cdot 3 \cdot 2 = 3 \cdot 2 \cdot 2.$$

What's unique about the prime factorization of 12 is that any product of primes equal to 12 will have exactly one 3 and two 2's. This means that if we *sort* the primes by size, then the product really will be unique.

Let's state this more carefully. A sequence of numbers is *weakly decreasing* when each number in the sequence is at least as big as the numbers after it. Note that a sequence of just one number as well as a sequence of no numbers—the empty sequence—is weakly decreasing by this definition.

Theorem 9.4.1. *[Fundamental Theorem of Arithmetic] Every positive integer is a product of a* unique *weakly decreasing sequence of primes.*

For example, 75237393 is the product of the weakly decreasing sequence of primes

$$23, 17, 17, 11, 7, 7, 7, 3,$$

and no other weakly decreasing sequence of primes will give 75237393.[8]

Notice that the theorem would be false if 1 were considered a prime; for example, 15 could be written as $5 \cdot 3$, or $5 \cdot 3 \cdot 1$, or $5 \cdot 3 \cdot 1 \cdot 1, \ldots$.

There is a certain wonder in unique factorization, especially in view of the prime number mysteries we've already mentioned. It's a mistake to take it for granted, even if you've known it since you were in a crib. In fact, unique factorization actually fails for many integer-like sets of numbers, such as the complex numbers of the form $n + m\sqrt{-5}$ for $m, n \in \mathbb{Z}$ (see Problem 9.25).

The Fundamental Theorem is also called the *Unique Factorization Theorem*, which is a more descriptive and less pretentious, name—but we really want to get your attention to the importance and non-obviousness of unique factorization.

[8] The "product" of just one number is defined to be that number, and the product of no numbers is by convention defined to be 1. So each prime p is uniquely the product of the primes in the length-one sequence consisting solely of p, and 1, which you will remember is not a prime, is uniquely the product of the empty sequence.

9.4.1 Proving Unique Factorization

The Fundamental Theorem is not hard to prove, but we'll need a couple of preliminary facts.

Lemma 9.4.2. *If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.*

Lemma 9.4.2 follows immediately from Unique Factorization: the primes in the product ab are exactly the primes from a and from b. But proving the lemma this way would be cheating: we're going to need this lemma to prove Unique Factorization, so it would be circular to assume it. Instead, we'll use the properties of gcd's and linear combinations to give an easy, noncircular way to prove Lemma 9.4.2.

Proof. One case is if $\gcd(a, p) = p$. Then the claim holds, because a is a multiple of p.

Otherwise, $\gcd(a, p) \neq p$. In this case $\gcd(a, p)$ must be 1, since 1 and p are the only positive divisors of p. Now $\gcd(a, p)$ is a linear combination of a and p, so we have $1 = sa + tp$ for some s, t. Then $b = s(ab) + (tb)p$, that is, b is a linear combination of ab and p. Since p divides both ab and p, it also divides their linear combination b. ■

A routine induction argument extends this statement to:

Lemma 9.4.3. *Let p be a prime. If $p \mid a_1 a_2 \cdots a_n$, then p divides some a_i.*

Now we're ready to prove the Fundamental Theorem of Arithmetic.

Proof. Theorem 2.3.1 showed, using the Well Ordering Principle, that every positive integer can be expressed as a product of primes. So we just have to prove this expression is unique. We will use Well Ordering to prove this too.

The proof is by contradiction: assume, contrary to the claim, that there exist positive integers that can be written as products of primes in more than one way. By the Well Ordering Principle, there is a smallest integer with this property. Call this integer n, and let

$$\begin{aligned} n &= p_1 \cdot p_2 \cdots p_j, \\ &= q_1 \cdot q_2 \cdots q_k, \end{aligned}$$

where both products are in weakly decreasing order and $p_1 \leq q_1$.

If $q_1 = p_1$, then n/q_1 would also be the product of different weakly decreasing sequences of primes, namely,

$$\begin{aligned} &p_2 \cdots p_j, \\ &q_2 \cdots q_k. \end{aligned}$$

Figure 9.1 Alan Turing

Since $n/q_1 < n$, this can't be true, so we conclude that $p_1 < q_1$.

Since the p_i's are weakly decreasing, all the p_i's are less than q_1. But

$$q_1 \mid n = p_1 \cdot p_2 \cdots p_j,$$

so Lemma 9.4.3 implies that q_1 divides one of the p_i's, which contradicts the fact that q_1 is bigger than all them. ■

9.5 Alan Turing

The man pictured in Figure 9.1 is Alan Turing, the most important figure in the history of computer science. For decades, his fascinating life story was shrouded by government secrecy, societal taboo, and even his own deceptions.

At age 24, Turing wrote a paper entitled *On Computable Numbers, with an Application to the Entscheidungsproblem*. The crux of the paper was an elegant way to model a computer in mathematical terms. This was a breakthrough, because it allowed the tools of mathematics to be brought to bear on questions of computation. For example, with his model in hand, Turing immediately proved that there exist problems that no computer can solve—no matter how ingenious the programmer. Turing's paper is all the more remarkable because he wrote it in 1936, a full decade

before any electronic computer actually existed.

The word "Entscheidungsproblem" in the title refers to one of the 28 mathematical problems posed by David Hilbert in 1900 as challenges to mathematicians of the 20th century. Turing knocked that one off in the same paper. And perhaps you've heard of the "Church-Turing thesis"? Same paper. So Turing was a brilliant guy who generated lots of amazing ideas. But this lecture is about one of Turing's less-amazing ideas. It involved codes. It involved number theory. And it was sort of stupid.

Let's look back to the fall of 1937. Nazi Germany was rearming under Adolf Hitler, world-shattering war looked imminent, and—like us —Alan Turing was pondering the usefulness of number theory. He foresaw that preserving military secrets would be vital in the coming conflict and proposed a way *to encrypt communications using number theory*. This is an idea that has ricocheted up to our own time. Today, number theory is the basis for numerous public-key cryptosystems, digital signature schemes, cryptographic hash functions, and electronic payment systems. Furthermore, military funding agencies are among the biggest investors in cryptographic research. Sorry, Hardy!

Soon after devising his code, Turing disappeared from public view, and half a century would pass before the world learned the full story of where he'd gone and what he did there. We'll come back to Turing's life in a little while; for now, let's investigate the code Turing left behind. The details are uncertain, since he never formally published the idea, so we'll consider a couple of possibilities.

9.5.1 Turing's Code (Version 1.0)

The first challenge is to translate a text message into an integer so we can perform mathematical operations on it. This step is not intended to make a message harder to read, so the details are not too important. Here is one approach: replace each letter of the message with two digits ($A = 01$, $B = 02$, $C = 03$, etc.) and string all the digits together to form one huge number. For example, the message "victory" could be translated this way:

	v	i	c	t	o	r	y
$\rightarrow$	22	09	03	20	15	18	25

Turing's code requires the message to be a prime number, so we may need to pad the result with some more digits to make a prime. The Prime Number Theorem indicates that padding with relatively few digits will work. In this case, appending the digits 13 gives the number 2209032015182513, which is prime.

Here is how the encryption process works. In the description below, m is the unencoded message (which we want to keep secret), $\widehat{m}$ is the encrypted message (which the Nazis may intercept), and k is the key.

Beforehand The sender and receiver agree on a *secret key*, which is a large prime k.

Encryption The sender encrypts the message m by computing:

$$\widehat{m} = m \cdot k$$

Decryption The receiver decrypts $\widehat{m}$ by computing:

$$\frac{\widehat{m}}{k} = m.$$

For example, suppose that the secret key is the prime number $k = 22801763489$ and the message m is "victory." Then the encrypted message is:

$$\begin{aligned}\widehat{m} &= m \cdot k \\ &= 2209032015182513 \cdot 22801763489 \\ &= 50369825549820718594667857\end{aligned}$$

There are a couple of basic questions to ask about Turing's code.

1. How can the sender and receiver ensure that m and k are prime numbers, as required?

   The general problem of determining whether a large number is prime or composite has been studied for centuries, and tests for primes that worked well in practice were known even in Turing's time. In the past few decades, very fast primality tests have been found as described in the text box below.

2. Is Turing's code secure?

   The Nazis see only the encrypted message $\widehat{m} = m \cdot k$, so recovering the original message m requires factoring $\widehat{m}$. Despite immense efforts, no really efficient factoring algorithm has ever been found. It appears to be a fundamentally difficult problem. So, although a breakthrough someday can't be ruled out, the conjecture that there is no efficient way to factor is widely accepted. In effect, Turing's code puts to practical use his discovery that there are limits to the power of computation. Thus, provided m and k are sufficiently large, the Nazis seem to be out of luck!

This all sounds promising, but there is a major flaw in Turing's code.

Primality Testing

It's easy to see that an integer n is prime iff it is not divisible by any number from 2 to $\lfloor \sqrt{n} \rfloor$ (see Problem 1.13). Of course this naive way to test if n is prime takes more than $\sqrt{n}$ steps, which is exponential in the *size* of n measured by the number of digits in the decimal or binary representation of n. Through the early 1970's, no prime testing procedure was known that would never blow up like this.

In 1974, Volker Strassen invented a simple, fast *probabilistic* primality test. Strassens's test gives the right answer when applied to any prime number, but has some probability of giving a wrong answer on a nonprime number. However, the probability of a wrong answer on any given number is so tiny that relying on the answer is the best bet you'll ever make.

Still, the theoretical possibility of a wrong answer was intellectually bothersome—even if the probability of being wrong was a lot less than the probability of an undetectable computer hardware error leading to a wrong answer. Finally in 2002, in a breakthrough paper beginning with a quote from Gauss emphasizing the importance and antiquity of primality testing, Manindra Agrawal, Neeraj Kayal, and Nitin Saxena presented an amazing, thirteen line description of a polynomial time primality test.

This definitively places primality testing way below the exponential effort apparently needed for SAT and similar problems. The polynomial bound on the Agrawal *et al.* test had degree 12, and subsequent research has reduced the degree to 5, but this is still too large to be practical, and probabilistic primality tests remain the method used in practice today. It's plausible that the degree bound can be reduced a bit more, but matching the speed of the known probabilistic tests remains a daunting challenge.

9.5.2 Breaking Turing's Code (Version 1.0)

Let's consider what happens when the sender transmits a *second* message using Turing's code and the same key. This gives the Nazis two encrypted messages to look at:

$$\widehat{m_1} = m_1 \cdot k \qquad \text{and} \qquad \widehat{m_2} = m_2 \cdot k$$

The greatest common divisor of the two encrypted messages, $\widehat{m_1}$ and $\widehat{m_2}$, is the secret key k. And, as we've seen, the gcd of two numbers can be computed very efficiently. So after the second message is sent, the Nazis can recover the secret key and read *every* message!

A mathematician as brilliant as Turing is not likely to have overlooked such a glaring problem, and we can guess that he had a slightly different system in mind, one based on *modular* arithmetic.

9.6 Modular Arithmetic

On the first page of his masterpiece on number theory, *Disquisitiones Arithmeticae*, Gauss introduced the notion of "*congruence*." Now, Gauss is another guy who managed to cough up a half-decent idea every now and then, so let's take a look at this one. Gauss said that *a is congruent to b modulo n* iff $n \mid (a - b)$. This is written

$$a \equiv b \pmod{n}.$$

For example:

$$29 \equiv 15 \pmod{7} \quad \text{because } 7 \mid (29 - 15).$$

It's not useful to allow a moduli $n \leq 0$, and so we will assume from now on that moduli are positive.

There is a close connection between congruences and remainders:

Lemma 9.6.1 (Remainder)**.**

$$a \equiv b \pmod{n} \quad \textit{iff} \quad \text{rem}(a, n) = \text{rem}(b, n).$$

Proof. By the Division Theorem 9.1.4, there exist unique pairs of integers q_1, r_1 and q_2, r_2 such that:

$$a = q_1 n + r_1$$
$$b = q_2 n + r_2,$$

where $r_1, r_2 \in [0..n)$. Subtracting the second equation from the first gives:

$$a - b = (q_1 - q_2)n + (r_1 - r_2),$$

where $r_1 - r_2$ is in the interval $(-n, n)$. Now $a \equiv b \pmod{n}$ if and only if n divides the left-hand side of this equation. This is true if and only if n divides the right-hand side, which holds if and only if $r_1 - r_2$ is a multiple of n. But the only multiple of n in $(-n, n)$ is 0, so $r_1 - r_2$ must in fact equal 0, that is, when $r_1 ::= \text{rem}(a,\ n) = r_2 ::= \text{rem}(b,\ n)$. ■

So we can also see that

$$29 \equiv 15 \pmod{7} \quad \text{because } \text{rem}(29,\ 7) = 1 = \text{rem}(15,\ 7).$$

Notice that even though "(mod 7)" appears on the end, the $\equiv$ symbol isn't any more strongly associated with the 15 than with the 29. It would probably be clearer to write $29 \equiv_{\text{mod } 7} 15$, for example, but the notation with the modulus at the end is firmly entrenched, and we'll just live with it.

The Remainder Lemma 9.6.1 explains why the congruence relation has properties like an equality relation. In particular, the following properties[9] follow immediately:

Lemma 9.6.2.

$$a \equiv a \pmod{n} \qquad \text{(reflexivity)}$$
$$a \equiv b \text{ IFF } b \equiv a \pmod{n} \qquad \text{(symmetry)}$$
$$(a \equiv b \text{ AND } b \equiv c) \text{ IMPLIES } a \equiv c \pmod{n} \qquad \text{(transitivity)}$$

We'll make frequent use of another immediate corollary of the Remainder Lemma 9.6.1:

Corollary 9.6.3.

$$a \equiv \text{rem}(a,\ n) \pmod{n}$$

Still another way to think about congruence modulo n is that it *defines a partition of the integers into n sets so that congruent numbers are all in the same set*. For example, suppose that we're working modulo 3. Then we can partition the integers into 3 sets as follows:

$$\begin{array}{llllllll} \{ & \ldots, & -6, & -3, & 0, & 3, & 6, & 9, & \ldots & \} \\ \{ & \ldots, & -5, & -2, & 1, & 4, & 7, & 10, & \ldots & \} \\ \{ & \ldots, & -4, & -1, & 2, & 5, & 8, & 11, & \ldots & \} \end{array}$$

[9] Binary relations with these properties are called *equivalence relations*, see Section 10.10.

according to whether their remainders on division by 3 are 0, 1, or 2. The upshot is that when arithmetic is done modulo n, there are really only n different kinds of numbers to worry about, because there are only n possible remainders. In this sense, modular arithmetic is a simplification of ordinary arithmetic.

The next most useful fact about congruences is that they are *preserved* by addition and multiplication:

Lemma 9.6.4 (Congruence)**.** *If* $a \equiv b \pmod{n}$ *and* $c \equiv d \pmod{n}$*, then*

$$a + c \equiv b + d \pmod{n}, \tag{9.7}$$

$$ac \equiv bd \pmod{n}. \tag{9.8}$$

Proof. Let's start with 9.7. Since $a \equiv b \pmod{n}$, we have by definition that $n \mid (b - a) = (b + c) - (a + c)$, so

$$a + c \equiv b + c \pmod{n}.$$

Since $c \equiv d \pmod{n}$, the same reasoning leads to

$$b + c \equiv b + d \pmod{n}.$$

Now transitivity (Lemma 9.6.2) gives

$$a + c \equiv b + d \pmod{n}.$$

The proof for 9.8 is virtually identical, using the fact that if n divides $(b - a)$, then it certainly also divides $(bc - ac)$. ■

9.7 Remainder Arithmetic

The Congruence Lemma 9.6.1 says that two numbers are congruent iff their remainders are equal, so we can understand congruences by working out arithmetic with remainders. And if all we want is the remainder modulo n of a series of additions, multiplications, subtractions applied to some numbers, we can take remainders at every step so that the entire computation only involves number in the range $[0..n)$.

General Principle of Remainder Arithmetic

To find the remainder on division by n of the result of a series of additions and multiplications, applied to some integers

- replace each integer operand by its remainder on division by n,
- keep each result of an addition or multiplication in the range $[0..n)$ by immediately replacing any result outside that range by its remainder on division by n.

For example, suppose we want to find

$$\text{rem}((44427^{3456789} + 15555858^{5555})403^{6666666},\ 36). \tag{9.9}$$

This looks really daunting if you think about computing these large powers and then taking remainders. For example, the decimal representation of $44427^{3456789}$ has about 20 million digits, so we certainly don't want to go that route. But remembering that integer exponents specify a series of multiplications, we follow the General Principle and replace the numbers being multiplied by their remainders. Since $\text{rem}(44427,\ 36) = 3, \text{rem}(15555858,\ 36) = 6$, and $\text{rem}(403,\ 36) = 7$, we find that (9.9) equals the remainder on division by 36 of

$$(3^{3456789} + 6^{5555})7^{6666666}. \tag{9.10}$$

That's a little better, but $3^{3456789}$ has about a million digits in its decimal representation, so we still don't want to compute that. But let's look at the remainders of the first few powers of 3:

$$\begin{aligned}
\text{rem}(3,\ 36) &= 3\\
\text{rem}(3^2,\ 36) &= 9\\
\text{rem}(3^3,\ 36) &= 27\\
\text{rem}(3^4,\ 36) &= 9.
\end{aligned}$$

We got a repeat of the second step, $\text{rem}(3^2,\ 36)$ after just two more steps. This means means that starting at 3^2, the sequence of remainders of successive powers of 3 will keep repeating every 2 steps. So a product of an odd number of at least three 3's will have the same remainder on division by 36 as a product of just three 3's. Therefore,

$$\text{rem}(3^{3456789},\ 36) = \text{rem}(3^3,\ 36) = 27.$$

What a win!

Powers of 6 are even easier because $\text{rem}(6^2,\ 36) = 0$, so 0's keep repeating after the second step. Powers of 7 repeat after six steps, but on the fifth step you get a 1, that is $\text{rem}(7^6,\ 36) = 1$, so (9.10) successively simplifies to be the remainders of the following terms:

$$\begin{aligned}&(3^{3456789} + 6^{5555})7^{6666666}\\&(3^3 + 6^2 \cdot 6^{5553})(7^6)^{1111111}\\&(3^3 + 0 \cdot 6^{5553})1^{1111111}\\&= 27.\end{aligned}$$

Notice that *it would be a disastrous blunder to replace an exponent by its remainder*. The general principle applies to numbers that are *operands* of plus and times, whereas the exponent is a number that controls how many multiplications to perform. Watch out for this.

9.7.1 The ring $\mathbb{Z}_n$

It's time to be more precise about the general principle and why it works. To begin, let's introduce the notation $+_n$ for doing an addition and then immediately taking a remainder on division by n, as specified by the general principle; likewise for multiplying:

$$\begin{aligned}i +_n j &::= \text{rem}(i + j,\ n),\\ i \cdot_n j &::= \text{rem}(ij,\ n).\end{aligned}$$

Now the General Principle is simply the repeated application of the following lemma.

Lemma 9.7.1.

$$\text{rem}(i + j,\ n) = \text{rem}(i,\ n) +_n \text{rem}(j,\ n), \tag{9.11}$$

$$\text{rem}(ij,\ n) = \text{rem}(i,\ n) \cdot_n \text{rem}(j,\ n). \tag{9.12}$$

Proof. By Corollary 9.6.3, $i \equiv \text{rem}(i,\ n)$ and $j \equiv \text{rem}(j,\ n)$, so by the Congruence Lemma 9.6.4

$$i + j \equiv \text{rem}(i,\ n) + \text{rem}(j,\ n) \pmod{n}.$$

By Corollary 9.6.3 again, the remainders on each side of this congruence are equal, which immediately gives (9.11). An identical proof applies to (9.12). ■

The set of integers in the range $[0..n)$ together with the operations $+_n$ and $\cdot_n$ is referred to as $\mathbb{Z}_n$, the *ring of integers modulo n*. As a consequence of Lemma 9.7.1, the familiar rules of arithmetic hold in $\mathbb{Z}_n$, for example:

$$(i \cdot_n j) \cdot_n k = i \cdot_n (j \cdot_n k).$$

These subscript-n's on arithmetic operations really clog things up, so instead we'll just write "$(\mathbb{Z}_n)$" on the side to get a simpler looking equation:

$$(i \cdot j) \cdot k = i \cdot (j \cdot k) \quad (\mathbb{Z}_n).$$

In particular, all of the following equalities[10] are true in $\mathbb{Z}_n$:

$$\begin{aligned}
(i \cdot j) \cdot k &= i \cdot (j \cdot k) && \text{(associativity of } \cdot\text{)},\\
(i + j) + k &= i + (j + k) && \text{(associativity of } +\text{)},\\
1 \cdot k &= k && \text{(identity for } \cdot\text{)},\\
0 + k &= k && \text{(identity for } +\text{)},\\
k + (-k) &= 0 && \text{(inverse for } +\text{)},\\
i + j &= j + i && \text{(commutativity of } +\text{)}\\
i \cdot (j + k) &= (i \cdot j) + (i \cdot k) && \text{(distributivity)},\\
i \cdot j &= j \cdot i && \text{(commutativity of } \cdot\text{)}
\end{aligned}$$

Associativity implies the familiar fact that it's safe to omit the parentheses in products:

$$k_1 \cdot k_2 \cdot \cdots \cdot k_m$$

comes out the same in $\mathbb{Z}_n$ no matter how it is parenthesized.

The overall theme is that remainder arithmetic is a lot like ordinary arithmetic. But there are a couple of exceptions we're about to examine.

9.8 Turing's Code (Version 2.0)

In 1940, France had fallen before Hitler's army, and Britain stood alone against the Nazis in western Europe. British resistance depended on a steady flow of sup-

[10] A set with addition and multiplication operations that satisfy these equalities is known as a *commutative ring*. In addition to $\mathbb{Z}_n$, the integers, rationals, reals, and polynomials with integer coefficients are all examples of commutative rings. On the other hand, the set $\{\mathbf{T}, \mathbf{F}\}$ of truth values with OR for addition and AND for multiplication is *not* a commutative ring because it fails to satisfy one of these equalities. The $n \times n$ matrices of integers are not a commutative ring because they fail to satisfy another one of these equalities.

plies brought across the north Atlantic from the United States by convoys of ships. These convoys were engaged in a cat-and-mouse game with German "U-boats" —submarines—which prowled the Atlantic, trying to sink supply ships and starve Britain into submission. The outcome of this struggle pivoted on a balance of information: could the Germans locate convoys better than the Allies could locate U-boats, or vice versa?

Germany lost.

A critical reason behind Germany's loss was not made public until 1974: Germany's naval code, *Enigma*, had been broken by the Polish Cipher Bureau,[11] and the secret had been turned over to the British a few weeks before the Nazi invasion of Poland in 1939. Throughout much of the war, the Allies were able to route convoys around German submarines by listening in to German communications. The British government didn't explain *how* Enigma was broken until 1996. When the story was finally released (by the US), it revealed that Alan Turing had joined the secret British codebreaking effort at Bletchley Park in 1939, where he became the lead developer of methods for rapid, bulk decryption of German Enigma messages. Turing's Enigma deciphering was an invaluable contribution to the Allied victory over Hitler.

Governments are always tight-lipped about cryptography, but the half-century of official silence about Turing's role in breaking Enigma and saving Britain may be related to some disturbing events after the war—more on that later. Let's get back to number theory and consider an alternative interpretation of Turing's code. Perhaps we had the basic idea right (multiply the message by the key), but erred in using *conventional* arithmetic instead of *modular* arithmetic. Maybe this is what Turing meant:

Beforehand The sender and receiver agree on a large number n, which may be made public. (This will be the modulus for all our arithmetic.) As in Version 1.0, they also agree that some prime number $k < n$ will be the secret key.

Encryption As in Version 1.0, the message m should be another prime in $\mathbf{[}0\mathbf{..}n\mathbf{)}$. The sender encrypts the message m to produce $\widehat{m}$ by computing mk, but this time modulo n:

$$\widehat{m} ::= m \cdot k \ (\mathbb{Z}_n) \tag{9.13}$$

Decryption (Uh-oh.)

The decryption step is a problem. We might hope to decrypt in the same way as before by dividing the encrypted message $\widehat{m}$ by the key k. The difficulty is that $\widehat{m}$

[11] See http://en.wikipedia.org/wiki/Polish_Cipher_Bureau.

is the *remainder* when mk is divided by n. So dividing $\widehat{m}$ by k might not even give us an integer!

This decoding difficulty can be overcome with a better understanding of when it is ok to divide by k in modular arithmetic.

9.9 Multiplicative Inverses and Cancelling

The *multiplicative inverse* of a number x is another number x^{-1} such that

$$x^{-1} \cdot x = 1.$$

From now on, when we say "inverse," we mean *multiplicative* (not relational) inverse.

For example, over the rational numbers, $1/3$ is, of course, an inverse of 3, since,

$$\frac{1}{3} \cdot 3 = 1.$$

In fact, with the sole exception of 0, every rational number n/m has an inverse, namely, m/n. On the other hand, over the integers, only 1 and -1 have inverses. Over the ring $\mathbb{Z}_n$, things get a little more complicated. For example, 2 is a multiplicative inverse of 8 in $\mathbb{Z}_{15}$, since

$$2 \cdot 8 = 1 \ (\mathbb{Z}_{15}).$$

On the other hand, 3 does not have a multiplicative inverse in $\mathbb{Z}_{15}$. We can prove this by contradiction: suppose there was an inverse j for 3, that is

$$1 = 3 \cdot j \ (\mathbb{Z}_{15}).$$

Then multiplying both sides of this equality by 5 leads directly to the contradiction $5 = 0$:

$$\begin{aligned} 5 &= 5 \cdot (3 \cdot j) \\ &= (5 \cdot 3) \cdot j \\ &= 0 \cdot j \ = 0 \ (\mathbb{Z}_{15}). \end{aligned}$$

So there can't be any such inverse j.

So some numbers have inverses modulo 15 and others don't. This may seem a little unsettling at first, but there's a simple explanation of what's going on.

9.9.1 Relative Primality

Integers that have no prime factor in common are called *relatively prime*.[12] This is the same as having no common divisor (prime or not) greater than 1. It's also equivalent to saying $\gcd(a, b) = 1$.

For example, 8 and 15 are relatively prime, since $\gcd(8, 15) = 1$. On the other hand, 3 and 15 are not relatively prime, since $\gcd(3, 15) = 3 \neq 1$. This turns out to explain why 8 has an inverse over $\mathbb{Z}_{15}$ and 3 does not.

Lemma 9.9.1. *If $k \in [0..n)$ is relatively prime to n, then k has an inverse in $\mathbb{Z}_n$.*[13]

Proof. If k is relatively prime to n, then $\gcd(n, k) = 1$ by definition of gcd. This means we can use the Pulverizer from section 9.2.2 to find a linear combination of n and k equal to 1:

$$sn + tk = 1.$$

So applying the General Principle of Remainder Arithmetic (Lemma 9.7.1), we get

$$(\text{rem}(s,\ n) \cdot \text{rem}(n,\ n)) + (\text{rem}(t,\ n) \cdot \text{rem}(k,\ n)) = 1\ (\mathbb{Z}_n).$$

But $\text{rem}(n,\ n) = 0$, and $\text{rem}(k,\ n) = k$ since $k \in [0..n)$, so we get

$$\text{rem}(t,\ n) \cdot k = 1\ (\mathbb{Z}_n).$$

Thus, $\text{rem}(t,\ n)$ is a multiplicative inverse of k. ■

By the way, it's nice to know that when they exist, inverses are unique. That is,

Lemma 9.9.2. *If i and j are both inverses of k in $\mathbb{Z}_n$, then $i = j$.*

Proof.

$$i = i \cdot 1 = i \cdot (k \cdot j) = (i \cdot k) \cdot j = 1 \cdot j = j\ (\mathbb{Z}_n).$$

■

So the proof of Lemma 9.9.1 shows that for any k relatively prime to n, the inverse of k in $\mathbb{Z}_n$ is simply the remainder of a coefficient we can easily find using the Pulverizer.

Working with a prime modulus is attractive here because, like the rational and real numbers, when p is prime, every nonzero number has an inverse in $\mathbb{Z}_p$. But arithmetic modulo a composite is really only a little more painful than working modulo a prime—though you may think this is like the doctor saying, "This is only going to hurt a little," before he jams a big needle in your arm.

[12] Other texts call them *coprime*.

[13] This works even in the extreme case that $n = 1$, because $0 \equiv 1 \pmod 1$, so it is consistent to define 0 to be its own inverse in $\mathbb{Z}_1$.

9.9.2 Cancellation

Another sense in which real numbers are nice is that it's ok to cancel common factors. In other words, if we know that $tr = ts$ for real numbers r, s, t, then as long as $t \neq 0$, we can cancel the t's and conclude that $r = s$. In general, cancellation is *not* valid in $\mathbb{Z}_n$. For example,

$$3 \cdot 10 = 3 \cdot 5 \ (\mathbb{Z}_{15}), \tag{9.14}$$

but cancelling the 3's leads to the absurd conclusion that 10 equals 5.

The fact that multiplicative terms cannot be cancelled is the most significant way in which $\mathbb{Z}_n$ arithmetic differs from ordinary integer arithmetic.

Definition 9.9.3. A number k is *cancellable* in $\mathbb{Z}_n$ iff

$$k \cdot a = k \cdot b \quad \text{implies} \quad a = b \ (\mathbb{Z}_n)$$

for all $a, b \in [0..n)$.

If a number is relatively prime to 15, it can be cancelled by multiplying by its inverse. So cancelling works for numbers that have inverses:

Lemma 9.9.4. *If k has an inverse in $\mathbb{Z}_n$, then it is cancellable.*

But 3 is not relatively prime to 15, and that's why it is not cancellable. More generally, if k is not relatively prime to n, then we can show it isn't cancellable in $\mathbb{Z}_n$ in the same way we showed that 3 is not cancellable in (9.14).

To summarize, we have

Theorem 9.9.5. *The following are equivalent for $k \in [0..n)$:*[14]

$$\gcd(k, n) = 1,$$
$$k \text{ has an inverse in } \mathbb{Z}_n,$$
$$k \text{ is cancellable in } \mathbb{Z}_n.$$

9.9.3 Decrypting (Version 2.0)

Multiplicative inverses are the key to decryption in Turing's code. Specifically, we can recover the original message by multiplying the encoded message by the $\mathbb{Z}_n$-inverse j of the key:

$$\widehat{m} \cdot j = (m \cdot k) \cdot j = m \cdot (k \cdot j) = m \cdot 1 = m \ (\mathbb{Z}_n).$$

So all we need to decrypt the message is to find an inverse of the secret key k, which will be easy using the Pulverizer—providing k has an inverse. But k is positive and less than the modulus n, so one simple way to ensure that k is relatively prime to the modulus is to have n be a prime number.

[14] This works even when $n = 1$—see the footnote following Lemma 9.9.1.

9.9.4 Breaking Turing's Code (Version 2.0)

The Germans didn't bother to encrypt their weather reports with the highly-secure Enigma system. After all, so what if the Allies learned that there was rain off the south coast of Iceland? But amazingly, this practice provided the British with a critical edge in the Atlantic naval battle during 1941.

The problem was that some of those weather reports had originally been transmitted using Enigma from U-boats out in the Atlantic. Thus, the British obtained both unencrypted reports and the same reports encrypted with Enigma. By comparing the two, the British were able to determine which key the Germans were using that day and could read all other Enigma-encoded traffic. Today, this would be called a *known-plaintext attack*.

Let's see how a known-plaintext attack would work against Turing's code. Suppose that the Nazis know both the plain text m and its

$$\widehat{m} = m \cdot k \ (\mathbb{Z}_n),$$

and since m is positive and less than the prime n, the Nazis can use the Pulverizer to find the $\mathbb{Z}_n$-inverse j of m. Now

$$j \cdot \widehat{m} = j \cdot (m \cdot k) = (j \cdot m) \cdot k = 1 \cdot k = k \ (\mathbb{Z}_n).$$

So by computing $j \cdot \widehat{m} = k \ (\mathbb{Z}_n)$, the Nazis get the secret key and can then decrypt any message!

This is a huge vulnerability, so Turing's hypothetical Version 2.0 code has no practical value. Fortunately, Turing got better at cryptography after devising this code; his subsequent deciphering of Enigma messages surely saved thousands of lives, if not the whole of Britain.

9.9.5 Turing Postscript

A few years after the war, Turing's home was robbed. Detectives soon determined that a former homosexual lover of Turing's had conspired in the robbery. So they arrested him—that is, they arrested Alan Turing—because at that time in Britain, homosexuality was a crime punishable by up to two years in prison. Turing was sentenced to a hormonal "treatment" for his homosexuality: he was given estrogen injections. He began to develop breasts.

Three years later, Alan Turing, the founder of computer science, was dead. His mother explained what happened in a biography of her own son. Despite her repeated warnings, Turing carried out chemistry experiments in his own home. Apparently, her worst fear was realized: by working with potassium cyanide while eating an apple, he poisoned himself.

However, Turing remained a puzzle to the very end. His mother was a devout woman who considered suicide a sin. And, other biographers have pointed out, Turing had previously discussed committing suicide by eating a poisoned apple. Evidently, Alan Turing, who founded computer science and saved his country, took his own life in the end, and in just such a way that his mother could believe it was an accident.

Turing's last project before he disappeared from public view in 1939 involved the construction of an elaborate mechanical device to test a mathematical conjecture called the Riemann Hypothesis. This conjecture first appeared in a sketchy paper by Bernhard Riemann in 1859 and is now one of the most famous unsolved problems in mathematics.

9.10 Euler's Theorem

The RSA cryptosystem examined in the next section, and other current schemes for encoding secret messages, involve computing remainders of numbers raised to large powers. A basic fact about remainders of powers follows from a theorem due to Euler about congruences.

Definition 9.10.1. For $n > 0$, define

$$\phi(n) ::= \text{the number of integers in } [0..n), \text{ that are relatively prime to } n.$$

This function ϕ is known as Euler's ϕ function.[15]

For example, $\phi(7) = 6$ because all 6 positive numbers in $[0..7)$ are relatively prime to the prime number 7. Only 0 is not relatively prime to 7. Also, $\phi(12) = 4$ since 1, 5, 7, and 11 are the only numbers in $[0..12)$ that are relatively prime to 12.[16]

More generally, if p is prime, then $\phi(p) = p - 1$ since every positive number in $[0..p)$ is relatively prime to p. When n is composite, however, the ϕ function gets a little complicated. We'll get back to it in the next section.

Euler's Theorem is traditionally stated in terms of congruence:

Theorem (*Euler's Theorem*)**.** *If n and k are relatively prime, then*

$$k^{\phi(n)} \equiv 1 \pmod{n}. \tag{9.15}$$

Things get simpler when we rephrase Euler's Theorem in terms of $\mathbb{Z}_n$.

[15] Some texts call it Euler's *totient function*.

[16] Also, $\phi(1) = 1$, but since we make no use of this fact, it only merits a footnote.

The Riemann Hypothesis

The formula for the sum of an infinite geometric series says:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Substituting $x = \frac{1}{2^s}$, $x = \frac{1}{3^s}$, $x = \frac{1}{5^s}$, and so on for each prime number gives a sequence of equations:

$$\begin{aligned}
1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \cdots &= \frac{1}{1-1/2^s} \\
1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \cdots &= \frac{1}{1-1/3^s} \\
1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + \cdots &= \frac{1}{1-1/5^s} \\
&\vdots
\end{aligned}$$

Multiplying together all the left-hand sides and all the right-hand sides gives:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \text{primes}} \left(\frac{1}{1-1/p^s} \right).$$

The sum on the left is obtained by multiplying out all the infinite series and applying the Fundamental Theorem of Arithmetic. For example, the term $1/300^s$ in the sum is obtained by multiplying $1/2^{2s}$ from the first equation by $1/3^s$ in the second and $1/5^{2s}$ in the third. Riemann noted that every prime appears in the expression on the right. So he proposed to learn about the primes by studying the equivalent, but simpler expression on the left. In particular, he regarded s as a complex number and the left side as a function $\zeta(s)$. Riemann found that the distribution of primes is related to values of s for which $\zeta(s) = 0$, which led to his famous conjecture:

Definition 9.9.6. The *Riemann Hypothesis*: Every nontrivial zero of the zeta function $\zeta(s)$ lies on the line $s = 1/2 + ci$ in the complex plane.

A proof would immediately imply, among other things, a strong form of the Prime Number Theorem.
Researchers continue to work intensely to settle this conjecture, as they have for over a century. It is another of the Millennium Problems whose solver will earn $1,000,000 from the Clay Institute.

Definition 9.10.2. Let $\mathbb{Z}_n^*$ be the integers in $[0..n)$, that are relatively prime to n:[17]

$$\mathbb{Z}_n^* ::= \{k \in [0..n) \mid \gcd(k, n) = 1\}. \tag{9.16}$$

Consequently,

$$\phi(n) = \left|\mathbb{Z}_n^*\right|.$$

Theorem 9.10.3 (Euler's Theorem for $\mathbb{Z}_n$)**.** *For all $k \in \mathbb{Z}_n^*$,*

$$k^{\phi(n)} = 1 \ (\mathbb{Z}_n). \tag{9.17}$$

Theorem 9.10.3 will follow from two very easy lemmas.
Let's start by observing that $\mathbb{Z}_n^*$ is closed under multiplication in $\mathbb{Z}_n$:

Lemma 9.10.4. *If $j, k \in \mathbb{Z}_n^*$, then $j \cdot_n k \in \mathbb{Z}_n^*$.*

There are lots of easy ways to prove this (see Problem 9.70).

Definition 9.10.5. For any element k and subset S of $\mathbb{Z}_n$, let

$$kS ::= \{k \cdot_n s \mid s \in S\}.$$

Lemma 9.10.6. *If $k \in \mathbb{Z}_n^*$ and $S \subseteq \mathbb{Z}_n$, then*

$$|kS| = |S|.$$

Proof. Since $k \in \mathbb{Z}_n^*$, by Theorem 9.9.5 it is cancellable. Therefore,

$$[ks = kt \ (\mathbb{Z}_n)] \quad \text{implies} \quad s = t.$$

So multiplying by k in $\mathbb{Z}_n$ maps all the elements of S to distinct elements of kS, which implies S and kS are the same size. ■

Corollary 9.10.7. *If $k \in \mathbb{Z}_n^*$, then*

$$k\mathbb{Z}_n^* = \mathbb{Z}_n^*.$$

Proof. A product of elements in $\mathbb{Z}_n^*$ remains in $\mathbb{Z}_n^*$ by Lemma 9.10.4. So if $k \in \mathbb{Z}_n^*$, then $k\mathbb{Z}_n^* \subseteq \mathbb{Z}_n^*$. But by Lemma 9.10.6, $k\mathbb{Z}_n^*$ and $\mathbb{Z}_n^*$ are the same size, so they must be equal. ■

Now we can complete the proof of Euler's Theorem 9.10.3 for $\mathbb{Z}_n$:

[17] Some other texts use the notation n^* for $\mathbb{Z}_n^*$.

Proof. Let

$$P ::= k_1 \cdot k_2 \cdots k_{\phi(n)} \ (\mathbb{Z}_n)$$

be the product in $\mathbb{Z}_n$ of all the numbers in $\mathbb{Z}_n^*$. Let

$$Q ::= (k \cdot k_1) \cdot (k \cdot k_2) \cdots (k \cdot k_{\phi(n)}) \ (\mathbb{Z}_n)$$

for some $k \in \mathbb{Z}_n^*$. Factoring out k's immediately gives

$$Q = k^{\phi(n)} P \ (\mathbb{Z}_n).$$

But Q is the same as the product of the numbers in $k\mathbb{Z}_n^*$, and $k\mathbb{Z}_n^* = \mathbb{Z}_n^*$, so we realize that Q is the product of the same numbers as P, just in a different order. Altogether, we have

$$P = Q = k^{\phi(n)} P \ (\mathbb{Z}_n).$$

Furthermore, $P \in \mathbb{Z}_n^*$ by Lemma 9.10.4, and so it can be cancelled from both sides of this equality, giving

$$1 = k^{\phi(n)} \ (\mathbb{Z}_n).$$

■

Euler's theorem offers another way to find inverses modulo n: if k is relatively prime to n, then $k^{\phi(n)-1}$ is a $\mathbb{Z}_n$-inverse of k, and we can compute this power of k efficiently using fast exponentiation. However, this approach requires computing $\phi(n)$. In the next section, we'll show that computing $\phi(n)$ is easy *if* we know the prime factorization of n. But we know that finding the factors of n is generally hard to do when n is large, and so the Pulverizer remains the best approach to computing inverses modulo n.

Fermat's Little Theorem

For the record, we mention a famous special case of Euler's Theorem that was known to Fermat a century earlier.

Corollary 9.10.8 (*Fermat's Little Theorem*)**.** *Suppose p is a prime and k is not a multiple of p. Then*

$$k^{p-1} \equiv 1 \pmod{p}.$$

9.10.1 Computing Euler's ϕ Function

RSA works using arithmetic modulo the product of two large primes, so we begin with an elementary explanation of how to compute $\phi(pq)$ for primes p and q:

Lemma 9.10.9.

$$\phi(pq) = (p-1)(q-1)$$

for primes $p \neq q$.

Proof. Since p and q are prime, any number that is not relatively prime to pq must be a multiple of p or a multiple of q. Among the pq numbers in $[0..pq)$, there are precisely q multiples of p and p multiples of q. Since p and q are relatively prime, the only number in $[0..pq)$ that is a multiple of both p and q is 0. Hence, there are $p+q-1$ numbers in $[0..pq)$ that are *not* relatively prime to pq. This means that

$$\begin{aligned}\phi(pq) &= pq - (p+q-1)\\ &= (p-1)(q-1),\end{aligned}$$

as claimed. ■

The following theorem provides a way to calculate $\phi(n)$ for arbitrary n.

Theorem 9.10.10.

(a) If p is a prime, then

$$\phi(p^k) = p^k\left(1-\frac{1}{p}\right) = p^k - p^{k-1}$$

for $k \geq 1$.

(b) If a and b are relatively prime, then $\phi(ab) = \phi(a)\phi(b)$.

Here's an example of using Theorem 9.10.10 to compute $\phi(300)$:

$$\begin{aligned}\phi(300) &= \phi(2^2\cdot 3\cdot 5^2)\\ &= \phi(2^2)\cdot\phi(3)\cdot\phi(5^2) && \text{(by Theorem 9.10.10.(b))}\\ &= (2^2-2^1)(3^1-3^0)(5^2-5^1) && \text{(by Theorem 9.10.10.(a))}\\ &= 80.\end{aligned}$$

Note that Lemma 9.10.9 also follows as a special case of Theorem 9.10.10.(b), since we know that $\phi(p) = p-1$ for any prime p.

To prove Theorem 9.10.10.(a), notice that every pth number among the p^k numbers in $[0..p^k)$ is divisible by p, and only these are divisible by p. So $1/p$ of these numbers are divisible by p and the remaining ones are not. That is,

$$\phi(p^k) = p^k - (1/p)p^k = p^k\left(1-\frac{1}{p}\right).$$

We'll leave a proof of Theorem 9.10.10.(b) to Problem 9.64.

As a consequence of Theorem 9.10.10, we have

Corollary 9.10.11. *For any number n, if $p_1, p_2, \ldots, p_j$ are the (distinct) prime factors of n, then*

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_j}\right).$$

Proof. Suppose $n = p_1^{e_1} p_2^{e_2} \cdots p_j^{e_j}$. Then

$$\begin{aligned}
\phi(n) &= \phi(p_1^{e_1})\phi(p_2^{e_2})\cdots\phi(p_j^{e_j}) && \text{(by Theorem 9.10.10.(b)),}\\
&= p_1^{e_1}\left(1-\frac{1}{p_1}\right) p_2^{e_2}\left(1-\frac{1}{p_2}\right)\cdots p_j^{e_j}\left(1-\frac{1}{p_j}\right) && \text{(by Theorem 9.10.10.(a)),}\\
&= p_1^{e_1} p_2^{e_2}\cdots p_j^{e_j}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_j}\right)\\
&= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_j}\right).
\end{aligned}$$

■

9.11 RSA Public Key Encryption

Turing's code did not work as he hoped. However, his essential idea—using number theory as the basis for cryptography—succeeded spectacularly in the decades after his death.

In 1977, Ronald Rivest, Adi Shamir, and Leonard Adleman at MIT proposed a highly secure cryptosystem, called **RSA**, based on number theory. The purpose of the RSA scheme is to transmit secret messages over public communication channels. As with Turing's codes, the messages transmitted are nonnegative integers of some fixed size.

Moreover, RSA has a major advantage over traditional codes: the sender and receiver of an encrypted message need not meet beforehand to agree on a secret key. Rather, the receiver has both a *private key*, which they guard closely, and a *public key*, which they distribute as widely as possible. A sender wishing to transmit a secret message to the receiver encrypts their message using the receiver's widely-distributed public key. The receiver can then decrypt the received message using their closely held private key. The use of such a *public key cryptography* system

allows you and Amazon, for example, to engage in a secure transaction without meeting up beforehand in a dark alley to exchange a key.

Interestingly, RSA does not operate modulo a prime, as Turing's hypothetical Version 2.0 may have, but rather modulo the product of *two* large primes—typically primes that are hundreds of digits long. Also, instead of encrypting by multiplication with a secret key, RSA exponentiates to a secret power—which is why Euler's Theorem is central to understanding RSA.

The scheme for RSA public key encryption appears in the box.

If the message m is relatively prime to n, then a simple application of Euler's Theorem implies that this way of decoding the encrypted message indeed reproduces the original unencrypted message. In fact, the decoding always works—even in (the highly unlikely) case that m is not relatively prime to n. The details are worked out in Problem 9.85.

Why is RSA thought to be secure? It would be easy to figure out the private key d if you knew p and q—you could do it the same way the **Receiver** does using the Pulverizer. But assuming the conjecture that it is hopelessly hard to factor a number that is the product of two primes with hundreds of digits, an effort to factor n is not going to break RSA.

Could there be another approach to reverse engineer the private key d from the public key that did not involve factoring n? Not really. It turns out that given just the private and the public keys, it is easy to factor n[18] (a proof of this is sketched in Problem 9.87). So if we are confident that factoring is hopelessly hard, then we can be equally confident that finding the private key just from the public key will be hopeless.

But even if we are confident that an RSA private key won't be found, this doesn't rule out the possibility of decoding RSA messages in a way that sidesteps the private key. It is an important unproven conjecture in cryptography that *any* way of cracking RSA—not just by finding the secret key—would imply the ability to factor. This would be a much stronger theoretical assurance of RSA security than is presently known.

But the real reason for confidence is that RSA has withstood all attacks by the world's most sophisticated cryptographers for nearly 40 years. Despite decades of these attacks, no significant weakness has been found—though the recommended key-length has had to be quadrupled in response to the billion-fold increase in computer speed over four decades. That's why the mathematical, financial, and intelligence communities are betting the family jewels on the security of RSA encryption.

You can hope that with more studying of number theory, you will be the first to

[18] For this reason, the public and private keys in practice should be randomly chosen so that neither is "too small."

The RSA Cryptosystem

A **Receiver** who wants to be able to receive secret numerical messages creates a *private key*, which they keep secret, and a *public key*, which they make publicly available. Anyone with the public key can then be a **Sender** who can publicly send secret messages to the **Receiver**—even if they have never communicated or shared any information besides the public key.
Here is how they do it:

Beforehand The **Receiver** creates a public key and a private key as follows.

1. Generate two distinct primes, p and q. These are used to generate the private key, and they must be kept hidden. (In current practice, p and q are chosen to be hundreds of digits long.)
2. Let $n ::= pq$.
3. Select an integer $e \in [0..n)$ such that $\gcd(e, (p-1)(q-1)) = 1$. The *public key* is the pair (e, n). This should be distributed widely.
4. Let the *private key* $d \in [0..n)$ be the inverse of e in the ring $\mathbb{Z}_{(p-1)(q-1)}$. This private key can be found using the Pulverizer. The private key d should be kept hidden!

Encoding To transmit a message $m \in [0..n)$ to **Receiver**, a **Sender** uses the public key to encrypt m into a numerical message

$$\widehat{m} ::= m^e \ (\mathbb{Z}_n).$$

The **Sender** can then publicly transmit $\widehat{m}$ to the **Receiver**.

Decoding The **Receiver** decrypts message $\widehat{m}$ back to message m using the private key:

$$m = \widehat{m}^d \ (\mathbb{Z}_n).$$

figure out how to factor numbers quickly and, among other things, break RSA. But be further warned that even Gauss worked on factoring for years without a lot to show for his efforts—and if you do figure it out, you might wind up confronting some humorless fellows working for a Federal agency in charge of security....

9.12 What has SAT got to do with it?

So why does society, or at least everybody's secret codes, fall apart if there is an efficient test for satisfiability (SAT), as we claimed in Section 3.5? To explain this, remember that RSA can be managed computationally because multiplication of two primes is fast, but factoring a product of two primes seems to be overwhelmingly demanding.

Let's begin with the observation from Section 3.2 that a digital circuit can be described by a bunch of propositional formulas of about the same total size as the circuit. So testing circuits for satisfiability is equivalent to the SAT problem for propositional formulas (see Problem 3.24).

Now designing digital multiplication circuits is completely routine. We can easily build a digital "product checker" circuit out of AND, OR, and NOT gates with 1 output wire and $4n$ digital input wires. The first n inputs are for the binary representation of an integer i, the next n inputs for the binary representation of an integer j, and the remaining $2n$ inputs for the binary representation of an integer k. The output of the circuit is 1 iff $ij = k$ and $i, j > 1$. A straightforward design for such a product checker uses proportional to n^2 gates.

Now here's how to factor any number m with a length $2n$ binary representation using a SAT solver. First, fix the last $2n$ digital inputs—the ones for the binary representation of k—so that k equals m.

Next, set the first of the n digital inputs for the representation of i to be 1. Do a SAT test to see if there is a satisfying assignment of values for the remaining $2n-1$ inputs used for the i and j representations. That is, see if the remaining inputs for i and j can be filled in to cause the circuit to give output 1. If there is such an assignment, fix the first i-input to be 1, otherwise fix it to be 0. So now we have set the first i-input equal to the first digit of the binary representations of an i such that $ij = m$.

Now do the same thing to fix the second of the n digital inputs for the representation of i, and then third, proceeding in this way through all the n inputs for the number i. At this point, we have the complete n-bit binary representation of an $i > 1$ such $ij = m$ for some $j > 1$. In other words, we have found an integer i that is a factor of m. We can now find j by dividing m by i.

So after n SAT tests, we have factored m. This means that if SAT for digital circuits with $4n$ inputs and about n^2 gates could be determined by a procedure taking a number of steps bounded above by a degree d polynomial in n, then $2n$ digit numbers can be factored in n times this many steps, that is, with a number of steps bounded by a polynomial of degree $d+1$ in n. So if SAT could be solved in polynomial time, then so could factoring, and consequently RSA would be "easy" to break.

9.13 References

[3], [45]

Problems for Section 9.1

Practice Problems

Problem 9.1.
Prove that a linear combination of linear combinations of integers $a_0, \ldots, a_n$ is a linear combination of $a_0, \ldots, a_n$.

Class Problems

Problem 9.2.
A number is *perfect* if it is equal to the sum of its positive divisors, other than itself. For example, 6 is perfect, because $6 = 1 + 2 + 3$. Similarly, 28 is perfect, because $28 = 1 + 2 + 4 + 7 + 14$. Explain why $2^{k-1}(2^k - 1)$ is perfect when $2^k - 1$ is prime.[19]

[19] Euclid proved this 2300 years ago. About 250 years ago, Euler proved the converse: *every* even perfect number is of this form (for a simple proof see http://primes.utm.edu/notes/proofs/EvenPerfect.html). It is not known if there are *any* odd perfect numbers at all. It is also not known if there are an infinite number of even perfect numbers. One of the charms of number theory is that simple results like those given in this problem lie at the brink of the unknown.

Problems for Section 9.2

Practice Problems

Problem 9.3.
Let

$$x ::= 21212121,$$
$$y ::= 12121212.$$

Use the Euclidean algorithm to find the GCD of x and y. *Hint:* Looks scary, but it's not.

Problem 9.4.
Let

$$x ::= 17^{88} \cdot 31^5 \cdot 37^2 \cdot 59^{1000}$$
$$y ::= 19^{(9^{22})} \cdot 37^{12} \cdot 53^{3678} \cdot 59^{29}.$$

(a) What is $\gcd(x, y)$?

(b) What is $\text{lcm}(x, y)$? ("lcm" is *least common multiple*.)

Problem 9.5.
Prove that

$$\gcd(a^5, b^5) = \gcd(a, b)^5$$

for all $a, b \in \mathbb{Z}$.

Problem 9.6.
Prove that $\gcd(m, n)$ is the minimum positive value of any integer linear combination of integers m and n.

Class Problems

Problem 9.7.
Use the Euclidean Algorithm to prove that

$$\gcd(13a + 8b, 5a + 3b) = \gcd(a, b).$$

Problem 9.8.

(a) Use the Pulverizer to find integers x, y such that

$$x30 + y22 = \gcd(30, 22).$$

(b) Now find integers x', y' with $0 \leq y' < 30$ such that

$$x'30 + y'22 = \gcd(30, 22)$$

Problem 9.9. (a) Use the Pulverizer to find $\gcd(84, 108)$

(b) Find integers x, y with $0 \leq y < 84$ such that

$$x \cdot 84 + y \cdot 108 = \gcd(84, 108).$$

(c) Is there a multiplicative inverse of 84 in $\mathbb{Z}_{108}$? If not briefly explain why, otherwise find it.

Problem 9.10.

Indicate **true** or **false** for the following statements about the greatest common divisor, and *provide counterexamples* for those that are **false**.

(a) If $\gcd(a, b) \neq 1$ and $\gcd(b, c) \neq 1$, then $\gcd(a, c) \neq 1$. **true** **false**

(b) If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$. **true** **false**

(c) $\gcd(a^n, b^n) = (\gcd(a, b))^n$ **true** **false**

(d) $\gcd(ab, ac) = a \gcd(b, c)$. **true** **false**

(e) $\gcd(1 + a, 1 + b) = 1 + \gcd(a, b)$. **true** **false**

(f) If an integer linear combination of a and b equals 1, then so does some integer linear combination of a and b^2. **true** **false**

(g) If no integer linear combination of a and b equals 2, then neither does any integer linear combination of a^2 and b^2. **true** **false**

Problem 9.11.
For nonzero integers a, b, prove the following properties of divisibility and GCD'S. You may use Theorem 9.2.2 that $\gcd(a, b)$ is an integer linear combination of a and b. You may *not* appeal to uniqueness of prime factorization Theorem 9.4.1 (because some of these properties are needed to *prove* unique factorization.)

(a) Every common divisor of a and b divides $\gcd(a, b)$.

(b) $\gcd(ka, kb) = k \cdot \gcd(a, b)$ for all $k \in \mathbb{N}$.

(c) If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

(d) If $p \mid bc$ for some prime p then $p \mid b$ or $p \mid c$.

(e) Let m be the smallest integer linear combination of a and b that is positive. Show that $m = \gcd(a, b)$.

Homework Problems

Problem 9.12.
Here is a game you can analyze with number theory and always beat me. We start with two distinct, positive integers written on a blackboard. Call them a and b. Now we take turns. (I'll let you decide who goes first.) On each turn, the player must write a new positive integer on the board that is the difference of two numbers that are already there. If a player cannot play, then they lose.

For example, suppose that 12 and 15 are on the board initially. Your first play must be 3, which is $15 - 12$. Then I might play 9, which is $12 - 3$. Then you might play 6, which is $15 - 9$. Then I can't play, so I lose.

(a) Show that every number on the board at the end of the game is a multiple of $\gcd(a, b)$.

(b) Show that every positive multiple of $\gcd(a, b)$ up to $\max(a, b)$ is on the board at the end of the game.

(c) Describe a strategy that lets you win this game every time.

Problem 9.13.

Define the Pulverizer State machine to have:

$$\begin{aligned}
\text{states} &::= \mathbb{N}^6 \\
\text{start state} &::= (a, b, 0, 1, 1, 0) && \text{(where } a \geq b > 0\text{)} \\
\text{transitions} &::= (x, y, s, t, u, v) \longrightarrow \\
&\qquad (y,\ \text{rem}(x,\ y),\ u - sq,\ v - tq,\ s,\ t) && \text{(for } q = \text{qcnt}(x, y), y > 0\text{).}
\end{aligned}$$

(a) Show that the following properties are preserved invariants of the Pulverizer machine:

$$\gcd(x, y) = \gcd(a, b), \tag{Inv1}$$

$$sa + tb = y, \text{ and} \tag{Inv2}$$

$$ua + vb = x. \tag{Inv3}$$

(b) Conclude that the Pulverizer machine is partially correct.

(c) Explain why the machine terminates after at most the same number of transitions as the Euclidean algorithm.

Problem 9.14.

The Euclidean state machine is defined by the rule

$$(x, y) \longrightarrow (y, \text{rem}(x,\ y)), \tag{9.18}$$

for $y > 0$.

Prove that the smallest positive integers $a \geq b$ for which, starting in state (a, b), the state machine will make n transitions are $F(n + 1)$ and $F(n)$, where $F(n)$ is the nth Fibonacci number.[20]

Hint: Induction.

Problem 9.15.

Let's extend the jug filling scenario of Section 9.1.3 to three jugs and a receptacle. Suppose the jugs can hold a, b and c gallons of water, respectively.

The receptacle can be used to store an unlimited amount of water, but has no measurement markings. Excess water can be dumped into the drain. Among the possible moves are:

[20] Problem 5.25 shows that $F(n) \leq \varphi^n$ where φ is the golden ratio $(1 + \sqrt{5})/2$. This implies that the Euclidean algorithm halts after at most $\log_\varphi(a)$ transitions, a somewhat smaller bound than $2 \log_2 a$ derived from equation (9.4).

1. fill a bucket from the hose,
2. pour from the receptacle to a bucket until the bucket is full or the receptacle is empty, whichever happens first,
3. empty a bucket to the drain,
4. empty a bucket to the receptacle, and
5. pour from one bucket to another until either the first is empty or the second is full.

(a) Model this scenario with a state machine. (What are the states? How does a state change in response to a move?)

(b) Prove that Bruce can get $k \in \mathbb{N}$ gallons of water into the receptacle using the above operations if $\gcd(a, b, c) \mid k$.

Problem 9.16.
The *Binary GCD* state machine computes the GCD of integers $a, b > 0$ using only division by 2 and subtraction, which makes it run very efficiently on hardware that uses binary representation of numbers. In practice, it runs more quickly than the more famous Euclidean algorithm described in Section 9.2.1.

$$\begin{aligned}
\text{states} &::= \mathbb{N}^3 \\
\text{start state} &::= (a, b, 1) \\
\text{transitions} &::= \text{if } \min(x, y) > 0, \text{ then } (x, y, e) \longrightarrow \\
&\qquad (x/2, y/2, 2e) && (\text{if } 2 \mid x \text{ and } 2 \mid y) && \text{(i1)} \\
&\qquad (x/2, y, e) && (\text{else if } 2 \mid x) && \text{(i2)} \\
&\qquad (x, y/2, e) && (\text{else if } 2 \mid y) && \text{(i3)} \\
&\qquad (x - y, y, e) && (\text{else if } x > y) && \text{(i4)} \\
&\qquad (x, y - x, e) && (\text{else if } y > x) && \text{(i5)} \\
&\qquad (1, 0, ex) && (\text{otherwise } (x = y)). && \text{(i6)}
\end{aligned}$$

(a) Use the Invariant Principle to prove that if this machine stops, that is, reaches a state (x, y, e) in which no transition is possible, then $e = \gcd(a, b)$.

(b) Prove that rule (i1)

$$(x, y, e) \to (x/2, y/2, 2e)$$

is never executed after any of the other rules is executed.

(c) Prove that the machine reaches a final state in at most $1 + 3(\log a + \log b)$ transitions.

Problem 9.17.
Extend the binary gcd procedure of Problem 9.16 to obtain a new pulverizer that uses only division by 2 and subtraction.

Hint: After the binary gcd procedure has factored out 2's, it starts computing the $\gcd(a, b)$ for numbers a, b at least one of which is odd. It does this by successively updating a pair of numbers (x, y) such that $\gcd(x, y) = \gcd(a, b)$. Extend the procedure to find and update coefficients u_x, v_x, u_y, v_y such that

$$u_x a + v_x b = x \text{ and } u_y a + v_y b = y.$$

To see how to update the coefficients when at least one of a and b is odd and $ua + vb$ is even, show that either u and v are both even, or else $u - b$ and $v + a$ are both even.

Problem 9.18.
For any set A of integers,

$$\gcd(A) ::= \text{the greatest common divisor of the elements of } A.$$

The following useful property of gcd's of sets is easy to take for granted:

Theorem.

$$\gcd(A \cup B) = \gcd(\gcd(A), \gcd(B)), \tag{AuB}$$

for all finite sets $A, B \subset \mathbb{Z}$.

Theorem (AuB) has an easy proof as a Corollary of the Unique Factorization Theorem. In this problem we develop a proof by induction just making repeated use of Lemma 9.2.6.b :

$$(d \mid a \text{ AND } d \mid b) \text{ IFF } d \mid \gcd(a, b). \tag{gcddiv}$$

The key to proving (AuB) will be generalizing (gcddiv) to finite sets.

Definition. For any subset $A \subseteq \mathbb{Z}$,

$$d \mid A ::= \forall a \in A.\ d \mid a. \tag{divdef}$$

Lemma.

$$d \mid A \text{ IFF } d \mid \gcd(A). \tag{A-iff-gcdA}$$

for all $d \in \mathbb{Z}$ and finite sets $A \subset \mathbb{Z}$.

(a) Prove that

$$\gcd(a, \gcd(b, c)) = \gcd(\gcd(a, b), c) \tag{gcd-assoc}$$

for all integers a, b, c.

From here on we write "$a \cup A$" as an abbreviation for "$\{a\} \cup A$."

(b) Prove that

$$d \mid (a \cup b \cup C) \text{ IFF } d \mid (\gcd(a, b) \cup C) \tag{abCgcd}$$

for all $a, b, d \in \mathbb{Z}$, and $C \subseteq \mathbb{Z}$.

Proof.

$$\begin{aligned} d \mid (a \cup b \cup C) &\text{ IFF } (d \mid a) \text{ AND } (d \mid b) \text{ AND } (d \mid C) && \text{(def (divdef) of divides)} \\ &\text{IFF } (d \mid \gcd(a, b)) \text{ AND } (d \mid C) && \text{by (gcddiv)} \\ &\text{IFF } d \mid (\gcd(a, b) \cup C) && \text{(def (divdef) of divides).} \end{aligned}$$

■

(c) Using parts (a) and (b), prove by induction on the size of A, that

$$d \mid (a \cup A) \quad \text{IFF} \quad d \mid \gcd(a, \gcd(A)), \tag{divauA}$$

for all integers a, d and finite sets $A \subset \mathbb{Z}$. Explain why this proves (A-iff-gcdA).

(d) Prove Theorem (AuB).

(e) Conclude that $\gcd(A)$ is an integer linear combination of the elements in A.

Exam Problems

Problem 9.19.

Prove that $\gcd(mb + r, b) = \gcd(b, r)$ for all integers m, b, r.

Problem 9.20.
The Stata Center's delicate balance depends on two buckets of water hidden in a secret room. The big bucket has a volume of 25 gallons, and the little bucket has a volume of 10 gallons. If at any time a bucket contains exactly 13 gallons, the Stata Center will collapse. There is an interactive display where tourists can remotely fill and empty the buckets according to certain rules. We represent the buckets as a state machine.

The state of the machine is a pair (b, l), where b is the volume of water in big bucket, and l is the volume of water in little bucket.

(a) We informally describe some of the legal operations tourists can perform below. Represent each of the following operations as a transition of the state machine. The first is done for you as an example.

1. Fill the big bucket.
$$(b, l) \longrightarrow (25, l).$$
2. Empty the little bucket.
3. Pour the big bucket into the little bucket. You should have two cases defined in terms of the state (b, l): if all the water from the big bucket fits in the little bucket, then pour all the water. If it doesn't, pour until the little jar is full, leaving some water remaining in the big jar.

(b) Use the Invariant Principle to show that, starting with empty buckets, the Stata Center will never collapse. That is, the state $(13, x)$ in unreachable. (In verifying your claim that the invariant is preserved, you may restrict to the representative transitions of part (a).)

Problem 9.21.
Let
$$m = 2^9 5^{24} 7^4 11^7,$$
$$n = 2^3 7^{22} 11^{211} 19^7,$$
$$p = 2^5 3^4 7^{6042} 19^{30}.$$

(a) What is the $\gcd(m, n, p)$?

(b) What is the *least common multiple* $\operatorname{lcm}(m, n, p)$?

Let $v_k(n)$ be the largest power of k that divides n, where $k > 1$. That is,
$$v_k(n) ::= \max\{i \mid k^i \text{ divides } n\}.$$

If A is a nonempty set of positive integers, define

$$v_k(A) ::= \{v_k(a) \mid a \in A\}.$$

(c) Express $v_k(\gcd(A))$ in terms of $v_k(A)$.

(d) Let p be a prime number. Express $v_p(\text{lcm}(A))$ in terms of $v_p(A)$.

(e) Give an example of integers a, b where $v_6(\text{lcm}(a,b)) > \max(v_6(a), v_6(b))$.

(f) Let $\prod A$ be the product of all the elements in A. Express $v_p(\prod A)$ in terms of $v_p(A)$.

(g) Let B also be a nonempty set of nonnegative integers. Conclude that

$$\gcd(A \cup B) = \gcd(\gcd(A), \gcd(B)). \tag{9.19}$$

Hint: Consider $v_p()$ of the left and right-hand sides of (9.19). You may assume

$$\min(A \cup B) = \min(\min(A), \min(B)). \tag{9.20}$$

Problems for Section 9.3

Homework Problems

Problem 9.22.
TBA: Chebyshvev lower bound in prime density, based on Shoup pp.75–76

Problems for Section 9.4

Practice Problems

Problem 9.23.
Let p be a prime number and $a_1, \ldots, a_n$ integers. Prove the following Lemma *by induction*:

Lemma.

$$\textit{If } p \textit{ divides a product } a_1 \cdot a_2 \cdots a_n, \textit{ then } p \textit{ divides some } a_i. \tag{*}$$

You may assume the case for $n = 2$ which was given by Lemma 9.4.2.

Be sure to clearly state and label your Induction Hypothesis, Base case(s), and Induction step.

Class Problems

Problem 9.24. **(a)** Let $m = 2^9 5^{24} 11^7 17^{12}$ and $n = 2^3 7^{22} 11^{211} 13^1 17^9 19^2$. What is the $\gcd(m,n)$? What is the *least common multiple* $\operatorname{lcm}(m,n)$ of m and n? Verify that

$$\gcd(m,n) \cdot \operatorname{lcm}(m,n) = mn. \tag{9.21}$$

(b) Describe in general how to find the $\gcd(m,n)$ and $\operatorname{lcm}(m,n)$ from the prime factorizations of m and n. Conclude that equation (9.21) holds for all positive integers m, n.

Homework Problems

Problem 9.25.

The set of complex numbers that are equal to $m + n\sqrt{-5}$ for some integers m, n is called $\mathbb{Z}[\sqrt{-5}]$. It will turn out that in $\mathbb{Z}[\sqrt{-5}]$, not all numbers have unique factorizations.

A sum or product of numbers in $\mathbb{Z}[\sqrt{-5}]$ is in $\mathbb{Z}[\sqrt{-5}]$, and since $\mathbb{Z}[\sqrt{-5}]$ is a subset of the complex numbers, all the usual rules for addition and multiplication are true for it. But some weird things do happen. For example, the prime 29 has factors:

(a) Find $x, y \in \mathbb{Z}[\sqrt{-5}]$ such that $xy = 29$ and $x \neq \pm 1 \neq y$.

On the other hand, the number 3 is still a "prime" even in $\mathbb{Z}[\sqrt{-5}]$. More precisely, a number $p \in \mathbb{Z}[\sqrt{-5}]$ is called *irreducible* in $\mathbb{Z}[\sqrt{-5}]$ iff when $xy = p$ for some $x, y \in \mathbb{Z}[\sqrt{-5}]$, either $x = \pm 1$ or $y = \pm 1$.

Claim. *The numbers* $3, 2 + \sqrt{-5}$, *and* $2 - \sqrt{-5}$ *are irreducible in* $\mathbb{Z}[\sqrt{-5}]$.

In particular, this Claim implies that the number 9 factors into irreducibles in $\mathbb{Z}[\sqrt{-5}]$ in two different ways:

$$3 \cdot 3 = 9 = (2 + \sqrt{-5})(2 - \sqrt{-5}).$$

So $\mathbb{Z}[\sqrt{-5}]$ is an example of what is called a *non-unique factorization* domain.

To verify the Claim, we'll appeal (without proof) to a familiar technical property of complex numbers given in the following Lemma.

Definition. For a complex number $c = r + si$ where $r, s \in \mathbb{R}$ and i is $\sqrt{-1}$, the *norm* $|c|$ of c is $\sqrt{r^2 + s^2}$.

Lemma. *For* $c, d \in \mathbb{C}$,

$$|cd| = |c|\,|d|.$$

(b) Prove that $|x|^2 \neq 3$ for all $x \in \mathbb{Z}[\sqrt{-5}]$.

(c) Prove that if $x \in \mathbb{Z}[\sqrt{-5}]$ and $|x| = 1$, then $x = \pm 1$.

(d) Prove that if $|xy| = 3$ for some $x, y \in \mathbb{Z}[\sqrt{-5}]$, then $x = \pm 1$ or $y = \pm 1$.

Hint: $|z|^2 \in \mathbb{N}$ for $z \in \mathbb{Z}[\sqrt{-5}]$.

(e) Complete the proof of the Claim.

Problems for Section 9.6

Practice Problems

Problem 9.26.
Prove that if $a \equiv b \pmod{14}$ and $a \equiv b \pmod{5}$, then $a \equiv b \pmod{70}$.

Problem 9.27.
Show that there is an integer x such that

$$ax \equiv b \pmod{n}$$

iff

$$\gcd(a, n) \mid b.$$

Class Problems

Problem 9.28. **(a)** Prove if n is not divisible by 3, then $n^2 \equiv 1 \pmod{3}$.

(b) Show that if n is odd, then $n^2 \equiv 1 \pmod{8}$.

(c) Conclude that if p is a prime greater than 3, then $p^2 - 1$ is divisible by 24.

Problem 9.29.
The values of polynomial $p(n) ::= n^2 + n + 41$ are prime for all the integers from 0 to 39 (see Section 1.1). Well, p didn't work, but are there any other polynomials whose values are always prime? No way! In fact, we'll prove a much stronger claim.

Definition. The set P of integer polynomial functions can be defined recursively:

Base cases:

- the identity function $\mathrm{Id}_{\mathbb{Z}}(x) ::= x$ is in P.
- for any integer m the constant function $c_m(x) ::= m$ is in P.

Constructor cases. If $r, s \in P$, then $r + s$ and $r \cdot s \in P$.

(a) Using the recursive definition of integer polynomial functions given above, prove by structural induction that for all $q \in P$,

$$j \equiv k \pmod{n} \quad \text{IMPLIES} \quad q(j) \equiv q(k) \pmod{n},$$

for all integers j, k, n where $n > 1$.

Be sure to clearly state and label your Induction Hypothesis, Base case(s), and Constructor step.

(b) We'll say that q *produces multiples* if, for every integer greater than one in the range of q, there are infinitely many different multiples of that integer in the range. For example, if $q(4) = 7$ and q produces multiples, then there are infinitely many different multiples of 7 in the range of q, and of course, except for 7 itself, none of these multiples is prime.

Prove that if q has positive degree and positive leading coefficient, then q produces multiples. You may assume that every such polynomial is strictly increasing for large arguments.

Part (b) implies that an integer polynomial with positive leading coefficient and degree has infinitely many nonprimes in its range. This fact no longer holds true for multivariate polynomials. An amazing consequence of Matiyasevich's [35] solution to Hilbert's Tenth Problem is that multivariate polynomials can be understood as *general purpose* programs for generating sets of integers. If a set of nonnegative integers can be generated by *any* program, then it equals the set of nonnegative integers in the range of a multivariate integer polynomial! In particular, there is an integer polynomial $p(x_1, \ldots, x_7)$ whose nonnegative values as $x_1, \ldots, x_7$ range over $\mathbb{N}$ are precisely the set of all prime numbers!

Problems for Section 9.7

Practice Problems

Problem 9.30.
List the numbers of all statements below that are *equivalent* to

$$a \equiv b \pmod{n},$$

where $n > 1$ and a and b are integers. Briefly explain your reasoning.

i) $2a \equiv 2b \pmod{n}$

ii) $2a \equiv 2b \pmod{2n}$

iii) $a^3 \equiv b^3 \pmod{n}$

iv) $\text{rem}(a,\ n) = \text{rem}(b,\ n)$

v) $\text{rem}(n,\ a) = \text{rem}(n,\ b)$

vi) $\gcd(a, n) = \gcd(b, n)$

vii) $\gcd(n, a - b) = n$

viii) $(a - b)$ is a multiple of n

ix) $\exists k \in \mathbb{Z}.\, a = b + nk$

Problem 9.31.
What is $\text{rem}(3^{101},\ 21)$?

Homework Problems

Problem 9.32.
Prove that congruence is preserved by arithmetic expressions. Namely, prove that

$$a \equiv b \pmod{n}, \tag{9.22}$$

then

$$\text{eval}(e, a) \equiv \text{eval}(e, b) \pmod{n}, \tag{9.23}$$

for all $e \in \text{Aexp}$ (see Section 7.4).

Problem 9.33.
A commutative ring is a set R of elements along with two binary operations $\oplus$ and $\otimes$ from $R \times R$ to R. There is an element in R called the zero-element, $\mathbf{0}$, and another element called the unit-element, $\mathbf{1}$. The operations in a commutative ring satisfy the following *ring axioms* for $r, s, t \in R$:

$$\begin{aligned}
(r \otimes s) \otimes t &= r \otimes (s \otimes t) && \text{(associativity of } \otimes\text{)},\\
(r \oplus s) \oplus t &= r \oplus (s \oplus t) && \text{(associativity of } \oplus\text{)},\\
r \oplus s &= s \oplus r && \text{(commutativity of } \oplus\text{)}\\
r \otimes s &= s \otimes r && \text{(commutativity of } \otimes\text{)},\\
\mathbf{0} \oplus r &= r && \text{(identity for } \oplus\text{)},\\
\mathbf{1} \otimes r &= r && \text{(identity for } \otimes\text{)},\\
\exists r' \in R.\, r \oplus r' &= \mathbf{0} && \text{(inverse for } \oplus\text{)},\\
r \otimes (s \oplus t) &= (r \otimes s) \oplus (r \otimes t) && \text{(distributivity)}.
\end{aligned}$$

(a) Show that the zero-element is unique, that is, show that if $z \in R$ has the property that

$$z \oplus r = r, \tag{9.24}$$

then $z = \mathbf{0}$.

(b) Show that additive inverses are unique, that is, show that

$$r \oplus r_1 = \mathbf{0} \quad \text{and} \tag{9.25}$$
$$r \oplus r_2 = \mathbf{0} \tag{9.26}$$

implies $r_1 = r_2$.

(c) Show that multiplicative inverses are unique, that is, show that

$$r \otimes r_1 = \mathbf{1}$$
$$r \otimes r_2 = \mathbf{1}$$

implies $r_1 = r_2$.

Problem 9.34.
This problem will use elementary properties of congruences to prove that every positive integer divides infinitely many Fibonacci numbers.

A function $f : \mathbb{N} \to \mathbb{N}$ that satisifies

$$f(n) = c_1 f(n-1) + c_2 f(n-2) + \cdots + c_d f(n-d) \tag{9.27}$$

for some $c_i \in \mathbb{N}$ and all $n \geq d$ is called *degree d linear-recursive*.

A function $f : \mathbb{N} \to \mathbb{N}$ *has a degree d repeat modulo m at n and k* when it satisfies the following *repeat congruences*:

$$\begin{aligned} f(n) &\equiv f(k) && \pmod{m}, \\ f(n-1) &\equiv f(k-1) && \pmod{m}, \\ &\vdots \\ f(n-(d-1)) &\equiv f(k-(d-1)) && \pmod{m}. \end{aligned}$$

for $k > n \geq d-1$.

For the rest of this problem, assume linear-recursive functions and repeats are degree $d > 0$.

(a) Prove that if a linear-recursive function has a repeat modulo m at n and k, then it has one at $n+1$ and $k+1$.

(b) Prove that for all $m > 1$, every linear-recursive function repeats modulo m at n and k for some $n, k \in [d-1 .. d+m^d)$.

(c) A linear-recursive function is *reverse-linear* if its dth coefficient $c_d = \pm 1$. Prove that if a reverse-linear-recursive function repeats modulo m at n and k for some $n \geq d$, then it repeats modulo m at $n-1$ and $k-1$.

(d) Conclude that every reverse-linear-recursive function must repeat modulo m at $d-1$ and $(d-1)+j$ for some $j > 0$.

(e) Conclude that if f is an reverse-linear-recursive function and $f(k) = 0$ for some $k \in [0..d)$, then every positive integer is a divisor of $f(n)$ for infinitely many n.

(f) Conclude that every positive integer is a divisor of infinitely many Fibonacci numbers.

Hint: Start the Fibonacci sequence with the values 0,1 instead of 1, 1.

Class Problems

Problem 9.35.
Find

$$\operatorname{rem}\left(9876^{3456789}\left(9^{99}\right)^{5555} - 6789^{3414259}, 14\right). \tag{9.28}$$

Problem 9.36.
The following properties of equivalence mod n follow directly from its definition and simple properties of divisibility. See if you can prove them without looking up the proofs in the text.

(a) If $a \equiv b \pmod{n}$, then $ac \equiv bc \pmod{n}$.

(b) If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

(c) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

(d) $\text{rem}(a, n) \equiv a \pmod{n}$.

Problem 9.37. (a) Why is a number written in decimal evenly divisible by 9 if and only if the sum of its digits is a multiple of 9? *Hint:* $10 \equiv 1 \pmod{9}$.

(b) Take a big number, such as 37273761261. Sum the digits, where every other one is negated:

$$3 + (-7) + 2 + (-7) + 3 + (-7) + 6 + (-1) + 2 + (-6) + 1 = -11$$

Explain why the original number is a multiple of 11 if and only if this sum is a multiple of 11.

Problem 9.38.
At one time, the Guinness Book of World Records reported that the "greatest human calculator" was a guy who could compute 13th roots of 100-digit numbers that were 13th powers. What a curious choice of tasks....

In this problem, we prove

$$n^{13} \equiv n \pmod{10} \tag{9.29}$$

for all n.

(a) Explain why (9.29) does not follow immediately from Euler's Theorem.

(b) Prove that

$$d^{13} \equiv d \pmod{10} \tag{9.30}$$

for $0 \leq d < 10$.

(c) Now prove the congruence (9.29).

Problem 9.39. (a) Ten pirates find a chest filled with gold and silver coins. There are twice as many silver coins in the chest as there are gold. They divide the gold coins in such a way that the difference in the number of coins given to any two pirates is not divisible by 10. They will only take the silver coins if it is possible to divide them the same way. Is this possible, or will they have to leave the silver behind? Prove your answer.

(b) There are also 3 sacks in the chest, containing 5, 49, and 51 rubies respectively. The treasurer of the pirate ship is bored and decides to play a game with the following rules:

- He can merge any two piles together into one pile, and
- he can divide a pile with an even number of rubies into two piles of equal size.

He makes one move every day, and he will finish the game when he has divided the rubies into 105 piles of one. Is it possible for him to finish the game?

Exam Problems

Problem 9.40.
Compute the remainder

$$\text{rem}(24989^{184637} \cdot 673459^{8447},\ 15), \tag{rem15}$$

carefully explaining the steps in your computation.

Problem 9.41.
The sum of the digits of the base 10 representation of an integer is congruent modulo 9 to that integer. For example,

$$763 \equiv 7 + 6 + 3 \pmod 9.$$

We can say that "9 is a *good modulus for base* 10."

More generally, we'll say "k is a good modulus for base b" when, for any nonnegative integer n, the sum of the digits of the base b representation of n is congruent to n modulo k. So 2 is *not* a good modulus for base 10 because

$$763 \not\equiv 7 + 6 + 3 \pmod 2.$$

(a) What integers $k > 1$ are good moduli for base 10?

(b) Show that if $b \equiv 1 \pmod k$, then k is good for base b.

(c) Prove conversely, that if k is good for some base $b \geq 2$, then $b \equiv 1 \pmod{k}$.

Hint: The base b representation of b.

(d) Exactly which integers $k > 1$ are good moduli for base 106?

Problem 9.42.
Let $p(x)$ be an integer polynomial, that is, $p(x) = \sum_{i=0}^{d} c_i x^i$ where the coefficients c_i are integers.

(a) Explain why $p(k) \equiv p(\text{rem}(k, n)) \pmod{n}$ for all integers k, n where $n > 1$.

Now let

$$q(x) ::= (x^2 - 4)(x^2 - 9),$$

and let $q(\mathbb{N}) ::= \{q(0), q(1), q(2), \ldots\}$.

(b) Verify that 3 divides every element of $q(\mathbb{N})$.

(c) Verify that 4 divides every element of $q(\mathbb{N})$.

(d) Prove that $\gcd(q(\mathbb{N})) = 12$.

Problem 9.43.
We define the sequence of numbers

$$a_n = \begin{cases} 1, & \text{for } n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}, & \text{for } n > 3. \end{cases}$$

Use *strong induction* to prove that $\text{rem}(a_n, 3) = 1$ for all $n \geq 0$.

Problems for Section 9.8

Exam Problems

Problem 9.44.

Definition. The set P of single variable integer polynomials can be defined recursively:

Base cases:

- the identity function, $\mathrm{Id}_{\mathbb{Z}}(x) ::= x$ is in P.
- for any integer m the constant function, $c_m(x) ::= m$ is in P.

Constructor cases. If $r, s \in P$, then $r + s$ and $r \cdot s \in P$.

Prove by structural induction that for all $q \in P$,

$$j \equiv k \pmod{n} \quad \text{IMPLIES} \quad q(j) \equiv q(k) \pmod{n},$$

for all integers j, k, n where $n > 1$.

Be sure to clearly state and label your Induction Hypothesis, Base case(s), and Constructor step.

Problems for Section 9.9

Practice Problems

Problem 9.45.

(a) Given inputs $m, n \in \mathbb{Z}^+$, the Pulverizer will produce $x, y \in \mathbb{Z}$ such that:

(b) Assume $n > 1$. Explain how to use the numbers x, y to find the inverse of m modulo n when there is an inverse.

Problem 9.46.

What is the multiplicative inverse (mod 7) of 2? *Reminder*: by definition, your answer must be an integer between 0 and 6.

Problem 9.47. **(a)** Find integer coefficients x, y such that $25x + 32y = \gcd(25, 32)$.

(b) What is the inverse (mod 25) of 32?

Problem 9.48. **(a)** Use the Pulverizer to find integers s, t such that

$$40s + 7t = \gcd(40, 7).$$

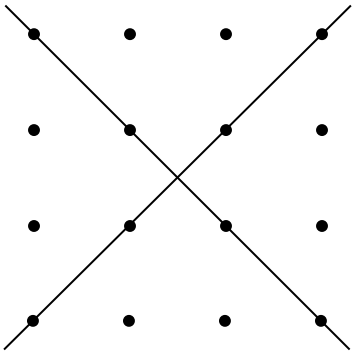
(b) Adjust your answer to part (a) to find an inverse modulo 40 of 7 in $[1..40)$.

Class Problems

Problem 9.49.
Two nonparallel lines in the real plane intersect at a point. Algebraically, this means that the equations

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

have a unique solution (x, y), provided $m_1 \neq m_2$. This statement would be false if we restricted x and y to the integers, since the two lines could cross at a noninteger point:



However, an analogous statement holds if we work over the integers *modulo a prime* p. Find a solution to the congruences

$$y \equiv m_1 x + b_1 \pmod{p}$$
$$y \equiv m_2 x + b_2 \pmod{p}$$

when $m_1 \not\equiv m_2 \pmod{p}$. Express your solution in the form $x \equiv ? \pmod{p}$ and $y \equiv ? \pmod{p}$ where the ?'s denote expressions involving m_1, m_2, b_1 and b_2. You may find it helpful to solve the original equations over the reals first.

Problems for Section 9.10

Practice Problems

Problem 9.50.
Prove that $k \in [0..n)$ has an inverse modulo n iff it has an inverse in $\mathbb{Z}_n$.

Problem 9.51.
What is rem(24^{79}, 79)?
Hint: You should not need to do any actual multiplications!

Problem 9.52. **(a)** Prove that 22^{12001} has a multiplicative inverse modulo 175.

(b) What is the value of $\phi(175)$, where ϕ is Euler's function?

(c) What is the remainder of 22^{12001} divided by 175?

Problem 9.53.
How many numbers between 1 and 6042 (inclusive) are relatively prime to 3780?
Hint: 53 is a factor.

Problem 9.54.
How many numbers between 1 and 3780 (inclusive) are relatively prime to 3780?

Problem 9.55.

(a) What is the probability that an integer from 1 to 360 selected with uniform probability is relatively prime to 360?

(b) What is the value of rem(7^{98}, 360)?

Class Problems

Problem 9.56.
Find the remainder of $26^{1818181}$ divided by 297.
Hint: $1818181 = (180 \cdot 10101) + 1$; use Euler's theorem.

Problem 9.57.
Find the last digit of 7^{7^7}.

Problem 9.58.

Prove that n and n^5 have the same last digit. For example:

$$\underline{2}^5 = 3\underline{2} \qquad\qquad 7\underline{9}^5 = 3077056399$$

Problem 9.59.
Use Fermat's theorem to find the inverse i of 13 modulo 23 with $1 \leq i < 23$.

Problem 9.60.
Let ϕ be Euler's function.

(a) What is the value of $\phi(2)$?

(b) What are three nonnegative integers $k > 1$ such that $\phi(k) = 2$?

(c) Prove that $\phi(k)$ is even for $k > 2$.

Hint: Consider whether k has an odd prime factor or not.

(d) Briefly explain why $\phi(k) = 2$ for exactly three values of k.

Problem 9.61.
Suppose a, b are relatively prime and greater than 1. In this problem you will prove the *Chinese Remainder Theorem*, which says that for all m, n, there is an x such that

$$x \equiv m \pmod{a}, \tag{9.31}$$
$$x \equiv n \pmod{b}. \tag{9.32}$$

Moreover, x is unique up to congruence modulo ab, namely, if x' also satisfies (9.31) and (9.32), then

$$x' \equiv x \pmod{ab}.$$

(a) Prove that for any m, n, there is some x satisfying (9.31) and (9.32).

Hint: Let b^{-1} be an inverse of b modulo a and define $e_a ::= b^{-1}b$. Define e_b similarly. Let $x = me_a + ne_b$.

(b) Prove that

$$[x \equiv 0 \pmod{a} \text{ AND } x \equiv 0 \pmod{b}] \quad \text{implies} \quad x \equiv 0 \pmod{ab}.$$

(c) Conclude that

$$\left[x \equiv x' \pmod{a} \text{ AND } x \equiv x' \pmod{b}\right] \quad \text{implies} \quad x \equiv x' \pmod{ab}.$$

(d) Conclude that the Chinese Remainder Theorem is true.

(e) What about the converse of the implication in part (c)?

Problem 9.62.
The *order* of $k \in \mathbb{Z}_n$ is the smallest positive m such that $k^m = 1$ $(\mathbb{Z}_n)$.

(a) Prove that

$$k^m = 1 \ (\mathbb{Z}_n) \text{ IMPLIES } \operatorname{ord}(k, n) \mid m.$$

Hint: Take the remainder of m divided by the order.

Now suppose $p > 2$ is a prime of the form $2^s + 1$. For example, $2^1 + 1, 2^2 + 1, 2^4 + 1$ are such primes.

(b) Conclude from part (a) that if $0 < k < p$, then $\operatorname{ord}(k, p)$ is a power of 2.

(c) Prove that $\operatorname{ord}(2, p) = 2s$ and conclude that s is a power of 2.[21]

Hint: $2^k - 1$ for $k \in [1..r]$ is positive but too small to equal 0 $(\mathbb{Z}_p)$.

Homework Problems

Problem 9.63.
This problem is about finding square roots modulo a prime p.

(a) Prove that $x^2 \equiv y^2 \pmod{p}$ if and only if $x \equiv y \pmod{p}$ or $x \equiv -y \pmod{p}$. *Hint:* $x^2 - y^2 = (x + y)(x - y)$

An integer x is called a *square root* of n mod p when

$$x^2 \equiv n \pmod{p}.$$

An integer with a square root is called a *square* mod p. For example, if n is congruent to 0 or 1 mod p, then n is a square and it is its own square root.

So let's assume that p is an odd prime and $n \not\equiv 0 \pmod{p}$. It turns out there is a simple test we can perform to see if n is a square mod p:

[21] Numbers of the form $2^{2^k} + 1$ are called *Fermat numbers*, so we can rephrase this conclusion as saying that any prime of the form $2^s + 1$ must actually be a Fermat number. The Fermat numbers are prime for $k = 1, 2, 3, 4$, but not for $k = 5$. In fact, it is not known if any Fermat number with $k > 4$ is prime.

Euler's Criterion

i. If n is a square modulo p, then $n^{(p-1)/2} \equiv 1 \pmod{p}$.

ii. If n is not a square modulo p then $n^{(p-1)/2} \equiv -1 \pmod{p}$.

(b) Prove Case (i) of Euler's Criterion. *Hint:* Use Fermat's theorem.

(c) Prove Case (ii) of Euler's Criterion. *Hint:* Use part (a)

(d) Suppose that $p \equiv 3 \pmod{4}$, and n is a square mod p. Find a simple expression in terms of n and p for a square root of n. *Hint:* Write p as $p = 4k + 3$ and use Euler's Criterion. You might have to multiply two sides of an equation by n at one point.

Problem 9.64.
Suppose a, b are relatively prime integers greater than 1. In this problem you will prove that Euler's function is *multiplicative*, that is, that

$$\phi(ab) = \phi(a)\phi(b).$$

The proof is an easy consequence of the Chinese Remainder Theorem (Problem 9.61).

(a) Conclude from the Chinese Remainder Theorem that the function $f : [0..ab) \to [0..a) \times [0..b)$ defined by

$$f(x) ::= (\text{rem}(x,\ a), \text{rem}(x,\ b))$$

is a bijection.

(b) For any positive integer k let $\mathbb{Z}_k^*$ be the integers in $[0..k)$ that are relatively prime to k. Prove that the function f from part (a) also defines a bijection from $\mathbb{Z}_{ab}^*$ to $\mathbb{Z}_a^* \times \mathbb{Z}_b^*$.

(c) Conclude from the preceding parts of this problem that

$$\phi(ab) = \phi(a)\phi(b). \tag{9.33}$$

(d) Prove Corollary 9.10.11: for any number $n > 1$, if $p_1, p_2, \ldots, p_j$ are the (distinct) prime factors of n, then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_j}\right).$$

Problem 9.65.

Definition. Define the *order* of k over $\mathbb{Z}_n$ to be

$$\text{ord}(k, n) ::= \min\{m > 0 \mid k^m = 1 \ (\mathbb{Z}_n)\}.$$

If no positive power of k equals 1 in $\mathbb{Z}_n$, then $\text{ord}(k, n) ::= \infty$.

(a) Show that $k \in \mathbb{Z}_n^*$ iff k has finite order in $\mathbb{Z}_n$.

(b) Prove that for every $k \in \mathbb{Z}_n^*$, the order of k over $\mathbb{Z}_n$ divides $\phi(n)$.

Hint: Let $m = \text{ord}(k, n)$. Consider the quotient and remainder of $\phi(n)$ divided by m.

Problem 9.66.

The general version of the Chinese Remainder Theorem (see Problem 9.61) extends to more than two relatively prime moduli. Namely,

Theorem (General Chinese Remainder)**.** *Suppose $a_1, \ldots, a_k$ are integers greater than 1 and each is relatively prime to the others. Let $n ::= a_1 \cdot a_2 \cdots a_k$. Then for any integers $m_1, m_2, \ldots, m_k$, there is a unique $x \in [0..n)$ such that*

$$x \equiv m_i \pmod{a_i},$$

for $1 \leq i \leq k$.

The proof is a routine induction on k using a fact that follows immediately from unique factorization: if a number is relatively prime to some other numbers, then it is relatively prime to their product.

The General Chinese Remainder Theorem is the basis for an efficient approach to performing a long series of additions and multiplications on "large" numbers.

Namely, suppose n was large, but each of the factors a_i was small enough to be handled by cheap and available arithmetic hardware units. Suppose a calculation requiring many additions and multiplications needs to be performed. To do a single multiplication or addition of two large numbers x and y in the usual way in this setting would involve breaking up the x and y into pieces small enough to be handled by the arithmetic units, using the arithmetic units to perform additions and multiplications on (many) pairs of small pieces, and then reassembling the pieces into an answer. Moreover, the order in which these operations on pieces can be performed is contrained by dependence among the pieces—because of "carries,"

for example. And this process of breakup and reassembly has to be performed for each addition and multiplication that needs to be performed on large numbers.

Explain how the General Chinese Remainder Theorem can be applied to perform a long series of additions and multiplications on "large" numbers much more efficiently than the usual way described above.

Problem 9.67.

In this problem we'll prove that for all integers a, m where $m > 1$,

$$a^m \equiv a^{m-\phi(m)} \pmod{m}. \tag{9.34}$$

Note that a and m need not be relatively prime.

Assume $m = p_1^{k_1} \cdots p_n^{k_n}$ for distinct primes, $p_1, \ldots, p_n$ and positive integers $k_1, \ldots, k_n$.

(a) Show that if p_i does not divide a, then

$$a^{\phi(m)} \equiv 1 \pmod{p_i^{k_i}}.$$

(b) Show that if $p_i \mid a$ then

$$a^{m-\phi(m)} \equiv 0 \pmod{p_i^{k_i}}. \tag{9.35}$$

(c) Conclude (9.34) from the facts above.

Hint: $a^m - a^{m-\phi(m)} = a^{m-\phi(m)}(a^{\phi(m)} - 1)$.

Problem 9.68.

The Generalized Postage Problem

Several other problems (2.7, 2.1, 5.32) work out which amounts of postage can be formed using two stamps of given denominations. In this problem, we generalize this to two stamps with arbitrary positive integer denominations a and b cents. Let's call an amount of postage that can be made from a and b cent stamps a *makeable* amount.

Lemma. *(Generalized Postage) If a and b are relatively prime positive integers, then any integer greater than $ab - a - b$ is makeable.*

To prove the Lemma, consider the following array with a infinite rows:

$$\begin{array}{rrrrl} 0 & a & 2a & 3a & \ldots \\ b & b+a & b+2a & b+3a & \ldots \\ 2b & 2b+a & 2b+2a & 2b+3a & \ldots \\ 3b & 3b+a & 3b+2a & 3b+3a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots \\ (a-1)b & (a-1)b+a & (a-1)b+2a & (a-1)b+3a & \ldots \end{array}$$

Note that every element in this array is clearly makeable.

(a) Suppose that n is at least as large as, and also congruent mod a to, the first element in some row of this array. Explain why n must appear in the array.

(b) Prove that every integer from 0 to $a-1$ is congruent modulo a to one of the integers in the first column of this array.

(c) Complete the proof of the Generalized Postage Lemma by using parts (a) and (b) to conclude that every integer $n > ab-a-b$ appears in the array, and hence is makeable.

Hint: Suppose n is congruent mod a to the first element in some row. Assume n is less than that element, and then show that $n \leq ab-a-b$.

(d) (Optional) What's more, $ab-a-b$ is not makeable. Prove it.

(e) Explain why the following even more general lemma follows directly from the Generalized Lemma and part (d).

Lemma. *(Generalized[2] Postage) If m and n are positive integers and $g ::= \gcd(m,n) > 1$, then with m and n cent stamps, you can only make amounts of postage that are multiples of g. You can actually make any amount of postage greater than $(mn/g)-m-n$ that is a multiple of g, but you cannot make $(mn/g)-m-n$ cents postage.*

(f) Optional and possibly unknown. Suppose you have three denominations of stamps, a, b, c and $\gcd(a,b,c) = 1$. Give a formula for the smallest number n_{abc} such that you can make every amount of postage $\geq n_{abc}$.

Exam Problems

Problem 9.69.

What is the remainder of 63^{9601} divided by 220?

Problem 9.70.
Prove that if k_1 and k_2 are relatively prime to n, then so is $k_1 \cdot_n k_2$,

(a) ...using the fact that k is relatively prime to n iff k has an inverse modulo n.

Hint: Recall that $k_1 k_2 \equiv k_1 \cdot_n k_2 \pmod{n}$.

(b) ...using the fact that k is relatively prime to n iff k is cancellable modulo n.

(c) ...using the Unique Factorization Theorem and the basic GCD properties such as Lemma 9.2.1.

Problem 9.71.

Circle **true** or **false** for the statements below, and *provide counterexamples* for those that are **false**. Variables, a, b, c, m, n range over the integers and $m, n > 1$.

(a) $\gcd(1+a, 1+b) = 1 + \gcd(a, b)$. **true** **false**

(b) If $a \equiv b \pmod{n}$, then $p(a) \equiv p(b) \pmod{n}$ for any polynomial $p(x)$ with integer coefficients. **true** **false**

(c) If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$. **true** **false**

(d) $\gcd(a^n, b^n) = (\gcd(a, b))^n$ **true** **false**

(e) If $\gcd(a, b) \neq 1$ and $\gcd(b, c) \neq 1$, then $\gcd(a, c) \neq 1$. **true** **false**

(f) If an integer linear combination of a and b equals 1, then so does some integer linear combination of a^2 and b^2. **true** **false**

(g) If no integer linear combination of a and b equals 2, then neither does any integer linear combination of a^2 and b^2. **true** **false**

(h) If $ac \equiv bc \pmod{n}$ and n does not divide c, then $a \equiv b \pmod{n}$. **true** **false**

(i) Assuming a, b have inverses modulo n, if $a^{-1} \equiv b^{-1} \pmod{n}$, then $a \equiv b \pmod{n}$. **true** **false**

(j) If $ac \equiv bc \pmod{n}$ and n does not divide c, then $a \equiv b \pmod{n}$. **true** **false**

(k) If $a \equiv b \pmod{\phi(n)}$ for $a, b > 0$, then $c^a \equiv c^b \pmod{n}$. **true** **false**

(l) If $a \equiv b \pmod{nm}$, then $a \equiv b \pmod{n}$. **true** **false**

(m) If $\gcd(m, n) = 1$, then
$[a \equiv b \pmod{m}$ AND $a \equiv b \pmod{n}]$ iff $[a \equiv b \pmod{mn}]$ **true** **false**

(n) If $\gcd(a, n) = 1$, then $a^{n-1} \equiv 1 \pmod{n}$ **true** **false**

(o) If $a, b > 1$, then
[a has a inverse mod b iff b has an inverse mod a]. **true** **false**

Problem 9.72.

Parts (a) to (j) offer a collection of number theoretic assertions. Several of the assertions require some side-conditions to be correct. A selection of possible side-conditions (**i**)–(**viii**) are listed below.

Next to each of assertions, write **True** if the assertion is correct as stated. Write **False** if none of the side-conditions ensure that the assertion is correct. Otherwise, write the number of a side-condition that ensures the assertion will be true; to get full credit your side-condition should not imply any other correct side-condition. For example, you would only get part credit for choosing $\gcd(a, b) > 2$ as a side-condition if $\gcd(a, b) > 1$ was all that was needed.

Assume all variables have appropriate integer values—for example, in the context of $\mathbb{Z}_k$, numbers are in $[0..k)$ and $k > 1$.

(i). NOT(a divides b)	**(ii)**. $\gcd(a, b) = 1$	**(iii)**. $\gcd(a, b) > 1$
(iv). $\gcd(a, b) > 2$	**(v)**. $\gcd(a, b) = 1 = \gcd(a, c)$	
(vi). a is prime	**(vii)**. b is prime	**(viii)**. a, b both prime

(a) If $a = b$ $(\mathbb{Z}_n)$, then $p(a) = p(b)$ $(\mathbb{Z}_n)$,
for any polynomial $p(x)$ with integer coefficients.

(b) If $a \mid bc$, then $a \mid c$.

(c) $\gcd(1 + a, 1 + b) = 1 + \gcd(a, b)$.

(d) No integer linear combination of a^2 and b^2 equals one.

(e) No integer linear combination of a^2 and b^2 equals two.

(f) If $ma = na$ $(\mathbb{Z}_b)$ then $m = n$ $(\mathbb{Z}_b)$.

(g) If $b^{-1} = c^{-1}$ $(\mathbb{Z}_a^*)$, then $b = c$ $(\mathbb{Z}_a^*)$.

(h) If $m = n$ $(\mathbb{Z}_{\phi(a)})$, then $b^m = b^n$ $(\mathbb{Z}_a)$.

(i) If $m = n$ $(\mathbb{Z}_{ab})$, then $m = n$ $(\mathbb{Z}_b)$.

(j) $a^b = a$ $(\mathbb{Z}_b)$.

Problem 9.73.
Indicate whether the following statements are **True** or **False**. For each of the false statements, **give counterexamples**. All variables range over the integers, $\mathbb{Z}$.

(a) For all a and b, there are x and y such that: $ax + by = 1$.

(b) $\gcd(mb + r, b) = \gcd(r, b)$ for all m, r and b.

(c) $k^{p-1} \equiv 1 \pmod{p}$ for every prime p and every k.

(d) For primes $p \neq q$, $\phi(pq) = (p-1)(q-1)$, where ϕ is Euler's totient function.

(e) If a and b are relatively prime to d, then

$$[ac \equiv bc \bmod d] \quad \text{IMPLIES} \quad [a \equiv b \bmod d].$$

Problem 9.74.
Find an integer $k > 1$ such that n and n^k agree in their last three digits whenever n is divisible by neither 2 nor 5. *Hint:* Euler's theorem.

Problem 9.75.

(a) Explain why $(-12)^{482}$ has a multiplicative inverse modulo 175.

(b) What is the value of $\phi(175)$, where ϕ is Euler's function?

(c) Call a number from 0 to 174 *powerful* iff some positive power of the number is congruent to 1 modulo 175. What is the probability that a random number from 0 to 174 is powerful?

(d) What is the remainder of $(-12)^{482}$ divided by 175?

Problem 9.76. (a) Calculate the remainder of 35^{86} divided by 29.

(b) Part (a) implies that the remainder of 35^{86} divided by 29 is not equal to 1. So there there must be a mistake in the following proof, where all the congruences are taken with modulus 29:

$$\begin{aligned} 1 &\not\equiv 35^{86} && \text{(by part (a))} && (9.36)\\ &\equiv 6^{86} && \text{(since } 35 \equiv 6 \pmod{29}\text{)} && (9.37)\\ &\equiv 6^{28} && \text{(since } 86 \equiv 28 \pmod{29}\text{)} && (9.38)\\ &\equiv 1 && \text{(by Fermat's Little Theorem)} && (9.39) \end{aligned}$$

Identify the exact line containing the mistake and explain the logical error.

Problem 9.77. (a) Show that if $p \mid n$ for some prime p and integer $n > 0$, then $(p-1) \mid \phi(n)$.

(b) Conclude that $\phi(n)$ is even for all $n > 2$.

Problem 9.78. (a) Calculate the value of $\phi(6042)$.

Hint: 53 is a factor of 6042.

(b) Consider an integer $k > 0$ that is relatively prime to 6042. Explain why $k^{9361} \equiv k \pmod{6042}$.

Hint: Use your solution to part (a).

Problem 9.79.
Let

$$S_k = 1^k + 2^k + \cdots + p^k,$$

where p is an odd prime and k is a positive multiple of $p-1$. Find $a \in [0..p)$ and $b \in ((-p)..0]$ such that

$$S_k \equiv a \equiv b \pmod{p}.$$

Problems for Section 9.11

Practice Problems

Problem 9.80.
Suppose a cracker knew how to factor the RSA modulus n into the product of distinct primes p and q. Explain how the cracker could use the public key-pair (e, n) to find a private key-pair (d, n) that would allow him to read any message encrypted with the public key.

Problem 9.81.
Suppose the RSA modulus $n = pq$ is the product of distinct 200 digit primes p and q. A message $m \in [0..n)$ is called *dangerous* if $\gcd(m, n) = p$, because such an m can be used to factor n and so crack RSA. Circle the best estimate of the fraction of messages in $[0..n)$ that are dangerous.

$$\frac{1}{200} \quad \frac{1}{400} \quad \frac{1}{200^{10}} \quad \frac{1}{10^{200}} \quad \frac{1}{400^{10}} \quad \frac{1}{10^{400}}$$

Problem 9.82.
Ben Bitdiddle decided to encrypt all his data using RSA. Unfortunately, he lost his private key. He has been looking for it all night, and suddenly a genie emerges from his lamp. He offers Ben a quantum computer that can perform exactly one procedure on large numbers e, d, n. Which of the following procedures should Ben choose to recover his data?

- Find $\gcd(e, d)$.
- Find the prime factorization of n.
- Determine whether n is prime.
- Find $\text{rem}(e^d, n)$.
- Find the inverse of e modulo n (the inverse of e in $\mathbb{Z}_n$).
- Find the inverse of e modulo $\phi(n)$.

Class Problems

Problem 9.83.
Let's try out RSA!

(a) Go through the **beforehand** steps.

- Choose primes p and q to be relatively small, say in the range 10–40. In practice, p and q might contain hundreds of digits, but small numbers are easier to handle with pencil and paper.
- Try $e = 3, 5, 7, \ldots$ until you find something that works. Use Euclid's algorithm to compute the gcd.
- Find d (using the Pulverizer).

When you're done, put your public key on the board prominentally labelled "Public Key." This lets another team send you a message.

(b) Now send an encrypted message to another team using their public key. Select your message m from the codebook below:

- 2 = Greetings and salutations!
- 3 = Yo, wassup?
- 4 = You guys are slow!
- 5 = All your base are belong to us.
- 6 = Someone on *our* team thinks someone on *your* team is kinda cute.
- 7 = You *are* the weakest link. Goodbye.

(c) Decrypt the message sent to you and verify that you received what the other team sent!

Problem 9.84. (a) Just as RSA would be trivial to crack knowing the factorization into two primes of n in the public key, explain why RSA would also be trivial to crack knowing $\phi(n)$.

(b) Show that if you knew n, $\phi(n)$, and that n was the product of two primes, then you could easily factor n.

Problem 9.85.
A critical fact about RSA is, of course, that decrypting an encrypted message always gives back the original message m. Namely, if $n = pq$ where p and q are distinct primes, $m \in [0..pq)$, and

$$d \cdot e \equiv 1 \pmod{(p-1)(q-1)},$$

then

$$\widehat{m}^d ::= \left(m^e\right)^d = m \ (\mathbb{Z}_n). \tag{9.40}$$

We'll now prove this.

(a) Explain why (9.40) follows very simply from Euler's theorem when m is *relatively prime to* n.

All the rest of this problem is about removing the restriction that m be relatively prime to n. That is, we aim to prove that equation (9.40) holds for *all* $m \in [0..n)$.

It is important to realize that there is no practical reason to worry about—or to bother to check for—this relative primality condition before sending a message m using RSA. That's because the whole RSA enterprise is predicated on the difficulty of factoring. If an m ever came up that wasn't relatively prime to n, then we could factor n by computing $\gcd(m, n)$. So believing in the security of RSA implies believing that the liklihood of a message m turning up that was not relatively prime to n is negligible.

But let's be pure, impractical mathematicians and get rid of this technically unnecessary relative primality side condition, even if it is harmless. One gain for doing this is that statements about RSA will be simpler without the side condition. More important, the proof below illustrates a useful general method of proving things about a number n by proving them separately for the prime factors of n.

(b) Prove that if p is prime and $a \equiv 1 \pmod{p-1}$, then

$$m^a = m \ (\mathbb{Z}_p). \tag{9.41}$$

(c) Give an elementary proof[22] that if $a \equiv b \pmod{p_i}$ for distinct primes p_i, then $a \equiv b$ modulo the product of these primes.

(d) Note that (9.40) is a special case of

Claim. *If n is a product of distinct primes and $a \equiv 1 \pmod{\phi(n)}$, then*

$$m^a = m \ (\mathbb{Z}_n).$$

Use the previous parts to prove the Claim.

[22] There is no need to appeal to the Chinese Remainder Theorem.

Homework Problems

Problem 9.86.

Although RSA has successfully withstood cryptographic attacks for a more than a quarter century, it is not known that breaking RSA would imply that factoring is easy.

In this problem we will examine the *Rabin cryptosystem* that does have such a security certification. Namely, if someone has the ability to break the Rabin cryptosystem efficiently, then they also have the ability to factor numbers that are products of two primes.

Why should that convince us that it is hard to break the cryptosystem efficiently? Well, mathematicians have been trying to factor efficiently for centuries, and they still haven't figured out how to do it.

What is the Rabin cryptosystem? The public key will be a number N that is a product of two very large primes p, q such that $p \equiv q \equiv 3 \pmod 4$. To send the message m, send $\text{rem}(m^2, N)$.[23]

The private key is the factorization of N, namely, the primes p, q. We need to show that if the person being sent the message knows p, q, then they can decode the message. On the other hand, if an eavesdropper who doesn't know p, q listens in, then we must show that they are very unlikely to figure out this message.

Say that s is a *square modulo* N if there is an $m \in [0, N)$ such that $s \equiv m^2 \pmod N$. Such an m is a *square root of* s *modulo* N.

(a) What are the squares modulo 5? For each square in the interval $[0, 5)$, how many square roots does it have?

(b) For each integer in $[1..15)$ that is relatively prime to 15, how many square roots (modulo 15) does it have? Note that all the square roots are *also* relatively prime to 15. We won't go through why this is so here, but keep in mind that this is a general phenomenon!

(c) Suppose that p is a prime such that $p \equiv 3 \pmod 4$. It turns out that squares modulo p have exactly 2 square roots. First show that $(p + 1)/4$ is an integer. Next figure out the two square roots of 1 modulo p. Then show that you can find a "square root mod a prime p" of a number by raising the number to the $(p + 1)/4$th power. That is, given s, to find m such that $s \equiv m^2 \pmod p$, you can compute $\text{rem}(s^{(p+1)/4}, p)$.

(d) The Chinese Remainder Theorem (Problem 9.61) implies that if p, q are dis-

[23] We will see soon, that there are other numbers that would be encrypted by $\text{rem}(m^2, N)$, so we'll have to disallow those other numbers as possible messages in order to make it possible to decode this cryptosystem, but let's ignore that for now.

tinct primes, then s is a square modulo pq if and only if s is a square modulo p and s is a square modulo q. In particular, if $s \equiv x^2 \equiv (x')^2 \pmod{p}$ where $x \neq x'$, and likewise $s \equiv y^2 \equiv (y')^2 \pmod{q}$ then s has exactly four square roots modulo N, namely,

$$s \equiv (xy)^2 \equiv (x'y)^2 \equiv (xy')^2 \equiv (x'y')^2 \pmod{pq}.$$

So, if you know p, q, then using the solution to part (c), you can efficiently find the square roots of s! Thus, given the private key, decoding is easy.

But what if you don't know p, q?

Let's assume that the evil message interceptor claims to have a program that can find all four square roots of any number modulo N. Show that he can actually use this program to efficiently find the factorization of N. Thus, unless this evil message interceptor is extremely smart and has figured out something that the rest of the scientific community has been working on for years, it is very unlikely that this efficient square root program exists!

Hint: Pick r arbitrarily from $[1..N)$. If $\gcd(N, r) > 1$, then you are done (why?) so you can halt. Otherwise, use the program to find all four square roots of r, call them $r, -r, r', -r'$. Note that $r^2 \equiv r'^2 \pmod{N}$. How can you use these roots to factor N?

(e) If the evil message interceptor knows that the message is the encoding one of two possible candidate messages (that is, either "meet at dome at dusk" or "meet at dome at dawn") and is just trying to figure out which of the two, then can he break this cryptosystem?

Problem 9.87.

You've seen how the RSA encryption scheme works, but why is it hard to break? In this problem, you will see that finding private keys is as hard as finding the prime factorizations of integers. Since there is a general consensus in the crypto community (enough to persuade many large financial institutions, for example) that factoring numbers with a few hundred digits requires astronomical computing resources, we can therefore be sure it will take the same kind of overwhelming effort to find RSA private keys of a few hundred digits. This means we can be confident the private RSA keys are not somehow revealed by the public keys[24].

[24] This is a very weak kind of "security" property, because it doesn't even rule out the possibility of deciphering RSA encoded messages by some method that did not require knowing the private key. Nevertheless, over twenty years experience supports the security of RSA in practice.

For this problem, assume that $n = p \cdot q$ where p, q are both *odd* primes and that e is the public key and d the private key of the RSA protocol.. Let $c ::= e \cdot d - 1$.

(a) Show that $\phi(n)$ divides c.

(b) Conclude that 4 divides c.

(c) Show that if $\gcd(r, n) = 1$, then $r^c \equiv 1 \pmod{n}$.

A *square root* of m modulo n is an integer $s \in [0.n)$ such that $s^2 \equiv m \pmod{n}$. Here is a nice fact to know: when n is a product of two odd primes, then every number m such that $\gcd(m, n) = 1$ has 4 square roots modulo n.

In particular, the number 1 has four square roots modulo n. The two trivial ones are 1 and $n - 1$ (which is $\equiv -1 \pmod{n}$). The other two are called the *nontrivial* square roots of 1.

(d) Since you know c, then for any integer r you can also compute the remainder y of $r^{c/2}$ divided by n. So $y^2 \equiv r^c \pmod{n}$. Now if r is relatively prime to n, then y will be a square root of 1 modulo n by part (c).

Show that if y turns out to be a *nontrivial* root of 1 modulo n, then you can factor n. *Hint:* From the fact that $y^2 - 1 = (y + 1)(y - 1)$, show that $y + 1$ must be divisible by exactly one of q and p.

(e) It turns out that at least half the positive integers $r < n$ that are relatively prime to n will yield y's in part (d) that are nontrivial roots of 1. Conclude that if, in addition to n and the public key e you also knew the private key d, then you can be sure of being able to factor n.

Exam Problems

Problem 9.88.

Suppose Alice and Bob are using the RSA cryptosystem to send secure messages. Each of them has a public key visible to everyone and a private key known only to themselves, and using RSA in the usual way, they are able to send secret messages to each other over public channels.

But a concern for Bob is how he knows that a message he gets is actually from Alice—as opposed to some imposter claiming to be Alice. This concern can be met by using RSA to add unforgeable "signatures" to messages. To send a message m to Bob with an unforgeable signature, Alice uses RSA encryption on her message m, but instead using Bob's public key to encrypt m, she uses her own *private* key to obtain a message m_1. She then sends m_1 as her "signed" message to Bob.

(a) Explain how Bob can read the original message m from Alice's signed message m_1. (Let (n_A, e_A) be Alice's public key and d_A her private key. Assume

$m \in [0..n_A)$.)

(b) Briefly explain why Bob can be confident, assuming RSA is secure, that m_1 came from Alice rather than some imposter.

(c) Notice that not only Bob, but *anyone* can use Alice's public key to reconstruct her message m from its signed version m_1. So how can Alice send a secret signed message to Bob over public channels?