15 Cardinality Rules

15.1 Counting One Thing by Counting Another

How do you count the number of people in a crowded room? You could count heads, since for each person there is exactly one head. Alternatively, you could count ears and divide by two. Of course, you might have to adjust the calculation if someone lost an ear in a pirate raid or someone was born with three ears. The point here is that you can often *count one thing by counting another*, though some fudging may be required. This is a central theme of counting, from the easiest problems to the hardest. In fact, we've already seen this technique used in Theorem 4.5.5, where the number of subsets of an n-element set was proved to be the same as the number of length-n bit-strings, by describing a bijection between the subsets and the bit-strings.

The most direct way to count one thing by counting another is to find a bijection between them, since if there is a bijection between two sets, then the sets have the same size. This important fact is commonly known as the *Bijection Rule*. We've already seen it as the Mapping Rules bijective case (4.7).

15.1.1 The Bijection Rule

The Bijection Rule acts as a magnifier of counting ability; if you figure out the size of one set, then you can immediately determine the sizes of many other sets via bijections. For example, let's look at the two sets mentioned at the beginning of Part III:

$$\begin{aligned} A &= \text{all ways to select a dozen donuts when five varieties are available} \\ B &= \text{all 16-bit sequences with exactly 4 ones} \end{aligned}$$

An example of an element of set A is:

$$\underbrace{0\;0}_{\text{chocolate}} \qquad \underbrace{\qquad}_{\text{lemon-filled}} \qquad \underbrace{0\;0\;0\;0\;0\;0}_{\text{sugar}} \qquad \underbrace{0\;0}_{\text{glazed}} \qquad \underbrace{0\;0}_{\text{plain}}$$

Here, we've depicted each donut with a 0 and left a gap between the different varieties. Thus, the selection above contains two chocolate donuts, no lemon-filled, six sugar, two glazed, and two plain. Now let's put a 1 into each of the four gaps:

$$\underbrace{0\;0}_{\text{chocolate}} \quad 1 \quad \underbrace{\qquad}_{\text{lemon-filled}} \quad 1 \quad \underbrace{0\;0\;0\;0\;0\;0}_{\text{sugar}} \quad 1 \quad \underbrace{0\;0}_{\text{glazed}} \quad 1 \quad \underbrace{0\;0}_{\text{plain}}$$

and close up the gaps:

$$0011000000100100\,.$$

We've just formed a 16-bit number with exactly 4 ones—an element of B!

This example suggests a bijection from set A to set B: map a dozen donuts consisting of:

$$c \text{ chocolate, } l \text{ lemon-filled, } s \text{ sugar, } g \text{ glazed, and } p \text{ plain}$$

to the sequence:

$$\underbrace{0\ldots0}_{c} \quad 1 \quad \underbrace{0\ldots0}_{l} \quad 1 \quad \underbrace{0\ldots0}_{s} \quad 1 \quad \underbrace{0\ldots0}_{g} \quad 1 \quad \underbrace{0\ldots0}_{p}$$

The resulting sequence always has 16 bits and exactly 4 ones, and thus is an element of B. Moreover, the mapping is a bijection: every such bit sequence comes from exactly one order of a dozen donuts. Therefore, $|A| = |B|$ by the Bijection Rule. More generally,

Lemma 15.1.1. *The number of ways to select n donuts when k flavors are available is the same as the number of binary sequences with exactly n zeroes and $k-1$ ones.*

This example demonstrates the power of the bijection rule. We managed to prove that two very different sets are actually the same size—even though we don't know exactly how big either one is. But as soon as we figure out the size of one set, we'll immediately know the size of the other.

This particular bijection might seem frighteningly ingenious if you've not seen it before. But you'll use essentially this same argument over and over, and soon you'll consider it routine.

15.2 Counting Sequences

The Bijection Rule lets us count one thing by counting another. This suggests a general strategy: get really good at counting just a few things, then use bijections to count everything else! This is the strategy we'll follow. In particular, we'll get really good at counting *sequences*. When we want to determine the size of some other set T, we'll find a bijection from T to a set of sequences S. Then we'll use our super-ninja sequence-counting skills to determine $|S|$, which immediately gives us $|T|$. We'll need to hone this idea somewhat as we go along, but that's pretty much it!

15.2.1 The Product Rule

The *Product Rule* gives the size of a product of sets. Recall that if $P_1, P_2, \ldots, P_n$ are sets, then

$$P_1 \times P_2 \times \cdots \times P_n$$

is the set of all sequences whose first term is drawn from P_1, second term is drawn from P_2 and so forth.

Rule 15.2.1 (Product Rule)**.** *If $P_1, P_2, \ldots P_n$ are finite sets, then:*

$$|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdots |P_n|$$

For example, suppose a *daily diet* consists of a breakfast selected from set B, a lunch from set L, and a dinner from set D where:

$$\begin{aligned} B &= \{\text{pancakes, bacon and eggs, bagel, Doritos}\} \\ L &= \{\text{burger and fries, garden salad, Doritos}\} \\ D &= \{\text{macaroni, pizza, frozen burrito, pasta, Doritos}\} \end{aligned}$$

Then $B \times L \times D$ is the set of all possible daily diets. Here are some sample elements:

$$\begin{gathered} (\text{pancakes, burger and fries, pizza}) \\ (\text{bacon and eggs, garden salad, pasta}) \\ (\text{Doritos, Doritos, frozen burrito}) \end{gathered}$$

The Product Rule tells us how many different daily diets are possible:

$$\begin{aligned} |B \times L \times D| &= |B| \cdot |L| \cdot |D| \\ &= 4 \cdot 3 \cdot 5 \\ &= 60. \end{aligned}$$

15.2.2 Subsets of an n-element Set

The fact that there are 2^n subsets of an n-element set was proved in Theorem 4.5.5 by setting up a bijection between the subsets and the length-n bit-strings. So the original problem about subsets was tranformed into a question about sequences—*exactly according to plan*! Now we can fill in the missing explanation of why there are 2^n length-n bit-strings: we can write the set of all n-bit sequences as a product of sets:

$$\{0,1\}^n ::= \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n \text{ terms}}.$$

Then Product Rule gives the answer:

$$|\{0,1\}^n| = |\{0,1\}|^n = 2^n.$$

15.2.3 The Sum Rule

Bart allocates his little sister Lisa a quota of 20 crabby days, 40 irritable days, and 60 generally surly days. On how many days can Lisa be out-of-sorts one way or another? Let set C be her crabby days, I be her irritable days, and S be the generally surly. In these terms, the answer to the question is $|C \cup I \cup S|$. Now assuming that she is permitted at most one bad quality each day, the size of this union of sets is given by the *Sum Rule*:

Rule 15.2.2 (Sum Rule)**.** *If $A_1, A_2, \ldots, A_n$ are* disjoint *sets, then:*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

Thus, according to Bart's budget, Lisa can be out-of-sorts for:

$$\begin{aligned} |C \cup I \cup S| &= |C| + |I| + |S| \\ &= 20 + 40 + 60 \\ &= 120 \text{ days} \end{aligned}$$

Notice that the Sum Rule holds only for a union of *disjoint* sets. Finding the size of a union of overlapping sets is a more complicated problem that we'll take up in Section 15.9.

15.2.4 Counting Passwords

Few counting problems can be solved with a single rule. More often, a solution is a flurry of sums, products, bijections, and other methods.

For solving problems involving passwords, telephone numbers, and license plates, the sum and product rules are useful together. For example, on a certain computer system, a valid password is a sequence of between six and eight symbols. The first symbol must be a letter (which can be lowercase or uppercase), and the remaining symbols must be either letters or digits. How many different passwords are possible?

Let's define two sets, corresponding to valid symbols in the first and subsequent positions in the password.

$$\begin{aligned} F &= \{a, b, \ldots, z, A, B, \ldots, Z\} \\ S &= \{a, b, \ldots, z, A, B, \ldots, Z, 0, 1, \ldots, 9\} \end{aligned}$$

In these terms, the set of all possible passwords is:[1]

$$(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)$$

[1] The notation S^5 means $S \times S \times S \times S \times S$.

Thus, the length-six passwords are in the set $F \times S^5$, the length-seven passwords are in $F \times S^6$, and the length-eight passwords are in $F \times S^7$. Since these sets are disjoint, we can apply the Sum Rule and count the total number of possible passwords as follows:

$$
\begin{aligned}
&|(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)| \\
&\qquad = |F \times S^5| + |F \times S^6| + |F \times S^7| && \text{Sum Rule} \\
&\qquad = |F| \cdot |S|^5 + |F| \cdot |S|^6 + |F| \cdot |S|^7 && \text{Product Rule} \\
&\qquad = 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7 \\
&\qquad \approx 1.8 \cdot 10^{14} \text{ different passwords.}
\end{aligned}
$$

15.3 The Generalized Product Rule

In how many ways can, say, a Nobel prize, a Japan prize, and a Pulitzer prize be awarded to n people? This is easy to answer using our strategy of translating the problem about awards into a problem about sequences. Let P be the set of n people taking the course. Then there is a bijection from ways of awarding the three prizes to the set $P^3 ::= P \times P \times P$. In particular, the assignment:

"Barack wins a Nobel, George wins a Japan, and Bill wins a Pulitzer prize"

maps to the sequence (Barack, George, Bill). By the Product Rule, we have $|P^3| = |P|^3 = n^3$, so there are n^3 ways to award the prizes to a class of n people. Notice that P^3 includes triples like (Barack, Bill, Barack) where one person wins more than one prize.

But what if the three prizes must be awarded to *different* students? As before, we could map the assignment to the triple (Bill, George, Barack) $\in P^3$. But this function is *no longer a bijection*. For example, no valid assignment maps to the triple (Barack, Bill, Barack) because now we're not allowing Barack to receive two prizes. However, there *is* a bijection from prize assignments to the set:

$$S = \{(x, y, z) \in P^3 \mid x,\ y \text{ and } z \text{ are different people}\}$$

This reduces the original problem to a problem of counting sequences. Unfortunately, the Product Rule does not apply directly to counting sequences of this type because the entries depend on one another; in particular, they must all be different. However, a slightly sharper tool does the trick.

Prizes for *truly exceptional* Coursework

Given everyone’s hard work on this material, the instructors considered awarding some prizes for truly exceptional coursework. Here are three possible prize categories:

Best Administrative Critique We asserted that the quiz was closed-book. On the cover page, one strong candidate for this award wrote, “There is no book.”

Awkward Question Award “Okay, the left sock, right sock, and pants are in an antichain, but how—even with assistance—could I put on all three at once?”

Best Collaboration Statement Inspired by a student who wrote “I worked alone” on Quiz 1.

Rule 15.3.1 (Generalized Product Rule)**.** *Let S be a set of length-k sequences. If there are:*

- *n_1 possible first entries,*
- *n_2 possible second entries for each first entry,*

  $\vdots$
- *n_k possible kth entries for each sequence of first $k-1$ entries,*

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

In the awards example, S consists of sequences (x, y, z). There are n ways to choose x, the recipient of prize #1. For each of these, there are $n-1$ ways to choose y, the recipient of prize #2, since everyone except for person x is eligible. For each combination of x and y, there are $n-2$ ways to choose z, the recipient of prize #3, because everyone except for x and y is eligible. Thus, according to the Generalized Product Rule, there are

$$|S| = n \cdot (n-1) \cdot (n-2)$$

ways to award the 3 prizes to different people.

15.3.1 Defective Dollar Bills

A dollar bill is *defective* if some digit appears more than once in the 8-digit serial number. If you check your wallet, you'll be sad to discover that defective bills are all-too-common. In fact, how common are *nondefective* bills? Assuming that the digit portions of serial numbers all occur equally often, we could answer this question by computing

$$\text{fraction of nondefective bills} = \frac{|\{\text{serial \#'s with all digits different}\}|}{|\{\text{serial numbers}\}|}. \tag{15.1}$$

Let's first consider the denominator. Here there are no restrictions; there are 10 possible first digits, 10 possible second digits, 10 third digits, and so on. Thus, the total number of 8-digit serial numbers is 10^8 by the Product Rule.

Next, let's turn to the numerator. Now we're not permitted to use any digit twice. So there are still 10 possible first digits, but only 9 possible second digits, 8 possible third digits, and so forth. Thus, by the Generalized Product Rule, there are

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = \frac{10!}{2} = 1{,}814{,}400$$

serial numbers with all digits different. Plugging these results into Equation 15.1, we find:

$$\text{fraction of nondefective bills} = \frac{1{,}814{,}400}{100{,}000{,}000} = 1.8144\%$$

15.3.2 A Chess Problem

In how many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column? A valid configuration is shown in Figure 15.1(a), and an invalid configuration is shown in Figure 15.1(b).

First, we map this problem about chess pieces to a question about sequences. There is a bijection from configurations to sequences

$$(r_P, c_P, r_N, c_N, r_B, c_B)$$

where r_P, r_N and r_B are distinct rows and c_P, c_N and c_B are distinct columns. In particular, r_P is the pawn's row c_P is the pawn's column r_N is the knight's row, etc. Now we can count the number of such sequences using the Generalized Product Rule:

- r_P is one of 8 rows

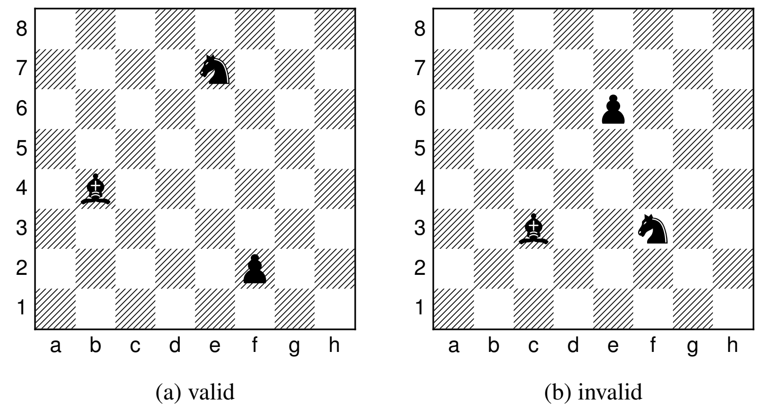


Figure 15.1 Two ways of placing a pawn (♙), a knight (♘), and a bishop (♗) on a chessboard. The configuration shown in (b) is invalid because the bishop and the knight are in the same row.

- c_P is one of 8 columns
- r_N is one of 7 rows (any one but r_P)
- c_N is one of 7 columns (any one but c_P)
- r_B is one of 6 rows (any one but r_P or r_N)
- c_B is one of 6 columns (any one but c_P or c_N)

Thus, the total number of configurations is $(8 \cdot 7 \cdot 6)^2$.

15.3.3 Permutations

A *permutation* of a set S is a sequence that contains every element of S exactly once. For example, here are all the permutations of the set $\{a, b, c\}$:

$$\begin{array}{ccc} (a,b,c) & (a,c,b) & (b,a,c) \\ (b,c,a) & (c,a,b) & (c,b,a) \end{array}$$

How many permutations of an n-element set are there? Well, there are n choices for the first element. For each of these, there are $n-1$ remaining choices for the second element. For every combination of the first two elements, there are $n-2$ ways to choose the third element, and so forth. Thus, there are a total of

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

permutations of an n-element set. In particular, this formula says that there are

$3! = 6$ permutations of the 3-element set $\{a, b, c\}$, which is the number we found above.

Permutations will come up again in this course approximately 1.6 bazillion times. In fact, permutations are the reason why factorial comes up so often and why we taught you Stirling's approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

15.4 The Division Rule

Counting ears and dividing by two is a silly way to count the number of people in a room, but this approach is representative of a powerful counting principle.

A *k-to-1 function* maps exactly k elements of the domain to every element of the codomain. For example, the function mapping each ear to its owner is 2-to-1. Similarly, the function mapping each finger to its owner is 10-to-1, and the function mapping each finger and toe to its owner is 20-to-1. The general rule is:

Rule 15.4.1 (Division Rule)**.** *If $f : A \to B$ is k-to-1, then $|A| = k \cdot |B|$.*

For example, suppose A is the set of ears in the room and B is the set of people. There is a 2-to-1 mapping from ears to people, so by the Division Rule, $|A| = 2 \cdot |B|$. Equivalently, $|B| = |A|/2$, expressing what we knew all along: the number of people is half the number of ears. Unlikely as it may seem, many counting problems are made much easier by initially counting every item multiple times and then correcting the answer using the Division Rule. Let's look at some examples.

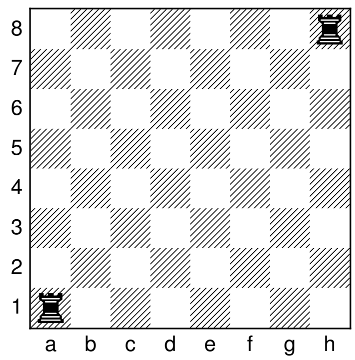
15.4.1 Another Chess Problem

In how many different ways can you place two identical rooks on a chessboard so that they do not share a row or column? A valid configuration is shown in Figure 15.2(a), and an invalid configuration is shown in Figure 15.2(b).

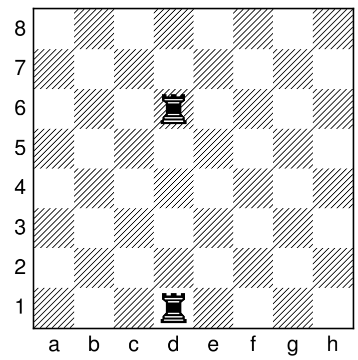
Let A be the set of all sequences

$$(r_1, c_1, r_2, c_2)$$

where r_1 and r_2 are distinct rows and c_1 and c_2 are distinct columns. Let B be the set of all valid rook configurations. There is a natural function f from set A to set B; in particular, f maps the sequence (r_1, c_1, r_2, c_2) to a configuration with one rook in row r_1, column c_1 and the other rook in row r_2, column c_2.



(a) valid



(b) invalid

Figure 15.2 Two ways to place 2 rooks (♖) on a chessboard. The configuration in (b) is invalid because the rooks are in the same column.

But now there's a snag. Consider the sequences:

$$(1, a, 8, h) \qquad \text{and} \qquad (8, h, 1, a)$$

The first sequence maps to a configuration with a rook in the lower-left corner and a rook in the upper-right corner. The second sequence maps to a configuration with a rook in the upper-right corner and a rook in the lower-left corner. The problem is that those are two different ways of describing the *same* configuration! In fact, this arrangement is shown in Figure 15.2(a).

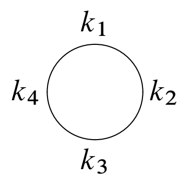
More generally, the function f maps exactly two sequences to *every* board configuration; f is a 2-to-1 function. Thus, by the quotient rule, $|A| = 2 \cdot |B|$. Rearranging terms gives:

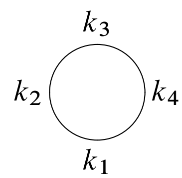
$$|B| = \frac{|A|}{2} = \frac{(8 \cdot 7)^2}{2}.$$

In the second equality, we've computed the size of A using the General Product Rule just as in the earlier chess problem.

15.4.2 Knights of the Round Table

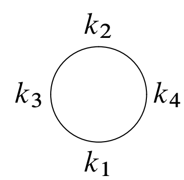
In how many ways can King Arthur arrange to seat his n different knights at his round table? A seating defines who sits where. Two seatings are considered to be the same *arrangement* if each knight sits with the same knight on his left in both seatings. An equivalent way to say this is that two seatings yield the same arrangement when they yield the same sequence of knights starting at knight number 1 and going clockwise around the table. For example, the following two seatings determine the same arrangement:





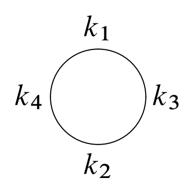
A seating is determined by the sequence of knights going clockwise around the table starting at the top seat. So seatings correspond to permutations of the knights, and there are $n!$ of them. For example,

$(k_2, k_4, k_1, k_3) \longrightarrow$



Two seatings determine the same arrangement if they are the same when the table is rotated so knight 1 is at the top seat. For example with $n = 4$, there are 4 different sequences that correspond to the seating arrangement:

$$\begin{array}{c} (k_2, k_4, k_1, k_3) \\ (k_4, k_1, k_3, k_2) \\ (k_1, k_3, k_2, k_4) \\ (k_3, k_2, k_4, k_1) \end{array} \longrightarrow$$



This mapping from seating to arrangments is actually an n-to-1 function, since all n cyclic shifts of the sequence of knights in the seating map to the same arrangement. Therefore, by the division rule, the number of circular seating arrangements is:

$$\frac{\#\,\text{seatings}}{n} = \frac{n!}{n} = (n-1)!\,.$$

15.5 Counting Subsets

How many k-element subsets of an n-element set are there? This question arises all the time in various guises:

- In how many ways can I select 5 books from my collection of 100 to bring on vacation?
- How many different 13-card bridge hands can be dealt from a 52-card deck?
- In how many ways can I select 5 toppings for my pizza if there are 14 available toppings?

This number comes up so often that there is a special notation for it:

$$\binom{n}{k} ::= \text{ the number of } k\text{-element subsets of an } n\text{-element set.}$$

The expression $\binom{n}{k}$ is read "n choose k." Now we can immediately express the answers to all three questions above:

- I can select 5 books from 100 in $\binom{100}{5}$ ways.
- There are $\binom{52}{13}$ different bridge hands.
- There are $\binom{14}{5}$ different 5-topping pizzas, if 14 toppings are available.

15.5.1 The Subset Rule

We can derive a simple formula for the n choose k number using the Division Rule. We do this by mapping any permutation of an n-element set $\{a_1, \dots, a_n\}$ into a k-element subset simply by taking the first k elements of the permutation. That is, the permutation $a_1 a_2 \dots a_n$ will map to the set $\{a_1, a_2, \dots, a_k\}$.

Notice that any other permutation with the same first k elements $a_1, \dots, a_k$ *in any order* and the same remaining elements $n - k$ elements *in any order* will also map to this set. What's more, a permutation can only map to $\{a_1, a_2, \dots, a_k\}$ if its first k elements are the elements $a_1, \dots, a_k$ in some order. Since there are $k!$ possible permutations of the first k elements and $(n-k)!$ permutations of the remaining elements, we conclude from the Product Rule that exactly $k!(n-k)!$ permutations of the n-element set map to the particular subset S. In other words, the mapping from permutations to k-element subsets is $k!(n-k)!$-to-1.

But we know there are $n!$ permutations of an n-element set, so by the Division Rule, we conclude that

$$n! = k!(n-k)!\binom{n}{k}$$

which proves:

Rule 15.5.1 (Subset Rule)**.** *The number of k-element subsets of an n-element set is*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Notice that this works even for 0-element subsets: $n!/(0!\,n!) = 1$. Here we use the fact that 0! is a *product* of 0 terms, which by convention[2] equals 1.

15.5.2 Bit Sequences

How many n-bit sequences contain exactly k ones? We've already seen the straightforward bijection between subsets of an n-element set and n-bit sequences. For example, here is a 3-element subset of $\{x_1, x_2, \ldots, x_8\}$ and the associated 8-bit sequence:

$$\begin{array}{cccccccccc} \{ & x_1, & & & x_4, & x_5 & & & & \} \\ (& 1, & 0, & 0, & 1, & 1, & 0, & 0, & 0 &) \end{array}$$

Notice that this sequence has exactly 3 ones, each corresponding to an element of the 3-element subset. More generally, the n-bit sequences corresponding to a k-element subset will have exactly k ones. So by the Bijection Rule,

Corollary 15.5.2. *The number of n-bit sequences with exactly k ones is* $\binom{n}{k}$.

Also, the bijection between selections of flavored donuts and bit sequences of Lemma 15.1.1 now implies,

Corollary 15.5.3. *The number of ways to select n donuts when $k > 0$ flavors are available is*

$$\binom{n+(k-1)}{n}.$$

15.6 Sequences with Repetitions

15.6.1 Sequences of Subsets

Choosing a k-element subset of an n-element set is the same as splitting the set into a pair of subsets: the first subset of size k and the second subset consisting of the remaining $n-k$ elements. So, the Subset Rule can be understood as a rule for counting the number of such splits into pairs of subsets.

[2] We don't use it here, but a *sum* of zero terms equals 0.

We can generalize this to a way to count splits into more than two subsets. Let A be an n-element set and $k_1, k_2, \ldots, k_m$ be nonnegative integers whose sum is n. A $(k_1, k_2, \ldots, k_m)$-*split of* A is a sequence

$$(A_1, A_2, \ldots, A_m)$$

where the A_i are disjoint subsets of A and $|A_i| = k_i$ for $i = 1, \ldots, m$.

To count the number of splits we take the same approach as for the Subset Rule. Namely, we map any permutation $a_1 a_2 \ldots a_n$ of an n-element set A into a $(k_1, k_2, \ldots, k_m)$-split by letting the 1st subset in the split be the first k_1 elements of the permutation, the 2nd subset of the split be the next k_2 elements, ..., and the mth subset of the split be the final k_m elements of the permutation. This map is a $k_1!\, k_2! \cdots k_m!$-to-1 function from the $n!$ permutations to the $(k_1, k_2, \ldots, k_m)$-splits of A, so from the Division Rule we conclude the *Subset Split Rule*:

Definition 15.6.1. For $n, k_1, \ldots, k_m \in \mathbb{N}$, such that $k_1 + k_2 + \cdots + k_m = n$, define the *multinomial coefficient*

$$\binom{n}{k_1, k_2, \ldots, k_m} ::= \frac{n!}{k_1!\, k_2! \ldots k_m!}.$$

Rule 15.6.2 (Subset Split Rule)**.** *The number of* $(k_1, k_2, \ldots, k_m)$*-splits of an* n*-element set is*

$$\binom{n}{k_1, \ldots, k_m}.$$

15.6.2 The Bookkeeper Rule

We can also generalize our count of n-bit sequences with k ones to counting sequences of n letters over an alphabet with more than two letters. For example, how many sequences can be formed by permuting the letters in the 10-letter word BOOKKEEPER?

Notice that there are 1 B, 2 O's, 2 K's, 3 E's, 1 P, and 1 R in BOOKKEEPER. This leads to a straightforward bijection between permutations of BOOKKEEPER and (1,2,2,3,1,1)-splits of $\{1, 2, \ldots, 10\}$. Namely, map a permutation to the sequence of sets of positions where each of the different letters occur.

For example, in the permutation BOOKKEEPER itself, the B is in the 1st position, the O's occur in the 2nd and 3rd positions, K's in 4th and 5th, the E's in the 6th, 7th and 9th, P in the 8th, and R is in the 10th position. So BOOKKEEPER maps to

$$(\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7, 9\}, \{8\}, \{10\}).$$

From this bijection and the Subset Split Rule, we conclude that the number of ways to rearrange the letters in the word BOOKKEEPER is:

$$\frac{\overbrace{10!}^{\text{total letters}}}{\underbrace{1!}_{\text{B's}}\ \underbrace{2!}_{\text{O's}}\ \underbrace{2!}_{\text{K's}}\ \underbrace{3!}_{\text{E's}}\ \underbrace{1!}_{\text{P's}}\ \underbrace{1!}_{\text{R's}}}$$

This example generalizes directly to an exceptionally useful counting principle which we will call the

Rule 15.6.3 (Bookkeeper Rule)**.** *Let $l_1, \dots, l_m$ be distinct elements. The number of sequences with k_1 occurrences of l_1, and k_2 occurrences of l_2, …, and k_m occurrences of l_m is*

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, \dots, k_m}.$$

For example, suppose you are planning a 20-mile walk, which should include 5 northward miles, 5 eastward miles, 5 southward miles, and 5 westward miles. How many different walks are possible?

There is a bijection between such walks and sequences with 5 N’s, 5 E’s, 5 S’s, and 5 W’s. By the Bookkeeper Rule, the number of such sequences is:

$$\frac{20!}{(5!)^4}.$$

A Word about Words

Someday you might refer to the Subset Split Rule or the Bookkeeper Rule in front of a roomful of colleagues and discover that they’re all staring back at you blankly. This is not because they’re dumb, but rather because we made up the name “Bookkeeper Rule.” However, the rule is excellent and the name is apt, so we suggest that you play through: “You know? The Bookkeeper Rule? Don’t you guys know *anything?*”

The Bookkeeper Rule is sometimes called the “formula for permutations with indistinguishable objects.” The size k subsets of an n-element set are sometimes called *k-combinations*. Other similar-sounding descriptions are “combinations with repetition, permutations with repetition, r-permutations, permutations with indistinguishable objects,” and so on. However, the counting rules we’ve taught you are sufficient to solve all these sorts of problems without knowing this jargon, so we won’t burden you with it.

15.6.3 The Binomial Theorem

Counting gives insight into one of the basic theorems of algebra. A *binomial* is a sum of two terms, such as $a + b$. Now consider its fourth power $(a + b)^4$.

By repeatedly using distributivity of products over sums to multiply out this 4th power expression completely, we get

$$\begin{aligned} (a+b)^4 = \quad & aaaa + aaab + aaba + aabb \\ + & abaa + abab + abba + abbb \\ + & baaa + baab + baba + babb \\ + & bbaa + bbab + bbba + bbbb \end{aligned}$$

Notice that there is one term for every sequence of a's and b's. So there are 2^4 terms, and the number of terms with k copies of b and $n - k$ copies of a is:

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

by the Bookkeeper Rule. Hence, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. So for $n = 4$, this means:

$$(a+b)^4 = \binom{4}{0} \cdot a^4b^0 + \binom{4}{1} \cdot a^3b^1 + \binom{4}{2} \cdot a^2b^2 + \binom{4}{3} \cdot a^1b^3 + \binom{4}{4} \cdot a^0b^4$$

In general, this reasoning gives the Binomial Theorem:

Theorem 15.6.4 (*Binomial Theorem*)**.** *For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The Binomial Theorem explains why the n choose k number is called a *binomial coefficient*.

This reasoning about binomials extends nicely to *multinomials*, which are sums of two or more terms. For example, suppose we wanted the coefficient of

$$bo^2k^2e^3pr$$

in the expansion of $(b + o + k + e + p + r)^{10}$. Each term in this expansion is a product of 10 variables where each variable is one of b, o, k, e, p or r. Now, the coefficient of $bo^2k^2e^3pr$ is the number of those terms with exactly 1 b, 2 o's, 2

k's, 3 e's, 1 p and 1 r. And the number of such terms is precisely the number of rearrangements of the word BOOKKEEPER:

$$\binom{10}{1,2,2,3,1,1} = \frac{10!}{1!\,2!\,2!\,3!\,1!\,1!}.$$

This reasoning extends to a general theorem:

Theorem 15.6.5 (Multinomial Theorem)**.** *For all $n \in \mathbb{N}$,*

$$(z_1 + z_2 + \cdots + z_m)^n = \sum_{\substack{k_1,\ldots,k_m \in \mathbb{N} \\ k_1+\cdots+k_m=n}} \binom{n}{k_1, k_2, \ldots, k_m} z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}.$$

But you'll be better off remembering the reasoning behind the Multinomial Theorem rather than this cumbersome formal statement.

15.7 Counting Practice: Poker Hands

Five-Card Draw is a card game in which each player is initially dealt a *hand* consisting of 5 cards from a deck of 52 cards.[3] The number of different hands in Five-Card Draw is the number of 5-element subsets of a 52-element set, which is

$$\binom{52}{5} = 2,598,960.$$

Let's get some counting practice by working out the number of hands with various special properties.

[3]There are 52 cards in a standard deck. Each card has a *suit* and a *rank*. There are four suits:

$$\spadesuit \text{ (spades)} \qquad \heartsuit \text{ (hearts)} \qquad \clubsuit \text{ (clubs)} \qquad \diamondsuit \text{ (diamonds)}$$

And there are 13 ranks, listed here from lowest to highest:

$$\overset{\text{Ace}}{A}, 2, 3, 4, 5, 6, 7, 8, 9, \overset{\text{Jack}}{J}, \overset{\text{Queen}}{Q}, \overset{\text{King}}{K}.$$

Thus, for example, $8\heartsuit$ is the 8 of hearts and $A\spadesuit$ is the ace of spades.

15.7.1 Hands with a Four-of-a-Kind

A *Four-of-a-Kind* is a set of four cards with the same rank. How many different hands contain a Four-of-a-Kind? Here are a couple examples:

$$\{8\spadesuit,\ 8\diamondsuit,\ Q\heartsuit,\ 8\heartsuit,\ 8\clubsuit\}$$
$$\{A\clubsuit,\ 2\clubsuit,\ 2\heartsuit,\ 2\diamondsuit,\ 2\spadesuit\}$$

As usual, the first step is to map this question to a sequence-counting problem. A hand with a Four-of-a-Kind is completely described by a sequence specifying:

1. The rank of the four cards.
2. The rank of the extra card.
3. The suit of the extra card.

Thus, there is a bijection between hands with a Four-of-a-Kind and sequences consisting of two distinct ranks followed by a suit. For example, the three hands above are associated with the following sequences:

$$(8, Q, \heartsuit) \leftrightarrow \{\ 8\spadesuit,\ 8\diamondsuit,\ 8\heartsuit,\ 8\clubsuit,\ Q\heartsuit\}$$
$$(2, A, \clubsuit) \leftrightarrow \{2\clubsuit,\ 2\heartsuit,\ 2\diamondsuit,\ 2\spadesuit,\ A\clubsuit\}$$

Now we need only count the sequences. There are 13 ways to choose the first rank, 12 ways to choose the second rank, and 4 ways to choose the suit. Thus, by the Generalized Product Rule, there are $13 \cdot 12 \cdot 4 = 624$ hands with a Four-of-a-Kind. This means that only 1 hand in about 4165 has a Four-of-a-Kind. Not surprisingly, Four-of-a-Kind is considered to be a very good poker hand!

15.7.2 Hands with a Full House

A *Full House* is a hand with three cards of one rank and two cards of another rank. Here are some examples:

$$\{2\spadesuit,\ 2\clubsuit,\ 2\diamondsuit,\ J\clubsuit,\ J\diamondsuit\}$$
$$\{5\diamondsuit,\ 5\clubsuit,\ 5\heartsuit,\ 7\heartsuit,\ 7\clubsuit\}$$

Again, we shift to a problem about sequences. There is a bijection between Full Houses and sequences specifying:

1. The rank of the triple, which can be chosen in 13 ways.
2. The suits of the triple, which can be selected in $\binom{4}{3}$ ways.
3. The rank of the pair, which can be chosen in 12 ways.
4. The suits of the pair, which can be selected in $\binom{4}{2}$ ways.

The example hands correspond to sequences as shown below:

$$(2, \{\spadesuit, \clubsuit, \diamondsuit\}, J, \{\clubsuit, \diamondsuit\}) \leftrightarrow \{2\spadesuit,\ 2\clubsuit,\ 2\diamondsuit,\ J\clubsuit,\ J\diamondsuit\}$$
$$(5, \{\diamondsuit, \clubsuit, \heartsuit\}, 7, \{\heartsuit, \clubsuit\}) \leftrightarrow \{5\diamondsuit,\ 5\clubsuit,\ 5\heartsuit,\ 7\heartsuit,\ 7\clubsuit\}$$

By the Generalized Product Rule, the number of Full Houses is:

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}.$$

We're on a roll—but we're about to hit a speed bump.

15.7.3 Hands with Two Pairs

How many hands have *Two Pairs*; that is, two cards of one rank, two cards of another rank, and one card of a third rank? Here are examples:

$$\{3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\}$$
$$\{9\heartsuit,\ 9\diamondsuit,\ 5\heartsuit,\ 5\clubsuit,\ K\spadesuit\}$$

Each hand with Two Pairs is described by a sequence consisting of:

1. The rank of the first pair, which can be chosen in 13 ways.
2. The suits of the first pair, which can be selected $\binom{4}{2}$ ways.
3. The rank of the second pair, which can be chosen in 12 ways.
4. The suits of the second pair, which can be selected in $\binom{4}{2}$ ways.
5. The rank of the extra card, which can be chosen in 11 ways.
6. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

Thus, it might appear that the number of hands with Two Pairs is:

$$13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4.$$

Wrong answer! The problem is that there is *not* a bijection from such sequences to hands with Two Pairs. This is actually a 2-to-1 mapping. For example, here are the pairs of sequences that map to the hands given above:

$$\begin{array}{rcl}
(3, \{\diamondsuit, \spadesuit\}, Q, \{\diamondsuit, \heartsuit\}, A, \clubsuit) & \searrow & \\
 & & \{3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\} \\
(Q, \{\diamondsuit, \heartsuit\}, 3, \{\diamondsuit, \spadesuit\}, A, \clubsuit) & \nearrow & \\
 & & \\
(9, \{\heartsuit, \diamondsuit\}, 5, \{\heartsuit, \clubsuit\}, K, \spadesuit) & \searrow & \\
 & & \{9\heartsuit,\ 9\diamondsuit,\ 5\heartsuit,\ 5\clubsuit,\ K\spadesuit\} \\
(5, \{\heartsuit, \clubsuit\}, 9, \{\heartsuit, \diamondsuit\}, K, \spadesuit) & \nearrow &
\end{array}$$

The problem is that nothing distinguishes the first pair from the second. A pair of 5's and a pair of 9's is the same as a pair of 9's and a pair of 5's. We avoided this difficulty in counting Full Houses because, for example, a pair of 6's and a triple of kings is different from a pair of kings and a triple of 6's.

We ran into precisely this difficulty last time, when we went from counting arrangements of *different* pieces on a chessboard to counting arrangements of two *identical* rooks. The solution then was to apply the Division Rule, and we can do the same here. In this case, the Division rule says there are twice as many sequences as hands, so the number of hands with Two Pairs is actually:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4}{2}.$$

Another Approach

The preceding example was disturbing! One could easily overlook the fact that the mapping was 2-to-1 on an exam, fail the course, and turn to a life of crime. You can make the world a safer place in two ways:

1. Whenever you use a mapping $f : A \to B$ to translate one counting problem to another, check that the same number of elements in A are mapped to each element in B. If k elements of A map to each of element of B, then apply the Division Rule using the constant k.

2. As an extra check, try solving the same problem in a different way. Multiple approaches are often available—and all had better give the same answer!

(Sometimes different approaches give answers that *look* different, but turn out to be the same after some algebra.)

We already used the first method; let's try the second. There is a bijection between hands with two pairs and sequences that specify:

1. The ranks of the two pairs, which can be chosen in $\binom{13}{2}$ ways.
2. The suits of the lower-rank pair, which can be selected in $\binom{4}{2}$ ways.
3. The suits of the higher-rank pair, which can be selected in $\binom{4}{2}$ ways.
4. The rank of the extra card, which can be chosen in 11 ways.
5. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

For example, the following sequences and hands correspond:

$$(\{3, Q\}, \{\diamondsuit, \spadesuit\}, \{\diamondsuit, \heartsuit\}, A, \clubsuit) \leftrightarrow \{3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\}$$
$$(\{9, 5\}, \{\heartsuit, \clubsuit\}, \{\heartsuit, \diamondsuit\}, K, \spadesuit) \leftrightarrow \{9\heartsuit,\ 9\diamondsuit,\ 5\heartsuit,\ 5\clubsuit,\ K\spadesuit\}$$

Thus, the number of hands with two pairs is:

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 11 \cdot 4.$$

This is the same answer we got before, though in a slightly different form.

15.7.4 Hands with Every Suit

How many hands contain at least one card from every suit? Here is an example of such a hand:

$$\{7\diamondsuit,\ K\clubsuit,\ 3\diamondsuit,\ A\heartsuit,\ 2\spadesuit\}$$

Each such hand is described by a sequence that specifies:

1. The ranks of the diamond, the club, the heart, and the spade, which can be selected in $13 \cdot 13 \cdot 13 \cdot 13 = 13^4$ ways.
2. The suit of the extra card, which can be selected in 4 ways.
3. The rank of the extra card, which can be selected in 12 ways.

For example, the hand above is described by the sequence:

$$(7, K, A, 2, \diamondsuit, 3) \leftrightarrow \{7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\}.$$

Are there other sequences that correspond to the same hand? There is one more! We could equally well regard either the $3\diamondsuit$ or the $7\diamondsuit$ as the extra card, so this is actually a 2-to-1 mapping. Here are the two sequences corresponding to the example hand:

$$\begin{array}{ll} (7, K, A, 2, \diamondsuit, 3) & \searrow \\ & \qquad \{7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\} \\ (3, K, A, 2, \diamondsuit, 7) & \nearrow \end{array}$$

Therefore, the number of hands with every suit is:

$$\frac{13^4 \cdot 4 \cdot 12}{2}.$$

15.8 The Pigeonhole Principle

Here is an old puzzle:

> A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

For example, picking out three socks is not enough; you might end up with one red, one green, and one blue. The solution relies on the

Pigeonhole Principle

If there are more pigeons than holes they occupy, then at least two pigeons must be in the same hole.

A rigorous statement of the Principle goes this way:

Rule 15.8.1 (Pigeonhole Principle)**.** *If $|A| > |B|$, then for every total function $f : A \to B$, there exist two different elements of A that are mapped by f to the same element of B.*

Stating the Principle this way may be less intuitive, but it should now sound familiar: it is simply the contrapositive of the Mapping Rules injective case (4.6). Here, the pigeons form set A, the pigeonholes are the set B, and f describes which hole each pigeon occupies.

Mathematicians have come up with many ingenious applications for the pigeonhole principle. If there were a cookbook procedure for generating such arguments, we'd give it to you. Unfortunately, there isn't one. One helpful tip, though: when you try to solve a problem with the pigeonhole principle, the key is to clearly identify three things:

1. The set A (the pigeons).
2. The set B (the pigeonholes).
3. The function f (the rule for assigning pigeons to pigeonholes).

The Pope's Pigeonholes

The town of Orvieto in Umbria, Italy, offered a refuge for medieval popes who might be forced to flee from Rome. It lies on top of a high plateau whose steep cliffs protected against attackers. Over centuries the townspeople excavated around 1200 underground rooms where vast flocks of pigeons were kept as a self-renewing food source to enable the town to withstand long sieges. Figure 15.3 shows a typical cave wall in which dozens pigeonholes have been carved.

15.8.1 Hairs on Heads

There are a number of generalizations of the pigeonhole principle. For example:

Rule 15.8.2 (Generalized Pigeonhole Principle)**.** *If $|A| > k \cdot |B|$, then every total function $f : A \to B$ maps at least $k+1$ different elements of A to the same element of B.*

For example, if you pick two people at random, surely they are extremely unlikely to have *exactly* the same number of hairs on their heads. However, in the remarkable city of Boston, Massachusetts, there is a group of *three* people who have exactly the same number of hairs! Of course, there are many completely bald people in Boston, and they all have zero hairs. But we're talking about non-bald people; say a person is non-bald if they have at least ten thousand hairs on their head.

Boston has about 500,000 non-bald people, and the number of hairs on a person's head is at most 200,000. Let A be the set of non-bald people in Boston, let $B =$

Figure 15.3 Pigeon holes in a cave under Orvieto.

$\{10,000, 10,001, \ldots, 200,000\}$, and let f map a person to the number of hairs on his or her head. Since $|A| > 2|B|$, the Generalized Pigeonhole Principle implies that at least three people have exactly the same number of hairs. We don't know who they are, but we know they exist!

15.8.2 Subsets with the Same Sum

For your reading pleasure, we have displayed ninety 25-digit numbers in Figure 15.4. Are there two different subsets of these 25-digit numbers that have the same sum? For example, maybe the sum of the last ten numbers in the first column is equal to the sum of the first eleven numbers in the second column?

Finding two subsets with the same sum may seem like a silly puzzle, but solving these sorts of problems turns out to be useful in diverse applications such as finding good ways to fit packages into shipping containers and decoding secret messages.

It turns out that it is hard to find different subsets with the same sum, which is why this problem arises in cryptography. But it is easy to prove that two such subsets *exist*. That's where the Pigeonhole Principle comes in.

Let A be the collection of all subsets of the 90 numbers in the list. Now the sum of any subset of numbers is at most $90 \cdot 10^{25}$, since there are only 90 numbers and every 25-digit number is less than 10^{25}. So let B be the the integer interval $[0..90 \cdot 10^{25}]$, and let f map each subset of numbers (in A) to its sum (in B).

We proved that an n-element set has 2^n different subsets in Section 15.2. Therefore:

$$|A| = 2^{90} \geq 1.237 \times 10^{27}$$

0020480135385502964448038	3171004832173501394113017
5763257331083479647409398	8247331000042995311646021
0489445991866915676240992	3208234421597368647019265
5800949123548989122628663	8496243997123475922766310
1082662032430379651370981	3437254656355157864869113
6042900801199280218026001	8518399140676002660747477
1178480894769706178994993	3574883393058653923711365
6116171789137737896701405	8543691283470191452333763
1253127351683239693851327	3644909946040480189969149
6144868973001582369723512	8675309258374137092461352
1301505129234077811069011	3790044132737084094417246
6247314593851169234746152	8694321112363996867296665
1311567111143866433882194	3870332127437971355322815
6814428944266874963488274	8772321203608477245851154
1470029452721203587686214	4080505804577801451363100
6870852945543886849147881	8791422161722582546341091
1578271047286257499433886	4167283461025702348124920
6914955508120950093732397	9062628024592126283973285
1638243921852176243192354	4235996831123777788211249
6949632451365987152423541	9137845566925526349897794
1763580219131985963102365	4670939445749439042111220
7128211143613619828415650	9153762966803189291934419
1826227795601842231029694	4815379351865384279613427
7173920083651862307925394	9270880194077636406984249
1843971862675102037201420	4837052948212922604442190
7215654874211755676220587	9324301480722103490379204
2396951193722134526177237	5106389423855018550671530
7256932847164391040233050	9436090832146695147140581
2781394568268599801096354	5142368192004769218069910
7332822657075235431620317	9475308159734538249013238
2796605196713610405408019	5181234096130144084041856
7426441829541573444964139	9492376623917486974923202
2931016394761975263190347	5198267398125617994391348
7632198126531809327186321	9511972558779880288252979
2933458058294405155197296	5317592940316231219758372
7712154432211912882310511	9602413424619187112552264
3075514410490975920315348	5384358126771794128356947
7858918664240262356610010	9631217114906129219461111
8149436716871371161932035	3157693105325111284321993
3111474985252793452860017	5439211712248901995423441
7898156786763212963178679	9908189853102753335981319
3145621587936120118438701	5610379826092838192760458
8147591017037573337848616	9913237476341764299813987
3148901255628881103198549	5632317555465228677676044
5692168374637019617423712	8176063831682536571306791

Figure 15.4 Ninety 25-digit numbers. Can you find two different subsets of these numbers that have the same sum?

On the other hand:

$$|B| = 90 \cdot 10^{25} + 1 \le 0.901 \times 10^{27}.$$

Both quantities are enormous, but $|A|$ is a bit greater than $|B|$. This means that f maps at least two elements of A to the same element of B. In other words, by the Pigeonhole Principle, two different subsets must have the same sum!

Notice that this proof gives no indication *which* two sets of numbers have the same sum. This frustrating variety of argument is called a *nonconstructive proof*.

The $100 prize for two same-sum subsets

To see if it was possible to actually *find* two different subsets of the ninety 25-digit numbers with the same sum, we offered a $100 prize to the first student who did it. We didn't expect to have to pay off this bet, but we underestimated the ingenuity and initiative of the students. One computer science major wrote a program that cleverly searched only among a reasonably small set of "plausible" sets, sorted them by their sums, and actually found a couple with the same sum. He won the prize. A few days later, a math major figured out how to reformulate the sum problem as a "lattice basis reduction" problem; then he found a software package implementing an efficient basis reduction procedure, and using it, he very quickly found lots of pairs of subsets with the same sum. He didn't win the prize, but he got a standing ovation from the class—staff included.

The $500 Prize for Sets with Distinct Subset Sums

How can we construct a set of n positive integers such that all its subsets have *distinct* sums? One way is to use powers of two:

$$\{1, 2, 4, 8, 16\}$$

This approach is so natural that one suspects all other such sets must involve larger numbers. (For example, we could safely replace 16 by 17, but not by 15.) Remarkably, there are examples involving *smaller* numbers. Here is one:

$$\{6, 9, 11, 12, 13\}$$

One of the top mathematicians of the Twentieth Century, Paul Erdős, conjectured in 1931 that there are no such sets involving *significantly* smaller numbers. More precisely, he conjectured that the largest number in such a set must be greater than $c2^n$ for some constant $c > 0$. He offered $500 to anyone who could prove or disprove his conjecture, but the problem remains unsolved.

15.8.3 A Magic Trick

A Magician sends an Assistant into the audience with a deck of 52 cards while the Magician looks away.

Five audience members each select one card from the deck. The Assistant then gathers up the five cards and holds up four of them so the Magician can see them. The Magician concentrates for a short time and then correctly names the secret, fifth card!

Since we don't really believe the Magician can read minds, we know the Assistant has somehow communicated the secret card to the Magician. Real Magicians and Assistants are not to be trusted, so we expect that the Assistant would secretly signal the Magician with coded phrases or body language, but for this trick they don't have to cheat. In fact, the Magician and Assistant could be kept out of sight of each other while some audience member holds up the 4 cards designated by the Assistant for the Magician to see.

Of course, without cheating, there is still an obvious way the Assistant can communicate to the Magician: he can choose any of the $4! = 24$ permutations of the 4 cards as the order in which to hold up the cards. However, this alone won't quite work: there are 48 cards remaining in the deck, so the Assistant doesn't have enough choices of orders to indicate exactly what the secret card is (though he could narrow it down to two cards).

15.8.4 The Secret

The method the Assistant can use to communicate the fifth card exactly is a nice application of what we know about counting and matching.

The Assistant has a second legitimate way to communicate: he can choose *which of the five cards to keep hidden*. Of course, it's not clear how the Magician could determine which of these five possibilities the Assistant selected by looking at the four visible cards, but there is a way, as we'll now explain.

The problem facing the Magician and Assistant is actually a bipartite matching problem. Each vertex on the left will correspond to the information available to the Assistant, namely, a *set* of 5 cards. So the set X of left-hand vertices will have $\binom{52}{5}$ elements.

Each vertex on the right will correspond to the information available to the Magician, namely, a *sequence* of 4 distinct cards. So the set Y of right-hand vertices will have $52 \cdot 51 \cdot 50 \cdot 49$ elements. When the audience selects a set of 5 cards, then the Assistant must reveal a sequence of 4 cards from that hand. This constraint is represented by having an edge between a set of 5 cards on the left and a sequence of 4 cards on the right precisely when every card in the sequence is also in the set. This specifies the bipartite graph. Some edges are shown in the diagram in

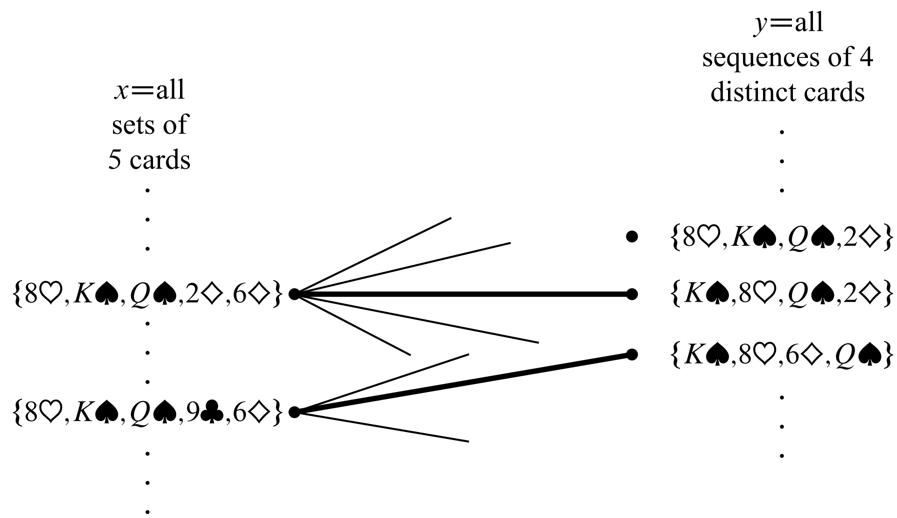


Figure 15.5 The bipartite graph where the nodes on the left correspond to *sets* of 5 cards and the nodes on the right correspond to *sequences* of 4 cards. There is an edge between a set and a sequence whenever all the cards in the sequence are contained in the set.

Figure 15.5.

For example,

$$\{8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit, 6\diamondsuit\} \tag{15.2}$$

is an element of X on the left. If the audience selects this set of 5 cards, then there are many different 4-card sequences on the right in set Y that the Assistant could choose to reveal, including $(8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit)$, $(K\spadesuit, 8\heartsuit, Q\spadesuit, 2\diamondsuit)$ and $(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit)$.

What the Magician and his Assistant need to perform the trick is a *matching* for the X vertices. If they agree in advance on some matching, then when the audience selects a set of 5 cards, the Assistant reveals the matching sequence of 4 cards. The Magician uses the matching to find the audience's chosen set of 5 cards, and so he can name the one not already revealed.

For example, suppose the Assistant and Magician agree on a matching containing the two bold edges in Figure 15.5. If the audience selects the set

$$\{8\heartsuit, K\spadesuit, Q\spadesuit, 9\clubsuit, 6\diamondsuit\}, \tag{15.3}$$

then the Assistant reveals the corresponding sequence

$$(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit). \tag{15.4}$$

Using the matching, the Magician sees that the hand (15.3) is matched to the sequence (15.4), so he can name the one card in the corresponding set not already revealed, namely, the 9♣. Notice that the fact that the sets are *matched*, that is, that different sets are paired with *distinct* sequences, is essential. For example, if the audience picked the previous hand (15.2), it would be possible for the Assistant to reveal the same sequence (15.4), but he better not do that; if he did, then the Magician would have no way to tell if the remaining card was the 9♣ or the 2♢.

So how can we be sure the needed matching can be found? The answer is that each vertex on the left has degree $5 \cdot 4! = 120$, since there are five ways to select the card kept secret and there are 4! permutations of the remaining 4 cards. In addition, each vertex on the right has degree 48, since there are 48 possibilities for the fifth card. So this graph is *degree-constrained* according to Definition 12.5.5, and so has a matching by Theorem 12.5.6.

In fact, this reasoning shows that the Magician could still pull off the trick if 120 cards were left instead of 48, that is, the trick would work with a deck as large as 124 different cards—without any magic!

15.8.5 The Real Secret

But wait a minute! It's all very well in principle to have the Magician and his Assistant agree on a matching, but how are they supposed to remember a matching with $\binom{52}{5} = 2,598,960$ edges? For the trick to work in practice, there has to be a way to match hands and card sequences mentally and on the fly.

We'll describe one approach. As a running example, suppose that the audience selects:

$$10\heartsuit \quad 9\diamondsuit \quad 3\heartsuit \quad Q\spadesuit \quad J\diamondsuit.$$

- The Assistant picks out two cards of the same suit. In the example, the assistant might choose the 3♡ and 10♡. This is always possible because of the Pigeonhole Principle—there are five cards and 4 suits so two cards must be in the same suit.

- The Assistant locates the ranks of these two cards on the cycle shown in Figure 15.6. For any two distinct ranks on this cycle, one is always between 1 and 6 hops clockwise from the other. For example, the 3♡ is 6 hops clockwise from the 10♡.

- The more counterclockwise of these two cards is revealed first, and the other becomes the secret card. Thus, in our example, the 10♡ would be revealed, and the 3♡ would be the secret card. Therefore:

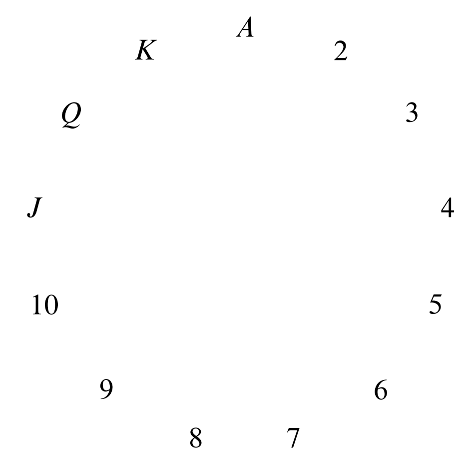


Figure 15.6 The 13 card ranks arranged in cyclic order.

- The suit of the secret card is the same as the suit of the first card revealed.
- The rank of the secret card is between 1 and 6 hops clockwise from the rank of the first card revealed.

- All that remains is to communicate a number between 1 and 6. The Magician and Assistant agree beforehand on an ordering of all the cards in the deck from smallest to largest such as:

$$A\clubsuit \; A\diamondsuit \; A\heartsuit \; A\spadesuit \; 2\clubsuit \; 2\diamondsuit \; 2\heartsuit \; 2\spadesuit \; \ldots \; K\heartsuit \; K\spadesuit$$

The order in which the last three cards are revealed communicates the number according to the following scheme:

$$\begin{array}{lccr}
(\text{ small}, & \text{medium}, & \text{large }) & = 1 \\
(\text{ small}, & \text{large}, & \text{medium}) & = 2 \\
(\text{medium}, & \text{small}, & \text{large }) & = 3 \\
(\text{medium}, & \text{large}, & \text{small }) & = 4 \\
(\text{ large}, & \text{small}, & \text{medium}) & = 5 \\
(\text{ large}, & \text{medium}, & \text{small }) & = 6
\end{array}$$

In the example, the Assistant wants to send 6 and so reveals the remaining three cards in large, medium, small order. Here is the complete sequence that the Magician sees:

$$10\heartsuit \quad Q\spadesuit \quad J\diamondsuit \quad 9\diamondsuit$$

- The Magician starts with the first card 10♡ and hops 6 ranks clockwise to reach 3♡, which is the secret card!

So that's how the trick can work with a standard deck of 52 cards. On the other hand, Hall's Theorem implies that the Magician and Assistant can *in principle* perform the trick with a deck of up to 124 cards. It turns out that there is a method which they could actually learn to use with a reasonable amount of practice for a 124-card deck, but we won't explain it here.[4]

15.8.6 The Same Trick with Four Cards?

Suppose that the audience selects only *four* cards and the Assistant reveals a sequence of *three* to the Magician. Can the Magician determine the fourth card?

Let X be all the sets of four cards that the audience might select, and let Y be all the sequences of three cards that the Assistant might reveal. Now, on one hand, we have

$$|X| = \binom{52}{4} = 270,725$$

by the Subset Rule. On the other hand, we have

$$|Y| = 52 \cdot 51 \cdot 50 = 132,600$$

by the Generalized Product Rule. Thus, by the Pigeonhole Principle, the Assistant must reveal the *same* sequence of three cards for at least

$$\left\lceil \frac{270,725}{132,600} \right\rceil = 3$$

different four-card hands. This is bad news for the Magician: if he sees that sequence of three, then there are at least three possibilities for the fourth card which he cannot distinguish. So there is no legitimate way for the Assistant to communicate exactly what the fourth card is!

15.9 Inclusion-Exclusion

How big is a union of sets? For example, suppose there are 60 math majors, 200 EECS majors, and 40 physics majors. How many students are there in these three

[4] See *The Best Card Trick* by Michael Kleber for more information.

departments? Let M be the set of math majors, E be the set of EECS majors, and P be the set of physics majors. In these terms, we're asking for $|M \cup E \cup P|$.

The Sum Rule says that if M, E and P are disjoint, then the sum of their sizes is

$$|M \cup E \cup P| = |M| + |E| + |P|.$$

However, the sets M, E and P might *not* be disjoint. For example, there might be a student majoring in both math and physics. Such a student would be counted twice on the right side of this equation, once as an element of M and once as an element of P. Worse, there might be a triple-major[5] counted *three* times on the right side!

Our most-complicated counting rule determines the size of a union of sets that are not necessarily disjoint. Before we state the rule, let's build some intuition by considering some easier special cases: unions of just two or three sets.

15.9.1 Union of Two Sets

For two sets, S_1 and S_2, the *Inclusion-Exclusion Rule* is that the size of their union is:

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| \tag{15.5}$$

Intuitively, each element of S_1 is accounted for in the first term, and each element of S_2 is accounted for in the second term. Elements in *both* S_1 and S_2 are counted *twice*—once in the first term and once in the second. This double-counting is corrected by the final term.

15.9.2 Union of Three Sets

So how many students are there in the math, EECS, and physics departments? In other words, what is $|M \cup E \cup P|$ if:

$$\begin{aligned} |M| &= 60 \\ |E| &= 200 \\ |P| &= 40. \end{aligned}$$

The size of a union of three sets is given by a more complicated Inclusion-Exclusion formula:

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| = {} & |S_1| + |S_2| + |S_3| \\ & - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| \\ & + |S_1 \cap S_2 \cap S_3|. \end{aligned}$$

[5] ... though not at MIT anymore.

Remarkably, the expression on the right accounts for each element in the union of S_1, S_2 and S_3 exactly once. For example, suppose that x is an element of all three sets. Then x is counted three times (by the $|S_1|$, $|S_2|$ and $|S_3|$ terms), subtracted off three times (by the $|S_1 \cap S_2|$, $|S_1 \cap S_3|$ and $|S_2 \cap S_3|$ terms), and then counted once more (by the $|S_1 \cap S_2 \cap S_3|$ term). The net effect is that x is counted just once.

If x is in two sets (say, S_1 and S_2), then x is counted twice (by the $|S_1|$ and $|S_2|$ terms) and subtracted once (by the $|S_1 \cap S_2|$ term). In this case, x does not contribute to any of the other terms, since $x \notin S_3$.

So we can't answer the original question without knowing the sizes of the various intersections. Let's suppose that there are:

4	math - EECS double majors
3	math - physics double majors
11	EECS - physics double majors
2	triple majors

Then $|M \cap E| = 4+2$, $|M \cap P| = 3+2$, $|E \cap P| = 11+2$, and $|M \cap E \cap P| = 2$. Plugging all this into the formula gives:

$$\begin{aligned}
|M \cup E \cup P| &= |M| + |E| + |P| - |M \cap E| - |M \cap P| - |E \cap P| \\
&\quad + |M \cap E \cap P| \\
&= 60 + 200 + 40 - 6 - 5 - 13 + 2 \\
&= 278
\end{aligned}$$

15.9.3 Sequences with 42, 04, or 60

In how many permutations of the set $\{0, 1, 2, \dots, 9\}$ do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively? For example, none of these pairs appears in:

$$(7, 2, 9, 5, 4, 1, 3, 8, 0, 6).$$

The 06 at the end doesn't count; we need 60. On the other hand, both 04 and 60 appear consecutively in this permutation:

$$(7, 2, 5, \underline{6}, \underline{0}, \underline{4}, 3, 8, 1, 9).$$

Let P_{42} be the set of all permutations in which 42 appears. Define P_{60} and P_{04} similarly. Thus, for example, the permutation above is contained in both P_{60} and P_{04}, but not P_{42}. In these terms, we're looking for the size of the set $P_{42} \cup P_{04} \cup P_{60}$.

First, we must determine the sizes of the individual sets, such as P_{60}. We can use a trick: group the 6 and 0 together as a single symbol. Then there is an immediate bijection between permutations of $\{0, 1, 2, \dots 9\}$ containing 6 and 0 consecutively and permutations of:

$$\{60, 1, 2, 3, 4, 5, 7, 8, 9\}.$$

For example, the following two sequences correspond:

$$(7, 2, 5, \underline{6}, \underline{0}, 4, 3, 8, 1, 9) \longleftrightarrow (7, 2, 5, \underline{60}, 4, 3, 8, 1, 9).$$

There are 9! permutations of the set containing 60, so $|P_{60}| = 9!$ by the Bijection Rule. Similarly, $|P_{04}| = |P_{42}| = 9!$ as well.

Next, we must determine the sizes of the two-way intersections, such as $P_{42} \cap P_{60}$. Using the grouping trick again, there is a bijection with permutations of the set:

$$\{42, 60, 1, 3, 5, 7, 8, 9\}.$$

Thus, $|P_{42} \cap P_{60}| = 8!$. Similarly, $|P_{60} \cap P_{04}| = 8!$ by a bijection with the set:

$$\{604, 1, 2, 3, 5, 7, 8, 9\}.$$

And $|P_{42} \cap P_{04}| = 8!$ as well by a similar argument. Finally, note that $|P_{60} \cap P_{04} \cap P_{42}| = 7!$ by a bijection with the set:

$$\{6042, 1, 3, 5, 7, 8, 9\}.$$

Plugging all this into the formula gives:

$$|P_{42} \cup P_{04} \cup P_{60}| = 9! + 9! + 9! - 8! - 8! - 8! + 7!.$$

15.9.4 Union of *n* Sets

The size of a union of n sets is given by the following rule.

Rule 15.9.1 (Inclusion-Exclusion)**.**

$$|S_1 \cup S_2 \cup \cdots \cup S_n| =$$

	the sum of the sizes of the individual sets
minus	*the sizes of all two-way intersections*
plus	*the sizes of all three-way intersections*
minus	*the sizes of all four-way intersections*
plus	*the sizes of all five-way intersections, etc.*

The formulas for unions of two and three sets are special cases of this general rule.

This way of expressing Inclusion-Exclusion is easy to understand and nearly as precise as expressing it in mathematical symbols, but we'll need the symbolic version below, so let's work on deciphering it now.

We already have a concise notation for the sum of sizes of the individual sets, namely,

$$\sum_{i=1}^{n} |S_i|.$$

A "two-way intersection" is a set of the form $S_i \cap S_j$ for $i \neq j$. We regard $S_j \cap S_i$ as the same two-way intersection as $S_i \cap S_j$, so we can assume that $i < j$. Now we can express the sum of the sizes of the two-way intersections as

$$\sum_{1 \le i < j \le n} |S_i \cap S_j|.$$

Similarly, the sum of the sizes of the three-way intersections is

$$\sum_{1 \le i < j < k \le n} |S_i \cap S_j \cap S_k|.$$

These sums have alternating signs in the Inclusion-Exclusion formula, with the sum of the k-way intersections getting the sign $(-1)^{k-1}$. This finally leads to a symbolic version of the rule:

Rule (Inclusion-Exclusion)**.**

$$\begin{aligned}
\left| \bigcup_{i=1}^{n} S_i \right| &= \sum_{i=1}^{n} |S_i| \\
&\quad - \sum_{1 \le i < j \le n} |S_i \cap S_j| \\
&\quad + \sum_{1 \le i < j < k \le n} |S_i \cap S_j \cap S_k| + \cdots \\
&\quad + (-1)^{n-1} \left| \bigcap_{i=1}^{n} S_i \right|.
\end{aligned}$$

While it's often handy express the rule in this way as a sum of sums, it is not necessary to group the terms by how many sets are in the intersections. So another way to state the rule is:

Rule (Inclusion-Exclusion-II)**.**

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right| \tag{15.6}$$

A proof of these rules using just highschool algebra is given in Problem 15.59.

If you're getting tired of all that nasty algebra, then good news is on the way. In the next section, we will show you how to prove some heavy-duty formulas without using any algebra at all. Just a few words and you are done. No kidding.

15.10 Combinatorial Proofs

Suppose you have n different T-shirts, but only want to keep k. You could equally well select the k shirts you want to keep or select the complementary set of $n-k$ shirts you want to throw out. Thus, the number of ways to select k shirts from among n must be equal to the number of ways to select $n-k$ shirts from among n. Therefore:

$$\binom{n}{k} = \binom{n}{n-k}.$$

This is easy to prove algebraically, since both sides are equal to:

$$\frac{n!}{k!\,(n-k)!}.$$

But we didn't really have to resort to algebra; we just used counting principles.

15.10.1 Pascal's Triangle Identity

Zach, famed Math for Computer Science Lecturer, has decided to try out for the city boxing team. After all, he's watched all of the *Rocky* movies and spent hours in front of a mirror sneering, “Yo, you wanna piece a' *me*?!” Zach figures that n people (including himself) are competing for spots on the team and only k will be selected. As part of maneuvering for a spot on the team, he needs to work out how many different teams are possible. There are two cases to consider:

- Zach *is* selected for the team, and his $k-1$ teammates are selected from among the other $n-1$ competitors. The number of different teams that can be formed in this way is:

$$\binom{n-1}{k-1}.$$

- Zach is *not* selected for the team, and all k team members are selected from among the other $n-1$ competitors. The number of teams that can be formed this way is:
$$\binom{n-1}{k}.$$

All teams of the first type contain Zach, and no team of the second type does; therefore, the two sets of teams are disjoint. Thus, by the Sum Rule, the total number of possible city boxing teams is:

$$\binom{n-1}{k-1} + \binom{n-1}{k}.$$

Albert, equally-famed co-Lecturer, thinks Zach isn't so tough, and so he might as well also try out. Albert reasons that n people (including himself) are trying out for k spots. Thus, the number of ways to select the team is simply:

$$\binom{n}{k}.$$

Albert and Zach each correctly counted the number of possible boxing teams. Thus, their answers must be equal. So we know:

Lemma 15.10.1 (Pascal's *Triangle Identity*)**.**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{15.7}$$

We proved *Pascal's Triangle Identity without any algebra!* Instead, we relied purely on counting techniques.

15.10.2 Giving a Combinatorial Proof

A *combinatorial proof* is an argument that establishes an algebraic fact by relying on counting principles. Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that $|S| = n$ by counting one way.
3. Show that $|S| = m$ by counting another way.
4. Conclude that $n = m$.

In the preceding example, S was the set of all possible city boxing teams. Zach computed

$$|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

by counting one way, and Albert computed

$$|S| = \binom{n}{k}$$

by counting another way. Equating these two expressions gave Pascal's Identity.

Checking a Combinatorial Proof

Combinatorial proofs are based on counting the same thing in different ways. This is fine when you've become practiced at different counting methods, but when in doubt, you can fall back on bijections and sequence counting to check such proofs.

For example, let's take a closer look at our combinatorial proof of Pascal's Identity (15.7). We assume the n competitors are numbered 1 to n. So the set S of things to be counted is the collection of all size-k subsets the integer interval $[1..n]$.

Albert counted S via the Subset Rule and found $|S| = \binom{n}{k}$. Zach had another way of counting. Giving himself the number n, Zach defined two collections of sets, Zach-chosen and Zach-not-chosen:

$$\begin{aligned} \text{Zach-chosen} &::= \{X \subseteq [1..n-1] \mid |X| = k-1\} \\ \text{Zach-not-chosen} &::= \{Y \subseteq [1..n-1] \mid |Y| = k\}. \end{aligned}$$

Clearly Zach-chosen and Zach-not-chosen are disjoint since the sets in Zach-chosen are smaller than those in Zach-not-chosen. So

$$|\text{Zach-chosen} \cup \text{Zach-not-chosen}| = |\text{Zach-chosen}| + |\text{Zach-not-chosen}|.$$

Also, by the Subset Rule

$$\begin{aligned} |\text{Zach-chosen}| &= \binom{n-1}{k-1}, \\ |\text{Zach-not-chosen}| &= \binom{n-1}{k}. \end{aligned}$$

Now the combinatorial proof of (15.7) is formalized by specifying a bijection

$$f : \text{Zach-chosen} \cup \text{Zach-not-chosen} \to S,$$

namely,

$$f(s) ::= \begin{cases} s \cup \{n\} & \text{if } |s| = k-1, \\ s & \text{if } |s| = k. \end{cases}$$

15.10.3 A Colorful Combinatorial Proof

The set that gets counted in a combinatorial proof in different ways is usually defined in terms of simple sequences or sets rather than an elaborate story about Teaching Assistants. Here is another colorful example of a combinatorial argument.

Theorem 15.10.2.

$$\sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r} = \binom{3n}{n}$$

Proof. We give a combinatorial proof. Let S be all n-card hands that can be dealt from a deck containing n different red cards and $2n$ different black cards. First, note that every $3n$-element set has

$$|S| = \binom{3n}{n}$$

n-element subsets.

From another perspective, the number of n-card hands with exactly r red cards is

$$\binom{n}{r}\binom{2n}{n-r}$$

since there are $\binom{n}{r}$ ways to choose the r red cards and $\binom{2n}{n-r}$ ways to choose the $n-r$ black cards. Since the number of red cards can be anywhere from 0 to n, the total number of n-card hands is:

$$|S| = \sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r}.$$

Equating these two expressions for $|S|$ proves the theorem. ■

Finding a Combinatorial Proof

Combinatorial proofs are almost magical. Theorem 15.10.2 looks pretty scary, but we proved it without any algebraic manipulations at all. The key to constructing a combinatorial proof is choosing the set S properly, which can be tricky. Generally, the simpler side of the equation should provide some guidance. For example, the right side of Theorem 15.10.2 is $\binom{3n}{n}$, which suggests that it will be helpful to choose S to be all n-element subsets of some $3n$-element set.

15.11 References

[6], [10], [17]

Problems for Section 15.2

Practice Problems

Problem 15.1.
Alice is thinking of a number between 1 and 1000.

What is the least number of yes/no questions you could ask her and be guaranteed to discover what it is? (Alice always answers truthfully.)

(a)

Problem 15.2.
In how many different ways is it possible to answer the next chapter's practice problems if:

- the first problem has four *true/false* questions,
- the second problem requires choosing one of four alternatives, and
- the answer to the third problem is an integer ≥ 15 and ≤ 20?

Problem 15.3.
How many total functions are there from set A to set B if $|A| = 3$ and $|B| = 7$?

Problem 15.4.
Let X be the six element set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$.

(a) How many subsets of X contain x_1?

(b) How many subsets of X contain x_2 and x_3 but do not contain x_6?

Class Problems

Problem 15.5.
A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)
- 2 characters—letters or numbers (big shot plate)

Let L be the set of all possible license plates.

(a) Express L in terms of

$$\mathcal{A} = \{A, B, C, \ldots, Z\}$$
$$\mathcal{D} = \{0, 1, 2, \ldots, 9\}$$

using unions ($\cup$) and set products ($\times$).

(b) Compute $|L|$, the number of different license plates, using the sum and product rules.

Problem 15.6. (a) How many of the billion numbers in the integer interval $[1..10^9]$ contain the digit 1? (*Hint:* How many don't?)

(b) There are 20 books arranged in a row on a shelf. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected, and 15-bit strings with exactly 6 ones.

Problem 15.7.

(a) Let $\mathcal{S}_{n,k}$ be the possible nonnegative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n. \tag{15.8}$$

That is

$$\mathcal{S}_{n,k} ::= \{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \mid (15.8) \text{ is true}\}.$$

Describe a bijection between $\mathcal{S}_{n,k}$ and the set of binary strings with n zeroes and k ones.

(b) Let $\mathcal{L}_{n,k}$ be the length k weakly increasing sequences of nonnegative integers $\leq n$. That is

$$\mathcal{L}_{n,k} ::= \{(y_1, y_2, \ldots, y_k) \in \mathbb{N}^k \mid y_1 \leq y_2 \leq \cdots \leq y_k \leq n\}.$$

Describe a bijection between $\mathcal{L}_{n,k}$ and $\mathcal{S}_{n,k}$.

Problem 15.8.
An n-vertex *numbered tree* is a tree whose vertex set is $[1..n]$ for some $n > 2$. We define the *code* of the numbered tree to be a sequence of $n - 2$ integers in $[1..n]$ obtained by the following recursive process:[6]

> If there are more than two vertices left, write down the *father* of the largest leaf, delete this *leaf*, and continue this process on the resulting smaller tree. If there are only two vertices left, then stop—the code is complete.

For example, the codes of a couple of numbered trees are shown in the Figure 15.7.

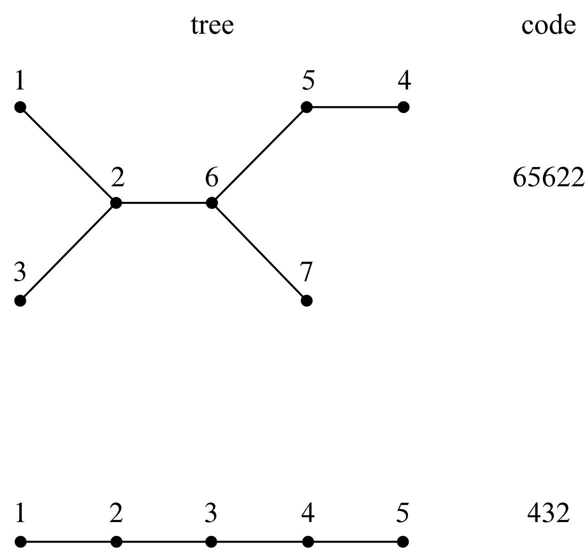


Figure 15.7

[6]The necessarily unique node adjacent to a leaf is called its *father*.

(a) Describe a procedure for reconstructing a numbered tree from its code.

(b) Conclude there is a bijection between the n-vertex numbered trees and sequences $(n-2)$ integers in $[1..n]$. State how many n-vertex numbered trees there are.

Problem 15.9.
Let X and Y be finite sets.

(a) How many binary relations from X to Y are there?

(b) Define a bijection between the set $[X \to Y]$ of all total functions from X to Y and the set $Y^{|X|}$. (Recall Y^n is the Cartesian product of Y with itself n times.) Based on that, what is $|\,[X \to Y]\,|$?

(c) Using the previous part, how many *functions*, not necessarily total, are there from X to Y? How does the fraction of functions vs. total functions grow as the size of X grows? Is it $O(1)$, $O(|X|)$, $O(2^{|X|})$,…?

(d) Show a bijection between the powerset $\text{pow}(X)$ and the set $[X \to \{0,1\}]$ of 0-1-valued total functions on X.

(e) Let X be a set of size n and B_X be the set of all bijections from X to X. Describe a bijection from B_X to the set of permutations of X.[7] This implies that there are how many bijections from X to X?

Problems for Section 15.4

Class Problems

Problem 15.10.
Use induction to prove that there are 2^n subsets of an n-element set (Theorem 4.5.5).

[7] A sequence in which all the elements of a set X appear exactly once is called a *permutation* of X (see Section 15.3.3).

Homework Problems

Problem 15.11.
Fermat's Little Theorem 9.10.8[8] asserts that

$$a^p \equiv a \pmod{p} \tag{15.9}$$

for all primes p and nonnegative integers a. This is immediate for $a = 0, 1$ so we assume that $a \geq 2$.

This problem offers a proof of (15.9) by counting strings over a fixed alphabet with a characters.

(a) How many length-k strings are there over an a-character alphabet?

(b) How many of these strings use more than one character?

Let z be a length-k string. The *length-n rotation* of z is the string yx, where $z = xy$ and the length $|x|$ of x is $\text{rem}(n, k)$.

(c) Verify that if u is the length-n rotation of z, and v is the length-m rotation of u, then v is the length-$(n + m)$ rotation of z.

(d) Let $\approx$ be the "is a rotation of" relation on strings. That is,

$$v \approx z \quad \text{IFF} \quad \exists n \in \mathbb{N}. [v \text{ is a length-}n \text{ rotation of } z].$$

Prove that $\approx$ is an equivalence relation.

(e) Prove that if $xy = yx$ then x and y each consist of repetitions of some string u. That is, if $xy = yx$, then $x, y \in u^*$ for some string u.

Hint: By induction on the length $|xy|$ of xy.

(f) Conclude that if p is prime and z is a length-p string containing at least two different characters, then z is equivalent under $\approx$ to exactly p strings (counting itself).

(g) Conclude from parts (a) and (f) that $p \mid (a^p - a)$, which proves Fermat's Little Theorem (15.9).

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[8] This Theorem is usually stated as

$$a^{p-1} \equiv 1 \pmod{p},$$

for all primes p and integers a not divisible by p. This follows immediately from (15.9) by canceling a.

Problems for Section 15.5

Practice Problems

Problem 15.12.
Eight students—Anna, Brian, Caine,…—are to be seated around a circular table in a circular room. Two seatings are regarded as defining the same *arrangement* if each student has the same student on his or her right in both seatings: it does not matter which way they face. We'll be interested in counting how many arrangements there are of these 8 students, given some restrictions.

(a) As a start, how many different arrangements of these 8 students around the table are there without any restrictions?

(b) How many arrangements of these 8 students are there with Anna sitting next to Brian?

(c) How many arrangements are there with if Brian sitting next to both Anna AND Caine?

(d) How many arrangements are there with Brian sitting next to Anna *OR* Caine?

Problem 15.13.
How many different ways are there to select an unordered bundle of three dozen colored roses if red, yellow, pink, white, purple and orange roses are available? Please explain your answer (you may leave it un-simplified).

Problem 15.14.
Suppose n books are lined up on a shelf. The number of selections of m of the books so that selected books are separated by at least three unselected books is the same as the number of *all* length k binary strings with exactly m ones.

(a) What is the value of k?

(b) Describe a bijection between between the set of all length k binary strings with exactly m ones and such book selections.

Problem 15.15.
Six women and nine men are on the faculty of a school's EECS department. The

individuals are distinguishable. How many ways are there to select a committee of 5 members if at least 1 woman must be on the committee?

Class Problems

Problem 15.16.
Your class tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your Teaching Assistant (TA) has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k-to-1 mapping for what k?

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j-to-1 for what j?

(c) How many group assignments are possible?

(d) In how many ways can $3n$ students be broken up into n groups of 3?

Problem 15.17.
A pizza house is having a promotional sale. Their commercial reads:

> We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's $22,369,621$ different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to $22,369,621$. Unfortunately, $(2^9)^3/3!$ can't be an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

Problem 15.18.
Answer the following quesions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

(b) An *r-permutation* of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{ccc} (a,b) & (a,c) & (a,d) \\ (b,a) & (b,c) & (b,d) \\ (c,a) & (c,b) & (c,d) \\ (d,a) & (d,b) & (d,c) \end{array}$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Problem 15.19. **(a)** There are 30 books arranged in a row on a shelf. In how many ways can eight of these books be selected so that there are at least two unselected books between any two selected books?

(b) How many nonnegative integer solutions are there for the following equality?

$$x_1 + x_2 + \cdots + x_m = k. \tag{15.10}$$

(c) How many nonnegative integer solutions are there for the following inequality?

$$x_1 + x_2 + \cdots + x_m \leq k. \tag{15.11}$$

(d) How many length m weakly increasing sequences of nonnegative integers $\leq k$ are there?

Problem 15.20.
How many ways are there to pick three distinct numbers from the integer interval $[1..15]$ such that the sum of the numbers is divisible by three?

Homework Problems

Problem 15.21.
This problem is about binary relations on the set of integers in the interval $[1..n]$ and digraphs whose vertex set is $[1..n]$.

(a) How many digraphs are there?

(b) How many simple graphs are there?

(c) How many asymmetric binary relations are there?

(d) How many linear strict partial orders are there?

Problem 15.22.
Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels a, e, i, o, u appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.

(b) How many ways are there to order the 26 letters of the alphabet so that there are *at least two* consonants immediately following each vowel?

(c) In how many different ways can $2n$ students be paired up?

(d) Two n-digit sequences of digits $0,1,\ldots,9$ are said to be of the *same type* if the digits of one are a permutation of the digits of the other. For $n = 8$, for example, the sequences 03088929 and 00238899 are the same type. How many types of n-digit sequences are there?

Problem 15.23.
In a standard 52-card deck, each card has one of thirteen *rank*s in the set R and one

of four *suits* in the set S where

$$R ::= \{A, 2, \ldots, 10, J, Q, K\},$$
$$S ::= \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

A 5-card *hand* is a set of five distinct cards from the deck.

For each part describe a bijection between a set that can easily be counted using the Product and Sum Rules of Ch. 15.1, and the set of hands matching the specification. *Give bijections, not numerical answers.*

For instance, consider the set of 5-card hands containing all 4 suits. Each such hand must have 2 cards of one suit. We can describe a bijection between such hands and the set $S \times R_2 \times R^3$ where R_2 is the set of two-element subsets of R. Namely, an element

$$(s, \{r_1, r_2\}, (r_3, r_4, r_5)) \in S \times R_2 \times R^3$$

indicates

1. the repeated suit $s \in S$,
2. the set $\{r_1, r_2\} \in R_2$ of ranks of the cards of suit s and
3. the ranks (r_3, r_4, r_5) of the remaining three cards, listed in increasing suit order where
$$\clubsuit \prec \diamondsuit \prec \heartsuit \prec \spadesuit.$$

For example,

$$(\clubsuit, \{10, A\}, (J, J, 2)) \longleftrightarrow \{A\clubsuit, 10\clubsuit, J\diamondsuit, J\heartsuit, 2\spadesuit\}.$$

(a) A single pair of the same rank (no 3-of-a-kind, 4-of-a-kind, or second pair).

(b) Three or more aces.

Problem 15.24.

Suppose you have seven dice—each a different color of the rainbow; otherwise the dice are standard, with faces numbered 1 to 6. A *roll* is a sequence specifying a value for each die in rainbow (ROYGBIV) order. For example, one roll is $(3, 1, 6, 1, 4, 5, 2)$ indicating that the red die showed a 3, the orange die showed 1, the yellow 6,. . . .

For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized Product, and similar rules. Then write a simple arithmetic formula, possibly involving factorials and binomial coefficients, for the size of the set of rolls. You do not need to prove that the correspondence between sets you describe is a bijection, and you do not need to simplify the expression you come up with.

For example, let A be the set of rolls where 4 dice come up showing the same number, and the other 3 dice also come up the same, but with a different number. Let R be the set of seven rainbow colors and $S ::= [1..6]$ be the set of dice values.

Define $B ::= P_{S,2} \times R_3$, where $P_{S,2}$ is the set of 2-permutations of S and R_3 is the set of size-3 subsets of R. Then define a bijection from A to B by mapping a roll in A to the sequence in B whose first element is a pair consisting of the number that came up three times followed by the number that came up four times, and whose second element is the set of colors of the three matching dice.

For example, the roll

$$(4, 4, 2, 2, 4, 2, 4) \in A$$

maps to

$$((2, 4), \{\text{yellow,green,indigo}\}) \in B.$$

Now by the Bijection rule $|A| = |B|$, and by the Generalized Product and Subset rules,

$$|B| = 6 \cdot 5 \cdot \binom{7}{3}.$$

(a) For how many rolls do *exactly* two dice have the value 6 and the remaining five dice all have different values? Remember to describe a bijection and write a simple arithmetic formula.

Example: $(6, 2, 6, 1, 3, 4, 5)$ is a roll of this type, but $(1, 1, 2, 6, 3, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

(b) For how many rolls do two dice have the same value and the remaining five dice all have different values? Remember to describe a bijection and write a simple arithmetic formula.

Example: $(4, 2, 4, 1, 3, 6, 5)$ is a roll of this type, but $(1, 1, 2, 6, 1, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

(c) For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice a third value? Remember to describe a bijection and write a simple arithmetic formula.

Example: $(6, 1, 2, 1, 2, 6, 6)$ is a roll of this type, but $(4, 4, 4, 4, 1, 3, 5)$ and $(5, 5, 5, 6, 6, 1, 2)$ are not.

Problem 15.25 (Counting trees)**.**
What is the number T_n of different trees that can be formed from a set of n distinct vertices?[9] Cayley's formula gives the answer

$$T_n = n^{n-2}.$$

One way to derive this appears in Problem 15.8. This and three additional derivations are given by Aigner & Ziegler (1998), who comment that "the most beautiful of them all" is a counting argument due to Jim Pitman that we now describe.

Pitman's derivation counts in two different ways the number of different sequences of edges that can be added to an empty graph on n vertices to form a rooted tree. One way to form such a sequence is to start with one of the T_n possible unrooted trees, choose one of its n vertices as root, and choose one of the $(n-1)!$ possible sequences in which to add its $n-1$ edges. Therefore, the total number of sequences that can be formed in this way is

$$T_n n(n-1)! = T_n n!\,.$$

Another way to count these edge sequences is to start with the empty graph and build up a spanning forest of rooted trees by adding edges in sequence. When $n-k$ edges have been added, the graph with these edges will be a spanning forest consisting of k rooted trees. To add the next edge, we choose any vertex to be the root of a new tree. Then we add an edge between this new root and the root of any one of the $k-1$ subtrees that did not include the chosen vertex. So the next edge can be chosen in $n(k-1)$ ways to form a new spanning forest consisting of $k-1$ rooted trees.

Therefore, if one multiplies together the number of choices from the first step, the second step, etc., the total number of choices is

$$\prod_{k=2}^{n} n(k-1) = n^{n-1}(n-1)! = n^{n-2}n!\,.$$

Equating these two formulas for the number of edge sequences, we get $T_n n! = n^{n-2}n!$, and cancelling $n!$ we arrive at Cayley's formula

$$T_n = n^{n-2}.$$

Generalize Pitman's derivation to count the number of spanning forests consisting of k rooted trees on n vertices.

[9]From Double counting, wikipedia, Aug. 30, 2014. See also Prüfer Sequences

Exam Problems

Problem 15.26.
Suppose that two identical 52-card decks are mixed together. Write a simple formula for the number of distinct permutations of the 104 cards.

Problems for Section 15.6

Class Problems

Problem 15.27.
The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word $BOOKKEEPER$.

(a) In how many ways can you arrange the letters in the word $POKE$?

(b) In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

(c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	...
BO_2O_1K	
...	

(d) This is a k-to-1 mapping, young grasshopper? What is k?

(e) In light of the Division Rule, how many arrangements are there of $BOOK$?

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

(h) What kind of mapping is this?

(i) So how many arrangements are there of the letters in $KEEPER$?

Now you are ready to face the BOOKKEEPER!

(j) How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

(m) How many arrangements of $BOOKKEEPER$ are there?

Remember well what you have learned: subscripts on, subscripts off.
This is the Tao of Bookkeeper.

(n) How many arrangements of $VOODOODOLL$ are there?

(o) How many length 52 sequences of digits contain exactly 17 two's, 23 fives, and 12 nines?

Problems for Section 15.6

Practice Problems

Problem 15.28.
How many different permutations are there of the sequence of letters in "MISSISSIPPI"?

Class Problems

Problem 15.29.
Find the coefficients of

(a) x^5 in $(1+x)^{11}$

(b) x^8y^9 in $(3x+2y)^{17}$

(c) a^6b^6 in $(a^2+b^3)^5$

Problem 15.30.
Let p be a **prime number**.

(a) Explain why the multinomial coefficient

$$\binom{p}{k_1, k_2, \ldots, k_n}$$

is divisible by p if all the k_i's are nonnegative integers less than p.

(b) Conclude from part (a) that

$$(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p}. \tag{15.12}$$

(Do not prove this using Fermat's "little" Theorem. The point of this problem is to offer an independent proof of Fermat's theorem.)

(c) Explain how (15.12) immediately proves Fermat's Little Theorem 9.10.8:

$$n^{p-1} \equiv 1 \pmod{p}$$

when n is not a multiple of p.

Homework Problems

Problem 15.31.
The *degree sequence* of a simple graph is the weakly decreasing sequence of degrees of its vertices. For example, the degree sequence for the 5-vertex numbered tree pictured in the Figure 15.7 in Problem 15.8 is $(2, 2, 2, 1, 1)$ and for the 7-vertex tree it is $(3, 3, 2, 1, 1, 1, 1)$.

We're interested in counting how many numbered trees there are with a given degree sequence. We'll do this using the bijection defined in Problem 15.8 between n-vertex numbered trees and length $n-2$ code words whose characters are integers between 1 and n.

The *occurrence number* for a character in a word is the number of times that the character occurs in the word. For example, in the word `65622`, the occurrence number for `6` is two, and the occurrence number for `5` is one. The *occurrence sequence* of a word is the weakly decreasing sequence of occurrence numbers of characters in the word. The occurrence sequence for this word is $(2, 2, 1)$ because it has two occurrences of each of the characters `6` and `2`, and one occurrence of `5`.

(a) There is a simple relationship between the degree sequence of an n-vertex numbered tree and the occurrence sequence of its code. Describe this relationship and explain why it holds. Conclude that counting n-vertex numbered trees with a given degree sequence is the same as counting the number of length $n-2$ code words with a given occurrence sequence.

Hint: How many times does a vertex of degree d occur in the code?

For simplicity, let's focus on counting 9-vertex numbered trees with a given degree sequence. By part (a), this is the same as counting the number of length 7 code words with a given occurrence sequence.

Any length 7 code word has a *pattern*, which is another length 7 word over the alphabet `a,b,c,d,e,f,g` that has the same occurrence sequence.

(b) How many length 7 patterns are there with three occurrences of `a`, two occurrences of `b`, and one occurrence of `c` and `d`?

(c) How many ways are there to assign occurrence numbers to integers $1, 2, \ldots, 9$ so that a code word with those occurrence numbers would have the occurrence sequence $3, 2, 1, 1, 0, 0, 0, 0, 0$?

In general, to find the pattern of a code word, list its characters in decreasing order by *number of occurrences*, and list characters with the same number of occurrences in decreasing order. Then replace successive characters in the list by successive letters `a,b,c,d,e,f,g`. The code word `2468751`, for example, has the pattern `fecabdg`, which is obtained by replacing its characters `8,7,6,5,4,2,1` by `a,b,c,d,e,f,g`, respectively. The code word `2449249` has pattern `caabcab`, which is obtained by replacing its characters `4,9,2` by `a,b,c`, respectively.

(d) What length 7 code word has three occurrences of `7`, two occurrences of `8`, one occurrence each of `2` and `9`, and pattern `abacbad`?

(e) Explain why the number of 9-vertex numbered trees with degree sequence $(4, 3, 2, 2, 1, 1, 1, 1, 1)$ is the product of the answers to parts (b) and (c).

Problem 15.32.

Let G be a simple graph with 6 vertices and an edge between every pair of vertices (that is, G is a *complete* graph). A length-3 cycle in G is called a *triangle*.

A set of two edges that share a vertex is called an *incident pair* (i.p.); the shared vertex is called the *center* of the i.p. That is, an i.p. is a set,

$$\{\langle u\text{—}v\rangle\,, \langle v\text{—}w\rangle\},$$

where u, v and w are distinct vertices, and its center is v.

(a) How many triangles are there?

(b) How many incident pairs are there?

Now suppose that every edge in G is colored either red or blue. A triangle or i.p. is called *multicolored* when its edges are not all the same color.

(c) Map the i.p.

$$\{\langle u—v\rangle, \langle v—w\rangle\}$$

to the triangle

$$\{\langle u—v\rangle, \langle v—w\rangle, \langle u—w\rangle\}.$$

Notice that multicolored i.p.'s map to multicolored triangles. Explain why this mapping is 2-to-1 on these multicolored objects.

(d) Show that at most six multicolored i.p.'s can have the same center. Conclude that there are at most 36 possible multicolored i.p.'s.

Hint: A vertex incident to r red edges and b blue edges is the center of $r \cdot b$ different multicolored i.p.'s.

(e) If two people are not friends, they are called *strangers*. If every pair of people in a group are friends, or if every pair are strangers, the group is called *uniform*.

Explain why parts (a), (c), and (d) imply that

> Every set of six people includes *two* uniform three-person groups.

Exam Problems

Problem 15.33.
There is a robot that steps between integer positions in 3-dimensional space. Each step of the robot increments one coordinate and leaves the other two unchanged.

(a) How many paths can the robot follow going from the origin $(0, 0, 0)$ to $(3, 4, 5)$?

(b) How many paths can the robot follow going from the origin (i, j, k) to (m, n, p)?

Problems for Section 15.7

Practice Problems

Problem 15.34.
Indicate how many 5-card hands there are of each of the following kinds.

(a) A **Sequence** is a hand consisting of five consecutive cards of any suit, such as

$$5\heartsuit - 6\heartsuit - 7\spadesuit - 8\diamondsuit - 9\clubsuit.$$

Note that an ace may either be high (as in 10-J-Q-K-A), or low (as in A-2-3-4-5), but can't go "around the corner" (that is, Q-K-A-2-3 is *not* a sequence).

How many different **Sequence** hands are possible?

(b) A **Matching Suit** is a hand consisting of cards that are all of the same suit in any order.

How many different **Matching Suit** hands are possible?

(c) A **Straight Flush** is a hand that is both a *Sequence* and a *Matching Suit*.

How many different **Straight Flush** hands are possible?

(d) A **Straight** is a hand that is a *Sequence* but not a *Matching Suit*.

How many possible **Straights** are there?

(e) A **Flush** is a hand that is a *Matching Suit* but not a *Sequence*.

How many possible **Flushes** are there?

Class Problems

Problem 15.35.
Here are the solutions to the next 7 short answer questions, in no particular order. Indicate the solutions for the questions and briefly explain your answers.

$$1.\quad \frac{n!}{(n-m)!} \qquad 2.\quad \binom{n+m}{m} \qquad 3.\quad (n-m)! \quad 4.\quad m^n$$

$$5.\quad \binom{n-1+m}{m} \quad 6.\quad \binom{n-1+m}{n} \quad 7.\quad 2^{mn} \qquad 8.\quad n^m$$

(a) How many length m words can be formed from an n-letter alphabet, if no letter is used more than once?

(b) How many length m words can be formed from an n-letter alphabet, if letters can be reused?

(c) How many binary relations are there from set A to set B when $|A| = m$ and $|B| = n$?

(d) How many total injective functions are there from set A to set B, where $|A| = m$ and $|B| = n \geq m$?

(e) How many ways are there to place a total of m distinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls?

(f) How many ways are there to place a total of m indistinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls?

(g) How many ways are there to put a total of m distinguishable balls into n distinguishable urns with at most one ball in each urn?

Exam Problems

Problem 15.36. **(a)** How many solutions over the *positive* integers are there to the inequality:

$$x_1 + x_2 + \ldots + x_{10} \le 100$$

(b) In how many ways can Mr. and Mrs. Grumperson distribute 13 identical pieces of coal to their three children for Christmas so that each child gets at least one piece?

Problem 15.37.
Let C_{41} be the graph with vertices $\{0, 1, \ldots, 40\}$ and edges

$$\langle 0\text{—}1\rangle, \langle 1\text{—}2\rangle, \ldots, \langle 39\text{—}40\rangle, \langle 40\text{—}0\rangle,$$

and let K_{41} be the *complete graph* on the same set of 41 vertices.

You may answer the following questions with formulas involving exponents, binomial coefficients, and factorials.

(a) How many edges are there in K_{41}?

(b) How many isomorphisms are there from K_{41} to K_{41}?

(c) How many isomorphisms are there from C_{41} to C_{41}?

(d) What is the chromatic number $\chi(K_{41})$?

(e) What is the chromatic number $\chi(C_{41})$?

(f) How many edges are there in a spanning tree of K_{41}?

(g) A graph is created by adding a single edge between nonadjacent vertices of a tree with 41 vertices. What is the largest number of cycles the graph might have?

(h) What is the smallest number of leaves possible in a spanning tree of K_{41}?

(i) What is the largest number of leaves possible in a in a spanning tree of K_{41}?

(j) How many spanning trees does C_{41} have?

(k) How many spanning trees does K_{41} have?

(l) How many length-10 paths are there in K_{41}?

(m) How many length-10 cycles are there in K_{41}?

Problems for Section 15.8

Practice Problems

Problem 15.38.
Below is a list of properties that a group of people might possess.

For each property, either give the minimum number of people that must be in a group to ensure that the property holds, or else indicate that the property need not hold even for arbitrarily large groups of people.

(Assume that every year has exactly 365 days; ignore leap years.)

(a) At least 2 people were born on the same day of the year (ignore year of birth).

(b) At least 2 people were born on January 1.

(c) At least 3 people were born on the same day of the week.

(d) At least 4 people were born in the same month.

(e) At least 2 people were born exactly one week apart.

Class Problems

Problem 15.39.
Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) In a certain Institute of Technology, every ID number starts with a 9. Suppose that each of the 75 students in a class sums the nine digits of their ID number. Explain why two people must arrive at the same sum.

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.

(c) For any five points inside a unit square (not on the boundary), there are two points at distance *less than* $1/\sqrt{2}$.

(d) Show that if $n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$, two must be consecutive, that is, equal to k and $k + 1$ for some k.

Problem 15.40. **(a)** Prove that every positive integer divides a number such as 70, 700, 7770, 77000, whose decimal representation consists of one or more 7's followed by one or more 0's.

Hint: $7, 77, 777, 7777, \ldots$

(b) Conclude that if a positive number is not divisible by 2 or 5, then it divides a number whose decimal representation is all 7's.

Problem 15.41.
The aim of this problem is to prove that there exist a natural number n such that 3^n has at least 2013 consecutive zeros in its decimal expansion.

(a) Prove that there exist a nonnegative integer n such that

$$3^n \equiv 1 pmod\, 10^{2014}.$$

Hint: Use pigeonhole principle or Euler's theorem.

(b) Conclude that there exist a natural number n such that 3^n has at least 2013 consecutive zeros.

Problem 15.42. **(a)** Show that the Magician could not pull off the trick with a deck larger than 124 cards.

Hint: Compare the number of 5-card hands in an n-card deck with the number of 4-card sequences.

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of 124 cards.

Hint: Hall's Theorem and degree-constrained (12.5.5) graphs.

Problem 15.43.
The Magician can determine the 5th card in a poker hand when his Assisant reveals the other 4 cards. Describe a similar method for determining 2 hidden cards in a hand of 9 cards when your Assisant reveals the other 7 cards.

Problem 15.44.
Suppose $2n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 4n\}$. Using the Pigeonhole Principle, show that there must be two selected numbers whose difference is 2. Clearly indicate what are the pigeons, holes, and rules for assigning a pigeon to a hole.

Problem 15.45.
Let

$$k_1, k_2, \ldots, k_{101}$$

be a sequence of 101 integers. A sequence

$$k_{m+1}, k_{m+2}, \ldots, k_n$$

where $0 \leq m < n \leq 101$ is called a *subsequence*. Prove that there is a subsequence whose elements sum to a number divisible by 100.

Homework Problems

Problem 15.46. **(a)** Show that any odd integer x in the range $10^9 < x < 2 \cdot 10^9$ containing all ten digits $0, 1, \ldots, 9$ must have consecutive even digits.

Hint: What can you conclude about the parities of the first and last digit?

(b) Show that there are 2 vertices of equal degree in any finite undirected graph with $n \geq 2$ vertices.

Hint: Cases conditioned upon the existence of a degree zero vertex.

Problem 15.47.
Suppose $n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$. Using the Pigeonhole Principle, show that there must be two selected numbers whose quotient is a power of two. Clearly indicate what are the pigeons, holes, and rules for assigning a pigeon to a hole.

Hint: Factor each number into the product of an odd number and a power of 2.

Problem 15.48. **(a)** Let R be an 82×4 rectangular matrix each of whose entries are colored red, white or blue. Explain why at least two of the 82 rows in R must have identical color patterns.

(b) Conclude that R contains four points with the same color that form the corners of a rectangle.

(c) Now show that the conclusion from part (b) holds even when R has only 19 rows.

Hint: How many ways are there to pick two positions in a row of length four and color them the same?

Problem 15.49.

Section 15.8.6 explained why it is not possible to perform a four-card variant of the hidden-card magic trick with one card hidden. But the Magician and her Assistant are determined to find a way to make a trick like this work. They decide to change the rules slightly: instead of the Assistant lining up the three unhidden cards for the Magician to see, he will line up all four cards with one card face down and the other three visible. We'll call this the *face-down four-card trick*.

For example, suppose the audience members had selected the cards $9\heartsuit$, $10\diamondsuit$, $A\clubsuit$, $5\clubsuit$. Then the Assistant could choose to arrange the 4 cards in any order so long as one is face down and the others are visible. Two possibilities are:

$A\clubsuit$	?	$10\diamondsuit$	$5\clubsuit$
?	$5\clubsuit$	$9\heartsuit$	$10\diamondsuit$

(a) Explain how to model this face-down four-card trick as a matching problem, and show that there must be a bipartite matching which theoretically will allow the Magician and Assistant to perform the trick.

(b) There is actually a simple way to perform the face-down four-card trick.[10]

[10]This elegant method was devised in Fall '09 by student Katie E Everett.

> **Case 1**. *there are two cards with the same suit*: Say there are two ♠ cards. The Assistant proceeds as in the original card trick: he puts one of the ♠ cards *face up as the first card*. He will place the second ♠ card *face down*. He then uses a permutation of the face down card and the remaining two face up cards to code the offset of the face down card from the first card.
>
> **Case 2**. *all four cards have different suits*: Assign numbers $0, 1, 2, 3$ to the four suits in some agreed upon way. The Assistant computes s the sum modulo 4 of the ranks of the four cards, and chooses the card with suit s to be placed *face down as the first card*. He then uses a permutation of the remaining three face-up cards to code the rank of the face down card.

Explain how in Case 2. the Magician can determine the face down card from the cards the Assistant shows her.

(c) Explain how any method for performing the face-down four-card trick can be adapted to perform the regular (5-card hand, show 4 cards) with a 52-card deck consisting of the usual 52 cards along with a 53rd card called the *joker*.

Problem 15.50.
Suppose $2n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 4n\}$. Using the Pigeonhole Principle, show that for any positive integer j that divides $2n$, there must be two selected numbers whose difference is j. Clearly indicate what are the pigeons, holes, and rules for assigning a pigeon to a hole.

Problem 15.51.
Let's start by marking a point on a circle of length one. Next, mark the point that is distance $\sqrt{2}$ clockwise around the circle. So you wrap around once and actually mark the point at distance $\sqrt{2} - 1$ clockwise from the start. Now repeat with the newly marked point as the starting point. In other words, the marked points are those at clockwise distances

$$0, \sqrt{2},\ 2\sqrt{2},\ 3\sqrt{2}, \ldots,\ n\sqrt{2}, \ldots,$$

from the start.

We will use a pigeonhole argument to prove that marked points are *dense* on the circle: for any point p on the circle, and any $\epsilon > 0$, there is a marked point within distance ϵ of p.

(a) Prove that no point gets marked twice. That is, the points at clockwise distance $k\sqrt{2}$ and $m\sqrt{2}$ are the same iff $k = m$.

(b) Prove that among the first $n > 1$ marked points, there have to be two that are at most distance $1/n$ from each other.

(c) Prove that every point on the circle is within $1/n$ of a marked point. This implies the claim that the marked points are dense on the circle.

Exam Problems

Problem 15.52.
A standard 52 card deck has 13 cards of each suit. Use the Pigeonhole Principle to determine the smallest k such that every set of k cards from the deck contains five cards of the same suit (called a *flush*). Clearly indicate what are the pigeons, holes, and rules for assigning a pigeon to a hole.

Problem 15.53.
Use the Pigeonhole Principle to determine the smallest nonnegative integer n such that every set of n integers is guaranteed to contain three integers that are congruent mod 211. Clearly indicate what are the pigeons, holes, and rules for assigning a pigeon to a hole, and give the value of n.

Problems for Section 15.9

Practice Problems

Problem 15.54.
Let A_1, A_2, A_3 be sets with $|A_1| = 100$, $|A_2| = 1,000$, and $|A_3| = 10,000$.

Determine $|A_1 \cup A_2 \cup A_3|$ in each of the following cases, or give an example showing that the value cannot be determined.

(a) $A_1 \subset A_2 \subset A_3$.

(b) The sets are pairwise disjoint.

(c) For any two of the sets, there is exactly one element in both.

(d) There are two elements common to each pair of sets and one element in all three sets.

Problem 15.55.
The working days in the next year can be numbered 1, 2, 3, ..., 300. I'd like to avoid as many as possible.

- On even-numbered days, I'll say I'm sick.
- On days that are a multiple of 3, I'll say I was stuck in traffic.
- On days that are a multiple of 5, I'll refuse to come out from under the blankets.

In total, how many work days will I *avoid* in the coming year?

Problem 15.56.
Twenty people work at CantorCorp, a small, unsuccessful start-up. A single six-person committee is to be formed. (It will be charged with the sole task of working to prove the Continuum Hypothesis.) Employees appointed to serve on the committee join as equals—they do not get assigned distinct roles or ranks.

(a) Let D denote the set of all possible committees. Find $|D|$.

(b) Two of the workers, Aleph and Beth, will be unhappy if they are to serve together.

Let P denote the set of all possible committees on which Aleph and Beth would serve together. Find $|P|$.

(c) Beth will also be unhappy if she has to serve with **both** Ferdinand and Georg.

Let Q denote the set of all possible committees on which Beth, Ferdinand, and Georg would all serve together. Find $|Q|$.

(d) Find $|P \cap Q|$.

(e) Let S denote the set of all possible committees on which there is at least one unhappy employee. Express S in terms of P and Q **only**.

(f) Find $|S|$.

(g) If we want to form a committee with no unhappy employees, how many choices do we have to choose from?

(h) Suddenly, we realize that it would be better to have two six-person committees instead of one. (One committee would work on proving the Continuum Hypothesis, while the other would work to disprove it!) Each employee can serve on at most one committee. How many ways are there to form such a pair of committees, if employee happiness is **not** taken into consideration?

Class Problems

Problem 15.57.
To ensure password security, a company requires their employees to choose a password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password can be a cword which does not contain any of the subwords "fails", "failed", or "drop."

For example, the following two words are passwords: adefiloprs, srpolifeda, but the following three cwords are not: a**drop**eflis, **failedrop**s, **drop**e**fails**.

(a) How many cwords contain the subword "drop"?

(b) How many cwords contain both "drop" and "fails"?

(c) Use the Inclusion-Exclusion Principle to find a simple arithmetic formula involving factorials for the number of passwords.

Problem 15.58.
We want to count step-by-step paths between points in the plane with integer coordinates. Only two kinds of step are allowed: a right-step which increments the x coordinate, and an up-step which increments the y coordinate.

(a) How many paths are there from $(0, 0)$ to $(20, 30)$?

(b) How many paths are there from $(0, 0)$ to $(20, 30)$ that go through the point $(10, 10)$?

(c) How many paths are there from $(0, 0)$ to $(20, 30)$ that do *not* go through either of the points $(10, 10)$ and $(15, 20)$?

Hint: Let P be the set of paths from $(0, 0)$ to $(20, 30)$, N_1 be the paths in P that go through $(10, 10)$ and N_2 be the paths in P that go through $(15, 20)$.

Problem 15.59.
Let's develop a proof of the Inclusion-Exclusion formula using high school algebra.

(a) Most high school students will get freaked by the following formula, even though they actually know the rule it expresses. How would you explain it to them?

$$\prod_{i=1}^{n}(1+x_i) = \sum_{I \subseteq \{1,\dots,n\}} \prod_{j \in I} x_j. \tag{15.13}$$

Hint: Show them an example.

Now let $S_1, S_2, \dots, S_n$ be a sequence of finite sets, and let $U^n S_i$ be their union. The Inclusion-Exclusion rule 15.6 says

$$|U| = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right| \tag{I-E}$$

Now to start proving, let M_S be the *membership* function for any set S:

$$M_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases}$$

and abbreviate M_{S_i} as M_i. Set complements are defined relative to U, that is,

$$\overline{T} ::= U - T,$$

for $T \subseteq U$.

(b) Verify that for $T \subseteq U$ and $I \subseteq [1..n]$

$$M_{\overline{T}} = 1 - M_T, \tag{15.14}$$

$$M_{(\bigcap_{i \in I} S_i)} = \prod_{i \in I} M_i, \tag{15.15}$$

$$M_{(\bigcup_{i \in I} S_i)} = 1 - \prod_{i \in I}(1 - M_i). \tag{15.16}$$

Note that (15.15) holds when I is empty because, by convention, an empty product equals 1, and an empty intersection $\bigcap_{i \in \emptyset} S_i$ by definition equals U.

(c) Use (15.13) and (15.16) to prove

$$M_U = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j. \tag{15.17}$$

(d) Prove that

$$|T| = \sum_{u \in U} M_T(u), \tag{15.18}$$

for all $T \subseteq U$.

(e) Now use the previous parts to prove equation (I-E).

(f) Finally, explain why (I-E) immediately implies the usual form of the Inclusion-Exclusion Principle:

$$|U| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{\substack{I \subseteq [1..n] \\ |I|=i}} \left| \bigcap_{j \in I} S_j \right|. \tag{15.19}$$

Homework Problems

Problem 15.60.

A *derangement* is a permutation $(x_1, x_2, \ldots, x_n)$ of the set $\{1, 2, \ldots, n\}$ such that $x_i \neq i$ for all i. For example, $(2, 3, 4, 5, 1)$ is a derangement, but $(2, 1, 3, 5, 4)$ is not because 3 appears in the third position. The objective of this problem is to count derangements.

It turns out to be easier to start by counting the permutations that are *not* derangements. Let S_i be the set of all permutations $(x_1, x_2, \ldots, x_n)$ that are not derangements because $x_i = i$. So the set of non-derangements is

$$\bigcup_{i=1}^{n} S_i.$$

(a) What is $|S_i|$?

(b) What is $\left|S_i \cap S_j\right|$ where $i \neq j$?

(c) What is $\left|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}\right|$ where $i_1, i_2, \ldots, i_k$ are all distinct?

(d) Use the inclusion-exclusion formula to express the number of non-derangements in terms of sizes of possible intersections of the sets $S_1, \ldots, S_n$.

(e) How many terms in the expression in part (d) have the form

$$\left|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}\right|?$$

(f) Combine your answers to the preceding parts to prove the number of non-derangements is:

$$n!\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots\pm\frac{1}{n!}\right).$$

Conclude that the number of derangements is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots\pm\frac{1}{n!}\right).$$

(g) As n goes to infinity, the number of derangements approaches a constant fraction of all permutations. What is that constant? *Hint:*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Problem 15.61.

How many of the numbers up to n are prime? The Inclusion-Exclusion Principle offers a useful way to calculate the answer when n is large. Actually, we will use Inclusion-Exclusion to count the number of *composite* (nonprime) positive integers up to n. Subtracting this from n then gives the number of primes.

Let C_n be the set of composite numbers in $(1..n]$, and let A_m be the set of numbers in the interval $(m..n]$ that are divisible by m. Note that $A_m ::= \emptyset$ for $m \geq n$, by definition. So

$$C_n = \bigcup_{i=2}^{n-1} A_i. \tag{15.20}$$

(a) Verify that if $m \mid k$, then $A_m \supseteq A_k$.

(b) Explain why the right-hand side of (15.20) equals

$$\bigcup_{\text{primes } p \leq \sqrt{n}} A_p. \tag{15.21}$$

(c) Explain why $|A_m| = \lfloor n/m \rfloor - 1$ for $m \geq 2$.

(d) Consider any two relatively prime numbers $a, b \leq n$. What is the one number in $(A_a \cap A_b) - A_{ab}$?

(e) Let $\mathcal{P}$ be a finite set of at least two primes. Give a simple formula for

$$\left|\bigcap_{p \in \mathcal{P}} A_p\right|.$$

(f) Use the Inclusion-Exclusion principle to obtain a formula for $|C_{150}|$ in terms the sizes of intersections among the sets $A_2, A_3, A_5, A_7, A_{11}$. (Omit the intersections that are empty; for example, any intersection of more than three of these sets must be empty.)

(g) Use this formula to find the number of primes up to 150.

Problem 15.62.
Inclusion-Exclusion provides an alternative derivation of the formula for Euler's function given in Corollary 9.10.11:

$$\phi(n) = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right), \tag{15.22}$$

where $p_1, p_2, \ldots, p_m$ are the distinct prime factors of n.

The proof hinges on an algebraic formula about a product of simple sums:

$$\prod_{i=1}^{n} (1 + x_i) = \sum_{I \subseteq \{1,\ldots,n\}} \prod_{j \in I} x_j. \tag{15.23}$$

(a) Verify (15.23) in the case $n = 3$.

(b) Now briefly explain why (15.23) is true.

To prove (15.22), let S be the set of integers in $[0..n)$ that are *not* relatively prime to n, so $\phi(n) = n - |S|$.

(c) Let C_a be the set of integers in $[0..n)$ that are divisible by a:

$$C_a ::= \{k \in [0..n) \mid \quad a \text{ is a factor of } k\}.$$

Explain why

$$S = \bigcup_{i=1}^{m} C_{p_i}. \tag{15.24}$$

We'll be able to find the size of the union (15.24) using Inclusion-Exclusion because the intersections of the C_{p_i}'s are easy to count.

(d) Suppose p, q, r are distinct prime divisors of n. Prove that

$$|C_p \cap C_q \cap C_r| = \frac{n}{pqr}.$$

Of course the solution to part (d) extends to arbitrary intersections of C_p's. Accordingly, we now assume

$$\left|\bigcap_{j \in I} C_{p_j}\right| = \frac{n}{\prod_{j \in I} p_j}, \tag{15.25}$$

for any nonempty set $I \subseteq [1..m]$. In fact, equation (15.25) even holds for $I = \emptyset$ since, by convention, an empty intersection of subsets of $[0..n)$ equals $[0..n)$, and an empty product equals 1.

(e) Justify each of the steps in the following derivation of a formula for the size of S.

$$\begin{aligned}
|S| &= \left|\bigcup_{i=1}^{m} C_{p_i}\right| \\
&= \sum_{\emptyset \neq I \subseteq [1..m]} (-1)^{|I|+1} \left|\bigcap_{i \in I} C_{p_i}\right| \\
&= \left[\sum_{I \subseteq [1..m]} (-1)^{|I|+1} \left|\bigcap_{i \in I} C_{p_i}\right|\right] + n \\
&= n - \sum_{I \subseteq [1..m]} (-1)^{|I|} \left|\bigcap_{i \in I} C_{p_i}\right| \\
&= n - \sum_{I \subseteq [1..m]} (-1)^{|I|} \frac{n}{\prod_{j \in I} p_j} \\
&= n - n \sum_{I \subseteq [1..m]} \frac{1}{\prod_{j \in I} (-p_j)} \\
&= n - n \sum_{I \subseteq [1..m]} \prod_{j \in I} \left(-\frac{1}{p_j}\right) \\
&= n - n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right),
\end{aligned}$$

(f) Use part (e) to complete the proof of (15.22).

Exam Problems

Problem 15.63.
We want to count the number of length-n binary strings in which the substring `011` occurs in various places. For example, the length-14 string

0010**011**0**011**101,

has `011` in the 4th position and the 8th position. (Note that by convention, a length-n string starts with position zero and ends with position $n-1$.) Assume $n \geq 7$.

(a) Let r be the number of length-n binary strings in which `011` occurs starting at the 4th position. Write a formula for r in terms of n.

(b) Let A_i be the set of length-n binary strings in which `011` occurs starting at the ith position. (A_i is empty for $i > n-3$.) For any $i \neq j$, the intersection $A_i \cap A_j$ is either empty or of size s. Write a formula for s in terms of n.

(c) Let t be the number of pairs (i, j) such that $A_i \cap A_j$ is nonempty, where $0 \leq i < j$. Write a binomial coefficient for t in terms of n.

(d) How many length 9 binary strings are there that contain the substring `011`? You should express your answer as an integer or as a simple expression which may include the above constants r, s and t for $n = 9$.

Hint: Inclusion-exclusion for $\left|\bigcup_0^8 A_i\right|$.

Problem 15.64.
There are 10 students $A, B, \ldots, J$ who will be lined up left to right according to the some rules below.

Rule I: Student A must not be rightmost.

Rule II: Student B must be adjacent to C (directly to the left or right of C).

Rule III: Student D is always second.

You may answer the following questions with a numerical formula that may involve factorials.

(a) How many possible lineups are there that satisfy all three of these rules?

(b) How many possible lineups are there that satisfy at least one of these rules?

Problem 15.65.
A robot on a point in the 3-D integer lattice can move a unit distance in one positive direction at a time. That is, from position (x, y, z), it can move to either $(x + 1, y, z)$, $(x, y + 1, z)$ or $(x, y, z + 1)$. For any two points P and Q in space, let $n(P, Q)$ denote the number of distinct paths the spacecraft can follow to go from P to Q.

Let

$$A = (0, 10, 20), B = (30, 50, 70), C = (80, 90, 100), D = (200, 300, 400).$$

(a) Express $n(A, B)$ as a **single multinomial coefficient**.

Answer the following questions with arithmetic expressions involving terms $n(P, Q)$ for $P, Q \in \{A, B, C, D\}$. Do not use numbers.

(b) How many paths from A to C go through B?

(c) How many paths from B to D do *not* go through C?

(d) How many paths from A to D go through **neither** B **nor** C?

Problem 15.66.
In a standard 52-card deck (13 ranks and 4 suits), a hand is a 5-card subset of the set of 52 cards. Express the answer to each part as a formula using factorial, binomial, or multinomial notation.

(a) Let H be the set of all hands. What is $|H|$?

(b) Let H_{NP} be the set of all hands that include no pairs; that is, no two cards in the hand have the same rank. What is $|H_{NP}|$?

(c) Let H_S be the set of all hands that are straights, that is, the ranks of the five cards are consecutive. The order of the ranks is $(A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)$; note that A appears twice.
What is $|H_S|$?

(d) Let H_F be the set of all hands that are flushes, that is, the suits of the five cards are identical. What is $|H_F|$?

(e) Let H_{SF} be the set of all straight flush hands, that is, the hand is both a straight and a flush. What is $|H_{SF}|$?

(f) Let H_{HC} be the set of all high-card hands; that is, hands that do not include pairs, are not straights, and are not flushes. Write a formula for $|H_{HC}|$ in terms of $|H_{NP}|, |H_S|, |H_F|, |H_{SF}|$.

Problems for Section 15.10

Practice Problems

Problem 15.67.
Prove the following identity by algebraic manipulation and by giving a combinatorial argument:

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

Problem 15.68.
Give a combinatorial proof for this identity:

$$\sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} \binom{n}{i,\, j,\, k} = 3^n$$

Class Problems

Problem 15.69.
According to the Multinomial theorem, $(w + x + y + z)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, r_3, r_4} w^{r_1} x^{r_2} y^{r_3} z^{r_4}.$$

(a) How many terms are there in the sum?

(b) The sum of these multinomial coefficients has an easily expressed value. What is it?

$$\sum_{\substack{r_1+r_2+r_3+r_4=n, \\ r_i\in\mathbb{N}}} \binom{n}{r_1,\ r_2,\ r_3,\ r_4} = ? \tag{15.26}$$

Hint: How many terms are there when $(w + x + y + z)^n$ is expressed as a sum of monomials in w, x, y, z *before* terms with like powers of these variables are collected together under a single coefficient?

Problem 15.70.

(a) Give a combinatorial proof of the following identity by letting S be the set of all length-n sequences of letters a, b and a single c and counting $|S|$ in two different ways.

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} \tag{15.27}$$

(b) Now prove (15.27) algebraically by applying the Binomial Theorem to $(1 + x)^n$ and taking derivatives.

Problem 15.71.
What do the following expressions equal? Give both algebraic and combinatorial proofs for your answers.

(a)

$$\sum_{i=0}^{n} \binom{n}{i}$$

(b)

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i$$

Hint: Consider the bit strings with an even number of ones and an odd number of ones.

Problem 15.72.
When an integer k occurs as the kth element of a sequence, we'll say it is "in place" in the sequence. For example, in the sequence

12453678

precisely the integers $1, 2, 6, 7$ and 8 occur in place. We're going to classify the sequences of distinct integers from 1 to n, that is the permutations of $[1..n]$, according to which integers do not occur "in place." Then we'll use this classification to prove the combinatorial identity[11]

$$n! = 1 + \sum_{k=1}^{n} (k-1) \cdot (k-1)!\,. \tag{15.28}$$

If π is a permutation of $[1..n]$, let $\text{mnp}\,(\pi)$ be the *maximum* integer in $[1..n]$ that does not occur in place in π. For example, for $n = 8$,

$$\begin{aligned}
\text{mnp}\,(12345687) &= 8,\\
\text{mnp}\,(21345678) &= 2,\\
\text{mnp}\,(23145678) &= 3.
\end{aligned}$$

(a) For how many permutations of $[1..n]$ is every element in place?

(b) How many permutations π of $[1..n]$ have $\text{mnp}\,(\pi) = 1$?

(c) How many permutations π of $[1..n]$ have $\text{mnp}\,(\pi) = k$?

(d) Conclude the equation (15.28).

Problem 15.73.
Give a combinatorial proof for this identity:

$$\sum_{r=0}^{n} \binom{n}{r}\binom{m}{k-r} = \binom{n+m}{k}$$

Homework Problems

Problem 15.74. (a) Find a combinatorial (*not* algebraic) proof that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

(b) Below is a combinatorial proof of an equation. What is the equation?

[11] This problem is based on "Use of everywhere divergent generating function," mathoverflow, response 8,147 by Aaron Meyerowitz, Nov. 12, 2010.

Proof. Stinky Peterson owns n newts, t toads, and s slugs. Conveniently, he lives in a dorm with $n + t + s$ other students. (The students are distinguishable, but creatures of the same variety are not distinguishable.) Stinky wants to put one creature in each neighbor's bed. Let W be the set of all ways in which this can be done.

On one hand, he could first determine who gets the slugs. Then, he could decide who among his remaining neighbors has earned a toad. Therefore, $|W|$ is equal to the expression on the left.

On the other hand, Stinky could first decide which people deserve newts and slugs and then, from among those, determine who truly merits a newt. This shows that $|W|$ is equal to the expression on the right.

Since both expressions are equal to $|W|$, they must be equal to each other. ■

(Combinatorial proofs are real proofs. They are not only rigorous, but also convey an intuitive understanding that a purely algebraic argument might not reveal. However, combinatorial proofs are usually less colorful than this one.)

Problem 15.75.

Give a combinatorial proof for this identity:

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

Hint: Let S_i be the set of binary sequences with exactly n zeroes, $k+1$ ones, and a total of exactly i occurrences of zeroes appearing before the rightmost occurrence of a one.

Problem 15.76.

According to the Multinomial Theorem 15.6.5, $(x_1 + x_2 + \cdots + x_k)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}.$$

(a) How many terms are there in the sum?

(b) The sum of these multinomial coefficients has an easily expressed value:

$$\sum_{\substack{r_1+r_2+\cdots+r_k=n,\\ r_i\in\mathbb{N}}} \binom{n}{r_1,\ r_2,\ \ldots,\ r_k} = k^n \tag{15.29}$$

Give a combinatorial proof of this identity.

Hint: How many terms are there when $(x_1 + x_2 + \cdots + x_k)^n$ is expressed as a sum of monomials in x_i *before* terms with like powers of these variables are collected together under a single coefficient?

Problem 15.77.
You want to choose a team of m people for your startup company from a pool of n applicants, and from these m people you want to choose k to be the team managers. You took a Math for Computer Science subject, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}.$$

Before doing the reasonable thing—dump on your CFO or Harvard Business School—you decide to check his answer against yours.

(a) Give a *combinatorial proof* that your CFO's formula agrees with yours.

(b) Verify this combinatorial proof by giving an *algebraic* proof of this same fact.

Exam Problems

Problem 15.78.
Each day, an MIT student selects a breakfast from among b possibilities, lunch from among l possibilities, and dinner from among d possibilities. In each case one of the possibilities is Doritos. However, a legimate daily menu may include Doritos for at most one meal. Give a combinatorial (not algebraic) proof based on the number of legimate daily menus that

$$\begin{aligned} &bld - [(b-1) + (l-1) + (d-1) + 1] \\ &\qquad = (l-1)(d-1) + (b-1)(d-1) + (b-1)(l-1) + (b-1)(l-1)(d-1). \end{aligned}$$

Problem 15.79.
Give a combinatorial proof of

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = 2\binom{n+1}{3}$$

Hint: Classify sets of three numbers from the integer interval $[0..n]$ by their maximum element.