The IRS Compactification of Moduli Space

Part I

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Outline

Two parts:

1. **Part I: Introduction**
2. Part II: Description of the IRS compactification of moduli space

What is an IRS?

IRS stands for **I**nvariant **R**andom **S**ubgroup

Intuition: Let G be a group. An **invariant** random subgroup of G is a random variable taking values in the set of subgroups $\mathrm{Sub}(G)$ of G **with conjugation invariant law**.

Q: How can we formalize this?

A: Need a σ-algebra on $\mathrm{Sub}(G)$

$\rightsquigarrow$ Borel σ-algebra induced by the Chabauty topology on $\mathrm{Sub}(G)$!

The space $\mathrm{Sub}(G)$

Let G be a locally compact second countable group.

The set of all **closed** subgroups

$$\mathrm{Sub}(G) = \{H \leq G \text{ closed subgroup}\}$$

can be equipped with the *Chabauty topology*.

Properties

1. $\mathrm{Sub}(G)$ *is* **compact** *and metrizable with respect to the Chabauty topology.*
2. G *acts* **continuously** *on* $\mathrm{Sub}(G)$ *via conjugation,*

$$g * H := gHg^{-1}$$

for every $g \in G$ *and every* $H \in \mathrm{Sub}(G)$*.*

Formal definition of IRS

Definition (IRS)

An **invariant random subgroup** of G is a conjugation invariant Borel probability measure on $\mathrm{Sub}(G)$.

Corollary

The space of all invariant random subgroups $\mathrm{IRS}(G) = \mathrm{Prob}(\mathrm{Sub}(G))^G$ *is* **compact** *with respect to its weak*-topology.*

Where do IRSs come from?

A **large** source of examples:

Let $G \curvearrowright (X, \nu)$ be a probability measure preserving (PMP) action.

Fact

Then the stabilizer map

$$\begin{aligned} \mathrm{stab} \colon X &\longrightarrow \mathrm{Sub}(G), \\ x &\longmapsto G_x, \end{aligned}$$

pushes ν to an invariant random subgroup $\mu := \mathrm{stab}_*(\nu) \in \mathrm{IRS}(G)$.

Theorem (Abért–Glasner–Virág [AGV14], Abért–Bergeron–Biringer–Gelander–Nikolov–Raimbault–Samet [Abé+17])

Every IRS arises as a "stabilizer IRS" of some PMP action $G \curvearrowright (X, \nu)$.

An important special case: Lattices as IRSs

Let $\Gamma \leq G$ be a lattice. Then G acts on the quotient $\Gamma\backslash G$ via

$$h * (\Gamma g) = \Gamma g h^{-1} \quad \forall h \in G \, \forall \Gamma g \in \Gamma\backslash G,$$

and there is a unique invariant probability measure ν_Γ on $\Gamma\backslash G$.

In this case, the stabilizer of $\Gamma g \in \Gamma\backslash G$ is

$$\mathrm{stab}(\Gamma g) = g^{-1}\Gamma g.$$

As before, we obtain an IRS $\mu_\Gamma := (\varphi_\Gamma)_*(\nu_\Gamma)$ as the push-forward measure of ν_Γ along

$$\mathrm{stab} = \varphi_\Gamma \colon \Gamma\backslash G \longrightarrow \mathrm{Sub}(G), \quad \Gamma g \longmapsto g^{-1}\Gamma g.$$

Idea (Gelander [Gel15])

Use this construction to compactify the moduli space!

The moduli space of finite-area hyperbolic surfaces

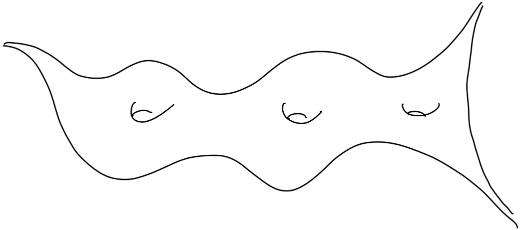
Let Σ be an oriented surface possibly punctured without boundary and negative Euler characteristic $\chi(\Sigma) < 0$.

Definition (Moduli Space)

The **moduli space** of finite-area hyperbolic surfaces homeomorphic to Σ is

$$\mathcal{M}(\Sigma) := \{X \text{ hyp. surf. with finite area} \,|\, X \cong \Sigma\}/\text{isometry}.$$

For example:

$X =$ 

Convention

From now on:

$$G := \mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$$

The moduli space and lattices in $G = \mathrm{PSL}(2, \mathbb{R})$

Two observations:

1. Every $X \in \mathcal{M}(\Sigma)$ is isometric to a quotient $X \cong \Gamma\backslash\mathbb{H}^2$ where $\Gamma < G$ is a lattice.
2. If $\Gamma_1\backslash\mathbb{H}^2, \Gamma_2\backslash\mathbb{H}^2 \in \mathcal{M}(\Sigma)$ are isometric then the isometry

$$\varphi\colon \Gamma_1\backslash\mathbb{H}^2 \longrightarrow \Gamma_2\backslash\mathbb{H}^2$$

lifts to an isometry

$$g := \tilde{\varphi}\colon \mathbb{H}^2 \longrightarrow \mathbb{H}^2 \in G$$

conjugating Γ_1 and $\Gamma_2 = g\Gamma_1 g^{-1}$.

The moduli space and lattices in $G = \mathrm{PSL}(2,\mathbb{R})$

Proposition

There is a one-to-one correspondence between $\mathcal{M}(\Sigma)$ and the set $G\backslash\mathcal{L}(\Sigma)$ of conjugacy classes of lattices via

$$G\backslash\mathcal{L}(\Sigma) \longrightarrow \mathcal{M}(\Sigma), \quad [\Gamma] \longmapsto \Gamma\backslash\mathbb{H}^2,$$

where

$$\mathcal{L}(\Sigma) \coloneqq \{\Gamma \leq G \text{ torsion-free lattice} \,|\, \Gamma\backslash\mathbb{H}^2 \cong \Sigma\} \subseteq \mathrm{Sub}(G).$$

This map is a homeomorphism.

Embedding the moduli space in IRS(G)

Proposition

The map

$$\begin{aligned} \iota\colon \mathcal{M}(\Sigma) &\longrightarrow \mathrm{IRS}(G), \\ X = \Gamma\backslash\mathbb{H}^2 &\longmapsto \mu_\Gamma, \end{aligned}$$

is a topological embedding.

Remark

The IRS μ_Γ *depends only on the conjugacy class* $[\Gamma] \in G\backslash\mathcal{L}(\Sigma)$.

The IRS compactification of moduli space

Definition (IRS Compactification; Gelander [Gel15])

The closure of its image

$$\overline{\mathcal{M}}^{\mathsf{IRS}}(\Sigma) \coloneqq \overline{\iota(\mathcal{M}(\Sigma))} \subset \mathsf{IRS}(G)$$

is called **the IRS compactification of** $\mathcal{M}(\Sigma)$.

Recall the corollary from the beginning:

Corollary

The space of all invariant random subgroups $\mathsf{IRS}(G) = \mathsf{Prob}(\mathsf{Sub}(G))^G$ *is* **compact**.

What is the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$?

We will answer this question in

Part II

Thank you!

References

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