EVERY COUNTABLE GROUP IS A SUBGROUP OF THE MAPPING CLASS GROUP OF THE LOCH NESS MONSTER SURFACE

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Abstract. In this short note we prove that every countable group is a subgroup of the mapping class group of the Loch Ness monster surface.

1. Introduction

A big Mapping Class Group is the Mapping Class Group (MCG) of a surface of infinite type. Big MCGs present intriguing similarities and differences from MCGs, in both algebraic and geometric terms. This paper focuses on subgroups of big MCGs.

For MCGs, and in general auotomorphisms of finite type surfaces, several results are known: Allcock [All06] proved that for any finite group F, there is a surface equipped with a hyperbolic structure X whose isometry group $\mathrm{Isom}(X)$ is isomorphic to F. Expanding on Allcock's ideas, Farb and Margalit [FM12, Theorem 7.12] proved that every finite group embeds in some MCG. Moreover, McCarthy [McC85] showed that MCGs satisfy the strong Tits alternative: every subgroup of a MCG is either virtually abelian or contains a free group.

In contrast, subgroups of big MCGs exhibit no rigidity: Lanier and Loving [LL20] proved that for any countable group C there exists a surface of infinite type Σ such that C embeds into $\mathrm{MGC}(\Sigma)$.

We expand on their result showing that there is a single surface of infinite type whose mapping class group contains every countable group as a subgroup: the Loch Ness monster surface L of [Ghy95]; see Figure 1.

Theorem 1.1. *Every countable group G is a subgroup of* $\mathrm{MCG}(L)$ *where L denotes the Loch Ness monster surface.*

2. Proof of Theorem 1.1

By the Higman–Neumann–Neumann embedding theorem [HNN49, Theorem IV] any countable group G is a subgroup of a finitely generated group H'. Thus, G is also a subgroup of the product $H := H' \times \mathbb{Z} \times \mathbb{Z}$.

Let S' denote a finite generating set for H'. Then $S = \{(s', 0, 0) \,|\, s' \in S'\} \cup \{(e, 1, 0), (e, 0, 1)\} \subseteq H' \times \mathbb{Z} \times \mathbb{Z}$ is a generating set of H. Observe that the Cayley graph $\Gamma := \mathrm{Cay}_S(H)$ is one-ended and has degree at least 4. Therefore, the surface Σ that is obtained by thickening the Cayley graph Γ of H is one-ended, too. Moreover, one may check that Σ has infinite genus. By the classification of topological surfaces of infinite type [Ker23; Ric63] Σ is thus homeomorphic to the Loch Ness monster L.

As a subgroup of H the group G acts freely on Γ via homeomorphisms. This action gives rise to a free action of G on $\Sigma \cong L$ by orientation preserving homeomorphisms, $\varphi\colon G \hookrightarrow \mathrm{Homeo}_+(L)$. Moreover, one obtains a continuous G-equivariant surjection $\pi\colon L \to \Gamma$ by projecting the thickening L to Γ.

For the monomorphism φ to pass to the mapping class group $\mathrm{MCG}(L) = \mathrm{Homeo}_+(L)/\mathrm{Homeo}_+^\circ(L)$ we will check that $\varphi(G) \cap \mathrm{Homeo}_+^\circ(L) = \{\mathrm{id}\}$. Assume to the contrary that there is a non-trivial element $g \in G$ such that $\varphi(g)$ is isotopic to the identity. Let e be an edge in the identity coset $\mathbb{Z}^2$ and observe that $ge \neq e$. Note that $\Gamma - \{e, ge\}$ is connected: Both cosets $\mathbb{Z}^2$ and $g \cdot \mathbb{Z}^2$ remain connected (as subgraphs of Γ) after removing the edges e and ge, respectively. As any vertex in Γ can be connected to any $\mathbb{Z}^2$-coset via a path with edges in $S' \subseteq H'$, the graph $\Gamma - \{e, ge\}$ is connected. Let γ be the core curve in $\pi^{-1}(e^\circ) \cong S^1 \times (0,1)$. Then $g\gamma$ needs to be isotopic to the core curve of, and contained in, $\pi^{-1}(ge^\circ)$. However, γ and $g\gamma$ cannot be isotopic: Otherwise, a component of $L - \{\gamma, g\gamma\}$ would be an annulus. But $L - \{\gamma, g\gamma\}$ is connected, since $\Gamma - \{e, ge\}$ is connected; a contradiction. ☐

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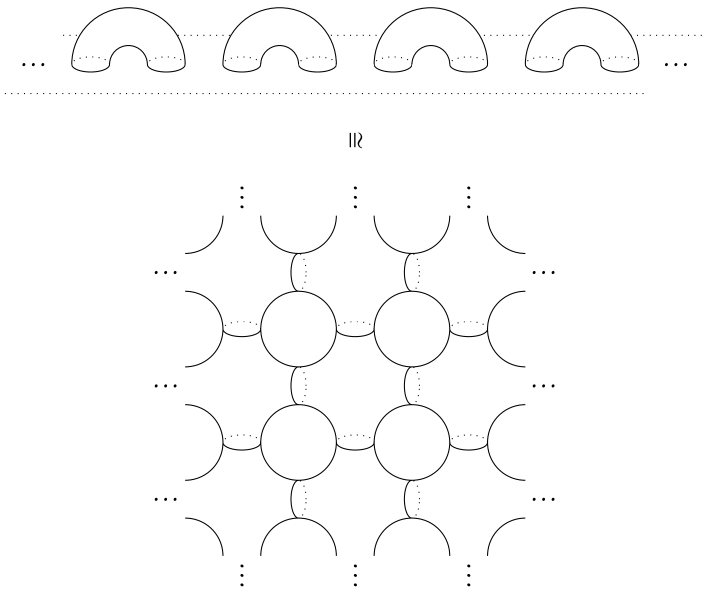

Figure 1. The Loch Ness monster surface L, also known as the infinite prison window.

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