A NOTE ON SUBGROUPS OF THE LOCH NESS MONSTER SURFACE'S MAPPING CLASS GROUP

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Abstract. In this short note we give an elementary proof of the fact that every countable group is a subgroup of the mapping class group of the Loch Ness monster surface.

1. Introduction

The following theorem follows from a deep result of Aougab–Patel–Vlamis [APV21] about hyperbolic isometry groups of infinite type surfaces:

Theorem 1.1. *Every countable group G is a subgroup of* $\mathrm{MCG}(L)$ *where L denotes the Loch Ness monster surface.*

The purpose of this short note is to provide an elementary proof of this fact.

2. Proof of Theorem 1.1

By the Higman–Neumann–Neumann embedding theorem [HNN49, Theorem IV] any countable group G is a subgroup of a finitely generated group H'. Thus, G is also a subgroup of the product $H := H' \times \mathbb{Z} \times \mathbb{Z}$. Let S' denote a finite generating set for H'. Then $S = \{(s', 0, 0) \mid s' \in S'\} \cup \{(e, 1, 0), (e, 0, 1)\} \subseteq H' \times \mathbb{Z} \times \mathbb{Z}$ is a generating set of H. Observe that the Cayley graph $\Gamma := \mathrm{Cay}_S(H)$ is one-ended and has degree at least 4. Therefore, the surface Σ that is obtained by thickening the Cayley graph Γ of H is one-ended, too. Moreover, one may check that Σ has infinite genus. By the classification of topological surfaces of infinite type [Ker23; Ric63] Σ is thus homeomorphic to the Loch Ness monster surface L; see Figure 1.

As a subgroup of H, the group G acts freely on Γ via graph automorphisms. Thus, we can choose an invariant hyperbolic metric on Σ so that this action gives rise to a free action of G on $\Sigma \cong L$ by orientation preserving isometries. As every non-trivial isometry is a homotopically non-trivial homeomorphism [Nor89], we obtain an injection $G \hookrightarrow \mathrm{Homeo}_+(L)/\mathrm{Homeo}_+^\circ(L) = \mathrm{MCG}(L)$. $\square$

Remark 2.1. The same result holds for the blooming Cantor tree surface instead of the Loch Ness monster surface. This is obtained by replacing the group H with the group $K = H' * \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ in the proof.

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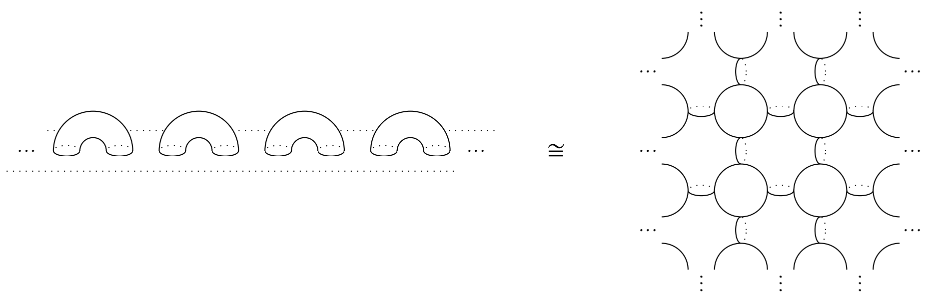

Figure 1. The Loch Ness monster surface L of [PS81; Ghy95], also known as the infinite prison window.

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