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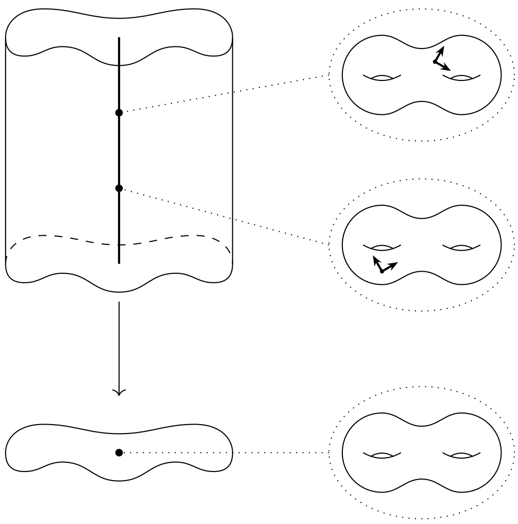
The IRS Compactification of Moduli Space

by Yannick Krifka



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The IRS Compactification of Moduli Space

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Dedicated to my parents

Dr. Petra Krifka and Dr. Franz-Josef Krifka

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Contents

Abstract

The space of invariant random subgroups $\mathrm{IRS}(G)$ of a locally compact Hausdorff group G is compact with respect to its weak*-topology. In [Gel15] Gelander observed that the moduli space of finite-area hyperbolic structures $\mathcal{M}(\Sigma)$ on an oriented topological surface Σ can be embedded in $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$. Taking the closure in $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ he defined the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ and proposed to analyze this new compactification. This is the main focus of this thesis.

Our approach is to relate the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ to the classical Deligne–Mumford compactification $\widehat{\mathcal{M}}(\Sigma)$. We show that there is a continuous finite-to-one surjection $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$, that identifies the IRS compactification with a quotient of $\widehat{\mathcal{M}}(\Sigma)$. Moreover, we give a uniform upper bound on the number of elements in a fiber of $\widehat{\Phi}$ in terms of the Euler characteristic, the number of punctures, and the genus of Σ.

As an application we describe a geometric construction of a convex compact subset of $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ with dense extreme points.

In addition to that we use Gelander's embedding of $\mathcal{M}(\Sigma_g)$ in $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ to define the genus g Weil–Petersson IRS $\mu_g^{\mathrm{WP}} \in \mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ as the barycenter of $\mathcal{M}(\Sigma_g) \subseteq \mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ with respect to its finite Weil–Petersson volume. We show that μ_g^{WP} converges to the trivial IRS $\delta_{\{\mathrm{id}\}} \in \mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ as the genus g tends to infinity. This result was obtained independently and at the same time by Monk [Mon20].

Zusammenfassung

Der Raum der invarianten zufälligen Untergruppen $\mathrm{IRS}(G)$ einer lokal kompakten Hausdorffschen Gruppe G ist kompakt bezüglich seiner Schwach-*-Topologie. In [Gel15] kam Gelander zu der Erkenntnis, dass der Modulraum $\mathcal{M}(\Sigma)$ der hyperbolischen Strukturen endlichen Flächeninhaltes auf einer orientierbaren topologischen Fläche Σ in $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ eingebettet werden kann. Mithilfe dieser Einbettung definierte er die IRS-Kompaktifizierung $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ als den Abschluss des Modulraums in $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$, und schlug vor diese neue Kompaktifizierung zu untersuchen. Das ist der Schwerpunkt dieser Dissertation.

Unser Ansatz ist die IRS-Kompaktifizierung $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ mit der klassischen Deligne–Mumford Kompaktifizierung $\widehat{\mathcal{M}}(\Sigma)$ in Verbindung zu bringen. Wir zeigen, dass es eine stetige endlich-zu-eins Surjektion $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ gibt, welche die IRS-Kompaktifizierung mit einem Quotienten von $\widehat{\mathcal{M}}(\Sigma)$ identifiziert. Darüber hinaus zeigen wir, dass es eine uniforme obere Schrank an die Anzahl der Element in einem Urbild von $\widehat{\Phi}$ gibt, welche von der Euler-Charakteristik, der Anzahl an markierten Punkten und dem Geschlecht von Σ abhängt.

Als Anwendung beschreiben wir eine geometrische Konstruktion einer konvexen und kompakten Teilmenge von $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ deren Extrempunkte dicht liegen.

Man kann Gelanders Einbettung von $\mathcal{M}(\Sigma_g)$ in $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ auch benutzen um das Weil–Petersson IRS $\mu_g^{\mathrm{WP}} \in \mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ als den Schwerpunkt des Modulraums $\mathcal{M}(\Sigma_g) \subseteq \mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ bezüglich seines endlichen Weil–Petersson Masses zu definieren. Wir zeigen, dass μ_g^{WP} gegen das triviale IRS $\delta_{\{\mathrm{id}\}} \in \mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$ konvergiert, wenn das Geschlecht g gegen Unendlich geht. Dieses Resultat wurde unabhängig und zur gleichen Zeit von Monk [Mon20] bewiesen.

1 Introduction

Although implicitly studied in [BG04; Ver12], invariant random subgroups (IRSs) were introduced only recently in [AGV14; Abé+17] and they have attracted considerable research activity since then [AGV14; Bow15; Gel15; Abé+17; Abé+18; GL18a; GL18b; LBMB18; Gel19; BLT19; Zhe19]. As we will see, they generalize lattices and have connections with many other areas of mathematics. For instance, it was observed by Gelander [Gel15] that the space of IRSs of $\mathrm{PSL}(2,\mathbb{R})$ can be used to obtain a new compactification of the moduli space of finite-area hyperbolic surfaces. This *IRS compactification* and its relation with the Deligne–Mumford compactification is the main focus of this thesis.

Before describing our results in more details in section 1.5 we will start with a geometric introduction to the theory of IRSs. This perspective reveals some of the fascinating connections of IRSs with other areas in mathematics. In particular, we will see that there is an interesting relationship between Benjamini–Schramm convergence of graphs and discrete IRSs, which extends even beyond the discrete case.

The material presented in sections 1.1, 1.2, 1.3 and 1.4 is well-known and can be found in [AGV14; Abé+17; Gel15; Gel19]. Our exposition is inspired by Gelander's articles [Gel15; Gel19], which we recommend for further reading.

1.1 Invariant Random Subgroups

Let us start with an informal discussion of *invariant random subgroups*. Given a locally compact Hausdorff group G a natural interpretation of a "random subgroup of G" is as a random variable taking values in the set $\mathrm{Sub}(G)$ of closed subgroups of G. Bearing in mind that G acts via conjugation on $\mathrm{Sub}(G)$, one is immediately led to an intuitive definition of invariant random subgroups: An invariant random subgroup is a random variable taking values in $\mathrm{Sub}(G)$ with a conjugation invariant probability distribution.

This "definition" has some measure theoretic issues. Indeed, in order to talk about random variables we need a notion of measurability, i.e. we need a σ-algebra on $\mathrm{Sub}(G)$. It turns out that $\mathrm{Sub}(G)$ carries a natural topology – the Chabauty topology [Cha50] – with respect to which it is a compact Hausdorff space; see section 2.7 for more details. Thus, we can consider the associated Borel σ-algebra on $\mathrm{Sub}(G)$.

Now, we could define an invariant random subgroup of G to be a Borel measurable map (a random variable) from some probability space to $\mathrm{Sub}(G)$ with conjugation invariant probability distribution. Conflating such random variables with their probability distributions we arrive at the following formal definition of invariant random subgroups.

Definition 1.1.1 (Invariant Random Subgroup)**.** Let G be a locally compact Hausdorff group and denote by $\mathrm{Sub}(G)$ its space of closed subgroups. An *invariant random subgroup (IRS)* of G is a conjugation invariant Borel probability measure on $\mathrm{Sub}(G)$.

We will denote by $\mathrm{IRS}(G) := \mathrm{Prob}(\mathrm{Sub}(G))^G$ the set of IRSs of G and equip it with the weak*-topology turning it into a compact Hausdorff space; see Lemma 4.1.2.

From our informal discussion one can see that IRSs arise naturally in a context, in which there is a way of sampling subgroups "randomly". This is the case when G admits a probability measure preserving (PMP) action on a (countably separated) probability space (X,ν). Indeed, by sampling random points $x \in X$ with probability distribution ν the corresponding stabilizer subgroups[1] $\mathrm{stab}(x) \in \mathrm{Sub}(G)$ give rise to an IRS of G. We will refer to this IRS as the stabilizer IRS with respect to the action $G \curvearrowright (X,\nu)$. Curiously, this construction is generic: Every IRS of a locally compact second countable group arises as the stabilizer IRS with respect to some PMP action; see [AGV14, Proposition 14] and [Abé+17, Theorem 2.6].

An important class of examples for PMP actions come from lattices. Recall that a lattice $\Gamma \leq G$ is a discrete subgroup such that the quotient $\Gamma\backslash G$ admits a unique invariant[2] Borel probability measure ν_Γ. In particular, the action $G \curvearrowright (\Gamma\backslash G, \nu_\Gamma)$ is PMP. Observe that the stabilizer of a point $\Gamma h \in \Gamma\backslash G$ is given by $\mathrm{stab}(\Gamma h) = h^{-1}\Gamma h$. Therefore, the stabilizer IRS μ_Γ with respect to the action $G \curvearrowright \Gamma\backslash G$ is given by the push-forward measure

[1] By a result of Varadarajan the stabilizers are closed subgroups; see [Zim84, Corollary 2.1.20].

[2] Here invariance is to be understood with respect to the action $G \curvearrowright \Gamma\backslash G$ given by $g \cdot (\Gamma h) = \Gamma h g^{-1}$ for $g \in G$ and $\Gamma h \in \Gamma\backslash G$.

$\mu_\Gamma \coloneqq (\varphi_\Gamma)_*(\nu_\Gamma)$ along the map

$$\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow \mathrm{Sub}(G), \quad \Gamma h \longmapsto h^{-1}\Gamma h.$$

In this way one may think of lattices as IRSs, which motivated a lot of research in the field. Indeed, many results about lattices find natural generalizations to results about IRSs. For example, there is a version of the Borel Density Theorem [Abé+17; GL18a] and the Kazhdan–Margulis Theorem [Gel18] for IRSs.

Remarkably, the proof of the latter is very elegant and makes use of the weak*-topology of IRS(G). That taking a probabilistic point of view facilitates certain arguments has already been observed in mathematics before. For example, probabilistic methods (as introduced by Erdős–Rényi [ER59]) have proven very successful in Graph Theory. IRSs can be seen as another instance of this phenomenon.

1.2 Benjamini–Schramm Convergence and Discrete IRSs

In their seminal paper [BS01] Benjamini and Schramm introduced a notion of distributional limits of finite graphs. It turns out that there is a strong connection between the notion of Benjamini–Schramm convergence and the theory of IRSs of discrete groups. At its heart lies a combinatorial interpretation of the space of (closed) subgroups, which we will explain now. This will be the blueprint for a similar geometric interpretation of IRSs in the non-discrete setting that we will discuss in section 1.4.

Suppose that G is a discrete group with a finite symmetric generating set $S \subseteq G$. We may think of any subgroup $H \in \mathrm{Sub}(G)$ in terms of its *Schreier graph* $\mathrm{Sch}_S(H\backslash G)$. The vertices of $\mathrm{Sch}_S(H\backslash G)$ are given by the cosets $H\gamma \in H\backslash G$ and there is a directed s-labeled edge from $H\gamma$ to $H\gamma s$ for every generator $s \in S$. Notice that Schreier graphs allow for a canonical choice of a root vertex, namely the identity coset $He \in H\backslash G$.

An important special case is the *Cayley graph* $\mathrm{Cay}(G,S)$, which is the Schreier graph of the trivial subgroup $\{e\} \in \mathrm{Sub}(G)$. One should think of every Schreier graph as being locally modeled on a Cayley graph. In fact, for every subgroup $H \in \mathrm{Sub}(G)$ one obtains its Schreier graph $\mathrm{Sch}_S(H\backslash G) \cong H\backslash \mathrm{Cay}(G,S)$ as the quotient of $\mathrm{Cay}(G,S)$ by the left-multiplication action $H \curvearrowright \mathrm{Cay}(G,S)$.

On the set $\mathcal{G}^S_\bullet$ of isomorphism classes of rooted directed S-edge-labeled graphs there is a natural notion of distance. It makes sense to think of two (isomorphism classes of) graphs $[\Gamma_1, v_1]$ and $[\Gamma_2, v_2]$ as "close", if "large" balls about their respective roots v_1 and v_2 are isomorphic. This intuition translates into the following metric on $\mathcal{G}_\bullet^S$

$$d([\Gamma_1, v_1], [\Gamma_2, v_2]) := \inf\{2^{-r} \mid r > 0, [B_{v_1}(r), v_1] = [B_{v_2}(r), v_2]\},$$

where the infimum is taken over all real numbers $r > 0$ such that the r-balls $B_{v_1}(r) \subseteq \Gamma_1$, $B_{v_2}(r) \subseteq \Gamma_2$ about their roots are isomorphic as rooted directed S-edge-labeled graphs.

This metric induces a topology on the subset of Schreier graphs rooted at their identity coset

$$\mathrm{SC}_S(G) := \{[\mathrm{Sch}_S(H\backslash G), He] \mid H \in \mathrm{Sub}(G)\} \subseteq \mathcal{G}_\bullet^S.$$

A fundamental observation due to Abért–Glasner–Virág [AGV14] is that this topology is compatible with the topology of $\mathrm{Sub}(G)$: The map

$$\psi\colon\ \mathrm{Sub}(G) \longrightarrow \mathrm{SC}_S(G), \quad H \longmapsto [\mathrm{Sch}_S(H\backslash G), He],$$

is a homeomorphism. This allows us to think of a closed subgroup $H \in \mathrm{Sub}(G)$ as its rooted Schreier graph $[\mathrm{Sch}_S(H\backslash G), He] \in \mathrm{SC}_S(G)$.

In the light of this identification the conjugation action of G on $\mathrm{Sub}(G)$ translates into a change of the root vertex in the Schreier graph. Indeed, for two conjugate subgroups H, $g^{-1}Hg \in \mathrm{Sub}(G)$, $g \in G$, there is a graph isomorphism induced by the map

$$H\backslash G \longrightarrow (g^{-1}Hg)\backslash G, \quad H\gamma \longmapsto (g^{-1}Hg)g^{-1}\gamma.$$

This isomorphism maps the vertex $Hg \in H\backslash G$ to $(g^{-1}Hg)e \in (g^{-1}Hg)\backslash G$, such that $[\mathrm{Sch}_S((g^{-1}Hg)\backslash G), (g^{-1}Hg)e] = [\mathrm{Sch}_S(H\backslash G), Hg]$.

Remark 1.2.1. In [AL07] Aldous and Lyons introduced *unimodular random networks*. Using the above interpretation of the conjugation action one can show that a probability measure $\mu \in \mathrm{Prob}(\mathrm{Sub}(G))$ is conjugation invariant if and only if $\psi_*(\mu) \in \mathrm{Prob}(\mathrm{SC}_S(G))$ is a unimodular random network; see [AGV14, Proposition 14].

Let us apply this graph theoretical interpretation of discrete IRSs to two examples:

Example 1.2.2 (IRS concentrated at a normal subgroup). Let $N \trianglelefteq G$ be a normal subgroup. Then clearly the Dirac measure δ_N concentrated at N is an IRS of G. Via the identification $\mathrm{Sub}(G) \cong \mathrm{SC}_S(G)$ we obtain a Borel probability measure on $\mathrm{SC}_S(G)$, that is concentrated at the Schreier graph $[\mathrm{Sch}_S(N\backslash G), Ne]$.

Example 1.2.3 (IRSs from lattices). We have already seen that lattices amount naturally to IRSs. Because G is discrete, the lattices of G are precisely its finite-index subgroups. Applying our construction of the IRS μ_Λ to a finite-index subgroup $\Lambda \leq G$ we obtain

$$\mu_\Lambda = \frac{1}{|\Lambda : G|} \sum_{\Lambda g \in \Lambda \backslash G} \delta_{g^{-1}\Lambda g}.$$

By the above reasoning we may think of the conjugate $g^{-1}\Lambda g \in \mathrm{Sub}(G)$, $g \in G$, as the Schreier graph $\mathrm{Sch}_S(\Lambda\backslash G)$ with root at the coset $\Lambda g \in \Lambda\backslash G$. Hence, we get that

$$\psi_*(\mu_\Lambda) = \frac{1}{|\Lambda : G|} \sum_{\Lambda g \in \Lambda \backslash G} \delta_{[\mathrm{Sch}_S(\Lambda\backslash G), \Lambda g]}.$$

Recall that $\mathrm{Sch}_S(\Lambda\backslash G)$ is a *finite* graph with $|\Lambda : G|$-many vertices. Therefore, the measure $\psi_*(\mu_\Lambda)$ can be interpreted as sampling the finite graph $\mathrm{Sch}_S(\Lambda\backslash G)$ with a uniformly distributed root vertex.

This last example rediscovers Benjamini and Schramm's construction for finite graphs. In [BS01][3] they associated to every finite (directed, S-edge-labeled, connected) graph $\Gamma = (V, E)$ a Borel probability measure on the space of rooted graphs $\mathcal{G}_\bullet^S$ via

$$\lambda_\Gamma := \frac{1}{|V|} \sum_{v \in V} \delta_{[\Gamma, v]} \in \mathrm{Prob}(\mathcal{G}_\bullet^S).$$

This amounts to an injection

$$j \colon \mathcal{G}_{\mathrm{fin}}^S \longrightarrow \mathrm{Prob}(\mathcal{G}_\bullet^S), \quad \Gamma \longmapsto \lambda_\Gamma,$$

of the set of (isomorphism classes of) finite graphs $\mathcal{G}_{\mathrm{fin}}^S$ into the space of Borel probability measures $\mathrm{Prob}(\mathcal{G}_\bullet^S)$. The latter is equipped with the weak*-topology, which allowed Benjamini and Schramm to define a notion of *distributional limits of finite graphs*: A sequence of finite graphs $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{G}_{\mathrm{fin}}^S$ *Benjamini–Schramm converges* to a probability measure $\lambda \in \mathrm{Prob}(\mathcal{G}_\bullet^S)$, if λ is the limit of the corresponding sequence of probability measures $(\lambda_{\Gamma_n})_{n\in\mathbb{N}}$. A limiting measure for a sequence of finite graphs is called a *Benjamini–Schramm limit*.

Revisiting Example 1.2.3 we see that $\psi_*(\mu_\Lambda) = \lambda_{\mathrm{Sch}_S(\Lambda\backslash G)}$. In particular, given a se-

[3] Actually, Benjamini and Schramm were working with unlabeled graphs, but their definition carries over to this setting verbatim; see also Aldous–Lyons [AL07].

quence of finite-index subgroups $(\Lambda_n)_{n\in\mathbb{N}} \subseteq \mathrm{Sub}(G)$, the sequence $(\mu_{\Lambda_n})_{n\in\mathbb{N}} \subseteq \mathrm{IRS}(G)$ converges to $\mu \in \mathrm{IRS}(G)$, if and only if their Schreier graphs $\mathrm{Sch}_S(\Lambda_n \backslash G)$ Benjamini–Schramm converge to $\psi_*(\mu) \in \mathrm{Prob}(\mathrm{SC}_S(G))$.

It is useful to think about the Benjamini–Schramm construction in terms of the forgetful map

$$F\colon \mathcal{G}^S_\bullet \longrightarrow \mathcal{G}^S, \quad [\Gamma, v] \longmapsto \Gamma,$$

from $\mathcal{G}^S_\bullet$ to the set of (isomorphism classes of) directed S-edge-labeled graphs $\mathcal{G}^S$. For every finite graph $\Gamma \in \mathcal{G}^S_{\mathrm{fin}}$ the fiber $F^{-1}(\Gamma)$ over Γ consists of the rooted isomorphism classes $[\Gamma, v]$ for different choices of the root vertex $v \in V(\Gamma)$. By sampling the root vertex uniformly at random from $V(\Gamma)$ one obtains the probability measure $\lambda_\Gamma \in \mathrm{Prob}(\mathcal{G}^S_\bullet)$, which is supported on the fiber $F^{-1}(\Gamma)$.

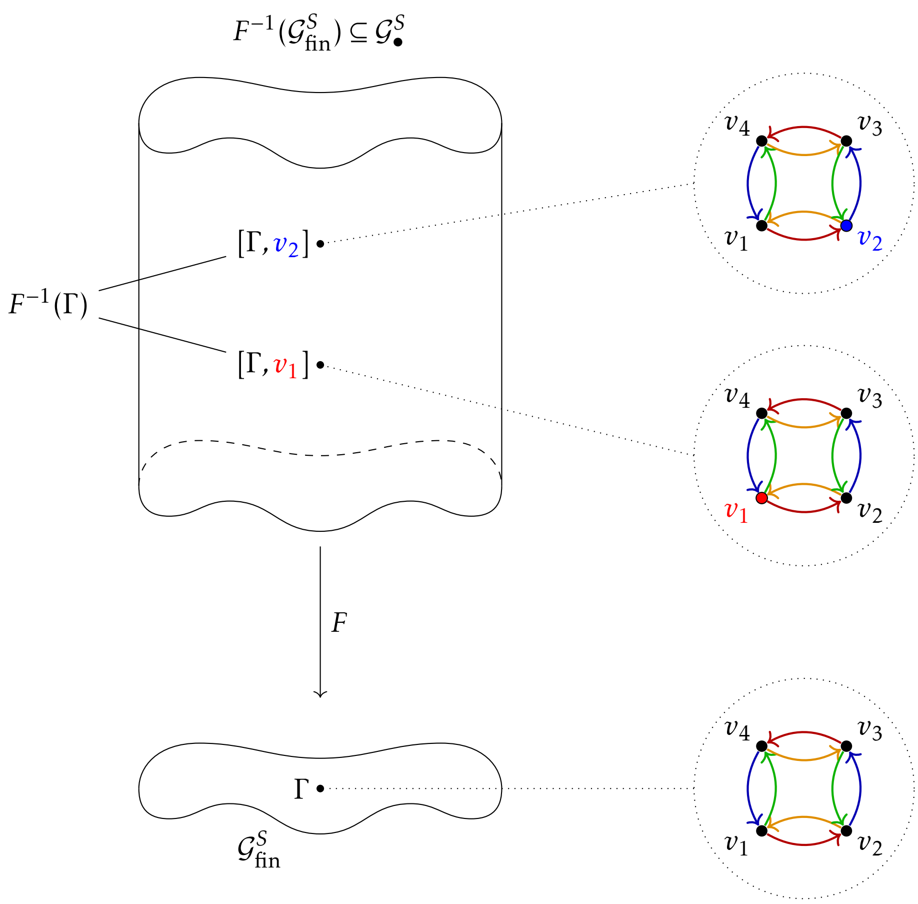


Figure 1.1

Example 1.2.4. In Figure 1.1 we see the graph $\Gamma \in \mathcal{G}^S_{\text{fin}}$ with vertices $V(\Gamma) = \{v_1, v_2, v_3, v_4\}$ and edges labeled/colored by $S = \{a, a^{-1}, b, b^{-1}\}$. Its fiber $F^{-1}(\Gamma) = \{[\Gamma, v_1], [\Gamma, v_2]\}$ consists only of two rooted graphs. Indeed, the map $v_1 \longmapsto v_3, v_2 \longmapsto v_4$ induces a graph automorphism preserving the edge labels, whence $[\Gamma, v_1] = [\Gamma, v_3]$ and $[\Gamma, v_2] = [\Gamma, v_4]$. On the other hand $[\Gamma, v_1] \neq [\Gamma, v_2]$, since v_1 has only an *outgoing* edge labeled by a, whereas v_2 has only an *incoming* edge labeled by a. Hence, $\lambda_\Gamma = \frac{1}{2}\delta_{[\Gamma, v_1]} + \frac{1}{2}\delta_{[\Gamma, v_2]}$.

1.3 Sofic Groups and Cosofic IRSs

In the previous section we have discussed the connection between Benjamini–Schramm convergence of Schreier graphs and IRSs of discrete groups. As we will see now, this amounts to an interpretation of sofic groups in terms of IRSs of the free group.

Definition 1.3.1 ([Wei00])**.** A group G with finite symmetric generating set S is called *sofic* if its Cayley graph $\mathrm{Cay}(G, S)$ is a Benjamini–Schramm limit, i.e. if there is a sequence of finite directed S-edge-labeled graphs that Benjamini–Schramm converges to the Dirac measure $\delta_{[\mathrm{Cay}(G,S),e]} \in \mathrm{Prob}(\mathcal{G}^S_\bullet)$.[4]

Sofic groups have been the stimulus of much research activity [Gro99; Wei00; ES04; ES05; ES06; Tho08; GR08; Bow10; ES11; Bow11; Hay16; Gle17; AP16; Arz+19]. They were first introduced by Gromov [Gro99] as "initially subamenable groups", generalizing both residually finite groups and amenable groups at the same time. Gromov proved that Gottschalk's conjecture holds for this new class of groups. Later, the term "sofic group" was coined by Weiss [Wei00]. In addition to Gottschalk's, many other conjectures are known to hold for sofic groups, such as Kaplansky's direct finiteness conjecture [ES04] and the Connes' embedding conjecture for group von Neumann algebras [ES05]. The latter in itself implies several others.

It is remarkable that up to this point no example of a non-sofic group is known, although its existence was already conjectured by Weiss:

"It is not likely that all groups are sofic – but I don't know of any definite example of a non sofic group." [Wei00, p. 351]

Using the dictionary between IRSs of discrete groups and Benjamini–Schramm convergence of Schreier graphs from section 1.2 one obtains a characterization of soficity in terms of cosofic IRSs:

[4] This definition does not depend on the finite generating set S; see [Wei00].

Definition 1.3.2. Let G be a locally compact Hausdorff group. An IRS $\mu \in \mathrm{IRS}(G)$ is called *cosofic* if it is the limit of a sequence of IRSs $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathrm{IRS}(G)$ supported on lattices in G.

Observe that a group G with finite symmetric generating set S can be written as a quotient $N\backslash\mathbb{F}$, where $\mathbb{F} = \langle S\rangle$ is the free group generated by S and $N \trianglelefteq \mathbb{F}$ is an appropriate normal subgroup. By definition the Schreier graph $\mathrm{Sch}_S(N\backslash\mathbb{F})$ is just the Cayley graph $\mathrm{Cay}(G,S)$. Using this observation one can show the following characterization of sofic groups in terms of cosofic IRSs of the free group $\mathbb{F}$:

Lemma 1.3.3 ([Gel19, Proposition 6.1], [AGN17, Lemma 16])**.** *Let $\mathbb{F}$ be a finitely generated free group and let $N \trianglelefteq \mathbb{F}$ be a normal subgroup. Then the quotient $N\backslash\mathbb{F}$ is sofic if and only if $\delta_N \in \mathrm{IRS}(\mathbb{F})$ is cosofic.*

Missing an example of a non-sofic group we do not know if there is a normal subgroup $N \trianglelefteq \mathbb{F}$ such that δ_N is not cosofic. More generally, one may ask if there are non-cosofic IRSs at all:

Question 1.3.4 ([Gel15, Question 2.6], [Gel19, Question 6.2])**.** Is every IRS of G cosofic?

In the case of a free group $G = \mathbb{F}$ this question is equivalent to a question of Aldous and Lyons [AL07, Question 10.1] asking whether every unimodular random network (supported on Schreier graphs) is a Benjamini–Schramm limit; see Remark 1.2.1. By Lemma 1.3.3 a positive answer would imply that every group is sofic.

However, the interest of Question 1.3.4 is, that one may ask it for other groups G as well. Because of the following general construction it is particularly appealing to ask it for groups that admit free groups as lattice subgroups.

Definition 1.3.5 ([Gel15, Section 2.1])**.** Let G be a locally compact Hausdorff group and let $\Gamma \leq G$ be a lattice. Then every $\mu \in \mathrm{IRS}(\Gamma)$ *induces* an IRS of G given by

$$\mathrm{ind}_\Gamma^G(\mu) = \int_{\Gamma\backslash G} (g^{-1})_*\mu \, d\nu_\Gamma(\Gamma g),$$

where ν_Γ denotes the unique invariant probability measure on $\Gamma\backslash G$, and $(g^{-1})_*$ denotes taking the push-forward measure with respect to conjugation by the element $g^{-1} \in G$.

One can verify that the map $\mathrm{ind}_\Gamma^G\colon \mathrm{IRS}(\Gamma) \longrightarrow \mathrm{IRS}(G)$ is continuous and maps IRSs supported on lattices of Γ to IRSs supported on lattices of G. In particular, ind_Γ^G maps cosofic IRSs to cosofic IRSs. Hence, if $N \trianglelefteq \Gamma$ is a normal subgroup and $\delta_N \in \mathrm{IRS}(\Gamma)$ is cosofic, then $\mathrm{ind}_\Gamma^G(\delta_N) \in \mathrm{IRS}(G)$ is cosofic. *Vice versa*, if $\mathrm{ind}_\Gamma^G(\delta_N) \in \mathrm{IRS}(G)$ is *not* cosofic then $\delta_N \in \mathrm{IRS}(\Gamma)$ is *not* cosofic. This shows the importance of studying cosofic IRSs in groups that admit free groups as lattice subgroups.

A prominent example of a Lie group that admits free groups as lattice subgroups is the group of orientation preserving isometries $G = \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}_+(\mathbb{H}^2)$ of the hyperbolic plane $\mathbb{H}^2$. Indeed, every torsion-free lattice $\Gamma \leq G$ with non-compact quotient $\Gamma\backslash G$ is a free group as the fundamental group of the punctured hyperbolic surface $\Gamma\backslash\mathbb{H}^2$. We will study cosofic IRSs of $\mathrm{PSL}(2,\mathbb{R})$, when we consider the IRS compactification of moduli space.

1.4 The IRS Compactification of Moduli Space

In [Gel15] Gelander first observed that one can embed the moduli space of finite-area hyperbolic surfaces in the space of invariant random subgroups $\mathrm{IRS}(\mathrm{PSL}(2,\mathbb{R}))$. This led him to define the IRS compactification of moduli space, which is the central object of this thesis. We will describe his construction in this section and highlight the similarities between Gelander's embedding of the moduli space and the Benjamini–Schramm embedding of finite graphs.

Recall that any oriented topological surface Σ with negative Euler characteristic admits a (complete) hyperbolic metric with finite area; see e.g. [Abi80]. However, this hyperbolic metric is far from being unique. In fact, there is an interesting *moduli space* $\mathcal{M}(\Sigma)$ of all isometry classes of finite-area hyperbolic surfaces homeomorphic to Σ. (Here and in the following we will only consider orientation preserving isometries.) The moduli space $\mathcal{M}(\Sigma)$ has already been studied for a long time in mathematics and traces back to Riemann, who studied the equivalent question of how many different complex structures the surface Σ admits.

For our purposes it will be advantageous to think about the moduli space $\mathcal{M}(\Sigma)$ as the set of conjugacy classes of certain torsion-free lattices in $\mathrm{Sub}(G)$. To this end, observe that (up to isometry) there is only one complete simply connected hyperbolic surface: the *hyperbolic plane* $\mathbb{H}^2$. Therefore, every hyperbolic surface $X \in \mathcal{M}(\Sigma)$ is covered by $\mathbb{H}^2$, whence X is isometric to a quotient $\Gamma\backslash\mathbb{H}^2$, where $\Gamma \leq \mathrm{Isom}_+(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R}) =: G$ is a discrete and torsion-free subgroup. In fact, the subgroup $\Gamma \leq G$ is a lattice, because X has finite area. Now, any isometry between two hyperbolic surfaces $X_1 = \Gamma_1\backslash\mathbb{H}^2$

and $X_2 = \Gamma_2\backslash\mathbb{H}^2$ lifts to an isometry $g\colon \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ conjugating the lattices Γ_1 and Γ_2. Vice versa, any element $g \in G$ conjugating Γ_1 and $\Gamma_2 = g^{-1}\Gamma_1 g$ descends to an isometry $\Gamma_1\backslash\mathbb{H}^2 \cong \Gamma_2\backslash\mathbb{H}^2$. Hence, if we denote

$$\mathcal{L}(\Sigma) := \{\Gamma \leq G \mid \Gamma \text{ torsion-free lattice such that } \Gamma\backslash\mathbb{H}^2 \cong \Sigma\} \subseteq \mathrm{Sub}(G)$$

and the quotient of $\mathcal{L}(\Sigma)$ by the conjugation action of G by $G\backslash\mathcal{L}(\Sigma)$, then the map

$$G\backslash\mathcal{L}(\Sigma) \longrightarrow \mathcal{M}(\Sigma), \quad [\Gamma] \longmapsto \Gamma\backslash\mathbb{H}^2,$$

is a bijection (and, in fact, a homeomorphism; see Proposition 2.8.4).

We saw in section 1.1 that every lattice $\Gamma \leq G$ gives rise to an IRS μ_Γ. Furthermore, this construction depends only on the conjugacy class of Γ; see Lemma 4.1.3. Hence, we obtain a map from the moduli space $\mathcal{M}(\Sigma)$ into the space of IRSs of $G = \mathrm{PSL}(2,\mathbb{R})$

$$\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G), \quad X = \Gamma\backslash\mathbb{H}^2 \longmapsto \mu_X := \mu_\Gamma.$$

It can be checked that ι is injective (and, in fact, a topological embedding; see Proposition 4.1.5), so that one may think of $\mathcal{M}(\Sigma)$ as a subset of $\mathrm{IRS}(G)$.

Recall that $\mathrm{IRS}(G)$ is a compact Hausdorff space, such that the closure $\overline{\iota(\mathcal{M}(\Sigma))}$ is compact, too. This is how Gelander defined the *IRS compactification* $\overline{\mathcal{M}}^{\mathit{IRS}}(\Sigma) := \overline{\iota(\mathcal{M}(\Sigma))}$ *of the moduli space* $\mathcal{M}(\Sigma)$ in [Gel15, Section 3.1]. There he also proposed the following problem:

Problem 1.4.1 ([Gel15, Problem 3.2])**.** Analyze the IRS compactification of $\mathcal{M}(\Sigma)$.

Remark 1.4.2. Observe that every point in the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is a cosofic IRS by definition. Hence, one can interpret Problem 1.4.1 as asking for a description of a certain subset of the cosofic IRSs of $G = \mathrm{PSL}(2,\mathbb{R})$.

We will give a detailed account of Gelander's Problem 1.4.1 in this thesis. But before we turn to our results in this direction, let us give a geometric interpretation of the injection ι. It will turn out that $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ can be thought of as a non-discrete version of Benjamini–Schramm's embedding $j\colon \mathcal{G}^S_{\mathrm{fin}} \longrightarrow \mathrm{Prob}(\mathcal{G}^S_\bullet)$ of finite graphs; see section 1.2.

As in the case of a discrete group this will rely on an interpretation of the space of closed subgroups of $G = \mathrm{PSL}(2,\mathbb{R})$. However, here we will only give a geometric description of the subspace of discrete and torsion-free subgroups $\mathrm{Sub}_{\mathrm{dtf}}(G) \subseteq \mathrm{Sub}(G)$. We will see that there is a homeomorphism between $\mathrm{Sub}_{\mathrm{dtf}}(G)$ and the space of framed hyperbolic surfaces $\mathcal{FM}(\mathbb{H}^2)$. This will allow us to interpret IRSs supported on discrete and torsion-free subgroups $\mathrm{IRS}_{\mathrm{dtf}}(G) \subseteq \mathrm{IRS}(G)$ as (certain) probability measures on the space of framed hyperbolic surfaces $\mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$.

A *framed hyperbolic surface* $(X, \mathbf{f})$ is a hyperbolic surface X together with a positively oriented orthonormal frame $\mathbf{f} = (f_1, f_2) \in \mathrm{OF}_+(X)$. We will identify two framed hyperbolic surfaces $(X, \mathbf{f}), (Y, \mathbf{g})$ if there is an (orientation preserving) isometry $\varphi \colon X \longrightarrow Y$ whose differential maps $\mathbf{f} = (f_1, f_2)$ to $\mathbf{g} = (g_1, g_2)$, i.e. $d\varphi(f_1) = g_1, d\varphi(f_2) = g_2$. The space $\mathcal{FM}(\mathbb{H}^2)$ consists of all equivalence classes of framed hyperbolic surfaces $[X, \mathbf{f}]$. One can put a topology on $\mathcal{FM}(\mathbb{H}^2)$ in the spirit of the topology on $\mathcal{G}_\bullet^S$ as defined in section 1.2. Indeed, it makes sense to think of two framed hyperbolic surfaces $[X, \mathbf{f}], [Y, \mathbf{g}] \in \mathcal{FM}(\mathbb{H}^2)$ as being "close" if there are "large" balls $B_1 \subseteq X$ resp. $B_2 \subseteq Y$ about the base points of $\mathbf{f}$ resp. $\mathbf{g}$ and an "approximate isometry" $\varphi \colon B_1 \longrightarrow B_2$, such that $d\varphi(\mathbf{f})$ is "close" to $\mathbf{g}$; see section 2.7 for a precise definition.

Let us fix a positively oriented orthonormal frame $\mathbf{e} \in \mathrm{OF}_+(\mathbb{H}^2)$. For any discrete and torsion-free subgroup $\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G)$ one obtains a framed hyperbolic surface $[\Gamma\backslash\mathbb{H}^2, d\pi_\Gamma(\mathbf{e})]$, where $\pi_\Gamma \colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ is the corresponding quotient map. Then the map

$$\psi \colon \mathrm{Sub}_{\mathrm{dtf}}(G) \longrightarrow \mathcal{FM}(\mathbb{H}^2), \quad \Gamma \longmapsto [\Gamma\backslash\mathbb{H}^2, d\pi_\Gamma(\mathbf{e})],$$

is a homeomorphism; see Proposition 2.7.15.

Via this identification the conjugation action of G on $\mathrm{Sub}_{\mathrm{dtf}}(G)$ amounts to a change of base frame in $\mathcal{FM}(\mathbb{H}^2)$. Indeed, if $\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G)$ and $g \in G$ then

$$\Gamma\backslash\mathbb{H}^2 \longrightarrow (g^{-1}\Gamma g)\backslash\mathbb{H}^2, \quad \pi_\Gamma(x) \longmapsto \pi_{g^{-1}\Gamma g}(g^{-1}x),$$

is an isometry and maps $d\pi_\Gamma(dg(\mathbf{e}))$ to $d\pi_{g^{-1}\Gamma g}(\mathbf{e})$. Hence,

$$[\Gamma\backslash\mathbb{H}^2, d\pi_\Gamma(dg(\mathbf{e}))] = [(g^{-1}\Gamma g)\backslash\mathbb{H}^2, d\pi_{g^{-1}\Gamma g}(\mathbf{e})].$$

Let us compare this to the case of a lattice/finite-index subgroup H of a discrete group G as discussed in section 1.2. Here the hyperbolic plane $\mathbb{H}^2$ takes the place of the Cayley graph $\mathrm{Cay}(G, S)$ and the role of the Schreier graph $\mathrm{Sch}_S(H\backslash G) \cong H\backslash\mathrm{Cay}(G, S)$ is played by the quotient surface $\Gamma\backslash\mathbb{H}^2$. Furthermore, we keep track of an orthonormal frame $\mathbf{f} \in \mathrm{OF}_+(\Gamma\backslash\mathbb{H}^2)$ instead of a root vertex. Indeed, the conjugation action amounts to a change of the orthonormal frame here, whereas it changes the root vertex in the discrete case. In contrast to the discrete situation, where we consider the entire space of subgroups $\mathrm{Sub}(G)$, the subspace of discrete and torsion-free subgroups $\mathrm{Sub}_{\mathrm{dtf}}(\mathrm{PSL}(2,\mathbb{R}))$ is not compact. However, using the Borel Density Theorem for IRSs one can show that $\mathrm{IRS}_{\mathrm{dtf}}(\mathrm{PSL}(2,\mathbb{R}))$ is still compact; see [AB16, Proposition 5.6].

As for discrete groups we can use the homeomorphism $\psi\colon \mathrm{Sub}_{\mathrm{dtf}}(G) \longrightarrow \mathcal{FM}(\mathbb{H}^2)$ to obtain a geometric description of $\mu_\Gamma \in \mathrm{IRS}(G)$ for a torsion-free lattice $\Gamma \leq G = \mathrm{PSL}(2,\mathbb{R})$. Recall that μ_Γ is the push-forward measure of the unique invariant probability measure ν_Γ on $\Gamma\backslash G$ under the map

$$\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow \mathrm{Sub}_{\mathrm{dtf}}(G), \quad \Gamma g \longmapsto g^{-1}\Gamma g.$$

Notice that one can identify $\Gamma\backslash G$ with the orthonormal frame bundle[5] $\mathrm{OF}_+(\Gamma\backslash\mathbb{H}^2)$ using the map

$$\Gamma\backslash G \longrightarrow \mathrm{OF}_+(\Gamma\backslash\mathbb{H}^2), \quad \Gamma g \longmapsto d\pi_\Gamma(dg(\mathbf{e})).$$

Via this identification the probability measure ν_Γ on $\Gamma\backslash G$ translates into the canonical (normalized) volume form $\overline{\nu}_{\Gamma\backslash\mathbb{H}^2}$ on $\mathrm{OF}_+(\Gamma\backslash\mathbb{H}^2)$. In analogy to the Benjamini–Schramm construction for finite graphs, we can use this probability measure $\overline{\nu}_{\Gamma\backslash\mathbb{H}^2}$ to sample a frame uniformly at random. Indeed, one obtains a probability measure $\lambda_{\Gamma\backslash\mathbb{H}^2} := (\overline{\varphi}_{\Gamma\backslash\mathbb{H}^2})_*(\overline{\nu}_{\Gamma\backslash\mathbb{H}^2}) \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ on the space of framed hyperbolic surfaces as the push-forward of $\overline{\nu}_{\Gamma\backslash\mathbb{H}^2}$ along the map

$$\overline{\varphi}_{\Gamma\backslash\mathbb{H}^2}\colon \mathrm{OF}_+(\Gamma\backslash\mathbb{H}^2) \longrightarrow \mathcal{FM}(\mathbb{H}^2), \quad \mathbf{f} \longmapsto [\Gamma\backslash\mathbb{H}^2, \mathbf{f}].$$

Altogether we obtain the following commutative diagram:

$$\begin{tikzcd}
(\Gamma\backslash G, \nu_\Gamma) \arrow[r, "\varphi_\Gamma"] \arrow[d, "\wr"'] & (\mathrm{Sub}_{\mathrm{dtf}}(G), \mu_\Gamma) \arrow[d, "\wr"', "\psi"] \\
(\mathrm{OF}_+(\Gamma\backslash\mathbb{H}^2), \overline{\nu}_{\Gamma\backslash\mathbb{H}^2}) \arrow[r, "\overline{\varphi}_{\Gamma\backslash\mathbb{H}^2}"] & (\mathcal{FM}(\mathbb{H}^2), \lambda_{\Gamma\backslash\mathbb{H}^2}).
\end{tikzcd}$$

In particular, the IRS μ_Γ corresponds to the measure $\lambda_{\Gamma\backslash\mathbb{H}^2} = \psi_*(\mu_\Gamma) \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ under the identification $\psi\colon \mathrm{Sub}_{\mathrm{dtf}}(G) \longrightarrow \mathcal{FM}(\mathbb{H}^2)$.

[5]Alternatively, one can identify $\Gamma\backslash G$ with the unit tangent bundle $T^1(\Gamma\backslash\mathbb{H}^2)$. Indeed, in dimension two the second vector of a positively oriented orthonormal frame is uniquely determined by the first.

Let us summarize this discussion. We can define a map

$$j\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2)), \quad X \longmapsto \lambda_X,$$

that associates to every finite-area hyperbolic surface X the measure $\lambda_X \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ obtained by sampling frames in $\mathrm{OF}_+(X)$ uniformly at random. Under the identification $\psi\colon \mathrm{Sub}_{\mathrm{dtf}}(G) \longrightarrow \mathcal{FM}(\mathbb{H}^2)$ the map $j\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ corresponds to the embedding $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$, i.e. the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{IRS}_{\mathrm{dtf}}(G) \arrow[d, "\psi_*"] \\
\mathcal{M}(\Sigma) \arrow[ur, "\iota"] \arrow[dr, "j"'] & \\
 & \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2)).
\end{tikzcd}$$

In analogy to the case of finite graphs we can define *Benjamini–Schramm convergence of hyperbolic surfaces*. A sequence of hyperbolic surfaces with finite area $(X_n)_{n\in\mathbb{N}}$ is said to *Benjamini–Schramm converge* to a probability measure $\lambda \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$, if their corresponding measures $(\lambda_{X_n})_{n\in\mathbb{N}}$ converge to $\lambda \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ in the weak*-topology. Therefore, given a sequence of torsion-free lattices $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathrm{Sub}_{\mathrm{dtf}}(G)$, the sequence $(\mu_{\Gamma_n})_{n\in\mathbb{N}} \subseteq \mathrm{IRS}_{\mathrm{dtf}}(G)$ converges to $\mu \in \mathrm{IRS}_{\mathrm{dtf}}(G)$, if and only if the hyperbolic surfaces $(\Gamma_n\backslash\mathbb{H}^2)_{n\in\mathbb{N}}$ Benjamini–Schramm converge to $\psi_*(\mu) \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$.

As in the case of graphs we can think about this construction in terms of the forgetful map

$$F\colon \mathcal{FM}(\mathbb{H}^2) \longrightarrow \mathcal{M}, \quad [X,\mathbf{f}] \longmapsto X,$$

from $\mathcal{FM}(\mathbb{H}^2)$ to the space of isometry classes of hyperbolic surfaces $\mathcal{M}$. For every hyperbolic surface $X \in \mathcal{M}$ the fiber $F^{-1}(X)$ over X consists of the framed isometry classes $[X,\mathbf{f}]$ for different choices of the base frame $\mathbf{f} \in \mathrm{OF}_+(X)$. By sampling the base frame uniformly at random from $\mathrm{OF}_+(X)$ one obtains the probability measure $\lambda_X \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$, which is supported on the fiber $F^{-1}(X)$; see Figure 1.2.

It is tempting to think of $F^{-1}(\mathcal{M}(\Sigma)) \subseteq \mathcal{FM}(\mathbb{H}^2)$ as a fiber bundle over the moduli space $\mathcal{M}(\Sigma)$ via the forgetful map F. However, this is not quite correct. By definition of $\mathcal{FM}(\mathbb{H}^2)$ two framed hyperbolic surfaces $[X,\mathbf{f}],[X,\mathbf{g}]$ are identified if there is an isometry $\varphi \in \mathrm{Isom}(X)$, such that $d\varphi(\mathbf{f}) = \mathbf{g}$; $X \in \mathcal{M}(\Sigma)$. Therefore, the fiber $F^{-1}(X)$

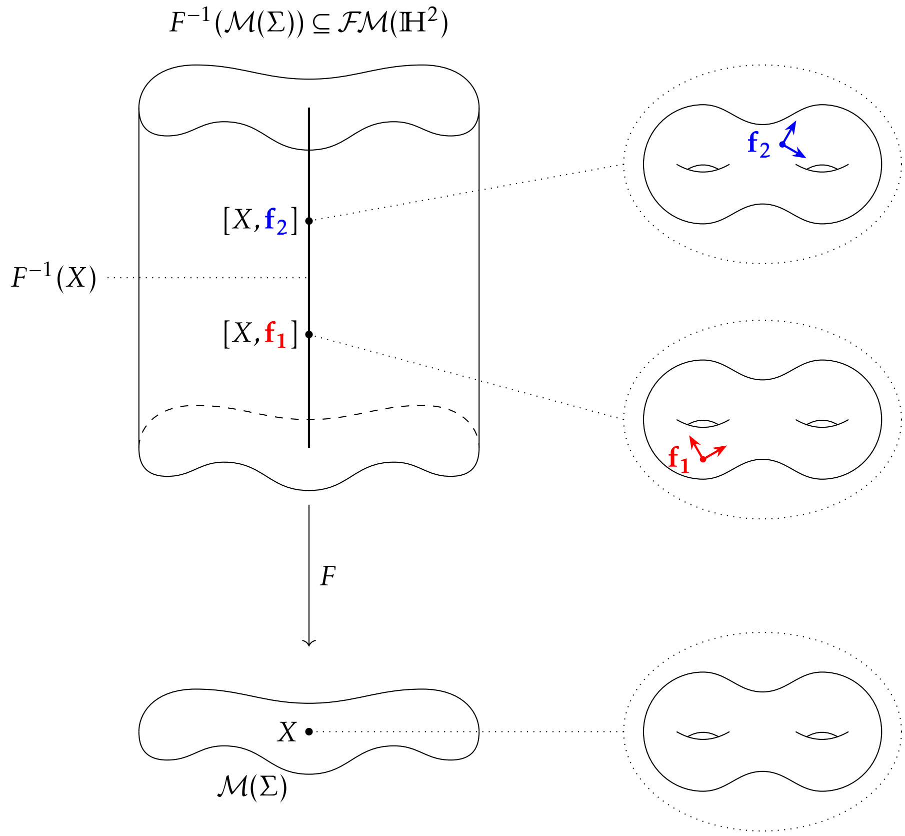


Figure 1.2

is homeomorphic to $\mathrm{Isom}(X)\backslash \mathrm{OF}_+(X)$. Although a generic point $X \in \mathcal{M}(\Sigma)$ has a trivial isometry group $\mathrm{Isom}(X)$, there exist highly symmetric hyperbolic surfaces. Hence, there are fibers of F that are not homeomorphic, such that $F^{-1}(\mathcal{M}(\Sigma))$ is not a fiber bundle over $\mathcal{M}(\Sigma)$. We have already observed the same phenomenon in the case of graphs; see Example 1.2.4.

Remark 1.4.3. With the IRS compactification in mind we focused on the case of $G = \mathrm{PSL}(2, \mathbb{R})$ and framed hyperbolic surfaces. However, there is an analogous picture for any semisimple Lie group G. In this case the symmetric space $\mathcal{X} = G/K$ associated to G takes the place of the hyperbolic plane $\mathbb{H}^2$, and the space of framed locally-$\mathcal{X}$ manifolds $\mathcal{FM}(\mathcal{X})$ plays the role of the space of framed hyperbolic surfaces $\mathcal{FM}(\mathbb{H}^2)$; see [Abé+17, Section 3.9] for details.

Remark 1.4.4. Instead of working with framed hyperbolic surfaces, one can just consider pointed hyperbolic surfaces as well. This approach is developed by Abért–Biringer in [AB16] and leads to the notion of *unimodular random manifolds* analogous to Aldous–Lyons' definition of unimodular random networks [AL07]; see Remark 1.2.1. It turns out that there is a dictionary between $\mathrm{IRS}_{\mathrm{dtf}}(G)$ and the space of unimodular random hyperbolic surfaces similar to the map $\psi_*\colon \mathrm{IRS}_{\mathrm{dtf}}(G) \longrightarrow \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ above; see [AB16, Proposition 2.9].

1.5 Results

The focus of this thesis is to address Gelander's Problem 1.4.1, and analyze the IRS compactification of the moduli space $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$.

In a first step we verify that the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is a compactification of the moduli space $\mathcal{M}(\Sigma)$ in a topological sense. Unlike the set of finite graphs $\mathcal{G}^S_{\mathrm{fin}}$, the moduli space $\mathcal{M}(\Sigma)$ already comes equipped with a natural topology as a quotient of the Teichmüller space $\mathcal{T}(\Sigma)$, and it is apriori not clear that this topology is compatible with the injection $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$. The following proposition shows that this is the case.

Proposition 1.5.1 (Proposition 4.1.5). *The map*

$$\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G), \quad X = \Gamma\backslash\mathbb{H}^2 \longmapsto \mu_X = \mu_\Gamma,$$

is a topological embedding.

The difficult part in the proof of Proposition 4.1.5 is to show that ι is continuous. In the case of a closed surface Σ this is proved by Gelander and Levit [GL18b, Proposition 11.2]. An important ingredient in our proof is Lemma 3.1.1, which shows that the truncated Dirichlet domain of a discrete and torsion-free subgroup depends continuously on the group; see chapter 3 for a precise statement.

In order to understand the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ we relate it to the classical Deligne–Mumford compactification $\widehat{\mathcal{M}}(\Sigma)$. We regard the Deligne–Mumford compactification as the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$; see section 2.6. Intuitively, a point $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ in the augmented moduli space can be thought of as a nodal surface, where a family of disjoint simple closed curves σ in Σ is collapsed to nodes and every complementary component $\Sigma_i \in c(\sigma) := \pi_0(\Sigma \setminus \sigma)$, $i = 1, \ldots, m$, carries a hyperbolic metric of finite area. These hyperbolic metrics give rise to a point $X_i \in \mathcal{M}(\Sigma_i)$ in the moduli space of each component $\Sigma_i \in c(\sigma)$, $i = 1, \ldots, m$, and we call the hyperbolic surfaces $X_1, \ldots, X_m$ the *parts* of $\mathbf{X}$. One can imagine that the parts $X_1, \ldots, X_m$ are glued according to some pairing of their punctures to form the nodal surface $\mathbf{X}$; see Figure 1.3.

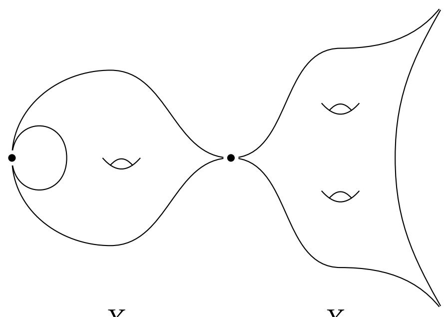


Figure 1.3: A nodal surface $\mathbf{X} \in \mathcal{M}(\Sigma_{4,2})$ with parts $X_1 \in \mathcal{M}(\Sigma_{1,2})$, $X_2 \in \mathcal{M}(\Sigma_{2,3})$.

Keeping this description in mind there is a rather natural map from the augmented moduli space to the IRS compactification

$$\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma).$$

Indeed, let $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ be a nodal surface with parts $X_i \in \mathcal{M}(\Sigma_i)$, $i = 1, \ldots, m$. Every part X_i is the quotient $\Gamma_i \backslash \mathbb{H}^2$ for a torsion-free lattice and amounts to an IRS $\mu_{X_i} = \mu_{\Gamma_i} = \iota(X_i)$, $i = 1, \ldots, m$. Moreover, the area of a hyperbolic surface is a topological invariant $\mathrm{vol}_{X_i}(X_i) = -2\pi\chi(\Sigma_i)$, $i = 1, \ldots, m$, and $\chi(\Sigma) = \sum_{i=1}^m \chi(\Sigma_i)$. In particular, it makes sense to think about the quotient $\chi(\Sigma_i)/\chi(\Sigma)$ as the proportion of area that the part X_i takes up in $\mathbf{X}$. This motivates the definition

$$\widehat{\Phi}(\mathbf{X}) := \sum_{i=1}^{m} \frac{\chi(\Sigma_i)}{\chi(\Sigma)} \cdot \mu_{X_i} \in \mathrm{IRS}(G),$$

which is a convex combination of IRSs.

An immediate observation is that the map $\widehat{\Phi}$ forgets about how the parts $X_1, \ldots, X_m$ are glued to form the nodal surface $\mathbf{X}$. Hence, it makes sense to pass to a quotient of the augmented moduli space that captures this phenomenon. To this end, we consider the set $|\widehat{\mathcal{M}}| := \mathbb{N}_0^{\bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')}$ of all functions from $\bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')$ to $\mathbb{N}_0$, where the disjoint union is taken over all oriented topological surfaces Σ' with negative Euler characteristic. There is a canonical map $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|$, that counts the parts of a nodal surface with their multiplicities. We call its image $|\widehat{\mathcal{M}}|(\Sigma) := Q(\widehat{\mathcal{M}}(\Sigma))$ equipped with the quotient topology the *moduli space of parts*; see section 2.6. Then the map $\widehat{\Phi}$ descends to a map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ given by

$$\Phi(\xi) = \sum_{X \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')} \frac{\chi(X)}{\chi(\Sigma)} \cdot \xi(X) \cdot \mu_X,$$

for all $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$.

Our main result is the following theorem.

Theorem 1.5.2 (Theorem 4.2.2)**.** *The map $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$ is a continuous surjection that extends the embedding $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$. Moreover, $\widehat{\Phi}$ descends to a homeomorphism $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$:*

$$\begin{tikzcd} & \widehat{\mathcal{M}}(\Sigma) \arrow[dl, two heads, "Q"'] \arrow[dr, two heads, "\widehat{\Phi}"] & \\ |\widehat{\mathcal{M}}|(\Sigma) \arrow[rr, "\Phi", "\sim"'] & & \overline{\mathcal{M}}^{IRS}(\Sigma) \end{tikzcd}$$

There is a uniform upper bound on the number of elements in each fiber of $\widehat{\Phi}$,

$$\#\widehat{\Phi}^{-1}(\mu) \leq B(\Sigma) := \binom{3|\chi|}{p} \cdot \frac{(2(|\chi|+g-1))!}{(|\chi|+g-1)! \cdot 2^{(|\chi|+g-1)}} \qquad \forall \mu \in \overline{\mathcal{M}}^{IRS}(\Sigma),$$

where $\chi = \chi(\Sigma)$, $g = g(\Sigma)$, and $p = p(\Sigma)$ denote the Euler characteristic, the genus, and the number of punctures of Σ, respectively.

In particular, this theorem shows that the IRS compactification is isomorphic to the moduli space of parts $|\widehat{\mathcal{M}}|(\Sigma)$.

The upper bound on the cardinality of the fibers of $\widehat{\Phi}$ is obtained by finding an upper bound for the quotient map $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|(\Sigma)$. This amounts to estimating in how many ways one may glue a given collection of hyperbolic surfaces along their punctures to obtain a nodal surface in $\widehat{\mathcal{M}}(\Sigma)$. In Proposition 2.6.15 we solve this combinatorial problem by counting the number of possible pairings of punctures. However, the bound $B(\Sigma)$ is not sharp; see Example 2.6.16. Also, computing the precise cardinality of the fiber $Q^{-1}(\xi)$ for some $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$ is delicate, because the isometry groups of the different parts play a role; see Example 2.6.17.

One can use Theorem 4.2.2 to explore the space of IRSs of $G = \mathrm{PSL}(2,\mathbb{R})$. In section 5.1 we show that IRS(G) contains a convex compact subset with dense extreme points by a geometric argument. This is not the first construction of such a subset. By different methods Bowen [Bow15, Remark 1] showed that IRS(G) contains the Poulsen simplex. Recall that the Poulsen simplex is the unique metrizable Choquet simplex whose extreme points are dense [LOS78].

Finally, we consider another natural construction that uses the embedding

$$\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G), \quad X \longmapsto \mu_X.$$

The moduli space $\mathcal{M}_g := \mathcal{M}(\Sigma_g)$ of closed hyperbolic surfaces of genus $g \geq 2$ carries a natural finite Borel measure: the Weil–Petersson volume $\mathrm{vol}_g^{\mathrm{WP}}$; see section 2.4. One can normalize this finite measure to obtain a probability measure $\mathbb{P}_g^{\mathrm{WP}} := \mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g)^{-1} \cdot \mathrm{vol}_g^{\mathrm{WP}}$. Using this probability measure to average the IRSs μ_X, $X \in \mathcal{M}_g$, one arrives at the definition of the *(genus g) Weil–Petersson IRS*

$$\mu_g^{\mathrm{WP}} := \int_{\mathcal{M}_g} \mu_X \, d\mathbb{P}_g^{\mathrm{WP}}(X) \in \mathrm{IRS}(G).$$

We can interpret the Weil–Petersson IRS μ_g^{WP} geometrically in terms of random framed hyperbolic surfaces. Indeed, the push-forward measure $\lambda_g^{\mathrm{WP}} := \psi_*(\mu_g^{\mathrm{WP}}) \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ chooses a random framed hyperbolic surface $[X, \mathbf{f}]$ by first sampling a $\mathbb{P}_g^{\mathrm{WP}}$-random hyperbolic surface $X \in \mathcal{M}_g$ and then picking a positively oriented orthonormal frame in $\mathbf{f} \in \mathrm{OF}_+(X)$ uniformly at random; see section 5.2.

In view of Question 1.3.4 it is natural to ask what the accumulation points of the genus g Weil–Petersson IRS is as the genus g tends to infinity. Using Mirzakhani's results [Mir13] on the Weil–Petersson volume of the moduli space $\mathcal{M}_g$ we answer this question. In joint work with Philip Engel we proved that the Weil–Petersson IRS μ_g^{WP} converges to the trivial IRS $\delta_{\{\mathrm{id}\}}$ as $g \to \infty$:

Theorem 1.5.3 (Theorem 5.2.3)**.** *The genus g Weil–Petersson IRS tends to the trivial IRS as $g \to \infty$, i.e.*

$$\mu_g^{\mathrm{WP}} \to \delta_{\{\mathrm{id}\}} \quad (g \to \infty).$$

Remark 1.5.4. This result was obtained independently and at the same time by Monk [Mon20].

Theorem 1.5.3 can be interpreted geometrically, as well. Notice that $\psi_*(\delta_{\{\mathrm{id}\}}) = \delta_{[\mathbb{H}^2,\mathbf{e}]}$, whence $\lambda_g^{\mathrm{WP}} \to \delta_{[\mathbb{H}^2,\mathbf{e}]}$ as $g \to \infty$. This is equivalent to saying that the injectivity radius $\mathrm{inj}_X(x)$ about the base point of the frame $\mathbf{f} \in \mathrm{OF}_+(X)_x$ is asymptotically almost surely greater than any $R > 0$ for a λ_g^{WP}-random framed hyperbolic surface $[X,\mathbf{f}]$ as $g \to \infty$; see Proposition 5.2.5.

1.6 Outline

This thesis is structured as follows.

In chapter 2 we review some background material. We recall the basic notions of the geometry of hyperbolic surfaces in section 2.2. Section 2.3 introduces the Teichmüller space. We take an algebraic point of view and regard Teichmüller space as a space of conjugacy classes of admissible representations. In section 2.4 we define the moduli space of hyperbolic surfaces as the quotient of Teichmüller space with respect to the mapping class group action. Section 2.5 then introduces the augmented Teichmüller space and we describe its topology in terms of representations. The augmented Teichmüller space is used in section 2.6 to define the augmented moduli space, which is central to our discussion of the IRS compactification. In section 2.7 we collect some properties of the space of closed subgroups. Section 2.8 then introduces the geometric topology on the set of admissible representations. We give a self-contained proof that the geometric topology coincides with the usual algebraic topology in Proposition 2.8.2. This result goes into the proof of Proposition 2.8.4, which in turn is used in the proof of Proposition 4.1.5.

In chapter 3 we prove Lemma 3.1.1, which lies at the heart of our proof of Proposition 4.1.5 and Theorem 4.2.2. More precisely, we show that the characteristic function of a truncated Dirichlet domain depends continuously on the group. On our way we prove Lemma 3.1.8 which concerns the hyperbolic area of the thin part of a hyperbolic surface. This is used again later in our discussion of the Weil–Petersson IRS in section 5.2.

In chapter 4 we prove our results about the IRS compactification. We show that the moduli space embeds in the space of IRSs of $\mathrm{PSL}(2,\mathbb{R})$ via the map ι in section 4.1. In section 4.2 we prove Theorem 4.2.2 relating the IRS compactification to the augmented moduli space using Lemma 3.1.1.

In chapter 5 we apply the embedding of the moduli space and Theorem 4.2.2 to explore the space of IRSs of $G = \mathrm{PSL}(2, \mathbb{R})$. We construct a convex compact subset of $\mathrm{IRS}(G)$ with dense extreme points via a geometric argument in section 5.1. Finally, we consider the genus g Weil-Petersson IRS and prove that it converges to the trivial IRS as the genus g tends to infinity in section 5.2.

2 Preliminaries

In this chapter we will lay the foundations of this thesis and introduce some basic objects of our investigations.

2.1 Notational Conventions

Throughout this thesis we will use the following notations:

We denote by Σ an oriented surface of genus g with p punctures, no boundary, and negative Euler characteristic $\chi(\Sigma) < 0$. Further, $G := \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ denotes the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2$, if not otherwise specified.

2.2 Hyperbolic Surfaces

We will recall some notions of hyperbolic surfaces in this section. The material presented here is well-known and can be found in [Bea83; Kat92; Mar16; BP92; Bus10].

A *hyperbolic surface* is a complete connected oriented 2-dimensional Riemannian manifold of constant sectional curvature -1. Any hyperbolic surface X is a quotient $X = \Gamma\backslash\mathbb{H}^2$ of the hyperbolic 2-space $\mathbb{H}^2$ by a discrete and torsion-free subgroup $\Gamma \leq G = \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$. The quotient map $\pi_\Gamma \colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ is a Riemannian covering map and the action $\Gamma \curvearrowright \mathbb{H}^2$ is via deck transformations. In this way, one may identify the fundamental group $\pi_1(X)$ with Γ after fixing a base point. If $\Gamma \cong \pi_1(X)$ is finitely generated, we say that X is of *finite type*.

If X has finite area, then one can compute its area in terms of its Euler characteristic

$$\mathrm{vol}_X(X) = 2\pi|\chi(X)|.$$

Note that $\Gamma \leq G$ is a lattice if and only if $X = \Gamma\backslash\mathbb{H}^2$ has finite area, and $\Gamma \leq G$ is a cocompact lattice if and only if $X = \Gamma\backslash\mathbb{H}^2$ is a closed surface.

The *limit set* of Γ is defined as $L(\Gamma) = \overline{\Gamma o} \cap \partial\mathbb{H}^2$ for some $o \in \mathbb{H}^2$ (this definition is independent of $o \in \mathbb{H}^2$), and it is the smallest closed Γ-invariant subset of $\partial\mathbb{H}^2$. Its complement $\Omega(\Gamma) = \partial\mathbb{H}^2 \setminus L(\Gamma)$ is called the *domain of discontinuity*. We define the convex Γ-invariant subset $\widetilde{C}(\Gamma) := \operatorname{conv}(L(\Gamma)) \cap \mathbb{H}^2$. Its quotient $C(\Gamma) := \Gamma\backslash\widetilde{C}(\Gamma) \subseteq \Gamma\backslash\mathbb{H}^2$ is called the *convex core of* $\Gamma\backslash\mathbb{H}^2$. If Γ is a lattice then $L(\Gamma) = \partial\mathbb{H}^2$, whence $\widetilde{C}(\Gamma) = \mathbb{H}^2$ and $C(\Gamma) = \Gamma\backslash\mathbb{H}^2$.

If $\Gamma \leq G$ is not a lattice then $\Omega(\Gamma) = \bigsqcup_{j\in\mathbb{N}} I_j$ is a non-empty disjoint union of open intervals $I_j \subseteq \partial\mathbb{H}^2 \cong \mathbb{S}^1$. The endpoints of each interval I_j are the fixed points of a unique primitive hyperbolic transformation $\gamma_j \in \Gamma$. Its axis $\operatorname{ax}(\gamma_j)$ abuts a half-space $H_j \subseteq \mathbb{H}^2$ bordering on the interval $I_j \subseteq \partial\mathbb{H}^2$ at infinity. The half-space H_j is preserved by $\langle\gamma_j\rangle \leq \Gamma$ and its quotient $F_j := \langle\gamma_j\rangle\backslash H_j$ is a hyperbolic funnel isometrically embedded in $\Gamma\backslash\mathbb{H}^2$. These funnels are also called *free ends* of $\Gamma\backslash\mathbb{H}^2$. The axis $\operatorname{ax}(\gamma_j)$ descends to a closed geodesic bounding the funnel F_j and is called its *waist geodesic*. It follows that one obtains the convex core $C(\Gamma) \subseteq \Gamma\backslash\mathbb{H}^2$ by cutting off all the funnels. Their waist geodesics amount to the geodesic boundary of $C(\Gamma)$.

Free ends are one of two possible types of ends of a hyperbolic surface $\Gamma\backslash\mathbb{H}^2$. The other type corresponds to parabolic transformations in Γ. Indeed, for every primitive parabolic transformation $\gamma_j \in \Gamma$ there is a horoball $B_j \subseteq \mathbb{H}^2$ based at its fixed point $\operatorname{Fix}(\gamma_j)$, such that the union $\bigcup_{j\in\mathbb{N}} B_j$ is disjoint and Γ-invariant. The horoball B_j is preserved by $\langle\gamma_j\rangle \leq \Gamma$ and its quotient $C_j := \langle\gamma_j\rangle\backslash\mathbb{H}^2$ is isometrically embedded in $\Gamma\backslash\mathbb{H}^2$. Any such C_j is called a *cusp* of $\Gamma\backslash\mathbb{H}^2$. In contrast to the free ends of $\Gamma\backslash\mathbb{H}^2$ the cusps have finite area. However, both funnels and cusps are topologically punctured discs. Thus, if $X = \Gamma\backslash\mathbb{H}^2$ is of finite type then X has only finitely many punctures such that there are only finitely many cusps and funnels.

Recall that one can identify the fundamental group $\pi_1(X)$ with Γ. This gives rise to a correspondence between free homotopy classes of closed curves in X and conjugacy classes of isometries in Γ. We will call any closed curve in X that is homotopic to a puncture *peripheral*. Moreover, any isometry $\gamma \in \Gamma$ corresponding to a peripheral curve in X will be called a *boundary element* or a *boundary transformation* of Γ. Closed curves in X that are neither null-homotopic nor peripheral will be called *essential*.

Any closed curve in X that does not correspond to a parabolic boundary element is freely homotopic to a closed geodesic in X. In this case the corresponding element $\gamma \in \Gamma$

is hyperbolic and its axis $\mathrm{ax}(\gamma)$ projects to said closed geodesic in X via $\pi_\Gamma\colon \mathbb{H}^2 \longrightarrow X = \Gamma\backslash\mathbb{H}^2$. If γ is primitive, its translation length

$$\ell(\gamma) = \inf\{d(x,\gamma x) \,|\, x \in \mathbb{H}^2\}$$

equals the length of the closed geodesic.

Let us denote by $\mathrm{inj}_X(x)$ the *injectivity radius* at a point $x \in X$, i.e. the supremum over all radii $r > 0$ such that the ball $B_x(r) \subseteq X$ centered at $x \in X$ is isometric to the ball $B_o(r) \subseteq \mathbb{H}^2$ in the hyperbolic plane for some (any) $o \in \mathbb{H}^2$. For every $\varepsilon > 0$ we denote by

$$X_{\leq\varepsilon} \coloneqq \{x \in X \,|\, \mathrm{inj}_X(x) \leq \varepsilon\}$$

the subset of all points $x \in X$ with injectivity radius less than ε. We call $X_{\leq\varepsilon}$ the *ε-thin part*. Likewise, we call

$$X_{\geq\varepsilon} \coloneqq \{x \in X \,|\, \mathrm{inj}_X(x) \geq \varepsilon\}$$

the *ε-thick part*. The ε-thick part is compact if and only if X has finite area.

For small enough $\varepsilon > 0$ the ε-thin part admits a very concrete description. There is a universal constant $\varepsilon_2 > 0$, the *Margulis' constant*, such that for every hyperbolic surface X and every $0 < \varepsilon \leq \varepsilon_2$ any connected component of $X_{\leq\varepsilon}$ is either a tube, a cusp or a closed geodesic of length ε. Here a *tube* $T \subseteq X$ (also called a *collar*) is a subset isometric to a quotient $T \cong \langle\gamma\rangle\backslash N_{w/2}(\mathrm{ax}(\gamma))$, where $\gamma \in \Gamma$ is a primitive hyperbolic transformation and $N_{w/2}(\mathrm{ax}(\gamma)) = \{x \in \mathbb{H}^2 \,|\, d(x, \mathrm{ax}(\gamma)) < \frac{w}{2}\}$ is the $\frac{w}{2}$-neighborhood of its axis, $w > 0$. We call $w > 0$ the *width* of the tube T. The axis of γ descends to a closed geodesic in T, which we call its *waist geodesic*.

The following *Collar Lemma* tells us how large we can choose the width of an embedded tube depending on the length of its waist geodesic.

Lemma 2.2.1 (Collar Lemma; see [Bus10, 4.1.1 Theorem])**.** *Let $\gamma \subseteq X$ be a closed geodesic. Then γ is the waist geodesic of an embedded tube of width*

$$w = 2\,\mathrm{arcsinh}\left(\frac{1}{\sinh\left(\frac{1}{2}\sinh(\ell(\gamma))\right)}\right).$$

In particular, the width w goes to infinity as the length $\ell(\gamma)$ of γ tends to 0.

A useful way to understand the action $\Gamma \curvearrowright \mathbb{H}^2$ is via fundamental domains.

Definition 2.2.2. Let Y be a metric space and let Γ be a group acting on Y via homeomorphisms. A *fundamental domain* for the action $\Gamma \curvearrowright Y$ is a closed domain $F \subseteq Y$, i.e. the closure of a non-empty open set $\mathrm{int}(F)$, such that

(i) $\bigcup_{\gamma\in\Gamma} \gamma \cdot F = Y$, and

(ii) $\mathrm{int}(F) \cap \gamma \cdot \mathrm{int}(F) = \varnothing$ for all $\gamma \in \Gamma \setminus \{e\}$.

If, additionally, Γ preserves a Borel measure μ on Y we further require that the boundary ∂F has measure zero.

A particularly nice choice for a fundamental domain is the Dirichlet domain.

Definition 2.2.3. Let $\Gamma < G$ be a discrete torsion-free subgroup and $o \in \mathbb{H}^2$. The *Dirichlet domain* of Γ with respect to $o \in \mathbb{H}^2$ is defined as

$$D_o(\Gamma) := \{x \in \mathbb{H}^2 \,|\, d(x,o) \leq d(x,\gamma o) \quad \forall \gamma \in \Gamma \setminus \{e\}\}.$$

If $X = \Gamma\backslash\mathbb{H}^2$ is compact then $D_o(\Gamma)$ is a convex polygon in $\mathbb{H}^2$ with finitely many sides. If $X = \Gamma\backslash\mathbb{H}^2$ is not compact with finite area then $D_o(\Gamma)$ is still a finite sided convex polygon with some ideal vertices on the boundary $\partial\mathbb{H}^2$. If $\Gamma\backslash\mathbb{H}^2$ has infinite area then $D_o(\Gamma)$ is a convex set with piecewise geodesic boundary. However, there are some sides ending on the boundary $\partial\mathbb{H}^2$ with no other side starting at this endpoint.

There is a partition of the sides of $D_o(\Gamma)$ into pairs such that any two paired sides are mapped to each other by a *side pairing transformation* in Γ. It can be shown, that these generate Γ, and that $X = \Gamma\backslash\mathbb{H}^2$ is of finite type if and only if $D_o(\Gamma)$ has finitely many sides. One can think of the surface $\Gamma\backslash\mathbb{H}^2$ as the Dirichlet domain with paired sides glued together via the respective side pairing transformations. The pairing is such that any two sides adjacent to an ideal vertex v on $\partial\mathbb{H}^2$ are paired by a parabolic transformation fixing v. In this way there is a one-to-one correspondence between ideal vertices of $D_o(\Gamma)$ and cusps of $\Gamma\backslash\mathbb{H}^2$. Similarly, for every funnel $F \subseteq \Gamma\backslash\mathbb{H}^2$ there are two sides of the Dirichlet domain ending on an interval $I \subseteq \partial\mathbb{H}^2$ that borders on a half-space $H \subseteq \mathbb{H}^2$ at infinity such that $F = \langle\gamma\rangle\backslash H$. These sides are then paired by the hyperbolic transformation γ.

In the case when $\Gamma\backslash\mathbb{H}^2$ has infinite area it is useful to consider the *truncated Dirichlet domain*

$$\widehat{D}_o(\Gamma) := D_o(\Gamma) \cap \widetilde{C}(\Gamma).$$

It is obtained from the Dirichlet domain by cutting off the parts contained in the half-spaces corresponding to funnels of $\Gamma\backslash\mathbb{H}^2$. In fact, $\widehat{D_o}(\Gamma)$ is a fundamental domain for the action of Γ on $\widetilde{C}(\Gamma)$ and we obtain the convex core as its quotient $C(\Gamma) = \pi_\Gamma(\widehat{D_o}(\Gamma))$. If Γ is of finite type then $\widehat{D_o}(\Gamma)$ is a convex polygon in $\mathbb{H}^2$ with finitely many sides and possibly some ideal vertices. Thus, $\widehat{D_o}(\Gamma)$ has finite area, whence the convex core $C(\Gamma) \subseteq \Gamma\backslash\mathbb{H}^2$ has finite area, too. Indeed, in this case $C(\Gamma)$ is a hyperbolic surface with finitely many cusps and geodesic boundary components, and by the Gauss–Bonnet formula

$$\mathrm{vol}_{\mathbb{H}^2}(\widehat{D_o}(\Gamma)) = \mathrm{vol}(C(\Gamma)) = 2\pi\,|\chi(C(\Gamma))| = 2\pi\left|\chi(\Gamma\backslash\mathbb{H}^2)\right|.$$

2.3 The Teichmüller Space

We will recall some basic notions of Teichmüller theory in the following in order to fix our notation. The material presented here is well-known and can be found in [Abi80; FLP12; Bus10; FM12; Mar16].

Teichmüller theory is concerned with deformations of hyperbolic surfaces. One way to think about this is to fix an oriented surface Σ of genus g with p punctures, no boundary and negative Euler characteristic $\chi(\Sigma) < 0$, and consider orientation preserving homeomorphisms $f : \Sigma \longrightarrow X$ between Σ and a hyperbolic surface X. Such a homeomorphism is called a *marking* and the tuple (f, X) is called a *marked hyperbolic surface*. Furthermore, one identifies two marked hyperbolic surfaces $(f_1, X_1), (f_2, X_2)$ if there is an orientation preserving isometry $\varphi\colon X_1 \longrightarrow X_2$ that is homotopic to $f_2 \circ f_1^{-1}$. The resulting set of (equivalence classes of) marked hyperbolic surfaces is then a model for the Teichmüller space $\mathrm{Teich}(\Sigma)$ of hyperbolic structures on Σ.

Remark 2.3.1. Notice that we do not require the hyperbolic surfaces to have finite area here. Instead, some authors prefer to consider Teichmüller spaces of finite-area hyperbolic surfaces with geodesic boundary components. However, one may pass freely between the two points of view. Indeed, by cutting of the hyperbolic funnels at their waist geodesics one obtains a finite-area hyperbolic surface with geodesic boundary. Vice versa, one may always attach hyperbolic funnels along the geodesic boundary components to get an element of $\mathrm{Teich}(\Sigma)$. We chose this approach because it avoids the (for our purposes) unnecessary distinction between boundary components and punctures. This uniform framework will facilitate our definition of the augmented Teichmüller space in section 2.5.

Instead of working directly with marked hyperbolic surfaces we will rather use an algebraic reformulation of $\mathrm{Teich}(\Sigma)$, which arises as follows. Any marking $f\colon \Sigma \longrightarrow X$ may be lifted to a homeomorphism $\widetilde{f}\colon \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$ between the universal covers after choosing some base points. Moreover, it induces an isomorphism between the fundamental groups $\rho := f_*\colon \pi_1(\Sigma) \longrightarrow \pi_1(X)$. Recall that both fundamental groups $\pi_1(\Sigma)$ and $\pi_1(X)$ act via deck transformations on $\widetilde{\Sigma}$ and $\mathbb{H}^2$, respectively. By definition the lift $\widetilde{f}\colon \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$ is equivariant with respect to these actions, i.e. $\widetilde{f}(\gamma \cdot x) = \rho(\gamma) \cdot \widetilde{f}(x)$ for all $x \in \widetilde{\Sigma}$ and every $\gamma \in \pi_1(\Sigma)$. In this way, we may interpret $\rho\colon \pi_1(\Sigma) \longrightarrow \pi_1(X) < G = \mathrm{Isom}_+(\mathbb{H}^2)$ as a discrete and faithful representation of $\pi_1(\Sigma)$ called a *holonomy representation* of f. This motivates the following definition.

Definition 2.3.2. A discrete and faithful representation

$$\rho\colon \pi_1(\Sigma) \longrightarrow G$$

is called *admissible* if it is a holonomy representation of an orientation preserving homeomorphism $f\colon \Sigma \longrightarrow X$ where X is a hyperbolic surface. The set of all such representations is denoted by $\mathcal{R}^*(\Sigma)$. If we additionally require that the hyperbolic surface X has finite area, we denote the resulting subset by $\mathcal{R}(\Sigma)$.

Remark 2.3.3. Because a hyperbolic surface of finite type has finite area if and only if every end is a cusp, an admissible representation $\rho \in \mathcal{R}^*(\Sigma)$ is in $\mathcal{R}(\Sigma)$ if and only if $\rho(\alpha) \in G$ is parabolic for every peripheral curve $\alpha \in \pi_1(\Sigma)$.

Note that a holonomy representation of a marking homeomorphism is not unique and depends on the choice of base points and the identification of the universal cover of X with $\mathbb{H}^2$. Moreover, points in the Teichmüller space of marked hyperbolic surfaces $\mathrm{Teich}(\Sigma)$ are actually equivalence classes of marked hyperbolic surfaces. Nevertheless, it turns out that there is a one-to-one correspondence between marked hyperbolic surfaces and *conjugacy classes* of admissible representations [FM12, Proposition 10.2]. This leads us to the following algebraic model of Teichmüller space.

Definition 2.3.4. The group G acts via conjugation on $\mathcal{R}(\Sigma)$ resp. $\mathcal{R}^*(\Sigma)$ from the left, and we denote the quotients by

$$\mathcal{T}(\Sigma) := G\backslash\mathcal{R}(\Sigma) \quad \text{resp.} \quad \mathcal{T}^*(\Sigma) := G\backslash\mathcal{R}^*(\Sigma).$$

We will refer to both as *Teichmüller spaces* of Σ.

One can use this model to put a topology on Teichmüller space in the following way. We begin by defining the algebraic topology on the set of admissible representations.

Definition 2.3.5. The group $\pi_1(\Sigma)$ admits a finite generating set S, and the map

$$i\colon \mathcal{R}^*(\Sigma) \hookrightarrow G^S, \rho \longmapsto (\rho(s))_{s\in S}$$

is injective. We equip G^S with the product topology and $\mathcal{R}^*(\Sigma)$ with the initial topology with respect to the injection i. This topology does not depend on the choice of generating set and is called the *algebraic topology*.

We equip $\mathcal{T}(\Sigma) = G\backslash\mathcal{R}(\Sigma)$ and $\mathcal{T}^*(\Sigma) = G\backslash\mathcal{R}^*(\Sigma)$ with the quotient topology. In this way, both $\mathcal{T}(\Sigma)$ and $\mathcal{T}^*(\Sigma)$ are Hausdorff [Mar07, Lemma 5.1.1].

Remark 2.3.6. A sequence of representations $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$ converges to $\rho \in \mathcal{R}^*(\Sigma)$ as $n \to \infty$ with respect to this topology if and only if $\rho_n(\gamma) \to \rho(\gamma)$ as $n \to \infty$ for every $\gamma \in \pi_1(\Sigma)$.

We will need the following definition.

Definition 2.3.7. Let $\alpha, \beta \in \pi_1(\Sigma)$. The *geometric intersection number* $i(\alpha,\beta)$ is defined as

$$i(\alpha,\beta) := \min_{c_1,c_2} \#(c_1 \cap c_2)$$

where the minimum is taken over all loops c_1, c_2 in the free homotopy classes of α, β respectively. We say that two loops c_1 and c_2 are in *minimal position* if they realize their geometric intersection number, i.e. $i([c_1],[c_2]) = \#(c_1 \cap c_2)$.

The following is an important consequence of the Collar Lemma 2.2.1 at the level of representations.

Lemma 2.3.8 (Collar Lemma; see [Bus10, Corollary 4.1.2])**.** *Let $\alpha, \beta \in \pi_1(\Sigma)$ such that α is primitive and $i(\alpha,\beta) > 0$. If $\rho \in \mathcal{R}^*(\Sigma)$, then*

$$\sinh\left(\frac{\ell(\rho(\alpha))}{2}\right) \cdot \sinh\left(\frac{\ell(\rho(\beta))}{2}\right) \geq 1.$$

In particular, if $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$ is a sequence such that $\ell(\rho_n(\alpha)) \to 0$ as $n \to \infty$, then $\ell(\rho_n(\beta)) \to \infty$ as $n \to \infty$.

The Teichmüller space $\mathcal{T}(\Sigma)$ parametrizes deformations of geometric objects: hyperbolic surfaces. Interestingly, it has a geometry in its own right. In fact, Teichmüller space carries more than one natural metric, e.g. the Teichmüller metric [Abi80; FM12], the Weil–Petersson metric [Wol07; Wol10], or the (asymmetric) Thurston metric [Thu98]. The Weil–Petersson metric is a non-complete Kähler metric on $\mathcal{T}(\Sigma)$ with negative sectional curvature. In particular, it induces the so called Weil–Petersson volume form on $\mathcal{T}(\Sigma)$. We refer to Wolpert [Wol07; Wol10] for further details.

2.4 The Moduli Space

In this section we will briefly recall the definition of the moduli space of finite-area hyperbolic surfaces. The material presented here is well-known and can be found in [FM12; Har77].

The moduli space of finite-area hyperbolic surfaces is the set of all isometry classes of finite-area hyperbolic surfaces. One may pass from the Teichmüller space of finite-area marked hyperbolic surfaces to the moduli space by forgetting the marking. This is achieved by taking the quotient by the mapping class group action.

Definition 2.4.1. The group

$$\mathrm{MCG}(\Sigma) := \mathrm{Homeo}^{+}(\Sigma)/\mathrm{Homeo}_{\circ}(\Sigma)$$

is called the *mapping class group of* Σ. Here $\mathrm{Homeo}_{\circ}(\Sigma)$ denotes the identity component in $\mathrm{Homeo}(\Sigma)$, i.e. all homeomorphisms isotopic to the identity. We will denote by $[f]$ the mapping class of an orientation preserving homeomorphism $f \in \mathrm{Homeo}^{+}(\Sigma)$.

Thus, the mapping class group acts via precomposition on $\mathrm{Teich}(\Sigma)$, $[f,X]\cdot[h] := [f\circ h, X]$, and one may define the moduli space as the resulting quotient. As in the case of Teichmüller space this construction allows for an algebraic reformulation.

Any mapping class amounts to an outer automorphism of $\pi_1(\Sigma)$ in the following way. Let $\varphi = [f] \in \mathrm{MCG}(\Sigma)$, let $p \in \Sigma$ and let $\beta\colon [0,1] \longrightarrow \Sigma$ be a path from $\beta(0) = p$ to $\beta(1) = f(p)$. Identifying $\pi_1(\Sigma) \cong \pi_1(\Sigma, p)$ we obtain an automorphism at the level of fundamental groups $f_*\colon \pi_1(\Sigma) \longrightarrow \pi_1(\Sigma)$ given by

$$f_*([c]) := [\beta\cdot(f\circ c)\cdot\beta^{-1}]$$

for every homotopy class $[c] \in \pi_1(\Sigma, p)$ of a closed loop c at p.

This construction depends on the choice of representative $f \in \varphi$ and the choice of path β. However, we obtain a well-defined outer automorphism:

Proposition and Definition 2.4.2 ([FM12, Chapter 8])**.** The map

$$\begin{aligned} \mathrm{MCG}(\Sigma) &\longrightarrow \mathrm{Out}(\pi_1(\Sigma)) = \mathrm{Aut}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma)), \\ \varphi = [f] &\longmapsto \varphi_* := [f_*] \end{aligned}$$

is a well-defined injective homomorphism. We denote its image by $\mathrm{Out}^*(\pi_1(\Sigma)) \leq \mathrm{Out}(\pi_1(\Sigma))$ and its preimage under the quotient map $\mathrm{Aut}(\pi_1(\Sigma)) \longrightarrow \mathrm{Out}(\pi_1(\Sigma))$ by $\mathrm{Aut}^*(\pi_1(\Sigma)) \leq \mathrm{Aut}(\pi_1(\Sigma))$. These (outer) automorphisms are called *geometric* or *admissible*.

This gives rise to the following definition of the moduli space.

Proposition and Definition 2.4.3. The group $\mathrm{Aut}^*(\pi_1(\Sigma))$ acts on $\mathcal{R}^*(\Sigma)$ from the right via precomposition, and induces a right-action of $\mathrm{Out}^*(\pi_1(\Sigma)) \cong \mathrm{MCG}(\Sigma)$ on $\mathcal{T}(\Sigma)$.

The quotient space

$$\mathcal{M}(\Sigma) := \mathcal{T}(\Sigma)/\mathrm{MCG}(\Sigma)$$

is called the *moduli space of* Σ. We will denote the $\mathrm{MCG}(\Sigma)$-equivalence class of $[\rho] \in \mathcal{T}(\Sigma)$ by $[[\rho]] \in \mathcal{M}(\Sigma)$. Emphasizing the geometric point of view one may identify an element $[[\rho]] \in \mathcal{M}(\Sigma)$ with (the isometry class of) the hyperbolic surface $X = \rho(\pi_1(\Sigma))\backslash\mathbb{H}^2$.

Moreover, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{R}(\Sigma) \arrow[r, "\mathrm{Aut}^*(\pi_1(\Sigma))"] \arrow[d, "G"] & \mathcal{R}(\Sigma)/\mathrm{Aut}^*(\pi_1(\Sigma)) \arrow[d, "G"] \\ \mathcal{T}(\Sigma) \arrow[r, "\mathrm{MCG}(\Sigma)\cong\mathrm{Out}^*(\pi_1(\Sigma))"] & \mathcal{M}(\Sigma), \end{tikzcd}$$

where every map is the quotient map with respect to the action of the annotated group.

Proof. Let $\rho \in \mathcal{R}^*(\Sigma)$ and $[\alpha] = [g_*] \in \mathrm{Out}^*(\pi_1(\Sigma))$, where $g \in \mathrm{Homeo}^+(\Sigma)$. We only need to check that $\rho \circ \alpha$ is admissible as well. Because ρ is admissible, it is a holonomy representation of an orientation preserving homeomorphism $f : \Sigma \longrightarrow X$. Thus $\rho \circ \alpha$ is a holonomy representation of $f \circ g : \Sigma \longrightarrow X$, whence $\rho \circ \alpha \in \mathcal{R}^*(\Sigma)$. $\square$

It turns out that the mapping class group action $\mathrm{MCG}(\Sigma) \curvearrowright \mathcal{T}(\Sigma)$ is isometric with respect to the Weil–Petersson metric, such that it descends to a metric on the moduli space $\mathcal{M}(\Sigma)$. In particular, the Weil–Petersson volume form descends as well and gives rise to a *finite* Borel measure $\mathrm{vol}^{\mathrm{WP}}$ on $\mathcal{M}(\Sigma)$, which we will call the *Weil–Petersson volume* [Wol07; Wol10]. This measure was studied very successfully by Mirzakhani [Mir07; Mir13]. We will use some of her results in our discussion of the Weil–Petersson IRS in section 5.2.

2.5 The Augmented Teichmüller Space

Following Harvey [Har74; Har77] and Abikoff [Abi80, Chapter 2, §3] we will now introduce the augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$ – a bordification of Teichmüller space $\mathcal{T}(\Sigma)$. The augmented Teichmüller space has been known for a long time, and is usually constructed in terms of Fenchel–Nielsen coordinates. We will go a different route and describe $\widehat{\mathcal{T}}(\Sigma)$ via representations. A similar approach to the augmented deformation space of convex real projective structures is taken by Loftin and Zhang [LZ18].

The idea behind the augmented Teichmüller space is to allow the lengths of (homotopically) disjoint simple closed curves to go to zero as one moves to infinity in $\mathcal{T}(\Sigma)$. This will be accounted for by attaching the Teichmüller spaces of the subsurfaces in the complement of the pinched curves. Thus the augmented Teichmüller space will admit a natural stratification in terms of the curve complex $\mathcal{C}(\Sigma)$.

Recall that the *curve complex* $\mathcal{C}(\Sigma)$ is a (combinatorial) simplicial complex and its vertices are given by homotopy classes of essential simple closed curves in Σ. A $(l-1)$-dimensional simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ is then given by a collection $\sigma = \{\alpha_1, \dots, \alpha_l\}$ of homotopy classes of essential simple closed curves, which are pairwise distinct and admit disjoint representatives. A maximal simplex $\widehat{\sigma} = \{\alpha_1, \dots, \alpha_N\}$, $N = 3g-3+p$, is a pairs of pants decomposition of Σ, such that the dimension of $\mathcal{C}(\Sigma)$ is $3g+p-4$; see [Har81].

In the following we will equip Σ with an auxiliary hyperbolic structure. Thus we may assume that every simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ consists of the (unique) closed geodesic representatives with respect to that hyperbolic structure. One can check a posteriori that the following definitions are independent of this choice up to natural isomorphisms.

Definition 2.5.1 (Augmented Teichmüller Space)**.** Let $\sigma \subseteq \mathcal{C}(\Sigma)$ be a simplex in the curve complex. We define

$$\mathcal{T}^*_\sigma(\Sigma) := \prod_{\Sigma' \in c(\sigma)} \mathcal{T}^*(\Sigma') \quad \text{and} \quad \mathcal{T}_\sigma(\Sigma) := \prod_{\Sigma' \in c(\sigma)} \mathcal{T}(\Sigma') \subseteq \mathcal{T}^*_\sigma(\Sigma),$$

where the product is taken over all *connected components* $c(\sigma)$ *of* $\Sigma \setminus \sigma$. The disjoint union over all simplices $\sigma \subseteq \mathcal{C}(\Sigma)$,

$$\widehat{\mathcal{T}}(\Sigma) := \bigsqcup_{\sigma \subseteq \mathcal{C}(\Sigma)} \mathcal{T}_\sigma(\Sigma),$$

is then called the *augmented Teichmüller space of* Σ.

It is important to note that for a point $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\Sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$ the simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ is implicit as is the decomposition of Σ into the components $\{\Sigma'\}_{\Sigma' \in c(\sigma)}$. Geometrically one may think of the points in the stratum $\mathcal{T}_\sigma(\Sigma)$ as marked nodal surfaces, where the curves in σ were collapsed to nodes and every complementary component carries a finite-area hyperbolic structure; see Figure 2.1 for an example.

We wish to equip the augmented Teichmüller space with a topology. In order to do so we will need *restriction maps*. Let $\sigma \subseteq \mathcal{C}(\Sigma)$ be a simplex in the curve complex and let $\Sigma' \in c(\sigma)$ be a connected component of $\Sigma \setminus \sigma$. Let $\pi \colon \mathbb{H}^2 \cong \widetilde{\Sigma} \longrightarrow \Sigma$ denote the universal covering; recall that Σ carries an auxiliary hyperbolic structure. Denote by $\widetilde{\sigma} := \pi^{-1}(\sigma) \subseteq \mathbb{H}^2$ the disjoint union of geodesics that project to $\sigma \subseteq \Sigma$. Further, let $\widetilde{\Sigma}' \subseteq \mathbb{H}^2 \setminus \widetilde{\sigma}$ be a connected component that projects to Σ', i.e. $\pi(\widetilde{\Sigma}') = \Sigma'$. Observe that $\widetilde{\Sigma}' \subseteq \mathbb{H}^2$ is a convex subset such that $\pi|_{\widetilde{\Sigma}'} \colon \widetilde{\Sigma}' \longrightarrow \Sigma'$ is a universal covering; see Figure 2.2. Thus, we obtain the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\Sigma}' \arrow[r, hook] \arrow[d, "\pi|_{\Sigma'}"] & \widetilde{\Sigma} \arrow[d, "\pi"] \\
\Sigma' \arrow[r, hook] & \Sigma
\end{tikzcd}$$

It follows that the homomorphism $\iota_{\Sigma'} \colon \pi_1(\Sigma') \longrightarrow \pi_1(\Sigma)$ induced by inclusion is injective, and identifies $\pi_1(\Sigma')$ with the subgroup of $\pi_1(\Sigma) \cong \mathrm{Deck}(\pi)$ that leaves the component $\widetilde{\Sigma}'$ invariant.

Remark 2.5.2. The monomorphism $\iota_{\Sigma'} \colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ depends on the choice of connected component $\widetilde{\Sigma}' \subseteq \pi^{-1}(\Sigma')$. Different choices amount to monomorphisms $\pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$, which are conjugate in $\pi_1(\Sigma)$.

Remark 2.5.3. Although we will not need this in the following, we want to mention that every simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ gives rise to a graph of groups structure on $\pi_1(\Sigma)$; see [Ser03]. Indeed, $\pi_1(\Sigma)$ is the fundamental group of the graph of groups whose vertices are the fundamental groups $\pi_1(\Sigma')$ of the components $\Sigma' \in c(\sigma)$. Identifying the peripheral subgroups corresponding to curves in σ then amounts to the edge homomorphisms.

Proposition and Definition 2.5.4. In the above situation, we obtain a well-defined *restriction map*

$$\begin{aligned} \mathrm{res}^{\Sigma}_{\Sigma'} \colon \mathcal{T}^*(\Sigma) &\longrightarrow \mathcal{T}^*(\Sigma'), \\ [\rho] &\longmapsto [\rho \circ \iota_{\Sigma'}]. \end{aligned}$$

For a face $\sigma' \subseteq \sigma \subseteq \mathcal{C}(\Sigma)$ these maps induce a restriction map

$$\begin{aligned} \mathrm{res}^{\sigma'}_{\sigma} \colon \mathcal{T}^*_{\sigma'}(\Sigma) &\longrightarrow \mathcal{T}^*_{\sigma}(\Sigma), \\ ([\rho_{\Sigma''}])_{\Sigma'' \in c(\sigma')} &\longmapsto \left(\mathrm{res}^{\Sigma''}_{\Sigma'}([\rho_{\Sigma''}])\right)_{\Sigma' \in c(\sigma)}, \end{aligned}$$

where on the right-hand-side Σ'' is the unique connected component that contains Σ'.

Proof. Let $[\rho] \in \mathcal{T}^*(\Sigma)$ with $\rho \in \mathcal{R}^*(\Sigma)$. Consider the composition $\rho \circ \iota_{\Sigma'} \colon \pi_1(\Sigma') \hookrightarrow G$. Then the image $\Gamma' := \rho(\iota_{\Sigma'}(\pi_1(\Sigma'))) \leq \Gamma := \rho(\pi_1(\Sigma))$ is discrete, such that $\rho \circ \iota_{\Sigma'}$ is a discrete and faithful representation. Our goal is to show that $\rho \circ \iota_{\Sigma'} \in \mathcal{R}^*(\Sigma')$. It will then be immediate that

$$\begin{aligned} \mathrm{res}^{\Sigma}_{\Sigma'} \colon \mathcal{T}^*(\Sigma) &\longrightarrow \mathcal{T}^*(\Sigma'), \\ [\rho] &\longmapsto [\rho \circ \iota_{\Sigma'}], \end{aligned}$$

is a well-defined map. Indeed, by Remark 2.5.2 the injective homomorphism $\iota_{\Sigma'}$ is well-defined only up to conjugation in $\pi_1(\Sigma)$. However, this issue is resolved after taking the quotient with respect to the conjugation action of G on $\mathcal{R}^*(\Sigma)$.

We are left to show that $\rho \circ \iota_{\Sigma'}$ is a holonomy representation of an orientation preserving homeomorphism $f' \colon \Sigma' \longrightarrow \Gamma' \backslash \mathbb{H}^2$. Let $f \colon \Sigma \longrightarrow X := \Gamma \backslash \mathbb{H}^2$ be an orientation preserving homeomorphism such that ρ is a holonomy representation of f. We may isotope f in such a way that it sends the curves in σ to geodesics $f(\sigma) \subseteq X$. Therefore, it sends Σ' to a connected component X' of $X \setminus f(\sigma)$. Consider a lift $\widetilde{f} \colon \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$ of $f \colon \Sigma \longrightarrow X$ with respect to the universal coverings $\pi \colon \widetilde{\Sigma} \longrightarrow \Sigma$ and $\pi_\Gamma \colon \mathbb{H}^2 \longrightarrow \Gamma \backslash \mathbb{H}^2$.

Then $\widetilde{X}' := \widetilde{f}(\widetilde{\Sigma}') \subseteq \mathbb{H}^2 \setminus \pi_\Gamma^{-1}(f(\sigma))$ is a connected component, and $\widetilde{f}|_{\widetilde{\Sigma}'}\colon \widetilde{\Sigma}' \longrightarrow \widetilde{X}'$ is $(\rho \circ \iota_{\Sigma'})$-equivariant. Thus, it descends to an orientation preserving homeomorphism $f'\colon \Sigma' \longrightarrow \Gamma'\backslash\widetilde{X}' \cong X'$, $\Gamma' := (\rho \circ \iota_{\Sigma'})(\pi_1(\Sigma'))$. The complement $\mathbb{H}^2 \setminus \widetilde{X}'$ is a disjoint union of closed half-spaces $\{H_j\}_{j\in\mathbb{N}}$ each bordering on a geodesic of $\widetilde{f}(\widetilde{\sigma})$ adjacent to $\widetilde{X}'$.

We want to understand the action of Γ' on each half-space $H_j, j \in \mathbb{N}$. Note that the disjoint union $\bigsqcup_{j\in\mathbb{N}} H_j$ is Γ'-invariant such that Γ' acts via permutations on $\{H_j\}_{j\in\mathbb{N}}$. Thus, if $\gamma \in \Gamma'$ is an element such that $\gamma H_j \cap H_j \neq \emptyset$ then $\gamma H_j = H_j$. If $I_j = \partial H_j \subseteq \partial\mathbb{H}^2 \cong \mathbb{S}^1$ denotes the interval that H_j borders on then γ has to fix I_j. Because Γ' is torsion-free it does not contain any elliptic elements such that γ must be a hyperbolic element that fixes the end points of I_j. By discreteness of Γ' there is a hyperbolic element $\gamma_j \in \Gamma'$ for every $j \in \mathbb{N}$ such that any element $\gamma \in \Gamma'$ satisfying $\gamma H_j \cap H_j \neq \emptyset$ is a power of γ_j. It follows that the quotient of H_j under the quotient map $\pi'\colon \mathbb{H}^2 \longrightarrow \Gamma'\backslash\mathbb{H}^2$ is a hyperbolic funnel $F_j := \pi'(H_j) \cong \langle\gamma_j\rangle\backslash H_j$.

Therefore, the complement of $\Gamma'\backslash\widetilde{X}'$ in $Y' = \Gamma'\backslash\mathbb{H}^2$ is a disjoint union of hyperbolic funnels. In particular, Y' deformation retracts to $\Gamma'\backslash\widetilde{X}'$, and we can easily modify f' to obtain an orientation preserving homeomorphism $f''\colon \Sigma' \longrightarrow Y'$ with holonomy $\rho \circ \iota_{\Sigma'}$. We conclude that $\rho \circ \iota_{\Sigma'} \in \mathcal{R}^*(\Sigma')$, and $\mathrm{res}^{\Sigma}_{\Sigma'}$ is well-defined. □

We may now define a topology on $\widehat{\mathcal{T}}(\Sigma)$.

Definition 2.5.5 (Topology on $\widehat{\mathcal{T}}(\Sigma)$)**.** For every $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma)$ we define a system of open neighborhoods by

$$(\mathrm{res}^{\sigma'}_{\sigma})^{-1}(U) \cap \mathcal{T}_{\sigma'}(\Sigma)$$

where $\sigma' \subseteq \sigma$ and $U \subseteq \mathcal{T}^*_\sigma(\Sigma) = \prod_{\Sigma'\in c(\sigma)} \mathcal{T}^*(\Sigma')$ runs over all open neighborhoods of $\mathfrak{r} \in \mathcal{T}^*_\sigma(\Sigma)$ in the product topology. This system of neighborhoods defines a topology on $\widehat{\mathcal{T}}(\Sigma)$.

In this topology a sequence $(\mathfrak{r}^{(n)})_{n\in\mathbb{N}} \subseteq \widehat{\mathcal{T}}(\Sigma)$ converges to $\mathfrak{r} \in \mathcal{T}_\sigma(\Sigma)$ if and only if $\mathfrak{r}^{(n)} \in \mathcal{T}_{\sigma_n}(\Sigma)$ with $\sigma_n \subseteq \sigma$ for large n, and

$$\mathrm{res}^{\sigma_n}_{\sigma}(\mathfrak{r}^{(n)}) \to \mathfrak{r} \qquad (n \to \infty)$$

in $\mathcal{T}^*_\sigma(\Sigma)$.

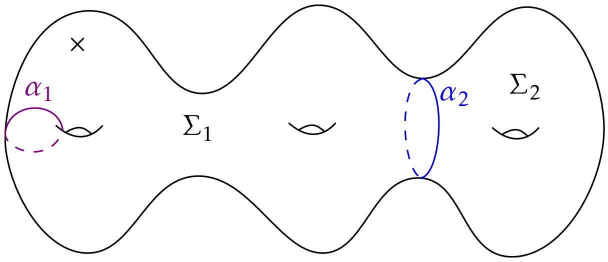
Notice that for $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma)$ every $[\rho_{\Sigma'}] \in \mathcal{T}(\Sigma')$ corresponds to a hyperbolic structure with finite area on $\Sigma' \in c(\sigma)$. Therefore, every peripheral curve

$\alpha \in \pi_1(\Sigma')$ is mapped to a parabolic boundary transformation $\rho_{\Sigma'}(\alpha)$. Suppose that $\mathfrak{r}^{(n)} = ([\rho^{(n)}_{\Sigma''_n}])_{\Sigma''_n \in c(\sigma_n)} \in \mathcal{T}_{\sigma_n}(\Sigma)$ converges to $\mathfrak{r} \in \mathcal{T}_\sigma(\Sigma)$. Then every curve $\alpha \in \pi_1(\Sigma)$, that is freely homotopic to a curve in σ but not in σ_n, is pinched as n tends to infinity. Indeed, up to conjugation $\rho^{(n)}_{\Sigma'_n}(\alpha)$ converges to a parabolic transformation for every component $\Sigma'_n \in c(\sigma_n)$ that contains α.

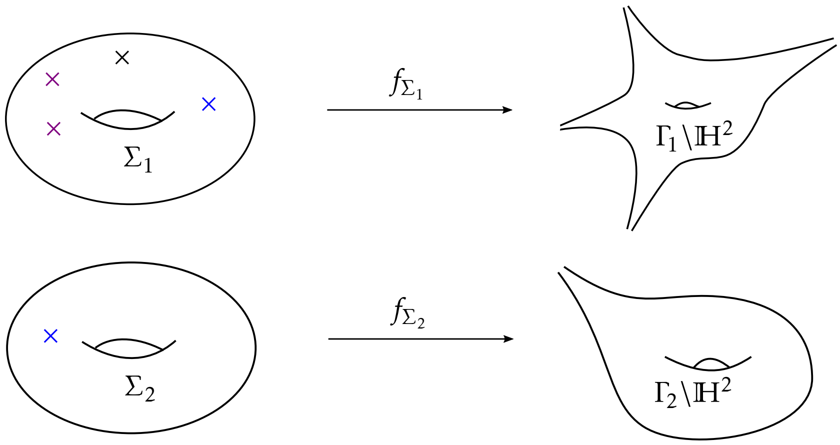
Remark 2.5.6. In [Abi80, Chapter 2, §3.4] the topology on augmented Teichmüller space is defined in terms of Fenchel–Nielsen coordinates. However, one can check that both topologies coincide. Indeed, every restriction map to a component corresponds to a projection onto an appropriate subset of length and twist parameters in Fenchel–Nielsen coordinates.

It is a result of Masur [Mas76] that the augmented Teichmüller space is the completion of Teichmüller space with respect to the Weil–Petersson metric; see also Wolpert [Wol07; Wol10]. In particular, the following holds.

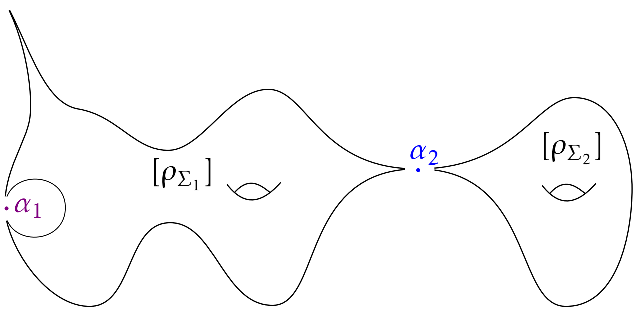
Proposition 2.5.7. *The augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$ is metrizable, and $\mathcal{T}(\Sigma) = \mathcal{T}_\varnothing(\Sigma)$ is an open and dense subset of $\widehat{\mathcal{T}}(\Sigma)$.*



(a) Let Σ be the surface of genus three with one puncture. We consider the simplex $\sigma = \{\alpha_1, \alpha_2\} \subseteq \mathcal{C}(\Sigma)$, with components $c(\sigma) = \{\Sigma_1, \Sigma_2\}$.



(b) Let $\mathfrak{r} = ([\rho_{\Sigma_1}], [\rho_{\Sigma_2}]) \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$ be a point in the augmented Teichmüller space, where $[\rho_{\Sigma_i}] \in \mathcal{T}(\Sigma_i)$, $i = 1, 2$. Because the representation ρ_{Σ_i} is admissible, it is a holonomy representation of an orientation preserving homeomorphism $f_{\Sigma_i} \colon \Sigma_1 \longrightarrow \Gamma_i\backslash\mathbb{H}^2$, $\Gamma_i := \rho_{\Sigma_i}(\pi_1(\Sigma_i))$, $i = 1, 2$.



(c) The above data is summarized in this picture.

Figure 2.1

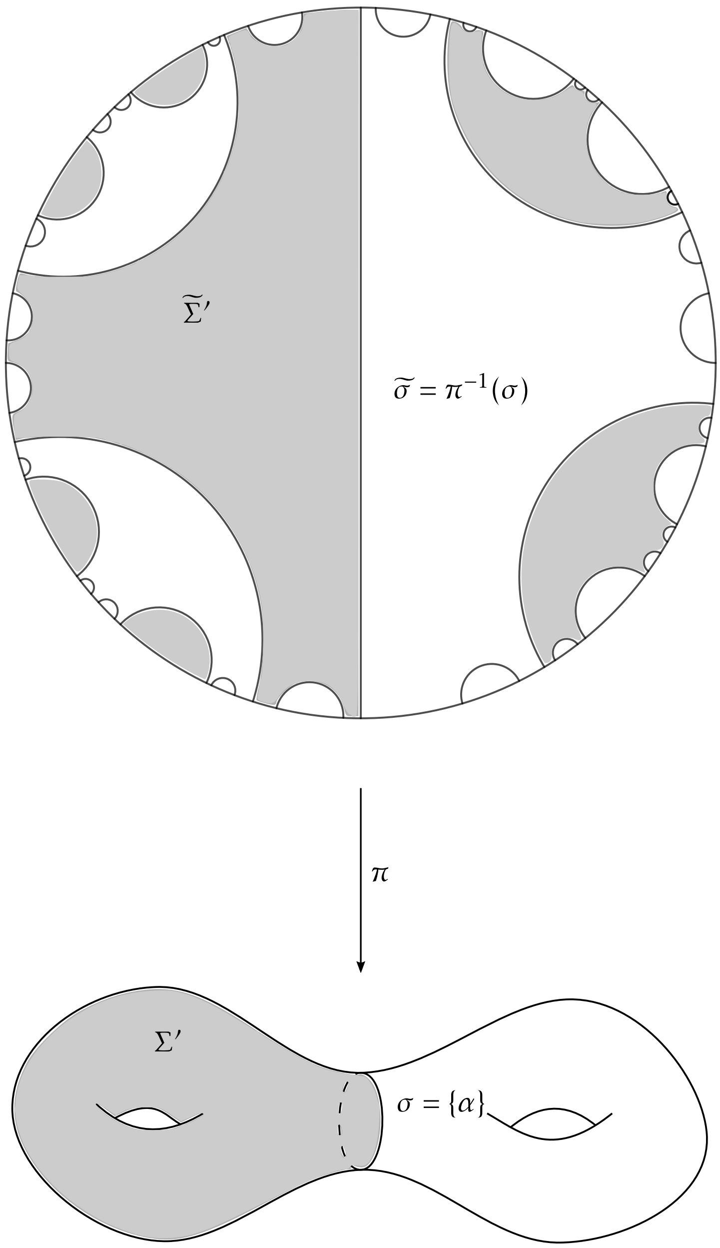


Figure 2.2: We are considering a genus two surface Σ and a simplex $\sigma = \{\alpha\} \subseteq \mathcal{C}(\Sigma)$ consisting of one separating curve $\alpha \subseteq \Sigma$. The preimage $\pi^{-1}(\Sigma')$ of the component $\Sigma' \in c(\sigma)$ is shaded. We chose one connected component $\widetilde{\Sigma}' \subseteq \mathbb{H}^2 \setminus \widetilde{\sigma}$.

2.6 The Augmented Moduli Space

We will now introduce the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ as a quotient of the augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$ by the extended mapping class group action; see Abikoff [Abi80, Chapter 2, §3.4].

The mapping class group action on Teichmüller space extends to $\widehat{\mathcal{T}}(\Sigma)$ in the following way. Let $\varphi = [f] \in \mathrm{MCG}(\Sigma)$ and $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$. The mapping class group acts simplicially on the curve complex $\mathcal{C}(\Sigma)$ such that $\varphi^{-1}(\sigma) \subseteq \mathcal{C}(\Sigma)$ is another simplex in the curve complex. Up to isotopy we may assume that f^{-1} sends σ to a geodesic representative of $f^{-1}(\sigma)$. Hence, f^{-1} induces a bijection between the components $c(\sigma)$ and $c(f^{-1}(\sigma)) = f^{-1}(c(\sigma))$, and acts from the right via restriction:

$$\mathfrak{r} \cdot \varphi := ([\rho_{f(\Sigma')} \circ (f|_{\Sigma'})_*])_{\Sigma' \in c(f^{-1}(\sigma))}.$$

The cutting homomorphism ensures that the action is well-defined; see [FM12, Section 3.6.3]. By definition this action extends the mapping class group action on $\mathcal{T}(\Sigma)$ such that the embedding $\mathcal{T}(\Sigma) \hookrightarrow \widehat{\mathcal{T}}(\Sigma)$ is MCG(Σ)-equivariant.

Note that this action permutes the different strata $\{\mathcal{T}_\sigma(\Sigma)\}_{\sigma \subseteq \mathcal{C}(\Sigma)}$ of the augmented Teichmüller space:

$$\mathcal{T}_\sigma(\Sigma) \xrightarrow{\varphi} \mathcal{T}_{\varphi^{-1}(\sigma)}(\Sigma), \qquad \varphi \in \mathrm{MCG}(\Sigma).$$

Definition 2.6.1. The quotient space $\widehat{\mathcal{M}}(\Sigma) := \widehat{\mathcal{T}}(\Sigma)/\mathrm{MCG}(\Sigma)$ of the augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$ by the mapping class group action is called the *augmented moduli space*.

We will denote the MCG(Σ)-equivalence class of $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$ by $[\mathfrak{r}] \in \widehat{\mathcal{M}}(\Sigma)$. Each $[[\rho_{\Sigma'}]] \in \mathcal{M}(\Sigma')$, $\Sigma' \in c(\sigma)$, will be called a *part* or *component* of $[\mathfrak{r}]$.[1]

In this way the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ may be interpreted as the space of all unmarked *nodal surfaces*, where every component carries a finite-area hyperbolic structure. If we want to emphasize this geometric point of view, we will denote a nodal surface in the augmented moduli space by a bold capital letter, e.g. $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$. Likewise, we will denote the parts of a nodal surface $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ by $X_{\Sigma'} \in \mathcal{M}(\Sigma')$, $\Sigma' \in c(\sigma)$, or $X_1, \dots, X_m$ if the components $c(\sigma) = \{\Sigma_1, \dots, \Sigma_m\}$ are enumerated.

[1] It is straight-forward to check that this definition depends only on the MCG(Σ)-equivalence class of $\mathfrak{r}$.

Remark 2.6.2. Changing perspective one can see the moduli space $\mathcal{M}(\Sigma)$ as the moduli space of smooth genus g curves with p marked points. In this setting, the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ corresponds to the Deligne–Mumford compactification of stable curves. We will not use this point of view in what follows and refer the reader to [Har74; Har77] and [HK14] for details.

The significance of this construction is that the augmented moduli space is *compact*.

Theorem 2.6.3 ([Abi80, Theorem, p. 104])**.** *The augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ is a compact Hausdorff space. The embedding $\mathcal{T}(\Sigma) \hookrightarrow \widehat{\mathcal{T}}(\Sigma)$ descends to an embedding $\mathcal{M}(\Sigma) \hookrightarrow \widehat{\mathcal{M}}(\Sigma)$ with open and dense image.*

Assembly Maps

We will now give an interpretation of how elements in $\widehat{\mathcal{M}}(\Sigma)$ may be assembled from elements in the moduli spaces of the components. Such gluing constructions are often used in Algebraic Geometry when studying the Deligne–Mumford compactification of stable curves [Arb+11, Chapter X, Section 7]. However, we could not find these constructions in the literature on the augmented moduli space. Therefore, we will present here a self-contained exposition of these ideas adapted to our setting.

In order to explain this, we will need some more notation.

Definition 2.6.4 (Pure Mapping Class Group)**.** The subgroup $\mathrm{PMCG}(\Sigma) \leq \mathrm{MCG}(\Sigma)$ of all mapping classes that fix each puncture of Σ individually is called the *pure mapping class group*.

Observe that there is the following short exact sequence

$$1 \longrightarrow \mathrm{PMCG}(\Sigma) \longrightarrow \mathrm{MCG}(\Sigma) \longrightarrow \mathrm{Sym}(p) \longrightarrow 1$$

coming from the action of the mapping class group on the p punctures of Σ. Here $\mathrm{Sym}(p)$ denotes the symmetric group on p elements. In particular, the pure mapping class group is a normal subgroup of the mapping class group of index $p! = \#\mathrm{Sym}(p)$.

We may now form a slightly larger moduli space of hyperbolic structures on Σ by keeping track of the punctures individually. Indeed, if we take the quotient of Teichmüller space by only the pure mapping class group, punctures are no longer allowed to be permuted by a mapping class. Thus, one may think of the elements of this new moduli space as hyperbolic surfaces with labeled punctures:

Definition 2.6.5. We define the *moduli space of (finite-area) hyperbolic surfaces with labeled punctures* as

$$\mathcal{M}^*(\Sigma) := \mathcal{T}(\Sigma)/\mathrm{PMCG}(\Sigma)$$

and denote the quotient map by $\pi_\Sigma \colon \mathcal{T}(\Sigma) \longrightarrow \mathcal{M}^*(\Sigma)$.

Let us now fix a simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ in the curve complex. We denote by

$$\mathrm{PMCG}_\sigma(\Sigma) := \{\varphi \in \mathrm{PMCG}(\Sigma) \colon \varphi(\alpha) = \alpha, \text{ for all } \alpha \in \sigma\} \leq \mathrm{PMCG}(\Sigma)$$

the subgroup of mapping classes fixing the homotopy class of each curve of σ individually. By definition $\mathrm{PMCG}_\sigma(\Sigma)$ acts on $\mathcal{T}_\sigma(\Sigma)$ and we define

$$p_\sigma \colon \mathcal{T}_\sigma(\Sigma) \longrightarrow \mathcal{M}^*_\sigma(\Sigma) := \mathcal{T}_\sigma(\Sigma)/\mathrm{PMCG}_\sigma(\Sigma).$$

Observe that if $f \colon \Sigma \longrightarrow \Sigma$ represents a mapping class $[f] \in \mathrm{PMCG}_\sigma(\Sigma)$ then we may isotope f so that it fixes each curve of σ individually. Because f is orientation preserving it follows that f fixes each component $\Sigma' \in c(\sigma)$ individually,

$$f|_{\Sigma'} \colon \Sigma' \longrightarrow \Sigma'.$$

Our next goal is to show that $\mathcal{M}^*_\sigma(\Sigma)$ can be identified with $\prod_{\Sigma' \in c(\sigma)} \mathcal{M}^*(\Sigma')$, which will then allow us to define the assembly map. To this end we will need the following lemma.

Lemma 2.6.6. *The map*

$$\begin{aligned} \varphi_\sigma \colon \mathrm{PMCG}_\sigma(\Sigma) &\longrightarrow \prod_{\Sigma' \in c(\sigma)} \mathrm{PMCG}(\Sigma'), \\ [f] &\longmapsto ([f|_{\Sigma'}])_{\Sigma' \in c(\sigma)} \end{aligned}$$

is a well-defined homomorphism. Moreover, φ_σ fits in the following short exact sequence

$$1 \longrightarrow \ker \varphi_\sigma \longrightarrow \mathrm{PMCG}_\sigma(\Sigma) \xrightarrow{\varphi_\sigma} \prod_{\Sigma' \in c(\sigma)} \mathrm{PMCG}(\Sigma') \longrightarrow 1$$

and $\ker \varphi_\sigma$ *is generated by Dehn twists about the curves in σ.*

Proof. The proof that φ_σ is well-defined with kernel generated by Dehn twists about the curves in σ is the same as for [FM12, Proposition 3.20].

We are left with proving that φ_σ surjects onto $\prod_{\Sigma'\in c(\sigma)} \mathrm{PMCG}(\Sigma')$. Let $[f_{\Sigma'}] \in \mathrm{PMCG}(\Sigma')$ for every $\Sigma' \in c(\sigma)$. Let $\{N_c : c \in \sigma\}$ be a collection of disjoint closed tubular neighborhoods about the curves in σ.

We define an orientation preserving homeomorphism $f\colon \Sigma \longrightarrow \Sigma$ in the following way. For every $x \in \Sigma' \setminus \bigsqcup_{c\in\sigma} N_c$, $\Sigma' \in c(\sigma)$, we set

$$f(x) = f_{\Sigma'}(x).$$

If N_c is a tubular neighborhood intersecting the components $\Sigma', \Sigma'' \in c(\sigma)$ we may interpolate continuously on N_c such that

$$\begin{aligned} f(x') &= f_{\Sigma'}(x') &\qquad& \forall x' \in \partial N_c \cap \Sigma', \\ f(x'') &= f_{\Sigma''}(x'') && \forall x'' \in \partial N_c \cap \Sigma'', \\ f(y) &= y && \forall y \in c. \end{aligned}$$

Then $[f] \in \mathrm{PMCG}_\sigma(\Sigma)$ by definition and $f|_{\Sigma'}$ coincides with $f_{\Sigma'}$ outside some disjoint discs about the punctures of Σ' for every $\Sigma' \in c(\sigma)$. Because the mapping class group of a punctured disc is trivial it follows that $[f|_{\Sigma'}] = [f_{\Sigma'}]$ for every $\Sigma' \in c(\sigma)$. $\square$

Proposition 2.6.7. *Define a map*

$$\begin{aligned} F_\sigma\colon \mathcal{T}_\sigma(\Sigma) = \prod_{\Sigma'\in c(\sigma)} \mathcal{T}(\Sigma') &\longrightarrow \prod_{\Sigma'\in c(\sigma)} \mathcal{M}^*(\Sigma'), \\ ([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)} &\longmapsto (\pi_{\Sigma'}([\rho_{\Sigma'}]))_{\Sigma'\in c(\sigma)}. \end{aligned}$$

Then F_σ descends to a homeomorphism:

$$\begin{tikzcd} & \mathcal{T}_\sigma(\Sigma) \arrow[dl, "p_\sigma"'] \arrow[dr, "F_\sigma"] & \\ \mathcal{M}^*_\sigma(\Sigma) \arrow[rr, dashed, "\overline{F}_\sigma", "\cong"'] & & \prod_{\Sigma'\in c(\sigma)} \mathcal{M}^*(\Sigma') \end{tikzcd}$$

Proof. Let us see that $\overline{F}_\sigma$ is well-defined. Let $([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}, ([\rho'_{\Sigma'}])_{\Sigma'\in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma)$ such that

$p_\sigma(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}) = p_\sigma(([\rho'_{\Sigma'}])_{\Sigma'\in c(\sigma)})$. Then there is $[f] \in \mathrm{PMCG}_\sigma(\Sigma)$ such that

$$([\rho'_{\Sigma'}])_{\Sigma'\in c(\sigma)} = ([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)} \cdot [f] = ([\rho_{\Sigma'} \circ (f|_{\Sigma'})_*])_{\Sigma'\in c(\sigma)} = ([\rho_{\Sigma'}] \cdot [f|_{\Sigma'}])_{\Sigma'\in c(\sigma)}.$$

Thus $F_\sigma(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}) = F_\sigma(([\rho'_{\Sigma'}])_{\Sigma'\in c(\sigma)})$ as required.

Let us see that $\overline{F}_\sigma$ is injective. Let $([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}, ([\rho'_{\Sigma'}])_{\Sigma'\in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma)$ such that

$$F_\sigma(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}) = F_\sigma(([\rho'_{\Sigma'}])_{\Sigma'\in c(\sigma)}).$$

Then there are $[f_{\Sigma'}] \in \mathrm{PMCG}(\Sigma')$ such that $[\rho'_{\Sigma'}] = [\rho_{\Sigma'}] \cdot [f_{\Sigma'}]$ for every $\Sigma' \in c(\sigma)$. By Lemma 2.6.6 there is an $[f] \in \mathrm{PMCG}_\sigma(\Sigma)$ such that $\varphi_\sigma([f]) = ([f|_{\Sigma'}])_{\Sigma'\in c(\sigma)} = ([f_{\Sigma'}])_{\Sigma'\in c(\sigma)}$. Thus, $([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)} \cdot [f] = ([\rho'_{\Sigma'}])_{\Sigma'\in c(\sigma)}$ and $p_\sigma(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}) = p_\sigma(([\rho'_{\Sigma'}])_{\Sigma'\in c(\sigma)})$.

Finally, $\overline{F}_\sigma$ is surjective because F_σ is. Moreover, because F_σ and p_σ are both open and continuous maps, also $\overline{F}_\sigma$ is open and continuous, whence $\overline{F}_\sigma$ is a homeomorphism. □

Projecting $\mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$ to the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ we obtain a continuous map $\mathcal{T}_\sigma(\Sigma) \longrightarrow \widehat{\mathcal{M}}(\Sigma)$. This map descends to a continuous map $\mathcal{M}^*_\sigma(\Sigma) \longrightarrow \widehat{\mathcal{M}}(\Sigma)$. Using the homeomorphism $\overline{F}_\sigma$ we obtain a continuous map $A_\sigma\colon \prod_{\Sigma'\in c(\sigma)} \mathcal{M}^*(\Sigma') \longrightarrow \widehat{\mathcal{M}}(\Sigma)$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{T}_\sigma(\Sigma) \arrow[rr] \arrow[d, "p_\sigma"] & & \widehat{\mathcal{M}}(\Sigma) \\
\mathcal{M}^*_\sigma(\Sigma) \arrow[urr] \arrow[r, "\overline{F}_\sigma", "\cong"'] & \prod_{\Sigma'\in c(\sigma)} & \mathcal{M}^*(\Sigma') \arrow[u, dashed, "A_\sigma"']
\end{tikzcd}$$

Definition 2.6.8. We will call the map $A_\sigma\colon \prod_{\Sigma'\in c(\sigma)} \mathcal{M}^*(\Sigma') \longrightarrow \widehat{\mathcal{M}}(\Sigma)$ the *assembly map with respect to* $\sigma \subseteq \mathcal{C}(\Sigma)$.

It is useful to know that the augmented moduli space is covered by the images of a *finite number* of assembly maps. The key here is the following observation due to Harvey [Har81].

Lemma 2.6.9 ([Har81, p. 247])**.** *The mapping class group* $\mathrm{MCG}(\Sigma)$ *acts on the curve complex* $\mathcal{C}(\Sigma)$ *simplicially and the quotient is a* finite *simplicial complex.*

One may think of this statement as the fact that one can dissect a given surface Σ only in finitely many different ways up to homeomorphism.

Proposition 2.6.10. *Let $\{\sigma_i : i \in I\}$ be a finite system of representatives for the simplices in the curve complex $\mathcal{C}(\Sigma)$ with respect to the mapping class group action. Then the augmented moduli space is covered by the images of the assembly maps A_{σ_i}, $i \in I$.*

Proof. Because $\{\sigma_i : i \in I\}$ is a system of representatives the projection

$$\bigsqcup_{i\in I} \mathcal{T}_{\sigma_i}(\Sigma) \longrightarrow \widehat{\mathcal{M}}(\Sigma)$$

is surjective. We obtain as above a commutative diagram:

$$\begin{tikzcd}
\bigsqcup_{i\in I} \mathcal{T}_{\sigma_i}(\Sigma) \arrow[rr, two heads] \arrow[d] & & \widehat{\mathcal{M}}(\Sigma) \\
\bigsqcup_{i\in I} \mathcal{M}^*_{\sigma_i}(\Sigma) \arrow[rru, two heads] \arrow[r, "\cong"] & \bigsqcup_{i\in I} \prod_{\Sigma'\in c(\sigma_i)} \mathcal{M}^*(\Sigma') \arrow[ru, two heads, "A"'] &
\end{tikzcd}$$

Here the map A is defined as A_{σ_i} on each $\prod_{\Sigma'\in c(\sigma_i)} \mathcal{M}^*(\Sigma')$, $i \in I$, such that the augmented moduli space is indeed covered by the images of all $\{A_{\sigma_i} : i \in I\}$. □

The previous formal treatment of the assembly map allows for a geometric interpretation. Every curve in σ corresponds to two punctures in some (possibly the same) component(s) $\Sigma', \Sigma'' \in c(\Sigma)$. In this way σ can be thought of as a pairing for the punctures of the components $c(\sigma)$. Given a collection of hyperbolic surfaces with labeled punctures $X_i \in \mathcal{M}^*(\Sigma_i)$, $i = 1, \ldots, m$, $c(\sigma) = \{\Sigma_1, \ldots, \Sigma_m\}$, we may "glue" them according to the pairing given by σ to obtain an element of $\widehat{\mathcal{M}}(\Sigma)$. This is exactly the image $A_\sigma(X_1, \ldots, X_m)$ of the assembly map.

Instead of starting out with a given topological surface Σ and a simplex $\sigma \subseteq \mathcal{C}(\Sigma)$, we may start with just a collection of surfaces and a pairing of their punctures. Indeed, let $\Sigma_1, \ldots, \Sigma_m$ be a collection of oriented topological surfaces with negative Euler characteristic $\chi(\Sigma_i) < 0$. We will think of the punctures as marked points instead of missing points, and denote by $P(\Sigma_i) \subseteq \Sigma_i$ the set of all punctures of Σ_i, $i = 1, \ldots, m$. Moreover, let $\mathcal{P}$ be a *pairing* of some of the punctures, i.e. $\mathcal{P}$ is a collection of two element subsets of $\bigsqcup_{i=1}^m P(\Sigma_i)$ such that $\{x_1, x_2\} \cap \{y_1, y_2\} = \varnothing$ for any two distinct $\{x_1, x_2\}, \{y_1, y_2\} \in \mathcal{P}$.

Using the pairing $\mathcal{P}$ we can assemble the surfaces $\Sigma_1, \ldots, \Sigma_m$ to a larger oriented topological surface $\Sigma(\Sigma_1, \ldots, \Sigma_m; \mathcal{P})$ with negative Euler characteristic in the following way.

For every puncture $x_j \in P(\Sigma_i)$, that occurs in the pairing $\mathcal{P}$, we choose a small open disc $x_j \in \mathbb{D}_j \subseteq \Sigma_i$. Up to shrinking these discs we may assume that their closures $\{\overline{\mathbb{D}}_j\}_j$ are pairwise disjoint. Next, we remove these discs from the surfaces $\Sigma_1, \dots, \Sigma_m$ to obtain a collection $\overline{\Sigma}_1, \dots, \overline{\Sigma}_m$ of surfaces with boundaries. For every pair $\{x_{j_1}, x_{j_2}\} \in \mathcal{P}$ we then identify the boundary $\partial\mathbb{D}_{j_1} \subseteq \partial\overline{\Sigma}_{i_1}$ with the boundary $\partial\mathbb{D}_{j_2} \subseteq \partial\overline{\Sigma}_{i_2}$ via an orientation reversing homeomorphism. In this way, we obtain an oriented topological surface $\Sigma = \Sigma(\Sigma_1, \dots, \Sigma_m; \mathcal{P})$. We call the pairing $\mathcal{P}$ *admissible*, if the surface Σ is connected.

Remark 2.6.11. Observe that $\chi(\Sigma) = \sum_{i=1}^{m} \chi(\Sigma_i) < 0$ by definition. However, the topological surface $\Sigma = \Sigma(\Sigma_1, \dots, \Sigma_m; \mathcal{P})$ depends on the pairing $\mathcal{P}$ and two different pairings $\mathcal{P}, \mathcal{P}'$ usually yield non-homeomorphic surfaces $\Sigma(\Sigma_1, \dots, \Sigma_m; \mathcal{P}) \ncong \Sigma(\Sigma_1, \dots, \Sigma_m; \mathcal{P}')$.

The pairing $\mathcal{P}$ amounts to a collection of closed curves $\sigma(\mathcal{P}) \subseteq \mathcal{C}(\Sigma)$ in the following way. For every pairing $p = \{x_{j_1}, x_{j_2}\} \in \mathcal{P}$ we identified the boundaries $\partial\mathbb{D}_{j_1}$ and $\partial\mathbb{D}_{j_2}$, which descend to a simple closed curve α_p in the quotient surface Σ. Setting $\sigma(\mathcal{P}) := \{\alpha_p \,|\, p \in \mathcal{P}\} \subseteq \mathcal{C}(\Sigma)$ one obtains a simplex in the curve complex. Moreover, we can canonically identify the surfaces $\Sigma_1, \dots, \Sigma_m$ with the components $c(\sigma(\mathcal{P}))$. Indeed, $c(\sigma(\mathcal{P})) = \{\overline{\Sigma}_1 \setminus \partial\overline{\Sigma}_1, \dots, \overline{\Sigma}_m \setminus \partial\overline{\Sigma}_m\}$ by construction, and $\Sigma_i \cong \overline{\Sigma}_i \setminus \partial\overline{\Sigma}_i$ via homeomorphisms supported on small neighborhoods about the punctures, $i = 1, \dots, m$. Therefore, we may define the assembly map with respect to an admissible pairing $\mathcal{P}$ as the assembly map with respect to the simplex $\sigma(\mathcal{P}) \subseteq \mathcal{C}(\Sigma(\Sigma_1, \dots, \Sigma_m; \mathcal{P}))$; see Figure 2.3 for an example.

Definition 2.6.12. Let $\Sigma_1, \dots, \Sigma_m$ be a collection of oriented topological surfaces with negative Euler characteristic $\chi(\Sigma_i) < 0$, let $\mathcal{P}$ be an admissible pairing and set $\Sigma := \Sigma(\Sigma_1, \dots, \Sigma_m; \mathcal{P})$ as above. Then the *assembly map with respect to the pairing* $\mathcal{P}$ is defined as

$$\begin{aligned} A_{\mathcal{P}} \colon \prod_{i=1}^{m} \mathcal{M}^*(\Sigma_i) &\longrightarrow \widehat{\mathcal{M}}(\Sigma), \\ (X_1, \dots, X_m) &\longmapsto A_{\sigma(\mathcal{P})}(X_1, \dots, X_m). \end{aligned}$$

We will also say that $X_1, \dots, X_m$ are *glued according to* $\mathcal{P}$ to form the nodal surface $\mathbf{X} = A_{\mathcal{P}}(X_1, \dots, X_m)$.

The following is an immediate consequence of Proposition 2.6.10.

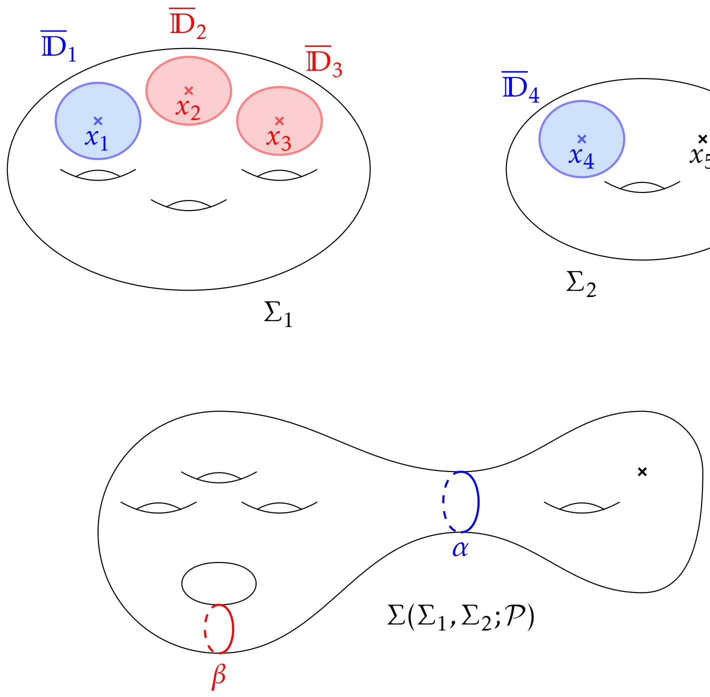


Figure 2.3: The topological surface $\Sigma(\Sigma_1, \Sigma_2; \mathcal{P})$ obtained from the surfaces Σ_1 and Σ_2 with respect to the pairing $\mathcal{P} = \{\{x_1, x_4\}, \{x_2, x_3\}\}$. The corresponding simplex is $\sigma(\mathcal{P}) = \{\alpha, \beta\}$.

Corollary 2.6.13. *For every nodal surface $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ there is a collection $X_1, \dots, X_m$, $m > 0$, of hyperbolic surfaces with labeled punctures and an admissible pairing of their punctures $\mathcal{P}$ such that $\mathbf{X} = A_{\mathcal{P}}(X_1, \dots, X_m)$.*

Forgetting the gluing

We have already seen that one may obtain nodal surfaces from hyperbolic surfaces with labeled punctures by gluing pairs of punctures. In this section we will go the other way and forget how the parts are glued.

Consider the set $|\widehat{\mathcal{M}}| := \mathbb{N}_0^{\bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')}$ of all maps from $\bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')$ to $\mathbb{N}_0$, where the disjoint union is taken over all oriented topological surfaces Σ' with negative Euler characteristic. There is a canonical map $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|$, that records each part of the nodal surface with its multiplicity. (Note that the parts of a nodal surface do *not* have labeled punctures.)

Formally, given a nodal surface $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ with parts $X_i \in \mathcal{M}(\Sigma_i)$, $i = 1,\dots,m$, we define

$$Q(\mathbf{X})(Y) := \#\{i \in \{1,\dots,m\} \mid X_i = Y\} \qquad \forall Y \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma').$$

Definition 2.6.14. We denote the image of $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|$ by $|\widehat{\mathcal{M}}|(\Sigma) := Q(\widehat{\mathcal{M}}(\Sigma))$ and call it the *moduli space of parts*. Furthermore, we equip $|\widehat{\mathcal{M}}|(\Sigma)$ with the quotient topology turning it into a compact topological space.

It turns out that the cardinalities of the fibers of Q admit a uniform upper bound, that depends only on the topology of the surface Σ.

Proposition 2.6.15. *Let $\chi = \chi(\Sigma)$, $g = g(\Sigma)$, and $p = p(\Sigma)$ denote the Euler characteristic, the genus, and the number of punctures of Σ, respectively. Then*

$$\#Q^{-1}(\xi) \leq B(\Sigma)$$

for all $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$, where

$$B(\Sigma) := \binom{3|\chi|}{p} \cdot \frac{(2(|\chi|+g-1))!}{(|\chi|+g-1)! \cdot 2^{(|\chi|+g-1)}}.$$

Proof. Let $\mathbf{X} = A_{\mathcal{P}_0}(X_1,\dots,X_m) \in \widehat{\mathcal{M}}(\Sigma)$ be a nodal surface glued from some $X_i \in \mathcal{M}^*(\Sigma_i)$, $i = 1,\dots,m$, according to some pairing of their punctures $\mathcal{P}_0$. Any other nodal surface $\mathbf{Y} \in Q^{-1}(Q(\mathbf{X}))$ can be obtained as a gluing $\mathbf{Y} = A_{\mathcal{P}}(X_1,\dots,X_m)$ according to some appropriate pairing $\mathcal{P}$. Therefore, it will suffice to show that there are at most $B(\Sigma)$-many admissible pairings $\mathcal{P}$ that yield a nodal surface in the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$. Notice that this imposes the requirement that the glued topological surface $\Sigma(\Sigma_1,\dots,\Sigma_m;\mathcal{P})$ is homeomorphic to the given surface Σ.

We will now estimate the number of such pairings. To this end let $P := \sum_{i=1}^{m} p(\Sigma_i)$ denote the total number of punctures of all the components $\Sigma_1,\dots,\Sigma_m$. We know that in the end $\Sigma(\Sigma_1,\dots,\Sigma_m;\mathcal{P})$ should have p punctures. Thus, in a first step, we will select p punctures from all of the punctures of the components $\Sigma_1,\dots,\Sigma_m$, that will not be glued. There are $\binom{P}{p}$-many possible choices. The remaining $P - p$ punctures will have to be glued.

By elementary combinatorics there are

$$\frac{(2k)!}{k! \cdot 2^k} \tag{2.1}$$

many ways of pairing $2k$ elements, $k \in \mathbb{N}$. Hence, there are

$$\frac{(P-p)!}{((P-p)/2)! \cdot 2^{(P-p)/2}}$$

many different ways of pairing the remaining $P-p$ punctures. Altogether there are at most

$$\binom{P}{p} \cdot \frac{(P-p)!}{((P-p)/2)! \cdot 2^{(P-p)/2}} \tag{2.2}$$

many pairings $\mathcal{P}$, that yield a topological surface $\Sigma(\Sigma_1,\dots,\Sigma_m;\mathcal{P})$ homeomorphic to Σ.

In order to obtain a uniform upper bound, let us first prove that $P \le 3|\chi|$: Let χ_i, g_i, and p_i denote the Euler characteristic, the genus, and the number of punctures of Σ_i, respectively. Observe that $\chi(\Sigma_i) = 2-2g_i-p_i \le -1$, $i=1,\dots,m$. Hence,

$$\chi = \sum_{i=1}^{m} \chi_i = \sum_{i=1}^{m} (2-2g_i-p_i). \tag{2.3}$$

We obtain

$$P = \sum_{i=1}^{m} p_i = |\chi| + \sum_{i=1}^{m} (2-2g_i) \le |\chi| + 2m.$$

Finally,

$$m \le \sum_{i=1}^{m} |\chi_i| = |\chi|,$$

because $\chi_i \le -1$. This yields $P \le 3|\chi|$, whence

$$\binom{P}{p} \le \binom{3|\chi|}{p}. \tag{2.4}$$

Next, observe that

$$2-2g-p = \sum_{i=1}^{m} (2-2g_i) - P$$

by (2.3), whence

$$P-p = \sum_{i=1}^{m} (2-2g_i) - (2-2g)$$

$$
\begin{aligned}
&= 2(m-1) + 2g - 2\sum_{i=1}^{m} g_i \\
&\leq 2(|\chi| + g - 1),
\end{aligned}
$$

and $\frac{1}{2}(P-p) \leq |\chi| + g - 1$.

It is straight-forward to check that (2.1) is increasing in $k \in \mathbb{N}$, whence we obtain

$$
\frac{(P-p)!}{((P-p)/2)! \cdot 2^{(P-p)/2}} \leq \frac{(2(|\chi|+g-1))!}{(|\chi|+g-1)! \cdot 2^{|\chi|+g-1}}. \tag{2.5}
$$

Using the upper bounds (2.4) and (2.5) in (2.2) we obtain that there are at most

$$
B(\Sigma) = \binom{3|\chi|}{p} \cdot \frac{(2(|\chi|+g-1))!}{(|\chi|+g-1)! \cdot 2^{|\chi|+g-1}}
$$

many pairings $\mathcal{P}$ such that $\Sigma \cong \Sigma(\Sigma_1, \ldots, \Sigma_m; \mathcal{P})$. This concludes the proof. □

We want to point out that the bound $B(\Sigma)$ is not sharp, as the following example shows.

Example 2.6.16. Consider the once punctured torus $\Sigma = \Sigma_{1,1}$. Observe that the boundary of $\widehat{\mathcal{M}}(\Sigma)$ consists of a single nodal surface $\mathbf{X}_0$, which is a degenerate hyperbolic pair of pants with two punctures glued. Since the quotient map $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|(\Sigma)$ is one-to-one on the moduli space $\mathcal{M}(\Sigma)$, it follows that the map Q is one-to-one on the entire augmented moduli space. However, we have that

$$
B(\Sigma_{1,1}) = \binom{3}{1} \cdot \frac{2!}{1! \cdot 2^1} = 3.
$$

For an arbitrary $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$ computing $\#Q^{-1}(\xi)$ can be delicate. Indeed, it is not sufficient to just count the number of pairings that yield the correct topological type of the glued surface, because the symmetries of the different parts play a role, too. This is demonstrated by the following example.

Example 2.6.17. Consider the closed surface $\Sigma = \Sigma_3$ of genus three, and let $X, Y \in \mathcal{M}(\Sigma_{1,2})$ be two hyperbolic surfaces of genus one with two punctures. We define an element $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$ by setting $\xi(X) = \xi(Y) = 1$ and $\xi(Z) = 0$ for all other surfaces $Z \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')$. Let us label the punctures of X as $\{x_1, x_2\}$ and the punctures of Y as $\{y_1, y_2\}$ to obtain points $X', Y' \in \mathcal{M}^*(\Sigma_{2,1})$. Then any point in the fiber $Q^{-1}(\xi)$ can be glued from X' and Y' by pairing the punctures appropriately. We have the following list of possible pairings:

$$\mathcal{P}_1 = \{\{x_1, y_1\}, \{x_2, y_2\}\},$$
$$\mathcal{P}_2 = \{\{x_1, y_2\}, \{x_2, y_1\}\},$$
$$\mathcal{P}_3 = \{\{x_1, x_2\}, \{y_1, y_2\}\}.$$

Note that the last pairing $\mathcal{P}_3$ is not admissible. Thus,

$$Q^{-1}(\xi) = \{A_{\mathcal{P}_1}(X', Y'), A_{\mathcal{P}_2}(X', Y')\}.$$

Moreover, if there is an isometry of X (or Y) that exchanges the punctures, then

$$A_{\mathcal{P}_1}(X', Y') = A_{\mathcal{P}_2}(X', Y') \in \widehat{\mathcal{M}}(\Sigma_3).$$

Thus, depending on the symmetries of X and Y, the preimage $Q^{-1}(\xi)$ has either one or two elements.

2.7 The Space of Closed Subgroups

In this section we will introduce the space of closed subgroups and briefly recall its properties. The material presented here is well-known and can be found in [CEM06; Gel15; Mac64; GL18b; BP92]. We also recommend the survey [Har08] for further reading.

The space of closed subgroups is essential for this thesis as every invariant random subgroup is a conjugation invariant probability measure on the space of closed subgroups by definition. Moreover, we will use it in section 2.8 to define the geometric topology on $\mathcal{R}^*(\Sigma)$. As before we will use the notation $G = \mathrm{PSL}(2, \mathbb{R})$ in the following.

Definition 2.7.1 (Space of closed subgroups $\mathrm{Sub}(G)$; [Abé+17, Section 2])**.** We denote by $\mathrm{Sub}(G)$ the set of closed subgroups of G. For open subsets $U \subseteq G$ and compact subsets $K \subseteq G$ we define the sets

$$\mathcal{O}(K) := \{A \in \mathrm{Sub}(G) : A \cap K = \varnothing\}, \qquad \mathcal{O}'(U) := \{A \in \mathrm{Sub}(G) : A \cap U \neq \varnothing\}.$$

The collection of all such subsets $\{\mathcal{O}(K) : K \subseteq G \text{ compact}\} \cup \{\mathcal{O}'(U) : U \subseteq G \text{ open}\}$ generates the *Chabauty topology* on $\mathrm{Sub}(G)$; see [Cha50].

The most important property of this topology is that $\mathrm{Sub}(G)$ is compact.

Lemma 2.7.2 ([CEM06, Proposition I.3.1.2]). *The space of closed subgroups* $\mathrm{Sub}(G)$ *is compact and metrizable.*

The following characterization is often useful.

Proposition 2.7.3 ([CEM06, Lemma I.3.1.3]). *A sequence* $(H_n)_{n\in\mathbb{N}} \subseteq \mathrm{Sub}(G)$ *converges to* $H \in \mathrm{Sub}(G)$ *if and only if the following two conditions are satisfied:*

(C1) For every $h \in H$ *there is a sequence* $(h_n)_{n\in\mathbb{N}} \subseteq G$ *such that* $h_n \in H_n$ *for every* $n \in \mathbb{N}$ *and* $h = \lim_{n\to\infty} h_n$.

(C2) If $h \in G$ *is the limit of a sequence* $(h_{n_k})_{k\in\mathbb{N}} \subseteq H$ *such that* $h_{n_k} \in H_{n_k}$ *for every* $k \in \mathbb{N}$, *then* $h \in H$.

The Chabauty topology is compatible with the conjugation action $G \curvearrowright \mathrm{Sub}(G)$:

Lemma 2.7.4 ([Abé+17, Section 2]). *The group G acts continuously on* $\mathrm{Sub}(G)$ *via conjugation*

$$\begin{aligned} G \times \mathrm{Sub}(G) &\longrightarrow \mathrm{Sub}(G) \\ (g, H) &\longmapsto gHg^{-1}. \end{aligned}$$

The following subsets will be of interest later on:

Definition 2.7.5. We define

$$\begin{aligned} \mathrm{Sub}_{\mathrm{d}}(G) &:= \{\Gamma \in \mathrm{Sub}(G) \,|\, \Gamma \text{ is discrete}\}, \\ \mathrm{Sub}_{\mathrm{dtf}}(G) &:= \{\Gamma \in \mathrm{Sub}(G) \,|\, \Gamma \text{ is discrete and torsion-free}\}, \end{aligned}$$

and equip these subsets with the subspace topology.

We will record some of their topological properties now:

Lemma 2.7.6 ([CEM06, Theorem I.3.1.4]). *Let* $\Gamma \in \mathrm{Sub}_{\mathrm{d}}(G)$. *Then there is an open neighborhood* $U \subseteq G$ *of the identity* $e \in G$ *and an open neighborhood* $\mathcal{U} \subseteq \mathrm{Sub}(G)$ *such that*

$$\Gamma' \cap U = \{e\}$$

for every $\Gamma' \in \mathcal{U}$.

Corollary 2.7.7. *The subset of discrete subgroups* $\mathrm{Sub}_{\mathrm{d}}(G) \subseteq \mathrm{Sub}(G)$ *is open.*

Not only discrete groups form an open subset of the space of closed subgroups but also cocompact lattices as the following lemma asserts.

Lemma 2.7.8 ([Mac64, Lemma 7.1], [GL18b, Proof of Proposition 9.6])**.** *Let* $\Gamma \leq G$ *be a cocompact lattice such that* $\Gamma \cdot K_0 = \mathbb{H}^2$ *for some compact subset* $K_0 \subseteq \mathbb{H}^2$*. Then we find for every neighborhood* $K_0 \subseteq K \subseteq \mathbb{H}^2$ *an open neighborhood* $\mathcal{U} \subseteq \mathrm{Sub}(G)$ *of* Γ *such that* $\Gamma' \cdot K = \mathbb{H}^2$ *for every* $\Gamma' \in \mathcal{U}$*.*

There is an even stronger result available, namely cocompact lattices are locally rigid in $\mathrm{Sub}(G)$.

Theorem 2.7.9 ([Mac64, Theorem 4], [GL18b, Proposition 9.6])**.** *Let* $\Gamma \leq G$ *be a cocompact lattice. Then there is an open neighborhood* $\mathcal{U} \subseteq \mathrm{Sub}(G)$ *of* Γ *such that every* $\Gamma' \in \mathcal{U}$ *is a cocompact lattice isomorphic to* Γ*.*

Remark 2.7.10. All of the above results hold for an arbitrary Lie group G as well. More generally, Gelander and Levit [GL18b] proved that Lemma 2.7.8 and Theorem 2.7.9 even hold for the isometry group of any proper geodesically complete CAT(0) space.

In the introduction we have informally explained how the subspace $\mathrm{Sub}_{\mathrm{dtf}}(G) \subseteq \mathrm{Sub}(G)$ is homeomorphic to the space of framed hyperbolic surfaces $\mathcal{FM}(\mathbb{H}^2)$. We will discuss this more rigorously, now.

Definition 2.7.11. Let X be a hyperbolic surface. We denote by $\mathrm{OF}_+(X) \subseteq TX \times TX$ the subbundle of all positively oriented orthonormal frames $\mathbf{f} = (f_1, f_2) \in T_xX \times T_xX$, $x \in X$.

Any (orientation preserving) isometry $\varphi\colon X \longrightarrow Y$ between two hyperbolic surfaces X, Y induces a bundle isomorphism

$$d\varphi\colon \mathrm{OF}_+(X) \longrightarrow \mathrm{OF}_+(Y), \quad \mathbf{f} = (f_1, f_2) \longmapsto d\varphi(\mathbf{f}) := (d\varphi(f_1), d\varphi(f_2))$$

between the respective positive orthonormal frame bundles $\mathrm{OF}_+(X)$ and $\mathrm{OF}_+(Y)$.

Definition 2.7.12. A *framed hyperbolic surface* $(X, \mathbf{f})$ is a hyperbolic surface X together with a positively oriented orthonormal frame $\mathbf{f} = (f_1, f_2) \in \mathrm{OF}_+(X)$. We will identify two framed hyperbolic surfaces $(X, \mathbf{f}), (Y, \mathbf{g})$ if there is an (orientation preserving) isometry $\varphi\colon X \longrightarrow Y$, such that $d\varphi(\mathbf{f}) = \mathbf{g}$.

We define $\mathcal{FM}(\mathbb{H}^2)$ to be the set of all equivalence classes of framed hyperbolic surfaces $[X, \mathbf{f}]$.

In order to define a topology on $\mathcal{FM}(\mathbb{H}^2)$ we will need the following notion.

Definition 2.7.13. Let $\Omega \subseteq \mathbb{H}^2$ be an open set, let $K \subseteq \Omega$ be a compact subset and let $f, g \colon \Omega \longrightarrow \mathbb{H}^2$ be two smooth maps. We define the *C^∞-distance between f and g on K* by

$$D_K(f,g) := \sum_{k=0}^{\infty} 2^{-k} \cdot \max\left(1, \sup_{\substack{x\in K\\ |\alpha|\le k}} \|\partial_\alpha f(x) - \partial_\alpha g(x)\|\right).$$

Here, we regard $\mathbb{H}^2$ as a subset of $\mathbb{R}^2$ via the upper half-plane model, and denote by

$$\partial_\alpha f = \partial_{i_1} \dots \partial_{i_r} f$$

the α-th partial derivative of f with respect to a multi-index $\alpha = (i_1, \dots, i_r)$ of degree $|\alpha| = r$.

Remark 2.7.14. Instead of appealing to the underlying Euclidean structure of $\mathbb{H}^2$ one could instead use the hyperbolic metric to define a C^∞-distance that is more adapted to the hyperbolic geometry of $\mathbb{H}^2$; see [BP92, p. 167]. Since we will not work with the C^∞-distance explicitly here, the above definition suffices for our purposes.

We fix an orthonormal frame $\mathbf{e} \in \mathrm{OF}_+(\mathbb{H}^2)$ based at some point $o \in \mathbb{H}^2$. After fixing this reference frame there is a unique covering map $\pi_{[X,\mathbf{f}]} \colon \mathbb{H}^2 \longrightarrow X$ such that $d\pi_{[X,\mathbf{f}]}(\mathbf{e}) = \mathbf{f}$ for every framed hyperbolic surface $[X,\mathbf{f}] \in \mathcal{FM}(\mathbb{H}^2)$.

We can now define a neighborhood of $[X,\mathbf{f}] \in \mathcal{FM}(\mathbb{H}^2)$ for every $r > 0, \varepsilon > 0$ as the set $\mathcal{N}_{r,\varepsilon}([X,\mathbf{f}]) \subseteq \mathcal{FM}(\mathbb{H}^2)$ of all framed hyperbolic surfaces $[X',\mathbf{f}']$, that satisfy the following: There are open neighborhoods $\Omega, \Omega' \supseteq \overline{B_o(r)}$ and a diffeomorphism $f : \Omega \longrightarrow \Omega'$ such that

(i) $f(o) = o$,

(ii) $\pi_{[X',\mathbf{f}']}(f(x)) = \pi_{[X',\mathbf{f}']}(f(y)) \iff \pi_{[X,\mathbf{f}]}(x) = \pi_{[X,\mathbf{f}]}(y)$, for every $x, y \in \Omega$, and

(iii) $D_{\overline{B_o(r)}}(f, \mathrm{id}) < \varepsilon$.

In particular, the diffeomorphism $f : \Omega \longrightarrow \Omega'$ descends to a diffeomorphism F mapping $\pi_{[X,\mathbf{f}]}(\Omega) \subseteq \Gamma\backslash\mathbb{H}^2$ to $\pi_{[X',\mathbf{f}']}(\Omega') \subseteq \Gamma'\backslash\mathbb{H}^2$:

$$\begin{array}{ccc} \Omega \subseteq \mathbb{H}^2 & \xrightarrow{\quad f \quad} & \Omega' \subseteq \mathbb{H}^2 \\ \downarrow{\scriptstyle \pi_{[X,\mathbf{f}]}} & & \downarrow{\scriptstyle \pi_{[X',\mathbf{f}']}} \\ \pi_{[X,\mathbf{f}]}(\Omega) \subseteq X & \xrightarrow{F} & \pi_{[X',\mathbf{f}']}(\Omega') \subseteq X' \end{array}$$

One can check that the neighborhoods $\{\mathcal{N}_{r,\varepsilon}([X,\mathbf{f}]) : r > 0, \varepsilon > 0, [X,\mathbf{f}] \in \mathcal{FM}(\mathbb{H}^2)\}$ form a neighborhood system for a topology of $\mathcal{FM}(\mathbb{H}^2)$; see [BP92, Section E.1].

Given a discrete and torsion-free subgroup $\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G)$ we obtain a framed hyperbolic surface

$$[\Gamma\backslash\mathbb{H}^2, d\pi_\Gamma(\mathbf{e})] \in \mathcal{FM}(\mathbb{H}^2),$$

where we denote by $\pi_\Gamma \colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ the quotient map. Notice that $\pi_{[\Gamma\backslash\mathbb{H}^2, d\pi_\Gamma(\mathbf{e})]} = \pi_\Gamma$ by definition.

It turns out that we can identify $\mathrm{Sub}_{\mathrm{dtf}}(G)$ with $\mathcal{FM}(\mathbb{H}^2)$ in this way.

Proposition 2.7.15 ([BP92, Theorem E.1.13])**.** *The map*

$$\psi\colon \mathrm{Sub}_{\mathrm{dtf}}(G) \longrightarrow \mathcal{FM}(\mathbb{H}^2), \quad \Gamma \longmapsto [\Gamma\backslash\mathbb{H}^2, d\pi_\Gamma(\mathbf{e})]$$

is a homeomorphism.

Informally, one can think of Proposition 2.7.15 as saying that “large” balls centered at the base points $\pi_\Gamma(o)$ in $\Gamma\backslash\mathbb{H}^2$ and $\pi_{\Gamma'}(o)$ in $\Gamma'\backslash\mathbb{H}^2$ are “almost isometric” if $\Gamma, \Gamma' \in \mathrm{Sub}_{\mathrm{dtf}}(G)$ are “close”.

2.8 The Geometric Topology

Previously, we have considered the algebraic topology on $\mathcal{R}(\Sigma)$. Using the Chabauty topology we will now introduce the *geometric topology*. This terminology is justified by its geometric implications; see Proposition 2.7.15.

Definition 2.8.1 (Geometric topology on $\mathcal{R}^*(\Sigma)$)**.** The *geometric topology* on $\mathcal{R}^*(\Sigma)$ is the initial topology with respect to the following two maps:

(i) $i\colon \mathcal{R}^*(\Sigma) \hookrightarrow G^S, \rho \longmapsto (\rho(s))_{s\in S}$, where $S \subseteq \pi_1(\Sigma)$ is a generating set for $\pi_1(\Sigma)$, and

(ii) $\mathrm{im}\colon \mathcal{R}^*(\Sigma) \longrightarrow \mathrm{Sub}_{\mathrm{dtf}}(G), \rho \longmapsto \mathrm{im}\,\rho$, which sends every (discrete) representation $\rho \in \mathcal{R}^*(\Sigma)$ to its image in $\mathrm{Sub}(G)$.

Recall that the algebraic topology on $\mathcal{R}^*(\Sigma)$ is the initial topology with respect to just the map $i\colon \mathcal{R}^*(\Sigma) \hookrightarrow G^S$, such that the geometric topology is apriori stronger than the algebraic topology. Using the fact that Σ is of finite type we will show that it is *not*. However, there are counterexamples for surfaces of infinite type; see [CEM06, Section I.3.1.10] for more details.

Proposition 2.8.2. *Consider $\mathcal{R}^*(\Sigma)$ with the algebraic topology. Then the map*

$$\mathrm{im}\colon \mathcal{R}^*(\Sigma) \longrightarrow \mathrm{Sub}_{\mathrm{dtf}}(G)$$

is a local homeomorphism onto its image $\mathcal{D}(\Sigma)$, which is the set of all discrete and torsion-free subgroups $\Gamma' < G$ such that there is an orientation preserving homeomorphism $f\colon \Sigma \longrightarrow \Gamma' \backslash \mathbb{H}^2$.

If we consider $\mathcal{R}(\Sigma)$ instead, the image of $\mathrm{im}\colon \mathcal{R}(\Sigma) \longrightarrow \mathrm{Sub}_{\mathrm{dtf}}(G)$ consists of all lattices $\Gamma \in \mathcal{D}(\Sigma)$. We denote this set by $\mathcal{L}(\Sigma)$.

Remark 2.8.3. In particular, Proposition 2.8.2 proves that the geometric topology coincides with the algebraic topology on $\mathcal{R}^*(\Sigma)$.

Skipping some details Harvey [Har77, Sections 2.3 and 2.4] gave a proof of Proposition 2.8.2. Another sketch of proof is given in [CEM06, Remark, p. 66]. However, we could not find a completely satisfactory proof in the literature. Therefore, we decided to include a careful proof using Proposition 2.7.15 in the following.

Proof. First, we shall prove that the map im is continuous. We need to prove that if a sequence $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$ converges to $\rho_\infty \in \mathcal{R}^*(\Sigma)$ in the algebraic topology then it converges to ρ_∞ in the geometric topology, too. To do so we will check (C1) and (C2) from Proposition 2.7.3 and see that $\mathrm{im}\,\rho_n \to \mathrm{im}\,\rho_\infty$ as $n \to \infty$.

(C1) If $\rho_\infty(\gamma) \in \mathrm{im}\,\rho_\infty$, $\gamma \in \pi_1(\Sigma)$, then $\rho_n(\gamma) \to \rho_\infty(\gamma)$ as $n \to \infty$ by algebraic convergence.

(C2) Let $(n_k)_{k\in\mathbb{N}}$ be a subsequence and $\gamma_{n_k} \in \pi_1(\Sigma)$ such that $\rho_{n_k}(\gamma_{n_k}) \in \mathrm{im}\,\rho_{n_k}$ converges to $g \in G$. Consider a standard generating set

$$S = \{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_p\} \subseteq \pi_1(\Sigma)$$

such that $\pi_1(\Sigma) = \langle S \mid \prod_{i=1}^g [a_i, b_i] \cdot c_1 \cdots c_p \rangle$. We set $\Gamma_n := \rho_n(\pi_1(\Sigma)) = \mathrm{im}\,\rho_n$, and denote by $\pi_n\colon \mathbb{H}^2 \longrightarrow \Gamma_n \backslash \mathbb{H}^2 =: X_n$ the respective covering maps for $n \in \mathbb{N} \cup \{\infty\}$. Now, choose $o \in \mathbb{H}^2$ and consider its images $x_n = \pi_n(o) \in X_n$. By construction the geodesics connecting o to $\rho_n(s)o$, $s \in S$, project to closed geodesics at x_n bounding a disc in X_n. This disc lifts to a geodesic polygon $P_n \subseteq \mathbb{H}^2$ with vertices

$$\left\{o, \rho_n(a_1)o, \rho_n(a_1 b_1)o, \rho_n(a_1 b_1 a_1^{-1})o, \dots, \rho_n(\textstyle\prod_{i=1}^g [a_i, b_i] \cdot c_1 \cdots c_{p-1})o\right\}.$$

Note that the interiors of the translates of the polygon P_n are pairwise disjoint, i.e. $\mathrm{int}(P_n) \cap \rho_n(\gamma)\mathrm{int}(P_n) \neq \emptyset$ for all $\gamma \in \pi_1(\Sigma) \setminus \{e\}$. Further, the orbit maps $\pi_1(\Sigma) \longrightarrow \mathbb{H}^2, \gamma \longmapsto \rho_n(\gamma)o$ induce homeomorphisms between the Cayley graph $\mathrm{Cay}(\pi_1(\Sigma), S)$ and $\rho_n(\pi_1(\Sigma)) \cdot \partial P_n$.

The vertices of P_n are determined by the elements $\rho_n(s)$, $s \in S$, and by algebraic convergence $\rho_n \to \rho_\infty$ it follows that $\mathrm{vol}_{\mathbb{H}^2}(P_n) \to \mathrm{vol}_{\mathbb{H}^2}(P_\infty)$ and $\mathrm{diam}(P_n) \to \mathrm{diam}(P_\infty)$ as $n \to \infty$.

Since $\rho_{n_k}(\gamma_{n_k})$ converges to g there is an upper bound

$$d(o, \rho_{n_k}(\gamma_{n_k})o) < r.$$

Hence, $\rho_{n_k}(\gamma_{n_k})o$ is adjacent to a $\rho_{n_k}(\pi_1(\Sigma))$-translate of P_{n_k} contained in the ball of radius $r + \mathrm{diam}(P_{n_k})$ about o. Since the interiors of the translates are pairwise disjoint, the number of such translates is bounded from above by

$$\frac{\mathrm{vol}_{\mathbb{H}^2}(B_o(r + \mathrm{diam}(P_{n_k})))}{\mathrm{vol}_{\mathbb{H}^2}(P_{n_k})}.$$

Further, every translate of P_{n_k} has $4g + p$ sides. This yields the following rough estimate

$$\ell_S(\gamma_{n_k}) \leq (4g + p) \cdot \frac{\mathrm{vol}_{\mathbb{H}^2}(B_o(r + \mathrm{diam}(P_{n_k}))}{\mathrm{vol}_{\mathbb{H}^2}(P_{n_k})},$$

where $\ell_S(\gamma)$ denotes the word length of an element $\gamma \in \pi_1(\Sigma)$ with respect to the generating set S. Because $\mathrm{diam}(P_{n_k}) \to \mathrm{diam}(P_\infty)$ and $\mathrm{vol}_{\mathbb{H}^2}(P_{n_k}) \to \mathrm{vol}_{\mathbb{H}^2}(P_\infty) > 0$ there is an upper bound $\ell_S(\gamma_{n_k}) \leq L$ for every $k \in \mathbb{N}$.

The set $\{\gamma \in \pi_1(\Sigma) | \ell_S(\pi_1(\Sigma)) \leq L\}$ is finite such that we may assume that $\gamma_{n_k} = \gamma' \in \pi_1(\Sigma)$ for every $k \in \mathbb{N}$, up to a subsequence. Then by algebraic convergence

$$g = \lim_{k \to \infty} \rho_{n_k}(\gamma_{n_k}) = \lim_{k \to \infty} \rho_{n_k}(\gamma') = \rho_\infty(\gamma') \in \mathrm{im}\, \rho_\infty.$$

Therefore, the map $\mathrm{im}\colon \mathcal{R}^*(\Sigma) \longrightarrow \mathrm{Sub}_{\mathrm{dtf}}(G)$ is continuous.

Next, let us see that im is locally injective. Let $\rho \in \mathcal{R}^*(\Sigma)$. By Lemma 2.7.6 there is an open neighborhood $U \subseteq G$ of the identity and an open neighborhood $\mathcal{V} \subseteq \mathrm{Sub}(G)$ of $\Gamma := \mathrm{im}\, \rho$ such that $\Gamma' \cap U = \{e\}$ for every $\Gamma' \in \mathcal{V}$. Consider the open preimage $\mathcal{U}' := \mathrm{im}^{-1}(\mathcal{V}) \subseteq \mathcal{R}^*(\Sigma)$ and let $V \subseteq U$ be an open neighborhood of the identity such that

$V^{-1}V \subseteq U$. Let $\mathcal{U} \subseteq \mathcal{U}'$ be a smaller open neighborhood consisting of all $\rho' \in \mathcal{U}$ such that $\rho'(s) \in \rho(s)V$ for every $s \in S$.

Let $\rho_1, \rho_2 \in \mathcal{U}$ with $\operatorname{im}\rho_1 = \operatorname{im}\rho_2$. Then for every $s \in S$ there is $v_i = v_i(s) \in V$ such that $\rho_i(s) = \rho(s)v_i$, $i = 1, 2$. Hence $\rho_2(s)^{-1}\rho_1(s) = v_2^{-1}v_1 \in V^{-1}V \subseteq U$. Since $\operatorname{im}\rho_1 = \operatorname{im}\rho_2 \in \mathcal{V}$ we have that $\rho_2(s)^{-1}\rho_1(s) \in \operatorname{im}\rho_1 \cap U = \{e\}$ for every $s \in S$. Therefore $\rho_1 = \rho_2$. This shows that $\operatorname{im}|_{\mathcal{U}}$ is injective.

Finally, we want to prove that $\operatorname{im}|_{\operatorname{im}(\mathcal{U})}^{-1} \colon \operatorname{im}(\mathcal{U}) \longrightarrow \mathcal{U}$ is continuous. Let $\Gamma_n = \operatorname{im}\rho_n \in \operatorname{im}(\mathcal{U})$ converge to $\Gamma = \operatorname{im}\rho \in \operatorname{im}(\mathcal{U})$ as $n \to \infty$. Note that $\Gamma\backslash\mathbb{H}^2$ is a hyperbolic surface of finite type such that there is a compact connected subset $C \subseteq \Gamma\backslash\mathbb{H}^2$ onto which $\Gamma\backslash\mathbb{H}^2$ deformation retracts. For example, we can take its convex core $C(\Gamma)$ and cut off the cusps. Let $o \in \widetilde{C} = \pi_\Gamma^{-1}(C)$. By Proposition 2.7.15 we find sequences $\varepsilon_n \to 0, r_n \to \infty$ as $n \to \infty$, open neighborhoods Ω_n, Ω_n' of $\overline{B}_o(r_n)$ and diffeomorphisms $f_n \colon \Omega_n \longrightarrow \Omega_n'$ such that

(i) $f_n(o) = o$,

(ii) $\pi_{\Gamma_n}(f_n(x)) = \pi_{\Gamma_n}(f_n(y)) \iff \pi_\Gamma(x) = \pi_\Gamma(y)$, for every $x, y \in \Omega_n$, and

(iii) $D_{\overline{B}_o(r_n)}(f_n, \mathrm{id}) < \varepsilon_n$.

In the following we will abbreviate $\pi_n = \pi_{\Gamma_n}, \pi = \pi_\Gamma$. Let us denote by $F_n \colon \pi(\Omega_n) \longrightarrow \pi_n(\Omega_n'), \pi(x) \longmapsto f_n(\pi(x))$ the induced diffeomorphisms. Since the diameter of C is finite $\pi(\overline{B}_o(r_n))$ contains C for large n. Recall that $\Gamma\backslash\mathbb{H}^2$ is homotopy equivalent to C such that we may identify the fundamental group of C with Γ. In this way $F_n \colon C \longrightarrow \Gamma_n\backslash\mathbb{H}^2$ induce homomorphisms $\sigma_n \colon \Gamma \longrightarrow \Gamma_n$ at the level of fundamental groups. Let us consider the lifts $\widetilde{F}_n \colon \widetilde{C} \longrightarrow \mathbb{H}^2$ of $F_n \colon C \longrightarrow \Gamma_n\backslash\mathbb{H}^2$ such that $\widetilde{F}_n(o) = o$. These are equivariant meaning that

$$\widetilde{F}_n(\gamma x) = \sigma_n(\gamma)\widetilde{F}_n(x)$$

for every $\gamma \in \Gamma$, $x \in \widetilde{C}$. Moreover, $\widetilde{F}_n$ coincides with f_n on $\overline{B}_o(r_n) \cap \widetilde{C}$ by uniqueness of lifts.

We want to show that $\sigma_n(\rho(s)) \to \rho(s)$ as $n \to \infty$ for every $s \in S$. This is equivalent to convergence of the differentials $D_o\sigma_n(\rho(s)) \to D_o\rho(s)$ as $n \to \infty$. There is $R > 0$ such that $\rho(s)o \in B_o(R)$ for every $s \in S$, and $\overline{B}_o(R) \subseteq \overline{B}_o(r_n)$ for large n. By equivariance

$$D_o\sigma_n(\rho(s)) = D_o(f_n \circ \rho(s) \circ f_n^{-1}) = D_{\rho(s)o}f_n \circ D_o\rho(s) \circ D_of_n^{-1}.$$

Since f_n converges to id on $\overline{B}_o(R)$ in the C^∞-distance it follows that

$$D_o\sigma_n(\rho(s)) \to D_o\rho(s) \qquad (n \to \infty).$$

This implies that $\sigma_n(\rho(s)) \to \rho(s)$ as $n \to \infty$ for every $s \in S$.

Next, let us see that $\sigma_n \colon \Gamma \longrightarrow \Gamma_n$ is injective for large n. To this end it suffices to prove that $F_n \colon C \longrightarrow \Gamma_n \backslash \mathbb{H}^2$ embeds C as a subsurface with homotopically non-trivial peripheral curves. Indeed, such subsurface embeddings are π_1-injective; see the definition of the restriction maps to subsurfaces in section 2.5. Let $\gamma_1 = \rho(c_1), \dots, \gamma_p = \rho(c_p) \in \Gamma \cong \pi_1(\Gamma\backslash\mathbb{H}^2)$ correspond to the peripheral elements $\{c_1, \dots, c_p\} \subseteq \pi_1(\Sigma)$. Then $\sigma_n(\gamma_i) \to \rho(c_i) \neq e$ as $n \to \infty$ such that $(F_n)_*(\gamma_i) = \sigma_n(\gamma_i)$ are homotopically non-trivial for large n, $i = 1, \dots, p$.

More is true by the above. The diffeomorphism F_n embeds $C \cong \Sigma$ as a subsurface with homotopically non-trivial peripheral curves in $\Gamma_n\backslash\mathbb{H}^2 \cong \Sigma$. This is only possible if F_n sends peripheral curves to peripheral curves. Therefore, $\Gamma_n\backslash\mathbb{H}^2$ deformation retracts to $F_n(\mathrm{int}(C))$. Because $\Gamma\backslash\mathbb{H}^2$ deformation retracts to $\mathrm{int}(C)$ we may extend F_n to a diffeomorphism $\Gamma\backslash\mathbb{H}^2 \longrightarrow \Gamma_n\backslash\mathbb{H}^2$ inducing $\sigma_n \colon \Gamma \longrightarrow \Gamma_n$. In particular, σ_n is an isomorphism induced by an orientation preserving diffeomorphism.

It follows that $\sigma_n \circ \rho \in \mathcal{R}^*(\Sigma)$. Moreover, $\sigma_n \circ \rho \to \rho$ as $n \to \infty$ such that $\sigma_n \circ \rho \in \mathcal{U}$ for large n. But also $\rho_n \in \mathcal{U}$ and $\mathrm{im}\,\rho_n = \Gamma_n = \mathrm{im}(\sigma_n \circ \rho)$. By injectivity of $\mathrm{im} \colon \mathcal{U} \longrightarrow \mathrm{im}(\mathcal{U})$ we have that $\rho_n = \sigma_n \circ \rho$ for large n, and $\rho_n = \sigma_n \circ \rho \to \rho$ as $n \to \infty$. Thus $\mathrm{im}|_{\mathcal{U}}^{-1}$ is indeed continuous.

That $\mathrm{im}(\mathcal{R}^*(\Sigma)) = \mathcal{D}(\Sigma)$ and $\mathrm{im}(\mathcal{R}(\Sigma)) = \mathcal{L}(\Sigma)$ follows at once from the definition of $\mathcal{R}^*(\Sigma)$ and $\mathcal{R}(\Sigma)$, respectively. □

Proposition 2.8.2 allows us to identify the moduli space $\mathcal{M}(\Sigma)$ with the space $G\backslash\mathcal{L}(\Sigma)$ of conjugacy classes of lattices.

Proposition 2.8.4. *The space $\mathcal{L}(\Sigma)$ is invariant under the conjugation action of G and we may identify its quotient $G\backslash\mathcal{L}(\Sigma)$ with the moduli space $\mathcal{M}(\Sigma)$ via the following homeomorphism*

$$\psi \colon \mathcal{M}(\Sigma) \longrightarrow G\backslash\mathcal{L}(\Sigma), \quad [[\rho]] \longmapsto [\mathrm{im}\,\rho].$$

Proof. Let $\Gamma \in \mathcal{L}(\Sigma)$ and $g \in G$. Then $\Gamma' := g\Gamma g^{-1}$ is a torsion-free lattice, too. Moreover, the element $g \in G \cong \mathrm{Isom}_+(\mathbb{H}^2)$ induces an orientation preserving isometry

$$\begin{aligned} g\colon \Gamma\backslash\mathbb{H}^2 &\longrightarrow \Gamma'\backslash\mathbb{H}^2, \\ \Gamma x &\longmapsto \Gamma' g x, \end{aligned}$$

whence $\Gamma' = g\Gamma g^{-1} \in \mathcal{L}(\Sigma)$.

Let us consider the right-action $\mathrm{Aut}^*(\pi_1(\Sigma)) \curvearrowright \mathcal{R}(\Sigma)$. We claim that the map

$$\mathrm{im}\colon \mathcal{R}(\Sigma) \longrightarrow \mathcal{L}(\Sigma)$$

induces a homeomorphism:

$$\begin{tikzcd} \mathcal{R}(\Sigma) \arrow[d] \arrow[dr, two heads, "\mathrm{im}"] & \\ \mathcal{R}(\Sigma)/\mathrm{Aut}^*(\pi_1(\Sigma)) \arrow[r, dashed, "\varphi"'] & \mathcal{L}(\Sigma) \end{tikzcd}$$

Clearly, if $\rho_1 = \rho_2 \circ \alpha$ for $\rho_1, \rho_2 \in \mathcal{R}(\Sigma)$ and $\alpha \in \mathrm{Aut}^*(\pi_1(\Sigma))$ then $\mathrm{im}\,\rho_1 = \mathrm{im}\,\rho_2$.

On the other hand, suppose $\Gamma = \mathrm{im}\,\rho_1 = \mathrm{im}\,\rho_2$ for some $\rho_1, \rho_2 \in \mathcal{R}(\Sigma)$. There are orientation preserving homeomorphisms $f_1, f_2\colon \Sigma \longrightarrow \Gamma\backslash\mathbb{H}^2$ such that ρ_1, ρ_2 are holonomy representations of f_1, f_2, respectively. Then $\rho_2^{-1} \circ \rho_1$ is induced by $f_2^{-1} \circ f_1$ such that $\alpha := \rho_2^{-1} \circ \rho_1 \in \mathrm{Aut}^*(\pi_1(\Sigma))$, and $\rho_1 = \rho_2 \circ \alpha$. Hence, φ is a bijection.

By definition of the quotient topology φ is continuous. Finally, im is a local homeomorphism such that φ^{-1} is continuous, too. This shows that φ is a homeomorphism.

Observe that φ is equivariant with respect to the conjugation action of G both on $\mathcal{R}(\Sigma)/\mathrm{Aut}^*(\pi_1(\Sigma))$ and on $\mathcal{L}(\Sigma)$. Therefore, taking the quotient by the conjugation actions yields a homeomorphism

$$\begin{tikzcd} \mathcal{R}(\Sigma)/\mathrm{Aut}^*(\pi_1(\Sigma)) \arrow[r, "\varphi", "\cong"'] \arrow[d] & \mathcal{L}(\Sigma) \arrow[d] \\ \mathcal{T}(\Sigma)/\mathrm{Out}^*(\pi_1(\Sigma)) \arrow[r, dashed, "\psi", "\cong"'] & G\backslash\mathcal{L}(\Sigma) \end{tikzcd}$$

given by $\psi([[\rho]]) = [\mathrm{im}\,\rho]$ for every $[[\rho]] \in \mathcal{M}(\Sigma) = \mathcal{T}(\Sigma)/\mathrm{Out}^*(\Sigma)$. □

We conclude this section with the following lemma, that restricts the kind of closed subgroups that arise in the closure of the G-orbit of a lattice $\Gamma \in \mathcal{L}(\Sigma)$ in $\mathrm{Sub}(G)$.

Lemma 2.8.5. *Let $\Gamma \in \mathcal{L}(\Sigma)$ and let $(g_n)_{n\in\mathbb{N}} \subseteq G$ be a sequence of elements such that*

$$g_n^{-1}\Gamma g_n \to H \qquad (n \to \infty)$$

converges to $H \in \mathrm{Sub}(G)$. Then, either

(i) H is abelian, or

(ii) H is a conjugate of Γ.

Proof. Let $\pi\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ denote the quotient map and $o \in \mathbb{H}^2$. There are two cases to consider:

a) The sequence $\pi(g_n o)$ is contained in a compact set $K \subseteq \Gamma\backslash\mathbb{H}^2$, or

b) The sequence $\pi(g_n o)$ goes to infinity in $\Gamma\backslash\mathbb{H}^2$.

In case a) we may find a compact set $K' \subseteq \mathbb{H}^2$ such that $\pi(K') = K$, and therefore elements $\gamma_n \in \Gamma$ such that $\gamma_n g_n o \in K'$. Then there is a convergent subsequence $\gamma_{n_k} g_{n_k} \to g$ as $k \to \infty$, such that

$$H = \lim_{n\to\infty} g_n^{-1}\Gamma g_n = \lim_{k\to\infty} g_{n_k}^{-1}\gamma_{n_k}^{-1}\Gamma\gamma_{n_k} g_{n_k} = g^{-1}\Gamma g.$$

In case b) there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\pi(g_{n_k} o) \in C$ is contained in a cusp region C of the thin part of $\Gamma\backslash\mathbb{H}^2$. Let $B \subseteq \pi^{-1}(C)$ be a horoball in the preimage of C centered at $\xi \in \partial\mathbb{H}^2$, and let $\widehat{\gamma}_{n_k} \in \Gamma$ be such that $\widehat{\gamma}_{n_k} g_{n_k} o \in B$ for every $k \in \mathbb{N}$, and

$$\xi = \lim_{k\to\infty} \widehat{\gamma}_{n_k} g_{n_k} o.$$

Thus, if we set $\widehat{g_k} := \widehat{\gamma}_{n_k} g_{n_k}$, then $\xi = \lim_{k\to\infty} \widehat{g_k} o$, and

$$H = \lim_{k\to\infty} g_{n_k}^{-1}\Gamma g_{n_k} = \lim_{k\to\infty} (\widehat{g_k})^{-1}(\widehat{\gamma}_{n_k})^{-1}\Gamma\widehat{\gamma}_{n_k}\widehat{g_k} = \lim_{k\to\infty} (\widehat{g_k})^{-1}\Gamma\widehat{g_k}.$$

Let $h \in H$ and let $\gamma_k \in \Gamma$ such that $(\widehat{g_k})^{-1}\gamma_k\widehat{g_k} \to h$ as $k \to \infty$. Note that

$$d(ho, o) = \lim_{k\to\infty} d((\widehat{g_k})^{-1}\gamma_k\widehat{g_k} o, o) = \lim_{k\to\infty} d(\gamma_k\widehat{g_k} o, \widehat{g_k} o).$$

Hence there is $D > 0$, such that

$$d(\gamma_k \widehat{g_k} o, \widehat{g_k} o) < D$$

for all $k \in \mathbb{N}$. Thus $\gamma_k B \cap B \neq \emptyset$ and $\gamma_k \in P = \mathrm{stab}_\xi(\Gamma)$ for large k. Because P is abelian, this implies that H is abelian. $\square$

3 L^1-Convergence of Truncated Dirichlet Domains

In chapter 2 we have introduced the space of discrete subgroups

$$\mathcal{D}(\Sigma) = \{\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G) \,|\, \Gamma\backslash\mathbb{H}^2 \cong \Sigma\},$$

whose quotient $\Gamma\backslash\mathbb{H}^2$ is homeomorphic to Σ. Moreover, we have defined the truncated Dirichlet domain

$$\widehat{D}_o(\Gamma) = D_o(\Gamma) \cap \widetilde{C}(\Gamma)$$

given a point $o \in \mathbb{H}^2$. It is natural to ask, whether the truncated Dirichlet domain $\widehat{D}_o(\Gamma)$ depends continuously on $\Gamma \in \mathcal{D}(\Sigma)$, and if so in what sense.

The goal of this chapter is to answer this question in the form of the following lemma.

Lemma 3.1.1. *Let $o \in \mathbb{H}^2$. The map*

$$\mathcal{D}(\Sigma) \longrightarrow L^1(\mathbb{H}^2), \quad \Gamma \longmapsto \mathbb{1}_{\widehat{D}_o(\Gamma)},$$

is continuous, where $\mathbb{1}_{\widehat{D}_o(\Gamma)}$ denotes the indicator function of $\widehat{D}_o(\Gamma) \subseteq \mathbb{H}^2$.

Although this lemma seems classical, we could not find a proof in the literature. Since it is an essential ingredient for Proposition 4.1.5 and Theorem 4.2.2, we will give a complete proof using elementary hyperbolic geometry here. The proof is split into several sublemmas, some of which might be of individual interest. In particular, we will use Lemma 3.1.8 again in our discussion of the Weil–Petersson IRS in section 5.2.

The first two lemmas are concerned with the pointwise convergence of the characteristic functions $\mathbb{1}_{\widetilde{C}(\Gamma_n)}$ and $\mathbb{1}_{D_o(\Gamma_n)}$ outside of a set of measure zero.

Lemma 3.1.2. *Let $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\Sigma)$ be a sequence converging to $\Gamma \in \mathcal{D}(\Sigma)$. Then*

$$\mathbb{1}_{\widetilde{C}(\Gamma_n)}(x) \to \mathbb{1}_{\widetilde{C}(\Gamma)}(x) \qquad (n \to \infty)$$

for every $x \in \mathbb{H}^2 \setminus \partial\widetilde{C}(\Gamma)$.

Proof. By Proposition 2.8.2 we may choose $\rho, \rho_n \in \mathcal{R}^*(\Sigma)$ such that $\operatorname{im}\rho = \Gamma$, $\operatorname{im}\rho_n = \Gamma_n$ for large n, and $\rho_n \to \rho$ as $n \to \infty$.

First, let $x \in \mathbb{H}^2 \setminus \widetilde{C}(\Gamma)$. Let $\gamma \in \Gamma$ be a hyperbolic element, whose axis bounds an open half-space $H(\gamma)$ such that $H(\gamma) \cap \widetilde{C}(\Gamma) = \varnothing$ and $x \in H(\gamma)$. Let $c \in \pi_1(\Sigma)$ such that $\rho(c) = \gamma$. Then $\rho_n(c) = \gamma_n$ are boundary elements converging to γ. But then $x \in H(\gamma_n)$ such that $x \in \mathbb{H}^2 \setminus \widetilde{C}(\Gamma_n)$ for large $n \in \mathbb{N}$.

Let $x \in \operatorname{int}(\widetilde{C}(\Gamma))$ and suppose that there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $x \notin \widetilde{C}(\Gamma_{n_k})$ for all $k \in \mathbb{N}$. Then there are primitive hyperbolic boundary elements $\gamma_{n_k} \in \Gamma_{n_k}$, whose axes $\operatorname{ax}(\gamma_{n_k})$ bound half-spaces $H(\gamma_{n_k}) \subseteq \mathbb{H}^2$ such that $x \in H(\gamma_{n_k})$.

We claim that there is $D \geq 0$ such that $d(x, \operatorname{ax}(\gamma_{n_k})) \leq D$. Suppose to the contrary that there is a further subsequence, also denoted by $(n_k)_{k\in\mathbb{N}}$, such that $d(x, \operatorname{ax}(\gamma_{n_k})) \to \infty$ as $k \to \infty$. Let $\eta, \eta' \in \Gamma$ be two non-commuting elements and let $\eta_{n_k}, \eta'_{n_k} \in \Gamma_{n_k}$, that converge to $\eta, \eta' \in \Gamma$, respectively. Then $\eta_{n_k}x, \eta'_{n_k}x \in H(\gamma_{n_k})$ for large k. However, this means that η_{n_k} and η'_{n_k} must leave the entire half-space $H(\gamma_{n_k})$ invariant. Thus they commute having the same axis $\operatorname{ax}(\gamma_{n_k})$. But then also η and η' commute; in contradiction to our assumption.

Because γ_{n_k} are primitive boundary elements their translation length $\ell(\gamma_{n_k})$ is uniformly bounded from above by the maximal length L of a boundary curve in $C(\Gamma_{n_k})$. Moreover, there is a lower bound $\varepsilon_0 > 0$ such that $d(x, \gamma_{n_k}(x)) \geq \varepsilon_0$, because the Γ_{n_k} converge to a discrete group Γ and $\gamma_{n_k} \neq e$ for all k. It follows from elmentary hyperbolic geometry that $d(x, \operatorname{ax}(\gamma_{n_k})) \leq D$ and $d(x, \gamma_{n_k}(x)) \geq \varepsilon_0$ implies $\ell(\gamma_{n_k}) \geq \varepsilon_0'$ for some $\varepsilon_0' > 0$. The subset

$$C = \{g \in \mathrm{PSL}_2(\mathbb{R}) \,|\, g \text{ is hyperbolic}, \varepsilon_0' \leq \ell(g) \leq L, \operatorname{ax}(g) \cap \overline{B}_x(D) \neq \varnothing\} \subseteq G$$

is compact, such that $\gamma_{n_k} \to \gamma \in \Gamma$ up to a subsequence.

Let $c \in \pi_1(\Sigma)$ such that $\rho(c) = \gamma$. Because $\gamma_{n_k} \to \gamma$ and $\rho_{n_k}(c) \to \gamma$ as $k \to \infty$, we have that $\gamma_{n_k} = \rho_{n_k}(c)$ by Lemma 2.7.6. Hence, $\gamma = \rho(c)$ is a boundary element. Since $x \in H(\gamma_{n_k})$ and $\gamma_{n_k} \to \gamma$, it follows that $x \in \overline{H(\gamma)}$ contradicting $x \in \operatorname{int}(\widetilde{C}(\Gamma))$. $\square$

Lemma 3.1.3. *Let $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\Sigma)$ be a sequence converging to $\Gamma \in \mathcal{D}(\Sigma)$. Then*

$$\mathbb{1}_{D_o(\Gamma_n)}(x) \to \mathbb{1}_{D_o(\Gamma)}(x) \qquad (n \to \infty)$$

for every $x \in \mathbb{H}^2 \setminus \partial D_o(\Gamma)$.

Proof. Let $x \in \operatorname{int}(D_o(\Gamma))$. Then $d(x,o) < d(x,\gamma o)$ for every $\gamma \in \Gamma \setminus \{e\}$. We want to show that $x \in D_o(\Gamma_n)$ for large n. Assume to the contrary that there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $x \notin D_o(\Gamma_{n_k})$, i.e. there are $\gamma_{n_k} \in \Gamma_{n_k} \setminus \{e\}$ such that $d(x,o) > d(x,\gamma_{n_k} o)$ for every $k \in \mathbb{N}$. Up to passing to a subsequence we may assume that $\gamma_{n_k} \to \gamma \in \Gamma$, and $\gamma \neq e$ by Lemma 2.7.6. But

$$d(x,o) \geq \lim_{k\to\infty} d(x,\gamma_{n_k} o) = d(x,\gamma o)$$

contradicting $x \in \operatorname{int}(D_o(\Gamma))$.

Let $x \in \mathbb{H}^2 \setminus D_o(\Gamma)$. Then there is $\gamma \in \Gamma \setminus \{e\}$ such that $d(x,o) > d(x,\gamma o)$. We want to show that $x \in \mathbb{H}^2 \setminus D_o(\Gamma_n)$ for large n. Let $\gamma_n \in \Gamma_n$ such that $\gamma_n \to \gamma \neq e$ as $n \to \infty$. Then $\gamma_n \neq e$ and $d(x,\gamma_n o) < d(x,o)$, such that $x \notin D_o(\Gamma_n)$ for large n. □

We give a characterization of peripheral curves, now.

Lemma 3.1.4. *Let $\mu = \{\gamma_1,\ldots,\gamma_r\} \subseteq \Sigma$ be a filling collection of essential simple closed curves such that*

(i) γ_i, γ_j are in minimal position for all $i,j \in \{1,\ldots,r\}$,

(ii) the curves in μ are pairwise non-isotopic, and

(iii) for distinct triples $i,j,k \in \{1,\ldots,r\}$ at least one of the intersections $\gamma_i\cap\gamma_j, \gamma_j\cap\gamma_k, \gamma_i\cap\gamma_k$ is empty.

Let $\alpha \subseteq \Sigma$ be a homotopically non-trivial closed curve. Then

$$\alpha \text{ is peripheral} \iff i(\alpha,\gamma_i) = 0 \text{ for every } i = 1,\ldots,r.$$

Proof. Suppose $\alpha \subseteq \Sigma$ is peripheral. Then α is homotopic to one of the punctures of Σ. Since μ fills Σ there is a punctured disk $\mathbb{D}^\times \subseteq \Sigma \setminus \mu$ surrounding this puncture. Thus we may homotope α into $\mathbb{D}^\times$ such that $i(\alpha,\gamma_i) = 0$ for every $i = 1,\ldots,r$.

If $i(\alpha,\gamma_i) = 0$ for every $i = 1,\ldots,r$, then there are isotopies moving γ_i to $\widetilde{\gamma_i}$ such that $\widetilde{\gamma_i} \cap \alpha = \varnothing$. Because our system satisfies the hypotheses (i)-(iii), there is an isotopy of Σ

moving $\bigcup_{i=1}^{r} \gamma_i$ to $\bigcup_{i=1}^{r} \widetilde{\gamma}_i$; see [FM12, Lemma 2.9]. The collection $\widetilde{\mu} = \{\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_r\}$ is still filling and α is a homotopically non-trivial closed curve in $\Sigma \setminus \widetilde{\mu}$. Thus α is contained in a punctured disk $\mathbb{D}^\times \subseteq \Sigma \setminus \mu$. Therefore, α is homotopic to a puncture, i.e. α is peripheral. $\square$

Using the previous lemma we will prove next that there is a lower bound for the lengths of essential curves with respect to a convergent sequence of representations:

Lemma 3.1.5. *Let $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$ be a sequence converging to $\rho \in \mathcal{R}^*(\Sigma)$. Then there is $\varepsilon > 0$ such that*

$$\ell(\rho_n(\alpha)) < \varepsilon \implies \alpha \text{ is peripheral}$$

for every $\alpha \in \pi_1(\Sigma)$ and all $n \in \mathbb{N}$.

Proof. Suppose to the contrary that there is a subsequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and non-peripheral elements $\alpha_{n_k} \in \pi_1(\Sigma)$ such that $\ell(\rho_{n_k}(\alpha_{n_k})) \to 0$ as $k \to \infty$. Choose a collection of curves $\mu = \{\gamma_1, \ldots, \gamma_r\} \subseteq \pi_1(\Sigma)$ as in Lemma 3.1.4. Then there is $j_k \in \{1, \ldots, r\}$ for every $k \in \mathbb{N}$ such that $i(\gamma_{j_k}, \alpha_{n_k}) \neq 0$. Up to a passing to another subsequence we may assume that $\gamma_{j_k} = \widehat{\gamma} \in \mu$ is constant. But then by the Collar Lemma 2.3.8

$$\ell(\rho_{n_k}(\widehat{\gamma})) \to \infty \qquad (k \to \infty).$$

This contradicts the fact that $\ell(\rho_{n_k}(\widehat{\gamma})) \to \ell(\rho(\widehat{\gamma}))$ as $k \to \infty$. $\square$

The following lemma shows how to obtain an upper bound on the diameter of a connected subset C of a hyperbolic surface given a lower bound ε for the injectivity radius and an upper bound for the volume of the ε-neighborhood of C.

Lemma 3.1.6. *Let X be a hyperbolic surface and let $\varepsilon > 0$. Further, let $C \subseteq X$ be a path-connected Borel set, and suppose that $\operatorname{inj}_X(x) \geq \varepsilon$ for every $x \in C$. Then*

$$\operatorname{diam}_X(C) \leq 4\varepsilon \cdot \left(\frac{\operatorname{vol}_X(N_\varepsilon(C))}{\operatorname{vol}_{\mathbb{H}^2}(B_o(\varepsilon))} + 1 \right),$$

where $N_\varepsilon(C) = \{x \in X \mid d_X(x, C) < \varepsilon\}$ is the open ε-neighborhood of C in X and $o \in \mathbb{H}^2$.

Proof. Note that for every $x \in C$ the ball $B_x(\varepsilon) \subseteq N_\varepsilon(C)$ is embedded and has the same measure as a ball of radius ε in the hyperbolic plane.

Let us now consider

$$S = \{Y \subseteq C : d(y_1, y_2) \geq 2\varepsilon \quad \forall y_1, y_2 \in Y, y_1 \neq y_2\}.$$

By Zorn's Lemma we may choose a maximal element $Y_0 \in S$ with respect to inclusion $\subseteq$. We claim that the collection of balls $\{B_y(2\varepsilon) : y \in Y_0\}$ covers C. Indeed, if there is $y' \in C$, which is not in any $\{B_y(2\varepsilon)\}_{y \in Y_o}$, it has distance greater or equal than 2ε from any point $y \in Y_0$. But then $\{y'\} \cup Y_0 \in S$, which contradicts the maximality of Y_0.

On the other hand the balls of radius ε centered at $y \in Y_0$ are disjoint by definition of S whence

$$\bigsqcup_{y \in Y_0} B_y(\varepsilon) \subseteq N_\varepsilon(C),$$

such that

$$\mathrm{vol}_X(N_\varepsilon(C)) \geq \sum_{y \in Y_0} \mathrm{vol}_X(B_y(\varepsilon)) = \# Y_0 \cdot \mathrm{vol}_{\mathbb{H}^2}(B_o(\varepsilon)).$$

It follows that

$$\# Y_0 \leq \frac{\mathrm{vol}_X(N_\varepsilon(C))}{\mathrm{vol}_{\mathbb{H}^2}(B_o(\varepsilon))}.$$

Let $x, y \in C$, and let $c \colon [0,1] \longrightarrow C$ be a path from x to y. Then c is covered by $\{B_y(2\varepsilon)\}_{y \in Y_0}$ and we may shorten c to a path c', that intersects any ball $B_y(2\varepsilon)$, $y \in Y_0$, at most once. Covering c' by $\{B_y(2\varepsilon)\}_{y \in Y_0}$ we obtain a sequence of pairwise distinct $y_1, \ldots, y_m \in Y_0$, such that $x \in B_{y_1}(2\varepsilon)$, $B_{y_i}(2\varepsilon) \cap B_{y_{i+1}}(2\varepsilon) \neq \emptyset$ and $y \in B_{y_m}(2\varepsilon)$. Thus,

$$d(x,y) \leq d(x, y_1) + \sum_{i=1}^{m} d(y_i, y_{i+1}) + d(y_m, y) \leq 2\varepsilon + 4\varepsilon \cdot \# Y_0 + 2\varepsilon \leq 4\varepsilon \cdot \left(\frac{\mathrm{vol}_X(N_\varepsilon(C))}{\mathrm{vol}_{\mathbb{H}^2}(B_o(\varepsilon))} + 1 \right).$$

Because $x, y \in C$ were arbitrary, this proves the assertion. □

Using the previous lemmas we obtain an upper bound on the diameter of the thick part of the convex core, as follows.

Lemma 3.1.7. *Let $(\Gamma_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(\Sigma)$ be a sequence converging to $\Gamma \in \mathcal{D}(\Sigma)$. Then there is $\varepsilon > 0$ such that for every $0 < \varepsilon' < \varepsilon$ the ε'-thick part of the convex core*

$$C(\Gamma_n)_{\geq \varepsilon'} := C(\Gamma_n) \cap (\Gamma_n \backslash \mathbb{H}^2)_{\geq \varepsilon'} = \{x \in C(\Gamma_n) \mid \mathrm{inj}_{\Gamma_n \backslash \mathbb{H}^2}(x) \geq \varepsilon'\}$$

is path-connected for every $n \in \mathbb{N}$.

In particular, for every $0 < \varepsilon' < \varepsilon$ *there is* $R = R(\varepsilon') > 0$ *such that*

$$\operatorname{diam}_{\Gamma_n \backslash \mathbb{H}^2}(C(\Gamma_n)_{\geq \varepsilon'}) \leq R$$

for every $n \in \mathbb{N}$.

Proof. We may choose $\rho_n \to \rho \in \mathcal{R}^*(\Sigma)$ such that $\operatorname{im} \rho_n = \Gamma_n$ and $\operatorname{im} \rho = \Gamma$. Let $\varepsilon > 0$ be as in Lemma 3.1.5. Without loss of generality we may assume that ε is smaller than the Margulis constant. Let $0 < \varepsilon' < \varepsilon$, let $n \in \mathbb{N}$, and let $T \subseteq (\Gamma_n \backslash \mathbb{H}^2)_{<\varepsilon'}$ be a tube component of the ε'-thin part. Let $\alpha_n \in \pi_1(\Sigma)$ such that $\rho_n(\alpha_n) \in \Gamma_n$ corresponds to the waist geodesic of T. Then $\ell(\rho_n(\alpha_n)) < \varepsilon' < \varepsilon$ such that α_n is peripheral. Therefore, all tube components of the ε'-thin part are peripheral such that $C(\Gamma_n)_{\geq \varepsilon'}$ is path-connected.

We want to apply Lemma 3.1.6 to $C(\Gamma_n)_{\geq \varepsilon'}$. Note that $N_{\varepsilon'}(C(\Gamma_n)_{\geq \varepsilon'}) \subseteq N_\varepsilon(C(\Gamma_n))$. Further, $N_\varepsilon(C(\Gamma_n)) \setminus C(\Gamma_n)$ consists of half-collars of width ε about the boundary curves of $C(\Gamma_n)$. Since the lengths of the boundary curves converge there is a uniform bound $V > 0$ such that $\operatorname{vol}_{\Gamma_n \backslash \mathbb{H}^2}(N_\varepsilon(C(\Gamma_n)) \backslash C(\Gamma_n)) \leq V$ for all $n \in \mathbb{N}$. Recall that $\operatorname{vol}_{\Gamma_n \backslash \mathbb{H}^2}(C(\Gamma_n)) = 2\pi|\chi(\Sigma)|$, such that

$$\begin{aligned} \operatorname{vol}_{\Gamma_n \backslash \mathbb{H}^2}(N_{\varepsilon'}(C(\Gamma_n)_{\geq \varepsilon'})) &\leq \operatorname{vol}_{\Gamma_n \backslash \mathbb{H}^2}(C(\Gamma_n)) + \operatorname{vol}_{\Gamma_n \backslash \mathbb{H}^2}(N_\varepsilon(C(\Gamma_n)) \setminus C(\Gamma_n)) \\ &\leq 2\pi|\chi(\Sigma)| + V. \end{aligned}$$

Setting

$$R(\varepsilon') := 4\varepsilon' \cdot \left(\frac{2\pi|\chi(\Sigma)| + V}{\operatorname{vol}_{\mathbb{H}^2}(B_o(\varepsilon'))} + 1 \right)$$

the assertion follows from Lemma 3.1.6. □

Finally, we will need to know the area of the residual thin parts.

Lemma 3.1.8.

(i) *Let* $\delta > 0$ *and let* $\gamma \in G = \operatorname{Isom}^+(\mathbb{H}^2)$ *be defined by* $\gamma(z) = z + 1$ *for every* $z \in \mathbb{H}^2$. *Consider the fundamental domain for the corresponding cusp region*

$$C_\delta := \{z = x + iy \in \mathbb{H}^2 \,|\, 0 \leq x \leq 1, d(z, \gamma(z)) \leq \delta\},$$

that consists of all the points that are moved less than δ *by* γ.

Then

$$\mathrm{vol}_{\mathbb{H}^2}(C_\delta) = 2\sinh(\delta/2).$$

(ii) Let $\delta > \delta_0 > 0$ and $\gamma \in G = \mathrm{Isom}^+(\mathbb{H}^2)$ be defined by $\gamma(z) = e^{\delta_0} z$ for every $z \in \mathbb{H}^2$. Consider the fundamental domain for the corresponding funnel

$$F_\delta := \{z = x + iy \in \mathbb{H}^2 \mid x \geq 0, 1 \leq |z| \leq e^{\delta_0}, d(z, \gamma(z)) \leq \delta\}$$

to the right of the axis $\mathrm{ax}\,\gamma = i\mathbb{R}$*, that consists of all the points that are moved less than δ by γ.*

Then

$$\mathrm{vol}_{\mathbb{H}^2}(F_\delta) \leq 2\sinh(\delta/2).$$

Proof. Recall the following formulas from hyperbolic geometry:

$$\sinh(d(z,w)/2) = \frac{|z-w|}{2\sqrt{\mathrm{Im}(z)\,\mathrm{Im}(w)}}$$

for every $z, w \in \mathbb{H}^2$, and

$$\mathrm{vol}_{\mathbb{H}^2}(A) = \int_A \frac{1}{y^2}\, dx\, dy$$

for every Borel set $A \subseteq \mathbb{H}^2$; see [Bea83, Theorem 7.2.1 (iii)] and [Bus10, (1.1.1), p. 2], respectively.

(i) We compute that $z = x + iy \in C_\delta$ if and only if

$$\sinh(\delta/2) \geq \sinh(d(z, z+1)/2) = \frac{1}{2y} \iff y \geq \frac{1}{2\sinh(\delta/2)} =: y_\delta.$$

Hence,

$$\mathrm{vol}_{\mathbb{H}^2}(C_\delta) = \int_0^1 \int_{y_\delta}^\infty \frac{1}{y^2}\, dy\, dx = \frac{1}{y_\delta} = 2\sinh(\delta/2).$$

(ii) For $z = re^{i\alpha} \in \mathbb{H}^2$, $r > 0, \alpha \in (0, \pi)$, we have that $d(z, \gamma(z)) \leq \delta$ if and only if

$$\sinh(\delta/2) \geq \sinh(d(z, e^{\delta_0} z)/2) = \frac{\left|re^{i\alpha} - re^{\delta_0} e^{i\alpha}\right|}{2\sqrt{r\sin(\alpha) \cdot re^{\delta_0}\sin(\alpha)}}$$

$$
\begin{aligned}
&= \frac{1}{\sin(\alpha)} \frac{e^{\delta_0} - 1}{2e^{\delta_0/2}} = \frac{\sinh(\delta_0/2)}{\sin(\alpha)} \\
\iff & \sin(\alpha) \geq \frac{\sinh(\delta_0/2)}{\sinh(\delta/2)}.
\end{aligned}
$$

There is a unique $\alpha_\delta \in (0, \pi/2)$ such that $\sin(\alpha_\delta) = \frac{\sinh(\delta_0/2)}{\sinh(\delta/2)}$.

Using polar coordinates we obtain

$$
\begin{aligned}
\operatorname{vol}_{\mathbb{H}^2}(F_\delta) &= \int_{F_\delta} \frac{1}{y^2}\, dx\, dy = \int_{\alpha_\delta}^{\pi/2} \int_1^{e^{\delta_0}} \frac{r}{r^2 \sin^2 \varphi}\, dr\, d\varphi \\
&= \delta_0 \cdot \int_{\alpha_\delta}^{\pi/2} \frac{1}{\sin^2 \varphi}\, d\varphi = \delta_0 \cdot [\cot \varphi]_{\varphi = \alpha_\delta}^{\pi/2} \\
&= \delta_0 \cdot \cot(\alpha_\delta) = \delta_0 \cdot \frac{\cos(\alpha_\delta)}{\sin(\alpha_\delta)} \leq \frac{\delta_0}{\sin(\alpha_\delta)} \\
&= \frac{\delta_0}{\sinh(\delta_0/2)} \sinh(\delta/2) \leq 2 \sinh(\delta/2),
\end{aligned}
$$

where we used in the last inequality that $x \leq \sinh(x)$ for all $x \geq 0$.

$\square$

We are ready to prove Lemma 3.1.1 now.

Proof of Lemma 3.1.1. By definition $\widehat{D}_o(\Gamma) = \widetilde{C}(\Gamma) \cap D_o(\Gamma)$ such that

$$
\mathbb{1}_{\widehat{D}_o(\Gamma)} = \mathbb{1}_{\widetilde{C}(\Gamma)} \cdot \mathbb{1}_{D_o(\Gamma)}.
$$

By Lemma 3.1.2 and Lemma 3.1.3 we have that

$$
\mathbb{1}_{\widetilde{C}(\Gamma_n)}(x) \cdot \mathbb{1}_{D_o(\Gamma_n)}(x) \to \mathbb{1}_{\widetilde{C}(\Gamma)}(x) \cdot \mathbb{1}_{D_o(\Gamma)}(x) \qquad (n \to \infty)
$$

for every $x \in \mathbb{H}^2 \setminus (\partial \widetilde{C}(\Gamma) \cup \partial D_o(\Gamma))$. Note that $\partial \widetilde{C}(\Gamma) \cup \partial D_o(\Gamma)$ has measure zero.

Let $\varepsilon > 0$ be as in Lemma 3.1.7 and let $0 < \varepsilon' < \varepsilon$. Then there is $R = R(\varepsilon') > 0$ such that $\pi_n(\overline{B}_o(R))$ contains $C(\Gamma_n)_{\geq \varepsilon'}$ for every $n \in \mathbb{N}$, where $\pi_n \colon \mathbb{H}^2 \longrightarrow \Gamma_n \backslash \mathbb{H}^2$ is the quotient map. The complement $C(\Gamma_n) \setminus \pi_n(\overline{B}_o(R)) = \bigsqcup_{k=1}^{l} W_k$ is a disjoint union of subsets $W_1, \ldots, W_l$ of peripheral cusp or tube components of the ε'-thin part. If Σ has genus g

and p punctures there are at most $l \leq p$ such components. By Lemma 3.1.8 we have that

$$\mathrm{vol}_{\Gamma_n \backslash \mathbb{H}^2}(W_k) \leq 2\sinh(\varepsilon'/2)$$

for every $k = 1, \ldots, l$, such that

$$\mathrm{vol}_{\Gamma_n \backslash \mathbb{H}^2}(C(\Gamma_n) \setminus \pi_n(\overline{B}_o(R))) \leq 2p\sinh(\varepsilon'/2) \to 0 \qquad (\varepsilon' \to 0).$$

Hence,

$$\mathrm{vol}_{\mathbb{H}^2}(\widehat{D}_o(\Gamma_n) \setminus \overline{B}_o(R)) \leq 2p\sinh(\varepsilon'/2) \to 0 \qquad (\varepsilon' \to 0).$$

Let $\varepsilon'' > 0$ be arbitrary, and choose $\varepsilon' > 0$ and $R = R(\varepsilon') > 0$ above, so that

$$\mathrm{vol}_{\mathbb{H}^2}(\widehat{D}_o(\Gamma_n) \setminus \overline{B}_o(R)) \leq \frac{\varepsilon''}{3} \qquad \text{and} \qquad \mathrm{vol}_{\mathbb{H}^2}(\widehat{D}_o(\Gamma) \setminus \overline{B}_o(R)) \leq \frac{\varepsilon''}{3}.$$

Then we compute that

$$\begin{aligned}
&\int_{\mathbb{H}^2} \left| \mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x) \right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) \\
\leq &\int_{\mathbb{H}^2} \left| \mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma_n) \cap \overline{B}_o(R)}(x) \right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) \\
+ &\int_{\mathbb{H}^2} \left| \mathbb{1}_{\widehat{D}_o(\Gamma_n) \cap \overline{B}_o(R)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma) \cap \overline{B}_o(R)}(x) \right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) \\
+ &\int_{\mathbb{H}^2} \left| \mathbb{1}_{\widehat{D}_o(\Gamma) \cap \overline{B}_o(R)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x) \right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) \\
\leq &\frac{2}{3}\varepsilon'' + \int_{\mathbb{H}^2} \mathbb{1}_{\overline{B}_o(R)}(x) \cdot \left| \mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x) \right| d\,\mathrm{vol}_{\mathbb{H}^2}(x).
\end{aligned}$$

Observe that the function $\mathbb{1}_{\overline{B}_o(R)}(x) \cdot \left| \mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x) \right|$ converges pointwise almost everywhere to 0 and is dominated by the L^1-function $\mathbb{1}_{\overline{B}_o(R)}(x)$. Hence, by the Dominated Convergence Theorem we conclude that

$$\int_{\mathbb{H}^2} \left| \mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x) \right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) < \varepsilon''$$

for large n.

Because $\varepsilon'' > 0$ was arbitrary, the asserted convergence in $L^1(\mathbb{H}^2)$ follows. □

4 The IRS Compactification of Moduli Space

We proved in chapter 3 that truncated Dirichlet domains depend continuously on the group (Lemma 3.1.1) and we will apply this result to study Gelander's IRS compactification in the following.

In section 4.1 we will show that Gelander's IRS compactification is a compactification in the topological sense. We will then prove our main result (Theorem 4.2.2) relating the IRS compactification to the augmented Teichmüller space in section 4.2.

4.1 Embedding Moduli Space

In section 1.4 we have explained how Gelander defined the IRS compactification of the moduli space $\mathcal{M}(\Sigma)$. Let us briefly recall his construction.

To any lattice $\Gamma \leq G$ we can associate an IRS $\mu_\Gamma \in \mathrm{IRS}(G)$ in the following way. Let ν_Γ denote the (unique) right-invariant Borel probability measure on $\Gamma\backslash G$. Then the orbit map $G \longrightarrow \mathrm{Sub}(G), g \longmapsto g^{-1}\Gamma g$ descends to the map

$$\begin{aligned} \varphi_\Gamma \colon \Gamma\backslash G &\longrightarrow \mathrm{Sub}(G), \\ \Gamma g &\longmapsto g^{-1}\Gamma g. \end{aligned}$$

We obtain $\mu_\Gamma = (\varphi_\Gamma)_*(\nu_\Gamma) \in \mathrm{IRS}(G)$ as the push-forward measure of ν_Γ along φ_Γ.

By Proposition 2.8.4 there is a one-to-one correspondence between conjugacy classes of lattices in $[\Gamma] \in G\backslash\mathcal{L}(\Sigma)$ and hyperbolic surfaces $X = \Gamma\backslash\mathbb{H}^2 \in \mathcal{M}(\Sigma)$. Thus, we can use the above construction of an IRS to obtain a map from the moduli space $\mathcal{M}(\Sigma)$ to the space of IRSs of $G = \mathrm{PSL}(2,\mathbb{R})$

$$\iota\colon \mathcal{M}(\Sigma) \longmapsto \mathrm{IRS}(G), \quad [\Gamma] \longmapsto \mu_\Gamma.$$

For this map to be well-defined the IRS μ_Γ must only depend on the conjugacy class $[\Gamma] \in G\backslash\mathcal{L}(\Sigma)$. We will see this in Lemma 4.1.3. Making use of the identification $\mathcal{M}(\Sigma) \cong G\backslash\mathcal{L}(\Sigma)$ we will also use the notation $\mu_X = \mu_\Gamma$ for a hyperbolic surface $X = \Gamma\backslash\mathbb{H}^2 \in \mathcal{M}(\Sigma)$.

Gelander defined the IRS compactification of the moduli space as follows.

Definition 4.1.1 ([Gel15, Section 3.1])**.** The *IRS compactification* $\overline{\mathcal{M}}^{IRS}(\Sigma)$ *of the moduli space* $\mathcal{M}(\Sigma)$ is defined as the closure

$$\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) := \overline{\iota(\mathcal{M}(\Sigma))} \subseteq \mathrm{IRS}(G).$$

That $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is indeed compact, follows from the following well-known lemma.

Lemma 4.1.2 ([Abé+17, Section 2])**.** *The space of invariant random subgroups* $\mathrm{IRS}(G)$ *of a locally compact Hausdorff group G is compact.*

We include the short proof for completeness.

Proof. Since $\mathrm{Sub}(G)$ is compact, so is $\mathrm{Prob}(\mathrm{Sub}(G))$ by Banach–Alaoglu's Theorem. The space of invariant random subgroups is a closed subspace

$$\mathrm{Prob}(\mathrm{Sub}(G))^G = \bigcap_{g\in G}\{\mu \in \mathrm{Prob}(\mathrm{Sub}(G))\,|\,g_*\mu = \mu\}.$$

□

Before proving that $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is a topological embedding let us see that ι is well-defined and injective. This will follow from Lemma 4.1.3 and Lemma 4.1.4.

Lemma 4.1.3. *Let* $\Gamma \leq G$ *be a lattice and denote by* $N(\Gamma) \leq G$ *its normalizer. Then:*

(i) The map $\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow G * \Gamma$ *is G-equivariant and descends to a G-equivariant homeomorphism*

$$\varphi_{N(\Gamma)}\colon N(\Gamma)\backslash G \longrightarrow G * \Gamma.$$

(ii) The measure $\mu_\Gamma \in \mathrm{IRS}(G)$ *is an ergodic IRS, which depends only on the conjugacy class of* $\Gamma \leq G$.

(iii) The support of μ_Γ *is the orbit closure of* Γ *in* $\mathrm{Sub}(G)$*:* $\mathrm{supp}(\mu_\Gamma) = \overline{G * \Gamma}$.

Proof. (i) The stabilizer of Γ for the conjugation action of G on $\mathrm{Sub}(G)$ is its normalizer $N(\Gamma)$ by definition. Hence, the orbit map $G \longrightarrow \mathrm{Sub}(G), g \longmapsto g^{-1}\Gamma g$ descends to a continuous bijection $\varphi_{N(\Gamma)}\colon N(\Gamma)\backslash G \longrightarrow G * \Gamma \subseteq \mathrm{Sub}(G)$ by the Orbit Stabilizer Theorem. Notice that the orbit map is equivariant with respect to the right-translation action on G and the conjugation action on $\mathrm{Sub}(G)$, such that both $\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow \mathrm{Sub}(G)$ and $\varphi_{N(\Gamma)}\colon N(\Gamma)\backslash G \longrightarrow \mathrm{Sub}(G)$ are equivariant as well. In order to show that this map is a homeomorphism, we will show that it is proper.

To this end suppose that $(g_n^{-1}\Gamma g_n)_{n\in\mathbb{N}} \subseteq G*\Gamma$ converges to some $\Gamma' \in G*\Gamma$. Because Γ' is discrete there is an open identity neighborhood $U \subseteq G$ such that $g_n^{-1}\Gamma g_n \cap U = \{e\}$ for all $n \in \mathbb{N}$ by Lemma 2.7.6. In particular, there is an $\varepsilon > 0$ such that $\pi_\Gamma(g_n o) \in \Gamma\backslash\mathbb{H}^2$ is in the ε-thick part for some base point $o \in \mathbb{H}^2$, where we used the notation $\pi_\Gamma\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ for the quotient map. Because the ε-thick part $(\Gamma\backslash\mathbb{H}^2)_{\geq\varepsilon}$ is compact, there is a compact set $K \subseteq \mathbb{H}^2$ and $\gamma_n \in \Gamma$ such that $\gamma_n g_n o \in K$. Thus there is a subsequence $(\gamma_{n_k} g_{n_k})_{k\in\mathbb{N}}$ converging to some $g \in G$ and $N(\Gamma)g_{n_k} = N(\Gamma)\gamma_{n_k} g_{n_k}$ converges to $N(\Gamma)g$ in $N(\Gamma)\backslash G$. Because the convergent sequence $(g_n^{-1}\Gamma g_n)_{n\in\mathbb{N}} \subseteq G*\Gamma$ was arbitrary, we conclude that $\varphi_{N(\Gamma)}$ is proper.

(ii) Because $\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow \mathrm{Sub}(G)$ is equivariant and ν_Γ is a G-invariant probability measure on $\Gamma\backslash G$, it follows that $\mu_\Gamma = (\varphi_\Gamma)_*(\nu_\Gamma)$ is G-invariant as well.

Regarding ergodicity, let $f\colon \mathrm{Sub}(G) \longrightarrow \mathbb{R}$ be a conjugation invariant Borel measurable function. Then f is constant on the oribt $G*\Gamma$. By definition $G*\Gamma \subseteq \mathrm{Sub}(G)$ is a subset of full measure on which f is constant. Hence, μ_Γ is ergodic.

We will show that μ_Γ depends only on the conjugacy class of $\Gamma \leq G$ now. Let $\Gamma' = h\Gamma h^{-1}$ be a conjugate of Γ, $h \in G$. Then left multiplication by h induces a G-equivariant homeomorphism

$$\widehat{h}\colon \Gamma\backslash G \longrightarrow \Gamma'\backslash G, \quad \Gamma g \longmapsto \Gamma' h g,$$

which makes the following diagram commute:

$$\begin{tikzcd}
\Gamma\backslash G \arrow[dr, "\varphi_\Gamma"] \arrow[dd, "\widehat{h}"'] & \\
& \mathrm{Sub}(G) \\
\Gamma'\backslash G \arrow[ur, "\varphi_{\Gamma'}"] &
\end{tikzcd}$$

Moreover, $\widehat{h}$ is G-equivariant such that $\widehat{h}_* \nu_\Gamma$ is another G-invariant probability measure and by uniqueness $\widehat{h}_* \nu_\Gamma = \nu_{\Gamma'}$. Thus $(\varphi_\Gamma)_* \nu_\Gamma = \mu_\Gamma = (\varphi_{\Gamma'})_* \nu_{\Gamma'}$, whence μ_Γ depends only on the conjugacy class of Γ.

(iii) Note that $\overline{G * \Gamma} = \overline{\varphi_\Gamma(\Gamma\backslash G)}$ by definition. Further, recall that $H \in \mathrm{Sub}(G)$ is in the support $\mathrm{supp}(\mu_\Gamma)$ if and only if every open neighborhood $\mathcal{U} \subseteq \mathrm{Sub}(G)$ of H has positive mass $\mu_\Gamma(\mathcal{U}) > 0$.

Let $H \notin \mathrm{supp}(\mu_\Gamma)$. We want to show that $H \notin \overline{\varphi_\Gamma(\Gamma\backslash G)}$, i.e. there is an open neighborhood $\mathcal{U} \subseteq \mathrm{Sub}(G)$ of H such that $\mathcal{U} \cap \varphi_\Gamma(\Gamma\backslash G) = \varnothing$. Because $H \notin \mathrm{supp}(\mu_\Gamma)$ there is an open neighborhood $\mathcal{U} \subseteq \mathrm{Sub}(G)$ such that $\mu_\Gamma(\mathcal{U}) = 0$. Then $V = \varphi_\Gamma^{-1}(\mathcal{U}) \subseteq \Gamma\backslash G$ is an open subset such that

$$\nu_\Gamma(V) = \nu_\Gamma(\varphi_\Gamma^{-1}(\mathcal{U})) = \mu_\Gamma(\mathcal{U}) = 0.$$

Because ν_Γ has full support on $\Gamma\backslash G$ the set V must be empty. Therefore, $\mathcal{U} \cap \varphi_\Gamma(\Gamma\backslash G) = \varnothing$.

Vice versa, let $H \in \mathrm{supp}(\mu_\Gamma)$, and let $\mathcal{U} \subseteq \mathrm{Sub}(G)$ be an open neighborhood of H. Then

$$0 < \mu_\Gamma(\mathcal{U}) = \nu_\Gamma(\varphi_\Gamma^{-1}(\mathcal{U})).$$

Hence, $V = \varphi_\Gamma^{-1}(\mathcal{U}) \neq \varnothing \subseteq \Gamma\backslash G$ is a non-empty open subset, and $\mathcal{U} \cap \varphi_\Gamma(\Gamma\backslash G) \neq \varnothing$. Because $\mathcal{U}$ was an arbitrary open neighborhood of H, it follows that $H \in \overline{\varphi_\Gamma(\Gamma\backslash G)} = \overline{G * \Gamma}$.

□

Lemma 4.1.4. *Let $[\Gamma_1], \dots, [\Gamma_m]$ be pairwise distinct conjugacy classes of lattices in G. Then the associated invariant random subgroups $\mu_{\Gamma_1}, \dots, \mu_{\Gamma_m} \in \mathrm{IRS}(G) \subseteq C(\mathrm{Sub}(G))^*$ are linearly independent.*

Proof. Let $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$0 = \sum_{j=1}^{m} \lambda_j \cdot \mu_{\Gamma_j}.$$

For all $i, j \in \{1, \dots, m\}, i \neq j$, there is an open neighborhood $U_{i,j} \subseteq \mathrm{Sub}(G)$ of Γ_i such that $U_{i,j} \cap G*\Gamma_j = \varnothing$. Indeed, otherwise there would be $i \neq j$ and a sequence $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that $g_n\Gamma_j g_n^{-1} \to \Gamma_i$ as $n \to \infty$. Because Γ_i is not abelian this implies that Γ_i is a conjugate of Γ_j by Lemma 2.8.5; contradiction.

We set

$$U_i = \bigcap_{\substack{j=1\\ j\neq i}}^{m} U_{i,j},$$

such that U_i is an open neighborhood of Γ_i satisfying

$$\varnothing = U_i \cap \overline{G * \Gamma_j} = U_i \cap \operatorname{supp}(\mu_{\Gamma_j})$$

for every $j \neq i$. Then

$$0 = \left(\sum_{j=1}^{m} \lambda_j \cdot \mu_{\Gamma_j}\right)(U_i) = \lambda_i \cdot \underbrace{\mu_{\Gamma_i}(U_i)}_{>0}$$

such that $\lambda_i = 0$ for every $i = 1, \dots, m$. □

From Lemma 4.1.3 (ii) it follows that ι is well-defined and Lemma 4.1.4 implies that ι is injective. Our next goal is to show that it is a topological embedding:

Proposition 4.1.5. *The map $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is a topological embedding.*

Remark 4.1.6. In the case where Σ is compact it is proved in [GL18b, Proposition 11.2] that the map ι is continuous.

Before we attempt a proof let us understand the measure $\mu_\Gamma \in \mathrm{IRS}(G)$, $\Gamma \in \mathcal{L}(\Sigma)$, in terms of a Haar measure ν on G. For any continuous function $F \in C(\mathrm{Sub}(G))$ we have that

$$\begin{aligned}\int_{\mathrm{Sub}(G)} F(H)\, d\mu_\Gamma(H) &= \int_{\Gamma\backslash G} F(g^{-1}\Gamma g)\, d\nu_\Gamma(g\Gamma)\\ &= \nu(D)^{-1}\int_G \mathbb{1}_D(g)\cdot F(g^{-1}\Gamma g)\, d\nu(g), \qquad (\star)\end{aligned}$$

where $D \subseteq G$ is a fundamental domain for the action $\Gamma \curvearrowright G$.

We shall pick a particularly convenient Haar measure ν on G now. Recall that $G = \mathrm{Isom}_+(\mathbb{H}^2) \cong \mathrm{PSL}(2,\mathbb{R})$. Then the map $p\colon G \longrightarrow \mathbb{H}^2, g \longmapsto g\cdot i$ is surjective and induces an identification $G/K \cong \mathbb{H}^2$, where $K = \mathrm{stab}_G(i) \cong \mathrm{SO}(2,\mathbb{R})$. Thus the hyperbolic area measure $\mathrm{vol}_{\mathbb{H}^2}$ on $\mathbb{H}^2$ amounts via this identification to a G-invariant measure on G/K, which we shall denote by v. Further, we choose a normalized Haar measure η on K

such that $\eta(K) = 1$. By Weil's quotient formula we obtain a Haar measure ν on G such that

$$\int_G f(g)\,d\nu(g) = \int_{G/K}\int_K f(gk)\,d\eta(k)\,d\nu(gK)$$

for every $f \in L^1(G,\nu)$. With this choice we have that $\nu(p^{-1}(B)) = v(B)$ for every measurable subset $B \subseteq \mathbb{H}^2$.

With this choice of measures we obtain the following lemma.

Lemma 4.1.7. *The map*

$$\begin{aligned} p^*\colon L^1(G/K,v) &\longrightarrow L^1(G,\nu),\\ f &\longmapsto f\circ p, \end{aligned}$$

is a linear isometry. In particular, p^* is continuous.

Proof. Let $f \in L^1(G/K,v)$. By Weil's quotient formula we get that

$$\begin{aligned} \|f\circ p\|_{L^1(G)} &= \int_G |f(p(g))|\,d\nu(g)\\ &= \int_{G/K}\int_K |f(p(gk))|\,d\eta(k)\,dv(gK)\\ &= \int_{G/K}\int_K |f(gK)|\,d\eta(k)\,dv(gK)\\ &= \int_{G/K} \eta(K)\cdot|f(gK)|\,dv(gK)\\ &= \int_{G/K} |f(gK)|\,dv(gK) = \|f\|_{L^1(G/K)}. \end{aligned}$$

□

In view of (⋆) we will need a fundamental domain for the action $\Gamma \curvearrowright G$. Given a point $o \in \mathbb{H}^2$ we have already introduced the Dirichlet fundamental domain $D_o(\Gamma) \subseteq \mathbb{H}^2$ for the action $\Gamma \curvearrowright \mathbb{H}^2$ in section 2.2. We can use the Dirichlet domain to obtain a fundamental domain for the action $\Gamma \curvearrowright G$. Indeed, it is straight-forward to check that $F_o(\Gamma) := p^{-1}(D_o(\Gamma))$ is a fundamental domain for $\Gamma \curvearrowright G$. Likewise, we may define

$$\widehat{F}_o(\Gamma) := p^{-1}(\widehat{D}_o(\Gamma))$$

as the preimage of the truncated Dirichlet domain $\widehat{D}_o(\Gamma) \subseteq \mathbb{H}^2$.

Recall that the truncated Dirichlet domain $\widehat{D}_o(\Gamma)$ is a fundamental domain for $\Gamma \curvearrowright \widetilde{C}(\Gamma)$. The quotient $C(\Gamma) = \Gamma\backslash\widetilde{C}(\Gamma)$ is the convex core of $\Gamma\backslash\mathbb{H}^2$, which one obtains from $\Gamma\backslash\mathbb{H}^2$ by cutting off its funnels at their waist geodesics. Furthermore, we saw that $\mathrm{vol}_{\mathbb{H}^2}(\widehat{D}_o(\Gamma)) = 2\pi\left|\chi(\Gamma\backslash\mathbb{H}^2)\right|$, such that we have

$$\nu(\widehat{F}_o(\Gamma)) = 2\pi\left|\chi(\Gamma\backslash\mathbb{H}^2)\right|$$

by our choice of Haar measure.

In Lemma 3.1.1 we showed that the indicator function of the truncated Dirichlet domain $\mathbb{1}_{\widehat{D}_o(\Gamma)} \in L^1(\mathbb{H}^2)$ depends continuously on $\Gamma \in \mathcal{D}(\Sigma)$. Applying Lemma 4.1.7 we obtain the following important corollary.

Corollary 4.1.8. *Let $o \in \mathbb{H}^2$. The map*

$$\mathcal{D}(\Sigma) \longrightarrow L^1(G), \quad \Gamma \longmapsto \mathbb{1}_{\widehat{F}_o(\Gamma)}$$

is continuous, where $\mathbb{1}_{\widehat{F}_o(\Gamma)} = \mathbb{1}_{\widehat{D}_o(\Gamma)} \circ p$ denotes the indicator function of $\widehat{F}_o(\Gamma) \subseteq G$.

Corollary 4.1.8 will play an essential role in our proofs of Proposition 4.1.5 and Theorem 4.2.2.

Finally, we record the following consequence of the Dominated Convergence Theorem for future reference.

Lemma 4.1.9. *Let (X,μ) be a measure space, let $(f_n)_{n\in\mathbb{N}} \subseteq L^1(X,\mu)$ and let $(g_n)_{n\in\mathbb{N}} \subseteq L^\infty(X,\mu)$. Assume that there is $f \in L^1(X,\mu)$ such that*

$$\|f_n - f\|_{L^1} \to 0 \quad (n \to \infty),$$

and that there is $C > 0$ and $g \in L^\infty(X,\mu)$ such that $\|g_n\|_{L^\infty} \leq C$, for every $n \in \mathbb{N}$, and

$$g_n(x) \to g(x) \quad (n \to \infty)$$

for μ-almost-every $x \in X$.

Then

$$\|f_n \cdot g_n - f \cdot g\|_{L^1} \to 0 \quad (n \to \infty).$$

Proof. We compute

$$\begin{aligned}\|f_n \cdot g_n - f \cdot g\|_{L^1} &\le \|f_n \cdot g_n - f \cdot g_n\|_{L^1} + \|f \cdot g_n - f \cdot g\|_{L^1} \\ &\le C \cdot \|f_n - f\|_{L^1} + \int_X |f(x)| \cdot |g_n(x) - g(x)| \, d\mu(x).\end{aligned}$$

Note that

$$|f(x)| \cdot |g_n(x) - g(x)| \to 0 \quad (n \to \infty)$$

for μ-almost-every $x \in X$ and the functions $|f(x)| \cdot |g_n(x) - g(x)|$ are μ-almost-everywhere dominated by the integrable function $2C\,|f(x)|$. By the Dominated Convergence Theorem we conclude that

$$\int_X |f(x)| \cdot |g_n(x) - g(x)| \, d\mu(x) \to 0 \quad (n \to \infty),$$

which in turn implies $\|f_n \cdot g_n - f \cdot g\|_{L^1} \to 0$ as $n \to \infty$. □

After these preparations we are ready to prove Proposition 4.1.5.

Proof of Proposition 4.1.5. We want to show that $\iota\colon G\backslash\mathcal{L}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is a topological embedding.

First, let us prove that ι is continuous. Let $([\Gamma_n])_{n\in\mathbb{N}} \subseteq G\backslash\mathcal{L}(\Sigma)$ be a convergent sequence with limit $[\Gamma] \in G\backslash\mathcal{L}(\Sigma)$. Up to taking conjugates we may assume that $\Gamma_n \to \Gamma$ in $\mathcal{L}(\Sigma)$. Let $o \in \mathbb{H}^2$ and we consider the fundamental domains $F_o(\Gamma_n) = p^{-1}(D_o(\Gamma_n))$ for $\Gamma_n \curvearrowright G$. Since Γ_n is a lattice, we have that $\widetilde{C}(\Gamma_n) = \mathbb{H}^2$ and $\widehat{D}_o(\Gamma_n) = D_o(\Gamma_n)$.

Let $f \in C(\mathrm{Sub}(G))$. Then

$$\begin{aligned}&\left|\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) - \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma}(H)\right| \\ &= \left|\nu(F_o(\Gamma_n))^{-1} \cdot \int_G \mathbb{1}_{F_o(\Gamma_n)}(g) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g) - \nu(F_o(\Gamma))^{-1} \cdot \int_G \mathbb{1}_{F_o(\Gamma)}(g) \cdot f(g^{-1}\Gamma g)\, d\nu(g)\right| \\ &\le \frac{1}{2\pi\,|\chi(\Sigma)|} \cdot \left\|\mathbb{1}_{F_o(\Gamma_n)} \cdot \bar{f}_n - \mathbb{1}_{F_o(\Gamma)} \cdot \bar{f}\right\|_{L^1},\end{aligned}$$

where we set $\bar{f}_n(g) := f(g^{-1}\Gamma_n g), \bar{f}(g) := f(g^{-1}\Gamma g)$ for every $g \in G$. Note that f is uniformly bounded because $\mathrm{Sub}(G)$ is compact, so that $(\bar{f}_n)_{n\in\mathbb{N}}$ are uniformly bounded, too. Moreover,

$$\bar{f}_n(g) = f(g^{-1}\Gamma_n g) \to \bar{f}(g) = f(g^{-1}\Gamma g) \qquad (n \to \infty)$$

for every $g \in G$ by continuity. By Corollary 4.1.8 we know that

$$\|\mathbb{1}_{F_o(\Gamma_n)} - \mathbb{1}_{F_o(\Gamma)}\|_{L^1(G,\nu)} \to 0 \qquad (n \to \infty).$$

Thus we may apply Lemma 4.1.9 and conclude that

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) \to \int_{\mathrm{Sub}(G)} f(H)\, d\mu_\Gamma(H) \qquad (n \to \infty).$$

This shows that ι is continuous.

Finally, let $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(\Sigma)$ and let $\Gamma \in \mathcal{L}(\Sigma)$, such that $\mu_{\Gamma_n} \to \mu_\Gamma$ as $n \to \infty$. We want to show that $[\Gamma_n] \to [\Gamma]$ as $n \to \infty$. Let $\mathcal{U} \subseteq \mathrm{Sub}(G)$ be an open neighborhood of Γ, and let $\overline{\mathcal{V}} \subseteq \mathcal{U}$ be a compact neighborhood of Γ. By Urysohn's Lemma we find a continuous function $f\colon \mathrm{Sub}(G) \longrightarrow [0,1]$ such that $f|_{\overline{\mathcal{V}}} \equiv 1$ and $f|_{\mathcal{U}^c} \equiv 0$. Because $\mu_{\Gamma_n} \to \mu_\Gamma$ we have that

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) \to \int_{\mathrm{Sub}(G)} f(H)\, d\mu_\Gamma(H) \qquad (n \to \infty).$$

Because

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_\Gamma(H) \geq \int_{\overline{\mathcal{V}}} f(H)\, d\mu_\Gamma(H) = \mu_\Gamma(\overline{\mathcal{V}}) > 0,$$

also

$$\nu_{\Gamma_n}(\varphi_{\Gamma_n}^{-1}(\mathcal{U})) = \mu_{\Gamma_n}(\mathcal{U}) \geq \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) > 0$$

for large $n \in \mathbb{N}$. Therefore, $\varphi_{\Gamma_n}^{-1}(\mathcal{U}) \subseteq \Gamma\backslash G$ is a non-empty open subset, whence there are $g_n \in G$ such that $\varphi_{\Gamma_n}(\Gamma_n g_n) = g_n^{-1}\Gamma_n g_n \in \mathcal{U}$. Because $\mathcal{U}$ was an arbitrary open neighborhood of Γ it follows that $[\Gamma_n] \to [\Gamma]$ as $n \to \infty$.

This shows that $\iota\colon G\backslash\mathcal{L}(\Sigma) \hookrightarrow \mathrm{IRS}(G)$ is a topological embedding. □

4.2 The Augmented Moduli Space and the IRS Compactification

In the previous section we have seen how the moduli space can be embedded in the space of invariant random subgroups $\iota\colon \mathcal{M}(\Sigma) \hookrightarrow \mathrm{IRS}(G)$. This gave rise to the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) = \overline{\iota(\mathcal{M}(\Sigma))}$. In [Gel15] Gelander proposed the problem to analyze the IRS compactification further; see Problem 1.4.1. This is the goal of this section.

Our strategy is to relate the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ to the augmented moduli space. In fact, we shall construct an extension $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ of the topological embedding $\iota\colon \mathcal{M}(\Sigma) \hookrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) \subseteq \mathrm{IRS}(G)$:

$$\begin{tikzcd} & \widehat{\mathcal{M}}(\Sigma) \arrow[dd, dashed, "\widehat{\Phi}"] \\ \mathcal{M}(\Sigma) \arrow[ur, hook] \arrow[dr, hook, "\iota"'] & \\ & \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) \end{tikzcd}$$

Instead of defining the map $\widehat{\Phi}$ directly on the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ we will first define a map on the augmented Teichmüller space $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$.

Definition 4.2.1. We define $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ by

$$\widetilde{\Phi}(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}) := \sum_{\Sigma'\in c(\sigma)} \frac{\chi(\Sigma')}{\chi(\Sigma)} \cdot \mu_{\operatorname{im}\rho_{\Sigma'}}$$

for every $([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma), \sigma \subseteq \mathcal{C}(\Sigma)$.

Observe that $\widetilde{\Phi}$ is well-defined. Indeed, $\widetilde{\Phi}(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)})$ is a convex combination of IRSs, since $\chi(\Sigma) = \sum_{\Sigma'\in c(\sigma)} \chi(\Sigma') < 0$. Here the coefficient $\chi(\Sigma')/\chi(\Sigma)$ should be interpreted as the proportion of the area that the component Σ' takes up in the whole surface Σ. This makes sense, because the area of a hyperbolic surface $X \in \mathcal{M}(\Sigma)$ is a topological invariant $\mathrm{vol}(X) = 2\pi|\chi(\Sigma)|$ by the Gauss–Bonnet Theorem.

Moreover, notice that $\widetilde{\Phi}(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)})$ depends only on the conjugacy classes $[\operatorname{im}\rho_{\Sigma'}] \in \mathcal{M}(\Sigma')$ and how often any one of them arises. Both the marking and even the gluing of the different parts do not affect the image. Hence, $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ descends to a map $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$, that descends further down to a map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ from the moduli space of parts $|\widehat{\mathcal{M}}|(\Sigma)$ to $\mathrm{IRS}(G)$.

$$\begin{tikzcd} & \widehat{\mathcal{M}}(\Sigma) \arrow[dl, "Q"'] \arrow[dr, "\widehat{\Phi}"] & \\ |\widehat{\mathcal{M}}|(\Sigma) \arrow[rr, dashed, "\Phi"] & & \mathrm{IRS}(G) \end{tikzcd}$$

Concretely, the map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ takes the form

$$\Phi(\xi) = \sum_{X \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')} \frac{\chi(X)}{\chi(\Sigma)} \cdot \xi(X) \cdot \mu_X$$

for every $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$. Recall that elements of $|\widehat{\mathcal{M}}|(\Sigma)$ are functions $\xi\colon \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma') \longrightarrow \mathbb{N}_0$, which take non-zero values only for a finite number of hyperbolic surfaces. For a nodal surface $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ the corresponding $\xi = Q(\mathbf{X}) \in |\widehat{\mathcal{M}}|(\Sigma)$ simply records how many times any one hyperbolic surface arises as a part of $\mathbf{X}$; see section 2.6.

It turns out, that $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ induces a homeomorphism $|\widehat{\mathcal{M}}|(\Sigma) \cong \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$. More precisely, we have the following theorem.

Theorem 4.2.2. *The map $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is continuous and descends to a continuous surjection $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$ that extends the embedding $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$. Moreover, $\widehat{\Phi}$ induces a homeomorphism $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$.*

There is a uniform upper bound on the number of elements in each fiber of $\widehat{\Phi}$,

$$\#\widehat{\Phi}^{-1}(\mu) \leq B(\Sigma) := \binom{3|\chi|}{p} \cdot \frac{(2(|\chi| + g - 1))!}{(|\chi| + g - 1)! \cdot 2^{(|\chi|+g-1)}} \qquad \forall \mu \in \overline{\mathcal{M}}^{IRS}(\Sigma),$$

where $\chi = \chi(\Sigma)$, $g = g(\Sigma)$, and $p = p(\Sigma)$ denote the Euler characteristic, the genus, and the number of punctures of Σ, respectively.

Remark 4.2.3. Observe that the upper bound $B(\Sigma)$ depends only on the topology of the surface Σ.

The main difficulty in the proof of Theorem 4.2.2 is to show that the map $\widetilde{\Phi}$ is continuous. As in the proof of continuity of the embedding $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ in Proposition 4.1.5 we will use the L^1-convergence of fundamental domains (Corollary 4.1.8). However, this time we will have to deal with curves that may degenerate along a convergent sequence in $\widehat{\mathcal{T}}(\Sigma)$. We will overcome this issue by dissecting a fundamental domain into convergent truncated fundamental domains for each subsurface appropriately. The following lemmas will help us to make this discussion precise.

Recall that for every component $\Sigma' \in c(\sigma)$, $\sigma \subseteq \mathcal{C}(\Sigma)$, we obtain a monomorphism $\iota_{\Sigma'}\colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ induced by the inclusion $\Sigma' \subseteq \Sigma$, which is well-defined up to conjugation; see Remark 2.5.2.

Lemma 4.2.4. *Let $\rho \in \mathcal{R}(\Sigma)$, let $\sigma \subseteq \mathcal{C}(\Sigma)$ be a simplex in the curve complex, let $\Sigma' \in c(\sigma)$ be a component, and let $\iota_{\Sigma'}\colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ be an inclusion monomorphism. Denote $\Gamma = \operatorname{im}\rho$, $\Gamma' = \operatorname{im}(\rho \circ \iota_{\Sigma'})$ and let $\pi\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ be the quotient map. Let $f\colon \Sigma \longrightarrow \Gamma\backslash\mathbb{H}^2$ be an orientation preserving homeomorphism such that ρ is a holonomy representation of f, and suppose that $f(\sigma) = \tau$ is a collection of closed geodesics.*

Then $\widetilde{\tau} := \pi^{-1}(\tau) \subseteq \mathbb{H}^2$ is a collection of disjoint geodesics and $\widetilde{C}(\Gamma') \subseteq \mathbb{H}^2$ is the closure of a connected component of $\mathbb{H}^2 \setminus \widetilde{\tau}$.

Proof. Let $\widetilde{f}\colon \widetilde{\Sigma} \cong \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ be a lift of $f\colon \Sigma \longrightarrow \Gamma\backslash\mathbb{H}^2$, that is ρ-equivariant. Let $\widetilde{\Sigma}' \subseteq \mathbb{H}^2 \setminus \widetilde{\sigma}$ be a connected component over Σ' such that the inclusion $\widetilde{\Sigma}' \hookrightarrow \widetilde{\Sigma} \cong \mathbb{H}^2$ is $\iota_{\Sigma'}$-equivariant; see Proposition and Definition 2.5.4. We set $\widetilde{X}' := \widetilde{f}(\widetilde{\Sigma}')$, and we want to show that $\widetilde{C}(\Gamma') = \overline{\widetilde{X}'}$. We will do so by showing that

$$\partial\widetilde{X}' \subseteq L(\Gamma') \subseteq \overline{\partial\widetilde{X}'}.$$

Note that $\widetilde{X}'$ is $\Gamma' = \rho(\iota_{\Sigma'}(\pi_1(\Sigma')))$-invariant. Therefore $\overline{\partial\widetilde{X}'}$ is a closed Γ'-invariant subset of $\partial\mathbb{H}^2$ that must contain the limit set $L(\Gamma')$ because the limit set is the smallest such subset.

Let $\xi \in \partial\widetilde{X}'$. If ξ is fixed by a parabolic element $\eta \in \Gamma'$ then

$$\xi = \lim_{n\to\infty} \eta^n o \in L(\Gamma'), \qquad o \in \mathbb{H}^2.$$

Hence, let us assume that ξ is not fixed by any parabolic element in Γ'. Let $\gamma \subseteq \widetilde{X}'$ be a geodesic from $\widetilde{p} = \gamma(0)$ to $\xi = \gamma(\infty)$. Further, let $\{P_j\}_{j\in\mathbb{N}}$ be a system of disjoint horoballs centered at the fixed points $\{\xi_j\}_{j\in\mathbb{N}}$ of parabolic elements in Γ', respectively. Then there is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$ and $\gamma(t_n) \notin \bigsqcup_{j\in\mathbb{N}} P_j$. Indeed, otherwise there would be a $T > 0$ and $j_0 \in \mathbb{N}$ such that $\gamma(t) \in P_{j_0}$ for all $t \geq T$. This in turn would imply that $\gamma(\infty) = \xi_{j_0} = \xi$; contradicting our assumption.

Observe that Γ' acts coboundedly on $\widetilde{X}' \setminus \bigsqcup_{j\in\mathbb{N}} P_j$. Therefore, there is $r > 0$, $o \in \widetilde{X}' \setminus \bigsqcup_{j\in\mathbb{N}} P_j$, and $\gamma_n \in \Gamma'$, such that $d(\gamma(t_n), \gamma_n \cdot o) \leq r$ for all $n \in \mathbb{N}$. Hence,

$$\xi = \lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} \gamma_n \cdot o \in L(\Gamma').$$

□

Lemma 4.2.5. *Let $\rho \in \mathcal{R}(\Sigma)$ and let $\sigma \subseteq C(\Sigma)$ be a simplex in the curve complex. Further, let $\iota_{\Sigma'}\colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ be an inclusion monomorphism for every component $\Sigma' \in c(\sigma)$. Denote $\Gamma = \operatorname{im}\rho$ and $\Gamma(\Sigma') = \operatorname{im}(\rho\circ\iota_{\Sigma'})$ for every $\Sigma' \in c(\sigma)$. Let $\{p(\Sigma') \in \mathbb{H}^2 \,|\, \Sigma' \in c(\sigma)\}$ be a collection of points. Then*

$$\bigcup_{\Sigma'\in c(\sigma)} \widehat{D}_{p(\Sigma')}(\Gamma(\Sigma'))$$

is a fundamental domain for the action of Γ on $\mathbb{H}^2$, and $\widehat{D}_{p(\Sigma')}(\Gamma(\Sigma')) \cap \widehat{D}_{p(\Sigma'')}(\Gamma(\Sigma''))$ has measure zero for distinct $\Sigma', \Sigma'' \in c(\sigma)$.

Proof. For simplicity we enumerate $\{\Sigma_i' \colon i = 1,\dots,l\} = c(\sigma)$ and set $p_i = p(\Sigma_i')$, $\Gamma(\Sigma_i') = \Gamma_i'$ for every $i = 1,\dots,l$. Further, denote by $q\colon \widetilde{\Sigma} \longrightarrow \Sigma$ and $\pi\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ the usual universal coverings, and let $f\colon \Sigma \longrightarrow \Gamma\backslash\mathbb{H}^2$ be an orientation preserving homeomorphism such that ρ is a holonomy representation of f and $\tau := f(\sigma)$ is a collection of closed geodesics. Let $\widetilde{f}\colon \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$ be a ρ-equivariant lift of f to the universal cover. We set $\widetilde{\sigma} := q^{-1}(\sigma)$ and $\widetilde{\tau} := \pi^{-1}(\tau) = \widetilde{f}(\widetilde{\sigma})$. Let $\widetilde{\Sigma}_i' \subseteq \widetilde{\Sigma} \setminus \widetilde{\sigma}$ be a connected component such that the inclusion $\widetilde{\Sigma}_i' \hookrightarrow \widetilde{\Sigma}$ is $\iota_{\Sigma_i'}$-equivariant. Because $\widetilde{\Sigma} = \bigcup_{i=1}^l \pi_1(\Sigma)\cdot\overline{\widetilde{\Sigma}_i'}$, we have that $\mathbb{H}^2 = \bigcup_{i=1}^l \Gamma\cdot\overline{\widetilde{f}(\widetilde{\Sigma}_i')}$. By Lemma 4.2.4 $\overline{\widetilde{f}(\widetilde{\Sigma}_i')} = \widetilde{C}(\Gamma_i)$ such that

$$\mathbb{H}^2 = \bigcup_{i=1}^{l} \Gamma\cdot\widetilde{C}(\Gamma_i).$$

Because $\widehat{D}_{p_i}(\Gamma_i')$ is a fundamental domain for the Γ_i'-action on $\widetilde{C}(\Gamma_i')$, it is readily verified that $\bigcup_{i=1}^l \widehat{D}_{p_i}(\Gamma_i')$ is a fundamental domain for the Γ-action on $\mathbb{H}^2$.

Finally, $\widehat{D}_{p_i}(\Gamma_i') \cap \widehat{D}_{p_j}(\Gamma_j') \subseteq \widetilde{\tau}$ for every $i \neq j$, which has measure zero. □

Lemma 4.2.6. *Let $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$, let $\sigma \subseteq C(\Sigma)$, let $\Sigma' \in c(\sigma)$ be a component, let $\iota_{\Sigma'}\colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ be an inclusion monomorphism, and suppose that*

$$\rho_n \circ \iota_{\Sigma'} \to \rho' \in \mathcal{R}(\Sigma') \qquad (n \to \infty).$$

Denote $\Gamma_n := \operatorname{im}\rho_n$, $\Gamma_n(\Sigma') := \operatorname{im}(\rho_n \circ \iota_{\Sigma'})$ and $\Gamma' := \operatorname{im}\rho' \in \mathcal{L}(\Sigma')$.

Then

$$\Gamma_n \to \Gamma' \qquad (n \to \infty).$$

Proof. We shall check conditions (C1) and (C2) from Proposition 2.7.3.

(C1) Let $\gamma' = \rho(c) \in \Gamma' = \operatorname{im}\rho'$ for some $c \in \pi_1(\Sigma')$. Then $\rho_n(\iota_{\Sigma'}(c)) \in \Gamma_n(\Sigma') \subseteq \Gamma_n$ converges to γ' as $n \to \infty$.

(C2) Let $(\gamma_{n_k})_{k\in\mathbb{N}}$ be a convergent sequence with limit $g \in G$ and $\gamma_{n_k} \in \Gamma_{n_k}$. We need to show that $g \in \Gamma'$. By Proposition 2.8.2 we know that $\Gamma_n(\Sigma') = \operatorname{im}(\rho_n \circ \iota_{\Sigma'}) \to \Gamma'$ as $n \to \infty$. Thus it will be sufficient to prove that $\gamma_{n_k} \in \Gamma_{n_k}(\Sigma')$ for large k.

Assume to the contrary that (up to a subsequence) $\gamma_{n_k} = \rho_{n_k}(\alpha_{n_k})$ where $\alpha_{n_k} \notin \iota_{\Sigma'}(\pi(\Sigma')) \cong \pi(\Sigma')$. We denote by $\partial\Sigma' \subseteq \sigma$ the curves that are adjacent to $\Sigma' \subseteq \Sigma$. Let $\{c_i : i \in \mathbb{N}\} \subseteq \pi_1(\Sigma')$ be the set of elements whose conjugacy classes correspond to some curve in $\partial\Sigma'$. Then the axes of $\{\rho_{n_k}(c_i) : i \in \mathbb{N}\}$ bound $\widetilde{C}(\Gamma_{n_k}(\Sigma'))$; see Lemma 4.2.4.

Let $w \in \mathbb{H}^2$ be a point such that $w \in \widetilde{C}(\Gamma_{n_k}(\Sigma'))$ for large $k \in \mathbb{N}$. Because $\gamma_{n_k} \notin \Gamma_{n_k}(\Sigma')$ it must send $w \in \widetilde{C}(\Gamma_{n_k}(\Sigma'))$ to $\gamma_{n_k} w \in \mathbb{H}^2 \setminus \widetilde{C}(\Gamma_{n_k}(\Sigma'))$ in some complementary region. Note that the geodesic segment σ_{n_k} from w to $\gamma_{n_k} w$ has to cross at least one axis of the $\{\rho_{n_k}(c_i) : i \in \mathbb{N}\}$.

Because $\Gamma_{n_k}(\Sigma')$ converges to a lattice $\Gamma' \leq G$ the lengths of all boundary curves $\ell(\rho_{n_k}(c_i))$ go to 0 as $k \to \infty$. Note that this convergence is uniform in i because the length $\ell(\rho_{n_k}(c_i))$ depends only on the conjugacy class of c_i and there are only finitely many curves in $\partial\Sigma' \subseteq \sigma$. By the Collar Lemma 2.2.1 the widths $w_{n_k}^{(i)}$ of the corresponding embedded collars tend to infinity (uniformly in i). Therefore,

$$d(\gamma_{n_k} w, w) = L(\sigma_{n_k}) \geq \min_{i\in\mathbb{N}}\{w_{n_k}^{(i)}\} \to \infty$$

as $k \to \infty$; in contradiction to $\gamma_{n_k} \to g$ as $k \to \infty$.

□

After these preparations, we are ready to prove Theorem 4.2.2.

Proof of Theorem 4.2.2. Let us prove that $\widetilde{\Phi}$ is continuous. We will first prove this for a sequence $([\rho_n])_{n\in\mathbb{N}} \subseteq \mathcal{T}(\Sigma)$ converging to $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma), \sigma \subseteq \mathcal{C}(\Sigma)$. By definition of the topology of $\widehat{\mathcal{T}}(\Sigma)$ we know that for every $\Sigma' \in c(\sigma)$ we have $[\rho_n \circ \iota_{\Sigma'}] \to [\rho_{\Sigma'}]$ as $n \to \infty$, i.e. there are $g_n(\Sigma') \in G$ such that

$$g_n(\Sigma')^{-1} \cdot (\rho_n \circ \iota_{\Sigma'}) \cdot g_n(\Sigma') \to \rho_{\Sigma'} \qquad (n \to \infty).$$

In particular,

$$g_n(\Sigma')^{-1} \cdot \Gamma_n(\Sigma') \cdot g_n(\Sigma') \to \Gamma(\Sigma') \qquad (n \to \infty),$$

where we set $\Gamma_n(\Sigma') := \mathrm{im}(\rho_n \circ \iota_{\Sigma'})$, $\Gamma(\Sigma') := \mathrm{im}(\rho \circ \iota_{\Sigma'})$.

Let $o \in \mathbb{H}^2$. By Lemma 4.2.5 the set

$$D_n := \bigcup_{\Sigma' \in c(\sigma)} \widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma'))$$

is a fundamental domain for the action of $\Gamma_n = \mathrm{im}\,\rho_n$ on $\mathbb{H}^2$.

Let $f \in C(\mathrm{Sub}(G))$. We have that

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) = \nu(p^{-1}(D_n))^{-1} \cdot \int_G \mathbb{1}_{D_n}(go) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g).$$

Observe that $\nu(p^{-1}(D_n)) = \mathrm{vol}_{\mathbb{H}^2}(D_n) = 2\pi|\chi(\Sigma)|$. Further,

$$\int_G \mathbb{1}_{D_n}(go) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g) = \sum_{\Sigma' \in c(\sigma)} \int_G \mathbb{1}_{\widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma'))}(go) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g).$$

Let $\Sigma' \in c(\sigma)$. Then

$$\begin{aligned}
&\int_G \mathbb{1}_{\widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma'))}(go) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g) \\
&= \int_G \mathbb{1}_{\widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma'))}(g_n(\Sigma')go) \cdot f(g^{-1}g_n(\Sigma')^{-1}\Gamma_n g_n(\Sigma')g)\, d\nu(g) \\
&= \int_G \mathbb{1}_{\widehat{D}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma'))}(go) \cdot f(g^{-1}g_n(\Sigma')^{-1}\Gamma_n g_n(\Sigma')g)\, d\nu(g) \\
&= \int_G \mathbb{1}_{\widehat{F}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma'))}(g) \cdot \bar{f}_{n,\Sigma'}(g)\, d\nu(g),
\end{aligned}$$

where we used the left-invariance of the Haar measure ν and the fact that

$$\widehat{D}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma')) = g_n(\Sigma')^{-1} \cdot \widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma')).$$

Moreover, we set

$$\bar{f}_{n,\Sigma'}(g) := f(g^{-1}g_n(\Sigma')^{-1}\Gamma_n g_n(\Sigma')g) \qquad \forall g \in G.$$

Note that $\|\bar{f}_{n,\Sigma'}\|_{L^\infty} \leq \|f\|_\infty < \infty$, $n \in \mathbb{N}$, and

$$g_n(\Sigma')^{-1} \cdot \Gamma_n \cdot g_n(\Sigma') \to \Gamma(\Sigma') \quad (n \to \infty)$$

by Lemma 4.2.6. Thus, if we set

$$\bar{f}_{\Sigma'}(g) := f(g^{-1}\Gamma(\Sigma')g) \qquad \forall g \in G,$$

then $\bar{f}_{n,\Sigma'}(g) \to \bar{f}(g)$ as $n \to \infty$ for every $g \in G$ by continuity. Moreover,

$$\left\| \mathbb{1}_{\widehat{F}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma'))} - \mathbb{1}_{\widehat{F}_o(\Gamma(\Sigma'))} \right\|_{L^1(G,\nu)} \to 0 \quad (n \to \infty)$$

by Corollary 4.1.8.

It follows from Lemma 4.1.9 that

$$\int_G \mathbb{1}_{\widehat{F}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma'))}(g) \cdot \bar{f}_{n,\Sigma'}(g)\, d\nu(g) \to \int_G \mathbb{1}_{\widehat{F}_o(\Gamma(\Sigma'))}(g) \cdot \bar{f}_{\Sigma'}(g)\, d\nu(g)$$

as $n \to \infty$.

All in all, we obtain that the integral

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H)$$

tends to

$$\begin{aligned}
&(2\pi|\chi(\Sigma)|)^{-1} \sum_{\Sigma' \in c(\sigma)} \int_G \mathbb{1}_{\widehat{F}_o(\Gamma(\Sigma'))}(g) \cdot f(g^{-1}\Gamma(\Sigma')g)\, d\nu(g) \\
&= \sum_{\Sigma' \in c(\sigma)} \frac{2\pi|\chi(\Sigma')|}{2\pi|\chi(\Sigma)|} \cdot \nu(\widehat{F}_o(\Gamma(\Sigma')))^{-1} \int_G \mathbb{1}_{\widehat{F}_o(\Gamma(\Sigma'))}(g) \cdot f(g^{-1}\Gamma(\Sigma')g)\, d\nu(g) \\
&= \sum_{\Sigma' \in c(\sigma)} \frac{\chi(\Sigma')}{\chi(\Sigma)} \cdot \int_G f(H)\, d\mu_{\mathrm{im}\,\rho_{\Sigma'}}(H) = \int_G f(H)\, d\widetilde{\Phi}(\mathfrak{r})(H)
\end{aligned}$$

as $n \to \infty$.

In general, let $\mathfrak{r}_n = ([\rho^{(n)}_{\Sigma''}])_{\Sigma'' \in c(\sigma_n)} \subseteq \widehat{\mathcal{T}}(\Sigma)$ converge to $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\sigma)}$ as $n \to \infty$. Then $\sigma_n \subseteq \sigma$ for large n. Because the simplex σ has only finitely many faces we may assume without loss of generality[1] that $\sigma_n = \sigma'$ for large n. Applying our previous discussion

[1] Just pass to a subsequence and treat every face separately.

to every component $\Sigma'' \in c(\sigma')$ we obtain

$$\begin{aligned}
&\int_{\mathrm{Sub}(G)} f(H)\, d\widetilde{\Phi}(\mathfrak{r})(H) \\
&= \sum_{\Sigma' \in c(\sigma)} \frac{\chi(\Sigma')}{\chi(\Sigma)} \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\mathrm{im}\,\rho_{\Sigma'}}(H) \\
&= \sum_{\Sigma'' \in c(\sigma')} \frac{\chi(\Sigma'')}{\chi(\Sigma)} \sum_{\substack{\Sigma' \in c(\sigma) \\ \Sigma' \subseteq \Sigma''}} \frac{\chi(\Sigma')}{\chi(\Sigma'')} \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\mathrm{im}\,\rho_{\Sigma'}}(H) \\
&= \sum_{\Sigma'' \in c(\sigma')} \frac{\chi(\Sigma'')}{\chi(\Sigma)} \lim_{n\to\infty} \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\mathrm{im}\,\rho^{(n)}_{\Sigma''}}(H) \\
&= \lim_{n\to\infty} \int_{\mathrm{Sub}(G)} f(H)\, d\widetilde{\Phi}(\mathfrak{r}_n)(H)
\end{aligned}$$

for every $f \in C(\mathrm{Sub}(G))$. This shows that $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is continuous.

Recall that the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ is a quotient of the augmented Teichmüller space, whence the induced map $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is continuous. In turn the moduli space of parts $|\widehat{\mathcal{M}}|(\Sigma)$ is a quotient of the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$, such that the induced map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is continuous, too.

Next, we want to show that $\widehat{\Phi}$ is surjective, i.e. $\widehat{\Phi}(\widehat{\mathcal{M}}(\Sigma)) = \overline{\iota(\mathcal{M}(\Sigma))}$. Since $\widehat{\mathcal{M}}(\Sigma)$ is compact and $\widehat{\Phi}$ is continuous the image $\widehat{\Phi}(\widehat{\mathcal{M}}(\Sigma))$ is compact and contains $\iota(\mathcal{M}(\Sigma)) = \widehat{\Phi}(\mathcal{M}(\Sigma))$. Because $\mathrm{IRS}(G)$ is Hausdorff, compact subsets are closed such that $\overline{\iota(\mathcal{M}(\Sigma))} \subseteq \widehat{\Phi}(\widehat{\mathcal{M}}(\Sigma))$. Vice versa, let $\mu \in \overline{\iota(\mathcal{M}(\Sigma))}$ and let $[[\rho_n]]_{n\in\mathbb{N}} \subseteq \mathcal{M}(\Sigma)$ be a sequence such that $\iota([[\rho_n]]) = \mu_{\mathrm{im}\,\rho_n}$ converges to μ as $n \to \infty$. Because $\widehat{\mathcal{M}}(\Sigma)$ is compact there is a convergent subsequence $[[\rho_{n_k}]] \to [\mathfrak{r}] \in \widehat{\mathcal{M}}(\Sigma)$ as $k \to \infty$. Because $\widehat{\Phi}$ is continuous it follows that

$$\mu = \lim_{k\to\infty} \widehat{\Phi}([[\rho_{n_k}]]) = \widehat{\Phi}([\mathfrak{r}]) \in \widehat{\Phi}(\widehat{\mathcal{M}}(\Sigma)).$$

Because $|\widehat{\mathcal{M}}|(\Sigma)$ is compact and $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is Hausdorff, it will suffice to prove that $\Phi : |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is a continuous bijection, in order to show that Φ is a homeomorphism. We have already seen that $\widehat{\Phi}$ is a continuous surjection, whence the induced map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is a continuous surjection, too. Thus, we only need to show that Φ is injective.

To this end let $\xi_1, \xi_2 \in |\widehat{\mathcal{M}}|(\Sigma)$, such that $\Phi(\xi_1) = \Phi(\xi_2)$. We have that

$$\Phi(\xi_1) = \sum_{X \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')} \frac{\chi(X)}{\chi(\Sigma)} \cdot \xi_1(X) \cdot \mu_X = \sum_{i=1}^{l} \frac{\chi(X_i)}{\chi(\Sigma)} \cdot \xi_1(X_i) \cdot \mu_{X_i},$$
$$\Phi(\xi_2) = \sum_{X \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')} \frac{\chi(X)}{\chi(\Sigma)} \cdot \xi_2(X) \cdot \mu_X = \sum_{j=1}^{m} \frac{\chi(Y_j)}{\chi(\Sigma)} \cdot \xi_2(Y_j) \cdot \mu_{Y_j},$$

for some pairwise non-isometric hyperbolic surfaces $X_1, \ldots, X_l$, and some pairwise non-isometric hyperbolic surfaces $Y_1, \ldots, Y_m$. By Lemma 4.1.4 the IRSs $\{\mu_{X_1}, \ldots, \mu_{X_l}\}$ and $\{\mu_{Y_1}, \ldots, \mu_{Y_m}\}$ are linearly independent, respectively. Thus, $\Phi(\xi_1) = \Phi(\xi_2)$ implies that $m = l$, $X_i = Y_i$ (up to relabelling), and

$$\frac{\chi(X_i)}{\chi(\Sigma)} \cdot \xi_1(X_i) = \frac{\chi(X_i)}{\chi(\Sigma)} \cdot \xi_2(X_i)$$

for all $i = 1, \ldots, m$. Thus, $\xi_1(X_i) = \xi_2(X_i)$ for all $i = 1, \ldots, m$. Since both ξ_1 and ξ_2 are zero for all other hyperbolic surfaces, it follows that $\xi_1 = \xi_2$. This shows that Φ is injective and that $\Phi \colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is a homeomorphism.

The upper bound on the number of elements in each fiber of $\widehat{\Phi}$ now follows from $\widehat{\Phi} = \Phi \circ Q$ and Proposition 2.6.15. Indeed, we have that

$$\#\widehat{\Phi}^{-1}(\mu) = \#Q^{-1}(\Phi^{-1}(\mu)) \leq B(\Sigma) = \binom{3|\chi|}{p} \cdot \frac{(2(|\chi| + g - 1))!}{(|\chi| + g - 1)! \cdot 2^{(|\chi|+g-1)}}$$

for every $\mu \in \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$. □

We conclude our current discussion with a minimal example that shows that there are points $\mu \in \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$, whose preimage $\widehat{\Phi}^{-1}(\mu) \subseteq \widehat{\mathcal{M}}(\Sigma)$ consists of more than one point.

Example 4.2.7. Let $\Sigma = \Sigma_{2,0}$ be a closed surface of genus two. Let $\sigma_1 = \{\alpha_1, \beta_1, \gamma_1\} \subseteq \mathcal{C}(\Sigma)$ be a pants decomposition of Σ where α_1, γ_1 are non-separating curves and β_1 is separating. Further, let $\sigma_2 = \{\alpha_2, \beta_2, \gamma_2\} \subseteq \mathcal{C}(\Sigma)$ be a pants decomposition where $\alpha_2, \beta_2, \gamma_2$ are all non-separating. Recall that the Teichmüller space $\mathcal{T}(\Sigma_{0,3}) = \{[\rho_0]\}$ of a thrice-punctured sphere is just one point. We consider the elements $\mathfrak{r}_1 = ([\rho_0])_{\Sigma' \in c(\sigma_1)} \in \mathcal{T}_{\sigma_1}(\Sigma)$, $\mathfrak{r}_2 = ([\rho_0])_{\Sigma' \in c(\sigma_2)} \in \mathcal{T}_{\sigma_1}(\Sigma)$ and their images $[\mathfrak{r}_1], [\mathfrak{r}_2] \in \widehat{\mathcal{M}}(\Sigma)$; see Figure 4.1. Clearly, $[\mathfrak{r}_1] \neq [\mathfrak{r}_2]$ because σ_1 and σ_2 are not in the same mapping class group orbit in $\mathcal{C}(\Sigma)$.

However,

$$\widehat{\Phi}([\mathfrak{r}_1]) = \mu_{\Gamma_0} = \widehat{\Phi}([\mathfrak{r}_2]) \in \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma),$$

where $\Gamma_0 = \operatorname{im} \rho_0$.

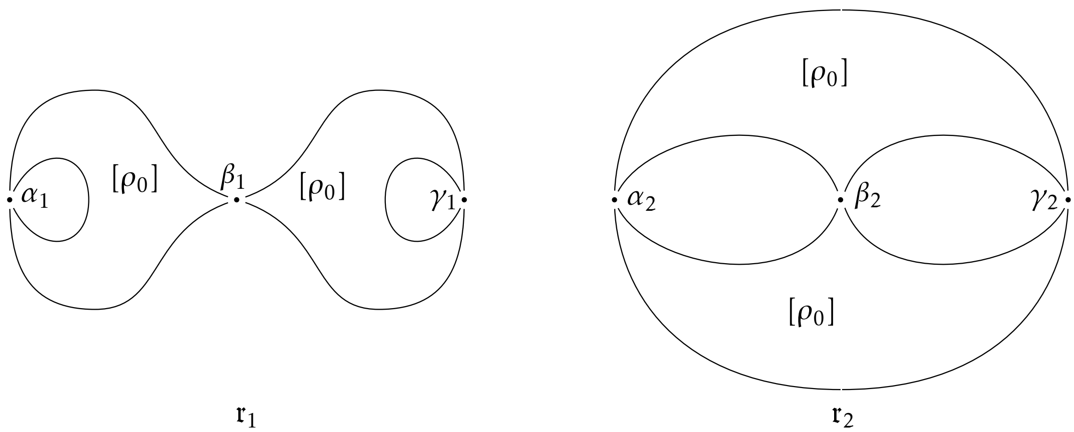


Figure 4.1: Both points $[\mathfrak{r}_1], [\mathfrak{r}_2] \in \widehat{\mathcal{M}}(\Sigma)$ are mapped to the same invariant random subgroup $\mu_{\Gamma_0} = \widehat{\Phi}([\mathfrak{r}_1]) = \widehat{\Phi}([\mathfrak{r}_2]) \in \mathrm{IRS}(G)$.

5 Limits of Invariant Random Subgroups

In this chapter we will use the embedding $\iota\colon \mathcal{M}(\Sigma) \hookrightarrow \mathrm{IRS}(G)$ of the moduli space and Theorem 4.2.2 to explore limits of IRSs of $G = \mathrm{PSL}(2,\mathbb{R})$.

In section 5.1 we will use Theorem 4.2.2 to give a geometric construction of a convex compact subset of $\mathrm{IRS}(G)$ with dense extreme points. Then we define the genus g Weil–Petersson IRSs $\mu_g^{\mathrm{WP}} \in \mathrm{IRS}(G)$ and show that they converge to the trivial IRS $\delta_{\{\mathrm{id}\}} \in \mathrm{IRS}(G)$ as g tends to infinity in section 5.2.

5.1 A Convex Compact Subset with Dense Extreme Points

Given a real vector space V and a convex subset $C \subseteq V$ we can consider its *set of extreme points* $\mathrm{ext}(C)$. An *extreme point* of C is a point $x \in C$ such that if x lies on the line segment between two points $x_1, x_2 \in C$ then $x = x_1$ or $x = x_2$, i.e. if there are $x_1, x_2 \in C$ and $t \in [0,1]$ such that $x = (1-t)x_1 + tx_2$ then $x = x_1$ or $x = x_2$. For example, every Euclidean ball $B \subseteq \mathbb{R}^{n+1}$ is convex and its set of extreme points is given by its boundary sphere $\mathrm{ext}(B) = \partial B \cong \mathbb{S}^n$.

Extreme points of convex subsets of infinite dimensional vector spaces arise naturally in Ergodic Theory: Let G be a locally compact second countable group acting continuously on a compact metric space X. Then the set of G-invariant probability measures $\mathrm{Prob}(X)^G$ is a convex and compact subset of the dual $C(X)^*$ via the Riesz Representation Theorem. Its extreme points $\mathrm{ext}(\mathrm{Prob}(X)^G)$ are precisely the ergodic measures for the action $G \curvearrowright X$ [EW11, Theorem 8.4].

In our case, where $G = \mathrm{PSL}(2,\mathbb{R})$, $X = \mathrm{Sub}(G)$ and $G \curvearrowright \mathrm{Sub}(G)$ via conjugation, the ergodic IRSs are exactly the extreme points of $\mathrm{IRS}(G) \subseteq C(\mathrm{Sub}(G))^*$. In particular, for every lattice $\Gamma \leq G$ the IRS μ_Γ is an extreme point; see Lemma 4.1.3.

As an application of Theorem 4.2.2 we will show in this section that there is a convex compact subset of $\mathrm{IRS}(G)$ with dense extreme points. Such a subset has already been found by Bowen [Bow15]. In fact, he proved that $\mathrm{IRS}(G)$ contains the Poulsen simplex, which is the unique metrizable Choquet simplex with dense extreme points [LOS78].

In order to streamline our notation we will denote $\mathcal{M}_{g,p} = \mathcal{M}(\Sigma_{g,p})$, where $\Sigma_{g,p}$ is the oriented surface of genus g with p punctures. We will drop the index p if $p = 0$. Moreover, we will use the embedding $\iota\colon \mathcal{M}_{g,p} \hookrightarrow \mathrm{IRS}(G)$ to tacitly think of $\mathcal{M}_{g,p}$ as a subset of $\mathrm{IRS}(G)$. In this context we will denote by $\overline{\mathcal{M}_{g,p}} = \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma_{g,p})$ its closure in $\mathrm{IRS}(G)$, i.e. its IRS compactification.

We will obtain a subset with dense extreme points in $\mathrm{IRS}(G)$ by considering the totality of all moduli spaces $\mathcal{M}_{g,p} \subseteq \mathrm{IRS}(G)$ of hyperbolic surfaces *with at least two punctures*. More precisely, we say that a *topological type* (g,p) is *admissible* if $p \geq 2$ and $\chi(\Sigma_{g,p}) = 2-2g-p < 0$, and denote the set of all admissible topological types by A.

Proposition 5.1.1. *The subset*

$$\overline{\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}} \subseteq \mathrm{IRS}(G),$$

is convex, compact and its extreme points are dense.

We will need the following lemma, whose proof relies on Theorem 4.2.2.

Lemma 5.1.2. *Let $m \geq 1$, and let $\mu_{X_i} \in \mathcal{M}_{g_i,p_i}$ for some admissible $(g_i,p_i) \in A$, $i = 1,\dots,m$. Further, let $\alpha_1,\dots,\alpha_m \in \mathbb{Q}_{\geq 0}$ be non-negative rational numbers summing to one, $\sum_{i=1}^m \alpha_i = 1$.*

Then there is an admissible $(\widehat{g},\widehat{p}) \in A$ such that

$$\sum_{i=1}^m \alpha_i \mu_{X_i} \in \overline{\mathcal{M}_{\widehat{g},\widehat{p}}}.$$

Proof of Lemma 5.1.2. Let us write

$$\alpha_i = \frac{k_i}{N}, \qquad i = 1,\dots,m,$$

with $N > 0, k_i \geq 0$ integers. From $\sum_{i=1}^m \alpha_i = 1$ it follows that $\sum_{i=1}^m k_i = N$.

Denote by $L = \mathrm{lcm}(|\chi(X_1)|,\dots,|\chi(X_m)|)$ the least common multiple of $|\chi(X_1)|,\dots,|\chi(X_m)|$. Then there are integers $b_i > 0$, $i = 1,\dots,m$, such that

$$b_i \cdot |\chi(X_i)| = L.$$

We set $l_i := k_i \cdot b_i$, $i = 1, \dots, m$.

Using the assembly maps from section 2.6 we may now construct a nodal surface $\mathbf{X}(l_1, \dots, l_m) \in \widehat{\mathcal{M}}_{\widehat{g},\widehat{p}}$ from the parts $X_1, \dots, X_m$, as follows. First, we take l_i-many copies of each part X_i, $i = 1, \dots, m$. Then we glue their punctures forming a chain-like nodal surface as in Figure 5.1.

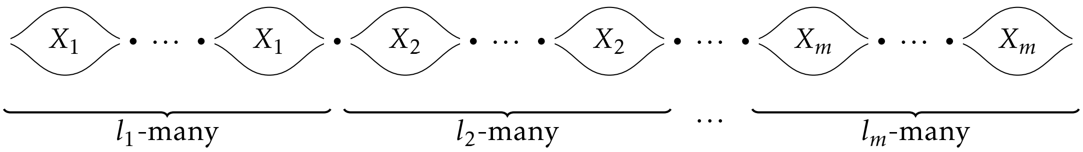


Figure 5.1: The gluing pattern for $\mathbf{X}(l_1, \dots, l_m) \in \widehat{\mathcal{M}}_{\widehat{g},\widehat{p}}$.

Note that $\widehat{p} \geq 2$ because we get at least one puncture from the first X_1 and one puncture from the last X_m. Moreover, the Euler characteristic of the underlying topological surface $\Sigma(l_1, \dots, l_m)$ is

$$\chi(\Sigma(l_1, \dots, l_m)) = \sum_{i=1}^{m} l_i \cdot \chi(X_i) < 0.$$

Hence, $(\widehat{g}, \widehat{p}) \in A$.

By Theorem 4.2.2 we obtain an element of the IRS compactification $\overline{\mathcal{M}_{\widehat{g},\widehat{p}}} = \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma_{\widehat{g},\widehat{p}})$ via the map $\widehat{\Phi} \colon \widehat{\mathcal{M}}_{\widehat{g},\widehat{p}} \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma_{\widehat{g},\widehat{p}})$,

$$\mu_{\mathbf{X}(l_1,\dots,l_m)} := \widehat{\Phi}(\mathbf{X}(l_1, \dots, l_m)) = \sum_{i=1}^{m} \frac{l_i \cdot \chi(X_i)}{\chi(\Sigma(l_1, \dots, l_m))} \cdot \mu_{X_i}.$$

Observe that

$$\frac{l_j \cdot \chi(X_j)}{\chi(\Sigma(l_1, \dots, l_m))} = \frac{k_j \cdot b_j \cdot |\chi(X_j)|}{\sum_{i=1}^{m} k_i \cdot b_i \cdot |\chi(X_i)|} = \frac{k_j \cdot L}{\sum_{i=1}^{m} k_i \cdot L} = \frac{k_j}{N} = \alpha_j,$$

for all $j = 1, \dots, m$.

Therefore,

$$\mu_{\mathbf{X}(l_1,\dots,l_m)} = \sum_{i=1}^{m} \alpha_i \mu_{X_i} \in \overline{\mathcal{M}_{\widehat{g},\widehat{p}}}.$$

□

Proof of Proposition 5.1.1. Let us first see that

$$\overline{\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}} = \overline{\bigcup_{(g,p)\in A} \overline{\mathcal{M}_{g,p}}}. \tag{5.1}$$

Obviously the left-hand-side is contained in the righ-hand-side. As to the other inclusion observe that $\overline{\mathcal{M}_{g',p'}} \subseteq \overline{\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}}$ for every $(g',p') \in A$. Therefore,

$$\bigcup_{(g,p)\in A} \overline{\mathcal{M}_{g,p}} \subseteq \overline{\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}},$$

which implies (5.1).

Next, we claim that

$$\overline{\mathrm{conv}}\left(\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}\right) = \overline{\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}}. \tag{5.2}$$

Note that

$$\overline{\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}} \subseteq \overline{\mathrm{conv}}\left(\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}\right)$$

because the right-hand-side is a closed set containing $\bigcup_{(g,p)\in A} \mathcal{M}_{g,p}$.

Let $\beta_1,\dots,\beta_m \in \mathbb{R}_{\geq 0}$ with $\sum_{i=1}^m \beta_i = 1$, and $\mu_{X_i} \in \mathcal{M}_{g_i,p_i}$ with $(g_i,p_i) \in A$, for $i = 1,\dots,m$. Further, let $\mathcal{U} \subseteq \mathrm{IRS}(G)$ be an open neighborhood of $\mu := \sum_{i=1}^m \beta_i \mu_{X_i}$. Because $\mathcal{U}$ is open, we can find rational numbers $\alpha_1,\dots,\alpha_m \in \mathbb{Q}_{\geq 0}$ with $\sum_{i=1}^m \alpha_i = 1$ such that

$$\sum_{i=1}^m \alpha_i \mu_{X_i} \in \mathcal{U}.$$

By Lemma 5.1.2, there is $(\widehat{g},\widehat{p}) \in A$ such that

$$\sum_{i=1}^m \alpha_i \mu_{X_i} \in \overline{\mathcal{M}_{\widehat{g},\widehat{p}}}.$$

The above argument shows that every open neighborhood of an element in

$\operatorname{conv}\left(\bigcup_{(g,p)\in A}\mathcal{M}_{g,p}\right)$ intersects $\bigcup_{(g,p)\in A}\overline{\mathcal{M}_{g,p}}$. Thus, by (5.1),

$$\operatorname{conv}\left(\bigcup_{(g,p)\in A}\mathcal{M}_{g,p}\right) \subseteq \overline{\bigcup_{(g,p)\in A}\overline{\mathcal{M}_{g,p}}} = \overline{\bigcup_{(g,p)\in A}\mathcal{M}_{g,p}}.$$

Taking the closure

$$\overline{\operatorname{conv}}\left(\bigcup_{(g,p)\in A}\mathcal{M}_{g,p}\right) \subseteq \overline{\bigcup_{(g,p)\in A}\mathcal{M}_{g,p}},$$

we obtain (5.2).

We conclude that $\overline{\bigcup_{(g,p)\in A}\mathcal{M}_{g,p}} \subseteq \mathrm{IRS}(G)$ is convex and compact. Moreover, every $\mu_X \in \mathcal{M}_{g,p}$ is ergodic by Lemma 4.1.3, such that

$$\bigcup_{(g,p)\in A}\mathcal{M}_{g,p} \subseteq \operatorname{ext}\left(\overline{\bigcup_{(g,p)\in A}\mathcal{M}_{g,p}}\right).$$

Therefore, the extreme points are dense. □

5.2 The Weil–Petersson IRS

Recall that the Weil–Petersson volume form gives rise to a *finite* Borel measure $\mathrm{vol}_g^{\mathrm{WP}}$ on the moduli space $\mathcal{M}_g$ of closed genus g hyperbolic surfaces; see section 2.4. By normalizing $\mathrm{vol}_g^{\mathrm{WP}}$ we obtain a Borel probability measure $\mathbb{P}_g^{\mathrm{WP}} := \mathrm{vol}_g^{\mathrm{WP}}/V_g$, where we denote by $V_g := \mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g)$ the Weil–Petersson volume of $\mathcal{M}_g$. In this way the moduli space $\mathcal{M}_g$ becomes a probability space, which gives rise to the notion of a $\mathbb{P}_g^{\mathrm{WP}}$-random hyperbolic surface.

In section 4.1 we used the embedding

$$\iota_g\colon \mathcal{M}_g \longrightarrow \mathrm{IRS}(G), \quad X = \Gamma\backslash\mathbb{H}^2 \longmapsto \mu_X = \mu_\Gamma,$$

to define the IRS compactification $\overline{\iota_g(\mathcal{M}_g)} \subseteq \mathrm{IRS}(G)$ of the moduli space $\mathcal{M}_g$. Because $\mathcal{M}_g$ is a probability space, we can also think of ι_g as a random variable taking values in $\mathrm{IRS}(G)$. By the Riesz Representation Theorem $\mathrm{IRS}(G)$ is a convex compact subset of the dual space $C(\mathrm{Sub}(G))^*$ equipped with its weak*-topology, such that it makes sense to consider the expected value $\mathbb{E}(\iota_g) \in \mathrm{IRS}(G)$. This leads to the following definition.

Definition 5.2.1. For every $g \geq 2$ we define the *(genus g) Weil–Petersson IRS* as

$$\mu_g^{\mathrm{WP}} := \mathbb{E}(\iota_g) = \int_{\mathcal{M}_g} \mu_X \, d\mathbb{P}_g^{\mathrm{WP}}(X) \in \mathrm{IRS}(G).$$

Remark 5.2.2. Alternatively, the Weil–Petersson IRS μ_g^{WP} can be interpreted as the barycenter of $\iota_g(\mathcal{M}_g) \subseteq \mathrm{IRS}(G)$ with respect to the push-forward of its Weil–Petersson measure:

$$\mu_g^{\mathrm{WP}} = \frac{1}{V_g} \int_{\mathcal{M}_g} \iota_g(X) \, d\,\mathrm{vol}_g^{\mathrm{WP}}(X).$$

We can use the geometric interpretation of IRSs supported on discrete and torsion-free subgroups $\mathrm{IRS}_{\mathrm{dtf}}(G)$ from section 1.4 to think of the Weil–Petersson IRS in terms of framed hyperbolic surfaces. Recall that if we fix a positively oriented orthonormal frame $\mathbf{e} \in \mathrm{OF}_+(\mathbb{H}^2)$ based at some point $o \in \mathbb{H}^2$, we can identify the space of discrete and torsion-free subgroups $\mathrm{Sub}_{\mathrm{dtf}}(G)$ with the space of framed hyperbolic surfaces $\mathcal{FM}(\mathbb{H}^2)$ via the homeomorphism

$$\psi\colon \mathrm{Sub}_{\mathrm{dtf}}(G) \longrightarrow \mathcal{FM}(\mathbb{H}^2), \quad \Gamma \longmapsto [\Gamma\backslash\mathbb{H}^2, d\pi_\Gamma(\mathbf{e})];$$

see Proposition 2.7.15. In particular, $\psi_*\colon \mathrm{IRS}_{\mathrm{dtf}}(G) \longrightarrow \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ is a homeomorphism onto its image.

In section 1.4 we saw that $\psi_*(\mu_X) = \lambda_X \in \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ for every $X \in \mathcal{M}_g$, where λ_X is obtained by sampling an orthonormal frame $\mathbf{f} \in \mathrm{OF}_+(X)$ uniformly at random. Recall that $\mathrm{OF}_+(X) \cong T^1X$ is a $\mathbb{S}^1$-bundle over X and carries a canonical volume form, which is locally the product of the volume form vol_X on X and the unique rotation invariant probability measure on $\mathbb{S}^1$. Identifying ν_X with its induced finite Borel measure and normalizing it, we obtain a Borel probability measure $\overline{\nu}_X := \nu_X/(2\pi(2g-2))$ on $\mathrm{OF}_+(X)$. The measure $\lambda_X = (\overline{\varphi}_X)_*(\overline{\nu}_X)$ is then obtained as the push-forward of $\overline{\nu}_X$ along the map

$$\overline{\varphi}_X\colon \mathrm{OF}_+(X) \longrightarrow \mathcal{FM}(\mathbb{H}^2), \quad \mathbf{f} \longmapsto [X, \mathbf{f}].$$

Applying the map $\psi_*\colon \mathrm{IRS}_{\mathrm{dtf}}(G) \longrightarrow \mathrm{Prob}(\mathcal{FM}(\mathbb{H}^2))$ to the Weil–Petersson IRS μ_g^{WP} we obtain

$$\lambda_g^{\mathrm{WP}} := \psi_*\left(\mu_g^{\mathrm{WP}}\right) = \psi_*\left(\int_{\mathcal{M}_g} \mu_X \, d\mathbb{P}_g^{\mathrm{WP}}(X)\right) = \int_{\mathcal{M}_g} \psi_*(\mu_X) \, d\mathbb{P}_g^{\mathrm{WP}}(X) = \int_{\mathcal{M}_g} \lambda_X \, d\mathbb{P}_g^{\mathrm{WP}}(X).$$

Thus, a λ_g^{WP}-random framed hyperbolic surface $[X,\mathbf{f}] \in \mathcal{FM}(\mathbb{H}^2)$ is obtained by first sampling a $\mathbb{P}_g^{\mathrm{WP}}$-random hyperbolic surface $X \in \mathcal{M}_g$ and then sampling a $\overline{\nu}_X$-random orthonormal frame $\mathbf{f} \in \mathrm{OF}_+(X)$.

It is natural to ask whether the sequence $(\mu_g^{\mathrm{WP}})_{g\geq 2} \subseteq \mathrm{IRS}(G)$ converges and, if so, what the limiting IRS is. In joint work with Philip Engel we answered this question in the following theorem.

Theorem 5.2.3. *The genus g Weil–Petersson IRS tends to the trivial IRS as $g \to \infty$, i.e.*

$$\mu_g^{\mathrm{WP}} \to \delta_{\{\mathrm{id}\}} \quad (g \to \infty).$$

Remark 5.2.4. This result was obtained independently and at the same time by Monk [Mon20].

In order to prove Theorem 5.2.3 we will use the following characterization of convergence to the trivial IRS.

Proposition 5.2.5 ([Abé+17, Proposition 3.2])**.** *For a discrete and torsion-free subgroup $\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G)$ we denote by $\pi_\Gamma \colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ the quotient map.*

Then a sequence of IRSs $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathrm{IRS}_{\mathrm{dtf}}(G)$ converges to the trivial IRS $\delta_{\{\mathrm{id}\}}$, if and only if

$$\mu_n\left(\left\{\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G) \,\middle|\, \mathrm{inj}_{\Gamma\backslash\mathbb{H}^2}(\pi_\Gamma(o)) \leq R\right\}\right) \to 0 \quad (n \to \infty) \tag{5.3}$$

for every $R > 0$, where $\mathrm{inj}_{\Gamma\backslash\mathbb{H}^2}(x)$ denotes the injectivity radius of $x \in \Gamma\backslash\mathbb{H}^2$.

Let us consider condition (5.3) for the Weil–Petersson IRSs μ_g^{WP}:

$$\begin{aligned}
&\mu_g^{\mathrm{WP}}\left(\left\{\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G) \,\middle|\, \mathrm{inj}_{\Gamma\backslash\mathbb{H}^2}(\pi_\Gamma(o)) \leq R\right\}\right)\\
&= \int_{\mathcal{M}_g}\int_{\mathrm{Sub}_{\mathrm{dtf}}(G)} \mathbb{1}_{[0,R]}(\mathrm{inj}_{\Gamma'\backslash\mathbb{H}^2}(\pi_{\Gamma'}(o)))\, d\mu_X(\Gamma')\, d\mathbb{P}_g^{\mathrm{WP}}(X)\\
&= \int_{\mathcal{M}_g}\int_{\mathcal{FM}(\mathbb{H}^2)} \mathbb{1}_{[0,R]}(\mathrm{inj}_{X'}(p(\mathbf{f}')))\, d\lambda_X([X',\mathbf{f}'])\, d\mathbb{P}_g^{\mathrm{WP}}(X)\\
&= \int_{\mathcal{M}_g}\int_{\mathrm{OF}_+(X)} \mathbb{1}_{[0,R]}(\mathrm{inj}_{X}(p(\mathbf{f})))\, d\overline{\nu}_X(\mathbf{f})\, d\mathbb{P}_g^{\mathrm{WP}}(X)\\
&= \frac{1}{2\pi(2g-2)} \cdot \int_{\mathcal{M}_g} \mathrm{vol}_X\left(\{x \in X \mid \mathrm{inj}_X(x) \leq R\}\right)\, d\mathbb{P}_g^{\mathrm{WP}}(X)\\
&= \frac{1}{2\pi(2g-2)} \cdot \int_{\mathcal{M}_g} \mathrm{vol}_X(X_{\leq R})\, d\mathbb{P}_g^{\mathrm{WP}}(X).
\end{aligned}$$

Here we denoted by $p\colon \mathrm{OF}_+(X) \longrightarrow X$ the map, that sends every orthonormal frame $\mathbf{f} \in \mathrm{OF}_+(X)_x$ to its base point $x \in X$, and used the fact that $p_*(\overline{\nu}_X) = \mathrm{vol}_X/(2\pi(2g-2))$.

Therefore, we will need to control the size of the R-thin part of $X \in \mathcal{M}_g$. This is achieved in terms of the number of geodesic loops of length $\leq 2R$ on X by the following lemma.

Definition 5.2.6. We denote by $N(X,L)$ the number of simple closed geodesics of length $\leq L$ for $X \in \mathcal{M}_g$.

Lemma 5.2.7. *Let $g \geq 2$, let $L > 0$, and let $X \in \mathcal{M}_g$. Then*

$$\mathrm{vol}_X(X_{\leq L/2}) = \mathrm{vol}_X(\{x \in X \mid \mathrm{inj}_X(x) \leq \tfrac{L}{2}\}) \leq 4N(X,L)\sinh(\tfrac{L}{2}).$$

Proof. Let $N = N(X,L)$ and let $\{\alpha_1,\dots,\alpha_N\}$ be the set of closed geodesics in X of length less than or equal to L. Denote by $\pi\colon \mathbb{H}^2 \longrightarrow X$ the universal covering. We lift every geodesic $\alpha_i \subseteq X$ to a geodesic $\widetilde{\alpha}_i \subseteq \mathbb{H}^2$ and denote by $\gamma_i \in \mathrm{Isom}^+(\mathbb{H}^2)$ the corresponding hyperbolic deck transformation, $i = 1,\dots,N$. We obtain sets

$$\widetilde{T}_i := \{x \in \mathbb{H}^2 \mid d(x,\gamma_i(x)) \leq L\}$$

that project to neighborhoods $T_i := \pi(\widetilde{T}_i) \subseteq X$ about the closed geodesics α_i, $i = 1,\dots,N$. Up to applying an isometry we can assume that $\widetilde{\alpha}_i$ coincides with the imaginary axis and the (conjugated) hyperbolic deck transformation has the form $\gamma_i(z) = e^{\ell_i} \cdot z$ where $\ell_i := \ell_X(\alpha_i)$ is its translation length. The set $\widetilde{T}_i$ is then a mirror-symmetric neighborhood about the imaginary axis and a fundamental domain for the $\langle\gamma_i\rangle$-action on $\widetilde{T}_i$ is given by

$$\bar{F}_i := \{z = x + iy \in \mathbb{H}^2 \mid 1 \leq |z| \leq e^{\ell_i}, d(z,\gamma_i(z)) \leq L\}.$$

Thus, the imaginary axis divides the set $\bar{F}_i$ into two sets isometric to

$$F_L = \{z = x + iy \in \mathbb{H}^2 \mid x \geq 0, 1 \leq |z| \leq e^{\ell_i}, d(z,\gamma_i(z)) \leq L\}$$

as defined in Lemma 3.1.8 (ii). Hence,

$$\mathrm{vol}_X(T_i) \leq \mathrm{vol}_{\mathbb{H}^2}(\widetilde{T}_i) = 2\,\mathrm{vol}_{\mathbb{H}^2}(F_L) \leq 4\sinh\left(\tfrac{L}{2}\right)$$

by Lemma 3.1.8 (ii).

Moreover, for every $x \in X$ with $\mathrm{inj}_X(x) \leq \frac{L}{2}$ there is an $i \in \{1,\dots N\}$ such that $x \in T_i$. Indeed, for any such $x \in X$ there is a broken geodesic loop $c_x \subseteq X$ at x of length $\ell(c_x) \leq L$.

Further, there is a unique geodesic $\alpha_x \subseteq X$ in the homotopy class of c_x. Note that $\ell(\alpha_x) \leq \ell(c_x) \leq L$, whence $\alpha_x = \alpha_i$ for some $i \in \{1, \ldots, N\}$. Now, consider the lift $\widetilde{\alpha_i} \subseteq \mathbb{H}^2$. We may lift c_x to a bi-infinite broken geodesic $\widetilde{c_x} \subseteq \mathbb{H}^2$ on which the deck transformation γ_i acts via translation. Set $\widetilde{x} := \widetilde{c_x}(0)$. Then $\pi(\widetilde{x}) = x$ and $d(\widetilde{x}, \gamma_i(\widetilde{x})) = \ell(c_x) \leq L$. Therefore $\widetilde{x} \in \widetilde{T_i}$, and $x = \pi(\widetilde{x}) \in T_i = \pi(\widetilde{T_i})$.

Altogether, we obtain

$$\mathrm{vol}_X(\{x \in X \mid \mathrm{inj}_X(x) \leq \tfrac{L}{2}\}) \leq \mathrm{vol}_X\left(\bigcup_{i=1}^{N} T_i\right) \leq 4N \sinh(\tfrac{L}{2}).$$

□

Thus, we are left to understand the number of short geodesics on $\mathbb{P}_g^{\mathrm{WP}}$-random hyperbolic surfaces of large genus. This problem has already been solved by Mirzakhani [Mir13].

Definition 5.2.8 ([Mir13, Section 4.4]). We define the following subset of $\mathcal{M}_g$

$$\mathcal{A}_g := \left\{X \in \mathcal{M}_g \,|\, N(X, \log(g)/3) \leq g^{1/3+1/4}\right\}.$$

Lemma 5.2.9 ([Mir13, Section 4.4, (4.14)]). *Then*

$$\mathbb{P}_g^{\mathrm{WP}}(\mathcal{M}_g \setminus \mathcal{A}_g) = \frac{\mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g \setminus \mathcal{A}_g)}{\mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g)} \to 0 \qquad (g \to \infty).$$

Proof. This would follow from

$$\frac{\mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g \setminus \mathcal{A}_g)}{\mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g)} = \mathcal{O}\left(g^{-1/4}\right),$$

which is precisely statement (4.14) in [Mir13]. However, there is a missing factor $\log(g)$ and the correct statement should be

$$\frac{\mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g \setminus \mathcal{A}_g)}{\mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g)} = \mathcal{O}\left(\log(g) g^{-1/4}\right). \tag{5.4}$$

Nevertheless, this expression still goes to 0 as $g \to \infty$.

Following Mirzakhani's original approach we will give a proof of (5.4), now:

Let $X \in \mathcal{M}_g$ and let $L > 0$. We denote by $N_0(X,L)$ the number of *non-separating* simple closed geodesics of length $\leq L$ in X. Likewise, we denote by $N_k(X,L)$ the number of *separating* simple closed geodesics $\alpha \subseteq X$ of lenght $\leq L$ such that α separates X into a surface of genus k and a surface of genus $g-k$ for every $1 \leq k \leq g-1$.

Every geodesic is either non-separating or separating. In the separating case it divides X in two subsurfaces of genus k and $g-k$ for some $1 \leq k \leq \lfloor g/2 \rfloor$, respectively. Therefore, we have

$$N(X,L) = \sum_{k=0}^{\lfloor g/2 \rfloor} N_k(X,L). \tag{5.5}$$

Mirzakhani [Mir13] computed the Weil–Petersson integrals of the functions $N_k(X,L)$ as $g \to \infty$: For every $1 \leq k \leq g-1$ we have that

$$\int_{\mathcal{M}_g} N_0(X,L)\, d\,\mathrm{vol}^{\mathrm{WP}}(X) = \mathcal{O}\Big(\big(e^L - 1\big) \cdot L \cdot V_g\Big), \tag{5.6}$$

$$\int_{\mathcal{M}_g} N_k(X,L)\, d\,\mathrm{vol}^{\mathrm{WP}}(X) = \mathcal{O}\Big(\big(e^L - 1\big) \cdot L \cdot V_{k,1} \cdot V_{g-k,1}\Big), \tag{5.7}$$

where we use the notation $V_{g,n} := \mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_{g,n})$ and the implied constants are independent of k, L and g. Indeed, assertion (5.6) is precisely [Mir13, (4.1) in Lemma 4.1 with $k=1$], and assertion (5.7) is precisely [Mir13, (4.3) in Lemma 4.1]. Moreover, we will need

$$\sum_{k=1}^{\lceil g/2 \rceil} V_{k,1} \cdot V_{g-k,1} = \mathcal{O}\left(\frac{V_g}{g}\right), \tag{5.8}$$

which is [Mir13, (3.19) with $r=0$].

Combining (5.7) and (5.8) we get

$$\sum_{k=1}^{\lfloor g/2 \rfloor} \int_{\mathcal{M}_g} N_k(X,L)\, d\,\mathrm{vol}_g^{\mathrm{WP}}(X) = \mathcal{O}\left(\big(e^L - 1\big) L \frac{V_g}{g}\right).$$

In conjunction with (5.5) and (5.6) this yields

$$\int_{\mathcal{M}_g} N(X,L)\, d\,\mathrm{vol}_g^{\mathrm{WP}}(X) = \mathcal{O}\Big(\big(e^L - 1\big) L V_g\Big). \tag{5.9}$$

Notice that we obtain $\mathcal{O}\big((e^L-1) L V_g\big)$ instead of $\mathcal{O}\big((e^L-1) V_g\big)$, which is where the missing factor of $\log(g)$ will come from.

By Markov's inequality we obtain

$$\begin{aligned}
\mathrm{vol}_g^{\mathrm{WP}}\left(\mathcal{M}_g \setminus \mathcal{A}_g\right) &= \mathrm{vol}_g^{\mathrm{WP}}\left(\left\{X \in \mathcal{M}_g \,|\, N(X, \log(g)/6) > g^{1/3+1/4}\right\}\right) \\
&\leq \frac{1}{g^{1/3+1/4}} \int_{\mathcal{M}_g} N(X, \log(g)/3)\, d\,\mathrm{vol}_g^{\mathrm{WP}}(X) \\
&= \mathcal{O}\left(\frac{\left(e^{\log(g)/3} - 1\right)(\log(g)/3)V_g}{g^{1/3+1/4}}\right) = \mathcal{O}\left(\log(g) g^{-1/4} V_g\right).
\end{aligned}$$

Dividing both sides by $V_g = \mathrm{vol}_g^{\mathrm{WP}}(\mathcal{M}_g)$ yields (5.4). □

We are now ready to prove Theorem 5.2.3.

Proof of Theorem 5.2.3. Let $R > 0$. Recall that we need to prove that

$$\frac{1}{2\pi(2g-2)} \cdot \int_{\mathcal{M}_g} \mathrm{vol}_X(X_{\leq R})\, d\mathbb{P}_g^{\mathrm{WP}}(X) \to 0 \qquad (g \to \infty).$$

We split the integral into

$$\begin{aligned}
&\frac{1}{2\pi(2g-2)} \cdot \int_{\mathcal{M}_g} \mathrm{vol}_X(X_{\leq R})\, d\mathbb{P}_g^{\mathrm{WP}}(X) \\
= &\underbrace{\int_{\mathcal{M}_g \setminus \mathcal{A}_g} \frac{\mathrm{vol}_X(X_{\leq R})}{2\pi(2g-2)} \cdot d\mathbb{P}_g^{\mathrm{WP}}(X)}_{\leq \mathbb{P}_g^{\mathrm{WP}}(\mathcal{M}_g \setminus \mathcal{A}_g) \to 0} + \frac{1}{2\pi(2g-2)} \cdot \int_{\mathcal{A}_g} \mathrm{vol}_X(X_{\leq R})\, d\mathbb{P}_g^{\mathrm{WP}}(X).
\end{aligned}$$

For the second term, observe that $\log(g)/6 \geq R$ for large enough g. By Lemma 5.2.7 we have

$$\begin{aligned}
\mathrm{vol}_X(X_{\leq R}) &\leq \mathrm{vol}_X(X_{\leq \log(g)/6}) \\
&\leq 4N(X, \log(g)/3) \sinh(\log(g)/6) \\
&\leq 4g^{1/3+1/4} g^{1/6} = 4g^{3/4}
\end{aligned}$$

for every $X \in \mathcal{A}_g$. Hence,

$$\frac{1}{2\pi(2g-2)} \cdot \int_{\mathcal{A}_g} \mathrm{vol}_X(X_{\leq R})\, d\mathbb{P}_g^{\mathrm{WP}}(X) \leq \frac{4g^{3/4}}{2\pi(2g-2)} \to 0 \qquad (g \to \infty).$$

This concludes the proof of Theorem 5.2.3. □

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