

# A NOTE ON SUBGROUPS OF THE LOCH NESS MONSTER SURFACE'S MAPPING CLASS GROUP

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**ABSTRACT.** In this short note we give an elementary proof of the fact that every countable group is a subgroup of the mapping class group of the Loch Ness monster surface.

## 1. INTRODUCTION

The following theorem follows from a deep result of Aougab–Patel–Vlamis [APV21] about hyperbolic isometry groups of infinite type surfaces:

**Theorem 1.1.** *Every countable group  $G$  is a subgroup of  $\text{MCG}(L)$  where  $L$  denotes the Loch Ness monster surface.*

The purpose of this short note is to provide an elementary proof of this fact.

## 2. PROOF OF THEOREM 1.1

By the Higman–Neumann–Neumann embedding theorem [HNN49, Theorem IV] any countable group  $G$  is a subgroup of a finitely generated group  $H'$ . Thus,  $G$  is also a subgroup of the product  $H := H' \times \mathbb{Z} \times \mathbb{Z}$ .

Let  $S'$  denote a finite generating set for  $H'$ . Then  $S = \{(s', 0, 0) | s' \in S'\} \cup \{(e, 1, 0), (e, 0, 1)\} \subseteq H' \times \mathbb{Z} \times \mathbb{Z}$  is a generating set of  $H$ . Observe that the Cayley graph  $\Gamma := \text{Cay}_S(H)$  is one-ended and has degree at least 4. Therefore, the surface  $\Sigma$  that is obtained by thickening the Cayley graph  $\Gamma$  of  $H$  is one-ended, too. Moreover, one may check that  $\Sigma$  has infinite genus. By the classification of topological surfaces of infinite type [Ker23; Ric63]  $\Sigma$  is thus homeomorphic to the Loch Ness monster  $L$ ; see Figure 1.

As a subgroup of  $H$  the group  $G$  acts freely on  $\Gamma$  via homeomorphisms. This action gives rise to a free action of  $G$  on  $\Sigma \cong L$  by orientation preserving homeomorphisms,  $\varphi: G \hookrightarrow \text{Homeo}_+(L)$ . Moreover, one obtains a continuous  $G$ -equivariant surjection  $\pi: L \rightarrow \Gamma$  by projecting the thickening  $L$  to  $\Gamma$ .

For the monomorphism  $\varphi$  to pass to the mapping class group  $\text{MCG}(L) = \text{Homeo}_+(L)/\text{Homeo}_+^\circ(L)$  we will check that  $\varphi(G) \cap \text{Homeo}_+^\circ(L) = \{\text{id}\}$ . Assume to the contrary that there is a non-trivial element  $g \in G$  such that  $\varphi(g)$  is isotopic to the identity. Let  $e$  be an edge in the identity coset  $\mathbb{Z}^2$  and observe that  $ge \neq e$ . Note that  $\Gamma - \{e, ge\}$  is connected: Both cosets  $\mathbb{Z}^2$  and  $g \cdot \mathbb{Z}^2$  remain connected (as subgraphs of  $\Gamma$ ) after removing the edges  $e$  and  $ge$ , respectively. As any vertex in  $\Gamma$  can be connected to any  $\mathbb{Z}^2$ -coset via a path with edges in  $S' \subseteq H'$ , the graph  $\Gamma - \{e, ge\}$  is connected. Let  $\gamma$  be the core curve in  $\pi^{-1}(e^\circ) \cong S^1 \times (0, 1)$ . Then  $g\gamma$  needs to be isotopic to the core curve of, and contained in,  $\pi^{-1}(ge^\circ)$ . However,  $\gamma$  and  $g\gamma$  cannot be isotopic: Otherwise, a component of  $L - \{\gamma, g\gamma\}$  would be an annulus. But  $L - \{\gamma, g\gamma\}$  is connected, since  $\Gamma - \{e, ge\}$  is connected; a contradiction.  $\square$

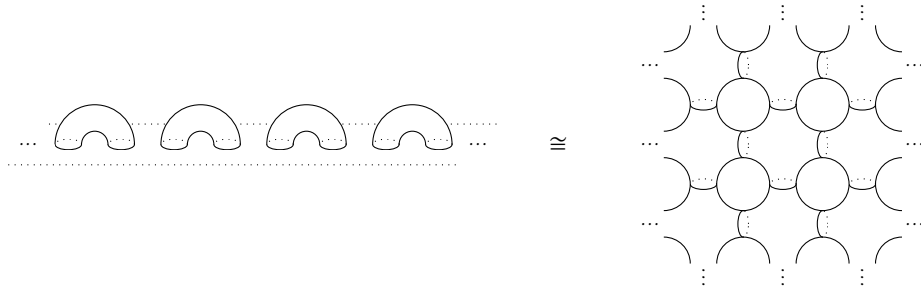


FIGURE 1. The Loch Ness monster surface  $L$  of [PS81; Ghy95], also known as the infinite prison window.

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