The IRS Compactification of Moduli Space

Part II

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Outline

Two parts:

1. Part I: Introduction to IRSs and construction of the IRS compactification of moduli space
2. **Part II: Description of the IRS compactification of moduli space**

Recap from Part I: Notation

- $G := \mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$
- $\mathrm{IRS}(G) = \mathrm{Prob}(\mathrm{Sub}(G))^G$ denotes space of invariant random subgroups of G
- Σ is an oriented topological surface possibly punctured with no boundary and negative Euler characteristic $\chi(\Sigma) < 0$
- $\mathcal{M}(\Sigma)$ denotes the moduli space of finite-area hyperbolic structures on Σ

Recap from Part I

- Any lattice $\Gamma \leq G$ amounts to an IRS μ_Γ.
- If ν_Γ is the unique invariant probability measure on $\Gamma \backslash G$ then $\mu_\Gamma := (\varphi_\Gamma)_*(\nu_\Gamma)$ is the push-forward along

$$\varphi_\Gamma \colon (\Gamma\backslash G, \nu_\Gamma) \longrightarrow (\mathrm{Sub}(G), \mu_\Gamma), \quad \Gamma g \longmapsto g^{-1}\Gamma g.$$

- The map

$$\iota \colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G), \quad X = \Gamma\backslash\mathbb{H}^2 \longmapsto \mu_\Gamma,$$

  is (well-defined and) a topological embedding.
- The *IRS compactification of* $\mathcal{M}(\Sigma)$ is defined as

$$\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) := \overline{\iota(\mathcal{M}(\Sigma))} \subseteq \mathrm{IRS}(G).$$

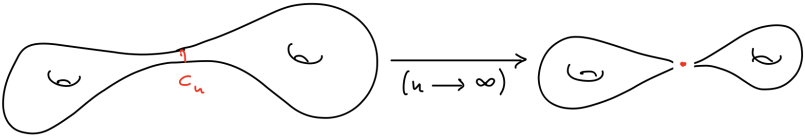
What is the IRS compactification $\overline{\mathcal{M}}^{\mathsf{IRS}}(\Sigma)$?

We will answer this question by relating $\overline{\mathcal{M}}^{\mathsf{IRS}}(\Sigma)$ to the **augmented moduli space** a.k.a. the **Deligne–Mumford compactification** $\widehat{\mathcal{M}}(\Sigma)$.

Augmented Moduli Space

Idea: Allow curves to collapse to nodes as we go to ∞ in $\mathcal{M}(\Sigma)$!

$X_n \in \mathcal{M}(\Sigma_2)$ $\qquad$ $X \in \hat{\mathcal{M}}(\Sigma_2)$



$\ell_{X_n}(c_n) \xrightarrow[(n\to\infty)]{} 0$

Augmented Moduli Space

"Definition" (Augmented Moduli Space)

The **augmented moduli space** $\widehat{\mathcal{M}}(\Sigma)$ is the space of nodal surfaces. A **nodal surface** $\mathbb{X} \in \widehat{\mathcal{M}}(\Sigma)$ comprises the following data:

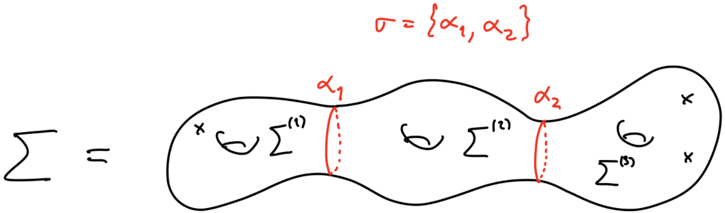
- a (possibly empty) family $\sigma \subset \Sigma$ of disjoint essential simple closed curves; and
- a finite-area hyperbolic structure $X^{(i)} \in \mathcal{M}(\Sigma^{(i)})$ for every connected component $\Sigma^{(i)} \subset \Sigma \setminus \sigma$.

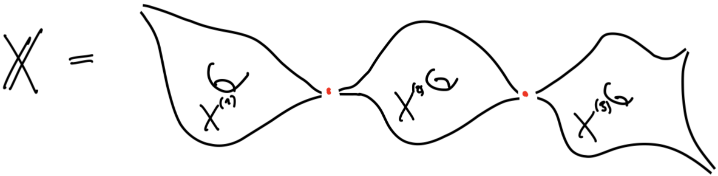
Remark

- *The family σ encodes how the $X^{(i)}$ fit together.*
- *The moduli space is a subset of the augmented moduli space and corresponds to those "nodal surfaces" with $\sigma = \varnothing$.*

Augmented Moduli Space

Example:





From $\widehat{\mathcal{M}}(\Sigma)$ to $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$

Let $\mathbb{X} \in \widehat{\mathcal{M}}(\Sigma)$ with components $X^{(i)} \in \mathcal{M}(\Sigma^{(i)}), i = 1, \dots, m$. There are lattices $\Gamma^{(i)} \leq G$ such that $X^{(i)} \cong \Gamma^{(i)} \backslash \mathbb{H}^2$.

We define

$$\mu_{\mathbb{X}} := \sum_{i=1}^{m} \underbrace{\frac{\chi(\Sigma^{(i)})}{\chi(\Sigma)}}_{\text{“proportion of area of } \Sigma^{(i)} \text{ in } \Sigma\text{”}} \cdot \mu_{\Gamma^{(i)}}.$$

Remark

Note that

$$\mathrm{vol}(X^{(i)}) = -2\pi \cdot \chi(\Sigma^{(i)}), \text{ and}$$

$$\chi(\Sigma) = \sum_{i=1}^{m} \chi(\Sigma^{(i)}).$$

Thus $\mu_{\mathbb{X}} \in \mathrm{IRS}(G)$ is a convex combination of IRSs.

Description of the IRS compactification

Theorem (K [Kri20])

The map

$$\Phi\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma), \quad \mathbb{X} \longmapsto \mu_{\mathbb{X}}.$$

is a finite-to-one continuous surjection extending the embedding $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$.

Moreover, there is an upper bound $B(\Sigma) > 0$ *on the cardinalities of its fibers* $\#\Phi^{-1}(\mu) \leq B(\Sigma)$, $\forall \mu \in \overline{\mathcal{M}}^{IRS}(\Sigma)$, *given by*

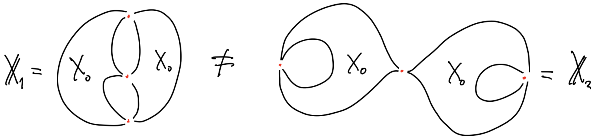
$$B(\Sigma) := \binom{3\,|\chi|}{p} \cdot \frac{(2\,(|\chi| + g - 1))!}{(|\chi| + g - 1)! \cdot 2^{(|\chi|+g-1)}},$$

where $\chi = \chi(\Sigma)$, $g = g(\Sigma)$, *and* $p = p(\Sigma)$ *denote the Euler characteristic, the number of punctures, and the genus of* Σ, *respectively.*

Example: Two nodal surfaces with the same image

Consider the three-punctured sphere $\Sigma_{0,3}$ and recall that $\mathcal{M}(\Sigma_{0,3}) = \{X_0 = \Gamma_0 \backslash \mathbb{H}^2\}$ has just one point.

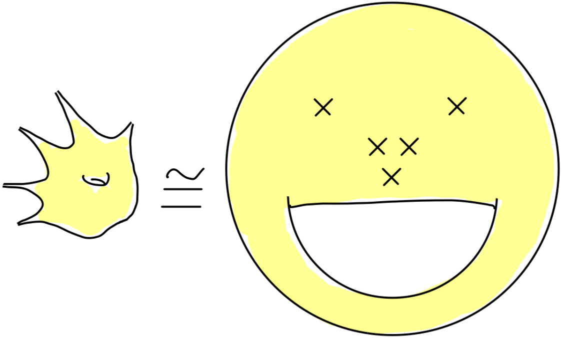
Gluing two copies of X_0 in two different ways we obtain two different points $\mathbb{X}_1, \mathbb{X}_2 \in \widehat{\mathcal{M}}(\Sigma_2)$:



However,

$$\Phi(\mathbb{X}_1) = \sum_{i=1}^{2} \frac{\chi(\Sigma_{0,3})}{\chi(\Sigma_2)} \cdot \mu_{\Gamma_0} = \mu_{\Gamma_0} = \Phi(\mathbb{X}_2).$$

Thank you!



References

Y. Krifka. *On the IRS compactification of moduli space*. 2020. arXiv: 2002.02279 [math.GT].