ON THE IRS COMPACTIFICATION OF MODULI SPACE

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Abstract. In 2015 Gelander introduced a new compactification of the moduli space of finite-area hyperbolic surfaces using invariant random subgroups (IRSs). The goal of this article is to give a complete description of this IRS compactification in terms of the classical augmented moduli space, also known as the Deligne–Mumford compactification. We construct a homeomorphism between the IRS compactification and the quotient of the augmented moduli space that is obtained by forgetting the gluing of each nodal surface's components. This quotient map is finite-to-one and we give an explicit upper bound on the cardinalities of its fibers that depends only on the topology of the underlying surface.

1. Introduction

Although implicitly studied in [BG04; Ver12], invariant random subgroups (IRSs) were introduced only recently in [AGV14; Abé+17] and they have attracted considerable research activity since then [Bow15; Gel15; Abé+18; GL18a; GL18b; LBMB18; Gel19; BLT19; Zhe19].

Definition 1.1.1. Let G be a locally compact Hausdorff group and denote by $\mathrm{Sub}(G)$ its space of closed subgroups; see section 2.8. An *invariant random subgroup (IRS) of G* is a conjugation invariant Borel probability measure μ on $\mathrm{Sub}(G)$. We denote by $\mathrm{IRS}(G)$ the set of all IRSs of G and equip it with its natural weak*-topology.

One motivation to study IRSs stems from the fact that they generalize lattices of G. In fact, many results about lattices find natural extensions to results about invariant random subgroups. For example, there is a version of the Borel density theorem, the Kazhdan–Margulis theorem and the Stuck–Zimmer rigidity theorem for invariant random subgroups; see [Gel15; Gel19].

Let us briefly recall how to associate an IRS $\mu_\Gamma \in \mathrm{IRS}(G)$ to a lattice $\Gamma \leq G$: Denote by ν_Γ the unique invariant probability measure on the homogeneous space $\Gamma\backslash G$. Then the orbit map $G \longrightarrow \mathrm{Sub}(G), g \longmapsto g^{-1}\Gamma g$ descends to the quotient $\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow \mathrm{Sub}(G)$. One

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obtains an IRS $\mu_\Gamma := (\varphi_\Gamma)_*(\nu_\Gamma)$ as the push-forward measure. Note that μ_Γ is supported on the conjugacy class $G * \Gamma$ and φ_Γ induces a homeomorphism $N(\Gamma)\backslash G \cong G * \Gamma$ where $N(\Gamma)$ denotes the normalizer of Γ in G (Lemma 4.1.3).

Gelander [Gel15] was the first to observe an interesting application of this construction to the moduli space $\mathcal{M}(\Sigma)$ of finite-area hyperbolic structures on an oriented surface Σ with negative Euler characteristic and no boundary: Every hyperbolic surface X with finite area is a quotient $X = \Gamma\backslash\mathbb{H}^2$ where $\Gamma \le G$ is a torsion-free lattice. Two such surfaces $\Gamma\backslash\mathbb{H}^2$ and $\Gamma'\backslash\mathbb{H}^2$ are isometric if and only if Γ and Γ' are conjugate. Therefore, isometry classes of finite-area hyperbolic surfaces are in one-to-one correspondence with conjugacy classes of torsion-free lattices in G. Because the IRS μ_Γ associated to a lattice Γ also depends only on its conjugacy class (Lemma 4.1.3), we obtain a well-defined map

$$\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G), \quad X = \Gamma\backslash\mathbb{H}^2 \longmapsto \mu_\Gamma.$$

In fact, we prove in Proposition 4.1.5 that ι is a topological embedding.

One of the key properties of the space of closed subgroups is that it is compact. Hence, its space of Borel probability measures $\mathrm{Prob}(\mathrm{Sub}(G))$ equipped with the weak* topology is compact, too, and it follows that $\mathrm{IRS}(G)$ is compact as a closed subset (Lemma 4.1.2). Therefore, one may take the closure $\overline{\iota(\mathcal{M}(\Sigma))}$ to obtain a compactification of the moduli space $\mathcal{M}(\Sigma)$:

Definition 1.1.2 ([Gel15, Section 3.1])**.** The *IRS compactification* $\overline{\mathcal{M}}^{IRS}(\Sigma)$ *of the moduli space* $\mathcal{M}(\Sigma)$ is defined as the closure

$$\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) := \overline{\iota(\mathcal{M}(\Sigma))} \subseteq \mathrm{IRS}(G).$$

It is natural to ask what can be said about this compactification:

Problem 1.1.3 ([Gel15, Problem 3.2])**.** Analyze the IRS compactification of $\mathcal{M}(\Sigma)$.

The objective of this article is to answer this question by relating the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ to the classical Deligne–Mumford compactification $\widehat{\mathcal{M}}(\Sigma)$.

We regard the Deligne–Mumford compactification as the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$; see section 2.5. Intuitively, a point $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ can be thought of as a nodal surface, where a family of disjoint simple closed curves σ in Σ is collapsed to nodes and every complementary component $\Sigma_i \in c(\sigma) := \pi_0(\Sigma \setminus \sigma)$, $i = 1,\ldots,m$, carries a hyperbolic metric of finite area. Each hyperbolic metric gives rise to a point $X_i \in \mathcal{M}(\Sigma_i)$ in the moduli space of the component $\Sigma_i \in c(\sigma)$, $i = 1,\ldots,m$, and we call the hyperbolic surfaces $X_1,\ldots,X_m$ the *parts* of $\mathbf{X}$. One can imagine that the parts $X_1,\ldots,X_m$ are glued according to some pairing of their punctures to form the nodal surface $\mathbf{X}$; see Figure 1. The topology on $\widehat{\mathcal{M}}(\Sigma)$ is such that a sequence of hyperbolic surfaces $(X_n)_{n\in\mathbb{N}}$ converges to a nodal surface $\mathbf{X}$ if there is a collection of curves in each X_n corresponding to the nodes of $\mathbf{X}$ such that their lengths go to 0 and the hyperbolic structures on the complementary subsurfaces converge to the respective parts of $\mathbf{X}$. More formally, $\widehat{\mathcal{M}}(\Sigma)$ is the quotient of the augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$; see section 2 for details.

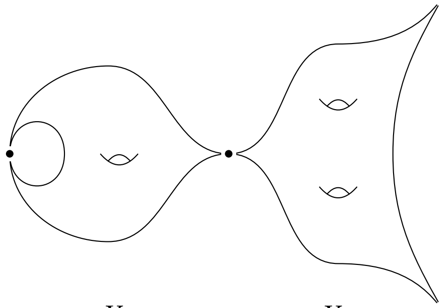

Figure 1. A nodal surface $\mathbf{X} \in \mathcal{M}(\Sigma_{4,2})$ with parts $X_1 \in \mathcal{M}(\Sigma_{1,2})$, $X_2 \in \mathcal{M}(\Sigma_{2,3})$.

Keeping this description in mind there is a natural map from the augmented moduli space to the IRS compactification

$$\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma).$$

Indeed, let $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ be a nodal surface with parts $X_i \in \mathcal{M}(\Sigma_i)$, $i = 1,\dots,m$. Every part X_i is the quotient $\Gamma_i\backslash\mathbb{H}^2$ for a torsion-free lattice and amounts to an IRS $\mu_{X_i} \coloneqq \mu_{\Gamma_i} = \iota(X_i)$, $i = 1,\dots,m$. Moreover, the area of a hyperbolic surface is a topological invariant $\mathrm{vol}_{X_i}(X_i) = -2\pi\chi(\Sigma_i)$, $i = 1,\dots,m$, and $\chi(\Sigma) = \sum_{i=1}^{m}\chi(\Sigma_i)$. In particular, it makes sense to think about the quotient $\chi(\Sigma_i)/\chi(\Sigma)$ as the proportion of area that the part X_i takes up in $\mathbf{X}$. This motivates the definition

$$\widehat{\Phi}(\mathbf{X}) \coloneqq \sum_{i=1}^{m} \frac{\chi(\Sigma_i)}{\chi(\Sigma)} \cdot \mu_{X_i} \in \mathrm{IRS}(G),$$

which is a convex combination of IRSs.

An immediate observation is that the map $\widehat{\Phi}$ forgets about how the parts $X_1,\dots,X_m$ are glued to form the nodal surface $\mathbf{X}$. Hence, it makes sense to pass to a quotient of the augmented moduli space that captures this phenomenon. To this end, we consider the set $|\widehat{\mathcal{M}}| \coloneqq \mathbb{N}_0^{\bigsqcup_{\Sigma'}\mathcal{M}(\Sigma')}$ of all functions from $\bigsqcup_{\Sigma'}\mathcal{M}(\Sigma')$ to $\mathbb{N}_0$, where the disjoint union is taken over all oriented topological surfaces Σ' with negative Euler characteristic. There is a canonical map $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|$, that counts the parts of a nodal surface with their multiplicities. We call its image $|\widehat{\mathcal{M}}|(\Sigma) \coloneqq Q(\widehat{\mathcal{M}}(\Sigma))$ equipped with the quotient topology the *moduli space of parts*; see section 2.5. Then the map $\widehat{\Phi}$ descends to a map

$\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ given by

$$\Phi(\xi) = \sum_{X \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')} \frac{\chi(X)}{\chi(\Sigma)} \cdot \xi(X) \cdot \mu_X,$$

for all $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$.

Our main result is the following theorem.

Theorem 1.1.4 (Theorem 4.2.2). *The map $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$ is a continuous surjection that extends the embedding $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$. Moreover, $\widehat{\Phi}$ descends to a homeomorphism $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$:*

$$\begin{tikzcd} & \widehat{\mathcal{M}}(\Sigma) \arrow[dl, two heads, "Q"'] \arrow[dr, two heads, "\widehat{\Phi}"] & \\ |\widehat{\mathcal{M}}|(\Sigma) \arrow[rr, "\Phi", "\sim"'] & & \overline{\mathcal{M}}^{IRS}(\Sigma) \end{tikzcd}$$

There is a uniform upper bound on the number of elements in each fiber of $\widehat{\Phi}$,

$$\#\widehat{\Phi}^{-1}(\mu) \leq B(\Sigma) := \binom{3|\chi|}{p} \cdot \frac{(2(|\chi|+g-1))!}{(|\chi|+g-1)! \cdot 2^{(|\chi|+g-1)}} \qquad \forall \mu \in \overline{\mathcal{M}}^{IRS}(\Sigma),$$

where $\chi = \chi(\Sigma)$, $g = g(\Sigma)$, and $p = p(\Sigma)$ denote the Euler characteristic, the genus, and the number of punctures of Σ, respectively.

In particular, this theorem shows that the IRS compactification is isomorphic to the moduli space of parts $|\widehat{\mathcal{M}}|(\Sigma)$.

The upper bound on the cardinality of the fibers of $\widehat{\Phi}$ is obtained by finding an upper bound for the quotient map $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|(\Sigma)$. This amounts to estimating in how many ways one may glue a given collection of hyperbolic surfaces along their punctures to obtain a nodal surface in $\widehat{\mathcal{M}}(\Sigma)$. In Proposition 2.7.2 we solve this combinatorial problem by counting the number of possible pairings of punctures. However, our bound is not sharp; see Example 2.7.3. In fact, computing the precise cardinality of the fiber $Q^{-1}(\xi)$ for some $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$ is delicate, because the isometry groups of the different parts play a role; see Example 2.7.4.

Outline. In section 2 we review some background material. We take an algebraic point of view and regard Teichmüller space as a space of conjugacy classes of admissible representations. In section 2.3 we define the moduli space of hyperbolic surfaces as the quotient of Teichmüller space with respect to the mapping class group action. Section 2.4 then introduces the augmented Teichmüller space and we describe its topology in terms of representations. The augmented Teichmüller space is used in section 2.5 to define the augmented moduli space, which is central to our discussion of the IRS compactification. In section 2.6 we discuss how to obtain a nodal surface by gluing finite-area hyperbolic surfaces at their punctures. By forgetting the gluing section 2.7 introduces the moduli space of parts. In section 2.8 we collect some properties of the space of closed subgroups. Section 2.9 then introduces the geometric topology on the set of admissible representations.

In section 3 we prove Lemma 3.1.1, which lies at the heart of our proof of Proposition 4.1.5 and Theorem 4.2.2. More precisely, we show that the characteristic function of a truncated Dirichlet domain depends continuously on the group.

In section 4 we prove our results about the IRS compactification. We show that the moduli space embeds in the space of IRSs of $\mathrm{PSL}(2,\mathbb{R})$ via the map ι in section 4.1. In section 4.2 we prove Theorem 4.2.2 relating the IRS compactification to the augmented moduli space using Lemma 3.1.1.

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2. Preliminaries

In this section we will present some background material and fix our notation.

2.1. **Notational Conventions.** We denote by Σ an oriented surface of genus g with p punctures, no boundary, and negative Euler characteristic $\chi(\Sigma) < 0$. Further, $G := \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ denotes the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2$, if not otherwise specified.

2.2. **The Teichmüller Space.** We will recall some basic notions of Teichmüller theory in the following.

Teichmüller theory is concerned with deformations of hyperbolic structures on surfaces. One way to think about this is to consider orientation preserving homeomorphisms $f : \Sigma \longrightarrow X$ between the topological surface Σ and a hyperbolic surface X. Such a homeomorphism is called a *marking* and the tuple (f, X) is called a *marked hyperbolic surface*. Furthermore, one identifies two marked hyperbolic surfaces $(f_1, X_1), (f_2, X_2)$ if there is an orientation preserving isometry $\varphi\colon X_1 \longrightarrow X_2$ that is homotopic to $f_2 \circ f_1^{-1}$. The resulting set of (equivalence classes of) marked hyperbolic surfaces is then a model for the Teichmüller space $\mathrm{Teich}(\Sigma)$ of hyperbolic structures on Σ.

Remark 2.2.1. Notice that we do not require the hyperbolic surfaces to have finite area here. Instead, some authors prefer to consider Teichmüller spaces of finite-area hyperbolic surfaces with geodesic boundary components. However, one may pass freely between the two points of view. Indeed, by cutting off the hyperbolic funnels at their waist geodesics one obtains a finite-area hyperbolic surface with geodesic boundary. Vice versa, one may always attach hyperbolic funnels along the geodesic boundary components to get an element of $\mathrm{Teich}(\Sigma)$. We chose this approach because it avoids the for our purposes unnecessary distinction between boundary components and punctures. This uniform framework will facilitate our definition of the augmented Teichmüller space in section 2.4.

Instead of working directly with marked hyperbolic surfaces we will rather use an algebraic reformulation of $\mathrm{Teich}(\Sigma)$, which arises as follows. Any marking $f\colon \Sigma \longrightarrow X$ may be lifted to a homeomorphism $\widetilde{f}\colon \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$ between the universal covers after choosing some base points. Moreover, it induces an isomorphism between the fundamental groups $\rho := f_*\colon \pi_1(\Sigma) \longrightarrow \pi_1(X)$. Recall that both fundamental groups $\pi_1(\Sigma)$ and $\pi_1(X)$ act via deck transformations on $\widetilde{\Sigma}$ and $\mathbb{H}^2$, respectively. By definition the lift $\widetilde{f}\colon \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$ is equivariant with respect to these actions, i.e. $\widetilde{f}(\gamma \cdot x) = \rho(\gamma) \cdot \widetilde{f}(x)$ for all $x \in \widetilde{\Sigma}$ and every $\gamma \in \pi_1(\Sigma)$. In this way, we may interpret $\rho\colon \pi_1(\Sigma) \longrightarrow \pi_1(X) < G = \mathrm{Isom}_+(\mathbb{H}^2)$ as a discrete and faithful representation of $\pi_1(\Sigma)$ called a *holonomy representation* of f. This motivates the following definition.

Definition 2.2.2. A discrete and faithful representation

$$\rho\colon \pi_1(\Sigma) \longrightarrow G$$

is called *admissible* if it is a holonomy representation of an orientation preserving homeomorphism $f\colon \Sigma \longrightarrow X$ where X is a hyperbolic surface. The set of all such representations is denoted by $\mathcal{R}^*(\Sigma)$. If we additionally require that the hyperbolic surface X has finite area, we denote the resulting subset by $\mathcal{R}(\Sigma)$.

Remark 2.2.3. Because a hyperbolic surface of finite type has finite area if and only if every end is a cusp, an admissible representation $\rho \in \mathcal{R}^*(\Sigma)$ is in $\mathcal{R}(\Sigma)$ if and only if $\rho(\alpha) \in G$ is parabolic for every peripheral curve $\alpha \in \pi_1(\Sigma)$.

Note that a holonomy representation of a marking homeomorphism is not unique and depends on the choice of base points and the identification of the universal cover of X with $\mathbb{H}^2$. Moreover, points in the Teichmüller space of marked hyperbolic surfaces $\mathrm{Teich}(\Sigma)$ are actually equivalence classes of marked hyperbolic surfaces. Nevertheless, it turns out that there is a one-to-one correspondence between marked hyperbolic surfaces and *conjugacy classes* of admissible representations [FM12, Proposition 10.2]. This leads us to the following algebraic model of Teichmüller space.

Definition 2.2.4. The group G acts via conjugation on $\mathcal{R}(\Sigma)$ resp. $\mathcal{R}^*(\Sigma)$ from the left, and we denote the quotients by

$$\mathcal{T}(\Sigma) := G\backslash\mathcal{R}(\Sigma) \quad \text{resp.} \quad \mathcal{T}^*(\Sigma) := G\backslash\mathcal{R}^*(\Sigma).$$

We will refer to both as *Teichmüller spaces* of Σ.

One can use this model to put a topology on Teichmüller space in the following way. We begin by defining the algebraic topology on the set of admissible representations.

Definition 2.2.5. The group $\pi_1(\Sigma)$ admits a finite generating set S, and the map

$$i\colon \mathcal{R}^*(\Sigma) \hookrightarrow G^S, \rho \longmapsto (\rho(s))_{s\in S}$$

is injective. We equip G^S with the product topology and $\mathcal{R}^*(\Sigma)$ with the initial topology with respect to the injection i. This topology does not depend on the choice of generating set and is called the *algebraic topology*.

The Teichmüller spaces $\mathcal{T}(\Sigma) = G\backslash\mathcal{R}(\Sigma)$ and $\mathcal{T}^*(\Sigma) = G\backslash\mathcal{R}^*(\Sigma)$ are then equipped with the quotient topology, respectively. In this way, both $\mathcal{T}(\Sigma)$ and $\mathcal{T}^*(\Sigma)$ are Hausdorff [Mar07, Lemma 5.1.1].

Remark 2.2.6. A sequence of representations $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$ converges to $\rho \in \mathcal{R}^*(\Sigma)$ as $n \to \infty$ with respect to this topology if and only if $\rho_n(\gamma) \to \rho(\gamma)$ as $n \to \infty$ for every $\gamma \in \pi_1(\Sigma)$.

We will need the following definitions.

Definition 2.2.7. The *translation length* of an element $g \in G$ is defined as

$$\ell(g) := \inf_{x\in\mathbb{H}^2} d(gx, x).$$

Lemma 2.2.8. *Let $g \in G$. Then*

$$\ell(g) = 2\operatorname{arcosh}\left(\tfrac{1}{2}\cdot\max\left(2, |\operatorname{tr}(g)|\right)\right).$$

In particular, $\mathcal{R}^*(\Sigma) \longrightarrow \mathbb{R}, \rho \longmapsto \ell(\rho(\gamma))$ is continuous for every $\gamma \in \pi_1(\Sigma)$.

Definition 2.2.9. Let $\alpha, \beta \in \pi_1(\Sigma)$. The *geometric intersection number $i(\alpha,\beta)$* is defined as

$$i(\alpha,\beta) := \min_{c_1,c_2} \#(c_1 \cap c_2)$$

where the minimum is taken over all loops c_1, c_2 in the free homotopy classes of α, β respectively. We say that two loops c_1 and c_2 are in *minimal position* if they realize their geometric intersection number, i.e. $i([c_1],[c_2]) = \#(c_1 \cap c_2)$.

Recall that any two closed geodesics are always in minimal position. Moreover, there is a unique closed geodesic $c \subseteq X$ in the free homotopy class of every essential closed curve $\gamma \in \pi_1(\Sigma)$. Its length $L(c)$ coincides with the translation length $\ell(\rho(\gamma))$ of the corresponding hyperbolic isometry.

It is well-known that short closed geodesics in a hyperbolic surface admit long collar neighborhoods; see [Bus10, Theorem 4.1.1]. As a consequence any two intersecting closed geodesics cannot both be short:

Lemma 2.2.10 (Collar Lemma; see [Bus10, Corollary 4.1.2])**.** *Let $\alpha, \beta \in \pi_1(\Sigma)$ such that α is primitive and $i(\alpha, \beta) > 0$. If $\rho \in \mathcal{R}^*(\Sigma)$, then*

$$\sinh\left(\frac{\ell(\rho(\alpha))}{2}\right) \cdot \sinh\left(\frac{\ell(\rho(\beta))}{2}\right) \geq 1.$$

In particular, if $(\rho_n)_{n \in \mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$ is a sequence such that $\ell(\rho_n(\alpha)) \to 0$ as $n \to \infty$, then $\ell(\rho_n(\beta)) \to \infty$ as $n \to \infty$.

The Teichmüller space $\mathcal{T}(\Sigma)$ parametrizes deformations of geometric objects: hyperbolic surfaces. Interestingly, it has a geometry in its own right. In fact, Teichmüller space carries more than one natural metric, e.g. the Teichmüller metric [Abi80; FM12], the Weil–Petersson metric [Wol07; Wol10], or the (asymmetric) Thurston metric [Thu98]. The Weil–Petersson metric is a non-complete Kähler metric on $\mathcal{T}(\Sigma)$ with negative sectional curvature.

2.3. **The Moduli Space.** In this section we will briefly recall the definition of the moduli space of finite-area hyperbolic surfaces. The material presented here is well-known and can be found in [FM12; Har77].

The moduli space of finite-area hyperbolic surfaces is the set of all isometry classes of finite-area hyperbolic surfaces. One may pass from the Teichmüller space of finite-area marked hyperbolic surfaces to the moduli space by forgetting the marking. This is achieved by taking the quotient by the mapping class group action.

Definition 2.3.1. The group

$$\mathrm{MCG}(\Sigma) := \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_\circ(\Sigma)$$

is called the *mapping class group of* Σ. Here $\mathrm{Homeo}_\circ(\Sigma)$ denotes the identity component in $\mathrm{Homeo}(\Sigma)$, i.e. all homeomorphisms isotopic to the identity. We will denote by $[f]$ the mapping class of an orientation preserving homeomorphism $f \in \mathrm{Homeo}^+(\Sigma)$.

Thus, the mapping class group acts on $\mathrm{Teich}(\Sigma)$ from the right via precomposition, $[f,X]\cdot[h] := [f\circ h, X]$, and one may define the moduli space as the resulting quotient. As in the case of Teichmüller space this construction allows for an algebraic reformulation.

Any mapping class amounts to an outer automorphism of $\pi_1(\Sigma)$ in the following way. Let $\varphi = [f] \in \mathrm{MCG}(\Sigma)$, let $p \in \Sigma$ and let $\beta\colon [0,1] \longrightarrow \Sigma$ be a path from $\beta(0) = p$ to $\beta(1) = f(p)$. Identifying $\pi_1(\Sigma) \cong \pi_1(\Sigma, p)$ we obtain an automorphism at the level of fundamental groups $f_*\colon \pi_1(\Sigma) \longrightarrow \pi_1(\Sigma)$ given by

$$f_*([c]) := [\beta \cdot (f \circ c) \cdot \beta^{-1}]$$

for every homotopy class $[c] \in \pi_1(\Sigma, p)$ of a closed loop c at p.

This construction depends on the choice of representative $f \in \varphi$ and the choice of path β. However, we obtain a well-defined outer automorphism:

Proposition and Definition 2.3.2 ([FM12, Chapter 8])**.** The map

$$\begin{aligned} \mathrm{MCG}(\Sigma) &\longrightarrow \mathrm{Out}(\pi_1(\Sigma)) = \mathrm{Aut}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma)), \\ \varphi = [f] &\longmapsto \varphi_* := [f_*] \end{aligned}$$

is a well-defined injective homomorphism. We denote its image by $\mathrm{Out}^*(\pi_1(\Sigma)) \le \mathrm{Out}(\pi_1(\Sigma))$ and its preimage under the quotient map $\mathrm{Aut}(\pi_1(\Sigma)) \longrightarrow \mathrm{Out}(\pi_1(\Sigma))$ by $\mathrm{Aut}^*(\pi_1(\Sigma)) \le \mathrm{Aut}(\pi_1(\Sigma))$. We call these (outer) automorphisms *geometric* or *admissible*.

This gives rise to the following definition of the moduli space.

Proposition and Definition 2.3.3. The group $\mathrm{Aut}^*(\pi_1(\Sigma))$ acts on $\mathcal{R}^*(\Sigma)$ from the right via precomposition, and induces a right-action of $\mathrm{Out}^*(\pi_1(\Sigma)) \cong \mathrm{MCG}(\Sigma)$ on $\mathcal{T}(\Sigma)$.

The quotient space

$$\mathcal{M}(\Sigma) := \mathcal{T}(\Sigma)/\mathrm{MCG}(\Sigma)$$

is called the *moduli space of* Σ. We will denote the $\mathrm{MCG}(\Sigma)$-equivalence class of $[\rho] \in \mathcal{T}(\Sigma)$ by $[[\rho]] \in \mathcal{M}(\Sigma)$. Emphasizing the geometric point of view one may identify an element $[[\rho]] \in \mathcal{M}(\Sigma)$ with (the isometry class of) the hyperbolic surface $X = \rho(\pi_1(\Sigma))\backslash\mathbb{H}^2$.

Moreover, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{R}(\Sigma) \arrow[r, "\mathrm{Aut}^*(\pi_1(\Sigma))"] \arrow[d, "G"] & \mathcal{R}(\Sigma)/\mathrm{Aut}^*(\pi_1(\Sigma)) \arrow[d, "G"] \\ \mathcal{T}(\Sigma) \arrow[r, "\mathrm{MCG}(\Sigma)\cong\mathrm{Out}^*(\pi_1(\Sigma))"] & \mathcal{M}(\Sigma), \end{tikzcd}$$

where every map is the quotient map with respect to the action of the annotated group.

Proof. Let $\rho \in \mathcal{R}^*(\Sigma)$ and $[\alpha] = [g_*] \in \mathrm{Out}^*(\pi_1(\Sigma))$, where $g \in \mathrm{Homeo}^+(\Sigma)$. We only need to check that $\rho \circ \alpha$ is admissible as well. Because ρ is admissible, it is a holonomy representation of an orientation preserving homeomorphism $f\colon \Sigma \longrightarrow X$. Thus $\rho \circ \alpha$ is a holonomy representation of $f \circ g\colon \Sigma \longrightarrow X$, whence $\rho \circ \alpha \in \mathcal{R}^*(\Sigma)$. $\square$

It turns out that the mapping class group action $\mathrm{MCG}(\Sigma) \curvearrowright \mathcal{T}(\Sigma)$ is isometric with respect to the Weil–Petersson metric, such that it descends to a metric on the moduli space $\mathcal{M}(\Sigma)$.

2.4. **The Augmented Teichmüller Space.** Following Harvey [Har74; Har77] and Abikoff [Abi80, Chapter 2, §3] we will now introduce the augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$ which is a bordification of Teichmüller space $\mathcal{T}(\Sigma)$. The augmented Teichmüller space has been known for a long time, and is usually constructed in terms of Fenchel–Nielsen coordinates. We will take a different route here and describe $\widehat{\mathcal{T}}(\Sigma)$ via representations. A similar approach to the augmented deformation space of convex real projective structures is taken by Loftin and Zhang [LZ18].

The idea behind the augmented Teichmüller space is to allow the lengths of (homotopically) disjoint simple closed curves to go to zero as one moves to infinity in $\mathcal{T}(\Sigma)$. This will be accounted for by attaching the Teichmüller spaces of the subsurfaces in the complement of the pinched curves. Thus the augmented Teichmüller space will admit a natural stratification in terms of the curve complex $\mathcal{C}(\Sigma)$.

Recall that the *curve complex* $\mathcal{C}(\Sigma)$ is a (combinatorial) simplicial complex and its vertices are given by homotopy classes of essential simple closed curves in Σ. A $(l-1)$-dimensional simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ is then given by a collection $\sigma = \{\alpha_1, \dots, \alpha_l\}$ of homotopy classes of essential simple closed curves, which are pairwise distinct and admit disjoint representatives. A maximal simplex $\widehat{\sigma} = \{\alpha_1, \dots, \alpha_N\}$, $N = 3g-3+p$, is a pairs of pants decomposition of Σ, such that the dimension of $\mathcal{C}(\Sigma)$ is $3g+p-4$; see [Har81].

In the following we will equip Σ with an auxiliary hyperbolic structure. Thus we may assume that every simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ consists of the (unique) closed geodesic representatives with respect to that hyperbolic structure. One can check a posteriori that the following definitions are independent of this choice up to natural isomorphisms.

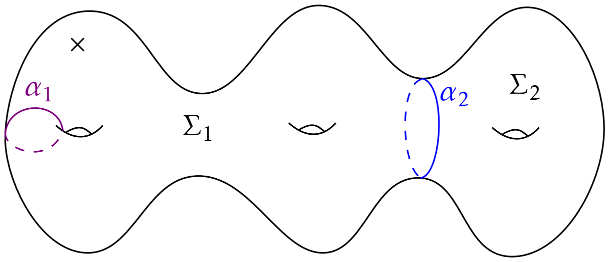
Definition 2.4.1 (Augmented Teichmüller Space)**.** Let $\sigma \subseteq \mathcal{C}(\Sigma)$ be a simplex in the curve complex. We define

$$\mathcal{T}^*_\sigma(\Sigma) := \prod_{\Sigma' \in c(\sigma)} \mathcal{T}^*(\Sigma') \quad \text{and} \quad \mathcal{T}_\sigma(\Sigma) := \prod_{\Sigma' \in c(\sigma)} \mathcal{T}(\Sigma') \subseteq \mathcal{T}^*_\sigma(\Sigma),$$

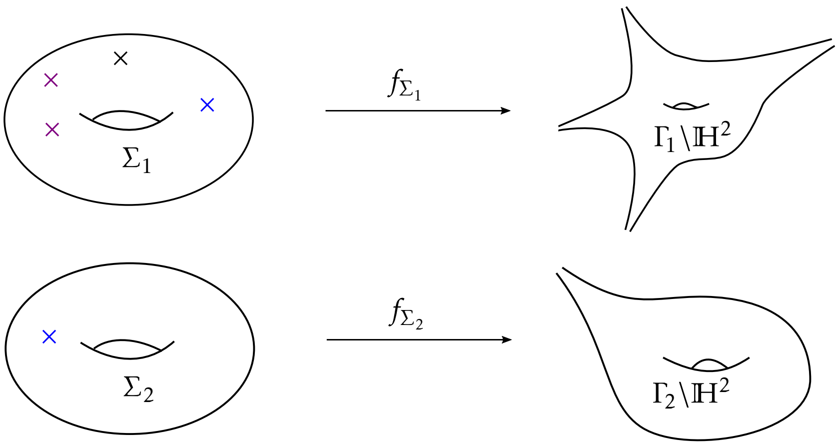
where the product is taken over all *connected components* $c(\sigma)$ *of* $\Sigma \setminus \sigma$. The disjoint union over all simplices $\sigma \subseteq \mathcal{C}(\Sigma)$,

$$\widehat{\mathcal{T}}(\Sigma) := \bigsqcup_{\sigma \subseteq \mathcal{C}(\Sigma)} \mathcal{T}_\sigma(\Sigma),$$

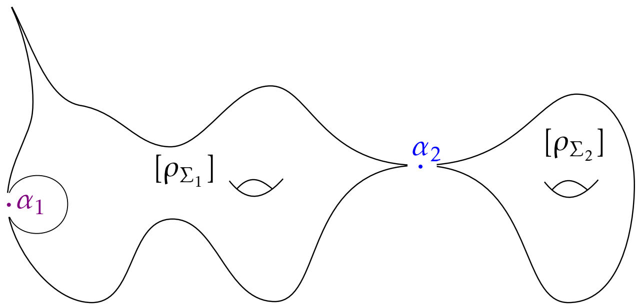
is then called the *augmented Teichmüller space of* Σ.



(A) Let Σ be the surface of genus three with one puncture. We consider the simplex $\sigma = \{\alpha_1, \alpha_2\} \subseteq \mathcal{C}(\Sigma)$, with components $c(\sigma) = \{\Sigma_1, \Sigma_2\}$.



(B) Let $\mathfrak{r} = ([\rho_{\Sigma_1}], [\rho_{\Sigma_2}]) \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$ be a point in the augmented Teichmüller space, where $[\rho_{\Sigma_i}] \in \mathcal{T}(\Sigma_i)$, $i = 1, 2$. Because the representation ρ_{Σ_i} is admissible, it is a holonomy representation of an orientation preserving homeomorphism $f_{\Sigma_i} \colon \Sigma_1 \longrightarrow \Gamma_i\backslash\mathbb{H}^2$, $\Gamma_i := \rho_{\Sigma_i}(\pi_1(\Sigma_i))$, $i = 1, 2$.



(C) The above data is summarized in this picture.

FIGURE 2

It is important to note that for a point $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\Sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$ the simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ is implicit as is the decomposition of Σ into the components $\{\Sigma'\}_{\Sigma' \in c(\sigma)}$. Geometrically one may think of the points in the stratum $\mathcal{T}_\sigma(\Sigma)$ as marked nodal surfaces, where the curves in σ were collapsed to nodes and every complementary component carries a finite-area hyperbolic structure; see Figure 2 for an example.

We wish to equip the augmented Teichmüller space with a topology. In order to do so we will need *restriction maps*. Let $\sigma \subseteq \mathcal{C}(\Sigma)$ be a simplex in the curve complex and let $\Sigma' \in c(\sigma)$ be a connected component of $\Sigma \setminus \sigma$. Let $\pi \colon \mathbb{H}^2 \cong \widetilde{\Sigma} \longrightarrow \Sigma$ denote the universal covering; recall that Σ carries an auxiliary hyperbolic structure. Denote by $\widetilde{\sigma} := \pi^{-1}(\sigma) \subseteq \mathbb{H}^2$ the disjoint union of geodesics that project to $\sigma \subseteq \Sigma$. Further, let $\widetilde{\Sigma}' \subseteq \mathbb{H}^2 \setminus \widetilde{\sigma}$ be a connected component that projects to Σ', i.e. $\pi(\widetilde{\Sigma}') = \Sigma'$. Observe that $\widetilde{\Sigma}' \subseteq \mathbb{H}^2$ is a convex subset such that $\pi|_{\widetilde{\Sigma}'} \colon \widetilde{\Sigma}' \longrightarrow \Sigma'$ is a universal covering; see Figure 3. Thus, we obtain the following commutative diagram:

$$\begin{tikzcd} \widetilde{\Sigma}' \arrow[r, hook] \arrow[d, "\pi|_{\Sigma'}"] & \widetilde{\Sigma} \arrow[d, "\pi"] \\ \Sigma' \arrow[r, hook] & \Sigma \end{tikzcd}$$

It follows that the homomorphism $\iota_{\Sigma'} \colon \pi_1(\Sigma') \longrightarrow \pi_1(\Sigma)$ induced by inclusion is injective, and identifies $\pi_1(\Sigma')$ with the subgroup of $\pi_1(\Sigma) \cong \mathrm{Deck}(\pi)$ that leaves the component $\widetilde{\Sigma}'$ invariant.

Remark 2.4.2. The monomorphism $\iota_{\Sigma'} \colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ depends on the choice of connected component $\widetilde{\Sigma}' \subseteq \pi^{-1}(\Sigma')$. Different choices amount to monomorphisms $\pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$, which are conjugate in $\pi_1(\Sigma)$.

Remark 2.4.3. Although we will not need this in the following, we want to mention that every simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ gives rise to a graph of groups structure on $\pi_1(\Sigma)$; see [Ser03]. Indeed, $\pi_1(\Sigma)$ is the fundamental group of the graph of groups whose vertices are the fundamental groups $\pi_1(\Sigma')$ of the components $\Sigma' \in c(\sigma)$. Identifying the peripheral subgroups corresponding to curves in σ then amounts to the edge homomorphisms.

Proposition and Definition 2.4.4. In the above situation, we obtain a well-defined *restriction map*

$$\begin{aligned} \mathrm{res}^{\Sigma}_{\Sigma'} \colon \mathcal{T}^*(\Sigma) &\longrightarrow \mathcal{T}^*(\Sigma'), \\ [\rho] &\longmapsto [\rho \circ \iota_{\Sigma'}]. \end{aligned}$$

For a face $\sigma' \subseteq \sigma \subseteq \mathcal{C}(\Sigma)$ these maps induce a restriction map

$$\mathrm{res}^{\sigma'}_{\sigma} \colon \mathcal{T}^*_{\sigma'}(\Sigma) \longrightarrow \mathcal{T}^*_{\sigma}(\Sigma),$$

$$([\rho_{\Sigma''}])_{\Sigma''\in c(\sigma')} \longmapsto \left(\mathrm{res}^{\Sigma''}_{\Sigma'}([\rho_{\Sigma''}])\right)_{\Sigma'\in c(\sigma)},$$

where on the right-hand-side Σ'' is the unique connected component that contains Σ'.

Proof. Let $[\rho] \in \mathcal{T}^*(\Sigma)$ with $\rho \in \mathcal{R}^*(\Sigma)$. Consider the composition $\rho \circ \iota_{\Sigma'} \colon \pi_1(\Sigma') \hookrightarrow G$. Then the image $\Gamma' := \rho(\iota_{\Sigma'}(\pi_1(\Sigma'))) \leq \Gamma := \rho(\pi_1(\Sigma))$ is discrete, such that $\rho \circ \iota_{\Sigma'}$ is a discrete and faithful representation. Our goal is to show that $\rho \circ \iota_{\Sigma'} \in \mathcal{R}^*(\Sigma')$. It will then be immediate that

$$\begin{aligned} \mathrm{res}^{\Sigma}_{\Sigma'} \colon \mathcal{T}^*(\Sigma) &\longrightarrow \mathcal{T}^*(\Sigma'), \\ [\rho] &\longmapsto [\rho \circ \iota_{\Sigma'}], \end{aligned}$$

is a well-defined map. Indeed, by Remark 2.4.2 the injective homomorphism $\iota_{\Sigma'}$ is well-defined only up to conjugation in $\pi_1(\Sigma)$. However, this issue is resolved after taking the quotient with respect to the conjugation action of G on $\mathcal{R}^*(\Sigma)$.

We are left to show that $\rho \circ \iota_{\Sigma'}$ is a holonomy representation of an orientation preserving homeomorphism $f' \colon \Sigma' \longrightarrow \Gamma' \backslash \mathbb{H}^2$. Let $f \colon \Sigma \longrightarrow X := \Gamma \backslash \mathbb{H}^2$ be an orientation preserving homeomorphism such that ρ is a holonomy representation of f. We may isotope f in such a way that it sends the curves in σ to geodesics $f(\sigma) \subseteq X$. Therefore, it sends Σ' to a connected component X' of $X \setminus f(\sigma)$. Consider a lift $\widetilde{f} \colon \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$ of $f \colon \Sigma \longrightarrow X$ with respect to the universal coverings $\pi \colon \widetilde{\Sigma} \longrightarrow \Sigma$ and $\pi_\Gamma \colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$. Then $\widetilde{X}' := \widetilde{f}(\widetilde{\Sigma}') \subseteq \mathbb{H}^2 \setminus \pi_\Gamma^{-1}(f(\sigma))$ is a connected component, and $\widetilde{f}|_{\widetilde{\Sigma}'} \colon \widetilde{\Sigma}' \longrightarrow \widetilde{X}'$ is $(\rho \circ \iota_{\Sigma'})$-equivariant. Thus, it descends to an orientation preserving homeomorphism $f' \colon \Sigma' \longrightarrow \Gamma' \backslash \widetilde{X}' \cong X'$, $\Gamma' := (\rho \circ \iota_{\Sigma'})(\pi_1(\Sigma'))$. The complement $\mathbb{H}^2 \setminus \widetilde{X}'$ is a disjoint union of closed half-spaces $\{H_j\}_{j\in\mathbb{N}}$ each bordering on a geodesic of $\widetilde{f}(\widetilde{\sigma})$ adjacent to $\widetilde{X}'$.

We want to understand the action of Γ' on each half-space $H_j, j \in \mathbb{N}$. Note that the disjoint union $\bigsqcup_{j\in\mathbb{N}} H_j$ is Γ'-invariant such that Γ' acts via permutations on $\{H_j\}_{j\in\mathbb{N}}$. Thus, if $\gamma \in \Gamma'$ is an element such that $\gamma H_j \cap H_j \neq \varnothing$ then $\gamma H_j = H_j$. If $I_j = \partial H_j \subseteq \partial \mathbb{H}^2 \cong \mathbb{S}^1$ denotes the interval that H_j borders on then γ has to fix I_j. Because Γ' is torsion-free it does not contain any elliptic elements such that γ must be a hyperbolic element that fixes the end points of I_j. By discreteness of Γ' there is a hyperbolic element $\gamma_j \in \Gamma'$ for every $j \in \mathbb{N}$ such that any element $\gamma \in \Gamma'$ satisfying $\gamma H_j \cap H_j \neq \varnothing$ is a power of γ_j. It follows that the quotient of H_j under the quotient map $\pi' \colon \mathbb{H}^2 \longrightarrow \Gamma'\backslash\mathbb{H}^2$ is a hyperbolic funnel $F_j := \pi'(H_j) \cong \langle \gamma_j \rangle \backslash H_j$.

Therefore, the complement of $\Gamma'\backslash\widetilde{X}'$ in $Y' = \Gamma'\backslash\mathbb{H}^2$ is a disjoint union of hyperbolic funnels. In particular, Y' deformation retracts to $\Gamma'\backslash\widetilde{X}'$, and we can easily modify f' to obtain an orientation preserving homeomorphism $f'' \colon \Sigma' \longrightarrow Y'$ with holonomy $\rho \circ \iota_{\Sigma'}$. We conclude that $\rho \circ \iota_{\Sigma'} \in \mathcal{R}^*(\Sigma')$, and $\mathrm{res}^{\Sigma}_{\Sigma'}$ is well-defined. □

We may now define a topology on $\widehat{\mathcal{T}}(\Sigma)$.

Definition 2.4.5 (Topology on $\widehat{\mathcal{T}}(\Sigma)$)**.** For every $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma)$ we define a system of open neighborhoods by

$$(\mathrm{res}_\sigma^{\sigma'})^{-1}(U) \cap \mathcal{T}_{\sigma'}(\Sigma)$$

where $\sigma' \subseteq \sigma$ and $U \subseteq \mathcal{T}_\sigma^*(\Sigma) = \prod_{\Sigma' \in c(\sigma)} \mathcal{T}^*(\Sigma')$ runs over all open neighborhoods of $\mathfrak{r} \in \mathcal{T}_\sigma^*(\Sigma)$ in the product topology. This system of neighborhoods defines a topology on $\widehat{\mathcal{T}}(\Sigma)$.

In this topology a sequence $(\mathfrak{r}^{(n)})_{n \in \mathbb{N}} \subseteq \widehat{\mathcal{T}}(\Sigma)$ converges to $\mathfrak{r} \in \mathcal{T}_\sigma(\Sigma)$ if and only if $\mathfrak{r}^{(n)} \in \mathcal{T}_{\sigma_n}(\Sigma)$ with $\sigma_n \subseteq \sigma$ for large n, and

$$\mathrm{res}_\sigma^{\sigma_n}(\mathfrak{r}^{(n)}) \to \mathfrak{r} \qquad (n \to \infty)$$

in $\mathcal{T}_\sigma^*(\Sigma)$.

Notice that for $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma)$ every $[\rho_{\Sigma'}] \in \mathcal{T}(\Sigma')$ corresponds to a hyperbolic structure with finite area on $\Sigma' \in c(\sigma)$. Therefore, every peripheral curve $\alpha \in \pi_1(\Sigma')$ is mapped to a parabolic boundary transformation $\rho_{\Sigma'}(\alpha)$. Suppose that $\mathfrak{r}^{(n)} = ([\rho^{(n)}_{\Sigma''_n}])_{\Sigma''_n \in c(\sigma_n)} \in \mathcal{T}_{\sigma_n}(\Sigma)$ converges to $\mathfrak{r} \in \mathcal{T}_\sigma(\Sigma)$. Then every curve $\alpha \in \pi_1(\Sigma)$, that is freely homotopic to a curve in σ but not in σ_n, is pinched as n tends to infinity. Indeed, up to conjugation $\rho^{(n)}_{\Sigma'_n}(\alpha)$ converges to a parabolic transformation for every component $\Sigma'_n \in c(\sigma_n)$ that contains α.

Remark 2.4.6. In [Abi80, Chapter 2, §3.4] the topology on augmented Teichmüller space is defined in terms of Fenchel–Nielsen coordinates. However, one can check that both topologies coincide. Indeed, every restriction map corresponds to a projection onto an appropriate subset of length and twist parameters in Fenchel–Nielsen coordinates.

It is a result of Masur [Mas76] that the augmented Teichmüller space is the completion of Teichmüller space with respect to the Weil–Petersson metric; see also Wolpert [Wol07; Wol10]. In particular, the following holds.

Proposition 2.4.7. *The augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$ is metrizable, and $\mathcal{T}(\Sigma) = \mathcal{T}_\emptyset(\Sigma)$ is an open and dense subset of $\widehat{\mathcal{T}}(\Sigma)$.*

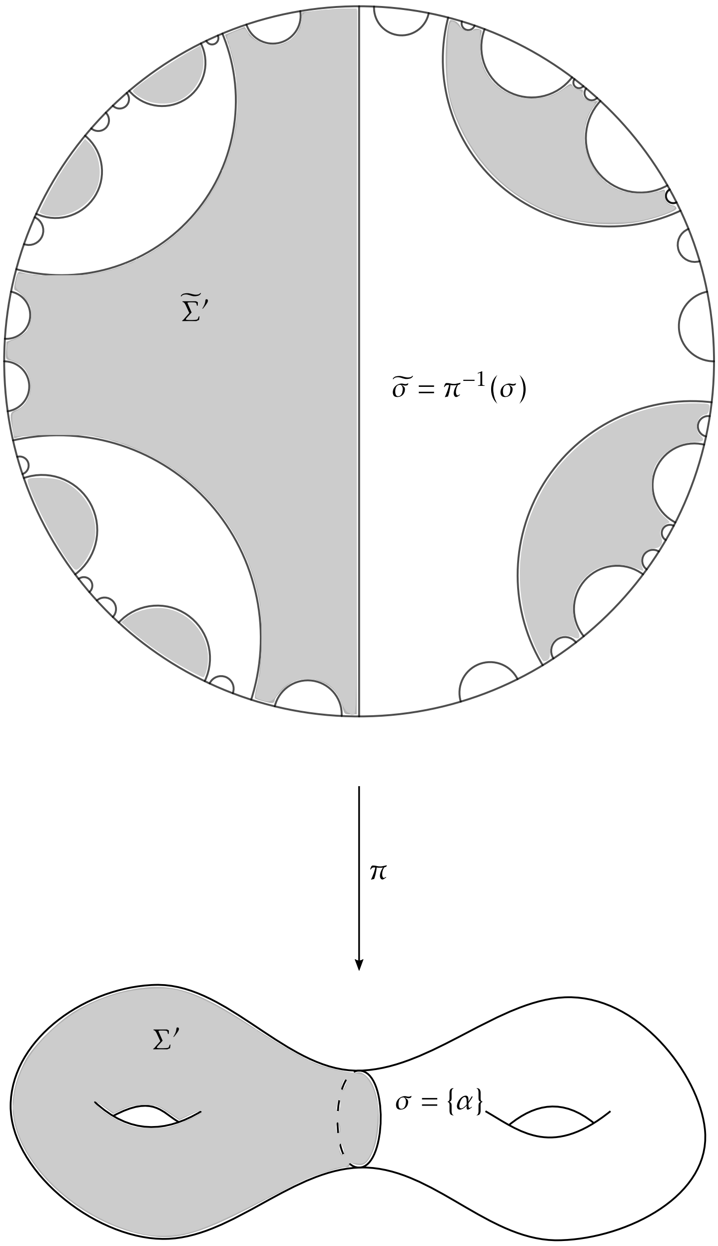


FIGURE 3. We are considering a genus two surface Σ and a simplex $\sigma = \{\alpha\} \subseteq \mathcal{C}(\Sigma)$ consisting of one separating curve $\alpha \subseteq \Sigma$. The preimages $\pi^{-1}(\Sigma')$ of the component $\Sigma' \in c(\sigma)$ are shaded. We chose one connected component $\widetilde{\Sigma}' \subseteq \mathbb{H}^2 \setminus \widetilde{\sigma}$.

2.5. **The Augmented Moduli Space.** We will now introduce the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ as a quotient of the augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$ by the extended mapping class group action; see Abikoff [Abi80, Chapter 2, §3.4].

The mapping class group action on Teichmüller space extends to $\widehat{\mathcal{T}}(\Sigma)$ in the following way. Let $\varphi = [f] \in \mathrm{MCG}(\Sigma)$ and $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$. Notice that the mapping class group acts simplicially on the curve complex $\mathcal{C}(\Sigma)$. Up to isotopy we may assume that f^{-1} sends σ to a geodesic representative of $f^{-1}(\sigma)$. Hence, f^{-1} induces a bijection between the components $c(\sigma)$ and $c(f^{-1}(\sigma)) = f^{-1}(c(\sigma))$, and acts from the right via restriction:

$$\mathfrak{r} \cdot \varphi := ([\rho_{f(\Sigma')} \circ (f|_{\Sigma'})_*])_{\Sigma' \in c(f^{-1}(\sigma))}.$$

The cutting homomorphism ensures that the action is well-defined; see [FM12, Section 3.6.3]. By definition this action extends the mapping class group action on $\mathcal{T}(\Sigma)$ such that the embedding $\mathcal{T}(\Sigma) \hookrightarrow \widehat{\mathcal{T}}(\Sigma)$ is $\mathrm{MCG}(\Sigma)$-equivariant.

Note that this action permutes the different strata $\{\mathcal{T}_\sigma(\Sigma)\}_{\sigma \subseteq \mathcal{C}(\Sigma)}$ of the augmented Teichmüller space:

$$\mathcal{T}_\sigma(\Sigma) \xrightarrow{\varphi} \mathcal{T}_{\varphi^{-1}(\sigma)}(\Sigma), \qquad \varphi \in \mathrm{MCG}(\Sigma).$$

Definition 2.5.1. The quotient space $\widehat{\mathcal{M}}(\Sigma) := \widehat{\mathcal{T}}(\Sigma)/\mathrm{MCG}(\Sigma)$ of the augmented Teichmüller space $\widehat{\mathcal{T}}(\Sigma)$ by the mapping class group action is called the *augmented moduli space*.

We will denote the $\mathrm{MCG}(\Sigma)$-equivalence class of $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma' \in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$ by $[\mathfrak{r}] \in \widehat{\mathcal{M}}(\Sigma)$. Each $[[\rho_{\Sigma'}]] \in \mathcal{M}(\Sigma')$, $\Sigma' \in c(\sigma)$, will be called a *part* or *component* of $[\mathfrak{r}]$.[1]

In this way the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ may be interpreted as the space of all unmarked *nodal surfaces*, where every component carries a finite-area hyperbolic structure. If we want to emphasize this geometric point of view, we will denote a nodal surface in the augmented moduli space by a bold capital letter, e.g. $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$. Likewise, we will denote the parts of a nodal surface $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ by $X_{\Sigma'} \in \mathcal{M}(\Sigma')$, $\Sigma' \in c(\sigma)$, or $X_1, \ldots, X_m$ if the components $c(\sigma) = \{\Sigma_1, \ldots, \Sigma_m\}$ are enumerated.

Remark 2.5.2. Changing perspective one can see the moduli space $\mathcal{M}(\Sigma)$ as the moduli space of smooth genus g curves with p marked points. In this setting, the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ corresponds to the Deligne–Mumford compactification of stable curves. We will not use this point of view in what follows and refer the reader to [Har74; Har77] and [HK14] for details.

The significance of this construction is that the augmented moduli space is *compact*.

[1] It is straight-forward to check that this definition depends only on the $\mathrm{MCG}(\Sigma)$-equivalence class of $\mathfrak{r}$.

Theorem 2.5.3 ([Abi80, Theorem, p. 104]). *The augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ is a compact Hausdorff space. The embedding $\mathcal{T}(\Sigma) \hookrightarrow \widehat{\mathcal{T}}(\Sigma)$ descends to an embedding $\mathcal{M}(\Sigma) \hookrightarrow \widehat{\mathcal{M}}(\Sigma)$ with open and dense image.*

2.6. **Assembly Maps.** We will now give an interpretation of how elements in $\widehat{\mathcal{M}}(\Sigma)$ may be assembled from elements in the moduli spaces of the components. Such gluing constructions are often used in Algebraic Geometry when studying the Deligne–Mumford compactification of stable curves [Arb+11, Chapter X, Section 7]. We will give a description of such gluing operations in the context of the augmented moduli space in what follows. A more detailed exposition may be found in [Kri21, Section 2.6].

We will need some more notation.

Definition 2.6.1 (Pure Mapping Class Group). The subgroup $\mathrm{PMCG}(\Sigma) \leq \mathrm{MCG}(\Sigma)$ of all mapping classes that fix each puncture of Σ individually is called the *pure mapping class group*.

This allows us to form a slightly larger moduli space of hyperbolic structures on Σ by keeping track of the punctures individually. Indeed, if we take the quotient of Teichmüller space by only the pure mapping class group, punctures are no longer allowed to be permuted by a mapping class. Thus, one may think of the elements of this new moduli space as hyperbolic surfaces with labeled punctures:

Definition 2.6.2. We define the *moduli space of (finite-area) hyperbolic surfaces with labeled punctures* as

$$\mathcal{M}^*(\Sigma) := \mathcal{T}(\Sigma)/\mathrm{PMCG}(\Sigma)$$

and denote the quotient map by $\pi_\Sigma \colon \mathcal{T}(\Sigma) \longrightarrow \mathcal{M}^*(\Sigma)$.

Let us now fix a simplex $\sigma \subseteq \mathcal{C}(\Sigma)$ in the curve complex. We denote by

$$\mathrm{PMCG}_\sigma(\Sigma) := \{\varphi \in \mathrm{PMCG}(\Sigma) \colon \varphi(\alpha) = \alpha, \text{ for all } \alpha \in \sigma\} \leq \mathrm{PMCG}(\Sigma)$$

the subgroup of mapping classes fixing the homotopy class of each curve of σ individually. By definition $\mathrm{PMCG}_\sigma(\Sigma)$ acts on $\mathcal{T}_\sigma(\Sigma)$ and we consider the quotient

$$\mathcal{M}^*_\sigma(\Sigma) := \mathcal{T}_\sigma(\Sigma)/\mathrm{PMCG}_\sigma(\Sigma).$$

One can show that the natural quotient maps $\mathcal{T}_\sigma(\Sigma) \longrightarrow \mathcal{M}^*_\sigma(\Sigma)$ and $\mathcal{T}(\Sigma') \longrightarrow \mathcal{M}^*(\Sigma')$, $\Sigma' \in c(\sigma)$, induce an identification $\mathcal{M}^*_\sigma(\Sigma) \cong \prod_{\Sigma' \in c(\sigma)} \mathcal{M}^*(\Sigma')$; see [Kri21, Proposition 2.6.7]:

$$\begin{tikzcd}
\mathcal{T}_\sigma(\Sigma) \arrow[r, equal] \arrow[d, two heads] & \prod_{\Sigma' \in c(\sigma)} \mathcal{T}(\Sigma') \arrow[d, two heads] \\
\mathcal{M}^*_\sigma(\Sigma) \arrow[r, dashed, "\cong"] & \prod_{\Sigma' \in c(\sigma)} \mathcal{M}^*(\Sigma')
\end{tikzcd}$$

Projecting $\mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma)$ to the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ we obtain a continuous map $\mathcal{T}_\sigma(\Sigma) \longrightarrow \widehat{\mathcal{M}}(\Sigma)$, which descends:

$$\mathcal{M}^*_\sigma(\Sigma) = \mathcal{T}_\sigma(\Sigma)/\mathrm{PMCG}_\sigma(\Sigma) \longrightarrow \widehat{\mathcal{M}}(\Sigma) = \widehat{\mathcal{T}}(\Sigma)/\mathrm{MCG}(\Sigma).$$

Hence, we obtain a continuous map A_σ as follows:

$$\begin{tikzcd}
\mathcal{T}_\sigma(\Sigma) \arrow[r] \arrow[d] & \widehat{\mathcal{M}}(\Sigma) \\
\mathcal{M}^*_\sigma(\Sigma) \arrow[r, "\cong"] \arrow[ur] & \prod_{\Sigma' \in c(\sigma)} \mathcal{M}^*(\Sigma') \arrow[u, dashed, "A_\sigma"']
\end{tikzcd}$$

Definition 2.6.3. We will call the map $A_\sigma \colon \prod_{\Sigma' \in c(\sigma)} \mathcal{M}^*(\Sigma') \longrightarrow \widehat{\mathcal{M}}(\Sigma)$ the *assembly map with respect to* $\sigma \subseteq \mathcal{C}(\Sigma)$.

Remark 2.6.4. The augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ is covered by the images of all the $\mathcal{T}_\sigma(\Sigma)$, $\sigma \in \mathcal{C}(\Sigma)$. Therefore, every nodal surface $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ is in the image of some assembly map A_σ for some simplex $\sigma \in \mathcal{C}(\Sigma)$. In fact, the images of *finitely* many assembly maps are sufficient to cover the augmented moduli space since the quotient of the curve complex by the mapping class group is a *finite* simplicial complex; see [Har81, p. 247].

The assembly map allows for the following geometric interpretation. Every curve in σ corresponds to two punctures in some (possibly the same) component(s) $\Sigma', \Sigma'' \in c(\Sigma)$. In this way σ can be thought of as a pairing for the punctures of the components $c(\sigma)$. Given a collection of hyperbolic surfaces with labeled punctures $X_i \in \mathcal{M}^*(\Sigma_i)$, $i = 1, \dots, m$, $c(\sigma) = \{\Sigma_1, \dots, \Sigma_m\}$, we may "glue" them according to the pairing given by σ to obtain an element of $\widehat{\mathcal{M}}(\Sigma)$. This is exactly the image $A_\sigma(X_1, \dots, X_m)$ of the assembly map.

Instead of starting out with a given topological surface Σ and a simplex $\sigma \subseteq \mathcal{C}(\Sigma)$, we may start with just a collection of surfaces and a pairing of their punctures. Indeed, let $\Sigma_1, \dots, \Sigma_m$ be a collection of oriented topological surfaces with negative Euler characteristic $\chi(\Sigma_i) < 0$. We will think of the punctures as marked points instead of missing points, and denote by $P(\Sigma_i) \subseteq \Sigma_i$ the set of all punctures of Σ_i, $i = 1, \dots, m$. Moreover, let $\mathcal{P}$ be a *pairing* of some of the punctures, i.e. $\mathcal{P}$ is a collection of two element subsets of $\bigsqcup_{i=1}^m P(\Sigma_i)$ such that $\{x_1,x_2\}\cap\{y_1,y_2\}=\varnothing$ for any two distinct $\{x_1,x_2\},\{y_1,y_2\}\in\mathcal{P}$.

Using the pairing $\mathcal{P}$ we can assemble the surfaces $\Sigma_1,\dots,\Sigma_m$ to a larger oriented topological surface $\Sigma=\Sigma(\Sigma_1,\dots,\Sigma_m;\mathcal{P})$ with negative Euler characteristic by replacing each puncture with a boundary component and gluing them according to the given pairing. We call the pairing $\mathcal{P}$ *admissible*, if the resulting surface Σ is connected.

In this construction the pairing $\mathcal{P}$ amounts to a collection of closed curves $\sigma(\mathcal{P})\subseteq\mathcal{C}(\Sigma)$ corresponding to the glued boundary components and the surfaces $\Sigma_1,\dots,\Sigma_m$ can be canonically identified with the components $c(\sigma(\mathcal{P}))$. Therefore, we may define the assembly map with respect to an admissible pairing $\mathcal{P}$ as the assembly map with respect to the simplex $\sigma(\mathcal{P})\subseteq\mathcal{C}(\Sigma(\Sigma_1,\dots,\Sigma_m;\mathcal{P}))$:

Definition 2.6.5. Let $\Sigma_1,\dots,\Sigma_m$ be a collection of oriented topological surfaces with negative Euler characteristic $\chi(\Sigma_i)<0$, let $\mathcal{P}$ be an admissible pairing of their punctures and set $\Sigma:=\Sigma(\Sigma_1,\dots,\Sigma_m;\mathcal{P})$ as above. Then the *assembly map with respect to the pairing* $\mathcal{P}$ is defined as

$$\begin{aligned} A_{\mathcal{P}}\colon \prod_{i=1}^m \mathcal{M}^*(\Sigma_i) &\longrightarrow \widehat{\mathcal{M}}(\Sigma),\\ (X_1,\dots,X_m) &\longmapsto A_{\sigma(\mathcal{P})}(X_1,\dots,X_m). \end{aligned}$$

We will also say that $X_1,\dots,X_m$ are *glued according to* $\mathcal{P}$ to form the nodal surface $\mathbf{X}=A_{\mathcal{P}}(X_1,\dots,X_m)$.

The following is an immediate consequence of Remark 2.6.4.

Corollary 2.6.6. *For every nodal surface $\mathbf{X}\in\widehat{\mathcal{M}}(\Sigma)$ there is a collection $X_1,\dots,X_m$, $m>0$, of hyperbolic surfaces with labeled punctures and an admissible pairing of their punctures $\mathcal{P}$ such that $\mathbf{X}=A_{\mathcal{P}}(X_1,\dots,X_m)$.*

2.7. **Forgetting the gluing.** We have already seen that one may obtain nodal surfaces from hyperbolic surfaces with labeled punctures by gluing pairs of punctures. In this section we will go the other way and forget how the parts are glued.

Consider the set $|\widehat{\mathcal{M}}|:=\mathbb{N}_0^{\bigsqcup_{\Sigma'}\mathcal{M}(\Sigma')}$ of all maps from $\bigsqcup_{\Sigma'}\mathcal{M}(\Sigma')$ to $\mathbb{N}_0$, where the disjoint union is taken over all oriented topological surfaces Σ' with negative Euler characteristic. There is a canonical map $Q\colon\widehat{\mathcal{M}}(\Sigma)\longrightarrow|\widehat{\mathcal{M}}|$, that records each part of the nodal surface with its multiplicity. (Note that the parts of a nodal surface do *not* have labeled punctures.)

Formally, given a nodal surface $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ with parts $X_i \in \mathcal{M}(\Sigma_i)$, $i = 1,\dots,m$, we define

$$Q(\mathbf{X})(Y) := \#\{i \in \{1,\dots,m\} \mid X_i = Y\} \qquad \forall Y \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma').$$

Definition 2.7.1. We denote the image of $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|$ by $|\widehat{\mathcal{M}}|(\Sigma) := Q(\widehat{\mathcal{M}}(\Sigma))$ and call it the *moduli space of parts*. Furthermore, we equip $|\widehat{\mathcal{M}}|(\Sigma)$ with the quotient topology turning it into a compact topological space.

It turns out that the cardinalities of the fibers of Q admit a uniform upper bound, that depends only on the topology of the surface Σ.

Proposition 2.7.2. *Let $\chi = \chi(\Sigma)$, $g = g(\Sigma)$, and $p = p(\Sigma)$ denote the Euler characteristic, the genus, and the number of punctures of Σ, respectively. Then*

$$\# Q^{-1}(\xi) \le B(\Sigma)$$

for all $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$, where

$$B(\Sigma) := \binom{3|\chi|}{p} \cdot \frac{(2(|\chi| + g - 1))!}{(|\chi| + g - 1)! \cdot 2^{(|\chi|+g-1)}}.$$

Proof. Let $\mathbf{X} = A_{\mathcal{P}_0}(X_1,\dots,X_m) \in \widehat{\mathcal{M}}(\Sigma)$ be a nodal surface glued from some $X_i \in \mathcal{M}^*(\Sigma_i)$, $i = 1,\dots,m$, according to some pairing of their punctures $\mathcal{P}_0$. Any other nodal surface $\mathbf{Y} \in Q^{-1}(Q(\mathbf{X}))$ can be obtained as a gluing $\mathbf{Y} = A_{\mathcal{P}}(X_1,\dots,X_m)$ according to some appropriate pairing $\mathcal{P}$. Therefore, it will suffice to show that there are at most $B(\Sigma)$-many admissible pairings $\mathcal{P}$ that yield a nodal surface in the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$. Notice that this imposes the requirement that the glued topological surface $\Sigma(\Sigma_1,\dots,\Sigma_m;\mathcal{P})$ is homeomorphic to the given surface Σ.

We will now estimate the number of such pairings. To this end let $P := \sum_{i=1}^m p(\Sigma_i)$ denote the total number of punctures of all the components $\Sigma_1,\dots,\Sigma_m$. We know that in the end $\Sigma(\Sigma_1,\dots,\Sigma_m;\mathcal{P})$ should have p punctures. Thus, in a first step, we will select p punctures from all of the punctures of the components $\Sigma_1,\dots,\Sigma_m$, that will not be glued. There are $\binom{P}{p}$-many possible choices and the remaining $P - p$ punctures will have to be glued.

By elementary combinatorics there are

$$\frac{(2k)!}{k! \cdot 2^k} \tag{1}$$

many ways of pairing $2k$ elements, $k \in \mathbb{N}$. Hence, there are

$$\frac{(P-p)!}{((P-p)/2)! \cdot 2^{(P-p)/2}}$$

many different ways of pairing the remaining $P-p$ punctures. Altogether there are at most

$$\binom{P}{p}\cdot\frac{(P-p)!}{((P-p)/2)!\cdot 2^{(P-p)/2}} \tag{2}$$

many pairings $\mathcal{P}$, that yield a topological surface $\Sigma(\Sigma_1,\ldots,\Sigma_m;\mathcal{P})$ homeomorphic to Σ.

In order to obtain a uniform upper bound, let us first prove that $P\le 3|\chi|$: Let χ_i, g_i, and p_i denote the Euler characteristic, the genus, and the number of punctures of Σ_i, respectively. Observe that $\chi(\Sigma_i)=2-2g_i-p_i\le -1$, $i=1,\ldots,m$. Hence,

$$\chi=\sum_{i=1}^{m}\chi_i=\sum_{i=1}^{m}(2-2g_i-p_i). \tag{3}$$

We obtain

$$P=\sum_{i=1}^{m}p_i=|\chi|+\sum_{i=1}^{m}(2-2g_i)\le|\chi|+2m.$$

Finally,

$$m\le\sum_{i=1}^{m}|\chi_i|=|\chi|,$$

because $\chi_i\le -1$. This yields $P\le 3|\chi|$, whence

$$\binom{P}{p}\le\binom{3|\chi|}{p}. \tag{4}$$

Next, observe that

$$2-2g-p=\sum_{i=1}^{m}(2-2g_i)-P$$

by (3), whence

$$\begin{aligned}P-p&=\sum_{i=1}^{m}(2-2g_i)-(2-2g)\\&=2(m-1)+2g-2\sum_{i=1}^{m}g_i\\&\le 2(|\chi|+g-1),\end{aligned}$$

and $\frac{1}{2}(P-p)\le|\chi|+g-1$.

It is straight-forward to check that (1) is increasing in $k\in\mathbb{N}$, whence we obtain

$$\frac{(P-p)!}{((P-p)/2)!\cdot 2^{(P-p)/2}}\le\frac{(2\,(|\chi|+g-1))!}{(|\chi|+g-1)!\cdot 2^{|\chi|+g-1}}. \tag{5}$$

Using the upper bounds (4) and (5) in (2) we obtain that there are at most

$$B(\Sigma) = \binom{3|\chi|}{p} \cdot \frac{(2(|\chi| + g - 1))!}{(|\chi| + g - 1)! \cdot 2^{|\chi|+g-1}}$$

many pairings $\mathcal{P}$ such that $\Sigma \cong \Sigma(\Sigma_1, \ldots, \Sigma_m; \mathcal{P})$. This concludes the proof. □

We want to point out that the bound $B(\Sigma)$ is not sharp, as the following example shows.

Example 2.7.3. Consider the once punctured torus $\Sigma = \Sigma_{1,1}$. Observe that the boundary of $\widehat{\mathcal{M}}(\Sigma)$ consists of a single nodal surface $\mathbf{X}_0$, which is a degenerate hyperbolic pair of pants with two punctures glued. Since the quotient map $Q\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow |\widehat{\mathcal{M}}|(\Sigma)$ is one-to-one on the moduli space $\mathcal{M}(\Sigma)$, it follows that the map Q is one-to-one on the entire augmented moduli space. However, we have that

$$B(\Sigma_{1,1}) = \binom{3}{1} \cdot \frac{2!}{1! \cdot 2^1} = 3.$$

For an arbitrary $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$ computing $\#Q^{-1}(\xi)$ can be delicate. Indeed, it is not sufficient to just count the number of pairings that yield the correct topological type of the glued surface, because the symmetries of the different parts play a role, too. This is demonstrated by the following example.

Example 2.7.4. Consider the closed surface $\Sigma = \Sigma_3$ of genus three, and let $X, Y \in \mathcal{M}(\Sigma_{1,2})$ be two hyperbolic surfaces of genus one with two punctures. We define an element $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$ by setting $\xi(X) = \xi(Y) = 1$ and $\xi(Z) = 0$ for all other surfaces $Z \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')$. Let us label the punctures of X as $\{x_1, x_2\}$ and the punctures of Y as $\{y_1, y_2\}$ to obtain points $X', Y' \in \mathcal{M}^*(\Sigma_{2,1})$. Then any point in the fiber $Q^{-1}(\xi)$ can be glued from X' and Y' by pairing the punctures appropriately. We have the following list of possible pairings:

$$\mathcal{P}_1 = \{\{x_1, y_1\}, \{x_2, y_2\}\},$$
$$\mathcal{P}_2 = \{\{x_1, y_2\}, \{x_2, y_1\}\},$$
$$\mathcal{P}_3 = \{\{x_1, x_2\}, \{y_1, y_2\}\}.$$

Note that the last pairing $\mathcal{P}_3$ is not admissible. Thus,

$$Q^{-1}(\xi) = \{A_{\mathcal{P}_1}(X', Y'), A_{\mathcal{P}_2}(X', Y')\}.$$

Moreover, if there is an isometry of X (or Y) that exchanges the punctures, then

$$A_{\mathcal{P}_1}(X', Y') = A_{\mathcal{P}_2}(X', Y') \in \widehat{\mathcal{M}}(\Sigma_3).$$

Thus, depending on the symmetries of X and Y, the preimage $Q^{-1}(\xi)$ has either one or two elements.

2.8. **The Space of Closed Subgroups.** In this section we will introduce the space of closed subgroups and briefly recall its properties. The material presented here is well-known and can be found in [CEM06; Gel15; Mac64; GL18b; BP92]. We also recommend the survey [Har08] for further reading.

The space of closed subgroups is essential for this article as every invariant random subgroup is a conjugation invariant probability measure on the space of closed subgroups by definition. Moreover, we will use it in section 2.9 to define the geometric topology on $\mathcal{R}^*(\Sigma)$. As before we will use the notation $G = \mathrm{PSL}(2,\mathbb{R})$ in the following.

Definition 2.8.1 (Space of closed subgroups $\mathrm{Sub}(G)$; [Abé+17, Section 2])**.** We denote by $\mathrm{Sub}(G)$ the set of closed subgroups of G. For open subsets $U \subseteq G$ and compact subsets $K \subseteq G$ we define the sets

$$\mathcal{O}(K) := \{A \in \mathrm{Sub}(G) : \; A \cap K = \varnothing\}, \qquad \mathcal{O}'(U) := \{A \in \mathrm{Sub}(G) : \; A \cap U \neq \varnothing\}.$$

The collection of all such subsets $\{\mathcal{O}(K) : \; K \subseteq G \text{ compact}\} \cup \{\mathcal{O}'(U) : \; U \subseteq G \text{ open}\}$ generates the *Chabauty topology* on $\mathrm{Sub}(G)$; see [Cha50].

The most important property of this topology is that $\mathrm{Sub}(G)$ is compact.

Lemma 2.8.2 ([CEM06, Proposition I.3.1.2])**.** *The space of closed subgroups* $\mathrm{Sub}(G)$ *is compact and metrizable.*

The following characterization is often useful.

Proposition 2.8.3 ([CEM06, Lemma I.3.1.3])**.** *A sequence* $(H_n)_{n\in\mathbb{N}} \subseteq \mathrm{Sub}(G)$ *converges to* $H \in \mathrm{Sub}(G)$ *if and only if the following two conditions are satisfied:*

- *(C1) For every $h \in H$ there is a sequence $(h_n)_{n\in\mathbb{N}} \subseteq G$ such that $h_n \in H_n$ for every $n \in \mathbb{N}$ and $h = \lim_{n\to\infty} h_n$.*
- *(C2) If $h \in G$ is the limit of a sequence $(h_{n_k})_{k\in\mathbb{N}} \subseteq H$ such that $h_{n_k} \in H_{n_k}$ for every $k \in \mathbb{N}$, then $h \in H$.*

The Chabauty topology is compatible with the conjugation action $G \curvearrowright \mathrm{Sub}(G)$:

Lemma 2.8.4 ([Abé+17, Section 2])**.** *The group G acts continuously on* $\mathrm{Sub}(G)$ *via conjugation*

$$\begin{aligned} G \times \mathrm{Sub}(G) &\longrightarrow \mathrm{Sub}(G) \\ (g,H) &\longmapsto gHg^{-1}. \end{aligned}$$

The following subsets will be of interest later on:

Definition 2.8.5. We define

$$\mathrm{Sub}_{\mathrm{d}}(G) := \{\Gamma \in \mathrm{Sub}(G) | \Gamma \text{ is discrete}\},$$

$$\mathrm{Sub}_{\mathrm{dtf}}(G) := \{\Gamma \in \mathrm{Sub}(G) \,|\, \Gamma \text{ is discrete and torsion-free}\},$$

and equip these subsets with the subspace topology.

We will record some of their topological properties now:

Lemma 2.8.6 ([CEM06, Theorem I.3.1.4])**.** *Let $\Gamma \in \mathrm{Sub}_{\mathrm{d}}(G)$. Then there is an open neighborhood $U \subseteq G$ of the identity $e \in G$ and an open neighborhood $\mathcal{U} \subseteq \mathrm{Sub}(G)$ such that*

$$\Gamma' \cap U = \{e\}$$

for every $\Gamma' \in \mathcal{U}$.

Corollary 2.8.7. *The subset of discrete subgroups $\mathrm{Sub}_{\mathrm{d}}(G) \subseteq \mathrm{Sub}(G)$ is open.*

In order to understand the Chabauty topology geometrically, the following proposition is very useful.

Proposition 2.8.8 ([BP92, Theorem E.1.13])**.** *Let $o \in \mathbb{H}^2$ and $\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G)$. Then the following holds:*

For every $r > 0, \varepsilon > 0$ there is an open neighborhood $\mathcal{U} \subseteq \mathrm{Sub}_{\mathrm{dtf}}(G)$ of Γ such that for every $\Gamma' \in \mathcal{U}$ there are open neighborhoods $\Omega, \Omega' \subseteq \mathbb{H}^2$ of the closed ball $\overline{B}_o(r)$ and a diffeomorphism $f\colon \Omega \longrightarrow \Omega'$ satisfying:

(i) $f(o) = o$,
(ii) $\pi_{\Gamma'}(f(x)) = \pi_{\Gamma'}(f(y)) \iff \pi_\Gamma(x) = \pi_\Gamma(y)$, *for every $x, y \in \Omega$, and*
(iii) $D_{\overline{B}_o(r)}(f, \mathrm{id}) < \varepsilon$,

where $\pi_\Gamma\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2, \pi_{\Gamma'}\colon \mathbb{H}^2 \longrightarrow \Gamma'\backslash\mathbb{H}^2$ are the respective quotient maps, and $D_K(f,g)$ denotes the C^∞-distance between two diffeomorphisms f, g defined on a neighborhood of a compact set $K \subseteq \mathbb{H}^2$.

In particular, the diffeomorphism $f\colon \Omega \longrightarrow \Omega'$ descends to a diffeomorphism F mapping $\pi_\Gamma(\Omega) \subseteq \Gamma\backslash\mathbb{H}^2$ to $\pi_{\Gamma'}(\Omega') \subseteq \Gamma'\backslash\mathbb{H}^2$:

$$\begin{tikzcd} \Omega \subseteq \mathbb{H}^2 \arrow[r, "f"] \arrow[d, "\pi_\Gamma"] & \Omega' \subseteq \mathbb{H}^2 \arrow[d, "\pi_{\Gamma'}"] \\ \pi_\Gamma(\Omega) \subseteq \Gamma\backslash\mathbb{H}^2 \arrow[r, "F"] & \pi_{\Gamma'}(\Omega') \subseteq \Gamma'\backslash\mathbb{H}^2 \end{tikzcd}$$

Informally, one can think of Proposition 2.8.8 as saying that "large" balls centered at the base points $\pi_\Gamma(o)$ in $\Gamma\backslash\mathbb{H}^2$ and $\pi_{\Gamma'}(o)$ in $\Gamma'\backslash\mathbb{H}^2$ are "almost isometric" if $\Gamma, \Gamma' \in \mathrm{Sub}_{\mathrm{dtf}}(G)$ are "close".

2.9. **The Geometric Topology.** Previously, we have considered the algebraic topology on $\mathcal{R}(\Sigma)$. Using the Chabauty topology we will now introduce the *geometric topology*. This terminology is justified by its geometric implications; see Proposition 2.8.8.

Definition 2.9.1 (Geometric topology on $\mathcal{R}^*(\Sigma)$)**.** The *geometric topology* on $\mathcal{R}^*(\Sigma)$ is the initial topology with respect to the following two maps:

(i) $i\colon \mathcal{R}^*(\Sigma) \hookrightarrow G^S, \rho \longmapsto (\rho(s))_{s\in S}$, where $S \subseteq \pi_1(\Sigma)$ is a generating set for $\pi_1(\Sigma)$, and

(ii) $\mathrm{im}\colon \mathcal{R}^*(\Sigma) \longrightarrow \mathrm{Sub}_{\mathrm{dtf}}(G), \rho \longmapsto \mathrm{im}\,\rho$, which sends every (discrete) representation $\rho \in \mathcal{R}^*(\Sigma)$ to its image in $\mathrm{Sub}(G)$.

Recall that the algebraic topology on $\mathcal{R}^*(\Sigma)$ is the initial topology with respect to just the map $i\colon \mathcal{R}^*(\Sigma) \hookrightarrow G^S$, such that the geometric topology is apriori stronger than the algebraic topology. Using the fact that Σ is of finite type it can be shown that it is *not*. However, there are counterexamples for surfaces of infinite type; see [CEM06, Section I.3.1.10] for more details.

Proposition 2.9.2. *Consider $\mathcal{R}^*(\Sigma)$ with the algebraic topology. Then the map*

$$\mathrm{im}\colon \mathcal{R}^*(\Sigma) \longrightarrow \mathrm{Sub}_{\mathrm{dtf}}(G)$$

is a local homeomorphism onto its image $\mathcal{D}(\Sigma)$, which is the set of all discrete and torsion-free subgroups $\Gamma' < G$ such that there is an orientation preserving homeomorphism $f\colon \Sigma \longrightarrow \Gamma'\backslash\mathbb{H}^2$.

If we consider $\mathcal{R}(\Sigma)$ instead, the image of $\mathrm{im}\colon \mathcal{R}(\Sigma) \longrightarrow \mathrm{Sub}_{\mathrm{dtf}}(G)$ *consists of all lattices $\Gamma \in \mathcal{D}(\Sigma)$. We denote this set by $\mathcal{L}(\Sigma)$.*

Remark 2.9.3. In particular, Proposition 2.9.2 implies that the geometric topology coincides with the algebraic topology on $\mathcal{R}^*(\Sigma)$.

Proof. This proposition is proved in [Har77, Sections 2.3 and 2.4]. However, his proof contains only the details for the case of a closed surface Σ. In the general case a sketch of proof is given in [CEM06, Remark, p. 66]. A detailed proof can be found in the author's PhD thesis [Kri21, Proposition 2.8.2]. □

Proposition 2.9.2 allows us to identify the moduli space $\mathcal{M}(\Sigma)$ with the space $G\backslash\mathcal{L}(\Sigma)$ of conjugacy classes of lattices.

Proposition 2.9.4. *The space $\mathcal{L}(\Sigma)$ is invariant under the conjugation action of G and we may identify its quotient $G\backslash\mathcal{L}(\Sigma)$ with the moduli space $\mathcal{M}(\Sigma)$ via the following homeomorphism*

$$\psi\colon \mathcal{M}(\Sigma) \longrightarrow G\backslash\mathcal{L}(\Sigma), \quad [[\rho]] \longmapsto [\mathrm{im}\,\rho].$$

Proof. Let $\Gamma \in \mathcal{L}(\Sigma)$ and $g \in G$. Then $\Gamma' := g\Gamma g^{-1}$ is a torsion-free lattice, too. Moreover, the element $g \in G \cong \mathrm{Isom}_+(\mathbb{H}^2)$ induces an orientation preserving isometry

$$g\colon \Gamma\backslash\mathbb{H}^2 \longrightarrow \Gamma'\backslash\mathbb{H}^2,$$

$$\Gamma x \longmapsto \Gamma' g x,$$

whence $\Gamma' = g\Gamma g^{-1} \in \mathcal{L}(\Sigma)$.

Let us consider the right-action $\mathrm{Aut}^*(\pi_1(\Sigma)) \curvearrowright \mathcal{R}(\Sigma)$. We claim that the map

$$\mathrm{im}\colon \mathcal{R}(\Sigma) \longrightarrow \mathcal{L}(\Sigma)$$

induces a homeomorphism:

$$\begin{tikzcd} \mathcal{R}(\Sigma) \arrow[d] \arrow[dr, two heads, "\mathrm{im}"] & \\ \mathcal{R}(\Sigma)/\mathrm{Aut}^*(\pi_1(\Sigma)) \arrow[r, dashed, "\varphi"'] & \mathcal{L}(\Sigma) \end{tikzcd}$$

Clearly, if $\rho_1 = \rho_2 \circ \alpha$ for $\rho_1, \rho_2 \in \mathcal{R}(\Sigma)$ and $\alpha \in \mathrm{Aut}^*(\pi_1(\Sigma))$ then $\mathrm{im}\,\rho_1 = \mathrm{im}\,\rho_2$.

On the other hand, suppose $\Gamma = \mathrm{im}\,\rho_1 = \mathrm{im}\,\rho_2$ for some $\rho_1, \rho_2 \in \mathcal{R}(\Sigma)$. There are orientation preserving homeomorphisms $f_1, f_2\colon \Sigma \longrightarrow \Gamma\backslash\mathbb{H}^2$ such that ρ_1, ρ_2 are holonomy representations of f_1, f_2, respectively. Then $\rho_2^{-1} \circ \rho_1$ is induced by $f_2^{-1} \circ f_1$ such that $\alpha := \rho_2^{-1} \circ \rho_1 \in \mathrm{Aut}^*(\pi_1(\Sigma))$, and $\rho_1 = \rho_2 \circ \alpha$. Hence, φ is a bijection.

By definition of the quotient topology φ is continuous. Finally, im is a local homeomorphism such that φ^{-1} is continuous, too. This shows that φ is a homeomorphism.

Observe that φ is equivariant with respect to the conjugation action of G both on $\mathcal{R}(\Sigma)/\mathrm{Aut}^*(\pi_1(\Sigma))$ and on $\mathcal{L}(\Sigma)$. Therefore, taking the quotient by the conjugation actions yields a homeomorphism

$$\begin{tikzcd} \mathcal{R}(\Sigma)/\mathrm{Aut}^*(\pi_1(\Sigma)) \arrow[r, "\varphi", "\cong"'] \arrow[d] & \mathcal{L}(\Sigma) \arrow[d] \\ \mathcal{T}(\Sigma)/\mathrm{Out}^*(\pi_1(\Sigma)) \arrow[r, dashed, "\psi", "\cong"'] & G\backslash\mathcal{L}(\Sigma) \end{tikzcd}$$

given by $\psi([[\rho]]) = [\mathrm{im}\,\rho]$ for every $[[\rho]] \in \mathcal{M}(\Sigma) = \mathcal{T}(\Sigma)/\mathrm{Out}^*(\Sigma)$. □

We conclude this section with the following lemma, that restricts the kind of closed subgroups that arise in the closure of the G-orbit of a lattice $\Gamma \in \mathcal{L}(\Sigma)$ in $\mathrm{Sub}(G)$.

Lemma 2.9.5. *Let $\Gamma \in \mathcal{L}(\Sigma)$ and let $(g_n)_{n\in\mathbb{N}} \subseteq G$ be a sequence of elements such that*

$$g_n^{-1}\Gamma g_n \to H \qquad (n \to \infty)$$

converges to $H \in \mathrm{Sub}(G)$. Then, either

(i) H is abelian, or
(ii) H is a conjugate of Γ.

Proof. Let $\pi\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ denote the quotient map and $o \in \mathbb{H}^2$. There are two cases to consider:

a) The sequence $\pi(g_n o)$ is contained in a compact set $K \subseteq \Gamma\backslash\mathbb{H}^2$, or
b) The sequence $\pi(g_n o)$ goes to infinity in $\Gamma\backslash\mathbb{H}^2$.

In case a) we may find a compact set $K' \subseteq \mathbb{H}^2$ such that $\pi(K') = K$, and therefore elements $\gamma_n \in \Gamma$ such that $\gamma_n g_n o \in K'$. Then there is a convergent subsequence $\gamma_{n_k} g_{n_k} \to g$ as $k \to \infty$, such that

$$H = \lim_{n\to\infty} g_n^{-1}\Gamma g_n = \lim_{k\to\infty} g_{n_k}^{-1}\gamma_{n_k}^{-1}\Gamma\gamma_{n_k} g_{n_k} = g^{-1}\Gamma g.$$

In case b) there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\pi(g_{n_k} o) \in C$ is contained in a cusp region C of the thin part of $\Gamma\backslash\mathbb{H}^2$. Let $B \subseteq \pi^{-1}(C)$ be a horoball in the preimage of C centered at $\xi \in \partial\mathbb{H}^2$, and let $\widehat{\gamma}_{n_k} \in \Gamma$ be such that $\widehat{\gamma}_{n_k} g_{n_k} o \in B$ for every $k \in \mathbb{N}$, and

$$\xi = \lim_{k\to\infty} \widehat{\gamma}_{n_k} g_{n_k} o.$$

Thus, if we set $\widehat{g}_k := \widehat{\gamma}_{n_k} g_{n_k}$, then $\xi = \lim_{k\to\infty} \widehat{g}_k o$, and

$$H = \lim_{k\to\infty} g_{n_k}^{-1}\Gamma g_{n_k} = \lim_{k\to\infty} (\widehat{g}_k)^{-1}(\widehat{\gamma}_{n_k})^{-1}\Gamma\widehat{\gamma}_{n_k}\widehat{g}_k = \lim_{k\to\infty} (\widehat{g}_k)^{-1}\Gamma\widehat{g}_k.$$

Let $h \in H$ and let $\gamma_k \in \Gamma$ such that $(\widehat{g}_k)^{-1}\gamma_k\widehat{g}_k \to h$ as $k \to \infty$. Note that

$$d(ho, o) = \lim_{k\to\infty} d((\widehat{g}_k)^{-1}\gamma_k\widehat{g}_k o, o) = \lim_{k\to\infty} d(\gamma_k\widehat{g}_k o, \widehat{g}_k o).$$

Hence there is $D > 0$, such that

$$d(\gamma_k\widehat{g}_k o, \widehat{g}_k o) < D$$

for all $k \in \mathbb{N}$. Thus $\gamma_k B \cap B \neq \emptyset$ and $\gamma_k \in P = \mathrm{stab}_\xi(\Gamma)$ for large k. Because P is abelian, this implies that H is abelian. □

3. L^1-Convergence of Truncated Dirichlet Domains

In Proposition 2.9.2 we have introduced the space of discrete subgroups

$$\mathcal{D}(\Sigma) = \{\Gamma \in \mathrm{Sub}_{\mathrm{dtf}}(G) \mid \Gamma\backslash\mathbb{H}^2 \cong \Sigma\},$$

whose quotient $\Gamma\backslash\mathbb{H}^2$ is homeomorphic to Σ. Given a group $\Gamma \in \mathcal{D}(\Sigma)$ and a point $o \in \mathbb{H}^2$ one can study the action of Γ on $\mathbb{H}^2$ via its *Dirichlet domain (based at o)*:

$$D_o(\Gamma) := \{x \in \mathbb{H}^2 \mid d(x, o) \leq d(x, \gamma o) \quad \forall \gamma \in \Gamma \setminus \{1\}\}.$$

In general, $D_o(\Gamma)$ will not have finite-area. However, Γ acts on the convex hull

$$\widetilde{C}(\Gamma) := \mathrm{conv}(L(\Gamma)),$$

of its limit set $L(\Gamma) := \overline{\Gamma \cdot o} \cap \partial\mathbb{H}^2$. Its quotient $C(\Gamma) := \Gamma\backslash\widetilde{C}(\Gamma)$ is called the *convex core* of $\Gamma\backslash\mathbb{H}^2$ and has finite area. Indeed, the convex core $C(\Gamma)$ is obtained from $\Gamma\backslash\mathbb{H}^2$ by cutting off its infinite-area hyperbolic funnels at their waist geodesics. Therefore, it is useful to study the *truncated Dirichlet domain*

$$\widehat{D_o}(\Gamma) := D_o(\Gamma) \cap \widetilde{C}(\Gamma),$$

which is always a finite-sided hyperbolic polygon. Note that the truncated Dirichlet domain coincides with the usual Dirichlet domain if Γ is a lattice. Indeed, in this case $L(\Gamma) = \partial\mathbb{H}^2$ and $\widetilde{C}(\Gamma) = \mathbb{H}^2$.

It is natural to ask, whether the truncated Dirichlet domain $\widehat{D_o}(\Gamma)$ depends continuously on $\Gamma \in \mathcal{D}(\Sigma)$, and if so in what sense. The goal of this section is to answer this question in the form of the following lemma.

Lemma 3.1.1. *Let $o \in \mathbb{H}^2$. The map*

$$\mathcal{D}(\Sigma) \longrightarrow L^1(\mathbb{H}^2), \quad \Gamma \longmapsto \mathbb{1}_{\widehat{D_o}(\Gamma)},$$

is continuous, where $\mathbb{1}_{\widehat{D_o}(\Gamma)}$ denotes the indicator function of $\widehat{D_o}(\Gamma) \subseteq \mathbb{H}^2$.

Although this lemma seems classical, we could not find a proof in the literature. Since it is an essential ingredient for Proposition 4.1.5 and Theorem 4.2.2, we will give a complete proof using elementary hyperbolic geometry here. The proof is split into several sublemmas, some of which might be of individual interest.

The first two lemmas are concerned with the pointwise convergence of the characteristic functions $\mathbb{1}_{\widetilde{C}(\Gamma_n)}$ and $\mathbb{1}_{D_o(\Gamma_n)}$ outside of a set of measure zero.

Lemma 3.1.2. *Let $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\Sigma)$ be a sequence converging to $\Gamma \in \mathcal{D}(\Sigma)$. Then*

$$\mathbb{1}_{\widetilde{C}(\Gamma_n)}(x) \to \mathbb{1}_{\widetilde{C}(\Gamma)}(x) \qquad (n \to \infty)$$

for every $x \in \mathbb{H}^2 \setminus \partial\widetilde{C}(\Gamma)$.

Proof. By Proposition 2.9.2 we may choose $\rho, \rho_n \in \mathcal{R}^*(\Sigma)$ such that $\operatorname{im}\rho = \Gamma$, $\operatorname{im}\rho_n = \Gamma_n$ for large n, and $\rho_n \to \rho$ as $n \to \infty$.

First, let $x \in \mathbb{H}^2 \setminus \widetilde{C}(\Gamma)$. Let $\gamma \in \Gamma$ be a hyperbolic element, whose axis bounds an open half-space $H(\gamma)$ such that $H(\gamma) \cap \widetilde{C}(\Gamma) = \emptyset$ and $x \in H(\gamma)$. Let $c \in \pi_1(\Sigma)$ such that $\rho(c) = \gamma$. Then $\rho_n(c) = \gamma_n$ are boundary elements converging to γ. But then $x \in H(\gamma_n)$ such that $x \in \mathbb{H}^2 \setminus \widetilde{C}(\Gamma_n)$ for large $n \in \mathbb{N}$.

Let $x \in \operatorname{int}(\widetilde{C}(\Gamma))$ and suppose that there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $x \notin \widetilde{C}(\Gamma_{n_k})$ for all $k \in \mathbb{N}$. Then there are primitive hyperbolic boundary elements $\gamma_{n_k} \in \Gamma_{n_k}$, whose axes $\operatorname{ax}(\gamma_{n_k})$ bound half-spaces $H(\gamma_{n_k}) \subseteq \mathbb{H}^2$ such that $x \in H(\gamma_{n_k})$.

We claim that there is $D \geq 0$ such that $d(x, \operatorname{ax}(\gamma_{n_k})) \leq D$. Suppose to the contrary that there is a further subsequence, also denoted by $(n_k)_{k\in\mathbb{N}}$, such that $d(x, \operatorname{ax}(\gamma_{n_k})) \to \infty$ as $k \to \infty$. Let $\eta, \eta' \in \Gamma$ be two non-commuting elements and let $\eta_{n_k}, \eta'_{n_k} \in \Gamma_{n_k}$, that converge to $\eta, \eta' \in \Gamma$, respectively. Then $\eta_{n_k} x, \eta'_{n_k} x \in H(\gamma_{n_k})$ for large k. However, this means that η_{n_k} and η'_{n_k} must leave the entire half-space $H(\gamma_{n_k})$ invariant. Thus they commute having the same axis $\mathrm{ax}(\gamma_{n_k})$. But then also η and η' commute; in contradiction to our assumption.

Because γ_{n_k} are primitive boundary elements their translation length $\ell(\gamma_{n_k})$ is uniformly bounded from above by the maximal length L of a boundary component of $C(\Gamma_{n_k})$. Moreover, there is a lower bound $\varepsilon_0 > 0$ such that $d(x, \gamma_{n_k}(x)) \geq \varepsilon_0$, because the Γ_{n_k} converge to a discrete group Γ and $\gamma_{n_k} \neq e$ for all k. It follows from elmentary hyperbolic geometry that $d(x, \mathrm{ax}(\gamma_{n_k})) \leq D$ and $d(x, \gamma_{n_k}(x)) \geq \varepsilon_0$ implies $\ell(\gamma_{n_k}) \geq \varepsilon'_0$ for some $\varepsilon'_0 > 0$. The subset

$$C = \{g \in \mathrm{PSL}_2(\mathbb{R}) \,|\, g \text{ is hyperbolic}, \varepsilon'_0 \leq \ell(g) \leq L, \mathrm{ax}(g) \cap \overline{B}_x(D) \neq \emptyset\} \subseteq G$$

is compact, such that $\gamma_{n_k} \to \gamma \in \Gamma$ up to a subsequence.

Let $c \in \pi_1(\Sigma)$ such that $\rho(c) = \gamma$. Because $\gamma_{n_k} \to \gamma$ and $\rho_{n_k}(c) \to \gamma$ as $k \to \infty$, we have that $\gamma_{n_k} = \rho_{n_k}(c)$ by Lemma 2.8.6. Hence, $\gamma = \rho(c)$ is a boundary element. Since $x \in H(\gamma_{n_k})$ and $\gamma_{n_k} \to \gamma$, it follows that $x \in \overline{H(\gamma)}$ contradicting $x \in \mathrm{int}(\widetilde{C}(\Gamma))$. □

Lemma 3.1.3. *Let $(\Gamma_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(\Sigma)$ be a sequence converging to $\Gamma \in \mathcal{D}(\Sigma)$. Then*

$$\mathbb{1}_{D_o(\Gamma_n)}(x) \to \mathbb{1}_{D_o(\Gamma)}(x) \qquad (n \to \infty)$$

for every $x \in \mathbb{H}^2 \setminus \partial D_o(\Gamma)$.

Proof. Let $x \in \mathrm{int}(D_o(\Gamma))$. Then $d(x, o) < d(x, \gamma o)$ for every $\gamma \in \Gamma \setminus \{e\}$. We want to show that $x \in D_o(\Gamma_n)$ for large n. Assume to the contrary that there is a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $x \notin D_o(\Gamma_{n_k})$, i.e. there are $\gamma_{n_k} \in \Gamma_{n_k} \setminus \{e\}$ such that $d(x, o) > d(x, \gamma_{n_k} o)$ for every $k \in \mathbb{N}$. Up to passing to a subsequence we may assume that $\gamma_{n_k} \to \gamma \in \Gamma$, and $\gamma \neq e$ by Lemma 2.8.6. But

$$d(x, o) \geq \lim_{k \to \infty} d(x, \gamma_{n_k} o) = d(x, \gamma o)$$

contradicting $x \in \mathrm{int}(D_o(\Gamma))$.

Let $x \in \mathbb{H}^2 \setminus D_o(\Gamma)$. Then there is $\gamma \in \Gamma \setminus \{e\}$ such that $d(x, o) > d(x, \gamma o)$. We want to show that $x \in \mathbb{H}^2 \setminus D_o(\Gamma_n)$ for large n. Let $\gamma_n \in \Gamma_n$ such that $\gamma_n \to \gamma \neq e$ as $n \to \infty$. Then $\gamma_n \neq e$ and $d(x, \gamma_n o) < d(x, o)$, such that $x \notin D_o(\Gamma_n)$ for large n. □

We give a characterization of peripheral curves, now.

Lemma 3.1.4. *Let $\mu = \{\gamma_1, \ldots, \gamma_r\} \subseteq \Sigma$ be a filling collection of essential simple closed curves such that*

(i) γ_i, γ_j are in minimal position for all $i, j \in \{1, \ldots, r\}$,
(ii) the curves in μ are pairwise non-isotopic, and

(iii) for distinct triples $i,j,k \in \{1,\dots,r\}$ at least one of the intersections $\gamma_i \cap \gamma_j, \gamma_j \cap \gamma_k, \gamma_i \cap \gamma_k$ is empty.

Let $\alpha \subseteq \Sigma$ be a homotopically non-trivial closed curve. Then

$$\alpha \text{ is peripheral} \iff i(\alpha,\gamma_i) = 0 \text{ for every } i = 1,\dots,r.$$

Proof. Suppose $\alpha \subseteq \Sigma$ is peripheral. Then α is homotopic to one of the punctures of Σ. Since μ fills Σ there is a punctured disk $\mathbb{D}^\times \subseteq \Sigma \setminus \mu$ surrounding this puncture. Thus we may homotope α into $\mathbb{D}^\times$ such that $i(\alpha,\gamma_i) = 0$ for every $i = 1,\dots,r$.

If $i(\alpha,\gamma_i) = 0$ for every $i = 1,\dots,r$, then there are isotopies moving γ_i to $\widetilde{\gamma_i}$ such that $\widetilde{\gamma_i} \cap \alpha = \emptyset$. Because our system satisfies the hypotheses (i)-(iii), there is an isotopy of Σ moving $\bigcup_{i=1}^r \gamma_i$ to $\bigcup_{i=1}^r \widetilde{\gamma_i}$; see [FM12, Lemma 2.9]. The collection $\widetilde{\mu} = \{\widetilde{\gamma_1},\dots,\widetilde{\gamma_r}\}$ is still filling and α is a homotopically non-trivial closed curve in $\Sigma \setminus \widetilde{\mu}$. Thus α is contained in a punctured disk $\mathbb{D}^\times \subseteq \Sigma \setminus \mu$. Therefore, α is homotopic to a puncture, i.e. α is peripheral. $\square$

Using the previous lemma we will prove next that there is a lower bound for the lengths of essential curves with respect to a convergent sequence of representations:

Lemma 3.1.5. *Let $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$ be a sequence converging to $\rho \in \mathcal{R}^*(\Sigma)$. Then there is $\varepsilon > 0$ such that*

$$\ell(\rho_n(\alpha)) < \varepsilon \implies \alpha \text{ is peripheral}$$

for every $\alpha \in \pi_1(\Sigma)$ and all $n \in \mathbb{N}$.

Proof. Suppose to the contrary that there is a subsequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and non-peripheral elements $\alpha_{n_k} \in \pi_1(\Sigma)$ such that $\ell(\rho_{n_k}(\alpha_{n_k})) \to 0$ as $k \to \infty$. Choose a collection of curves $\mu = \{\gamma_1,\dots,\gamma_r\} \subseteq \pi_1(\Sigma)$ as in Lemma 3.1.4. Then there is $j_k \in \{1,\dots,r\}$ for every $k \in \mathbb{N}$ such that $i(\gamma_{j_k},\alpha_{n_k}) \neq 0$. Up to passing to another subsequence we may assume that $\gamma_{j_k} = \widehat{\gamma} \in \mu$ is constant. But then by the Collar Lemma 2.2.10

$$\ell(\rho_{n_k}(\widehat{\gamma})) \to \infty \qquad (k \to \infty).$$

This contradicts the fact that $\ell(\rho_{n_k}(\widehat{\gamma})) \to \ell(\rho(\widehat{\gamma}))$ as $k \to \infty$. $\square$

The following lemma shows how to obtain an upper bound on the diameter of a connected subset C of a hyperbolic surface given a lower bound ε for the injectivity radius and an upper bound for the volume of the ε-neighborhood of C.

Lemma 3.1.6. *Let X be a hyperbolic surface and let $\varepsilon > 0$. Further, let $C \subseteq X$ be a path-connected Borel set, and suppose that $\mathrm{inj}_X(x) \geq \varepsilon$ for every $x \in C$. Then*

$$\mathrm{diam}_X(C) \leq 4\varepsilon \cdot \left(\frac{\mathrm{vol}_X(N_\varepsilon(C))}{\mathrm{vol}_{\mathbb{H}^2}(B_o(\varepsilon))} + 1 \right),$$

where $N_\varepsilon(C) = \{x \in X \,|\, d_X(x, C) < \varepsilon\}$ *is the open* ε*-neighborhood of* C *in* X *and* $o \in \mathbb{H}^2$.

Proof. Note that for every $x \in C$ the ball $B_x(\varepsilon) \subseteq N_\varepsilon(C)$ is embedded and has the same measure as a ball of radius ε in the hyperbolic plane.

Let us now consider

$$S = \{Y \subseteq C : d(y_1, y_2) \geq 2\varepsilon \quad \forall y_1, y_2 \in Y, y_1 \neq y_2\}.$$

By Zorn's Lemma we may choose a maximal element $Y_0 \in S$ with respect to inclusion $\subseteq$. We claim that the collection of balls $\{B_y(2\varepsilon) : y \in Y_0\}$ covers C. Indeed, if there is $y' \in C$, which is not in any $\{B_y(2\varepsilon)\}_{y \in Y_o}$, it has distance greater or equal than 2ε from any point $y \in Y_0$. But then $\{y'\} \cup Y_0 \in S$, which contradicts the maximality of Y_0.

On the other hand the balls of radius ε centered at $y \in Y_0$ are disjoint by definition of S whence

$$\bigsqcup_{y \in Y_0} B_y(\varepsilon) \subseteq N_\varepsilon(C),$$

such that

$$\mathrm{vol}_X(N_\varepsilon(C)) \geq \sum_{y \in Y_0} \mathrm{vol}_X(B_y(\varepsilon)) = \#Y_0 \cdot \mathrm{vol}_{\mathbb{H}^2}(B_o(\varepsilon)).$$

It follows that

$$\#Y_0 \leq \frac{\mathrm{vol}_X(N_\varepsilon(C))}{\mathrm{vol}_{\mathbb{H}^2}(B_o(\varepsilon))}.$$

Let $x, y \in C$, and let $c\colon [0,1] \longrightarrow C$ be a path from x to y. Then c is covered by $\{B_y(2\varepsilon)\}_{y \in Y_0}$ and we may shorten c to a path c', that intersects any ball $B_y(2\varepsilon)$, $y \in Y_0$, at most once. Covering c' by $\{B_y(2\varepsilon)\}_{y \in Y_0}$ we obtain a sequence of pairwise distinct $y_1, \ldots, y_m \in Y_0$, such that $x \in B_{y_1}(2\varepsilon)$, $B_{y_i}(2\varepsilon) \cap B_{y_{i+1}}(2\varepsilon) \neq \varnothing$ and $y \in B_{y_m}(2\varepsilon)$. Thus,

$$d(x, y) \leq d(x, y_1) + \sum_{i=1}^{m} d(y_i, y_{i+1}) + d(y_m, y) \leq 2\varepsilon + 4\varepsilon \cdot \#Y_0 + 2\varepsilon \leq 4\varepsilon \cdot \left(\frac{\mathrm{vol}_X(N_\varepsilon(C))}{\mathrm{vol}_{\mathbb{H}^2}(B_o(\varepsilon))} + 1 \right).$$

Because $x, y \in C$ were arbitrary, this proves the assertion. $\square$

Using the previous lemmas we obtain an upper bound on the diameter of the thick part of the convex core, as follows.

Lemma 3.1.7. *Let* $(\Gamma_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(\Sigma)$ *be a sequence converging to* $\Gamma \in \mathcal{D}(\Sigma)$. *Then there is* $\varepsilon > 0$ *such that for every* $0 < \varepsilon' < \varepsilon$ *the* ε'*-thick part of the convex core*

$$C(\Gamma_n)_{\geq \varepsilon'} := C(\Gamma_n) \cap (\Gamma_n \backslash \mathbb{H}^2)_{\geq \varepsilon'} = \{x \in C(\Gamma_n) \,|\, \mathrm{inj}_{\Gamma_n \backslash \mathbb{H}^2}(x) \geq \varepsilon'\}$$

is path-connected for every $n \in \mathbb{N}$.

In particular, for every $0 < \varepsilon' < \varepsilon$ *there is* $R = R(\varepsilon') > 0$ *such that*

$$\mathrm{diam}_{\Gamma_n \backslash \mathbb{H}^2}(C(\Gamma_n)_{\geq \varepsilon'}) \leq R$$

for every $n \in \mathbb{N}$.

Proof. We may choose $\rho_n \to \rho \in \mathcal{R}^*(\Sigma)$ such that $\operatorname{im}\rho_n = \Gamma_n$ and $\operatorname{im}\rho = \Gamma$. Let $\varepsilon > 0$ be as in Lemma 3.1.5. Without loss of generality we may assume that ε is smaller than the Margulis constant. Let $0 < \varepsilon' < \varepsilon$, let $n \in \mathbb{N}$, and let $T \subseteq (\Gamma_n \backslash \mathbb{H}^2)_{<\varepsilon'}$ be a tube component of the ε'-thin part. Let $\alpha_n \in \pi_1(\Sigma)$ such that $\rho_n(\alpha_n) \in \Gamma_n$ corresponds to the waist geodesic of T. Then $\ell(\rho_n(\alpha_n)) < \varepsilon' < \varepsilon$ such that α_n is peripheral. Therefore, all tube components of the ε'-thin part are peripheral and $C(\Gamma_n)_{\geq\varepsilon'}$ is path-connected.

We want to apply Lemma 3.1.6 to $C(\Gamma_n)_{\geq\varepsilon'}$. Note that $N_{\varepsilon'}(C(\Gamma_n)_{\geq\varepsilon'}) \subseteq N_\varepsilon(C(\Gamma_n))$. Further, $N_\varepsilon(C(\Gamma_n)) \setminus C(\Gamma_n)$ consists of half-collars of width ε about the boundary curves of $C(\Gamma_n)$. Since the lengths of the boundary curves converge there is a uniform bound $V > 0$ such that $\operatorname{vol}_{\Gamma_n\backslash\mathbb{H}^2}(N_\varepsilon(C(\Gamma_n))\backslash C(\Gamma_n)) \leq V$ for all $n \in \mathbb{N}$. Recall that $\operatorname{vol}_{\Gamma_n\backslash\mathbb{H}^2}(C(\Gamma_n)) = 2\pi|\chi(\Sigma)|$, such that

$$\begin{aligned}\operatorname{vol}_{\Gamma_n\backslash\mathbb{H}^2}(N_{\varepsilon'}(C(\Gamma_n)_{\geq\varepsilon'})) &\leq \operatorname{vol}_{\Gamma_n\backslash\mathbb{H}^2}(C(\Gamma_n)) + \operatorname{vol}_{\Gamma_n\backslash\mathbb{H}^2}(N_\varepsilon(C(\Gamma_n)) \setminus C(\Gamma_n)) \\ &\leq 2\pi|\chi(\Sigma)| + V.\end{aligned}$$

Setting

$$R(\varepsilon') := 4\varepsilon' \cdot \left(\frac{2\pi|\chi(\Sigma)| + V}{\operatorname{vol}_{\mathbb{H}^2}(B_o(\varepsilon'))} + 1\right)$$

the assertion follows from Lemma 3.1.6. $\square$

Finally, we will need to know the area of the residual thin parts.

Lemma 3.1.8.

(i) Let $\delta > 0$ *and let* $\gamma \in G = \operatorname{Isom}^+(\mathbb{H}^2)$ *be defined by* $\gamma(z) = z + 1$ *for every* $z \in \mathbb{H}^2$. *Consider the fundamental domain for the corresponding cusp region*

$$C_\delta := \{z = x + iy \in \mathbb{H}^2 \mid 0 \leq x \leq 1, d(z, \gamma(z)) \leq \delta\},$$

that consists of all the points that are moved less than δ *by* γ.

Then

$$\operatorname{vol}_{\mathbb{H}^2}(C_\delta) = 2\sinh(\delta/2).$$

(ii) Let $\delta > \delta_0 > 0$ *and* $\gamma \in G = \operatorname{Isom}^+(\mathbb{H}^2)$ *be defined by* $\gamma(z) = e^{\delta_0}z$ *for every* $z \in \mathbb{H}^2$. *Consider the fundamental domain for the corresponding funnel*

$$F_\delta := \{z = x + iy \in \mathbb{H}^2 \mid x \geq 0, 1 \leq |z| \leq e^{\delta_0}, d(z, \gamma(z)) \leq \delta\}$$

to the right of the axis $\operatorname{ax}\gamma = i\mathbb{R}$, *that consists of all the points that are moved less than* δ *by* γ.

Then

$$\operatorname{vol}_{\mathbb{H}^2}(F_\delta) \leq 2\sinh(\delta/2).$$

Proof. Recall the following formulas from hyperbolic geometry:

$$\sinh(d(z,w)/2) = \frac{|z-w|}{2\sqrt{\mathrm{Im}(z)\,\mathrm{Im}(w)}}$$

for every $z, w \in \mathbb{H}^2$, and

$$\mathrm{vol}_{\mathbb{H}^2}(A) = \int_A \frac{1}{y^2}\,dx\,dy$$

for every Borel set $A \subseteq \mathbb{H}^2$; see [Bea83, Theorem 7.2.1 (iii)] and [Bus10, (1.1.1), p. 2], respectively.

(i) We compute that $z = x + iy \in C_\delta$ if and only if

$$\sinh(\delta/2) \geq \sinh(d(z,z+1)/2) = \frac{1}{2y} \iff y \geq \frac{1}{2\sinh(\delta/2)} =: y_\delta.$$

Hence,

$$\mathrm{vol}_{\mathbb{H}^2}(C_\delta) = \int_0^1 \int_{y_\delta}^{\infty} \frac{1}{y^2}\,dy\,dx = \frac{1}{y_\delta} = 2\sinh(\delta/2).$$

(ii) For $z = re^{i\alpha} \in \mathbb{H}^2$, $r > 0, \alpha \in (0,\pi)$, we have that $d(z,\gamma(z)) \leq \delta$ if and only if

$$\begin{aligned}
\sinh(\delta/2) \geq \sinh(d(z,e^{\delta_0}z)/2) &= \frac{\left|re^{i\alpha} - re^{\delta_0}e^{i\alpha}\right|}{2\sqrt{r\sin(\alpha)\cdot re^{\delta_0}\sin(\alpha)}} \\
&= \frac{1}{\sin(\alpha)}\frac{e^{\delta_0}-1}{2e^{\delta_0/2}} = \frac{\sinh(\delta_0/2)}{\sin(\alpha)} \\
\iff \sin(\alpha) \geq \frac{\sinh(\delta_0/2)}{\sinh(\delta/2)}&.
\end{aligned}$$

There is a unique $\alpha_\delta \in (0,\pi/2)$ such that $\sin(\alpha_\delta) = \frac{\sinh(\delta_0/2)}{\sinh(\delta/2)}$.

Using polar coordinates we obtain

$$\begin{aligned}
\mathrm{vol}_{\mathbb{H}^2}(F_\delta) &= \int_{F_\delta} \frac{1}{y^2}\,dx\,dy = \int_{\alpha_\delta}^{\pi/2}\int_1^{e^{\delta_0}} \frac{r}{r^2\sin^2\varphi}\,dr\,d\varphi \\
&= \delta_0 \cdot \int_{\alpha_\delta}^{\pi/2} \frac{1}{\sin^2\varphi}\,d\varphi = \delta_0 \cdot [\cot\varphi]_{\varphi=\alpha_\delta}^{\pi/2} \\
&= \delta_0 \cdot \cot(\alpha_\delta) = \delta_0 \cdot \frac{\cos(\alpha_\delta)}{\sin(\alpha_\delta)} \leq \frac{\delta_0}{\sin(\alpha_\delta)} \\
&= \frac{\delta_0}{\sinh(\delta_0/2)}\sinh(\delta/2) \leq 2\sinh(\delta/2),
\end{aligned}$$

where we used in the last inequality that $x \leq \sinh(x)$ for all $x \geq 0$.

□

We are ready to prove Lemma 3.1.1 now.

Proof of Lemma 3.1.1. Let $\Gamma \in \mathcal{D}(\Sigma)$ and let $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\Sigma)$ be a sequence converging to Γ. By definition $\widehat{D}_o(\Gamma) = \widetilde{C}(\Gamma) \cap D_o(\Gamma)$ such that

$$\mathbb{1}_{\widehat{D}_o(\Gamma)} = \mathbb{1}_{\widetilde{C}(\Gamma)} \cdot \mathbb{1}_{D_o(\Gamma)}.$$

By Lemma 3.1.2 and Lemma 3.1.3 we have that

$$\mathbb{1}_{\widetilde{C}(\Gamma_n)}(x) \cdot \mathbb{1}_{D_o(\Gamma_n)}(x) \to \mathbb{1}_{\widetilde{C}(\Gamma)}(x) \cdot \mathbb{1}_{D_o(\Gamma)}(x) \qquad (n \to \infty)$$

for every $x \in \mathbb{H}^2 \setminus (\partial\widetilde{C}(\Gamma) \cup \partial D_o(\Gamma))$. Note that $\partial\widetilde{C}(\Gamma) \cup \partial D_o(\Gamma)$ has measure zero.

Let $\varepsilon > 0$ be as in Lemma 3.1.7 and let $0 < \varepsilon' < \varepsilon$. Then there is $R = R(\varepsilon') > 0$ such that $\pi_n(\overline{B}_o(R))$ contains $C(\Gamma_n)_{\geq\varepsilon'}$ for every $n \in \mathbb{N}$, where $\pi_n\colon \mathbb{H}^2 \longrightarrow \Gamma_n\backslash\mathbb{H}^2$ is the quotient map. The complement $C(\Gamma_n) \setminus \pi_n(\overline{B}_o(R)) = \bigsqcup_{k=1}^{l} W_k$ is a disjoint union of subsets $W_1, \dots, W_l$ of peripheral cusp or tube components of the ε'-thin part. If Σ has genus g and p punctures there are at most $l \leq p$ such components. By Lemma 3.1.8 we have that

$$\mathrm{vol}_{\Gamma_n\backslash\mathbb{H}^2}(W_k) \leq 2\sinh(\varepsilon'/2)$$

for every $k = 1, \dots, l$, such that

$$\mathrm{vol}_{\Gamma_n\backslash\mathbb{H}^2}(C(\Gamma_n) \setminus \pi_n(\overline{B}_o(R))) \leq 2p\sinh(\varepsilon'/2) \to 0 \qquad (\varepsilon' \to 0).$$

Hence,

$$\mathrm{vol}_{\mathbb{H}^2}(\widehat{D}_o(\Gamma_n) \setminus \overline{B}_o(R)) \leq 2p\sinh(\varepsilon'/2) \to 0 \qquad (\varepsilon' \to 0).$$

Let $\varepsilon'' > 0$ be arbitrary, and choose $\varepsilon' > 0$ and $R = R(\varepsilon') > 0$ above, so that

$$\mathrm{vol}_{\mathbb{H}^2}(\widehat{D}_o(\Gamma_n) \setminus \overline{B}_o(R)) \leq \frac{\varepsilon''}{3} \qquad \text{and} \qquad \mathrm{vol}_{\mathbb{H}^2}(\widehat{D}_o(\Gamma) \setminus \overline{B}_o(R)) \leq \frac{\varepsilon''}{3}.$$

Then we compute that

$$\begin{aligned}
&\int_{\mathbb{H}^2} \left|\mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x)\right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) \\
\leq &\int_{\mathbb{H}^2} \left|\mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma_n)\cap\overline{B}_o(R)}(x)\right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) \\
+ &\int_{\mathbb{H}^2} \left|\mathbb{1}_{\widehat{D}_o(\Gamma_n)\cap\overline{B}_o(R)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)\cap\overline{B}_o(R)}(x)\right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) \\
+ &\int_{\mathbb{H}^2} \left|\mathbb{1}_{\widehat{D}_o(\Gamma)\cap\overline{B}_o(R)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x)\right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) \\
\leq &\frac{2}{3}\varepsilon'' + \int_{\mathbb{H}^2} \mathbb{1}_{\overline{B}_o(R)}(x) \cdot \left|\mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x)\right| d\,\mathrm{vol}_{\mathbb{H}^2}(x).
\end{aligned}$$

Observe that the function $\mathbb{1}_{\overline{B}_o(R)}(x) \cdot \left|\mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x)\right|$ converges pointwise almost everywhere to 0 and is dominated by the L^1-function $\mathbb{1}_{\overline{B}_o(R)}(x)$. Hence, by the Dominated

Convergence Theorem we conclude that

$$\int_{\mathbb{H}^2} \left| \mathbb{1}_{\widehat{D}_o(\Gamma_n)}(x) - \mathbb{1}_{\widehat{D}_o(\Gamma)}(x) \right| d\,\mathrm{vol}_{\mathbb{H}^2}(x) < \varepsilon''$$

for large n.

Because $\varepsilon'' > 0$ was arbitrary, the asserted convergence in $L^1(\mathbb{H}^2)$ follows. □

4. The IRS Compactification of Moduli Space

We proved in section 3 that truncated Dirichlet domains depend continuously on the group (Lemma 3.1.1) and we will apply this result to study Gelander's IRS compactification in the following.

In section 4.1 we will show that Gelander's IRS compactification is a compactification in the topological sense. We will then prove our main result (Theorem 4.2.2) relating the IRS compactification to the augmented moduli space in section 4.2.

4.1. Embedding the Moduli Space.

In section 1 we have explained how Gelander defined the IRS compactification of the moduli space $\mathcal{M}(\Sigma)$. Let us briefly recall his construction.

To any lattice $\Gamma \le G$ we can associate an IRS $\mu_\Gamma \in \mathrm{IRS}(G)$ in the following way. Let ν_Γ denote the (unique) right-invariant Borel probability measure on $\Gamma\backslash G$. Then the orbit map $G \longrightarrow \mathrm{Sub}(G), g \longmapsto g^{-1}\Gamma g$ descends to the map

$$\begin{aligned} \varphi_\Gamma \colon \Gamma\backslash G &\longrightarrow \mathrm{Sub}(G), \\ \Gamma g &\longmapsto g^{-1}\Gamma g. \end{aligned}$$

We obtain $\mu_\Gamma = (\varphi_\Gamma)_*(\nu_\Gamma) \in \mathrm{IRS}(G)$ as the push-forward measure of ν_Γ along φ_Γ.

By Proposition 2.9.4 there is a one-to-one correspondence between conjugacy classes of lattices in $[\Gamma] \in G\backslash\mathcal{L}(\Sigma)$ and hyperbolic surfaces $X = \Gamma\backslash\mathbb{H}^2 \in \mathcal{M}(\Sigma)$. Thus, we can use the above construction of an IRS to obtain a map from the moduli space $\mathcal{M}(\Sigma)$ to the space of IRSs of $G = \mathrm{PSL}(2,\mathbb{R})$

$$\iota \colon \mathcal{M}(\Sigma) \longmapsto \mathrm{IRS}(G), \quad [\Gamma] \longmapsto \mu_\Gamma.$$

For this map to be well-defined the IRS μ_Γ must only depend on the conjugacy class $[\Gamma] \in G\backslash\mathcal{L}(\Sigma)$. We will see this in Lemma 4.1.3. Making use of the identification $\mathcal{M}(\Sigma) \cong G\backslash\mathcal{L}(\Sigma)$ we will also use the notation $\mu_X = \mu_\Gamma$ for a hyperbolic surface $X = \Gamma\backslash\mathbb{H}^2 \in \mathcal{M}(\Sigma)$.

Gelander defined the IRS compactification of the moduli space as follows.

Definition 4.1.1 ([Gel15, Section 3.1])**.** The *IRS compactification* $\overline{\mathcal{M}}^{IRS}(\Sigma)$ *of the moduli space* $\mathcal{M}(\Sigma)$ is defined as the closure

$$\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) := \overline{\iota(\mathcal{M}(\Sigma))} \subseteq \mathrm{IRS}(G).$$

That $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is indeed compact, follows from the following well-known lemma.

Lemma 4.1.2 ([Abé+17, Section 2]). *The space of invariant random subgroups* $\mathrm{IRS}(G)$ *of a locally compact Hausdorff group G is compact.*

We include the short proof for completeness.

Proof. Since $\mathrm{Sub}(G)$ is compact, so is $\mathrm{Prob}(\mathrm{Sub}(G))$ by Banach–Alaoglu's Theorem. The space of invariant random subgroups is a closed subspace

$$\mathrm{Prob}(\mathrm{Sub}(G))^G = \bigcap_{g\in G}\{\mu \in \mathrm{Prob}(\mathrm{Sub}(G)) \,|\, g_*\mu = \mu\}.$$

□

Before proving that $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is a topological embedding let us see that ι is well-defined and injective. This will follow from Lemma 4.1.3 and Lemma 4.1.4.

Lemma 4.1.3. *Let $\Gamma \leq G$ be a lattice and denote by $N(\Gamma) \leq G$ its normalizer. Then:*

(i) The map $\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow G*\Gamma$ is G-equivariant and descends to a G-equivariant homeomorphism

$$\varphi_{N(\Gamma)}\colon N(\Gamma)\backslash G \longrightarrow G*\Gamma.$$

(ii) The measure $\mu_\Gamma \in \mathrm{IRS}(G)$ is an ergodic IRS, which depends only on the conjugacy class of $\Gamma \leq G$.

(iii) The support of μ_Γ is the orbit closure of Γ in $\mathrm{Sub}(G)$: $\mathrm{supp}(\mu_\Gamma) = \overline{G*\Gamma}$.

Proof. (i) The stabilizer of Γ for the conjugation action of G on $\mathrm{Sub}(G)$ is its normalizer $N(\Gamma)$ by definition. Hence, the orbit map $G \longrightarrow \mathrm{Sub}(G), g \longmapsto g^{-1}\Gamma g$ descends to a continuous bijection $\varphi_{N(\Gamma)}\colon N(\Gamma)\backslash G \longrightarrow G*\Gamma \subseteq \mathrm{Sub}(G)$ by the Orbit Stabilizer Theorem. Notice that the orbit map is equivariant with respect to the right-translation action on G and the conjugation action on $\mathrm{Sub}(G)$, such that both $\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow \mathrm{Sub}(G)$ and $\varphi_{N(\Gamma)}\colon N(\Gamma)\backslash G \longrightarrow \mathrm{Sub}(G)$ are equivariant as well. In order to show that this map is a homeomorphism, we will show that it is proper.

To this end suppose that $(g_n^{-1}\Gamma g_n)_{n\in\mathbb{N}} \subseteq G*\Gamma$ converges to some $\Gamma' \in G*\Gamma$. Because Γ' is discrete there is an open identity neighborhood $U \subseteq G$ such that $g_n^{-1}\Gamma g_n \cap U = \{e\}$ for all $n \in \mathbb{N}$ by Lemma 2.8.6. In particular, there is an $\varepsilon > 0$ such that $\pi_\Gamma(g_n o) \in \Gamma\backslash\mathbb{H}^2$ is in the ε-thick part for some base point $o \in \mathbb{H}^2$, where we used the notation $\pi_\Gamma\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ for the quotient map. Because the ε-thick part $(\Gamma\backslash\mathbb{H}^2)_{\geq\varepsilon}$ is compact, there is a compact set $K \subseteq \mathbb{H}^2$ and $\gamma_n \in \Gamma$ such that $\gamma_n g_n o \in K$. Thus there is a subsequence $(\gamma_{n_k} g_{n_k})_{k\in\mathbb{N}}$ converging to some $g \in G$ and $N(\Gamma)g_{n_k} = N(\Gamma)\gamma_{n_k} g_{n_k}$ converges to $N(\Gamma)g$ in $N(\Gamma)\backslash G$. Because the convergent sequence $(g_n^{-1}\Gamma g_n)_{n\in\mathbb{N}} \subseteq G * \Gamma$ was arbitrary, we conclude that $\varphi_{N(\Gamma)}$ is proper.

(ii) Because $\varphi_\Gamma\colon \Gamma\backslash G \longrightarrow \mathrm{Sub}(G)$ is equivariant and ν_Γ is a G-invariant probability measure on $\Gamma\backslash G$, it follows that $\mu_\Gamma = (\varphi_\Gamma)_*(\nu_\Gamma)$ is G-invariant as well.

Regarding ergodicity, let $f\colon \mathrm{Sub}(G) \longrightarrow \mathbb{R}$ be a conjugation invariant Borel measurable function. Then f is constant on the oribt $G*\Gamma$. By definition $G*\Gamma \subseteq \mathrm{Sub}(G)$ is a subset of full measure on which f is constant. Hence, μ_Γ is ergodic.

We will show that μ_Γ depends only on the conjugacy class of $\Gamma \leq G$ now. Let $\Gamma' = h\Gamma h^{-1}$ be a conjugate of Γ, $h \in G$. Then left multiplication by h induces a G-equivariant homeomorphism

$$\widehat{h}\colon \Gamma\backslash G \longrightarrow \Gamma'\backslash G, \quad \Gamma g \longmapsto \Gamma' h g,$$

which makes the following diagram commute:

$$\begin{tikzcd}
\Gamma\backslash G \arrow[dr, "\varphi_\Gamma"] \arrow[dd, "\widehat{h}"'] & \\
 & \mathrm{Sub}(G) \\
\Gamma'\backslash G \arrow[ur, "\varphi_{\Gamma'}"] &
\end{tikzcd}$$

Moreover, $\widehat{h}$ is G-equivariant such that $\widehat{h}_*\nu_\Gamma$ is another G-invariant probability measure and by uniqueness $\widehat{h}_*\nu_\Gamma = \nu_{\Gamma'}$. Thus $(\varphi_\Gamma)_*\nu_\Gamma = \mu_\Gamma = (\varphi_{\Gamma'})_*\nu_{\Gamma'}$, whence μ_Γ depends only on the conjugacy class of Γ.

(iii) Note that $\overline{G*\Gamma} = \overline{\varphi_\Gamma(\Gamma\backslash G)}$ by definition. Further, recall that $H \in \mathrm{Sub}(G)$ is in the support $\mathrm{supp}(\mu_\Gamma)$ if and only if every open neighborhood $\mathcal{U} \subseteq \mathrm{Sub}(G)$ of H has positive mass $\mu_\Gamma(\mathcal{U}) > 0$.

Let $H \notin \mathrm{supp}(\mu_\Gamma)$. We want to show that $H \notin \overline{\varphi_\Gamma(\Gamma\backslash G)}$, i.e. there is an open neighborhood $\mathcal{U} \subseteq \mathrm{Sub}(G)$ of H such that $\mathcal{U}\cap\varphi_\Gamma(\Gamma\backslash G) = \varnothing$. Because $H \notin \mathrm{supp}(\mu_\Gamma)$ there is an open neighborhood $\mathcal{U} \subseteq \mathrm{Sub}(G)$ such that $\mu_\Gamma(\mathcal{U}) = 0$. Then $V = \varphi_\Gamma^{-1}(\mathcal{U}) \subseteq \Gamma\backslash G$ is an open subset such that

$$\nu_\Gamma(V) = \nu_\Gamma(\varphi_\Gamma^{-1}(\mathcal{U})) = \mu_\Gamma(\mathcal{U}) = 0.$$

Because ν_Γ has full support on $\Gamma\backslash G$ the set V must be empty. Therefore, $\mathcal{U} \cap \varphi_\Gamma(\Gamma\backslash G) = \varnothing$.

Vice versa, let $H \in \mathrm{supp}(\mu_\Gamma)$, and let $\mathcal{U} \subseteq \mathrm{Sub}(G)$ be an open neighborhood of H. Then

$$0 < \mu_\Gamma(\mathcal{U}) = \nu_\Gamma(\varphi_\Gamma^{-1}(\mathcal{U})).$$

Hence, $V = \varphi_\Gamma^{-1}(\mathcal{U}) \neq \varnothing \subseteq \Gamma\backslash G$ is a non-empty open subset, and $\mathcal{U} \cap \varphi_\Gamma(\Gamma\backslash G) \neq \varnothing$. Because $\mathcal{U}$ was an arbitrary open neighborhood of H, it follows that $H \in \overline{\varphi_\Gamma(\Gamma\backslash G)} = \overline{G * \Gamma}$.

□

Lemma 4.1.4. *Let $[\Gamma_1], \ldots, [\Gamma_m]$ be pairwise distinct conjugacy classes of lattices in G. Then the associated invariant random subgroups $\mu_{\Gamma_1}, \ldots, \mu_{\Gamma_m} \in \mathrm{IRS}(G) \subseteq C(\mathrm{Sub}(G))^*$ are linearly independent.*

Proof. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

$$0 = \sum_{j=1}^{m} \lambda_j \cdot \mu_{\Gamma_j}.$$

For all $i, j \in \{1, \ldots, m\}, i \neq j$, there is an open neighborhood $U_{i,j} \subseteq \mathrm{Sub}(G)$ of Γ_i such that $U_{i,j} \cap G * \Gamma_j = \varnothing$. Indeed, otherwise there would be $i \neq j$ and a sequence $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that $g_n \Gamma_j g_n^{-1} \to \Gamma_i$ as $n \to \infty$. Because Γ_i is not abelian this implies that Γ_i is a conjugate of Γ_j by Lemma 2.9.5; contradiction.

We set

$$U_i = \bigcap_{\substack{j=1 \\ j\neq i}}^{m} U_{i,j},$$

such that U_i is an open neighborhood of Γ_i satisfying

$$\varnothing = U_i \cap \overline{G * \Gamma_j} = U_i \cap \mathrm{supp}(\mu_{\Gamma_j})$$

for every $j \neq i$. Then

$$0 = \left(\sum_{j=1}^{m} \lambda_j \cdot \mu_{\Gamma_j} \right)(U_i) = \lambda_i \cdot \underbrace{\mu_{\Gamma_i}(U_i)}_{>0}$$

such that $\lambda_i = 0$ for every $i = 1, \ldots, m$. □

From Lemma 4.1.3 (ii) it follows that ι is well-defined and Lemma 4.1.4 implies that ι is injective. Our next goal is to show that it is a topological embedding:

Proposition 4.1.5. *The map $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is a topological embedding.*

Remark 4.1.6. In the case where Σ is compact it is proved in [GL18b, Proposition 11.2] that the map ι is continuous.

Before we attempt a proof let us understand the measure $\mu_\Gamma \in \mathrm{IRS}(G)$, $\Gamma \in \mathcal{L}(\Sigma)$, in terms of a Haar measure ν on G. For any continuous function $F \in C(\mathrm{Sub}(G))$ we have that

$$\begin{aligned}\int_{\mathrm{Sub}(G)} F(H)\,d\mu_\Gamma(H) &= \int_{\Gamma\backslash G} F(g^{-1}\Gamma g)\,d\nu_\Gamma(g\Gamma)\\ &= \nu(D)^{-1}\int_G \mathbb{1}_D(g)\cdot F(g^{-1}\Gamma g)\,d\nu(g),\end{aligned}\tag{$\star$}$$

where $D \subseteq G$ is a fundamental domain for the action $\Gamma \curvearrowright G$.

We shall choose a conveniently normalized Haar measure ν on G now. Recall that $G = \mathrm{Isom}_+(\mathbb{H}^2) \cong \mathrm{PSL}(2,\mathbb{R})$. The map $p\colon G \longrightarrow \mathbb{H}^2, g \longmapsto g\cdot i$ is surjective and induces an identification $G/K \cong \mathbb{H}^2$, where $K = \mathrm{stab}_G(i) \cong \mathrm{SO}(2,\mathbb{R})$. Thus the hyperbolic area measure $\mathrm{vol}_{\mathbb{H}^2}$ on $\mathbb{H}^2$ amounts via this identification to a G-invariant measure on G/K, which we shall denote by v. Further, we choose a normalized Haar measure η on K such that $\eta(K) = 1$. By Weil's quotient formula we obtain a Haar measure ν on G such that

$$\int_G f(g)\,d\nu(g) = \int_{G/K}\int_K f(gk)\,d\eta(k)\,dv(gK)$$

for every $f \in L^1(G,\nu)$. With this choice we have that $\nu(p^{-1}(B)) = v(B)$ for every measurable subset $B \subseteq \mathbb{H}^2$.

Moreover, we have the following:

Lemma 4.1.7. *The map*

$$\begin{aligned} p^*\colon L^1(G/K,v) &\longrightarrow L^1(G,\nu),\\ f &\longmapsto f\circ p,\end{aligned}$$

is a linear isometry. In particular, p^* is continuous.

Proof. Let $f \in L^1(G/K,v)$. By Weil's quotient formula we get that

$$\begin{aligned}\|f\circ p\|_{L^1(G)} &= \int_G |f(p(g))|\,d\nu(g)\\ &= \int_{G/K}\int_K |f(p(gk))|\,d\eta(k)\,dv(gK)\\ &= \int_{G/K}\int_K |f(gK)|\,d\eta(k)\,dv(gK)\\ &= \int_{G/K} \eta(K)\cdot|f(gK)|\,dv(gK)\\ &= \int_{G/K} |f(gK)|\,dv(gK) = \|f\|_{L^1(G/K)}.\end{aligned}$$

□

In view of ($\star$) we will need a fundamental domain for the action $\Gamma \curvearrowright G$. Given a point $o \in \mathbb{H}^2$ we have already introduced the Dirichlet domain $D_o(\Gamma) \subseteq \mathbb{H}^2$ for the action $\Gamma \curvearrowright \mathbb{H}^2$ in section 3. We can use the Dirichlet domain to obtain a fundamental domain for the action $\Gamma \curvearrowright G$. Indeed, it is straight-forward to check that $F_o(\Gamma) := p^{-1}(D_o(\Gamma))$ is a fundamental domain for $\Gamma \curvearrowright G$. Likewise, we may define

$$\widehat{F}_o(\Gamma) := p^{-1}(\widehat{D}_o(\Gamma))$$

as the preimage of the truncated Dirichlet domain $\widehat{D}_o(\Gamma) \subseteq \mathbb{H}^2$.

Recall that the truncated Dirichlet domain $\widehat{D}_o(\Gamma)$ is a fundamental domain for $\Gamma \curvearrowright \widetilde{C}(\Gamma)$. The quotient $C(\Gamma) = \Gamma \backslash \widetilde{C}(\Gamma)$ is the convex core of $\Gamma \backslash \mathbb{H}^2$, which one obtains from $\Gamma \backslash \mathbb{H}^2$ by cutting off its funnels at their waist geodesics. Furthermore, $\mathrm{vol}_{\mathbb{H}^2}(\widehat{D}_o(\Gamma)) = 2\pi \left|\chi(\Gamma \backslash \mathbb{H}^2)\right|$ such that we have

$$\nu(\widehat{F}_o(\Gamma)) = 2\pi \left|\chi(\Gamma \backslash \mathbb{H}^2)\right|$$

by our choice of Haar measure.

In Lemma 3.1.1 we showed that the indicator function of the truncated Dirichlet domain $\mathbb{1}_{\widehat{D}_o(\Gamma)} \in L^1(\mathbb{H}^2)$ depends continuously on $\Gamma \in \mathcal{D}(\Sigma)$. Applying Lemma 4.1.7 we obtain the following important corollary.

Corollary 4.1.8. *Let $o \in \mathbb{H}^2$. The map*

$$\mathcal{D}(\Sigma) \longrightarrow L^1(G), \quad \Gamma \longmapsto \mathbb{1}_{\widehat{F}_o(\Gamma)}$$

is continuous, where $\mathbb{1}_{\widehat{F}_o(\Gamma)} = \mathbb{1}_{\widehat{D}_o(\Gamma)} \circ p$ denotes the indicator function of $\widehat{F}_o(\Gamma) \subseteq G$.

Corollary 4.1.8 will play an essential role in our proofs of Proposition 4.1.5 and Theorem 4.2.2.

Finally, we record the following consequence of the Dominated Convergence Theorem for future reference.

Lemma 4.1.9. *Let (X, μ) be a measure space, let $(f_n)_{n\in\mathbb{N}} \subseteq L^1(X, \mu)$ and let $(g_n)_{n\in\mathbb{N}} \subseteq L^\infty(X, \mu)$. Assume that there is $f \in L^1(X, \mu)$ such that*

$$\|f_n - f\|_{L^1} \to 0 \quad (n \to \infty),$$

and that there is $C > 0$ and $g \in L^\infty(X, \mu)$ such that $\|g_n\|_{L^\infty} \leq C$, for every $n \in \mathbb{N}$, and

$$g_n(x) \to g(x) \quad (n \to \infty)$$

for μ-almost-every $x \in X$.

Then

$$\|f_n \cdot g_n - f \cdot g\|_{L^1} \to 0 \quad (n \to \infty).$$

Proof. We compute

$$\begin{aligned}\|f_n \cdot g_n - f \cdot g\|_{L^1} &\le \|f_n \cdot g_n - f \cdot g_n\|_{L^1} + \|f \cdot g_n - f \cdot g\|_{L^1} \\ &\le C \cdot \|f_n - f\|_{L^1} + \int_X |f(x)| \cdot |g_n(x) - g(x)| \, d\mu(x).\end{aligned}$$

Note that

$$|f(x)| \cdot |g_n(x) - g(x)| \to 0 \quad (n \to \infty)$$

for μ-almost-every $x \in X$ and the functions $|f(x)| \cdot |g_n(x) - g(x)|$ are μ-almost-everywhere dominated by the integrable function $2C|f(x)|$. By the Dominated Convergence Theorem we conclude that

$$\int_X |f(x)| \cdot |g_n(x) - g(x)| \, d\mu(x) \to 0 \quad (n \to \infty),$$

which in turn implies $\|f_n \cdot g_n - f \cdot g\|_{L^1} \to 0$ as $n \to \infty$. $\square$

After these preparations we are ready to prove Proposition 4.1.5.

Proof of Proposition 4.1.5. We want to show that $\iota \colon G\backslash\mathcal{L}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is a topological embedding.

First, let us prove that ι is continuous. Let $([\Gamma_n])_{n\in\mathbb{N}} \subseteq G\backslash\mathcal{L}(\Sigma)$ be a convergent sequence with limit $[\Gamma] \in G\backslash\mathcal{L}(\Sigma)$. Up to taking conjugates we may assume that $\Gamma_n \to \Gamma$ in $\mathcal{L}(\Sigma)$. Let $o \in \mathbb{H}^2$ and we consider the fundamental domains $F_o(\Gamma_n) = p^{-1}(D_o(\Gamma_n))$ for $\Gamma_n \curvearrowright G$. Since Γ_n is a lattice, we have that $\widetilde{C}(\Gamma_n) = \mathbb{H}^2$ and $\widehat{D}_o(\Gamma_n) = D_o(\Gamma_n)$.

Let $f \in C(\mathrm{Sub}(G))$. Then

$$\begin{aligned}&\left|\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) - \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma}(H)\right| \\ &= \left|\nu(F_o(\Gamma_n))^{-1} \cdot \int_G \mathbb{1}_{F_o(\Gamma_n)}(g) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g) - \nu(F_o(\Gamma))^{-1} \cdot \int_G \mathbb{1}_{F_o(\Gamma)}(g) \cdot f(g^{-1}\Gamma g)\, d\nu(g)\right| \\ &\le \frac{1}{2\pi|\chi(\Sigma)|} \cdot \left\|\mathbb{1}_{F_o(\Gamma_n)} \cdot \bar{f}_n - \mathbb{1}_{F_o(\Gamma)} \cdot \bar{f}\right\|_{L^1},\end{aligned}$$

where we set $\bar{f}_n(g) := f(g^{-1}\Gamma_n g), \bar{f}(g) := f(g^{-1}\Gamma g)$ for every $g \in G$. Note that f is uniformly bounded because $\mathrm{Sub}(G)$ is compact, so that $(\bar{f}_n)_{n\in\mathbb{N}}$ are uniformly bounded, too. Moreover,

$$\bar{f}_n(g) = f(g^{-1}\Gamma_n g) \to \bar{f}(g) = f(g^{-1}\Gamma g) \qquad (n \to \infty)$$

for every $g \in G$ by continuity. By Corollary 4.1.8 we know that

$$\|\mathbb{1}_{F_o(\Gamma_n)} - \mathbb{1}_{F_o(\Gamma)}\|_{L^1(G,\nu)} \to 0 \qquad (n \to \infty).$$

Thus we may apply Lemma 4.1.9 and conclude that

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) \to \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma}(H) \qquad (n \to \infty).$$

This shows that ι is continuous.

Finally, let $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(\Sigma)$ and let $\Gamma \in \mathcal{L}(\Sigma)$, such that $\mu_{\Gamma_n} \to \mu_\Gamma$ as $n \to \infty$. We want to show that $[\Gamma_n] \to [\Gamma]$ as $n \to \infty$. Let $\mathcal{U} \subseteq \mathrm{Sub}(G)$ be an open neighborhood of Γ, and let $\overline{\mathcal{V}} \subseteq \mathcal{U}$ be a compact neighborhood of Γ. By Urysohn's Lemma we find a continuous function $f\colon \mathrm{Sub}(G) \longrightarrow [0,1]$ such that $f|_{\overline{\mathcal{V}}} \equiv 1$ and $f|_{\mathcal{U}^c} \equiv 0$. Because $\mu_{\Gamma_n} \to \mu_\Gamma$ we have that

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) \to \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma}(H) \qquad (n \to \infty).$$

Because

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma}(H) \geq \int_{\overline{\mathcal{V}}} f(H)\, d\mu_{\Gamma}(H) = \mu_\Gamma(\overline{\mathcal{V}}) > 0,$$

also

$$\nu_{\Gamma_n}(\varphi_{\Gamma_n}^{-1}(\mathcal{U})) = \mu_{\Gamma_n}(\mathcal{U}) \geq \int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) > 0$$

for large $n \in \mathbb{N}$. Therefore, $\varphi_{\Gamma_n}^{-1}(\mathcal{U}) \subseteq \Gamma\backslash G$ is a non-empty open subset, whence there are $g_n \in G$ such that $\varphi_{\Gamma_n}(\Gamma_n g_n) = g_n^{-1}\Gamma_n g_n \in \mathcal{U}$. Because $\mathcal{U}$ was an arbitrary open neighborhood of Γ it follows that $[\Gamma_n] \to [\Gamma]$ as $n \to \infty$.

This shows that $\iota\colon G\backslash\mathcal{L}(\Sigma) \hookrightarrow \mathrm{IRS}(G)$ is a topological embedding. $\square$

4.2. **The Augmented Moduli Space and the IRS Compactification.** In the previous section we have seen how the moduli space can be embedded in the space of invariant random subgroups $\iota\colon \mathcal{M}(\Sigma) \hookrightarrow \mathrm{IRS}(G)$. This gave rise to the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) = \overline{\iota(\mathcal{M}(\Sigma))}$. In [Gel15] Gelander proposed the problem to analyze the IRS compactification further; see Problem 1.1.3. This is the goal of this section.

Our strategy is to relate the IRS compactification $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ to the augmented moduli space. In fact, we shall construct an extension $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ of the topological embedding $\iota\colon \mathcal{M}(\Sigma) \hookrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) \subseteq \mathrm{IRS}(G)$:

$$\begin{tikzcd} & \widehat{\mathcal{M}}(\Sigma) \arrow[dd, dashed, "\widehat{\Phi}"] \\ \mathcal{M}(\Sigma) \arrow[ur, hook] \arrow[dr, hook, "\iota"'] & \\ & \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma) \end{tikzcd}$$

Instead of defining the map $\widehat{\Phi}$ directly on the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ we will first define a map on the augmented Teichmüller space $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$.

Definition 4.2.1. We define $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ by

$$\widetilde{\Phi}(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}) := \sum_{\Sigma'\in c(\sigma)} \frac{\chi(\Sigma')}{\chi(\Sigma)} \cdot \mu_{\operatorname{im}\rho_{\Sigma'}}$$

for every $([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma), \sigma \subseteq \mathcal{C}(\Sigma)$.

Observe that $\widetilde{\Phi}$ is well-defined. Indeed, $\widetilde{\Phi}(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)})$ is a convex combination of IRSs, since $\chi(\Sigma) = \sum_{\Sigma'\in c(\sigma)} \chi(\Sigma') < 0$. Here the coefficient $\chi(\Sigma')/\chi(\Sigma)$ should be interpreted as the proportion of the area that the component Σ' takes up in the whole surface Σ. This makes sense, because the area of a hyperbolic surface $X \in \mathcal{M}(\Sigma)$ is a topological invariant $\mathrm{vol}(X) = 2\pi|\chi(\Sigma)|$ by the Gauss–Bonnet Theorem.

Moreover, notice that $\widetilde{\Phi}(([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)})$ depends only on the conjugacy classes $[\operatorname{im}\rho_{\Sigma'}] \in \mathcal{M}(\Sigma')$ and how often any one of them arises. Both the marking and even the gluing of the different parts do not affect the image. Hence, $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ descends to a map $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$, that descends further down to a map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ from the moduli space of parts $|\widehat{\mathcal{M}}|(\Sigma)$ to $\mathrm{IRS}(G)$.

$$\begin{tikzcd} & \widehat{\mathcal{M}}(\Sigma) \arrow[dl, "Q"'] \arrow[dr, "\widehat{\Phi}"] & \\ |\widehat{\mathcal{M}}|(\Sigma) \arrow[rr, dashed, "\Phi"] & & \mathrm{IRS}(G) \end{tikzcd}$$

Concretely, the map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ takes the form

$$\Phi(\xi) = \sum_{X\in\bigsqcup_{\Sigma'}\mathcal{M}(\Sigma')} \frac{\chi(X)}{\chi(\Sigma)} \cdot \xi(X) \cdot \mu_X$$

for every $\xi \in |\widehat{\mathcal{M}}|(\Sigma)$. Recall that elements of $|\widehat{\mathcal{M}}|(\Sigma)$ are functions $\xi\colon \bigsqcup_{\Sigma'}\mathcal{M}(\Sigma') \longrightarrow \mathbb{N}_0$, which take non-zero values only for a finite number of hyperbolic surfaces. For a nodal surface $\mathbf{X} \in \widehat{\mathcal{M}}(\Sigma)$ the corresponding $\xi = Q(\mathbf{X}) \in |\widehat{\mathcal{M}}|(\Sigma)$ simply records how many times any one hyperbolic surface arises as a part of $\mathbf{X}$; see section 2.5.

It turns out, that $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ induces a homeomorphism $|\widehat{\mathcal{M}}|(\Sigma) \cong \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$. More precisely, we have the following theorem.

Theorem 4.2.2. *The map $\widetilde{\Phi}\colon \widehat{\mathcal{T}}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is continuous and descends to a continuous surjection $\widehat{\Phi}\colon \widehat{\mathcal{M}}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$ that extends the embedding $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$. Moreover, $\widehat{\Phi}$ induces a homeomorphism $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{IRS}(\Sigma)$.*

There is a uniform upper bound on the number of elements in each fiber of $\widehat{\Phi}$,

$$\#\widehat{\Phi}^{-1}(\mu) \leq B(\Sigma) := \binom{3|\chi|}{p} \cdot \frac{(2(|\chi|+g-1))!}{(|\chi|+g-1)! \cdot 2^{(|\chi|+g-1)}} \qquad \forall \mu \in \overline{\mathcal{M}}^{IRS}(\Sigma),$$

where $\chi = \chi(\Sigma)$*,* $g = g(\Sigma)$*, and* $p = p(\Sigma)$ *denote the Euler characteristic, the genus, and the number of punctures of* Σ*, respectively.*

Remark 4.2.3. Observe that the upper bound $B(\Sigma)$ depends only on the topology of the surface Σ.

The main difficulty in the proof of Theorem 4.2.2 is to show that the map $\widetilde{\Phi}$ is continuous. As in the proof of continuity of the embedding $\iota\colon \mathcal{M}(\Sigma) \longrightarrow \mathrm{IRS}(G)$ in Proposition 4.1.5 we will use the L^1-convergence of fundamental domains (Corollary 4.1.8). However, this time we will have to deal with curves that may degenerate along a convergent sequence in $\widehat{\mathcal{T}}(\Sigma)$. We will overcome this issue by considering a fundamental domain built from convergent truncated Dirichlet domains for each subsurface. The following lemmas will help us to make this discussion precise.

Recall that for every component $\Sigma' \in c(\sigma)$, $\sigma \subseteq \mathcal{C}(\Sigma)$, we obtain a monomorphism $\iota_{\Sigma'}\colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ induced by the inclusion $\Sigma' \subseteq \Sigma$, which is well-defined up to conjugation; see Remark 2.4.2.

Lemma 4.2.4. *Let* $\rho \in \mathcal{R}(\Sigma)$*, let* $\sigma \subseteq \mathcal{C}(\Sigma)$ *be a simplex in the curve complex, let* $\Sigma' \in c(\sigma)$ *be a component, and let* $\iota_{\Sigma'}\colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ *be an inclusion monomorphism. Denote* $\Gamma = \operatorname{im}\rho$*,* $\Gamma' = \operatorname{im}(\rho \circ \iota_{\Sigma'})$ *and let* $\pi\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ *be the quotient map. Let* $f\colon \Sigma \longrightarrow \Gamma\backslash\mathbb{H}^2$ *be an orientation preserving homeomorphism such that* ρ *is a holonomy representation of* f*, and suppose that* $f(\sigma) = \tau$ *is a collection of closed geodesics.*

Then $\widetilde{\tau} := \pi^{-1}(\tau) \subseteq \mathbb{H}^2$ *is a collection of disjoint geodesics and* $\widetilde{C}(\Gamma') \subseteq \mathbb{H}^2$ *is the closure of a connected component of* $\mathbb{H}^2 \setminus \widetilde{\tau}$*.*

Proof. Let $\widetilde{f}\colon \widetilde{\Sigma} \cong \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ be a lift of $f\colon \Sigma \longrightarrow \Gamma\backslash\mathbb{H}^2$, that is ρ-equivariant. Let $\widetilde{\Sigma}' \subseteq \mathbb{H}^2 \setminus \widetilde{\sigma}$ be a connected component over Σ' such that the inclusion $\widetilde{\Sigma}' \hookrightarrow \widetilde{\Sigma} \cong \mathbb{H}^2$ is $\iota_{\Sigma'}$-equivariant; see Proposition and Definition 2.4.4. We set $\widetilde{X}' := \widetilde{f}(\widetilde{\Sigma}')$, and we want to show that $\widetilde{C}(\Gamma') = \overline{\widetilde{X}'}$. We will do so by showing that

$$\partial\widetilde{X}' \subseteq L(\Gamma') \subseteq \overline{\partial\widetilde{X}'}.$$

Note that $\widetilde{X}'$ is $\Gamma' = \rho(\iota_{\Sigma'}(\pi_1(\Sigma')))$-invariant. Therefore $\overline{\partial\widetilde{X}'}$ is a closed Γ'-invariant subset of $\partial\mathbb{H}^2$ that must contain the limit set $L(\Gamma')$ because the limit set is the smallest such subset.

Let $\xi \in \partial\widetilde{X}'$. If ξ is fixed by a parabolic element $\eta \in \Gamma'$ then

$$\xi = \lim_{n\to\infty} \eta^n o \in L(\Gamma'), \qquad o \in \mathbb{H}^2.$$

Hence, let us assume that ξ is not fixed by any parabolic element in Γ'. Let $\gamma \subseteq \widetilde{X}'$ be a geodesic from $\widetilde{p} = \gamma(0)$ to $\xi = \gamma(\infty)$. Further, let $\{P_j\}_{j\in\mathbb{N}}$ be a system of disjoint horoballs centered at the fixed points $\{\xi_j\}_{j\in\mathbb{N}}$ of parabolic elements in Γ', respectively. Then there is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$ and $\gamma(t_n) \notin \bigsqcup_{j\in\mathbb{N}} P_j$. Indeed, otherwise there would be a $T > 0$ and $j_0 \in \mathbb{N}$ such that $\gamma(t) \in P_{j_0}$ for all $t \geq T$. This in turn would imply that $\gamma(\infty) = \xi_{j_0} = \xi$; contradicting our assumption.

Observe that Γ' acts coboundedly on $\widetilde{X}' \setminus \bigsqcup_{j\in\mathbb{N}} P_j$. Therefore, there is $r > 0$, $o \in \widetilde{X}' \setminus \bigsqcup_{j\in\mathbb{N}} P_j$, and $\gamma_n \in \Gamma'$, such that $d(\gamma(t_n), \gamma_n \cdot o) \leq r$ for all $n \in \mathbb{N}$. Hence,

$$\xi = \lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} \gamma_n \cdot o \in L(\Gamma').$$

□

Lemma 4.2.5. *Let $\rho \in \mathcal{R}(\Sigma)$ and let $\sigma \subseteq C(\Sigma)$ be a simplex in the curve complex. Further, let $\iota_{\Sigma'}\colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ be an inclusion monomorphism for every component $\Sigma' \in c(\sigma)$. Denote $\Gamma = \operatorname{im}\rho$ and $\Gamma(\Sigma') = \operatorname{im}(\rho\circ\iota_{\Sigma'})$ for every $\Sigma' \in c(\sigma)$. Let $\{p(\Sigma') \in \mathbb{H}^2 \,|\, \Sigma' \in c(\sigma)\}$ be a collection of points. Then*

$$\bigcup_{\Sigma'\in c(\sigma)} \widehat{D}_{p(\Sigma')}(\Gamma(\Sigma'))$$

is a fundamental domain for the action of Γ on $\mathbb{H}^2$, and $\widehat{D}_{p(\Sigma')}(\Gamma(\Sigma')) \cap \widehat{D}_{p(\Sigma'')}(\Gamma(\Sigma''))$ has measure zero for distinct $\Sigma', \Sigma'' \in c(\sigma)$.

Proof. For simplicity we enumerate $\{\Sigma'_i\colon i = 1,\ldots,l\} = c(\sigma)$ and set $p_i = p(\Sigma'_i)$, $\Gamma(\Sigma'_i) = \Gamma'_i$ for every $i = 1,\ldots,l$. Further, denote by $q\colon \widetilde{\Sigma} \longrightarrow \Sigma$ and $\pi\colon \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ the usual universal coverings, and let $f : \Sigma \longrightarrow \Gamma\backslash\mathbb{H}^2$ be an orientation preserving homeomorphism such that ρ is a holonomy representation of f and $\tau := f(\sigma)$ is a collection of closed geodesics. Let $\widetilde{f}\colon \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$ be a ρ-equivariant lift of f to the universal cover. We set $\widetilde{\sigma} := q^{-1}(\sigma)$ and $\widetilde{\tau} := \pi^{-1}(\tau) = \widetilde{f}(\widetilde{\sigma})$. Let $\widetilde{\Sigma}'_i \subseteq \widetilde{\Sigma} \setminus \widetilde{\sigma}$ be a connected component such that the inclusion $\widetilde{\Sigma}'_i \hookrightarrow \widetilde{\Sigma}$ is $\iota_{\Sigma'_i}$-equivariant. Because $\widetilde{\Sigma} = \bigcup_{i=1}^l \pi_1(\Sigma) \cdot \overline{\widetilde{\Sigma}'_i}$, we have that $\mathbb{H}^2 = \bigcup_{i=1}^l \Gamma \cdot \overline{\widetilde{f}(\widetilde{\Sigma}'_i)}$. By Lemma 4.2.4 $\overline{\widetilde{f}(\widetilde{\Sigma}'_i)} = \widetilde{C}(\Gamma_i)$ such that

$$\mathbb{H}^2 = \bigcup_{i=1}^{l} \Gamma \cdot \widetilde{C}(\Gamma_i).$$

Because $\widehat{D}_{p_i}(\Gamma'_i)$ is a fundamental domain for the Γ'_i-action on $\widetilde{C}(\Gamma'_i)$, it is readily verified that $\bigcup_{i=1}^l \widehat{D}_{p_i}(\Gamma'_i)$ is a fundamental domain for the Γ-action on $\mathbb{H}^2$.

Finally, $\widehat{D}_{p_i}(\Gamma'_i) \cap \widehat{D}_{p_j}(\Gamma'_j) \subseteq \widetilde{\tau}$ for every $i \neq j$, which has measure zero. □

Lemma 4.2.6. *Let $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}^*(\Sigma)$, let $\sigma \subseteq C(\Sigma)$, let $\Sigma' \in c(\sigma)$ be a component, let $\iota_{\Sigma'}\colon \pi_1(\Sigma') \hookrightarrow \pi_1(\Sigma)$ be an inclusion monomorphism, and suppose that*

$$\rho_n \circ \iota_{\Sigma'} \to \rho' \in \mathcal{R}(\Sigma') \qquad (n \to \infty).$$

Denote $\Gamma_n := \operatorname{im}\rho_n$, $\Gamma_n(\Sigma') := \operatorname{im}(\rho_n \circ \iota_{\Sigma'})$ and $\Gamma' := \operatorname{im}\rho' \in \mathcal{L}(\Sigma')$.

Then

$$\Gamma_n \to \Gamma' \qquad (n \to \infty).$$

Proof. We shall check conditions (C1) and (C2) from Proposition 2.8.3.

(C1) Let $\gamma' = \rho(c) \in \Gamma' = \operatorname{im}\rho'$ for some $c \in \pi_1(\Sigma')$. Then $\rho_n(\iota_{\Sigma'}(c)) \in \Gamma_n(\Sigma') \subseteq \Gamma_n$ converges to γ' as $n \to \infty$.

(C2) Let $(\gamma_{n_k})_{k\in\mathbb{N}}$ be a convergent sequence with limit $g \in G$ and $\gamma_{n_k} \in \Gamma_{n_k}$. We need to show that $g \in \Gamma'$. By Proposition 2.9.2 we know that $\Gamma_n(\Sigma') = \operatorname{im}(\rho_n \circ \iota_{\Sigma'}) \to \Gamma'$ as $n \to \infty$. Thus it will be sufficient to prove that $\gamma_{n_k} \in \Gamma_{n_k}(\Sigma')$ for large k.

Assume to the contrary that (up to a subsequence) $\gamma_{n_k} = \rho_{n_k}(\alpha_{n_k})$ where $\alpha_{n_k} \notin \iota_{\Sigma'}(\pi(\Sigma')) \cong \pi(\Sigma')$. We denote by $\partial\Sigma' \subseteq \sigma$ the curves that are adjacent to $\Sigma' \subseteq \Sigma$. Let $\{c_i : i \in \mathbb{N}\} \subseteq \pi_1(\Sigma')$ be the set of elements whose conjugacy classes correspond to some curve in $\partial\Sigma'$. Then the axes of $\{\rho_{n_k}(c_i) : i \in \mathbb{N}\}$ bound $\widetilde{C}(\Gamma_{n_k}(\Sigma'))$; see Lemma 4.2.4.

Let $w \in \mathbb{H}^2$ be a point such that $w \in \widetilde{C}(\Gamma_{n_k}(\Sigma'))$ for large $k \in \mathbb{N}$. Because $\gamma_{n_k} \notin \Gamma_{n_k}(\Sigma')$ it must send $w \in \widetilde{C}(\Gamma_{n_k}(\Sigma'))$ to $\gamma_{n_k} w \in \mathbb{H}^2 \setminus \widetilde{C}(\Gamma_{n_k}(\Sigma'))$ in some complementary region. Note that the geodesic segment from w to $\gamma_{n_k} w$ has to cross at least one axis of the $\{\rho_{n_k}(c_i) : i \in \mathbb{N}\}$.

Because $\Gamma_{n_k}(\Sigma')$ converges to a lattice $\Gamma' \leq G$ the lengths of all boundary curves $\ell(\rho_{n_k}(c_i))$ go to 0 as $k \to \infty$. Note that this convergence is uniform in i because the length $\ell(\rho_{n_k}(c_i))$ depends only on the conjugacy class of c_i and there are only finitely many curves in $\partial\Sigma' \subseteq \sigma$. By the Collar Lemma 2.2.10 it follows that

$$d(\gamma_{n_k} w, w) \geq \ell(\gamma_{n_k}) \geq 2\operatorname{arcsinh}\left(\frac{1}{\sinh\left(\ell(\rho_{n_k}(c_i)/2\right)}\right) \to \infty$$

as $k \to \infty$; in contradiction to $\gamma_{n_k} \to g$ as $k \to \infty$.

□

After these preparations, we are ready to prove Theorem 4.2.2.

Proof of Theorem 4.2.2. Let us prove that $\widetilde{\Phi}$ is continuous. We will first prove this for a sequence $([\rho_n])_{n\in\mathbb{N}} \subseteq \mathcal{T}(\Sigma)$ converging to $\mathfrak{r} = ([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)} \in \mathcal{T}_\sigma(\Sigma) \subseteq \widehat{\mathcal{T}}(\Sigma), \sigma \subseteq \mathcal{C}(\Sigma)$. By definition of the topology of $\widehat{\mathcal{T}}(\Sigma)$ we know that for every $\Sigma' \in c(\sigma)$ we have $[\rho_n \circ \iota_{\Sigma'}] \to [\rho_{\Sigma'}]$ as $n \to \infty$, i.e. there are $g_n(\Sigma') \in G$ such that

$$g_n(\Sigma')^{-1} \cdot (\rho_n \circ \iota_{\Sigma'}) \cdot g_n(\Sigma') \to \rho_{\Sigma'} \qquad (n \to \infty).$$

In particular,

$$g_n(\Sigma')^{-1} \cdot \Gamma_n(\Sigma') \cdot g_n(\Sigma') \to \Gamma(\Sigma') \qquad (n \to \infty),$$

where we set $\Gamma_n(\Sigma') := \operatorname{im}(\rho_n \circ \iota_{\Sigma'})$, $\Gamma(\Sigma') := \operatorname{im}(\rho \circ \iota_{\Sigma'})$.

Let $o \in \mathbb{H}^2$. By Lemma 4.2.5 the set

$$D_n := \bigcup_{\Sigma' \in c(\sigma)} \widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma'))$$

is a fundamental domain for the action of $\Gamma_n = \operatorname{im} \rho_n$ on $\mathbb{H}^2$.

Let $f \in C(\mathrm{Sub}(G))$. We have that

$$\int_{\mathrm{Sub}(G)} f(H)\, d\mu_{\Gamma_n}(H) = \nu(p^{-1}(D_n))^{-1} \cdot \int_G \mathbb{1}_{D_n}(go) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g).$$

Observe that $\nu(p^{-1}(D_n)) = \mathrm{vol}_{\mathbb{H}^2}(D_n) = 2\pi|\chi(\Sigma)|$. Further,

$$\int_G \mathbb{1}_{D_n}(go) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g) = \sum_{\Sigma' \in c(\sigma)} \int_G \mathbb{1}_{\widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma'))}(go) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g).$$

Let $\Sigma' \in c(\sigma)$. Then

$$\begin{aligned}
&\int_G \mathbb{1}_{\widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma'))}(go) \cdot f(g^{-1}\Gamma_n g)\, d\nu(g) \\
&= \int_G \mathbb{1}_{\widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma'))}(g_n(\Sigma')go) \cdot f(g^{-1}g_n(\Sigma')^{-1}\Gamma_n g_n(\Sigma')g)\, d\nu(g) \\
&= \int_G \mathbb{1}_{\widehat{D}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma'))}(go) \cdot f(g^{-1}g_n(\Sigma')^{-1}\Gamma_n g_n(\Sigma')g)\, d\nu(g) \\
&= \int_G \mathbb{1}_{\widehat{F}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma'))}(g) \cdot \bar{f}_{n,\Sigma'}(g)\, d\nu(g),
\end{aligned}$$

where we used the left-invariance of the Haar measure ν and the fact that

$$\widehat{D}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma')) = g_n(\Sigma')^{-1} \cdot \widehat{D}_{g_n(\Sigma')o}(\Gamma_n(\Sigma')).$$

Moreover, we set

$$\bar{f}_{n,\Sigma'}(g) := f(g^{-1}g_n(\Sigma')^{-1}\Gamma_n g_n(\Sigma')g) \qquad \forall g \in G.$$

Note that $\|\bar{f}_{n,\Sigma'}\|_{L^\infty} \leq \|f\|_\infty < \infty$, $n \in \mathbb{N}$, and

$$g_n(\Sigma')^{-1} \cdot \Gamma_n \cdot g_n(\Sigma') \to \Gamma(\Sigma') \quad (n \to \infty)$$

by Lemma 4.2.6. Thus, if we set

$$\bar{f}_{\Sigma'}(g) := f(g^{-1}\Gamma(\Sigma')g) \qquad \forall g \in G,$$

then $\bar{f}_{n,\Sigma'}(g) \to \bar{f}(g)$ as $n \to \infty$ for every $g \in G$ by continuity. Moreover,

$$\left\| \mathbb{1}_{\widehat{F}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma'))} - \mathbb{1}_{\widehat{F}_o(\Gamma(\Sigma'))} \right\|_{L^1(G,\nu)} \to 0 \quad (n \to \infty)$$

by Corollary 4.1.8.

It follows from Lemma 4.1.9 that

$$\int_G \mathbb{1}_{\widehat{F}_o(g_n(\Sigma')^{-1}\Gamma_n(\Sigma')g_n(\Sigma'))}(g)\cdot \bar{f}_{n,\Sigma'}(g)\,d\nu(g) \to \int_G \mathbb{1}_{\widehat{F}_o(\Gamma(\Sigma'))}(g)\cdot \bar{f}_{\Sigma'}(g)\,d\nu(g)$$

as $n\to\infty$.

All in all, we obtain that the integral

$$\int_{\mathrm{Sub}(G)} f(H)\,d\mu_{\Gamma_n}(H)$$

tends to

$$\begin{aligned}
&(2\pi|\chi(\Sigma)|)^{-1}\sum_{\Sigma'\in c(\sigma)}\int_G \mathbb{1}_{\widehat{F}_o(\Gamma(\Sigma'))}(g)\cdot f(g^{-1}\Gamma(\Sigma')g)\,d\nu(g)\\
&=\sum_{\Sigma'\in c(\sigma)}\frac{2\pi|\chi(\Sigma')|}{2\pi|\chi(\Sigma)|}\cdot\nu(\widehat{F}_o(\Gamma(\Sigma')))^{-1}\int_G \mathbb{1}_{\widehat{F}_o(\Gamma(\Sigma'))}(g)\cdot f(g^{-1}\Gamma(\Sigma')g)\,d\nu(g)\\
&=\sum_{\Sigma'\in c(\sigma)}\frac{\chi(\Sigma')}{\chi(\Sigma)}\cdot\int_G f(H)\,d\mu_{\mathrm{im}\,\rho_{\Sigma'}}(H)=\int_G f(H)\,d\widetilde{\Phi}(\mathfrak{r})(H)
\end{aligned}$$

as $n\to\infty$.

In general, let $\mathfrak{r}_n=([\rho^{(n)}_{\Sigma''}])_{\Sigma''\in c(\sigma_n)}\subseteq\widehat{\mathcal{T}}(\Sigma)$ converge to $\mathfrak{r}=([\rho_{\Sigma'}])_{\Sigma'\in c(\sigma)}$ as $n\to\infty$. Then $\sigma_n\subseteq\sigma$ for large n. Because the simplex σ has only finitely many faces we may assume without loss of generality[2] that $\sigma_n=\sigma'$ for large n. Applying our previous discussion to every component $\Sigma''\in c(\sigma')$ we obtain

$$\begin{aligned}
&\int_{\mathrm{Sub}(G)} f(H)\,d\widetilde{\Phi}(\mathfrak{r})(H)\\
&=\sum_{\Sigma'\in c(\sigma)}\frac{\chi(\Sigma')}{\chi(\Sigma)}\int_{\mathrm{Sub}(G)} f(H)\,d\mu_{\mathrm{im}\,\rho_{\Sigma'}}(H)\\
&=\sum_{\Sigma''\in c(\sigma')}\frac{\chi(\Sigma'')}{\chi(\Sigma)}\sum_{\substack{\Sigma'\in c(\sigma)\\ \Sigma'\subseteq\Sigma''}}\frac{\chi(\Sigma')}{\chi(\Sigma'')}\int_{\mathrm{Sub}(G)} f(H)\,d\mu_{\mathrm{im}\,\rho_{\Sigma'}}(H)\\
&=\sum_{\Sigma''\in c(\sigma')}\frac{\chi(\Sigma'')}{\chi(\Sigma)}\lim_{n\to\infty}\int_{\mathrm{Sub}(G)} f(H)\,d\mu_{\mathrm{im}\,\rho^{(n)}_{\Sigma''}}(H)\\
&=\lim_{n\to\infty}\int_{\mathrm{Sub}(G)} f(H)\,d\widetilde{\Phi}(\mathfrak{r}_n)(H)
\end{aligned}$$

for every $f\in C(\mathrm{Sub}(G))$. This shows that $\widetilde{\Phi}\colon\widehat{\mathcal{T}}(\Sigma)\longrightarrow\mathrm{IRS}(G)$ is continuous.

Recall that the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$ is a quotient of the augmented Teichmüller space, whence the induced map $\widehat{\Phi}\colon\widehat{\mathcal{M}}(\Sigma)\longrightarrow\mathrm{IRS}(G)$ is continuous. In turn

[2] Just pass to a subsequence and treat every face separately.

the moduli space of parts $|\widehat{\mathcal{M}}|(\Sigma)$ is a quotient of the augmented moduli space $\widehat{\mathcal{M}}(\Sigma)$, such that the induced map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \mathrm{IRS}(G)$ is continuous, too.

Next, we want to show that $\widehat{\Phi}$ is surjective, i.e. $\widehat{\Phi}(\widehat{\mathcal{M}}(\Sigma)) = \overline{\iota(\mathcal{M}(\Sigma))}$. Since $\widehat{\mathcal{M}}(\Sigma)$ is compact and $\widehat{\Phi}$ is continuous the image $\widehat{\Phi}(\widehat{\mathcal{M}}(\Sigma))$ is compact and contains $\iota(\mathcal{M}(\Sigma)) = \widehat{\Phi}(\mathcal{M}(\Sigma))$. Because $\mathrm{IRS}(G)$ is Hausdorff, compact subsets are closed such that $\overline{\iota(\mathcal{M}(\Sigma))} \subseteq \widehat{\Phi}(\widehat{\mathcal{M}}(\Sigma))$. Vice versa, let $\mu \in \overline{\iota(\mathcal{M}(\Sigma))}$ and let $[[\rho_n]]_{n\in\mathbb{N}} \subseteq \mathcal{M}(\Sigma)$ be a sequence such that $\iota([[\rho_n]]) = \mu_{\mathrm{im}\,\rho_n}$ converges to μ as $n \to \infty$. Because $\widehat{\mathcal{M}}(\Sigma)$ is compact there is a convergent subsequence $[[\rho_{n_k}]] \to [\mathfrak{r}] \in \widehat{\mathcal{M}}(\Sigma)$ as $k \to \infty$. Because $\widehat{\Phi}$ is continuous it follows that

$$\mu = \lim_{k\to\infty} \widehat{\Phi}([[\rho_{n_k}]]) = \widehat{\Phi}([\mathfrak{r}]) \in \widehat{\Phi}(\widehat{\mathcal{M}}(\Sigma)).$$

Because $|\widehat{\mathcal{M}}|(\Sigma)$ is compact and $\overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is Hausdorff, it will suffice to prove that $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is a continuous bijection, in order to show that Φ is a homeomorphism. We have already seen that $\widehat{\Phi}$ is a continuous surjection, whence the induced map $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is a continuous surjection, too. Thus, we only need to show that Φ is injective.

To this end let $\xi_1, \xi_2 \in |\widehat{\mathcal{M}}|(\Sigma)$, such that $\Phi(\xi_1) = \Phi(\xi_2)$. We have that

$$\Phi(\xi_1) = \sum_{X \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')} \frac{\chi(X)}{\chi(\Sigma)} \cdot \xi_1(X) \cdot \mu_X = \sum_{i=1}^{l} \frac{\chi(X_i)}{\chi(\Sigma)} \cdot \xi_1(X_i) \cdot \mu_{X_i},$$

$$\Phi(\xi_2) = \sum_{X \in \bigsqcup_{\Sigma'} \mathcal{M}(\Sigma')} \frac{\chi(X)}{\chi(\Sigma)} \cdot \xi_2(X) \cdot \mu_X = \sum_{j=1}^{m} \frac{\chi(Y_j)}{\chi(\Sigma)} \cdot \xi_2(Y_j) \cdot \mu_{Y_j},$$

for some pairwise non-isometric hyperbolic surfaces $X_1, \ldots, X_l$, and some pairwise non-isometric hyperbolic surfaces $Y_1, \ldots, Y_m$. By Lemma 4.1.4 the IRSs $\{\mu_{X_1}, \ldots, \mu_{X_l}\}$ and $\{\mu_{Y_1}, \ldots, \mu_{Y_m}\}$ are linearly independent, respectively. Thus, $\Phi(\xi_1) = \Phi(\xi_2)$ implies that $m = l$, $X_i = Y_i$ (up to relabelling), and

$$\frac{\chi(X_i)}{\chi(\Sigma)} \cdot \xi_1(X_i) = \frac{\chi(X_i)}{\chi(\Sigma)} \cdot \xi_2(X_i)$$

for all $i = 1, \ldots, m$. Thus, $\xi_1(X_i) = \xi_2(X_i)$ for all $i = 1, \ldots, m$. Since both ξ_1 and ξ_2 are zero for all other hyperbolic surfaces, it follows that $\xi_1 = \xi_2$. This shows that Φ is injective and that $\Phi\colon |\widehat{\mathcal{M}}|(\Sigma) \longrightarrow \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$ is a homeomorphism.

The upper bound on the number of elements in each fiber of $\widehat{\Phi}$ now follows from $\widehat{\Phi} = \Phi \circ Q$ and Proposition 2.7.2. Indeed, we have that

$$\#\widehat{\Phi}^{-1}(\mu) = \#Q^{-1}(\Phi^{-1}(\mu)) \leq B(\Sigma) = \binom{3|\chi|}{p} \cdot \frac{(2\,(|\chi| + g - 1))!}{(|\chi| + g - 1)! \cdot 2^{(|\chi|+g-1)}}$$

for every $\mu \in \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$. $\square$

We conclude our discussion with a minimal example that shows that there are points $\mu \in \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma)$, whose preimage $\widehat{\Phi}^{-1}(\mu) \subseteq \widehat{\mathcal{M}}(\Sigma)$ consists of more than one point.

Example 4.2.7. Let $\Sigma = \Sigma_{2,0}$ be a closed surface of genus two. Let $\sigma_1 = \{\alpha_1, \beta_1, \gamma_1\} \subseteq \mathcal{C}(\Sigma)$ be a pants decomposition of Σ where α_1, γ_1 are non-separating curves and β_1 is separating. Further, let $\sigma_2 = \{\alpha_2, \beta_2, \gamma_2\} \subseteq \mathcal{C}(\Sigma)$ be a pants decomposition where $\alpha_2, \beta_2, \gamma_2$ are all non-separating. Recall that the Teichmüller space $\mathcal{T}(\Sigma_{0,3}) = \{[\rho_0]\}$ of a thrice-punctured sphere is just one point. We consider the elements $\mathfrak{r}_1 = ([\rho_0])_{\Sigma' \in c(\sigma_1)} \in \mathcal{T}_{\sigma_1}(\Sigma)$, $\mathfrak{r}_2 = ([\rho_0])_{\Sigma' \in c(\sigma_2)} \in \mathcal{T}_{\sigma_1}(\Sigma)$ and their images $[\mathfrak{r}_1], [\mathfrak{r}_2] \in \widehat{\mathcal{M}}(\Sigma)$; see Figure 4. Clearly, $[\mathfrak{r}_1] \neq [\mathfrak{r}_2]$ because σ_1 and σ_2 are not in the same mapping class group orbit in $\mathcal{C}(\Sigma)$.

However,

$$\widehat{\Phi}([\mathfrak{r}_1]) = \mu_{\Gamma_0} = \widehat{\Phi}([\mathfrak{r}_2]) \in \overline{\mathcal{M}}^{\mathrm{IRS}}(\Sigma),$$

where $\Gamma_0 = \operatorname{im} \rho_0$.

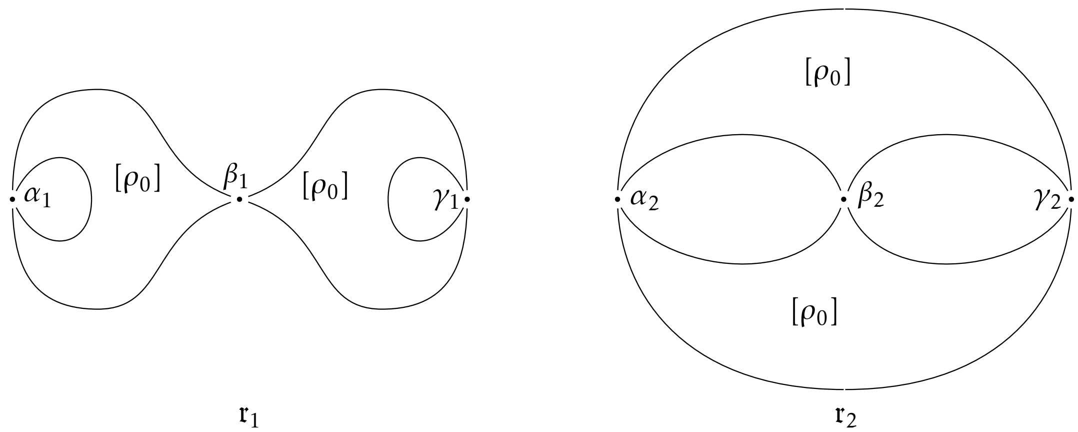


FIGURE 4. Both points $[\mathfrak{r}_1], [\mathfrak{r}_2] \in \widehat{\mathcal{M}}(\Sigma)$ are mapped to the same invariant random subgroup $\mu_{\Gamma_0} = \widehat{\Phi}([\mathfrak{r}_1]) = \widehat{\Phi}([\mathfrak{r}_2]) \in \mathrm{IRS}(G)$.

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