Nachdiplomvorlesung an der ETH Zürich im FS 2016

Geometric and topological aspects of Coxeter groups and buildings

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April 22, 2016

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Contents

1 Lecture One: Motivation and Examples

02.03.2016

1.1 Coxeter Groups

Before we give a proper definition of Coxeter groups let us give some examples.

1.1.1 One-dimensional examples

Example 1.1 (One-dimensional unit sphere)**.** Let us first consider the one-dimensional unit sphere centered at the origin and two lines through the origin with dihedral angle $\frac{\pi}{m}, (m \in \{2, 3, 4, \dots\})$; see Figure 1.1. Further let s_1 and s_2 denote the reflections across the lines respectively. Note that $s_1 s_2$ is rotation by $\frac{2\pi}{m}$. Hence the group $\langle s_1 s_2 \rangle$ generated by $s_1 s_2$ is cyclic of order m (i.e. isomorphic to C_m).

The full group $W = \langle s_1, s_2 \rangle$ is the dihedral group of order $2m$, denoted by D_{2m}, which has the presentation

$$W = \langle s_1, s_2 \,|\, s_i^2 = 1 \quad \forall i = 1, 2, \quad (s_1 s_2)^m = 1 \rangle$$

Example 1.2 (One-dimensional Euclidean space)**.** Let us now consider the real line and the reflections at 0 and 1, denoted by s_1 and s_2 respectively; see Figure 1.2. Hence $s_2 s_1$ is the translation by 2, such that $\langle s_2 s_1 \rangle \cong \mathbb{Z}$. The full group $W = \langle s_1, s_2 \rangle$ has in this case the presentation

$$W = \langle s_1, s_2 \,|\, s_i^2 \, \forall i = 1, 2, \quad (s_1 s_2)^\infty = 1 \rangle$$

1.1.2 Examples in dimension $n \geq 2$

Notation. $\mathbb{X}^n$ denotes either ...

- ... the n-dimensional sphere $\mathbb{S}^n$,
- ... the n-dimensional Euclidean space $\mathbb{E}^n$, or
- ... the n-dimensional hyperbolic space $\mathbb{H}^n$

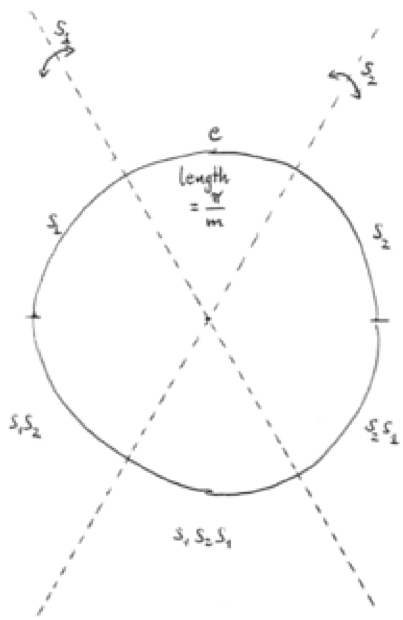

Figure 1.1: One-dimensional unit sphere

Definition 1.3. A *convex polytope* $P^n \subseteq \mathbb{X}^n$ is a convex, compact intersection of a finite number of half-spaces; e.g. see Figure 1.3.

The *link* $lk(v)$ of a vertex v of P^n is the $(n-1)$-dimensional spherical polytope obtained by intersecting P with a small sphere around v.

P is *simple* if for every vertex v of P its link $\mathrm{lk}(v)$ is a simplex.

Definition 1.4. Suppose $G \curvearrowright X$. A *fundamental domain* is a closed connected subset C of X, such that $Gx \cap C \neq \emptyset$ for every $x \in X$, and $|Gx \cap C| = 1$ for every $x \in \mathrm{int}(C)$. A fundamental domain C is called *strict*, if $Gx \cap C = \{x\}$ for every $x \in C$, i.e. C has exactly one point from each G-orbit. For example $[0,1]$ is a strict fundamental domain for $D_\infty \curvearrowright \mathbb{E}^1$, whereas $\langle s_1 s_2 \rangle \cong \mathbb{Z} \curvearrowright \mathbb{E}^1$ does not have a strict fundamental domain.

Theorem 1.5. *Let P^n be a simple convex polytope in $\mathbb{X}^n$ with codimension-one faces F_i. Suppose that $\forall i \neq j$ the dihedral angle between F_i and F_j, if $F_i \cap F_j \neq \emptyset$, is $\frac{\pi}{m_{ij}}$ for some $m_{ij} \in \{2,3,4,\ldots\}$. Set $m_{ii} = 1$ for every i, and $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$. Let s_i be the isometric reflection of $\mathbb{X}^n$ across the hyperplane supported by F_i and $W = \langle s_i \rangle$ the group generated by these reflections. Then:*

1. *$W = \langle s_i \mid s_i^2 = 1\,\forall i, \quad (s_i s_j)^{m_{ij}} = 1 \rangle$,*
2. *W is a discrete subgroup of Isom$(\mathbb{X}^n)$, and*
3. *P^n is a strict fundamental domain for the W action on $\mathbb{X}^n$ and P^n tiles $\mathbb{X}^n$.*

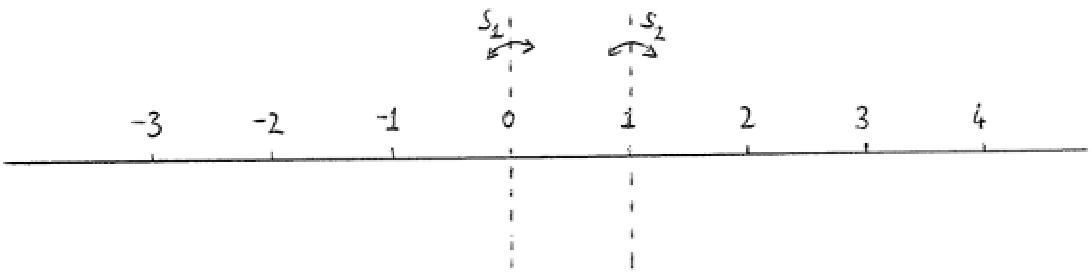

Figure 1.2: One-dimensional Euclidean space

Proof. Later ... □

Definition 1.6. A group W as in the one-dimensional examples or as in Theorem 1.5 is called a *geometric reflection group.*

Examples

Example 1.7 (Spherical)**.** For $n = 2$ we may project classical polytopes to the sphere and consider their symmetry groups; see Figure 1.4. For arbitrary n one may also consider the symmetry groups S_n = symmetries of $(n-1)$-simplex; see Figure 1.5.

Example 1.8 (Euclidean)**.**

1. Taking an equilateral triangle in $\mathbb{E}^2$ we get the situation depicted in Figure 1.6.

   This amounts to:

   $$m_{ij} = \begin{cases} 1 & \text{, if } i \neq j \\ 3 & \text{, else} \end{cases}$$

   $W = \langle s_0, s_1, s_2 \rangle = \langle s_i \mid s_i^2 = 1\,\forall i, \quad (s_i s_j)^3 = 1\,\forall i \neq j \rangle$

2. Similarly one may take another tesselation of $\mathbb{E}^2$ by triangles, i.e. P is one of the triangles depicted in Figure 1.7.

3. If we take P to be a square (see Figure 1.8) we get

   $$m_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, & \text{if } |i-j| = 1 \\ \infty, & \text{else} \end{cases}.$$

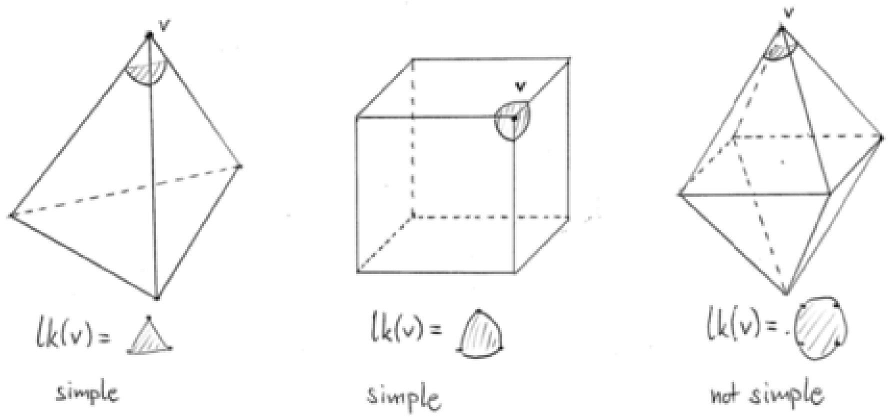

Figure 1.3: Three convex polytopes in $\mathbb{E}^n$. Two of them are simple and one is not.

Note that $m_{ij} = 2$ if and only if s_i and s_j commute. (W is an example of a right-angled Coxeter group).

Further note, that P is a product of simplices. This generalizes to the following theorem by Coxeter:

Theorem 1.9. *All Euclidean P^n are products of simplices.*

It is worth noting, that Coxeter actually classified all spherical and Euclidean P^n.

Example 1.10 (Hyperbolic)**.** 1. There are infinitely many triples (p, q, r) such that

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi.$$

Hence there are infinitely many hyperbolic triangle groups.

2. In $\mathbb{H}^2$ there are right-angled p-gons for $p \geq 5$. Here

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, & \text{if } |i - j| = 1 \\ \infty, & \text{else} \end{cases}$$

Now W induces a tessellation of $\mathbb{H}^2$; see Figure 1.9.

3. In $\mathbb{H}^3$, P could be a dodecahedron with all dihedral angles $\frac{\pi}{2}$; see Figure 1.10.

Definition 1.11 (Tits, 1950s)**.** Let $S = \{s_i\}_{i \in I}$ be a finite set. Let $M = (m_{ij})_{i,j \in I}$ be a matrix such that

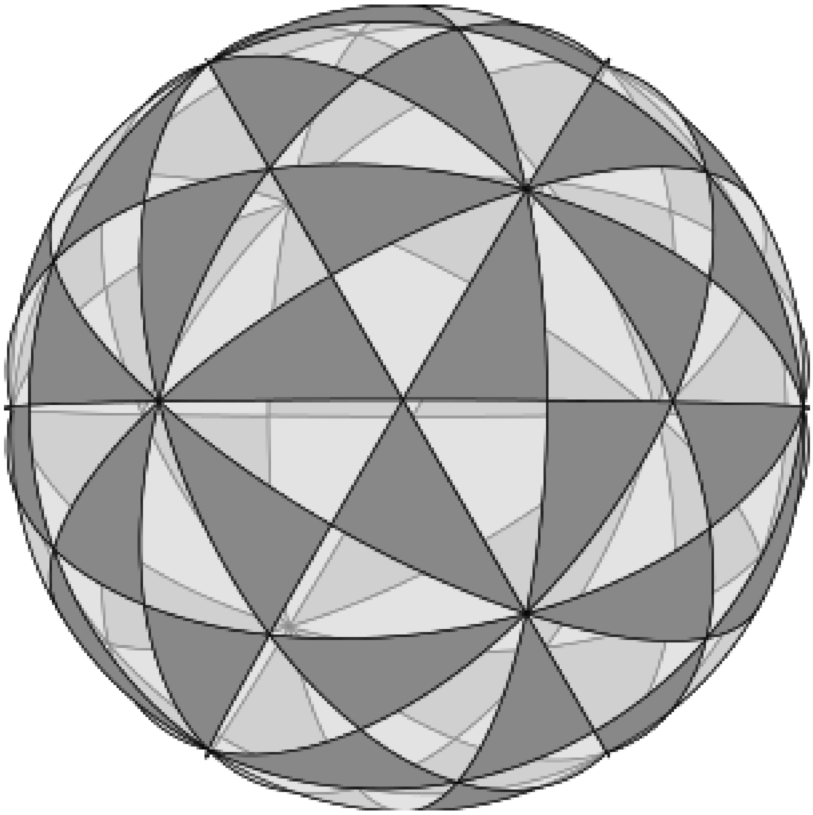

Figure 1.4: Spherical tiling induced by symmetry group of the icosahedron.

- $m_{ii} = 1 \; \forall i \in I$,
- $m_{ij} = m_{ji} \; \forall i \neq j$, and
- $m_{ij} \in \{2, 3, 4, \ldots\} \cup \{\infty\} \; \forall i \neq j$.

Then M is called a *Coxeter matrix.* The associated *Coxeter group* $W = W_M$ is defined by the presentation

$$W = \langle S \,|\, (s_i s_j)^{m_{ij}} \, \forall i, j \rangle.$$

The pair (W, S) is called a *Coxeter system.*

Remark 1.

1. Theorem 1.5 says that geometric reflection groups are Coxeter groups. So all examples above are Coxeter groups.
2. A finite Coxeter groups is sometimes called a spherical Coxeter group. The reason is, that all finite Coxeter groups can be realised as geometric reflection groups with $\mathbb{X}^n = \mathbb{S}^n$.

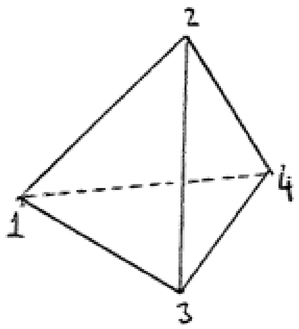

Figure 1.5: Simplex

3. In the next lecture we'll show:
   - all s_i's are pairwise distinct,
   - each s_i has order 2, and
   - each $s_i s_j$ has order m_{ij}.

   Also we'll construct an embedding $W \hookrightarrow \mathrm{GL}(N, \mathbb{R})$, where $N = |S|$. This gives us our first geometric realisation for a general Coxeter group.
4. Coxeter groups arise in Lie theory as Weyl groups of root systems, e.g.
   a) type A_2 root system has Weyl group
   $$W = \langle s_\alpha \,|\, \alpha \text{ in the root system}\rangle,$$
   where s_α is the reflection in the hyperplane orthogonal to α, i.e.
   $$W = \langle s_1, s_2 \,|\, s_i^2 = 1, (s_1 s_2)^3 = 1\rangle = D_6 \cong S_3.$$
   See for example Figure 1.11.
   b) Euclidean geometric groups can arise as "affine Weyl groups" for algebraic groups over local fields with a discrete valuation, e.g. $\mathrm{SL}_3(\mathbb{Q}_p)$.
   The affine Weyl group of type $\tilde{A}_2$ is
   $$W = \langle s_0, s_1, s_2\rangle = \langle s_1, s_2\rangle \ltimes \mathbb{Z}^2.$$
   See for example Figure 1.6. Hence, $\langle s_1, s_2\rangle$ is the subgroup of W which fixes the origin and $\mathbb{Z}^2$ is the subgroup of W consisting of translations.
   c) infinite non-euclidean Coxeter groups can arise as "Kac-Moody Weyl groups".
   A Coxeter matrix M satisfies the *crystallographic restriction* if $m_{ij} \in \{2, 3, 4, 6, \infty\}$ for $i \neq j$.
   Provided this restriction is satisfied, $W = W_M$ is the Weyl group for some Kac-Moody algebra.

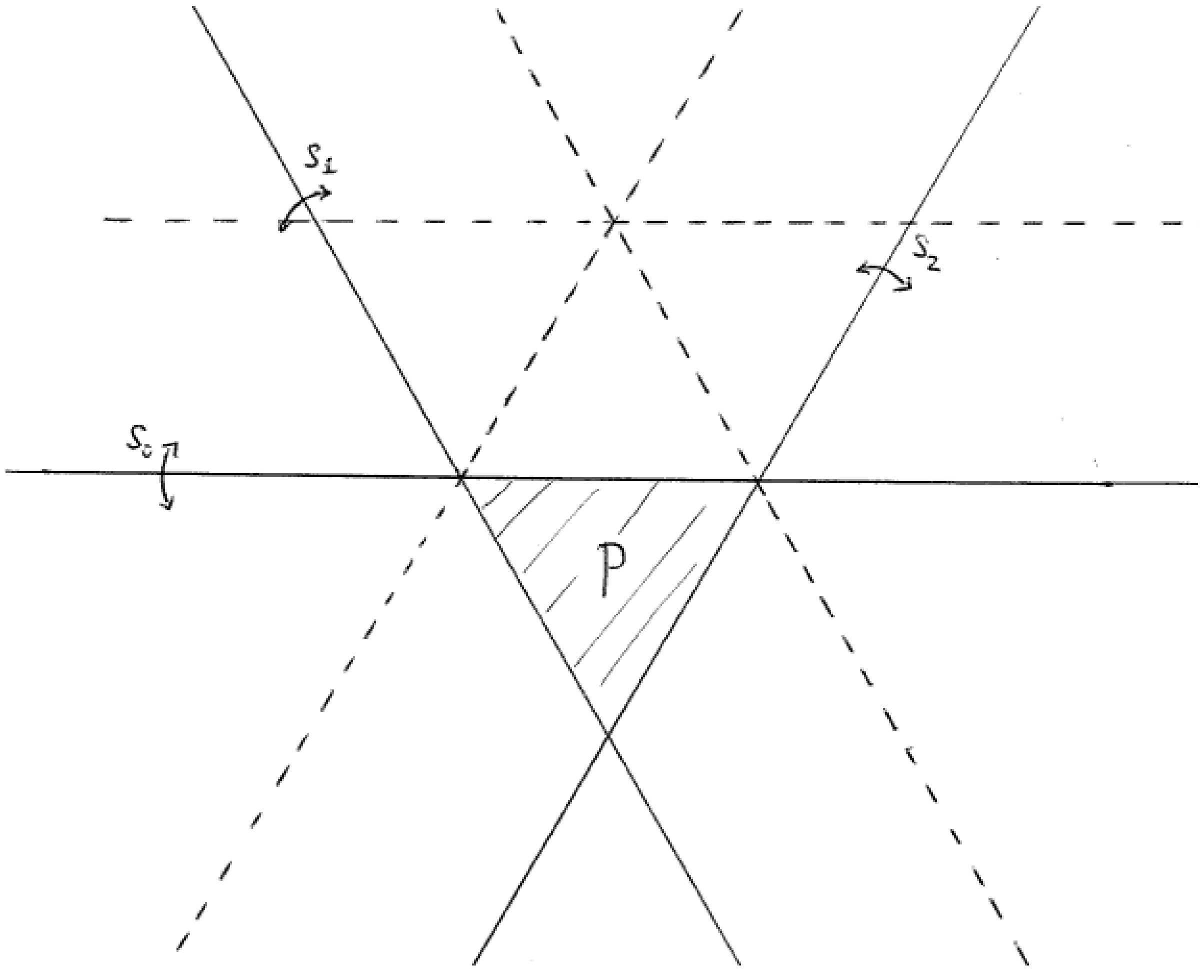

Figure 1.6: P is an equilateral triangle.

5. Tits formulated the general definition of a Coxeter group in order to formulate the definition of a building.

1.2 Buildings

Definition 1.12. A *polyhedral complex* is a finite-dimensional CW-complex in which each n-cell is metrised as a convex polytope in $\mathbb{X}^n$ ($\mathbb{X}^n$ should be the same for each cell) and the restriction of each attaching map to a codimension-one face is an isometry. We will discuss later conditions under which a polyhedral complex is a metric space.

Example 1.13.

- the tessellation of $\mathbb{X}^n$ by copies of P; see Figure 1.6 or Figure 1.8.
- a simplicial tree; see Figure 1.12.

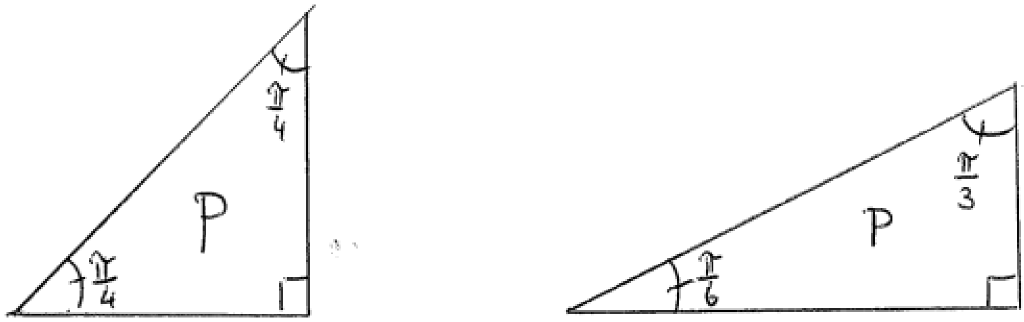

Figure 1.7: Each P tiles $\mathbb{E}^2$.

Definition 1.14. Let $P = P^n$ be as in Theorem 1.5 above, $S = \{s_i\}, W = \langle S \rangle$. A *building of type* (W, S) is a polyhedral complex Δ, which is a union of subcomplexes called *apartments*. Each apartment is isometric to the tiling of $\mathbb{X}^n$ by copies of P, and each such copy of P is called a *chamber*. The apartments and chambers satisfy:

1. Any two chambers are contained in a common apartment.
2. Given any two apartments A and A', there is an isometry $A \to A'$ fixing $A \cap A'$ pointwise.

Example 1.15.

1. A single copy of $\mathbb{X}^n$ tiled by copies of P is a *thin building*, i.e. there is a single apartment.
2. Spherical:

   Let us consider
   $$W = \langle s_1, s_2 \,|\, s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle \cong D_4.$$

   Then there is a (thin) spherical building of type (W, S) as depicted in Figure 1.13. Hereby each edge is a chamber and the only apartment is actually the complete bipartite graph $K_{2,2}$.

   However there is also a thick building of type (W, S) given by $K_{3,3}$; see Figure 1.14.

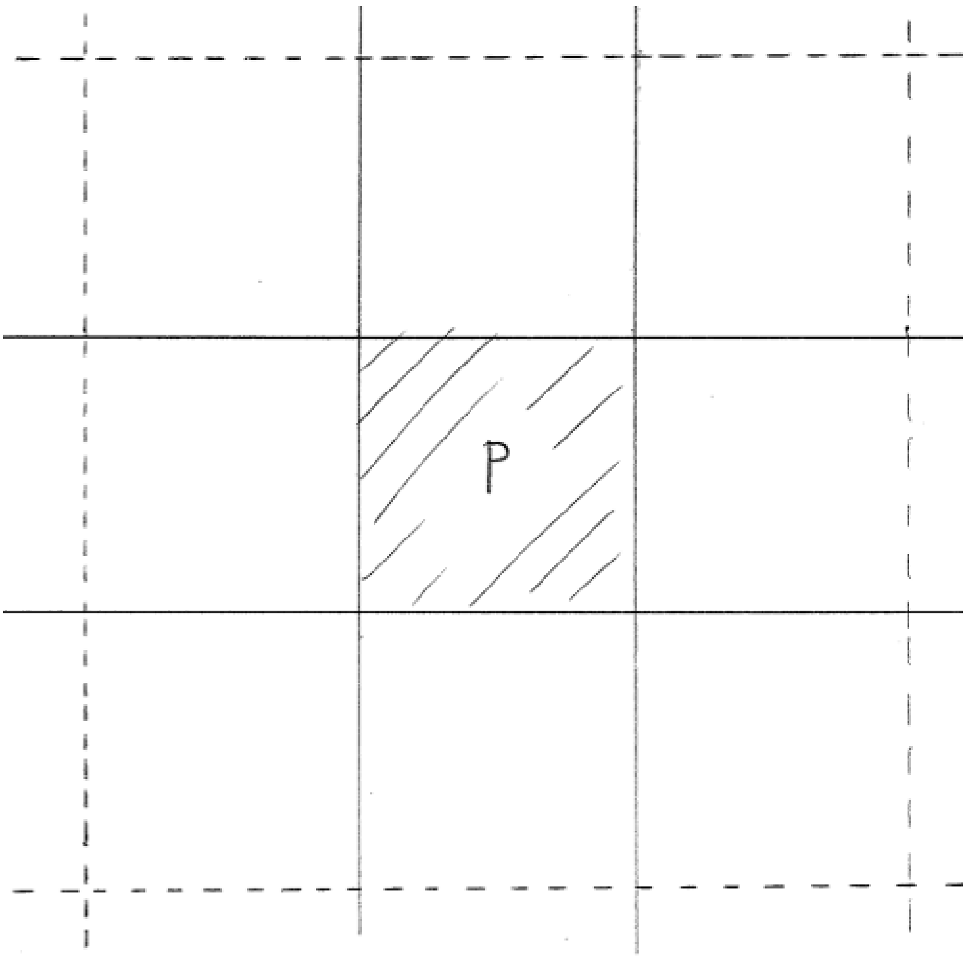

Figure 1.8: P is a square.

3. Euclidean:

   If we consider $W = D_\infty$ as in Example 1.2, we get an apartment as depicted in Figure 1.15. We can now put these together to the regular three-valent tree (see Figure 1.12) and get a Euclidean building.

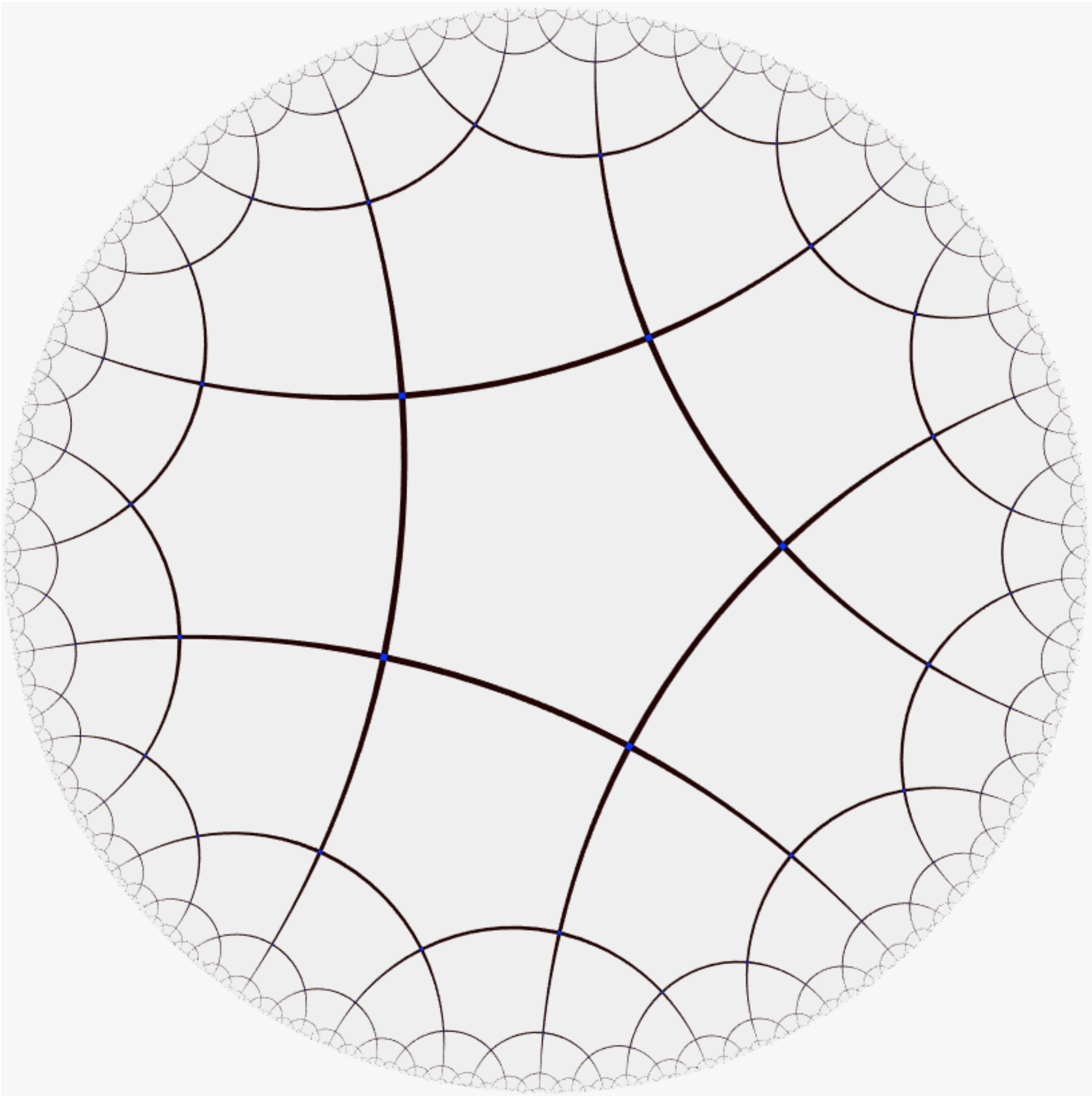

Figure 1.9: P is a right-angled pentagon in $\mathbb{H}^2$. This image was created by Jeff Weeks' free software KaleidoTile.

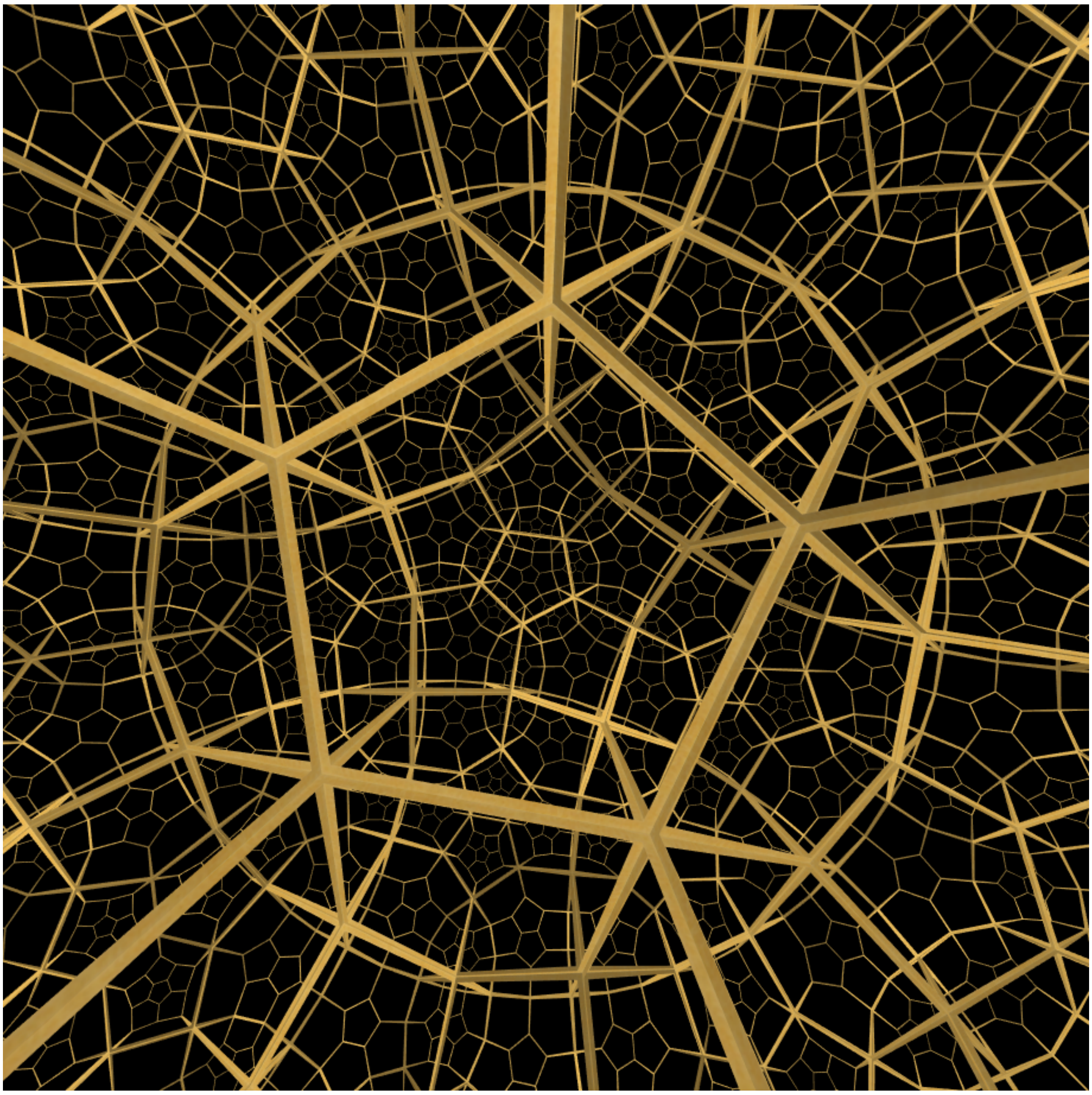

Figure 1.10: P is a dodecahedron in $\mathbb{H}^3$ with all dihedral angles $\pi/2$.. This image was created by Jeff Weeks' free software CurvedSpaces.

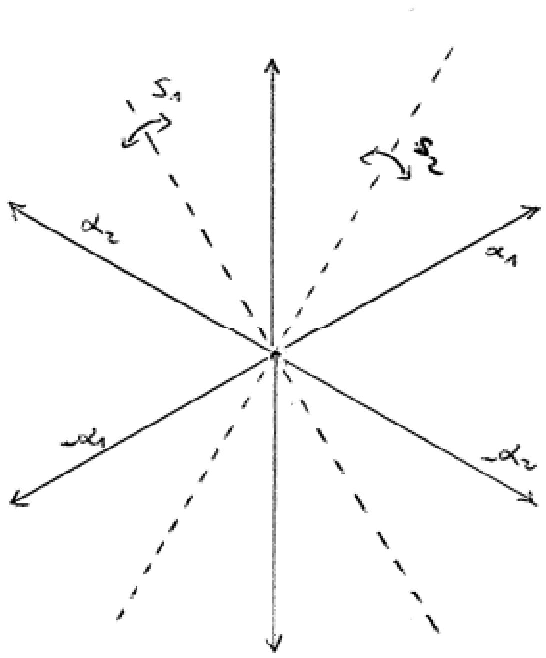

Figure 1.11: Coxeter groups as Weyl group of the root system A_2.

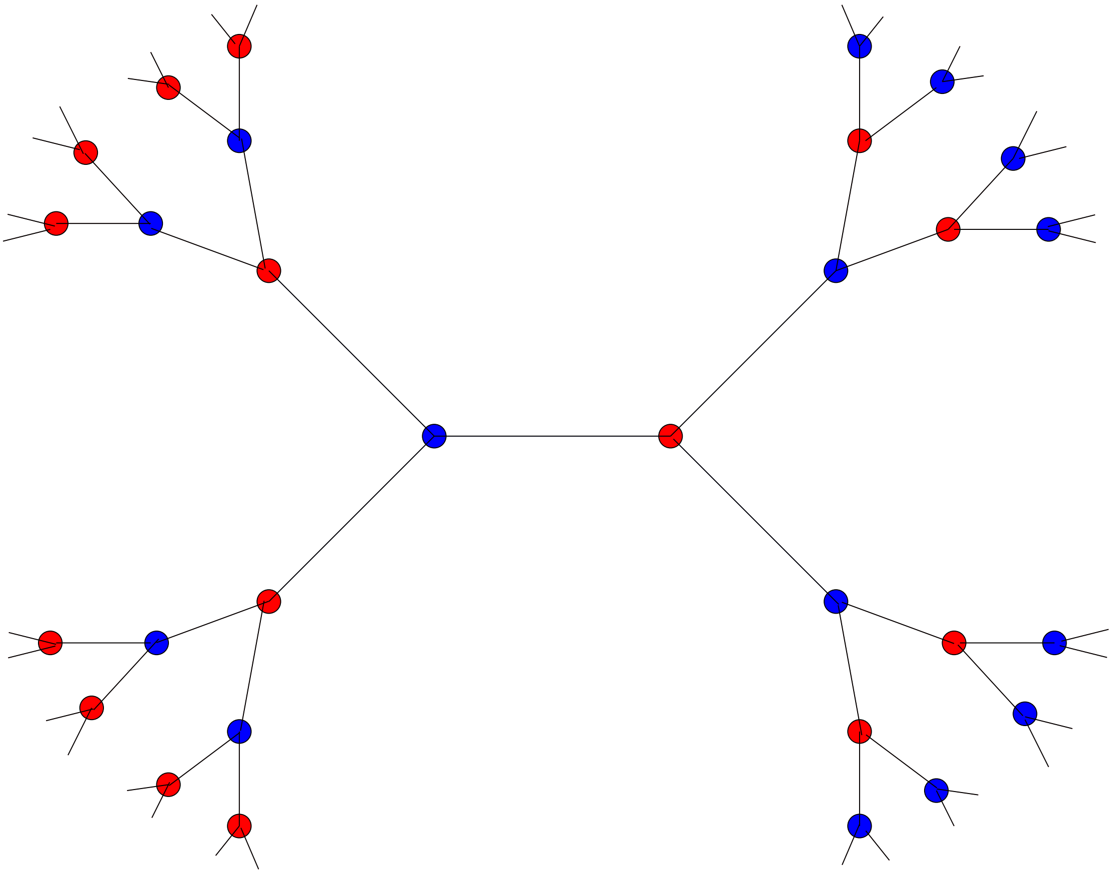

Figure 1.12: The three-valent regular tree.

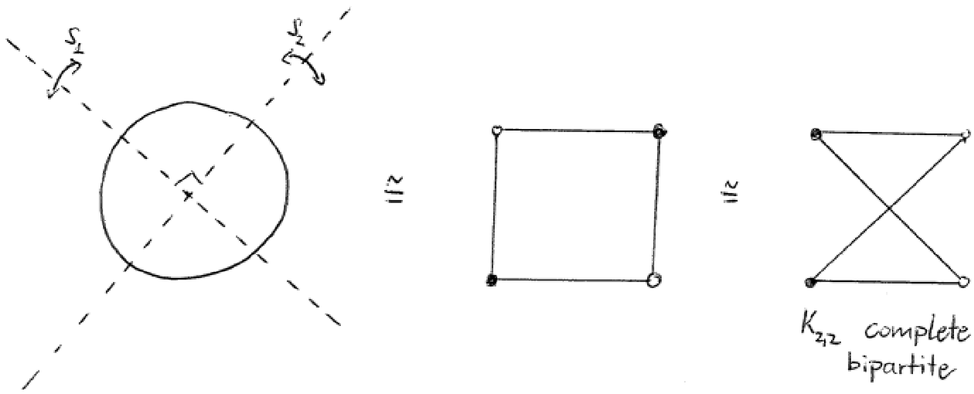

Figure 1.13: The graph $K_{2,2}$ as a spherical building.

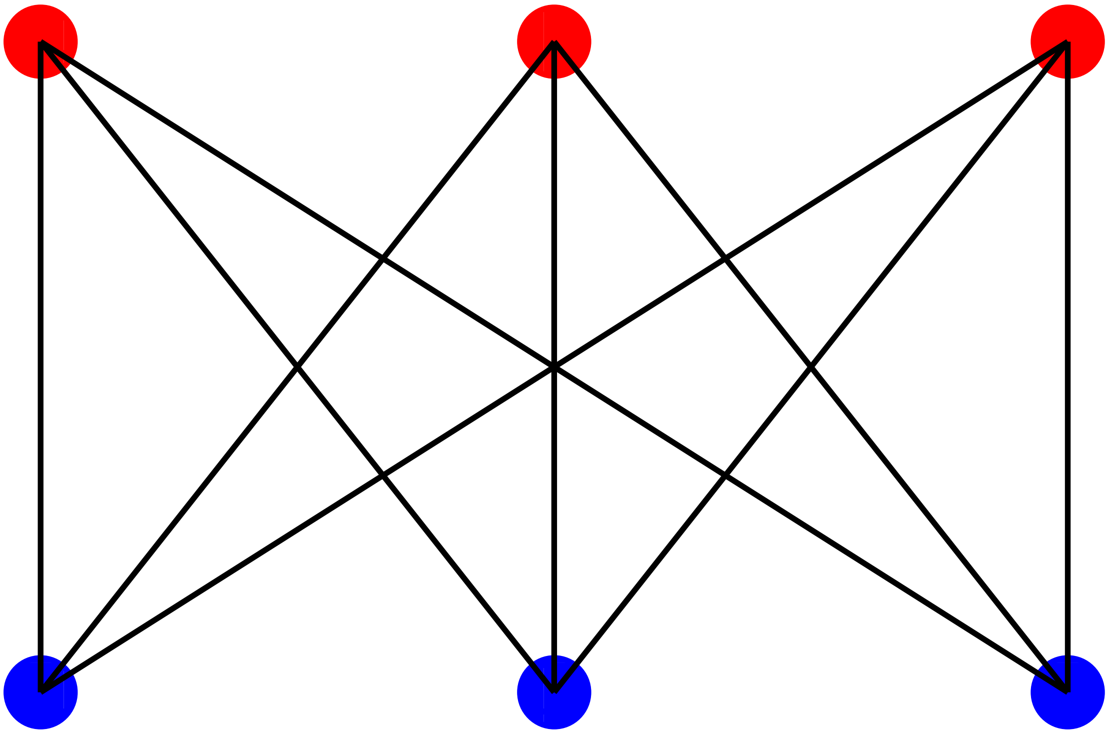

Figure 1.14: The graph $K_{3,3}$.

Figure 1.15: One apartment of the three-valent tree regarded as a euclidean building.

2 Lecture Two: Some combinatorial theory of Coxeter groups

09.03.2016

Let G be a group generated by a set S with $1 \notin S$.

2.1 Word metrics and Cayley graphs

Definition 2.1. The *word length* with respect to S is

$$\ell_S(g) = \min\{n \in \mathbb{N}_0 \mid \exists s_1, \dots, s_n \in S \cup S^{-1} \text{ such that } g = s_1 \dots s_n\}.$$

If $\ell_S(g) = n$ and $g = s_1 \dots s_n$ then the word $(s_1, \dots, s_n)$ is a *reduced expression* for g.
The *word metric* on G with respect to S is $d_S(g,h) = \ell_s(g^{-1}h)$.

Definition 2.2. The *Cayley graph Cay*(G,S) of G with respect to S is the graph with vertices $V = G$ and (directed) edges

$$E = \{(g, gs) \mid g \in G, s \in S\}.$$

However, if s is an involution (i.e. has order 2), we will put a single undirected edge labelled by s.

Example 2.3.
1. The Cayley graph of $D_6 = \langle s_1, s_2 \mid s_i^2 = 1, (s_1 s_2)^3 = 1\rangle$ is depicted in Figure 2.1.
2. The Cayley graph of $D_\infty = \langle s_1, s_2 \mid s_i^2 = 1\rangle$ is depicted in Figure 2.2.
3. If W is the $(3,3,3)$ triangle group, $\mathrm{Cay}(W,S)$ is the dual graph to the tesselation of $\mathbb{R}^2$ by equilateral triangles; see Figure 2.3.
4. If W is generated by the reflections in the sides of a square, $\mathrm{Cay}(W,S)$ is depicted in Figure 2.4.

Since S generates G, $\mathrm{Cay}(G,S)$ is connected. The word metric $d_S(\cdot,\cdot)$ on G extends to the path metric on $\mathrm{Cay}(G,S)$. Note that G acts on $\mathrm{Cay}(G,S)$ on the left by graph automorphisms.

This action is also isometric with respect to $d_S(\cdot,\cdot)$:

$$d_S(hg, hg') = \ell_S((hg)^{-1}hg') = \ell_S(g^{-1}g') = d_S(g,g')$$

If $s \in S$ is an involution, the group element gsg^{-1} flips the edge $(g,gs) \leftrightarrow (gs,g)$ onto itself. In fact, gsg^{-1} is the unique group element which does this, since $hg = gs$ if and only if $h = gsg^{-1}$.

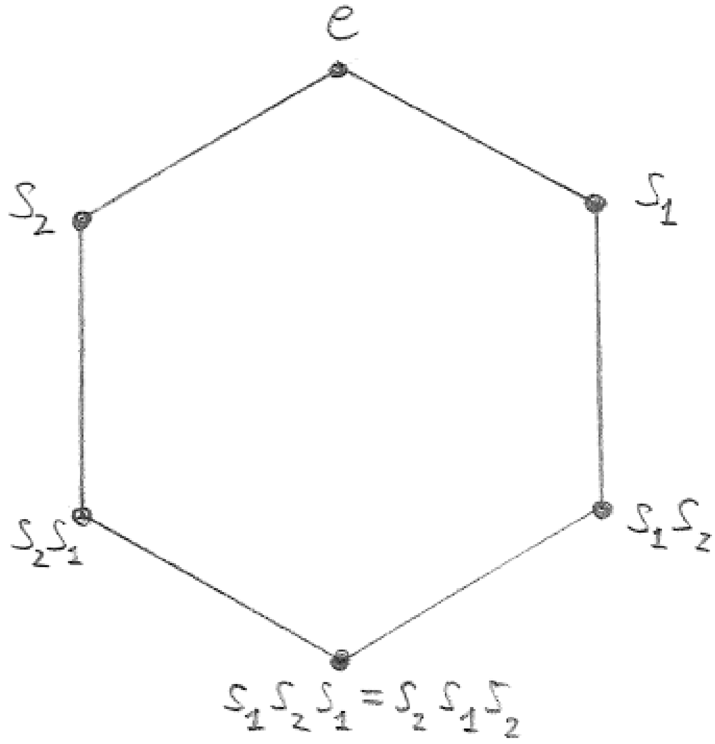

Figure 2.1: Cayley graph of D_6.

2.2 Coxeter systems

Recall from the first lecture the following definition (cf. Definition 1.11): A Coxeter matrix $M = (m_{ij})_{i,j \in I}$ has $m_{ii} = 1$, $m_{ij} = m_{ji} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ if $i \neq j$.

The corresponding Coxeter group is

$$W = \langle S = \{s_i\}_{i \in I} \,|\, (s_i s_j)^{m_{ij}} = 1 \rangle$$

and (W, S) is called a Coxeter system.

Lemma 2.4. Let (W, S) be a Coxeter system. Then there is an epimorphism

$$\varepsilon : W \to \{-1, 1\}$$

induced by $\varepsilon(s) = -1$ for all $s \in S$.

Corollary 2.5. Each $s \in S$ is an involution.

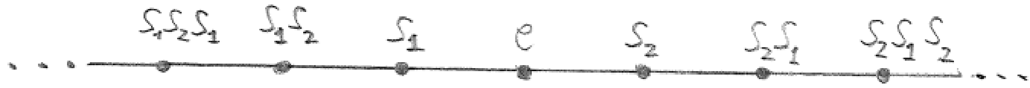

Figure 2.2: Cayley graph of D_∞.

Corollary 2.6. Write $\ell = \ell_S$. Then $\forall w \in W, s \in S : \ell(ws) = \ell(w) \pm 1$ and $\ell(sw) = \ell(w) \pm 1$.

Theorem 2.7 (Tits)**.** *Let (W, S) be a Coxeter system. Then there is a faithful representation*

$$\rho : W \to GL(N),$$

where $N = |S|$, such that:

- *$\rho(s_i) = \sigma_i$ is a linear involution with fixed set a hyperplane. (This is NOT necessarily a orthogonal reflection!)*
- *If s_i, s_j are distinct then $\sigma_i\sigma_j$ has order m_{ij}.*

Corollary 2.8. In a Coxeter system (W, S) the elements of S are distinct involutions in W.

Proof of Theorem 2.7. Let V be a vector space over $\mathbb{R}$ with basis $\{e_1, \ldots, e_N\}$. Now define a symmetric bilinear form B by

$$B(e_i, e_j) = \begin{cases} -\cos(\frac{\pi}{m_{ij}}), & \text{if } m_{ij} \text{ is finite} \\ -1, & \text{if } m_{ij} = \infty \end{cases}.$$

Note that $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ if $i \neq j$.

Let us consider the hyperplane $H_i = \{v \in V \mid B(e_i, v) = 0\}$, and $\sigma_i : V \to V$ given by

$$\sigma_i(v) = v - 2B(e_i, v)e_i.$$

It is easy to check, that $\sigma_i(e_i) = -e_i$, σ_i fixes H_i pointwise, $\sigma_i^2 = \mathrm{id}$, and that σ_i preserves $B(\cdot, \cdot)$. The theorem will then follow from the following two claims, whose proofs we postpone for now.

Claim 1: The map $s_i \mapsto \sigma_i$ extends to a homomorphism $\rho : W \to \mathrm{GL}(V)$.

Claim 2: ρ is faithful.

$\square$

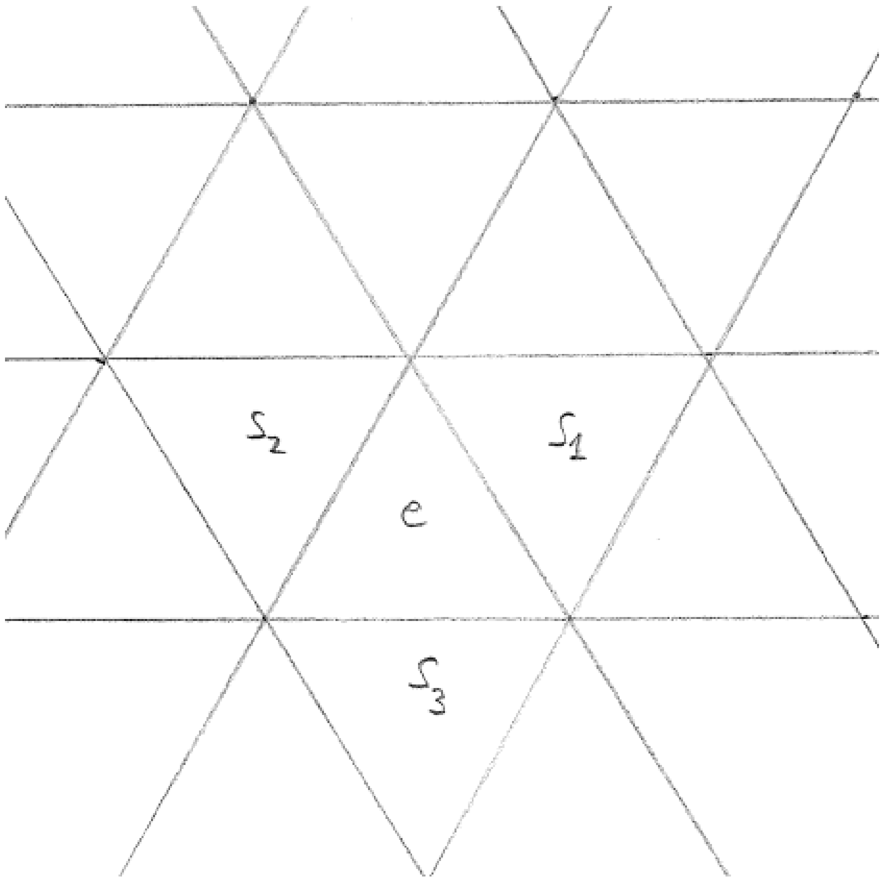

Figure 2.3: Tesselation of $\mathbb{R}^2$ by the $(3, 3, 3)$ triangle group.

2.3 Reflection Systems

Definition 2.9. A *pre-reflection system* for a group G is a pair (X, R), where X is a connected simplicial graph, G acts on X, and R is a subset of G, such that:

1. each $r \in R$ is an involution;
2. R is closed under conjugation, i.e. $\forall g \in G \, \forall r \in R : grg^{-1} \in R$;
3. R generates G;
4. for every edge e in X there is a unique $r = r_e \in R$, which flips e; and
5. for every $r \in R$ there is at least one edge e in X, which is flipped by r.

Example 2.10. If we consider again $W = D_6$, $X = \mathrm{Cay}(G, S)$, and set $R = \{s_1, s_2, s_1 s_2 s_1\}$, then we get the situation depicted in Figure 2.5.

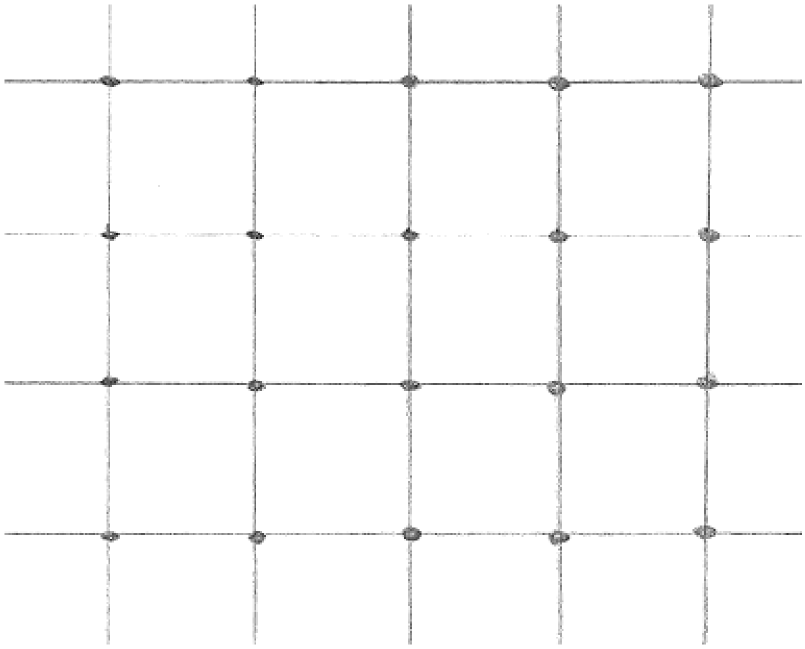

Figure 2.4: Cayley graph of the group generated by the reflection in the sides of a square.

Example 2.11. If (W, S) is any Coxeter system, let $X = \mathrm{Cay}(W, S)$ and set $R = \{wsw^{-1} \mid w \in W, s \in S\}$. Then (X, R) is a pre-reflection system; indeed wsw^{-1} flips the edge (w, ws).

Definition 2.12. Let (X, R) be a pre-reflection system. For each $r \in R$, the *wall* H_r is the set of midpoints of edges which are flipped by r. A pre-reflection system (X, R) is a *reflection system*, if in addition

6. for each $r \in R$, $X \backslash H_r$ has exactly two components. (These will be interchanged by r).

We call R the set of *reflections*.

Theorem 2.13. *Suppose a group W is generated by a set of distinct involutions S. Then the following are equivalent:*

1. (W, S) *is a Coxeter system;*
2. *if $X = Cay(W, S)$ and $R = \{wsw^{-1} \mid w \in W, s \in S\}$, then (X, R) is a reflection system;*
3. (W, S) *satisfies the* Deletion Condition*:*

   if $(s_1, \ldots, s_k)$ is a word in S with $\ell(s_1, \ldots, s_k) < k$, then there are $i < j$, such that

$$s_1 \ldots s_k = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_k,$$

   where $\hat{s_i}$ means, we delete this letter;

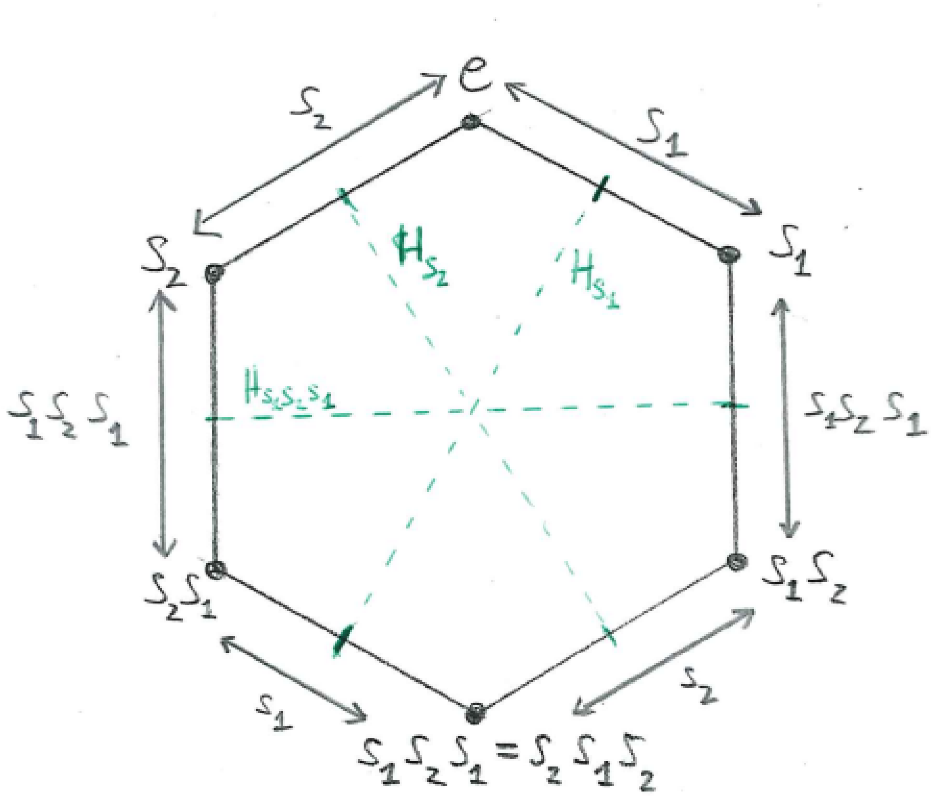

Figure 2.5: Cayley graph of D_6 with the reflections $R = \{s_1, s_2, s_1s_2s_1\}$ and corresponding walls H_r for $r \in R$.

4. (W, S) *satisfies the* Exchange Condition*:*
   if $(s_1, \ldots, s_k)$ *is a reduced expression for* $w \in W$ *and* $s \in S$*, either* $\ell(sw) = k + 1$ *or there is an index* i*, such that*

$$s_1 \ldots s_k = ss_1 \ldots \hat{s_i} \ldots s_k.$$

Proof. We will sketch 1. $\implies$ 2. $\implies$ 3. $\implies$ 4. $\implies$ 1.

1. $\implies$ 2.: There is a bijection

$$\{\text{words in } S\} \longleftrightarrow \{\text{paths in } X = \mathrm{Cay}(W, S) \text{ starting at } e\}$$

mapping a word $(s_1, \ldots, s_k)$ to the path with vertices e, s_1, s_1s_2, $s_1s_2s_3$, $\ldots$, $s_1s_2 \ldots s_k$; see Figure 2.6.

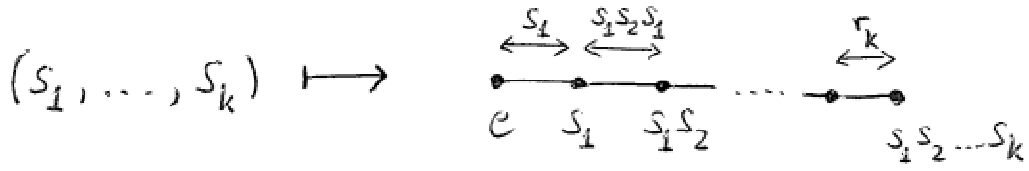

Figure 2.6: The map sending a word $(s_1, \ldots, s_k)$ to the path e, s_1, s_1s_2, $s_1s_2s_3$, $\ldots$, $s_1s_2 \ldots s_k$.

The word $(s_1, \ldots, s_k)$ has a canonical associated sequence of reflections

$$\begin{aligned} r_1 &= s_1 \\ r_2 &= s_1s_2s_1 \\ r_3 &= s_1s_2s_3s_2s_1 \\ &\vdots \end{aligned}$$

Further we have the following key lemma.

Lemma 2.14. If $w \in W$ and $r \in R$, any word for w crosses H_r the same number of times mod 2, i.e. if $\underline{s}, \underline{s}'$ are words for w, and $n(r, \underline{s}), n(r, \underline{s}')$ are the number of times, these paths cross H_r, then $(-1)^{n(r,\underline{s})} = (-1)^{n(r,\underline{s}')}$.

Proof. Define a homomorphism $\varphi : W \to \mathrm{Sym}(R \times \{-1, 1\})$ by extending $\varphi(s)(r, \varepsilon) = (srs, \varepsilon(-1)^{\delta_{rs}})$, $\varepsilon \in \{\pm 1\}$. $\square$

We can use this lemma to show that each H_r separates $\mathrm{Cay}(W, S)$; namely, w and w' are on the same side of H_r if and only if any path from w to w' crosses H_r an even number of times.

2. $\implies$ 3.:

Lemma 2.15. Let $(s_1, \ldots, s_k)$ be a word in S with associated reflections $(r_1, \ldots, r_k)$ as above. If $r_i = r_j$ for $i < j$, then $s_1 \ldots s_k = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_k$.

Proof. Let $r = r_i = r_j$ and let $w_k := s_1 \ldots s_k$ for each k. If we now apply r to the subpath from w_i to w_{j-1}, we get a new path of the type

$$(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j-1}, s_{j+1}, \ldots s_k),$$

as depicted in Figure 2.7.

$\square$

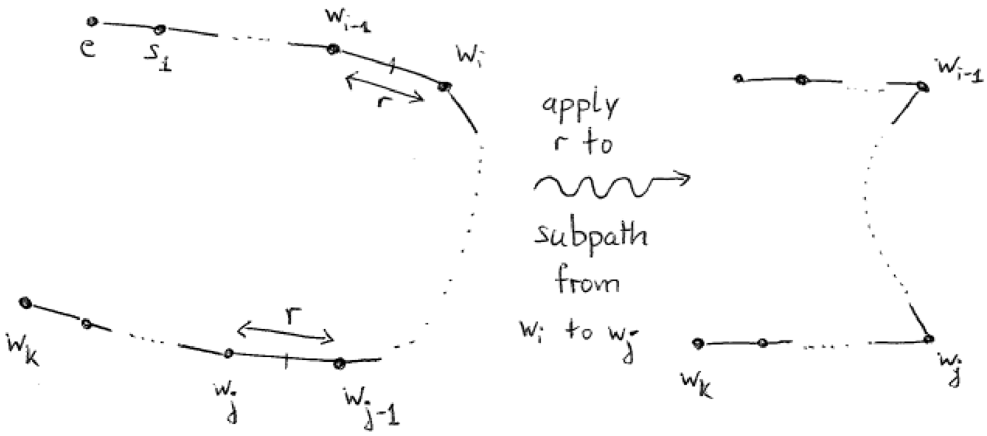

Figure 2.7: Applying the reflection r to the subpath from w_i to w_{j-1}.

Lemma 2.16. If $(s_1, \ldots, s_k)$ is a word in S with associated reflections $(r_1, \ldots, r_k)$ as above, then this word is a reduced expression if and only if the r_i are pairwise distinct.

Proof. If some $r_i = r_j$, the word is non-reduced by the previous lemma. Now let $w = s_1 \ldots s_k$ and $R(e, w) = \{r \mid H_r \text{ separates } e \text{ from } w\}$. Then

$$r \in R(e,w) \implies r = r_i \text{ for some } i \implies \ell(w) \geq |R(e,w)|.$$

Hence if all r_i are pairwise distinct, then $|R(e,w)| \geq k$. On the other hand $\ell(w) \leq k$, and so $\ell(w) = k$, i.e. the word is reduced. The Deletion Condition follows. □

3. $\implies$ 4.: Suppose $(s_1, \ldots, s_k)$ is a reduced word and $s \in S$. We set $w = s_1 \ldots s_k$.

If $\ell(sw) = k+1$, there is nothing to show. Hence let us assume that $\ell(sw) \leq k$. In this case $(s, s_1, \ldots, s_k)$ is non-reduced and by the Deletion Condition we can delete two letters. However $(s_1, \ldots, s_k)$ is reduced, such that one of the two letters has to be s. Thus

$$\begin{aligned} ss_1 \ldots s_k = s_1 \ldots \hat{s}_i \ldots s_k &\implies sw = s_1 \ldots \hat{s}_i \ldots s_k \\ &\implies sw = ss_1 \ldots \hat{s}_i \ldots s_k. \end{aligned}$$

4. $\implies$ 1.: We will use Tits' solution to the word problem in Coxeter groups.

Definition 2.17. Suppose W is generated by a set of distinct involutions S. If $s, t \in S$, $s \neq t$, let m_{st} be the order of st in W. If m_{st} is finite, a *braid move* on a word in S

replaces a subword $(s, t, s, \ldots)$ by a subword $(t, s, t, \ldots)$, where each subword has m_{st} letters. For example in D_6: $(s_1, s_2, s_1) \leftrightarrow (s_2, s_1, s_2)$.

Theorem 2.18 (Tits)**.** *Suppose a group W is generated by a set of distinct involutions and the Exchange Condition holds. Then:*

1. *A word $(s_1, \ldots, s_k)$ is reduced if and only if it cannot be shortened by a sequence of*
   - *deleting a subword (s, s), $s \in S$; or*
   - *carrying out a braid move.*
2. *Two reduced expressions represent the same group element if and only if they are related by a sequence of braid moves.*

Proof. Use induction on $\ell(w)$. Prove 2., then 1. □

To show (W, S) is a Coxeter system: Let m'_{ij} be the order of $s_i s_j$ in W. Further let W' be the Coxeter group with Coxeter matrix $M' = (m'_{ij})$. Finally use Theorem 2.18 to show that $W' \to W$ is injective. □

Definition 2.19. For each $T \subseteq S$, the *special subgroup* W_T of W is $W_T = \langle T \rangle$. Sometimes these are also called *parabolic subgroups* or *visual subgroups*. We shall also use the alternative notation: if $J \subseteq I$, $W_J = \langle s_j \mid j \in J \rangle$. If $T = \emptyset$, we define $W_\emptyset$ to be the trivial group.

Using Theorem 2.13 we can show, that for each $T \subseteq S$:

1. (W_T, T) is a Coxeter system.
2. For every $w \in W_T$, $\ell_T(w) = \ell_S(w)$, and any reduced expression for w only uses letters in T, i.e. $\mathrm{Cay}(W_T, T)$ embeds isometrically as a convex subgraph of $\mathrm{Cay}(W, S)$.
3. If $T, T' \subseteq S$, then $W_T \cap W_{T'} = W_{T \cap T'}$. There is a bijection

$$\{\text{subsets of } S\} \longleftrightarrow \{\text{special subgroups}\},$$

   which preserves inclusion.

3 Lecture Three: The Tits representation

16.03.2016

3.1 Proof of Tits' representation theorem

We will now return the proof of Theorem 2.7. Let us briefly recall the statement and what we have said so far.

Theorem (Theorem 2.7 (Tits))**.** *Let (W, S) be a Coxeter system. Then there is a faithful representation*

$$\rho : W \to GL(N, \mathbb{R}),$$

where $N = |S|$, such that:

- *$\rho(s_i) = \sigma_i$ is a linear involution with fixed set a hyperplane. (This is NOT necessarily a orthogonal reflection!)*
- *If s_i, s_j are distinct then $\sigma_i\sigma_j$ has order m_{ij}.*

Definition 3.1. This representation is called the *Tits representation*, or the standard (geometric) representation.

Continuation of the proof. So far we had the following. Let V be a vector space over $\mathbb{R}$ with basis $\{e_1, \ldots, e_N\}$. Now define a symmetric bilinear form B by

$$B(e_i, e_j) = \begin{cases} -\cos(\frac{\pi}{m_{ij}}), & \text{if } m_{ij} \text{ is finite} \\ -1, & \text{if } m_{ij} = \infty. \end{cases}$$

Note that $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ if $i \neq j$.

Let us consider the hyperplane $H_i = \{v \in V \mid B(e_i, v) = 0\}$, and $\sigma_i : V \to V$ given by

$$\sigma_i(v) = v - 2B(e_i, v)e_i;$$

see Figure 3.1.

It is easy to check that $\sigma_i(e_i) = -e_i$, σ_i fixes H_i pointwise, $\sigma_i^2 = \text{id}$, and that σ_i preserves $B(\cdot, \cdot)$. The theorem will then follow from the following two claims.

Claim 1: The map $s_i \mapsto \sigma_i$ extends to a homomorphism $\rho : W \to \text{GL}(V)$.

Proof of Claim 1. It is enough to show that $\sigma_i\sigma_j$ has order m_{ij}: Let V_{ij} be the subspace $\text{span}(e_i, e_j)$. Then σ_i and σ_j preserve V_{ij}, so we will consider the restriction of $\sigma_i\sigma_j$ to V_{ij}.

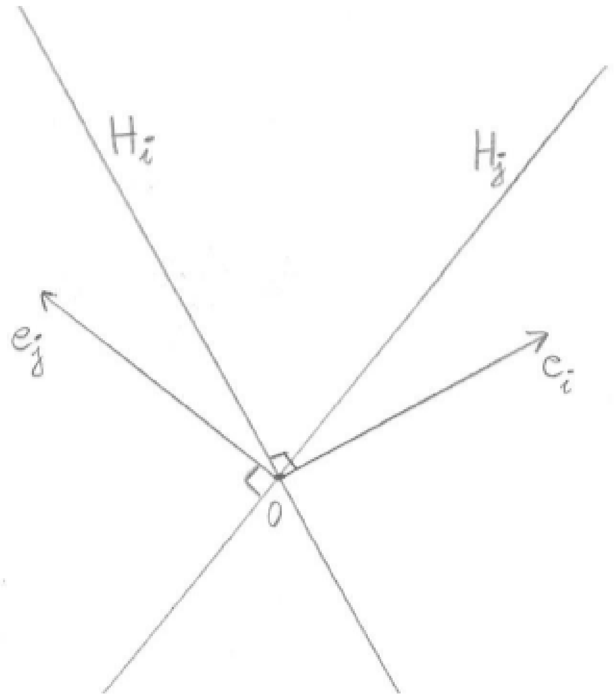

Figure 3.1: The basis vectors e_i, e_j and their corresponding hyperplanes H_i, H_j.

Case I (m_{ij} is finite): Let $v = \lambda_i e_i + \lambda_j e_j \in V_{ij}$. If $v \neq 0$ then

$$\begin{aligned} B(v,v) &= \lambda_i^2 - 2\lambda_i\lambda_j \cos\left(\frac{\pi}{m_{ij}}\right) + \lambda_j^2 \\ &= \left(\lambda_i - \lambda_j \cos\left(\frac{\pi}{m_{ij}}\right)\right)^2 + \lambda_j^2 \sin\left(\frac{\pi}{m_{ij}}\right)^2 > 0 \end{aligned}$$

So B is positive definite on V_{ij}, however not necessarily so on the whole of V. Thus we can identify V_{ij} with Euclidean two-space and $B|_{V_{ij}}$ with the standard inner product. The maps σ_i and σ_j are now orthogonal reflections. Since

$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right) = \cos\left(\pi - \frac{\pi}{m_{ij}}\right),$$

the angle between e_i and e_j (in V_{ij}) is $\pi - \frac{\pi}{m_{ij}}$. Hence the dihedral angle between H_i and H_j is $\frac{\pi}{m_{ij}}$ and so $\sigma_i\sigma_j$ is a rotation by the angle $2\frac{\pi}{m_{ij}}$. This shows that $\sigma_i\sigma_j$ has order m_{ij} when restricted to the subspace V_{ij}.

Let us now consider $V_{ij}^{\perp} = \{v' \in V \mid B(v', v) = 0 \quad \forall v \in V_{ij}\}$. Since B is positive definite on V_{ij},

$$V = V_{ij} \oplus V_{ij}^{\perp}.$$

Now $\sigma_i\sigma_j$ fixes $V_{ij}^{\perp}$ pointwise. Hence $\sigma_i\sigma_j$ has order m_{ij} on V.

Case II ($m_{ij} = \infty$): Again let $v = \lambda_i v_i + \lambda_j v_j \in V_{ij}$. Then

$$\begin{aligned} B(v,v) &= \lambda_i^2 - 2\lambda_i\lambda_j + \lambda_j^2 \\ &= (\lambda_i - \lambda_j)^2 \geq 0, \end{aligned}$$

with equality if and only if $\lambda_i = \lambda_j$. So B is positive semi-definite, but not positive definite on V_{ij}. Consider

$$\begin{aligned} \sigma_i\sigma_j(e_i) &= \sigma_i(e_i + 2e_j) \\ &= -e_i + 2(e_j + 2e_i) = e_i + 2(e_i + e_j). \end{aligned}$$

By induction we get for all $k \geq 1$:

$$(\sigma_i\sigma_j)^k(e_i) = e_i + 2k(e_i + e_j).$$

Thus $\sigma_i\sigma_j$ has infinite order on V_{ij} and hence also on the whole of V. This finishes the proof of our first claim and we have a representation $\rho : W \to GL(N, \mathbb{R})$. □

Before we move on to the proof of the faithfulness of ρ let us discuss the geometry of the second case above. Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

This is the matrix for $B|_{V_{ij}}$ in the basis $\{e_i, e_j\}$ of V_{ij} when $m_{ij} = \infty$. Since B is positive semi-definite, but not positive definite on V_{ij}, the matrix A has a one-dimensional nullspace of vectors v such that $B(v, v) = 0$:

$$\mathrm{null}(A) = \mathrm{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathrm{span}(e_i + e_j) = \{v \in V_{ij} \mid B(v, v) = 0\}.$$

Thus B induces a positive definite form on $V_{ij}/\mathrm{null}(A)$ and the latter can be identified with one-dimensional Euclidean space. Let $W_{ij} = \langle s_i, s_j \rangle \cong D_\infty$. Note:

1. W_{ij} acts faithfully on V_{ij}.
2. We have
$$\sigma_i(e_i + e_j) = \sigma_j(e_i + e_j) = e_i + e_j,$$
so W_{ij} fixes $\mathrm{null}(A)$ pointwise.

Now consider the dual vector space

$$V_{ij}^* = \{\text{linear functionals } \varphi : V_{ij} \to \mathbb{R}\}.$$

The group W_{ij} acts on V_{ij}^* via $(w \cdot \varphi)(v) = \varphi(w^{-1} \cdot v)$ ($w \in W_{ij}$, $\varphi \in V_{ij}^*$, $v \in V_{ij}$) and this action is faithful because the original one was. So we have a faithful action of D_∞.

Consider the codimension-one subspace of V_{ij}^*

$$Z = \{\varphi \in V_{ij}^* \,|\, \varphi(e_i + e_j) = 0\}.$$

Since W_{ij} fixes $e_i + e_j$, it preserves Z.

We may now identify

$$Z \longleftrightarrow (V_{ij}/\mathrm{null}(A))^* .$$

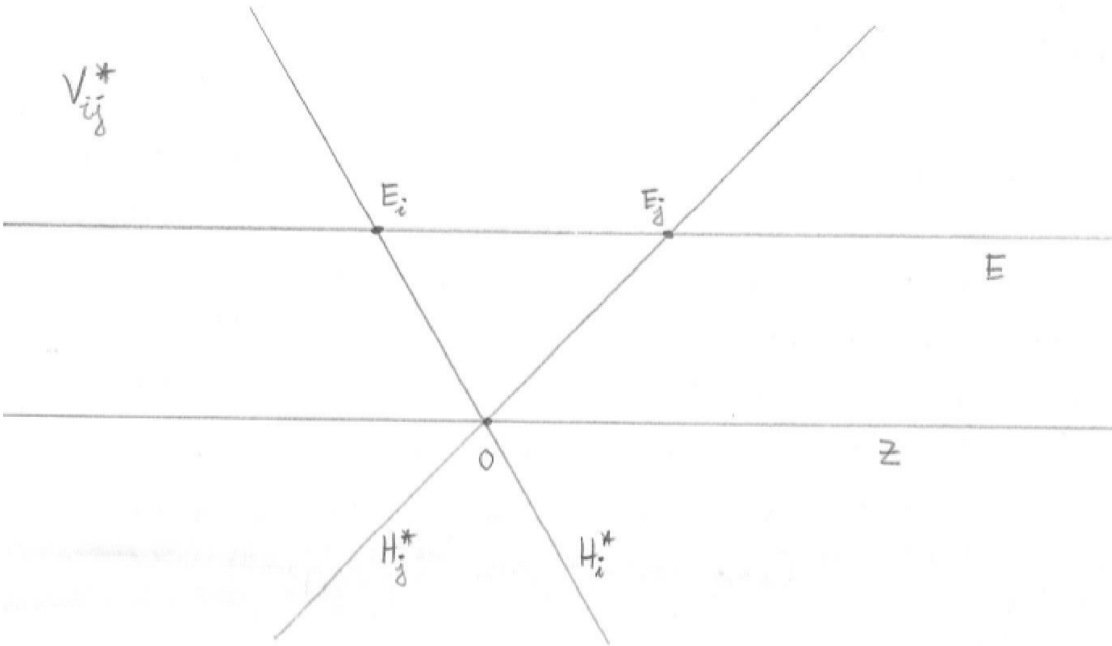

Figure 3.2: The dual space V_{ij}^* with the (affine) subspaces E and Z.

So Z has a one-dimensional Euclidean structure as well. Let E be the codimension-one affine subspace

$$E = \{\varphi \in V_{ij}^* \,|\, \varphi(e_i + e_j) = 1\} = Z + \mathbb{1}.$$

Therefore also E has a one-dimensional Euclidean structure. Since W_{ij} fixes $e_i + e_j$, it stabilizes E. Now E spans V_{ij}^* and W_{ij} acts faithfully on V_{ij}^*, so the W_{ij}-action on E is faithful. Let

$$H_i^* = \{\varphi \in V_{ij}^* \,|\, \varphi(e_i) = 0\}.$$

Then $H_i^* \neq Z$, so $H_i^* \cap E =: E_i \neq \emptyset$ is a codimension-one hyperplane of E. The same holds for j. Observe that $s_i \cdot e_i = -e_i$, so s_i acts on E as an isometric reflection with fixed hyperplane E_i. We get an isometric action of $W_{ij} \cong D_\infty$ on E generated by reflections.

Claim 2: ρ is faithful.

Sketch of proof of Claim 2. Consider the dual representation $\rho^* : W \to GL(V^*)$ given by

$$(\rho^*(w)(\varphi))(v) = \varphi(\rho(w^{-1})(v))$$

for all $\varphi \in V^*, w \in W, v \in V$.

Define elements $\varphi_i \in V^*$ by $\varphi_i(v) = B(e_i, v)$. Now define

$$H_i^* = \{\varphi \in V^* \mid \varphi(e_i) = 0\}.$$

Then $\sigma_i^* := \rho^*(s_i)$ is $\sigma_i^*(\varphi) = \varphi - 2\varphi(e_i)\varphi_i$. Using this it is easy to check that $\sigma_i^*(\varphi_i) = -\varphi_i$, $(\sigma_i^*)^2 = \mathrm{id}$ and that σ_i^* fixes H_i^* pointwise.

Define the *chamber* C by

$$C = \{\varphi \in V^* \mid \varphi(e_i) \geq 0 \quad \forall i\}.$$

Example 3.2. If $W = D_{2m}$ (Case I), V^* is $\mathbb{E}^2$; see Figure 3.3.

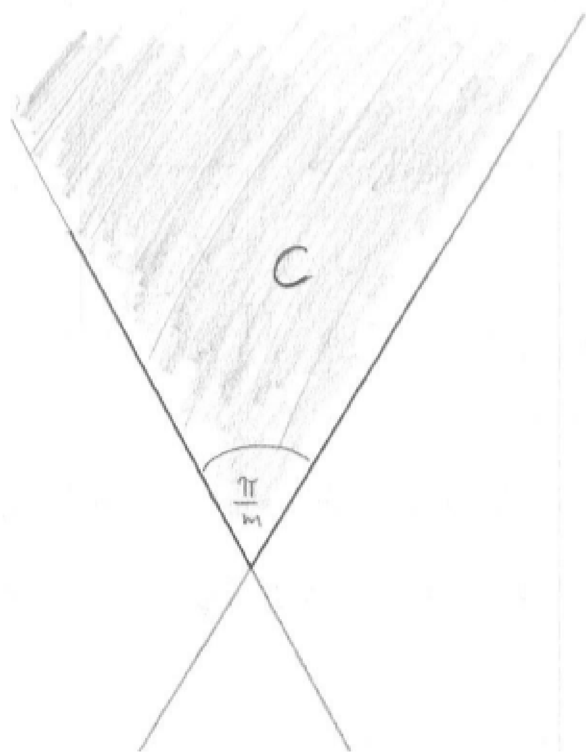

Figure 3.3: The chamber C for $W = D_{2m}$.

If $W = W_{ij} = D_\infty$ (Case II), we have the situation as in Figure 3.4.

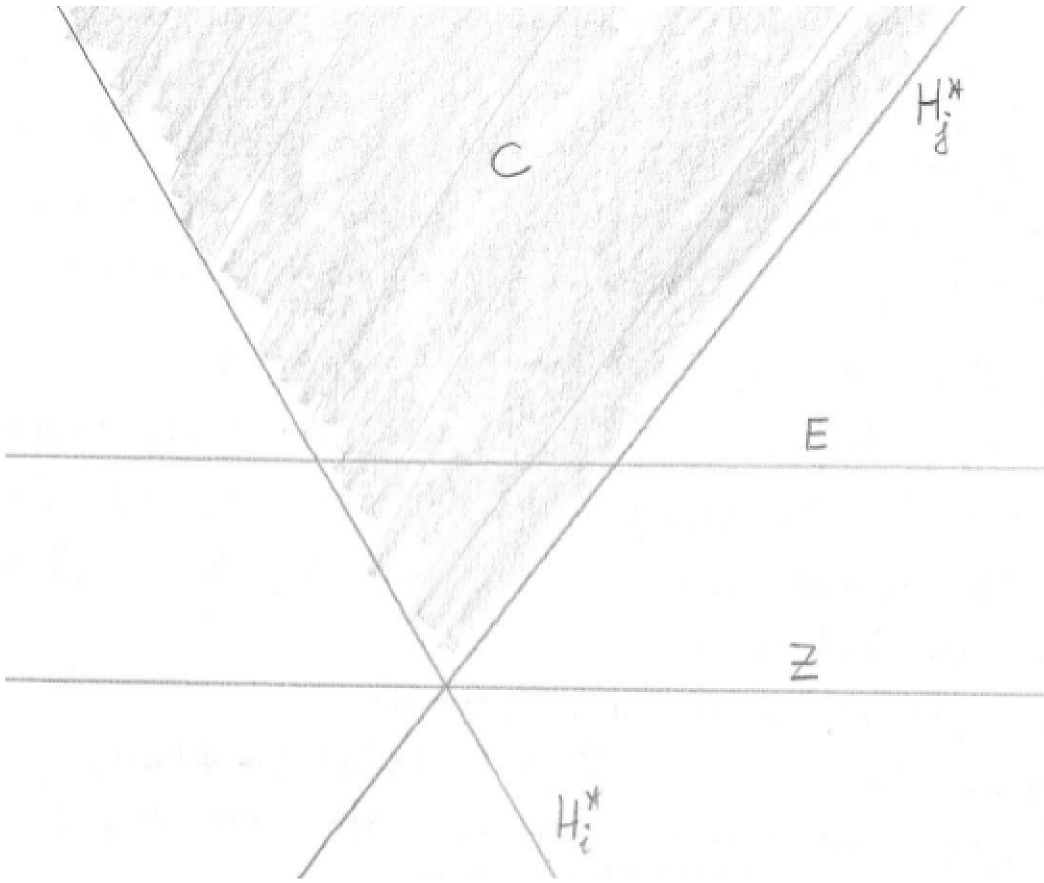

Figure 3.4: The chamber C for $W = D_\infty$.

This is the "simplicial cone" cut out by the hyperplanes H_i^*. Let

$$\mathring{C} = \mathrm{int}(C) = \{\varphi \in V^* \mid \varphi(e_i) > 0\}.$$

Theorem 3.3 (Tits)**.** *Let $w \in W$. If $w\mathring{C} \cap \mathring{C} \neq \emptyset$, then $w = 1$.*

Sketch. Holds for each $W_{ij} = \langle s_i, s_j \rangle$ by Cases I and II above. Use a combinatorial lemma of Tits to promote to W ... □

Corollary 3.4. ρ^* is faithful $\implies$ ρ is faithful.

□

This finished the proof of our second claim and hence Theorem 2.7 has been proven.
□

3.2 Some corollaries of Tits' representation theorem

Definition 3.5. The *Tits cone* of W is the subset of V^* given by $\bigcup_{w\in W} wC$ where C is the chamber defined above.

Example 3.6.

1. If $W = D_{2m}$, the Tits cone is all of $\mathbb{E}^2$.
2. If $W = D_\infty$, the Tits cone is $\{\varphi \in V_{ij}^* \,|\, \varphi(e_i + e_j) > 0\} \cup \{0\}$.

Corollary 3.7. $\rho(W)$ is a discrete subgroup of $GL(N, \mathbb{R})$.

Proof. Consider the W-action on the interior of the Tits cone. This action has finite point stabilisers. □

Definition 3.8. A group G is *linear* (over $\mathbb{R}$) if there is a faithful representation $\varphi : G \to GL(n, \mathbb{R})$ for some $n \in \mathbb{N}$.

Corollary 3.9. Coxeter groups and their subgroups are linear.

This is particularly nice because of the following two theorems on linear groups.

Theorem 3.10 (Selberg)**.** *Finitely generated linear groups are virtually torsion-free, i.e. they have a torsion-free subgroup with finite index.*

Theorem 3.11 (Malcev)**.** *Finitely generated linear groups are residually finite: For every $g \in G, g \neq 1$, there is a finite group H_g and a homomorphism $\varphi : G \to H_g$ such that $\varphi(g) \neq 1$.*

Definition 3.12. A Coxeter system (W, S) is *reducible* if $S = S' \sqcup S''$, $S' \neq \emptyset, S'' \neq \emptyset$, such that everything in S' commutes with everything in S'', i.e. $m_{ij} = 2\; \forall s_i \in S', s_j \in S''$. Then $W = \langle S' \rangle \times \langle S'' \rangle = W_{S'} \times W_{S''}$.
(W, S) is *irreducible* if it is not reducible.

Theorem 3.13. *Suppose (W, S) is irreducible and $n = |S|$. Then:*

1. *B is positive definite if and only if W is finite. In this case, W is a geometric reflection group (cf. Definition 1.6) generated by reflections in codimension-one faces of a simplex in $\mathbb{S}^n$ with dihedral angles $\frac{\pi}{m_{ij}}$.*
2. *If B is positive semi-definite, then W is a geometric reflection group on $\mathbb{E}^{n-1}$ generated by reflections in codimension-one faces of either an interval if $n = 2$ (D_∞), or a simplex if $n \geq 3$, with dihedral angles $\frac{\pi}{m_{ij}}$.*

Proof. To 2.: We can find a codimension-one affine Euclidean subspace E in V^* on which W acts by isometric reflections. If $n \geq 3$, H_i^* and H_j^* meet at an angle of $\frac{\pi}{m_{ij}}$ in E. The subspace E is a "slice" across the Tits core. □

Remark 2. The positive definite B, and the positive semi-definite B but not definite B, can be classified using graphs. This gives a classification of irreducible finite Coxeter groups W, and of irreducible affine Coxeter groups. This may be found in any book on Coxeter groups and was first done by Coxeter himself.

3.3 Geometry for W finite

Let W be finite, $C = \{\varphi \in V^* \mid \varphi(e_i) \geq 0\} \subseteq V^* \cong \mathbb{E}^n$. Now take $x \in \mathring{C}$ and act on x by W. The orbit then has $|W|$ points. By regarding its convex hull we get a convex Euclidean polytope (in general not regular), which is stabilised by W. In fact, its one-skeleton is isomorphic as a non-metric graph to $\mathrm{Cay}(W, S)$. This polytope is another geometric realisation for W. See for example Figure 3.5.

Later on, we will past together these polytopes to get a piecewise Euclidean geometric realisation for arbitrary W. This polytope is (depending on who you are talking to) called *Coxeter polytope*, *W-permutahedron*, *W-associahedron*, *weight polytope*.

3.4 Motivation for other geometric realisation

Let $W_n = \langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_{i+1})^2 = 1 \text{ for } i \in \mathbb{Z}/n\mathbb{Z}\rangle$. The Tits representation gives $W_n \hookrightarrow GL(n, \mathbb{R})$. But for $n \geq 5$, W_n is a two-dimensional hyperbolic reflection group, i.e. W_n is generated by reflections in the sides of a right-angled hyperbolic n-gon (see Figure 1.9). The finite special subgroups of W_n are all $C_2 \times C_2$ and the Coxeter polytope for this is a square.

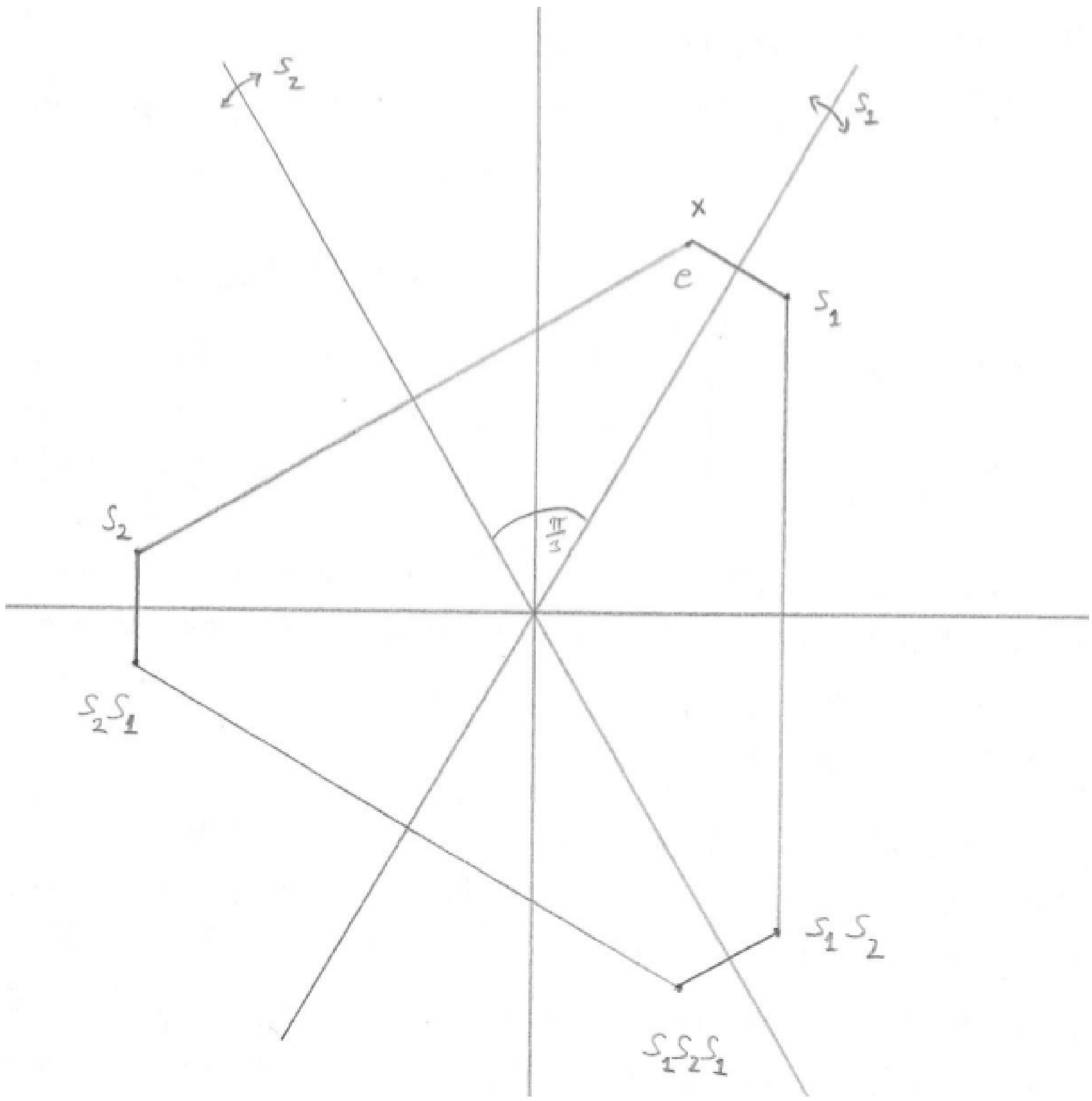

Figure 3.5: A Coxeter polytope for the group $W = D_6$.

4 Lecture Four: The basic construction of a geometric realisation

23.03.2016

The term "geometric realisation" is not a formally defined mathematical term. It gets used in various situations where W acts on some space X such that the elements of S are in some sense reflections. The action might not be by isometries.

Today we want to give a "universal" construction of geometric realisations for a Coxeter group.

4.1 Simplicial complexes

Definition 4.1. An *abstract simplicial complex* consists of a set V, possibly infinite, called the *vertex set* and a collection X of finite subsets of V such that

1. $\{v\} \in X$ for all $v \in V$,
2. If $\Delta \in X$ and $\Delta' \subseteq \Delta$ then $\Delta' \in X$.

An element of X is called a *simplex*. If Δ is a simplex and $\Delta' \subsetneq \Delta$ then Δ' is a *face* (sometimes "facet" for codimension one). The *dimension* of a simplex Δ is $\dim \Delta = |\Delta| - 1$. A *k-simplex* is a simplex of dimension k, a 0-simplex is a *vertex*, a 1-simplex is an *edge*. The *k-skeleton* $X^{(k)}$ consists of all simplices of dimension k. This is also a simplicial complex.

The dimension of X is $\dim X = \max\{\dim(\Delta) \mid \Delta \in X\}$ if this exists. A simplicial complex is called *pure* if all its maximal simplices have the same definition. We do not assume that X is pure. However we do assume that $\dim(X)$ is finite.

We will frequently identify an abstract simplicial complex X with the following simplicial cell complex X and refer to both as simplicial complexes. The *standard n-simplex* Δ^n is the convex hull of the $(n+1)$ points $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ in $\mathbb{R}^{n+1}$; see Figure 4.1.

For each n-simplex Δ in X, we identify Δ with Δ^n. This gives the n-cells in X. The attaching maps are obtained by gluing faces accordingly.

Conversely, define $V = V(X) = X^{(0)}$. Then $\Delta \subseteq V$ is in $X \iff \Delta$ spans a copy of Δ^n.

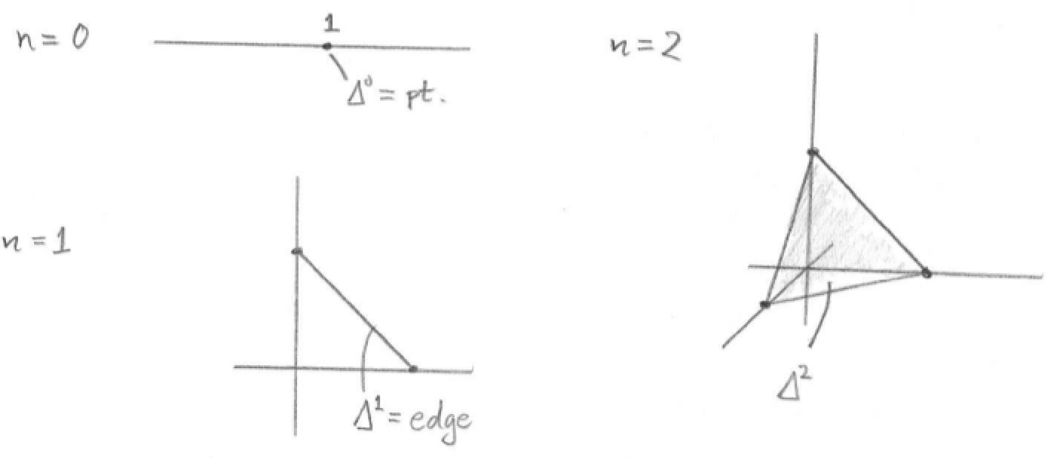

Figure 4.1: Simplices of different dimensions.

4.2 The "Basic construction"

Definition 4.2. Let (W, S) be any Coxeter system and let X be a connected, Hausdorff topological space. A *mirror structure* on X over S is a collection $(X_s)_{s\in X}$ where each X_s is a non-empty closed subspace of X. We call X_s the *s-mirror*.

Idea: The basic construction $\mathcal{U}(W, X)$ is a geometric realisation for W obtained by gluing together W-many copies of X along mirrors.

Example 4.3 (Examples of mirror structures). 1. Let X be the cone on $|S|$ vertices $\{\sigma_s \mid s \in S\}$. Put $X_s = \sigma_s$; see Figure 4.2.

2. Let X be the n-simplex where $|S| = n+1$, with codimension-one faces $\{\Delta_s \mid s \in S\}$. Put $X_s = \Delta_s$. E.g. $S = \{s, t, u\}$; see Figure 4.3.

   Note that we can view X as a cone on $X^{(n-1)}$, which is complete.

3. Let P^n be a simple convex polytope in $\mathbb{X}^n \in \{\mathbb{S}^n, \mathbb{E}^n, \mathbb{H}^n\}$, $n \geq 2$ with codimension-one faces $\{F_i\}_{i\in I}$ such that if $i \neq j$ and $F_i \cap F_j \neq \emptyset$ then the dihedral angle between them is $\frac{\pi}{m_{ij}}$ where $m_{ij} \geq 2$ is an integer. Put $m_{ii} = 1$ and $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$. Let (W, S) be the Coxeter system with Coxeter matrix (m_{ij}). Put $X = P^n$ and $X_{s_i} = F_i$. In the next lecture we will prove that $\mathcal{U}(W, P^n)$ is isometric to $\mathbb{X}^n$. This will then imply Theorem 1.5.

4. Let $C \subseteq V^*$ be the chamber associated to the Tits representation. Put $X = C$, $X_{s_i} = C \cap H_i^*$.

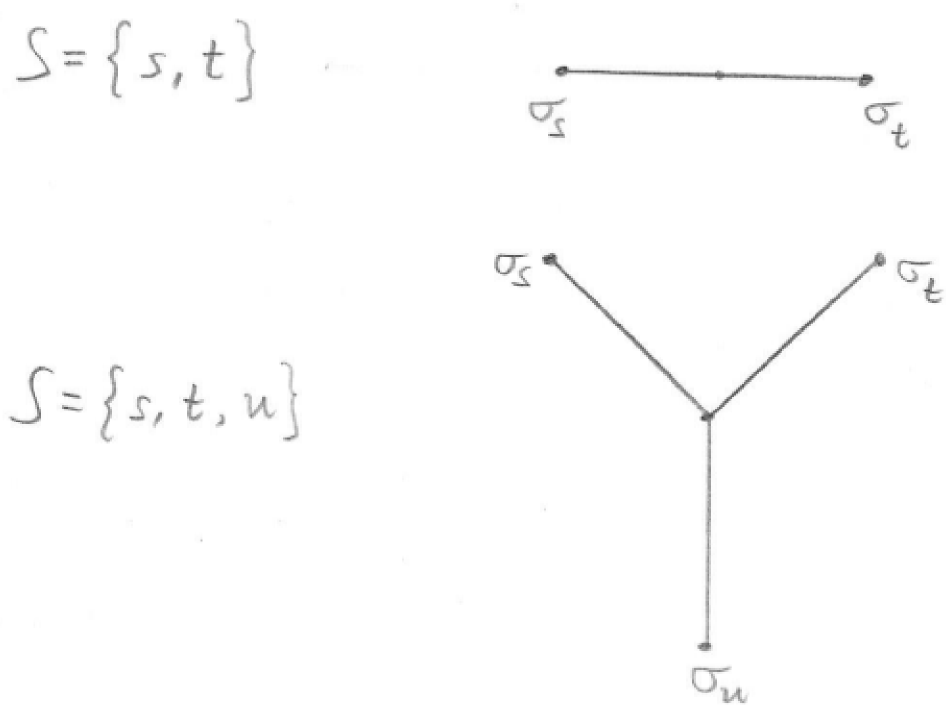

Figure 4.2: X is the cone on $|S|$ vertices $\{\sigma_s \,|\, s \in S\}$ and $X_s = \sigma_s$.

5. If W is finite, the Tits representation gives $\rho : W \to O(n, \mathbb{R})$ with $n = |S|$. Let $C = \{v \in \mathbb{R}^n \,|\, \langle v, e_i \rangle \geq 0 \quad \forall i\}$. Let $x \in \mathring{C}$ and take the convex hull of $W.x$, i.e. consider the associated Coxeter polytope. Put $X = C \cap$ Coxeter polytope, $X_{s_i} = X \cap H_i$; see Figure 4.4.
6. Recall: If the bilinear form B for the Tits representation is positive semi-definite and not definite, we get a tiling of $\mathbb{E}^{n-1}$ by intersecting the Tits cone with an affine subspace E. Put $X = C \cap E$, $X_{s_i} = X \cap H_i^*$.

Construction of $\mathcal{U}(W, X)$: For each $x \in X$, define $S(x) \subseteq S$ by

$$S(x) := \{s \in S \,|\, x \in X_s\}$$

.

Example 4.4. 1. In the first example of Example 4.3 above

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s \,|\, s \in S\} \\ \{s\}, & \text{if } x = \sigma_s \end{cases}$$

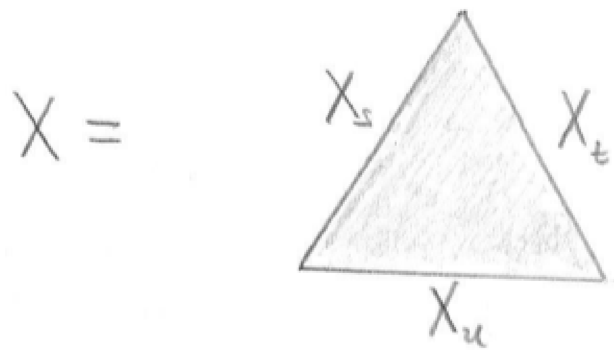

Figure 4.3: X is a 2-simplex, with codimension-one faces $\{\Delta_s \mid s \in S\}$ where $S = \{s, t, u\}$.

2. In the second example of Example 4.3 above

$$\{S(x) \mid x \in X\} = \{T \subsetneq S\}.$$

Recall: If $T \subseteq S$, the special subgroup W_T is $\langle T \rangle$ with $W_\emptyset = 1$.

Now let us define a relation on $W \times X$ by $(w, x) \sim (w', x') \iff x = x'$ and $w^{-1}w' \in W_{S(x)}$. Check: this is an equivalence relation. Equip W with the discrete topology and $W \times X$ with the product topology. Define

$$\mathcal{U}(W, X) = W \times X/\sim.$$

We write $[w, x]$ for the equivalence class of (w, x) and wX for the image of $\{w\} \times X$ in $\mathcal{U}(W, X)$. This is well-defined since $x \mapsto [w, x]$ is an embedding. Each wX is called a *chamber*.

Example 4.5. 1. Let $W = \langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle$, i.e. W is the $(3, 3, 3)$-triangle group. Let $X = \mathrm{Cone}\{\sigma_s, \sigma_t, \sigma_u\}$. Now

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s, \sigma_t, \sigma_u\} \\ \{s\}, \{t\}, \{u\}, & \text{as } x = \sigma_s, \sigma_t, \sigma_u \text{ resp.} \end{cases}$$

So $W_{S(x)}$ are either 1, $\{1, s\}$, $\{1, t\}$, or $\{1, u\}$. Thus if $x \notin \{\sigma_s, \sigma_t, \sigma_u\}$ then the equivalence class $[w, x] = \{(w, x)\}$. If $x = \sigma_s$ then $(w, \sigma_s) \sim (w', \sigma_s) \iff w^{-1}w' \in \{1, s\} \iff w = w'$ or $w' = ws$. So $[w, \sigma_s] = \{(w, \sigma_s), (ws, \sigma_s)\}$. Hence we glue wX and wsX along σ_s; see Figure 4.5.

The space $\mathcal{U}(W, X)$ is the Cayley graph $\mathrm{Cay}(W, S)$ up to subdivision. In general, for any Coxeter system (W, S): If $X = \mathrm{Cone}\{\sigma_s \mid s \in S\}$ and $X_s = \sigma_s$ then $\mathcal{U}(W, X) = \mathrm{Cay}(W, S)$ (up to subdivision).

2. Let W be the same as in 1. Let X be a two-simplex and $X_s = \Delta_s$ its codimension-one faces. Then $\mathcal{U}(W,X)$ is a tesselation of $\mathbb{E}^2$. If $x \in \Delta_s \cap \Delta_t$, then $W_{S(x)} = \langle s,t\rangle \cong D_6$; see Figure 4.6.

   For any Coxeter system (W,S): If X = simplex with codimension-one faces $\{\Delta_s \mid s \in S\}$, $X_s = \Delta_s$, then the simplicial complex $\mathcal{U}(W,X)$ is called the *Coxeter complex*. If (W,S) is irreducible affine, the Coxeter complex is the tessellation $E\cap$ Tits cone.

4.3 Properties of $\mathcal{U}(W,X)$

Lemma 4.6. $\mathcal{U}(W,X)$ is connected as a topological space.

Proof. Since $\mathcal{U}(W,X) = W \times X/\sim$ has the quotient topology, $A \subseteq \mathcal{U}(W,X)$ is open (resp. closed) if and only if $A \cap wX$ is open (resp. closed) for all chambers wX. Suppose $A \subseteq \mathcal{U}(W,X)$ is both open and closed. Assume $A \neq \emptyset$. Since X is connected, for any $w \in W$, $A\cap wX$ is either $\emptyset$ or wX. So A is a non-empty union of chambers $A = \bigcup_{v\in V} vX$ where $\emptyset \neq V \subseteq W$. Let $v \in V$ and $s \in S$. Since $X_s \neq \emptyset$, if $x \in X_s$ then any open neighbourhood of $[v,x] \in vX$ must contain $[vs,x] \in vsX$. So $V_S \subseteq V$. But S generates W, so $V = W$ and $A = \mathcal{U}(W,X)$. □

Definition 4.7. We say $\mathcal{U}(W,X)$ is *locally finite* if for every $[w,x] \in \mathcal{U}(W,X)$ there is an open neighbourhood of $[w,x]$ which meets only finitely many chambers.

Lemma 4.8. The following are equivalent:

- $\mathcal{U}(W,X)$ is locally finite;
- $\forall x \in X : W_{S(x)}$ is finite;
- $\forall T \subseteq S$ such that W_T is infinite we have $\bigcap_{x\in T} X_t = \emptyset$.

Example 4.9. Let $W = \langle s,t,u \mid s^2 = t^2 = u^2 = 1, (st)^3 = 1\rangle$. Then the Coxeter complex is not locally finite; see Figure 4.7.

Next time we will construct for a general Coxeter system (W,S) a chamber $X = K$ with mirror structure $(K_s)_{s\in S}$ such that $\mathcal{U}(W,K)$ is locally finite and contractible.

4.4 Action of W on $\mathcal{U}(W,X)$

The group W acts on $W \times X$ by $w' \cdot (w,x) = (w'w, x)$. Check: This action preserves the equivalence relation $\sim$, such that W acts on $\mathcal{U}(W,X) = W \times X/\sim$. This also induces an action on the set of chambers: $w \cdot w'X = (ww')X$. This action is transitive on the set of chambers, and is free on the set of chambers provided there is some point $x \in X$ which is not contained in any mirror. In this situation, the map $w \mapsto wX$ is a bijection from W to the set of chambers.

Stabilisers: The point $[w,x] \in \mathcal{U}(W,X)$ has stabiliser

$$\begin{aligned}\{w' \in W \mid w' \cdot (w,x) \sim (w,x)\} &= \{w' \in W \mid (w'w,x) \sim (w,x)\} \\ &= \{w' \in W \mid (w'w)^{-1}w \in W_{S(x)}\} \\ &= \{w' \in W \mid w^{-1}w'w \in W_{S(x)}\} = wW_{S(x)}w^{-1},\end{aligned}$$

i.e. the stabiliser of $[w,x]$ is a conjugate of $W_{S(x)}$.

Definition 4.10. The action by homeomorphisms of a discrete group G on a Hausdorff space Y (not necessarily locally compact) is called *properly discontinuous* if

1. Y/G is Hausdorff;
2. $\forall y \in Y : G_y = \mathrm{stab}_G(Y)$ is finite;
3. $\forall y \in Y$ there is an open neighbourhood U_y of y which is stabilised by G_y and $gU_y \cap U_y = \emptyset$ for all $g \in G \backslash G_y$.

Lemma 4.11. The W-action on $\mathcal{U}(W,X)$ is properly discontinuous if and only if $W_{S(x)}$ is finite for every $x \in X$.

Proof. Let us first assume that the W-action on $\mathcal{U}(W,X)$ is properly discontinuous. As we have seen before the stabiliser of a point $[w,x] \in \mathcal{U}(W,X)$ is $wW_{S(x)}w^{-1}$. Thus $W_{S(x)}$ is finite by 2.

Let us now assume that $W_{S(x)}$ is finite for every $x \in X$. All that needs to be seen is 3. Without loss of generality we consider $y = [1,x] \in \mathcal{U}(W,X)$. Let $V_x = X - \bigcup\{\text{mirrors which do not contain } x\}$. The sought for neighbourhood of y is then given by $U_y = W_{S(x)}V_x$. □

4.5 Universal property of $\mathcal{U}(W,X)$

$\mathcal{U}(W,X)$ satisfies the following universal property.

Theorem 4.12 (Vinberg)**.** *Let (W,S) be any Coxeter system. Suppose W acts by homeomorphisms on a connected Hausdorff space Y such that, for every $s \in S$, the fixed point set Y^s of s is non-empty. Suppose further that X is a connected Hausdorff space with a mirror structure $(X_s)_{s\in S}$. Then if $f : X \to Y$ is a continuous map such that $f(X_s) \subseteq Y^s$ for all $s \in S$, there is a unique extension of f to a W-equivariant map $\tilde{f} : \mathcal{U}(W,X) \to Y$ given by $\tilde{f}([w,x]) = w \cdot f(x)$.*

Next time we will apply the above theorem to Theorem 1.5.

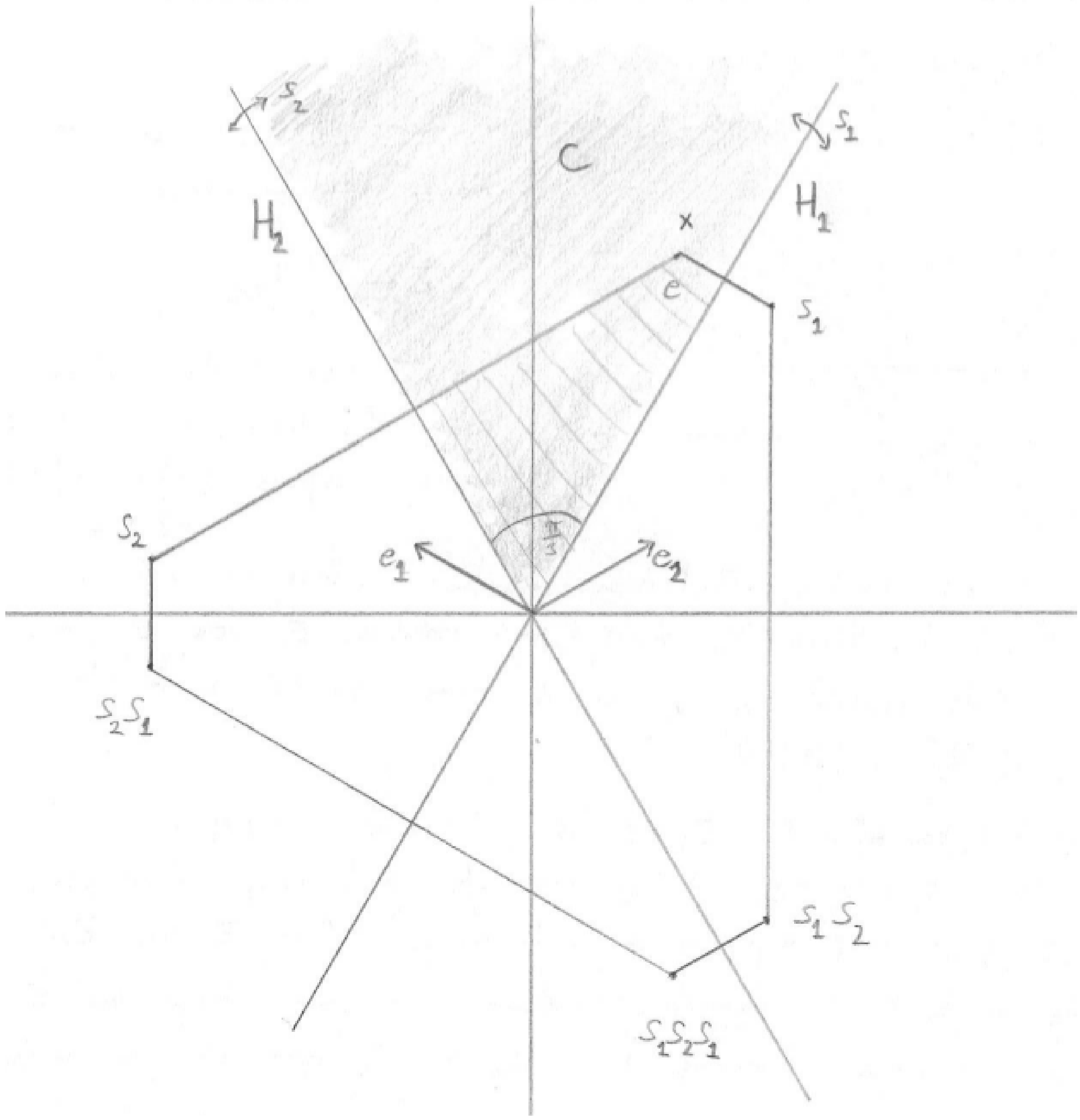

Figure 4.4: $W = D_6$, $X = C \cap$ Coxeter polytope, $X_{s_i} = X \cap H_i$.

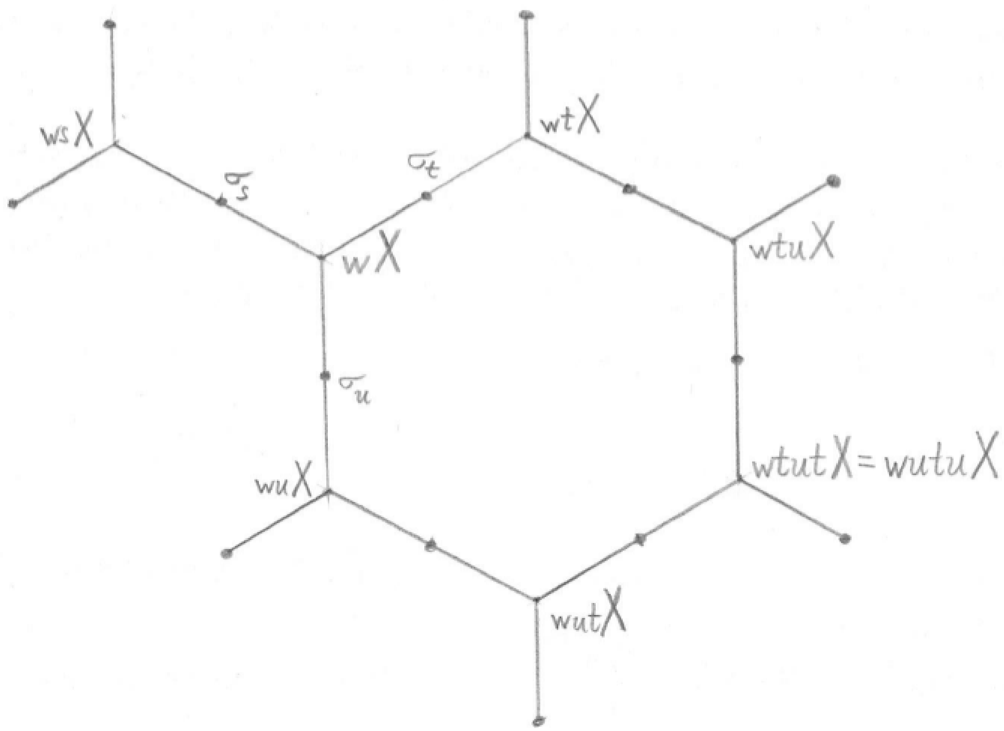

Figure 4.5: $\mathcal{U}(W, X)$ depicted for $W = D_6$ and $X = \text{Cone}\{\sigma_s, \sigma_t, \sigma_u\}$.

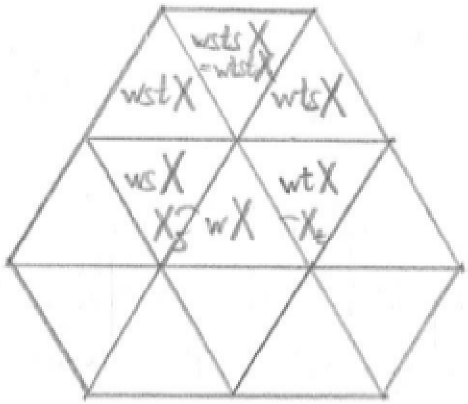

Figure 4.6: $\mathcal{U}(W, X)$ depicted for $W = D_6$ and $X =$ two-simplex, $X_s = \Delta_s$ its codimension-one faces.

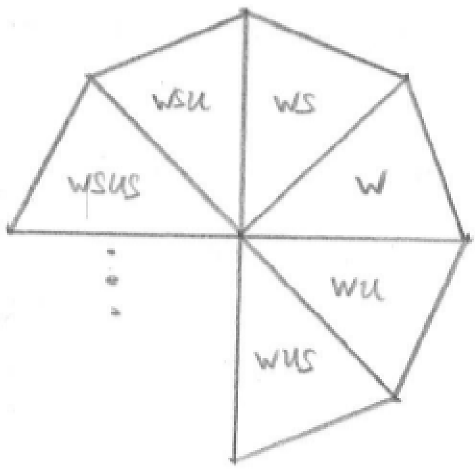

Figure 4.7: For $W = \langle s, t, u \,|\, s^2 = t^2 = u^2 = 1, (st)^3 = 1\rangle$ the Coxeter complex is not locally finite.

5 Lecture Five: Geometric Reflection Groups and the Davis complex

06.04.2016

5.1 Geometric Reflection Groups

Theorem 5.1 (this includes Theorem 1.5)**.** *Let $X = P^n$ be a simple convex polytope in $\mathbb{X}^n$ ($n \geq 2$), with codimension-one faces $\{F_i\}_{i\in I}$, such that if $i \neq j$ and $F_i \cap F_j \neq \emptyset$, then the dihedral angle between them is $\frac{\pi}{m_{ij}}$ where $m_{ij} \in \{2, 3, 4, \ldots\}$ is finite. Put $m_{ii} = 1$ and $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$.*

Let (W, S) be the abstract Coxeter system with Coxeter matrix $(m_{ij})_{i,j\in I}$.

Define a mirror structure on X by $X_{s_i} = F_i$. For each $i \in I$, let $\bar{s}_i \in Isom(\mathbb{X}^n)$ be the reflection in F_i. Let $\overline{W}$ be the subgroup of $Isom(\mathbb{X}^n)$ generated by the $\bar{s}_i$.

Then:

1. *there is an isomorphism $\varphi : W \to \overline{W}$ induced by $s_i \mapsto \bar{s}_i$;*
2. *the induced map $\mathcal{U}(W, P^n) \to \mathbb{X}^n$ is a homeomorphism;*
3. *the Coxeter group W acts properly discontinuously on $\mathbb{X}^n$ with strict fundamental domain P^n, so W is a discrete subgroup of $Isom(\mathbb{X}^n)$ and $\mathbb{X}^n$ is tiled by copies of P^n.*

Proof.

To 1: First we show $s_i \mapsto \bar{s}_i$ induces a homomorphism $W \to \overline{W}$.
Each s_i has order 2 in W and each $\bar{s}_i$ has order 2 in $\overline{W}$.
Also:

$$\begin{aligned} m_{ij} \text{ is finite } &\iff F_i \cap F_j \neq \emptyset \text{ and meet at dihedral angle } \frac{\pi}{m_{ij}} \\ &\iff \bar{s}_i\bar{s}_j \text{ has order } m_{ij}. \end{aligned}$$

Hence we have a homomorphism $\varphi : W \to \overline{W}$.

To 2: Since $\overline{W}$ acts by isometries on $\mathbb{X}^n$, also W acts by isometries on $\mathbb{X}^n$.

In the W-action, each s_i fixes (at least) the faces F_i. So by the universal property, the inclusion $f : P \to \mathbb{X}^n$ induces the (unique) W-equivariant map

$$\tilde{f} : \mathcal{U}(W, P) \to \mathbb{X}^n.$$

The injectivity of φ and 3 follow from the next claim.

Claim: $\tilde{f}$ is a homeomorphism.

Proof. We will prove the claim via a quite complicated induction scheme on the dimension n. Let us introduce some notation first.

Notation:

- (s_n) is the claim when $\mathbb{X}^n = \mathbb{S}^n$ and $P^n = \sigma^n$ is a spherical simplex with dihedral angles $\frac{\pi}{m_{ij}}$ $(n \geq 2)$.
- (c_n) is the claim when $\mathbb{X}^n$ is replaced by $B_x(r)$, the open ball of radius r about a point $x \in \mathbb{X}^n$, and P^n is replaced by $C_x(r)$, the open simplicial cone of radius r about x with dihedral angles $\frac{\pi}{m_{ij}}$.
- (t_n) is the claim in dimension n.

We will prove (c_2) and show that $\forall n \geq 2$, $(c_n) \implies (t_n)$ and $(s_n) \implies (c_{n+1})$. Then as (s_n) is a special case of (t_n), we get

$$(c_2) \implies (t_2) \implies (s_2) \implies (c_3) \implies (t_3) \implies \ldots \implies (t_n) \implies \ldots$$

Proof of (c_2)*:* In $\mathbb{X}^2$ let

$$W = \langle s_1, s_2 \mid s_i^2 = 1, (s_1 s_2)^{m_{12}} = 1 \rangle = D_{2m_{12}}.$$

The basic construction $\mathcal{U}(W, C_x(r))$ is $|W| = 2m_{12}$ copies of $C_x(r)$ glued along mirrors. This is homeomorphic to $B_x(r)$; see Figure 5.1.

Proof that $(s_n) \implies (c_{n+1})$*:* Let $S_x(r)$ be the sphere of radius r about x in $\mathbb{X}^{n+1}$. Regard $\mathbb{S}^n$ (unit-sphere) as living in $T_x\mathbb{X}^{n+1}$. Then the exponential map $\exp : T_x\mathbb{X}^{n+1} \to \mathbb{X}^{n+1}$ induces a homeomorphism from $\mathbb{S}^n \to S_x(1)$.

Let $\sigma^n \subset \mathbb{S}^n$ be the spherical simplex, such that $\exp(\sigma^n) = S_x(1) \cap \overline{C_x(1)}$. Then σ^n has dihedral angles $\frac{\pi}{m_{ij}}$, so the Coxeter group W associated to σ^n is the same as the one associated to the simplicial cone $C_x(1)$; see Figure 5.2.

Since (s_n) holds,

$$\begin{aligned}
&\mathcal{U}(W, \sigma^n) \to \mathbb{S}^n \text{ is a homeomorphism} \\
&\implies \mathcal{U}\left(W, S_x(1) \cap \overline{C_x(1)}\right) \to S_x(1) \text{ is a homeomorphism} \\
&\implies \mathcal{U}(W, \overline{C_x(1)}) \to \overline{B_x(1)} \text{ is a homeomorphism} \\
&\implies \mathcal{U}(W, C_x(1)) \to B_x(1) \text{ is a homeomorphism} \\
&\implies \mathcal{U}(W, C_x(r)) \to B_x(r) \text{ is a homeomorphism.}
\end{aligned}$$

This proves (c_{n+1}).

Proof that $(c_n) \implies (t_n)$*:*

Definition 5.2. A n-dimensional topological manifold M^n has an $\mathbb{X}^n$*-structure*, if it has an atlas of charts $\{\psi_\alpha : U_\alpha \to \mathbb{X}^n\}_{\alpha \in A}$ where $(U_\alpha)_{\alpha \in A}$ is an open cover of M^n, each ψ_α is a homeomorphism onto its image, and for all $\alpha, \beta \in A$ the map

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$$

is the restriction of an element of $\mathrm{Isom}(\mathbb{X}^n)$; see Figure 5.3. In particular an $\mathbb{X}^n$ structure turns M^n into a (smooth) Riemannian manifold.

Facts:

- An $\mathbb{X}^n$-structure on M^n induces one on its universal cover $\widetilde{M}^n$.
- There is a *developing map* $D : \widetilde{M}^n \to \mathbb{X}^n$ given by analytic continuation along paths.
- If M^n is metrically complete, D is a covering map.

Let $x \in P^n \subset \mathbb{X}^n$. Let $r = r_x > 0$ be the distance from x to the nearest F_i which does not contain x. Let $C_x(r) = B_x(r) \cap P^n$ be the open simplicial cone in $\mathbb{X}^n$ with vertex x.

Let $\mathcal{U}_x = \mathcal{U}(W_{S(x)}, C_x(r))$ where $S(x) = \{s_i \mid x \in F_i\}$. Then $\mathcal{U}_x$ is an open neighbourhood of $[1, x]$ in $\mathcal{U}(W, P^n)$. By (c_n), the map $\mathcal{U}_x \to B_x(r)$ is a homeomorphism. By equivariance, for all $w \in W$ the map

$$w\mathcal{U}_x \to \varphi(w)B_x(r)$$

is also a homeomorphism. Now $\varphi(w)$ is an isometry of $\mathbb{X}^n$, so $M^n = \mathcal{U}(W, P^n)$ has an $\mathbb{X}^n$-structure.

The W-action on $\mathcal{U}(W, P^n)$ is cocompact, so by a standard argument $\mathcal{U}(W, P^n)$ is metrically complete. Hence the developing map $D : \widetilde{\mathcal{U}(W, P^n)} \to \mathbb{X}^n$ is a covering map.

The map D is locally given by $\tilde{f}$, and since $\mathcal{U}(W, P^n)$ is connected and $\tilde{f}$ is globally defined, $\tilde{f}$ is also a covering map. But $\mathbb{X}^n$ is simply connected so $\tilde{f} = D$ is a homeomorphism.

□

This finishes the proof of the theorem.

□

5.2 The Davis complex – a first definition

Recall: if X has mirror structure $(X_s)_{s \in S}$, then $\mathcal{U}(W, X)$ is ...

- ... connected;
- ... locally finite $\iff$ $W_{S(x)}$ is finite $\forall x \in X$;

- ...the point stabilisers are given by:

$$\mathrm{stab}_W([w,x]) = wW_{S(x)}w^{-1};$$

- ...the W-action is properly discontinuous $\iff$ $W_{S(x)}$ is finite $\forall x \in X$.

The *Davis complex* $\Sigma = \Sigma(W,S)$ is $\mathcal{U}(W,K)$ where the chamber K has mirror structure $(K_s)_{s\in S}$ such that $\forall x \in K$, $W_{S(x)}$ is finite.

In order to define K: A subset $T \subseteq S$ is *spherical* if W_T is finite; we say W_T is a *spherical special subgroup*.

Consider

$$\{T \subseteq S \mid T \neq \emptyset, T \text{ is spherical}\}.$$

This collection is an abstract simplicial complex: if $\emptyset \neq T' \subseteq T$, and W_T is finite, then $W_{T'}$ is finite. Also $\{s\}$ is spherical for all $s \in S$.

This simplicial complex is called the *nerve* of (W,S), denoted by $L = L(W,S)$. Concretely: L has vertex set S, and a simplex σ_T spanning each $T \subseteq S$ such that $T \neq \emptyset$ and W_T is finite.

Example 5.3. 1. If W is finite, the nerve L is the full simplex on S.

2. If $W \cong D_\infty = \langle s,t \mid s^2 = t^2 = 1\rangle$, the nerve L consists exactly of the two vertices s and t.

3. If W is the $(3,3,3)$-triangle group,

$$W = \langle s,t,u \mid s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1\rangle,$$

then the nerve L is a triangle with vertices s,t,u but not filled in as W is not finite.

4. If W is a geometric reflection group with fundamental domain P then L can be identified with the boundary of P^*, the dual polytope of P. (This needs proof!)

5. If (W,S) is a reducible Coxeter system with

$$(W,S) = (W_1 \times W_2, S_1 \sqcup S_2),$$

then $T \subseteq S$ is spherical $\iff$ $T = T_1 \sqcup T_2$, with $T_i = T \cap S_i$, and both T_1 and T_2 are finite. Then $L(W,S)$ is the join of $L(W_1,S_1)$ and $L(W_2,S_2)$. See for example Figure 5.4.

6. (Right-angled Coxeter groups) Let Γ be a finite simplicial graph with vertex set $S = V(\Gamma)$ and edge set $E(\Gamma)$. The associated Coxeter group is

$$\begin{aligned} W_\Gamma &= \langle S \mid s^2 = 1 \,\forall s \in S, \quad st = ts \iff \{s,t\} \in E(\Gamma)\rangle \\ &= \langle S \mid s^2 = 1 \,\forall s \in S, \quad (st)^2 = 1 \iff \{s,t\} \in E(\Gamma)\rangle \end{aligned}$$

Then $\langle s,t\rangle$ is finite if and only if s and t are adjacent in Γ. Hence the nerve $L(W_\Gamma, S)$ has 1-skeleton equal to Γ.

Definition 5.4. A simplicial complex L is called a *flag complex* if each finite, non-empty set of vertices T spans a simplex in L if and only if any two elements of T span an edge/1-simplex in L.

A flag simplicial complex is completely determined by its 1-skeleton.

Lemma 5.5. If (W, S) is a right-angled Coxeter system, then $L(W, S)$ is a flag complex.

Proof. Suppose $T \subseteq S$, $T \neq \emptyset$ and any two vertices in T are connected by an edge in L. Then $W_T \cong (C_2)^{|T|}$ is finite, so T is spherical and σ_T is in L. □

Now we can define K and its mirror structure $(K_s)_{s\in S}$. Let $L = L(W, S)$ be the nerve of the Coxeter system (W, S) and let L' be its barycentric subdivision.

We define

$$K = \mathrm{Cone}(L').$$

For each $s \in S$, define K_s to be the closed star in L' of the vertex s. (The closed star of s is the union of the closed simplices in L' which contain s.)

Then $(K_s)_{s\in S}$ is a mirror structure on K. We have:

$$\begin{aligned}&\text{Two mirrors } K_s \text{ and } K_t \text{ intersect}\\ &\iff \text{there is an edge of } L \text{ between } s \text{ and } t\\ &\iff \langle s, t\rangle \text{ is finite.}\end{aligned}$$

Similarly,

$$\begin{aligned}\bigcap_{t\in T} K_t \neq \emptyset &\iff T \subseteq S \text{ is a non-empty spherical subset}\\ &\iff W_T \text{ is finite and non-trivial.}\end{aligned}$$

Hence $\forall x \in X$, $S(x) = \{s \in S \mid x \in K_s\}$ is spherical.

Example 5.6.
1. If W is finite and the nerve L is a simplex Δ on $|S|$ vertices then $K = \mathrm{Cone}(L')$ is a simplex of dimension one higher. So Σ will be the cone on a tesselation of the sphere induced by the W-action.
2. If W is a geometric reflection group with fundamental domain P, then $L = \partial P^*$ and so $L' = (\partial P^*)' = \partial P'$. Thus K is the cone on the barycentric subdivision of ∂P, so K is the barycentric subdivision of P and Σ is the barycentric subdivision of a tesselation of $\mathbb{X}^n$.

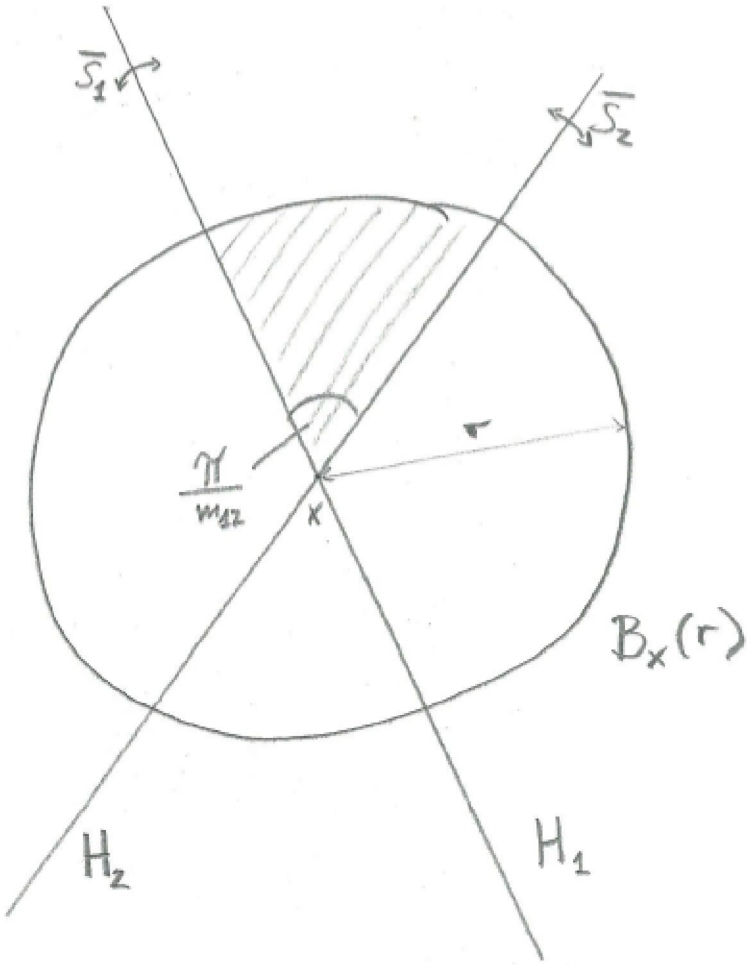

Figure 5.1: The basic construction $\mathcal{U}(W, C_x(r))$ for $W = \langle s_1, s_2 \mid s_i^2 = 1, (s_1 s_2)^{m_{12}} = 1\rangle = D_{2m_{12}}$ in dimension $n = 2$.

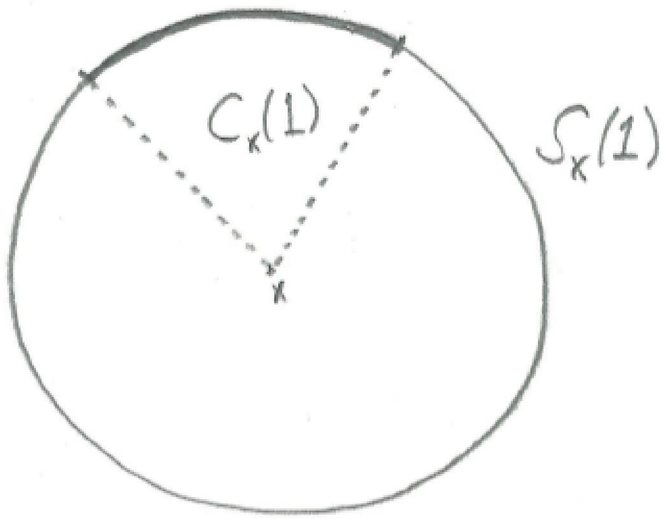

Figure 5.2: The simplicial cone $C_x(1)$.

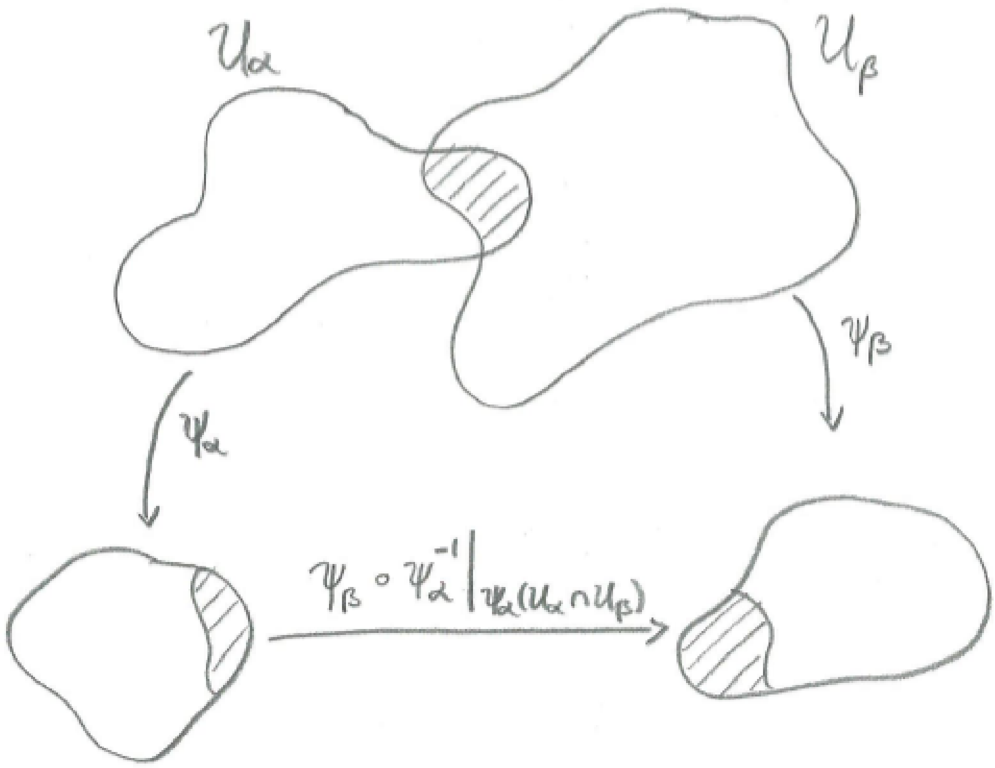

Figure 5.3: The transition map $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$.

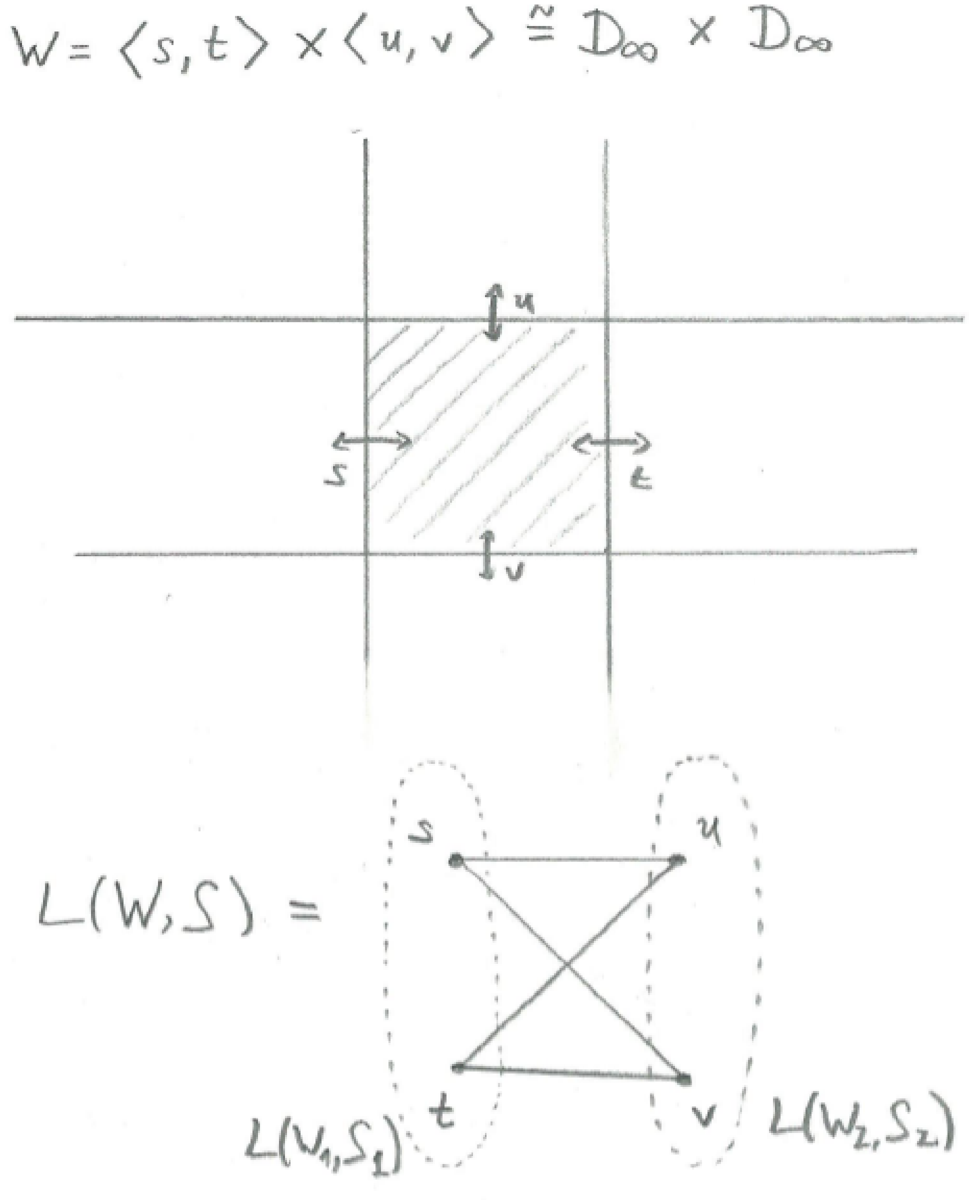

Figure 5.4: The Coxeter system (W, S) with $W = \langle s, t\rangle \times \langle u, v\rangle \cong D_\infty \times D_\infty$ and $S = \{s, t, u, v\}$ is reducible. Its nerve $L = L(W, S)$ is the join of $L(W_1, S_1)$ and $L(W_2, S_2)$ where $W_1 = \langle s, t\rangle$, $S_1 = \{s, t\}$, $W_2 = \langle u, v\rangle$, $S_2 = \{u, v\}$.

6 Lecture Six: Topology of the Davis complex

13.04.2016

For the rest of this lecture let (W, S) be a Coxeter system with corresponding Davis complex $\Sigma = \Sigma(W, S)$ given by $\Sigma = \mathcal{U}(W, K)$. Recall that:

- L is the nerve of (W, S), i.e. the simplicial complex with vertex set S, and a simplex σ_T spanned by $\emptyset \neq T \subseteq S$ is contained in L if and only if W_T is finite;
- L' denotes the barycentric subdivision of L;
- $K = \mathrm{Cone}(L')$ is called the chamber, and has mirrors $\{K_s\}_{s \in S}$ where K_s is the star of the vertex s in L';
- Given $x \in K$, define $S(x) = \{s \in S \,|\, x \in K_s\}$. Then $\Sigma = \mathcal{U}(W, K) = (W \times K)/\sim$ where
$$(w, x) \sim (w', x') \iff x = x' \text{ and } w^{-1}w' \in W_{S(x)}.$$
For example, chambers wK and wsK are glued together along the mirror K_s.

Example 6.1. Figures 6.1 to 6.4 illustrate the Davis complexes for certain Coxeter systems (W, S).

6.1 Contractibility of the Davis complex

In the last lecture we have seen that:

- Σ is connected and locally finite;
- the W-action $W \curvearrowright \Sigma$ is properly discontinuous and cocompact, and
$$\mathrm{stab}_W([w, x]) = wW_{S(x)}w^{-1}$$
is a finite group for every $w \in W$, $x \in K$.

Today we will prove that Σ is contractible.

Theorem 6.2 (Davis). *Σ is contractible.*

6.1.1 Some combinatorial preliminaries

For $w \in W$ define

$$\mathrm{In}(w) = \{s \in S \mid \ell(ws) < \ell(w)\},$$
$$\mathrm{Out}(w) = \{s \in S \mid \ell(ws) > \ell(w)\}.$$

Since $\ell(ws) = \ell(w) \pm 1$, we have $S = \mathrm{In}(w) \sqcup \mathrm{Out}(w)$.

Recall that we get by the Exchange Condition: if $\ell(ws) < \ell(w)$ and $(s_{i_1}, \ldots, s_{i_k})$ is a reduced expression for w, then $ws = s_{i_1} \ldots \hat{s}_{i_j} \ldots s_{i_k} \hat{s}$ for some j. Hence $w = s_{i_1} \ldots \hat{s}_{i_j} \ldots s_{i_k} s$, so there is a reduced expression for w which ends in s. So

$$\mathrm{In}(w) = \{s \in S \mid \text{a reduced expression for } w \text{ can end in } s\}.$$

Example 6.3. If $W = \langle s, t, u \mid s^2 = t^2 = u^2, (st)^2 = 1\rangle$ (the right-angled Coxeter group) then

$$\mathrm{In}(ust) = \{t, s\}, \qquad \mathrm{Out}(ust) = \{u\},$$
$$\mathrm{In}(us) = \{s\}, \qquad \mathrm{Out}(us) = \{u, t\}.$$

Proposition 6.4. For all $w \in W$, $\mathrm{In}(w)$ is a spherical subset, i.e. $W_{\mathrm{In}(w)}$ is finite.

Proof. A sufficient condition for a Coxeter group W to be finite is the following:

Lemma 6.5. If there is a $w_0 \in W$ such that $\ell(w_0 s) < \ell(w_0)$ for all $s \in S$, then W is finite.

Proof. Use the Exchange Condition to show by induction that for every reduced expression $(s_{i_1}, \ldots, s_{i_k})$ there is a reduced expression for w_0 which ends in $(s_{i_1}, \ldots, s_{i_k})$.

Then for any $w \in W$, we get

$$\ell(w_0) = \ell(w_0 w^{-1}) + \ell(w)$$

by ending a reduced expression for w_0 with a reduced expression for w. So $\ell(w_0) \geq \ell(w)$ for every $w \in W$.

Hence W is finite. □

We will also need:

Lemma 6.6. Let $T \subseteq S$ be a subset and suppose w is a minimal length element in the left coset wW_T. Then any $w' \in wW_T$ can be written as $w' = wa'$ where $a' \in W_T$ and $\ell(w') = \ell(w) + \ell(a')$.

Also wW_T has a unique minimal length element.

Proof. Existence of length additive factorisation: Deletion Condition.

Uniqueness of minimal length coset representative: Suppose w_1, w_2 are both minimal length elements in $w_1 W_T = w_2 W_T$. Then $w_1 = w_2 a'$ with $a' \in W_T$ and

$$\ell(w_1) = \ell(w_2) + \ell(a').$$

On the other hand $\ell(w_1) = \ell(w_2)$, so $a' = 1$ thus $w_1 = w_2$. □

To prove the proposition: Let $T = \mathrm{In}(w)$ and let u be a minimal length element in wW_T. By Lemma 6.6, w can be written uniquely as $w = ua'$ with $a' \in W_T$ and

$$\ell(w) = \ell(u) + \ell(a').$$

Let $s \in \mathrm{In}(w) = T$, so $\ell(ws) < \ell(w)$. Now $a's \in W_T$ so $ws = ua's$ and by Lemma 6.6 again,

$$\ell(ws) = \ell(u) + \ell(a's).$$

Hence $\ell(a's) < \ell(a')$ for every $s \in \mathrm{In}(w)$. By Lemma 6.5 with $a' = w_0$, $W_{\mathrm{In}(w)}$ is finite.

This finishes the proof of the proposition. □

6.1.2 Proof of Theorem 6.2

Enumerate the elements of W as $w_1, w_2, w_3, \ldots$ such that $\ell(w_k) \leq \ell(w_{k+1})$. For $n \geq 1$ let $U_n = \{w_1, \ldots, w_n\} \subseteq W$, so

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots \subseteq W$$

and $W = \bigcup_{n=1}^{\infty} U_n$. Further let

$$P_n = \bigcup_{w \in U_n} wK = \bigcup_{i=1}^{n} w_i K \subseteq \Sigma,$$

so $P_1 \subseteq P_2 \subseteq \ldots$ and $\Sigma = \bigcup_{n=1}^{\infty} P_n$.

Now P_n is obtained from P_{n-1} by gluing on a copy of K along some mirrors. To be precise: $P_n = P_{n-1} \cup w_n K$ where $w_n K$ is glued to P_{n-1} along the union of mirrors $\{K_s \,|\, \ell(w_n s) < \ell(w_n)\}$. That is, $w_n K$ is glued to P_{n-1} along the union of its mirrors of types $s \in \mathrm{In}(w)$.

By Proposition 6.4, $\mathrm{In}(w)$ is spherical. The theorem then follows from the next lemma.

Lemma 6.7.

1. K is contractible;
2. for all spherical $T \subseteq S$, the union of mirrors

$$K^T = \bigcup_{t \in T} K_t$$

is contractible.

Proof.

To 1: K is a cone.

To 2: We have a bijection

$$\{\text{simplices of } L\} \longleftrightarrow \{T \subseteq S \,|\, T \neq \emptyset, T \text{ is spherical}\}.$$

So

$$\{\text{vertices of } L'\} \longleftrightarrow \{T \subseteq S \,|\, T \neq \emptyset, T \text{ is spherical}\}.$$

Hence

$$\{\text{vertices of } K\} \longleftrightarrow \{T \subseteq S \,|\, T \text{ is spherical}\},$$

by identifying the cone point with $\emptyset$.

Moreover we can orient the edges in K by inclusion of types of their endpoints.

Let $\sigma'_T \subset L'$ be the barycentric subdivision of $\sigma_T \subseteq L$. Since K^T is the union of closed stars in L' of the vertices of σ_T, K^T is the first derived neighbourhood of σ'_T in L'. Since σ'_T is contractible, it is enough to construct a deformation retraction $r : K^T \to \sigma'_T$.

Define r by sending a simplex of K^T with vertex types $\{T'_0, \ldots, T'_k\}$ to the simplex with vertex types $\{T'_0 \cap T, \ldots, T'_k \cap T\}$. Check that this works. $\square$

Remark 3. As pointed out to us by Nir Lazarovich, the proof of Theorem 6.2 has a Morse-theoretic interpretation, and part 2 of Lemma 6.7 can be viewed as showing that the down-links are contractible.

6.1.3 Second definition of Σ

Let P be any poset (partially ordered set). A *chain* is a totally ordered subset of P. We can associate a simplicial complex $\Delta(P)$, called the *geometric realisation of* P via

$$\text{finite chain with } (n+1) \text{ elements} \quad \longrightarrow \quad n\text{-simplex};$$

see for example Figure 6.5.

Check: K is the geometric realisation of the poset $\{T \subseteq S \,|\, T \text{ spherical}\}$ ordered by inclusion; or equivalently, K is the geometric realisation of the poset $\{W_T \subseteq S \,|\, T \text{ spherical}\}$ ordered by inclusion.

The vertex types in K are preserved by the gluing which gives Σ. Also the W-action on Σ is type-preserving, and transitive on each type of vertex.

Note that for the action $W \curvearrowright \Sigma$ each vertex of type T has stabiliser a conjugate of W_T. Thus we can identify Σ with the geometric realisation of

$$\{wW_T \,|\, w \in W,\, W_T \text{ is spherical}\},$$

ordered by inclusion.

Cf.: The Coxeter complex is the geometric realisation of

$$\{wW_T \,|\, w \in W, T \subseteq S\}.$$

6.2 Applications to W

In the following denote

$$K^{\mathrm{Out}(w)} = \bigcup_{s \in \mathrm{Out}(w)} K_s.$$

If T is spherical, $W^T = \{w \in W \mid \mathrm{In}(w) = T\} \subseteq W$, and

$$W = \bigsqcup \{W^T \mid T \subseteq S, T \text{ spherical }\}.$$

Here $\mathbb{Z}W^T$ denotes the free abelian group with basis W^T.

Theorem 6.8 (Davis)**.**

$$\begin{aligned} H^i(W; \mathbb{Z}W) &\cong \bigoplus_{w \in W} H^i\left(K, K^{Out(w)}\right) \\ &\cong \bigoplus \left\{ \left(\mathbb{Z}W^T \otimes H^i\left(K, K^{S-T}\right)\right) \mid T \subseteq S, T \textit{ spherical } \right\} \\ &\cong \bigoplus \left\{ \left(\mathbb{Z}W^T \otimes \overline{H^{i-1}}\left(L - \sigma_T\right)\right)\right) \mid T \subseteq S, T \textit{ spherical } \right\} \end{aligned}$$

Theorem 6.8 is used for, e.g.:

- ends of W;
- determining when W is virtually free;
- virtual cohomological dimension of W;
- showing that any W is the fundamental group of a tree of groups with finite or 1-ended special subgroups as vertex groups, and finite special subgroups as edge groups.

Definition 6.9. Let G be any group. A *classifying space* for G, denoted by BG, is an aspherical CW-complex with fundamental group G (also called an Eilenberg-MacLane space or a $K(G, 1)$). Its universal cover, denoted by EG, is called the *universal space* for G.

Fact: BG is unique up to homotopy equivalence.
We can define the cohomology of G with coefficients in any $\mathbb{Z}G$-module A by

$$H^*(G; A) = H^*(BG; A),$$

where the latter is cellular cohomology.
Problem: If G has torsion, then no BG is finite dimensional!

Definition 6.10 (tom Dieck 1977)**.** Let G be a discrete group. A CW-complex X together with a proper, cocompact, cellular G-action is a *universal space for proper G-actions*, denoted by $\underline{E}G$, if for all finite subgroups F of G, the fixed set X^F is contractible.

Note:

- taking $F = \{1\}$ yields that X must be contractible;
- if $H \leq G$ is infinite, X^H is empty since the G-action is proper.

Theorem 6.11. *$\underline{E}G$ exists and is unique up to G-homotopy, and*

$$H^*(G; \mathbb{Z}G) = H_c^*(\underline{E}G),$$

where the latter is cohomology with compact support, i.e. "cohomology at infinity" of $\underline{E}G$.

We will prove next time that Σ is a (finite dimensional) $\underline{E}W$.
In order to prove Theorem 6.8 we use the following proposition:

Proposition 6.12 (Brown)**.** If a a discrete group G acts properly discontinuously and cocompactly on an acyclic CW-complex X then

$$H^*(G; \mathbb{Z}G) = H_c^*(X).$$

Proof of Theorem 6.8 (sketch). Enumerate the elements of W as $w_1, w_2, w_3, \ldots$ such that $\ell(w_k) \leq \ell(w_{k+1})$ and let

$$P_n = \bigcup_{i=1}^{n} w_i K.$$

Then $P_1 \subseteq P_2 \subseteq \ldots$ is an increasing sequence of compact subcomplexes of Σ so

$$H_c^*(\Sigma) = \varinjlim H^*(\Sigma, \Sigma - P_n).$$

If we write $\hat{P}_n = \bigcup_{i \geq n+1} w_i K$, i.e. $\hat{P} = \bigcup \{wK \mid w \notin \{w_1, \ldots, w_n\}\}$ then $\hat{P}_1 \supseteq \hat{P}_2 \supseteq \ldots$ and $H_c^*(\Sigma) = \varinjlim H^*(\Sigma, \hat{P}_n)$.

By considering the triples $(\Sigma, \hat{P}_{n-1}, \hat{P}_n)$, we get an exact sequence in cohomology

$$\cdots \longrightarrow H^*(\Sigma, \hat{P}_{n-1}) \longrightarrow H^*(\Sigma, \hat{P}_n) \longrightarrow H^*(\hat{P}_{n-1}, \hat{P}_n) \longrightarrow \cdots$$

By construction we have

$$H^*(\hat{P}_{n-1}, \hat{P}_n) \cong H^*(w_n K, w_n K^{\mathrm{Out}(w_n)}) \cong H^*\left(K, K^{\mathrm{Out}(w_n)}\right).$$

One can now show that the above sequence splits and we hence get

$$H^*(\Sigma, \hat{P}_n) \cong \bigoplus_{i=1}^{n} H^*\left(K, K^{\mathrm{Out}(w_i)}\right).$$

□

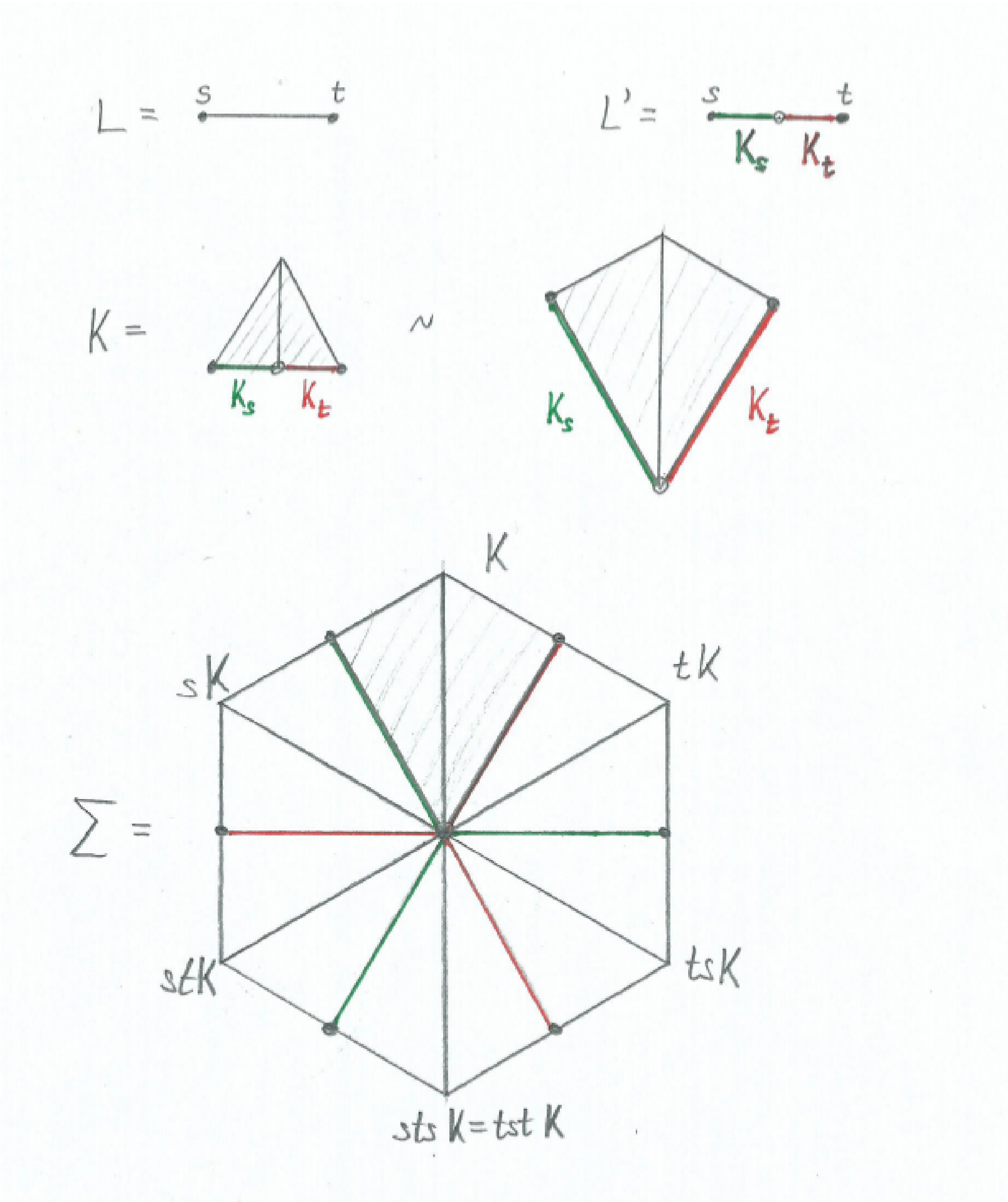

Figure 6.1: Davis complex for $W = \langle s, t \,|\, s^2 = t^2 = 1, (st)^3 = 1 \rangle \cong D_6$. Note that wK and wsK are glued along the s-mirror K_s.

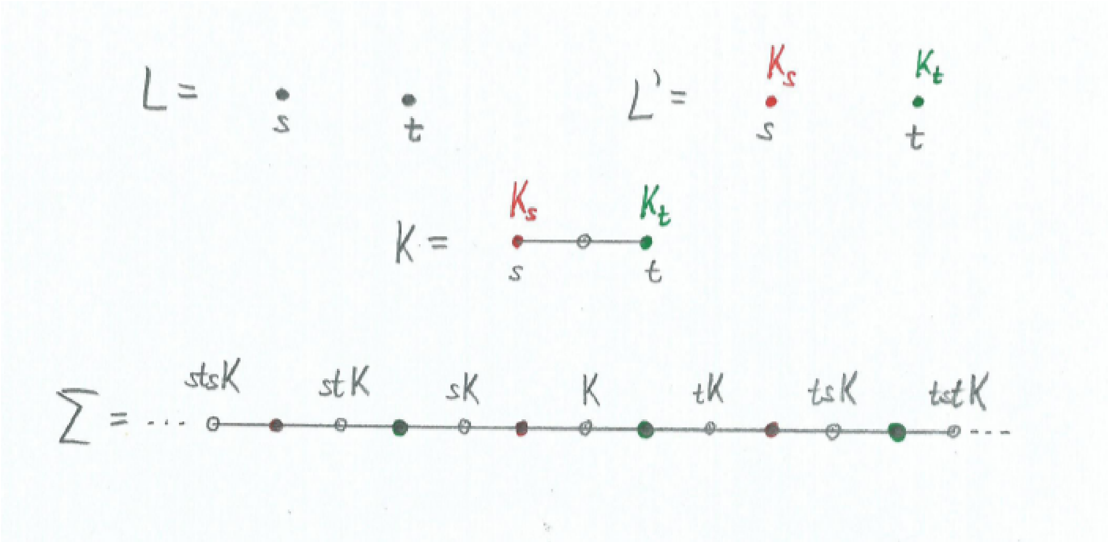

Figure 6.2: Davis complex for $W = \langle s, t \,|\, s^2 = t^2 = 1 \rangle \cong D_\infty$.

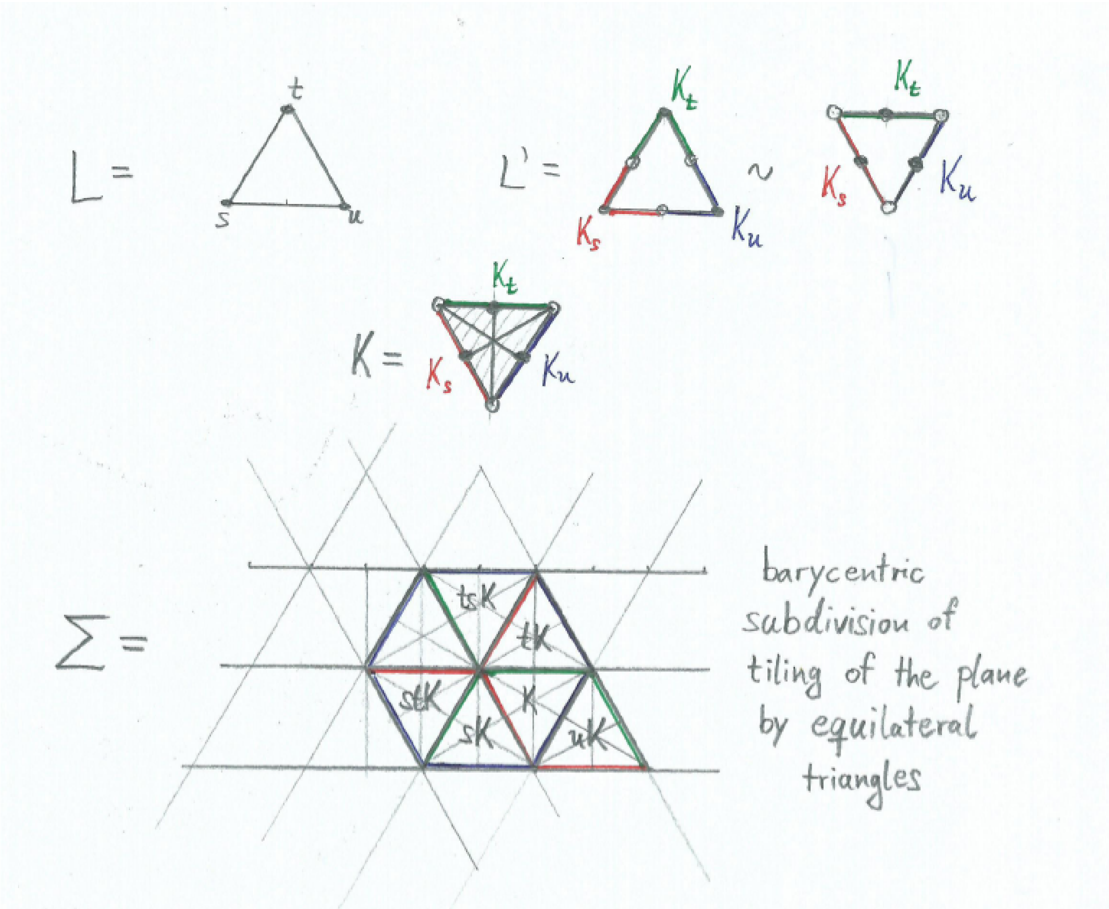

Figure 6.3: Davis complex for $W = \langle s, t, u \,|\, s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle$, i.e. the $(3, 3, 3)$-triangle group.

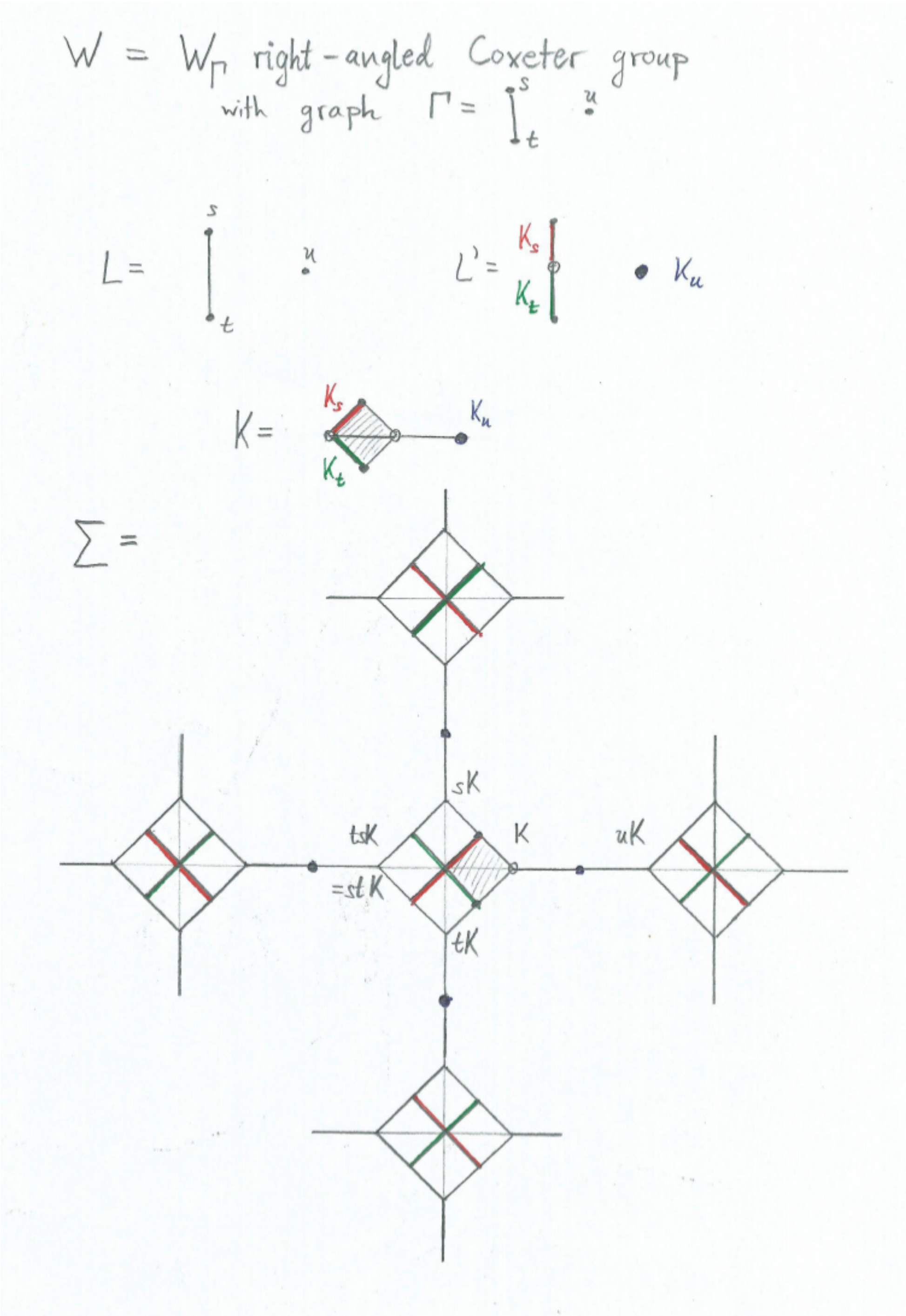

Figure 6.4: Davis complex for the right-angled Coxeter group $W = \langle s, t, u \,|\, s^2 = t^2 = u^2 = 1, (st)^2 = 1\rangle$ with corresponding graph Γ. Note that this yields a tree-like structure, since u does not commute with s and t.

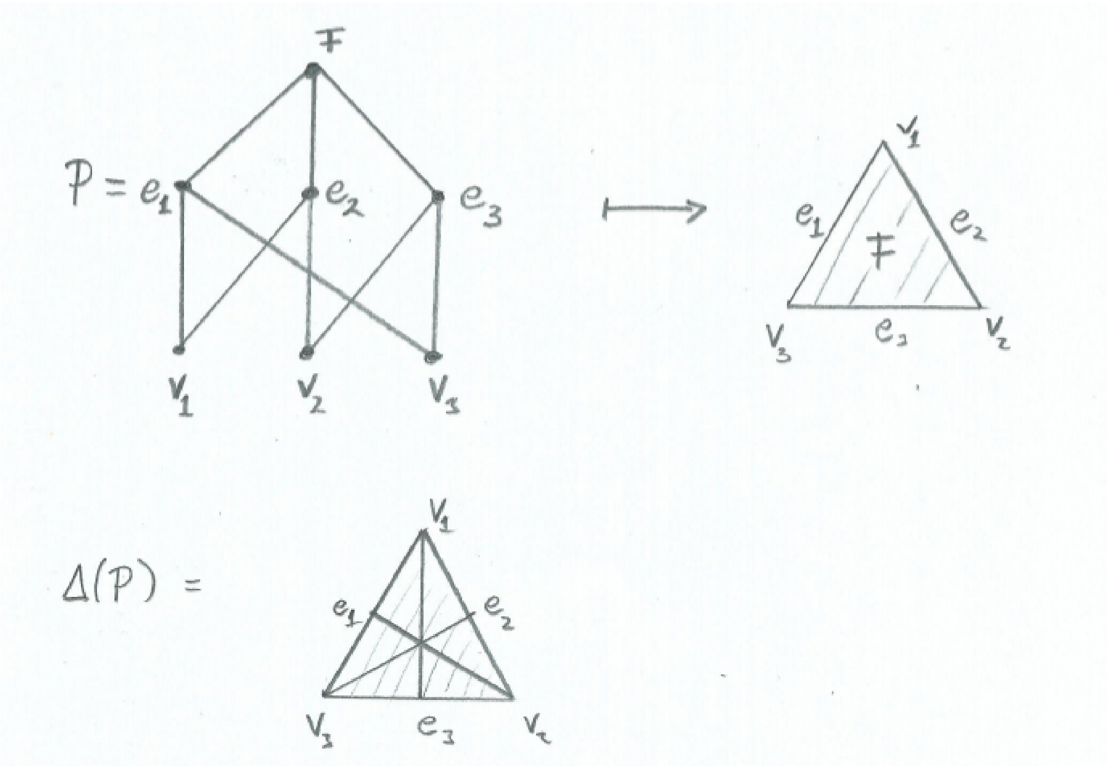

Figure 6.5: The poset P of cells, ordered by inclusion. $\Delta(P)$ is the barycentric subdivision of the corresponding cell complex.

7 Lecture Seven: Geometry of the Davis complex

20.04.2016

In the following let (W, S) be a Coxeter system and $\Sigma = \Sigma(W, S)$ the associated Davis complex with chambers wK $(w \in W)$.

Recall that we have the following bijections:

$$\{\text{vertices of } K\} \longleftrightarrow \{W_T \mid T \subseteq S, W_T \text{ is finite}\},$$
$$\{\text{vertices of } \Sigma\} \longleftrightarrow \{wW_T \mid T \subseteq S, W_T \text{ is finite}, w \in W\},$$

and the n-simplices in K (resp. Σ) correspond to $(n+1)$-chains. Figure 7.1 and Figure 7.2 illustrate this.

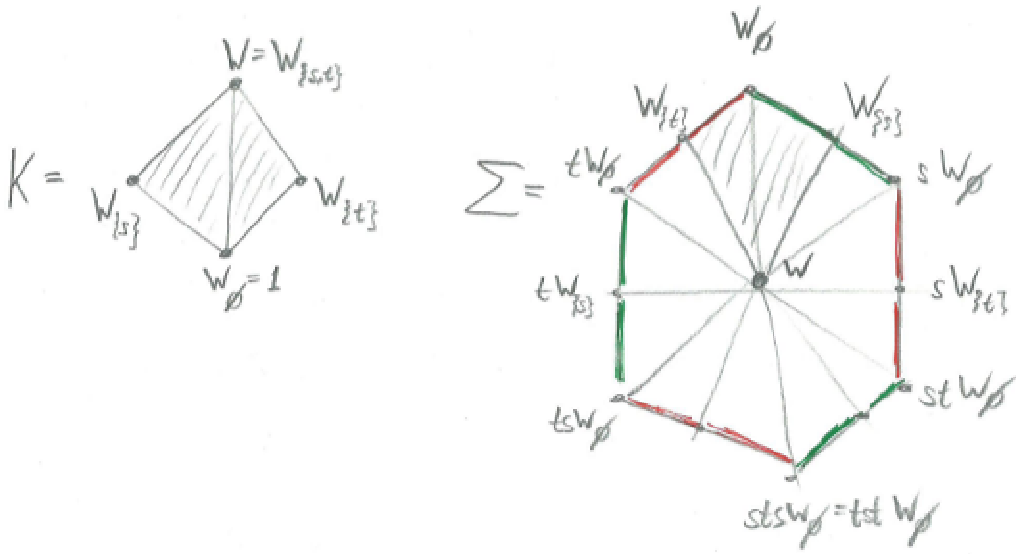

Figure 7.1: The Davis complex constructed from posets for $W = \langle s, t \mid s^2 = t^2 = 1, (st)^3 = 1\rangle \cong D_6$.

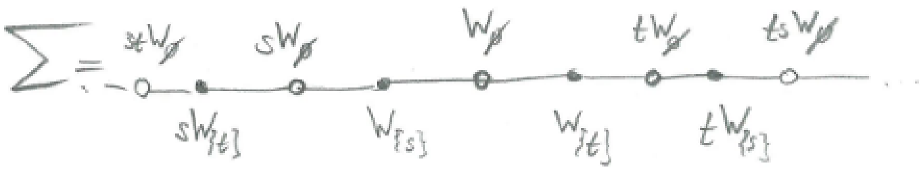

Figure 7.2: The Davis complex constructed from posets for $W = \langle s, t \,|\, s^2 = t^2 = 1 \rangle \cong D_\infty$.

7.1 Re-cellulation of Σ

We shall now equip Σ with a new cellular structure and denote the resulting CW-complex by Σ_{new}. The vertices of the new cellulation Σ_{new} are the cosets $wW_\emptyset$, i.e. the cosets of the trivial group. Thus the vertices of Σ_{new} are in bijection with the elements of W.

The edges of Σ_{new} are spanned by the cosets $wW_{\{s\}}$ ($w \in W, s \in S$). Now $wW_{\{s\}} = \{w, ws\}$, so the 1-skeleton of Σ_{new} is $\text{Cay}(W, S)$.

In general a subset $U \subseteq W$ is the vertex set of a cell in $\Sigma_{\text{new}} \iff U = wW_T$ where $w \in W$, W_T finite; see for example Figures 7.3 to 7.6. This eliminates the "topologically unimportant" additional cells coming from the barycentric subdivision in the previous definition of $\Sigma = \mathcal{U}(W, K)$.

So a third definition of Σ is that it is $\text{Cay}(W, S)$ with all cosets of finite special subgroups "filled in". From now on, we work with Σ_{new} and write Σ for it.

Lemma 7.1. Σ is simply-connected.

Proof. It is sufficient to consider the 2-skeleton $\Sigma^{(2)}$ and show that any loop in $\Sigma^{(1)} = \text{Cay}(W, S)$ is null-homotopic.

The 2-cells of Σ have vertex sets $wW_{\{s,t\}}$ with $W_{\{s,t\}}$ a finite dihedral group. This 2-cell has boundary word $(st)^m$ where $W_{\{s,t\}} \cong D_{2m}$. That is, any loop in $\Sigma^{(1)}$ can be filled in by conjugates of relators in the presentation of (W, S). So $\Sigma^{(2)}$ is simply-connected. $\square$

7.2 Coxeter polytopes

Recall: if W is finite, $|S| = n$, then a Coxeter polytope is the convex hull of a generic W-orbit in $\mathbb{R}^n$; see for example Figure 7.7 and Figure 7.8.

In $\Sigma = \Sigma_{\text{new}}$, the cell with vertex set wW_T is cellularly isomorphic to any Coxeter polytope for W_T.

Today we will metrise Σ by making each cell wW_T isometric to a (fixed) Coxeter polytope for W_T.

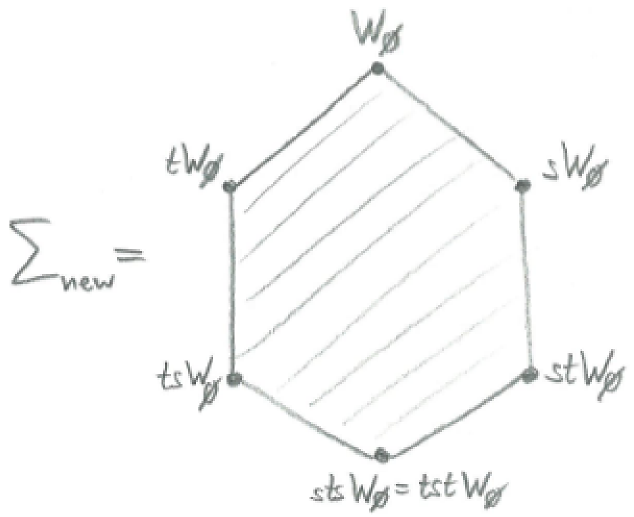

Figure 7.3: The new cellulation of the Davis complex for $W = \langle s, t \mid s^2 = t^2 = 1, (st)^3 = 2\rangle \cong D_6$.

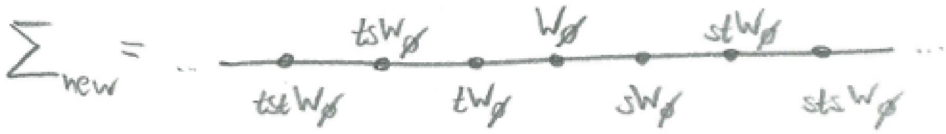

Figure 7.4: The new cellulation of the Davis complex for $W = \langle s, t \mid s^2 = t^2 = 1\rangle \cong D_\infty$.

7.3 Polyhedral complexes

Definition 7.2. A *polyhedral complex* is a finite-dimensional CW-complex in which each cell is metrised as a convex polytope in $\mathbb{X}^n$ (the same $\mathbb{X}^n$ for each n-dimensional cell), and the attaching maps are isometries on codimension-one faces.

Theorem 7.3 (Bridson)**.** *If a polyhedral complex X has finitely many isometry types of cells, then X is a geodesic metric space.*

Note: if we use the same Coxeter polytope for each W_T, Σ is a piecewise Euclidean geodesic metric space.

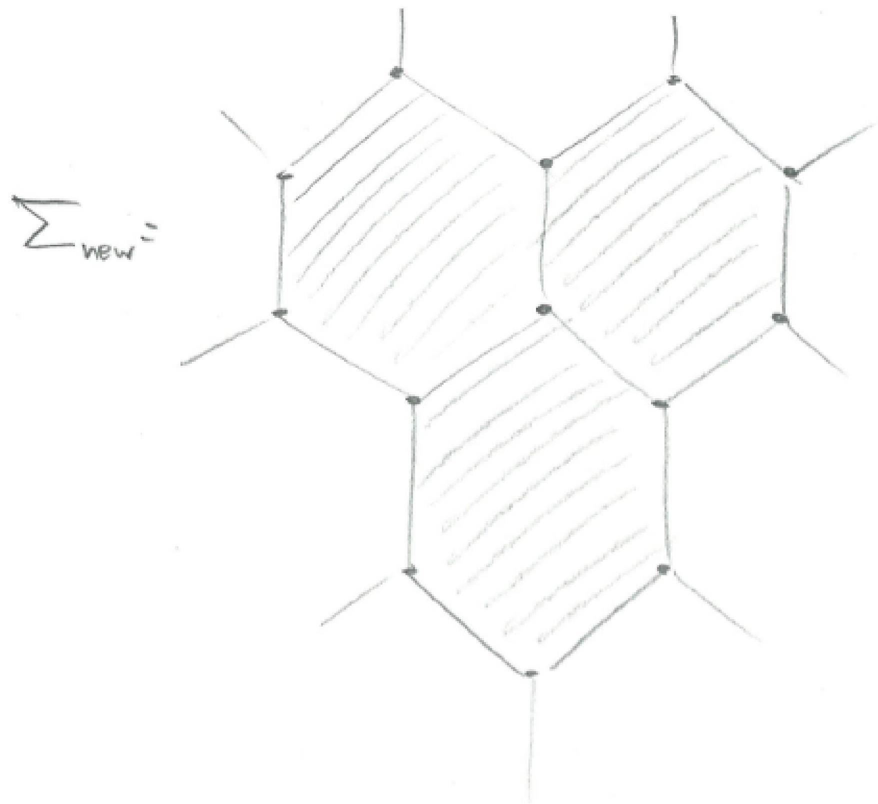

Figure 7.5: The new cellulation of the Davis complex for the (3, 3, 3)-triangle group W.

7.3.1 Metrisation of Σ

Pick $\underline{d} = (d_s)_{s \in S}$, $d_s > 0$. For W_T finite, let $\rho : W_T \to O(n, \mathbb{R})$, $n = |T|$ be the Tits representation. The fixed set of $\rho(t)$ is the hyperplane H_t with unit normalvector e_t, and the hyperplanes H_t, $H_{t'}$ meet at dihedral angle $\frac{\pi}{m}$ where $\langle t, t' \rangle \cong D_{2m}$; see for example Figure 7.9.

Let C be the chamber $\{x \in \mathbb{R}^n \mid \langle x, e_t \rangle \geq 0 \quad \forall t \in T\}$. Then there is a unique $x \in \mathrm{int}(C)$ such that $d(x, H_t) = d_t > 0$. We metrise each wW_T as a copy of the Coxeter polytope which is the convex hull of the W_T-orbit of this x.

Example 7.4. If $W = W_\Gamma$ is right-angled, then each finite W_T is $(C_2)^m$ so we are filling in right-angled euclidean polytopes.

7.3.2 Nonpositive curvature

Theorem 7.5. *When equipped with this piecewise Euclidean metric Σ is $CAT(0)$.*

Definition 7.6. A metric space X is CAT(0) if X is geodesic and geodesic triangles in X are "no-fatter" than triangles in $\mathbb{E}^2$.

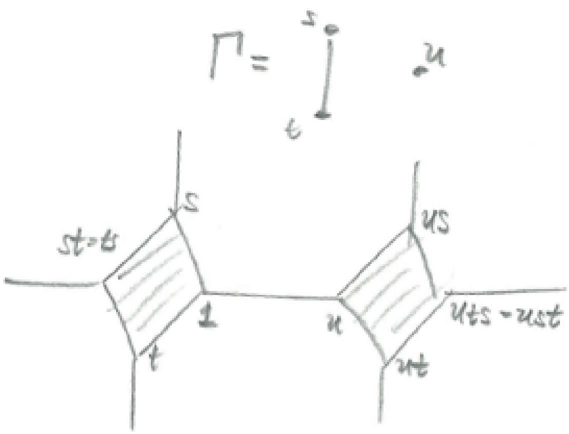

Figure 7.6: The new cellulation of the Davis complex for the right-angled Coxeter group $W = W_\Gamma$ with graph Γ as depicted.

That means: if $\Delta = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$ is a geodesic triangle in X with respective edge lengths l_1, l_2, l_3 then there is a so called *comparison triangle* $\overline{\Delta} = \{[\overline{x}_1\overline{x}_2], [\overline{x}_2\overline{x}_3], [\overline{x}_3\overline{x}_1]\}$ in $\mathbb{E}^2$ with the same respective edge lengths, i.e. $d_X(x_i, x_j) = d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j)$. (Here $[xy]$ denotes some geodesic segment from a point x to a point y.) Now Δ should not be "fatter" than the comparison triangle $\overline{\Delta}$, i.e. we must have

$$d_X(p, q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q}),$$

where p, q are arbitrary points on the sides of Δ and $\overline{p}, \overline{q}$ the corresponding points on $\overline{\Delta}$; see Figure 7.10.

Similarly, a geodesic metric space X is CAT(-1) if geodesic triangles in X are "no-fatter" than comparison triangles in $\mathbb{H}^2$.

A metric space X is CAT(1) if all points in X at distance $< \pi$ are connected by geodesics, and all triangles in X with perimeter $\leq 2\pi$ are "no-fatter" than comparison triangles in a hemisphere of $\mathbb{S}^2$.

Example 7.7. If X is a metric graph, then X is CAT(1) if and only if each embedded cycle has length $\geq 2\pi$.

Remark 4. The motivation for CAT(κ) is to give a notion of curvature which applies to symmetric spaces, buildings and many other (possibly) singular spaces.

The next proposition summarises some properties of CAT(0) spaces.

Proposition 7.8. Let X be a complete CAT(0) space. Then:

1. X is uniquely geodesic.

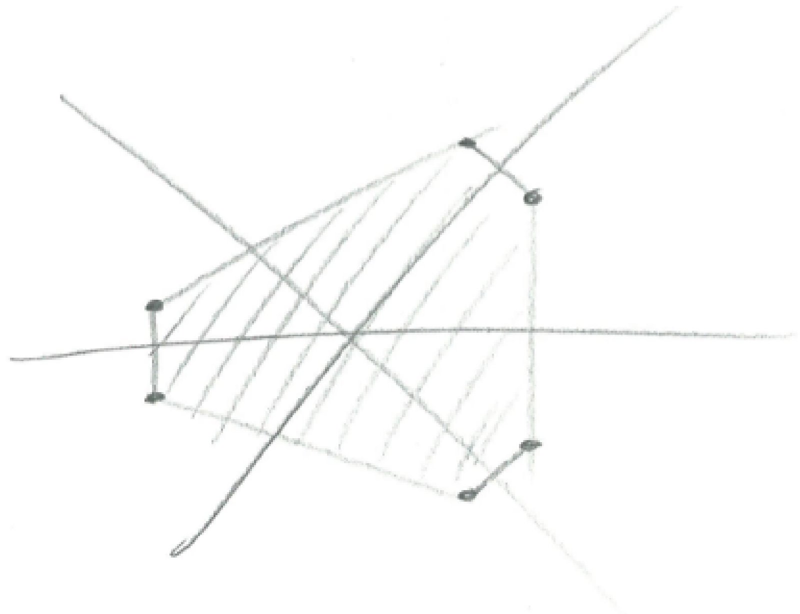

Figure 7.7: A Coxeter polytope for $W = \langle s, t \,|\, s^2 = t^2 = 1, (st)^3 = 2\rangle \cong D_6$.

2. X is contractible.

3. If G acts on X by isometries and H is a subgroup of G then X^H, the fixed set of H in X, if non-empty, is convex. In particular convex subsets of CAT(0) spaces are CAT(0), so every fixed set X^H is contractible by 2.

4. (Bruhat-Tits Fixed Point Theorem). If G acts on X by isometries and G has a bounded orbit, then $X^G \neq \emptyset$. In particular for every finite subgroup $H \leq G$: $X^H \neq \emptyset$.

5. If a group G acts properly and cocompactly by isometries on X then the "word problem" and the "conjugacy problem" are both decidable for G (in the sense of theoretical computer science).

We will only give sketchy proofs for some of the above results.

Proof.

To 1: Let $x, y \in X$ and let $\gamma = [xy]$ be some geodesic from x to y in X. Suppose γ' is another geodesic from x to y and let z be a point on γ. Then $\gamma \cup \gamma'$ forms a geodesic triangle with vertices x, y, z. However, a comparison triangle in $\mathbb{E}^2$ is degenerate; see Figure 7.11. Because X is CAT(0) and hence Euclidean comparison triangles are not fatter than geodesic triangles in X, the point z has to be in γ'. Since the point z was arbitrarily chosen, we get that every point of γ is a point of γ'; hence $\gamma = \gamma'$.

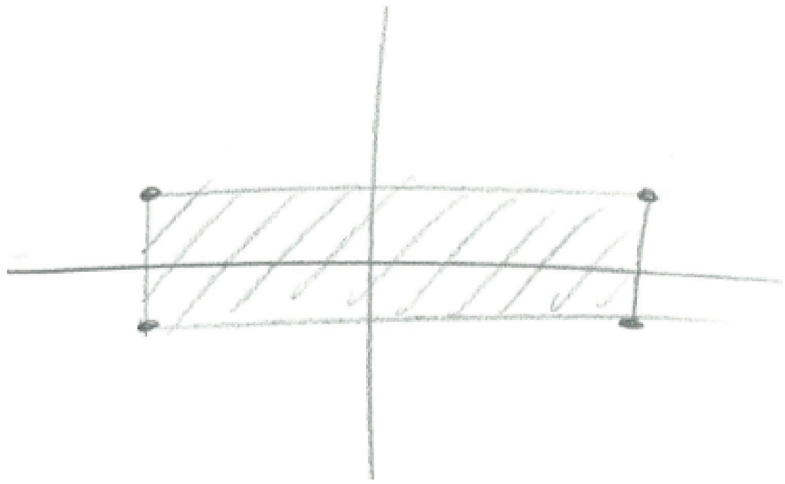

Figure 7.8: A Coxeter polytope for $W \cong (C_2)^n$.

To 2: By 1, X is uniquely geodesic. With a bit of work, one may now show that a contraction of X is given by sliding each point along its unique geodesic towards some point x_0 in X.

To 3: Let $x, y \in X^H$ be fixed by the H-action and let γ be the (unique) geodesic from x to y. Because H acts by isometries and isometries map geodesics to geodesics, also $h\gamma$ is a geodesic from $hx = x$ to $hy = y$. By uniqueness, we get $h\gamma = \gamma$, i.e. $\gamma \subseteq X^H$; see Figure 7.12. Hence X^H is (geodesically) convex.

To 4: If G has a bounded orbit Gx in X then we can consider the convex hull of the points in Gx which defines a finite polytope in X. The barycenter of this polytope is then a fixed point of G.

□

These, in combination with Theorem 7.5, prove:

1. Σ is a finite-dimensional $\underline{E}W$.
2. If $H \leq W$ is finite then there is a point $w \in W$ and a spherical subset $T \subseteq S$, such that $H \leq wW_Tw^{-1}$. (This was already earlier proved by Tits.)
3. The "conjugacy problem" for W is decidable. (This question still remains open for the "isomorphism problem".)

7.4 Proof of Theorem 7.5

Let us now give a proof of Theorem 7.5. We will need the following "Cartan-Hadamard Theorem for CAT(0) spaces" due to Gromov:

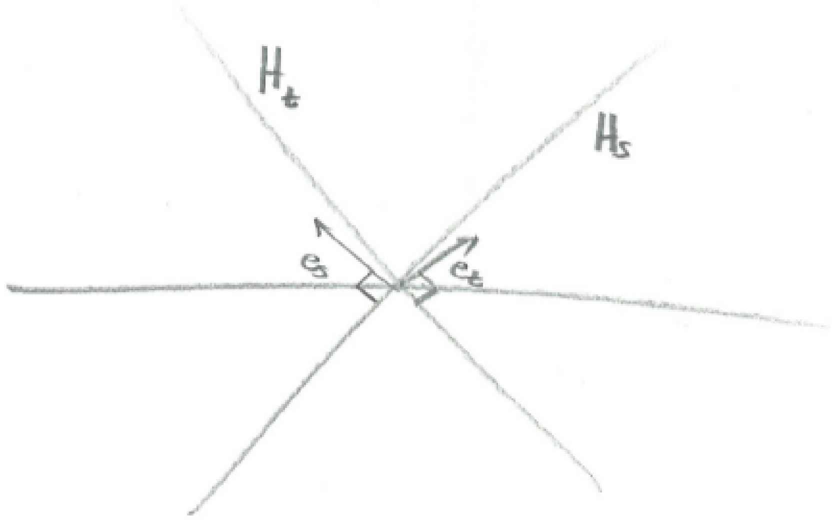

Figure 7.9: Two hyperplanes $H_s = e_s^\perp$ and $H_t = e_t^\perp$ in the Tits representation corresponding to a subgroup $\langle t, s\rangle \cong D_{2m}$.

Theorem 7.9 (Gromov)**.** *Let X be a complete, connected geodesic metric space. If X is locally CAT(0) then the universal cover of X is CAT(0).*

Since Σ is complete, connected and simply-connected, it is enough to show that Σ is locally CAT(0). For that we use Gromov's Link Condition:

Theorem 7.10 (Gromov Link Condition)**.** *If X is a piecewise Euclidean polyhedral complex then X is locally CAT(0) if and only if for every vertex v of X, the link of v is CAT(1).*

Before we proceed, let us give an example of an X as above which is not CAT(0) and for which the Link Condition does not hold.

Example 7.11. Consider X to be the 2-skeleton of a cube. The link of a vertex is depicted in Figure 7.13. Each arc of it has length $\frac{\pi}{2}$, so $\mathrm{lk}(v, X)$ is not CAT(1).
If we wanted to make X a CAT(0) space, we we would need to fill in the cube.

Therefore we need to investigate the links in Σ. Because W acts transitively on the vertices of Σ (by definition) it is enough to consider the link of the vertex $v = W_\emptyset = 1$. If W_T is finite, $\mathrm{lk}(v, \Sigma)$ contains a spherical simplex σ_T which is the link of v in the corresponding Coxeter polytope.
In the Coxeter polytope of Figure 7.14, σ_T is the spherical simplex with vertex set the unit normal vectors $\{-e_t\}_{t\in T}$. So we identify σ_T with the simplex with vertex set $\{e_t\}_{t\in T}$.

Corollary 7.12. In Σ, the link of $v = 1$ is L, the finite nerve, with each simplex σ_T of L metrised as the simplex in $\mathbb{S}^{|T|-1}$ with vertex set $\{e_t\}_{t\in T}$.

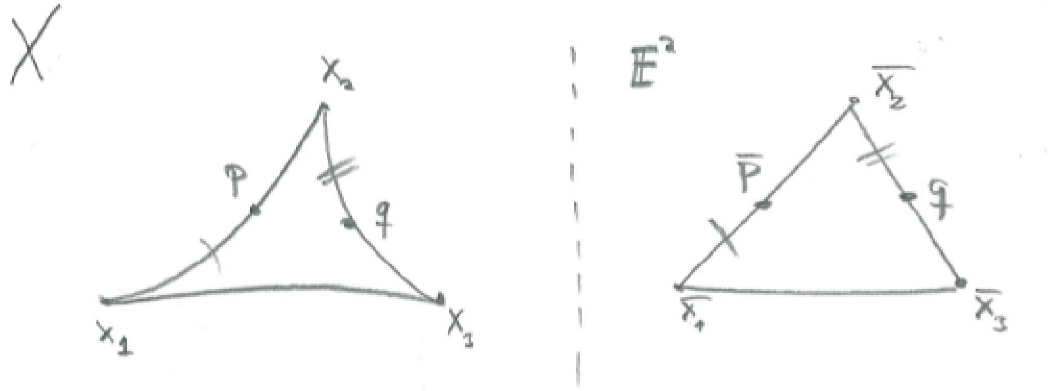

Figure 7.10: A geodesic triangle $\Delta = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$ in a CAT(0) space X with corresponding comparison triangle $\overline{\Delta} = \{[\overline{x}_1\overline{x}_2], [\overline{x}_2\overline{x}_3], [\overline{x}_3\overline{x}_1]\}$ in $\mathbb{E}^2$.

Example 7.13. Figure 7.15 gives an example of a CAT(1) link.

Hence we will be done, if we can show that L, with this piecewise spherical structure, is CAT(1). In the special case that $W = W_\Gamma$ is right-angled, L is the flag complex with 1-skeleton Γ; see Figure 7.16. This motivates the following lemma:

Lemma 7.14 (Gromov)**.** Suppose all simplices of a simplicial complex Δ are metrised as spherical simplices with edge lengths $\frac{\pi}{2}$. Then Δ is CAT(1) if and only if Δ is flag.

Corollary 7.15. If W_Γ is right-angled, Σ can be metrised as a CAT(0) cube complex (with proper, cocompact W_Γ-action).

In general a spherical simplicial complex Δ with an assignment of edge lengths is a *metric flag complex* if a pairwise connected subset of vertices spans a simplex in Δ if and only if there is a spherical simplex with these edge lengths.

Lemma 7.16 (Moussong)**.** Suppose a simplicial complex Δ is metrised as a spherical simplicial complex so that all edge lengths are $\geq \frac{\pi}{2}$.
Then Δ is CAT(1) if and only if Δ is a metric flag complex.

Corollary 7.17. Since all edge lengths in L are $\pi - \frac{\pi}{m}$ with $m \geq 2$, Σ is CAT(0).

This finishes the proof of Theorem 7.5.

□

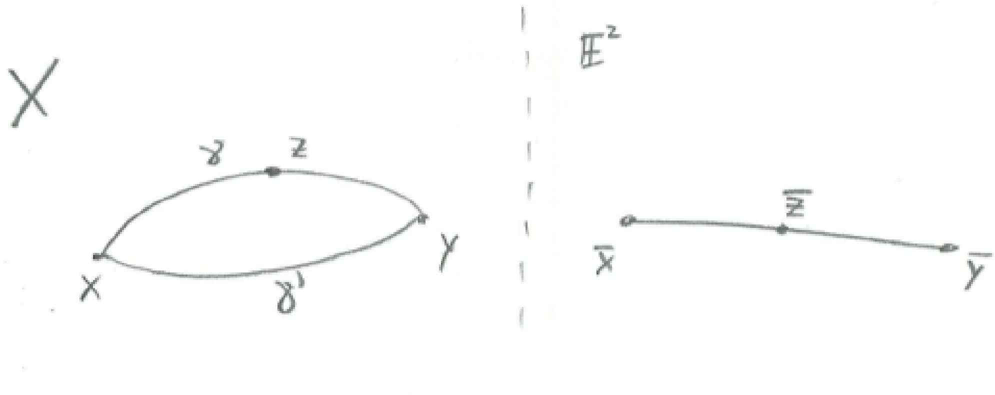

Figure 7.11: Illustration of the proof of assertion 1 in Proposition 7.8.

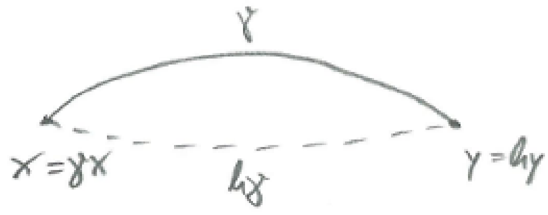

Figure 7.12: Illustration of the proof of assertion 3 in Proposition 7.8.

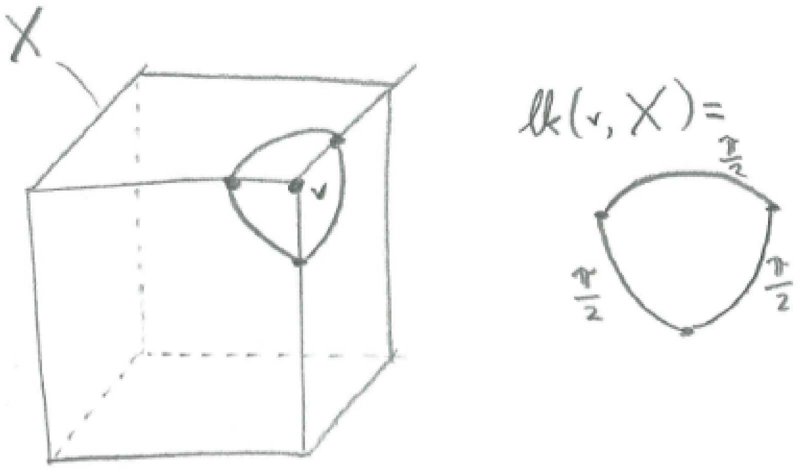

Figure 7.13: If X is the 2-skeleton of a cube its link is not CAT(1).

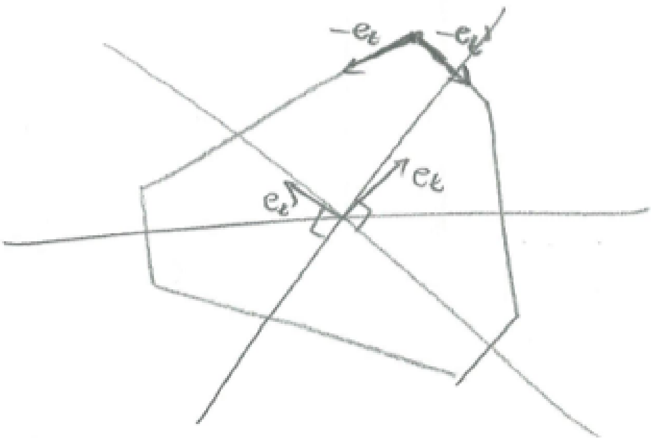

Figure 7.14: In this Coxeter polytope σ_T is the spherical simplex with vertex set the unit normal vectors $\{-e_t\}_{t \in T}$.

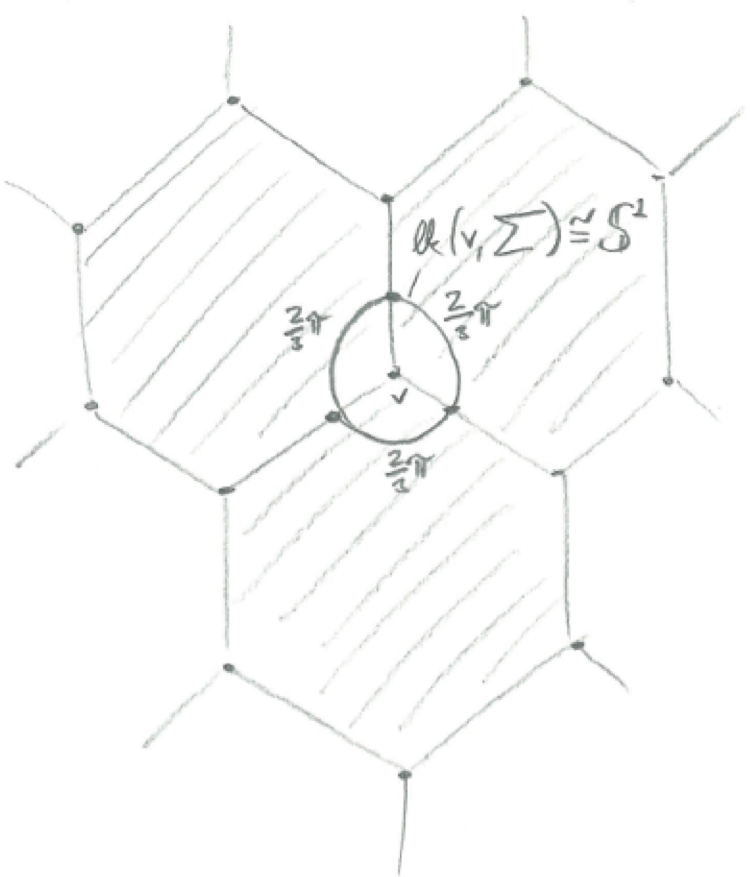

Figure 7.15: If W is the $(3,3,3)$-triangle group then $\mathrm{lk}(v,\Sigma)$ is isometric to $\mathbb{S}^1$.

Figure 7.16: If W_Γ is right-angled, each σ_T is a right-angled spherical simplex where all edges have length $\frac{\pi}{2}$.

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