VOLUME OF COMPLEMENT OF RANDOM GEODESICS

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Abstract: Given a filling primitive geodesic loop in a closed hyperbolic surface one obtains a hyperbolic three-manifold as the complement of the loop's canonical lift to the projective tangent bundle. In this paper we give the first known lower bound for the volume of these manifolds in terms of the length of generic loops. We show that estimating the volume from below can be reduced to a counting problem in the unit tangent bundle and solve it by applying an exponential multiple mixing result for the geodesic flow.

1. Introduction

1.1. **Volumes lift complements of curves.** Let $X \cong \Sigma_{g,k}$ be a hyperbolic surface, which is an orientable smooth surface of genus g with k punctures and negative Euler characteristic. Associated to X is the 3-manifold $\mathrm{PT}(X)$, the projectivised tangent bundle. For any finite collection of smooth essential closed curves $\mathcal{G}$ on X, there is a *canonical* lift $\widehat{\mathcal{G}}$ in $\mathrm{PT}(X)$ realized by the set of tangent lines to $\mathcal{G}$. Drilling $\widehat{\mathcal{G}}$ from $\mathrm{PT}(X)$ produces a 3-manifold $M_{\widehat{\mathcal{G}}} = \mathrm{PT}(X) \setminus \widehat{\mathcal{G}}$ and when $M_{\widehat{\mathcal{G}}}$ is hyperbolic, by Mostow Rigidity [BP91], any invariant of $M_{\widehat{\mathcal{G}}}$ naturally becomes a mapping class group invariant of $\mathcal{G}$. When $\mathcal{G}$ is filling, its components are primitive, and $\mathcal{G}$ is in minimal position, Foulon and Hasselblatt [FH13] first observed that $M_{\widehat{\mathcal{G}}}$ admits a complete hyperbolic metric of finite-volume, in particular $\mathrm{Vol}(M_{\widehat{\mathcal{G}}})$ is such an invariant. One should think of $\widehat{\mathcal{G}}$ as a weak version of a link diagram where "over" and "under" crossings are encoded by the tangent directions to $\mathcal{G}$. The most general such hyperbolicity result appears in [CRM18]. The authors show that if one takes a primitive filling system $\mathcal{G}$ in minimal position over a surface X and then drills a transverse lift $\overline{\mathcal{G}}$ in a Seifert-fibered manifold M then, the resulting manifold is hyperbolic. Transverse lifts of such systems will be called *topological lifts*.

In the rest of the paper we will use $\widehat{\mathcal{G}}$ to denote canonical lifts of $\mathcal{G} \subseteq S$ in $\mathrm{PT}(S)$ and $\overline{\mathcal{G}}$ to denote topological lifts of $\mathcal{G} \subseteq X$ in a Seifert-fibered manifold M.

Several upper and lower bounds for $\mathrm{Vol}(M_{\overline{\mathcal{G}}})$ in terms of invariants of $\mathcal{G}$ have been studied in recent literature, see [BPS17a, BPS19, RM20, Mig21, CRM18, CMY20].

Going back to the special sub-class that arises by considering M to be $\mathrm{PT}(X)$, or the unit tangent bundle, and using the bundle projection map $\pi\colon \mathrm{PT}(X) \to X$ to lift a filling geodesic γ to its canonical lift $\widehat{\gamma}$, i.e. $\dot{\gamma}$. In the case that the surface X is the modular surface $\mathrm{PT}(X)$ can be identified with the Trefoil complement in $\mathbb{S}^3$ and Ghys [Ghy07] showed that all Lorenz knots and links arise as canonical lifts of geodesics on the modular surface. Moreover, the setup of canonical lifts has been extensively studied in [Mig21, BPS17a, BPS19, RM20] and others.

In [CRM18] the authors gave an upper bound which is linear in terms of the self-intersection number of $\mathcal{G}$, a fact reminiscent of classical results in knot theory. In [BPS17b], it is shown that for every hyperbolic structure X on S, there is a constant C_X such that $\mathrm{Vol}(M_{\widehat{\mathcal{G}}}) \le C_X \ell_X(\mathcal{G})$, where $\ell_X(\mathcal{G})$ denotes the length of the geodesic representative. However, since $\mathrm{vol}(M_{\widehat{\mathcal{G}}})$ is independent of the choice of X, this bound demonstrates some odd behaviour, for example the volume is mapping class group invariant while the length and the hyperbolic structure X are not, see [RM20, Mig21] for many other interesting examples.

There are key differences between the volumes corresponding to canonical lifts or to topological lifts and also between simple filling systems and closed filling geodesics. In [CRM18, Corollary 1.6] the authors construct examples in which the volume is asymptotic to the self-intersection number $\iota(\gamma_n, \gamma_n)$ of the generating filling curves. By fixing a hyperbolic structure X on S and a result of Basmajian [Bas13], the self-intersection number is bounded above by $\ell_X(\gamma_n)^2$ which is in contrast with the general length upper bound for volumes of canonical lifts of [BPS17b].

The best known lower bound appears in [RM20, CRM18], where the bound is given in terms of the number of essential homotopy classes of arcs of $\mathcal{G}$ after cutting X open along any multi-curve $\mathfrak{m}$ and taking the maximum over such $\mathfrak{m}$. While this lower bound is shown to be sharp for some families of *non-simple* closed curves on the modular surface [RM20], it is always at most $6(3g+n)(3g-3+n)$ whenever $\mathcal{G}$ is composed entirely of *simple* closed curves. This is addressed in [CMY20].

In [CMY20] the authors study the setting in which one considers a filling collection of simple closed curve in minimal position instead of a primitive filling loop in minimal position. This is interesting because the only known lower bound is completely ineffective in this case. In [CMY20, Theorem A] the authors relate the volume of complement of the canonical lift of a pair of filling geodesics to pants distance in the pants graph of the surface.

1.2. **Upper bounds on length.** We now describe in more details the length upper bound and refer to some numerical evidence. The length upper bound of [BPS17b, Theorem 1.1] is:

Theorem. Let X be a hyperbolic surface. Then, there exists C_X such that for any filling primitive geodesic $\gamma \subseteq X$:

$$\mathrm{Vol}(M_{\widehat{\gamma}}) \le C_X \ell_X(\gamma)$$

and the result also works for multi-curves. The proof goes by showing the equivalent result in the case of the modular surface $Y = \mathbb{H}^2/SL_2(\mathbb{Z})$. Then, one shows that by taking branched coverings Z of Y and curves $\gamma \subseteq Y$ one can obtain all filling primitive systems on any $S_{g,n}$. Then, the constant C_X comes from considering the optimal quasi-conformal map from X to Z. This is in general a non-trivial problem.

Remark 1.1. There are large families of filling primitive geodesics for which the volume of the complements are uniformly bounded but whose length go to infinity. The easiest such example can be obtained by taking the mapping class group orbit of a fixed loop γ. However, there are also more interesting examples in which the loops γ_n are not in the same mapping class group orbit, see [RM20, Mig21]. Some of these example can be thought of as taking a filling curve γ say intersecting another loop α once and concatenating γ with powers of α. These are called *twist families* and will always give rise to bounded volumes families.

A sequence of random geodesics is, informally, a sequence of geodesics that gets more and more equidistributed with respect to the volume of $\mathrm{UT}(X)$ and converges to the Liouville measure of $\mathrm{UT}(X)$. See Subsection 2.4 for precise definitions.

A sequence of random geodesics in the modular surface has been considered by Duke in [Duk88]. This model is constructed via number theoretic techniques. In this paper we will construct another family of random geodesics using geometry and dynamics. It would be interesting to obtain a lower bound for the volume of Duke's random sequence. In [BPS19] the authors compute the volumes for finitely many terms of Duke's random sequence and then give numerical evidence of the linear volume growth as a function of geodesic length (see Figure 1.1).

However, by [Mig21, Corollary 1.2] there exist multi-curves whose volumes are asymptotic to $\frac{L}{W(L)}$ for L the length and $W(x)$ the Lambert function. The *Lambert function* is the principal branch of the inverse of $f(w) = w \log w$ which is asymptotic to $\log(x) - \log\log(x) + o(1)$.

Recently, Yarmola computed the volumes for all geodesics of length at most 16 in the modular surface. The graph clearly still shows a linear upper bound but the situation for the lower bound is more complicated due to the presence of twist families. Such a family can be seen in the lower left corner of Figure 1.2.

1.3. **Lower bound.** No geometric lower bounds are currently known and we only have combinatorial ones. Using work of Agol, Storm and Thurston [AST07] Rodriguez-Migueles showed in [RM20] that:

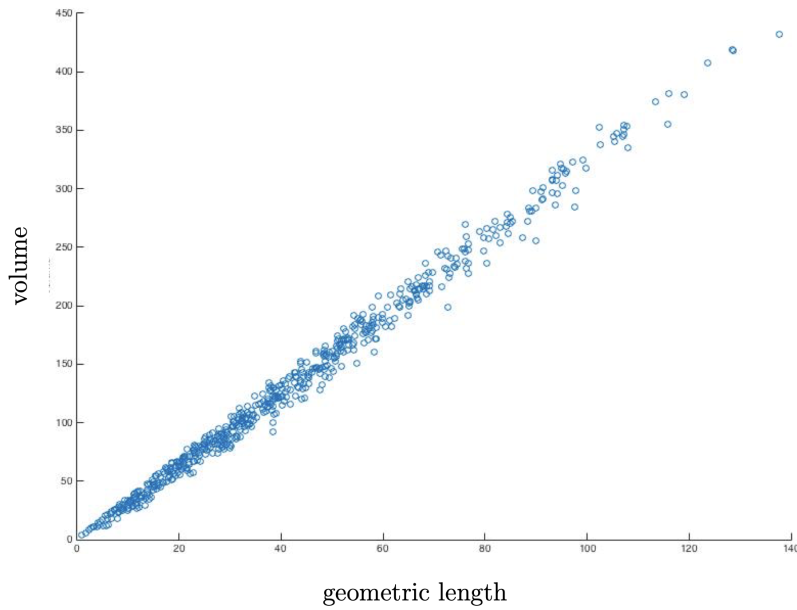

Figure 1.1. Figure 2 of [BPS19]

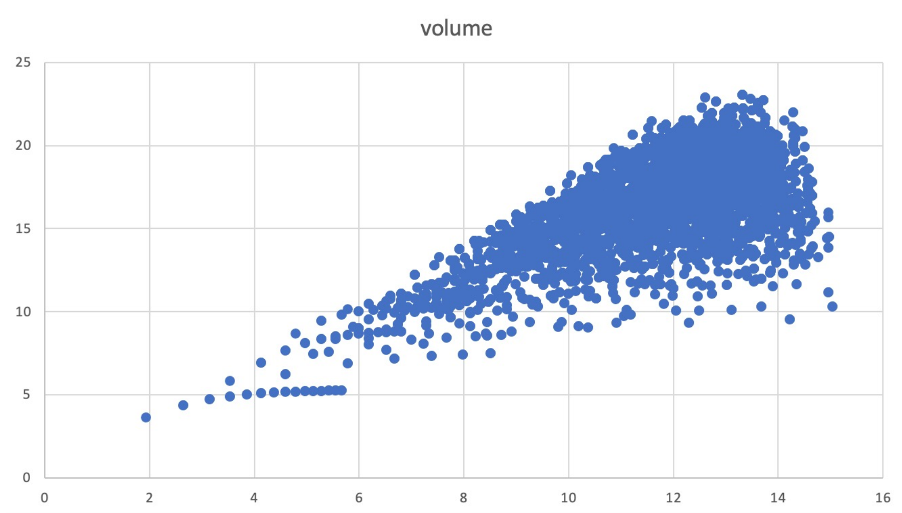

Figure 1.2. The volumes are the y axis and the x axis is the hyperbolic length.

Theorem. Let $\mathcal{P}$ be an essential surface decomposition of S and let γ be a filling primitive loop in minimal position with respect to $\partial\mathcal{P}$ and itself. Then:

$$\frac{v_3}{2}\sum_{Q\in\pi_0(\mathcal{P})}\#\{\gamma\text{-arcs in } Q\} \leq \mathrm{Vol}(\mathrm{PT}(S)\setminus\widehat{\gamma})$$

where v_3 is the volume of a regular ideal tetrahedra and we sum over all components of $\mathcal{P}$. For $Q \in \pi_0(\mathcal{P})$, the γ-*arcs* are the free homotopy classes of $\gamma \cap Q$ (see Figure 1.3).

For example if Q is a pair of pants and γ is a filling geodesic there are at most 6 simple γ-arcs in $\gamma \cap Q$.

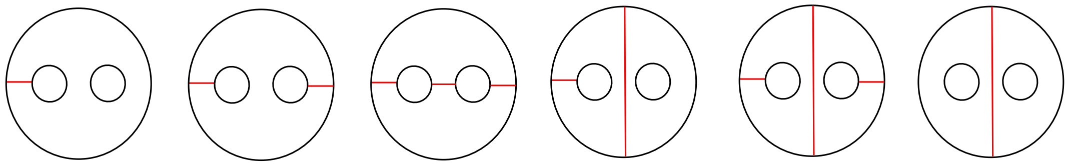

FIGURE 1.3. Up to free homotopy there are 6 γ-arcs configurations consisting of simple γ-arcs

In [CRM18] the authors show that the above lower bound also works in the setting of Seifert-fibered spaces.

1.4. **Statement of the theorem.** Because of the length upper bound it would be nice to have a lower bound also depending on length, possibly in a linear fashion. The possible linearity of a lower-bound is hinted by the experimental data of 1.1, consisting of a finite number of terms of a sequence of random geodesics that converge to the Liouville measure (see discussion on Duke's model in page 3).

However, as mentioned in Remark 1.1, one can construct sequences of geodesics γ_n such that $\ell_X(\gamma_n) \to \infty$ but the volume stay bounded. Similarly, there are examples of $\gamma_n' s$ where the growth is bounded above by a sub-linear function $f(t)$ (see [Mig21]). Therefore, we look for a lower bound that is true almost surely with respect to the normalized Liouville measure (see Definitions in Subsection 2.2).

In this regard our main result is the following. Let X be a hyperbolic structure on a closed surface X with an isometric pants decomposition $\mathcal{P}$ in which every pants P is glued without twits. Moreover, let $0 < \delta < 1$ be the Hausdorff dimension of the limit set of P.

Theorem 7.1. Let $\eta > 1 (> \delta > 0)$. There exist positive constants A, B, C, such that for μ-almost every $v \in \mathrm{PT}(X)$ the geodesic $\widehat{\gamma}_v^p := \widehat{\gamma}_v^p(t)$ for all $t \geq t(v)$ large enough satisfies:

$$A \cdot \left(\frac{C \cdot \ell_X(\widehat{\gamma}_v^p)}{W\left(C \cdot \ell_X(\widehat{\gamma}_v^p)\right)} \right)^{\delta/2\eta} - B \leq \mathrm{Vol}(\mathrm{PT}(X)_{\widehat{\gamma}_v^p}),$$

where W denotes the Lambert W function.

1.5. **Short outline.** We will build our random sequence of geodesics in the following way. For μ-almost every vector $v \in \mathrm{PT}(X)$, for all $\varepsilon > 0$, if we flow for a sufficiently long time (depending on v), we return close enough to v i.e. $d(g_t(v), v) < \varepsilon$. Then, we can close up via an arc that creates a piece-wise closed geodesic $\gamma_v = \gamma_v(t)$ with small defect. We will show that for t large enough, γ_v will converge to the Liouville measure and so will be in particular a filling geodesic. A more general model has been considered by Bonahon [Bon88, Page 151] (see also [Bri04, Theorem 3]), where no restrictions on the closing arc are imposed. Here we need further control on the defects. By using the fact that γ_v is nearly geodesic we will be able to control the homotopy to its geodesic representative $\widehat{\gamma}_v$ showing that, for a pants decomposition $\mathcal{P}$, we can control the number of $\widehat{\gamma}_v$-arcs via the number of γ_v-arcs.

Finally, we will use dynamical techniques involving higher order exponential mixing of the geodesic flow to estimate the number of γ_v-arcs by just considering the ones induced by the geodesic ray $g_{[0,t]}(v)$.

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2. Background

2.1. **Notation.** We will use the following notational conventions:

- $A \cong B$ denotes that the two topological spaces A, B are homeomorphic;
- $\alpha \simeq \beta$ denotes that α, β are homotopic maps or spaces, generally loops or arcs.

For two functions $f, g\colon \mathbb{T} \to \mathbb{R}$ we will use the following notations, where $\mathbb{T}$ stands for $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$:

- $f(t) \sim g(t)$ means that the two functions $f, g\colon \mathbb{T} \to \mathbb{R}$ are asymptotic, i.e.
$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1;$$
- $f(t) \ll g(t)$ means that there is a constant $C > 0$ such that $f(t) \leq Cg(t)$ for all $t \in \mathbb{T}$;
- $f(t) \asymp g(t)$ means that there are constants $C_1, C_2 > 0$ such that
$$C_1 f(t) \leq g(t) \leq C_2 f(t)$$
for all $t \in \mathbb{T}$.

2.2. **Dynamics.** We recall some concepts of dynamics that will play a key role in this paper. For a standard references see [KH95, Page 151], [Wal82].

Let X be a topological space equipped with a (Borel) probability measure μ. Suppose that $g_t\colon X \to X$ is a continuous *flow*, for $t \in \mathbb{R}$, i.e. a family of continuous maps so that g_0 is the identity and $g_{s+t} = g_s \circ g_t$ for all $s, t \in \mathbb{R}$. We say μ is g-invariant if for any Borel set $A \subseteq X$, $\mu(A) = \mu(g_t(A))$ for all $t \in \mathbb{R}$. There is an analogous picture for discrete time dynamical systems, where instead of a flow, we have a continuous map $T\colon X \to X$ (which need not be invertible) and μ is said to be T-invariant if $\mu(B) = \mu(T^{-1}(B))$ for every Borel set $B \subseteq X$. One can go from a continuous to a discrete setting by taking $T = g_1$, the *time-one flow map*, and we will be doing so throughout the paper. We will focus on two properties that a continuous or discrete dynamical system can have: ergodicity and mixing.

2.2.1. *Ergodicity.* The property of ergodicity will be used to construct our geometric random model. A flow $g_t\colon X \to X$ (resp. transformation $T\colon X \to X$) is *ergodic* with respect to μ if for every subset $A \subseteq X$ which is g-invariant $g_t(A) = A$ (resp. T-invariant $T^{-1}(A) = A$), then either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

If P_x is a property depending on a point $x \in X$, we say that P holds for *μ-almost every* $x \in X$ if it holds for a subset $B \subseteq X$ so that $\mu(B) = 1$.

Ergodic dynamical systems satisfy the following two properties that we will use.

Theorem 2.1 ([Wal82, Theorem 1.7])**.** Let X be a compact metric space, $T\colon X \to X$ a continuous transformation and μ a probability measure so that $\mu(U) > 0$ for all open set $U \subseteq X$. Suppose that μ is T-invariant and T is ergodic with respect to μ. Then for μ-almost every $x \in X$ the orbit of x, i.e. $\{T^n(x) : x \in X\}$ is dense.

Theorem 2.2 (Birkhoff's Ergodic Theorem; [Wal82, Theorem 1.14])**.** Let X be a topological space equipped with a probability measure μ. For $g_t\colon X \to X$ an ergodic flow with respect to μ, and f any measurable function, we have that for μ-almost every $x \in X$:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(g_t(x))dt = \int_X f(x)d\mu$$

For a discrete system, the flow is replaced by a transformation T, the integral by a sum, and the limit in the previous theorem takes the following form

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x)) \to \int f d\mu \qquad (N \to \infty)$$

for μ-almost every $x \in X$.

2.2.2. *Mixing.* We will need a stronger property than ergodicity to have enough control on our model. We say μ is *mixing* for the flow $g_t\colon X \to X$ if for all $f, g \in L^2(X,\mu)$, the *correlation function*

$$\rho(t) := \int (f \circ g_t) g d\mu - \int f d\mu \int g d\mu,$$

satisfies $\rho(t) \to 0$ as $t \to +\infty$. For discrete systems, the correlation is written as

$$\rho(n) := \int (f \circ T^n) g \, d\mu - \int f \, d\mu \int g \, d\mu,$$

and mixing means $\rho(n) \to 0$ as $n \to +\infty$.

We will be interested in extending this correlation function to more than two functions, as well as on quantifying its decay. This will lead to the notion of exponential k-mixing, see Appendix B.

Another equivalent way to phrase mixing is that, for all Borel sets $A, B \subseteq X$,

$$\lim_n \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$$

It follows that mixing implies ergodicity. Intuitively, mixing for a continuous transformation $T \colon X \to X$, means that for any set A the sequence of $T^{-n}(A)$ becomes independent of any set B. Ergodicity means that $T^{-n}(A)$ becomes independent of B on average, for any A, B sets.

In this paper, we will let g_t be the geodesic flow on the unit tangent bundle $\mathrm{UT}(X)$, and μ will be the normalized Liouville probability measure. T will be the time-one map of the geodesic flow. The geodesic flow is mixing with respect to μ (see [Bab02, Theorem 1]), and is by the above discussion, also ergodic.

2.3. **Topology.** In the following sections we recall some facts and definitions about the topology of surfaces and 3-manifolds. For references, see [Hem76, Hat07, Jac80].

Definition 2.3. A *knot* in M will be any embedding of $\mathbb{S}^1$ into a 3-manifold M.

Definition 2.4. Given a loop $\gamma \subseteq S$ and the projective tangent bundle $\mathrm{PT}(S)$ we define the *canonical lift* to be the knot $\widehat{\gamma} \subseteq \mathrm{PT}(S)$ obtained by taking the lift of γ given by the tangential line field.

For the arguments in Section 5 we will use the result of Hass and Scott in [HS94] that curves in non-minimal position contain bigons and/or monogons.

2.4. **Geometry.** A *geodesic current* is a positive finite Radon measure μ on $\mathrm{UT}(X)$ which is invariant under the geodesic flow, in the sense that $(g_t)_*(\mu) = \mu$ for all $t \in \mathbb{R}$, where the subscript $*$ denotes the push-forward of measures.

A closed geodesic γ can be seen as a geodesic current. Consider the canonical lift $\widehat{\gamma}$ of γ to $\mathrm{UT}(X)$; this is a periodic orbit of g_t. Let μ_γ be the length-normalized δ-function on this orbit. That is, for an open set U we set $\mu_\gamma(U)$ to be the total length of $\widetilde{\gamma} \cap U$ with respect to the natural Riemannian metric on $\mathrm{UT}(X)$. The geometric intersection number between closed geodesics extends continuously to a bilinear form $i(\cdot,\cdot)$ on geodesic currents [Bon86, Proposition 4.5]. The *Liouville current* $\mathcal{L}_X$ associated to the hyperbolic metric X is the Liouville volume of $\mathrm{UT}(X)$ normalized so that $i(\mathcal{L}_X, \gamma) = \ell_X(\gamma)$ for any closed geodesic γ.

A *sequence/family of random geodesics* is a sequence $\gamma_{n\in\mathbb{N}}$/family $\gamma_{t\in\mathbb{R}}$ of closed geodesics if, after normalization, the geodesic currents corresponding to γ_n/to γ_t converge in the weak*-topology to the Liouville current, i.e.

$$\lim_{n\to\infty} \frac{4\pi^2|\chi(S)|}{\ell(\gamma_n)} \int_{\mathrm{UT}(X)} f\, d\gamma_n \to \int_{\mathrm{UT}(X)} f\, d\mathcal{L}_X$$

for any continuous and compactly supported function $f \in C_c(\mathrm{UT}(X))$.

Let $\mathcal{P}$ be a geodesic pants decomposition of $X \cong S_{g,k}$. Pick a pant $P^j \in \mathcal{P}$ and let $\{o_i^j\}_{i\in\mathbb{N}}$ be the collection of orthogeodesics in P^j. We then define $U_i^j \subseteq \mathrm{UT}(X)$ to be the set of direction giving arcs homotopic to o_i^j in P^j. Then, we have the following decomposition:

$$\mathrm{UT}(X) = \bigcup_j \mathrm{UT}(P^j) \stackrel{m}{=} \bigcup_{i,j} U_i^j,$$

where $\stackrel{m}{=}$ means equality up to a measure zero set.

Consider the unit tangent bundle $\mathrm{UT}(X)$ and pick a vector $v \in \mathrm{UT}(X)$. Let g_t be the geodesic flow and r_θ, r_φ be rotation of angle θ and φ respectively, $\theta, \varphi \in [-\pi, \pi]$. Then, we get coordinates (θ, t, φ) in a neighbourhood of v by assigning $z = r_\varphi \circ g_t \circ r_\theta(v)$.

These coordinates correspond to the KAK-decomposition of $G = PSL(2, \mathbb{R})$. Indeed, let us fix the unit tangent vector $v_0 := (i, i) \in \mathrm{UT}(\mathbb{H}^2)$ in the upper half-plane model of $\mathbb{H}^2$. We can identify $\mathrm{UT}(\mathbb{H}^2)$ and G via the diffeomorphism $\psi\colon G \to \mathrm{UT}(\mathbb{H}^2), g \mapsto dg(v_0)$. Let us denote by

$$K := \{k_\theta \,|\, \theta \in [0, 2\pi]\} \cong SO(2, \mathbb{R}), \qquad k_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

the stabilizer subgroup of i, and by

$$A := \{a_t \,|\, t \in \mathbb{R}\}, \qquad a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

the diagonal subgroup. Every element of $PSL(2, \mathbb{R})$ admits a KAK-decomposition, i.e. for every $g \in G$ there are $\theta, \varphi \in [0, 2\pi]$ and $t \in \mathbb{R}$ such that $g = k_\theta a_t k_\varphi$.

It is straight-forward to check that the rotation r_φ about an angle $\varphi \in [0, 2\pi]$ and the geodesic flow g_t in $\mathrm{UT}(\mathbb{H}^2)$ for some time $t \in \mathbb{R}$ correspond via ψ to the right-multiplication by k_φ and a_t in G, respectively, i.e.

$$r_\varphi(\psi(g)) = \psi(gk_\varphi), \quad g_t(\psi(g)) = \psi(ga_t)$$

for every $g \in G$, $\varphi \in [0, 2\pi]$, $t \in \mathbb{R}$. Thus, for every vector $v \in \mathrm{UT}(\mathbb{H}^2)$ there is $g = k_\theta a_t k_\varphi \in PSL(2, \mathbb{R})$, $\theta, \varphi \in [0, 2\pi], t \in \mathbb{R}$, such that

$$v = \psi(g) = \psi(k_\theta a_t k_\varphi) = r_\varphi \circ g_t \circ r_\theta(v_0).$$

The global coordinates $(\varphi, t, \theta) \mapsto r_\varphi \circ g_t \circ r_\theta(v_0)$ on $\mathrm{UT}(\mathbb{H}^2)$ then produce local coordinates on $\mathrm{UT}(X) \cong \Gamma \backslash \mathrm{UT}(\mathbb{H}^2) \cong \Gamma \backslash G$ via the covering map $\pi \colon \mathbb{H}^2 \to X = \Gamma \backslash \mathbb{H}^2$.

2.5. **Counting arcs.** Let P be a hyperbolic pair of pants with totally geodesic boundary components. Given two (possibly the same) boundary components $C_-, C_+ \subseteq \partial P$ one may consider arcs starting on C_- and ending on C_+. In each relative homotopy class of such an arc, where we allow each endpoint to glide on the respecitve boundary component, there is a unique geodesic arc. It meets C_- and C_+ perpendicularly and is called an *orthogeodesic arc*.

One may now ask how many orthogeodesics running from C_- to C_+ of length $\leq \ell$ there are. Let us denote by $N_{C_-,C_+}(\ell)$ the number of such orthogeodesic arcs. In [PP16] Parkkonen and Paulin consider the same counting problem in the more general setting of pinched negatively curved manifolds and properly immersed closed locally convex subsets C_-, C_+. Applying their result [PP16, Theorem 1] to our situation we obtain the following corollary:

Corollary 2.1. Let $0 < \delta < 1$ denote the Hausdorff dimension of the limit set of P. Then there is a constant $C_0 > 0$ such that asymptotically

$$N_{C_-,C_+}(\ell) \sim C_0 \cdot e^{\delta \ell}$$

as $\ell \to \infty$.

We want to point out that similar counting results in varying generality were obtained before by different authors; see [PP13] and the references therein.

2.6. **Sobolev Norms.** Denote $G := PSL(2, \mathbb{R})$ and let $\Gamma \leq G$ be a cocompact lattice.

Let $d_G \colon G \times G \to \mathbb{R}_{\geq 0}$ be a left-invariant metric on G. This metric descends to a metric on the quotient $d_{\Gamma \backslash G} \colon \Gamma \backslash G \times \Gamma \backslash G \to \mathbb{R}_{\geq 0}$. We may assume that d_G is induced by a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on G. In particular, we can equip G with such a metric via the identification $G \cong \mathrm{UT}(\mathbb{H}^2)$. We denote by ν the corresponding (bi-)invariant Haar measure on G.

The left-action of Γ on G amounts to a quotient map $\pi \colon G \to \Gamma \backslash G, g \mapsto \Gamma g$. Because the action $\Gamma \curvearrowright G$ is isometric the metric d_G descends to a metric $d_{\Gamma \backslash G} \colon \Gamma \backslash G \times \Gamma \backslash G \to \mathbb{R}_{\geq 0}$, and we obtain a Riemannian metric on the quotient, which we shall denote by $\langle \cdot, \cdot \rangle$ as well, such that $\pi \colon G \to \Gamma \backslash G$ is a Riemannian covering map. The corresponding volume form amounts to a right-invariant quotient measure μ on $\Gamma \backslash G$. After possibly rescaling we may assume that μ is a probability measure. In this way μ coincides with the normalized Liouville measure on $\mathrm{UT}(\Gamma\backslash\mathbb{H}^2)$ via the usual identification $\Gamma\backslash G \cong \mathrm{UT}(\Gamma\backslash\mathbb{H}^2)$.

Note that the left-action

$$G \times \Gamma\backslash G \to G,$$
$$(g, \Gamma h) \mapsto g \cdot \Gamma h := \Gamma h g^{-1},$$

is probability measure preserving by right-invariance of the quotient probability measure μ. Thus the regular representation $\lambda\colon G \to \mathcal{U}(L^2(\Gamma\backslash G))$ is unitary, where we denote

$$(\lambda_g f)(\Gamma h) = f(g^{-1} \cdot \Gamma h) = f(\Gamma h g)$$

for every $g \in G$, $f \in L^2(\Gamma\backslash G, \mu)$.

More generally, whenever there is a smooth G-action $G \curvearrowright M$ on a smooth manifold M (e.g. $M = G$ or $M = \Gamma\backslash G$), there is an induced action of the universal envelopping algebra $\mathcal{U}(\mathfrak{g})$ on the space of smooth functions with compact support $C_c^\infty(M)$. This action is given via differentiation of the left regular representation $\lambda\colon G \to C_c^\infty(M)$ as follows

$$(X \cdot \varphi)(x) := \left.\frac{d}{dt}\right|_{t=0} (\lambda_{\exp(tX)}\varphi)(x) = \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp(-tX) \cdot x)$$

for all $X \in \mathfrak{g}$, $x \in M$, $\varphi \in C_c^\infty(M)$.

We interpret the Riemannian metric $\langle \cdot, \cdot \rangle$ as an inner product on the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ (of left-invariant vector fields), and pick an orthonormal basis E_1, E_2, E_3. Given a multi-index $\alpha = (i_1, \ldots, i_d) \in \{1, 2, 3\}^d$ of degree $|\alpha| := d$ we may define

$$E_\alpha \cdot \varphi := E_{i_1} E_{i_2} \cdots E_{i_d} \cdot \varphi$$

for every $\varphi \in C_c^\infty(M)$. We use the convention $\alpha = \emptyset$ iff $|\alpha| = 0$, and define $E_\emptyset \cdot \varphi = \varphi$.

This allows us to define the *(degree d) Sobolev norm*

$$\mathcal{S}_d(\varphi) := \sum_{0 \le |\alpha| \le d} \|E_\alpha \cdot \varphi\|_2$$

for all $\varphi \in C_c^\infty(\Gamma\backslash G)$. The *(degree d) Sobolev space* $H^d(\Gamma\backslash G)$ is by definition the completion of $C_c^\infty(\Gamma\backslash G)$ with respect to $\mathcal{S}_d$.

The following version of the Sobolev Embedding Theorem applies.

Theorem 2.2 ([Aub98, Theorem 2.10]). If $(d - r)/3 > 1/2$ then $H^d(\Gamma\backslash G) \subseteq C^r(\Gamma\backslash G)$ and the identity operator is continuous. Here $r \ge 0$ is an integer and $C^r(\Gamma\backslash G)$ is the space of C^r-functions with norm $\|\varphi\|_{C^r} := \max_{0 \le |\alpha| \le r} \|E_\alpha \cdot \varphi\|_\infty$, $\varphi \in C^r(\Gamma\backslash G)$.

Corollary 2.3. In particular, if the degree $d = 3$ and $r = 1$ then there is a *Sobolev constant* $K_{\mathrm{Sob}} > 0$ such that

$$\|\varphi\|_\infty \le \|\varphi\|_{C^1} \le K_{\mathrm{Sob}} \cdot \mathcal{S}(\varphi)$$

for all $\varphi \in H^3(\Gamma\backslash G)$, where we dropped the degree $d = 3$ in $\mathcal{S}(\varphi) = \mathcal{S}_3(\varphi)$.

Recall that $L^1(G)$ is a Banach algebra when we define multiplication by convolution

$$\begin{aligned}(f_1 * f_2)(g) &:= \int_G f_1(h) \cdot f_2(h^{-1}g)\, d\nu(h) \\ &= \int_G f_1(gh) \cdot f_2(h^{-1})\, d\nu(h) \qquad \forall g \in G \quad \forall f_1, f_2 \in L^1(G)\end{aligned}$$

There is a Banach algebra action of $L^1(G)$ on $L^2(\Gamma\backslash G)$ given by convolution

$$\begin{aligned}(\psi * f)(\Gamma g) &:= \int_G \psi(h) \cdot (\lambda_{h^{-1}} f)(\Gamma g)\, d\nu(h) \\ &= \int_G \psi(h) \cdot f(\Gamma g h)\, d\nu(h),\end{aligned}$$

for all $\Gamma g \in \Gamma\backslash G, \psi \in L^1(G), f \in L^2(\Gamma\backslash G)$. An application of Fubini shows that

$$\|\psi * f\|_2 \leq \|\psi\|_1 \cdot \|f\|_2.$$

Moreover, we have the following Lemma familiar from the situation in $\mathbb{R}^n$.

Lemma 2.4. Let $\varepsilon > 0$, let $\psi \in C_c^\infty(G)$, and let $f \in L^2(\Gamma\backslash G)$.
Then:

(1) $\psi * f$ is smooth;
(2) $E_\alpha \cdot (\psi * f) = (E_\alpha \cdot \psi) * f$ for all multi-indices α;
(3) $\mathrm{supp}(\psi * f) \subseteq N_\varepsilon(\mathrm{supp}(f))$, if $\mathrm{supp}(\psi) \subseteq B_\varepsilon(e)$.

Proof. (1) This will follow from (2).

(2) By induction on $|\alpha|$ it is enough to show this for $|\alpha| = 1$. Let $E := E_i$ be a basis vector. We compute:

$$\begin{aligned}(E \cdot (\psi * f))(\Gamma g) &= \frac{d}{dt}\bigg|_{t=0} \int_G \psi(h) \cdot f(\Gamma g \exp(tE) h)\, d\nu(h) \\ &= \frac{d}{dt}\bigg|_{t=0} \int_G \psi(\exp(-tE)h) \cdot f(\Gamma g h)\, d\nu(h) \\ &= \int_G (E \cdot \psi)(h) \cdot f(\Gamma g h)\, d\nu(h) \\ &= ((E \cdot \psi) * f)(\Gamma g)\end{aligned}$$

(3) Recall that

$$(\psi * f)(\Gamma g) = \int_G \psi(h) \cdot f(\Gamma g h)\, d\nu(h).$$

Thus, if $(\psi * f)(\Gamma g) \neq 0$ then there is $h \in \operatorname{supp}(\psi) \subseteq B_\varepsilon(e)$ such that $\Gamma g h = h^{-1} \cdot \Gamma g \in \operatorname{supp}(f)$. Hence,

$$d_{\Gamma\backslash G}(\Gamma g h, \Gamma g) \leq d_G(gh, g) = d_G(h, e) < \varepsilon.$$

This shows that $\Gamma g \in N_\varepsilon(\operatorname{supp}(f))$.

■

3. Outline of the Main theorem

We now outline the construction and proof of the main result where we fix a closed hyperbolic surface X with an isometric geodesic pant decomposition $\mathcal{P}$. Let $\gamma_v(t)$ denote a random geodesic ray starting at $v \in \mathrm{UT}(X)$, parameterised by arc length, and flowing for time t so that $d(g_t(v), v) < \varepsilon$. Let γ_t denote the *closure* of γ_v^t, i.e. the broken geodesic path obtained by following γ_v^t, and then closing the geodesic ray off by a short geodesic arc. Let $\widehat{\gamma_t}$ be the corresponding geodesic representative and let $\widehat{\gamma_t^p}$ be the associated primitive sub-curve of $\widehat{\gamma_t}$.

Sketch of the Main Theorem. (1) For any $\eta > 1$, Section 6 counts the number of regions visited by a ray of length t. It is a sub-linear function

$$F(t) := \left(\frac{\frac{2\beta}{3\eta} t}{W\left(\frac{2\beta}{3\eta} t\right)} \right)^{\delta/2\eta}$$

for a positive constant β and $0 < \delta < 1$.

(2) Section 4 constructs the random loop γ_t and by pulling it tight we obtain a random geodesic representative $\widehat{\gamma_t}$. We will also show that the homotopy has bounded height $K \approx 1.3$, and we have

$$\ell(\widehat{\gamma_t}) \leq \ell(\gamma_t) \leq t + \varepsilon.$$

Moreover, we have that the primitive geodesic representative $\widehat{\gamma_t^p}$ satisfies

$$\ell(\widehat{\gamma_t^p}) \leq \ell(\widehat{\gamma_t}).$$

(3) Section 5 shows that since $\widehat{\gamma_t}$ and γ_t are related by a bounded homotopy and γ_t is nearly in general position with respect to $\mathcal{P}$, we have that

$$\#\{\gamma_t - \text{arcs}\} - 2 \leq \#\{\widehat{\gamma_t} - \text{arcs}\} = \#\{\widehat{\gamma_t^p} - \text{arcs}\}.$$

(4) By (3) and Section 5, we obtain that asymptotically

$$\frac{v_3}{2} F(t) - v_3 \leq \frac{v_3}{2} \#\{\gamma_t - \text{arcs}\} - v_3 \leq \frac{v_3}{2} \#\{\widehat{\gamma_t} - \text{arcs}\} = \frac{v_3}{2} \#\{\widehat{\gamma_t^p} - \text{arcs}\} \leq \mathrm{Vol}(M_{\widehat{\gamma_t^p}}).$$

(5) We also have $\ell_X(\widehat{\gamma_v}) \leq t + \varepsilon = t + \frac{1}{t}$, hence there exists $B > 0$ such that:

$$\frac{v_3}{2}F(t+\varepsilon) - B \leq \frac{v_3}{2}F(t) - \frac{v_3}{2},$$

so that for large t we have:

$$\frac{v_3}{2}F(\ell_X(\widehat{\gamma_t})) - B \leq \frac{v_3}{2}F(t+\varepsilon) - B \leq \frac{v_3}{2}F(t) - v_3 \leq \mathrm{Vol}(M_{\widehat{\gamma}_t^p}).$$

(6) Putting all the steps together, we obtain

$$A \cdot \left(\frac{C \cdot \ell_X(\widehat{\gamma}_v^p)}{W\left(C \cdot \ell_X(\widehat{\gamma}_v^p)\right)} \right)^{\delta/2\eta} - B \leq \mathrm{Vol}(M_{\widehat{\gamma}_t^p})$$

for positive constants $A, B, C, 0 < \delta < 1$, and $\eta > 1$.

■

4. Random model

In this section we construct our random model and prove some preliminary facts about the random geodesics we will consider. We will use $\pi : \mathrm{UT}(X) \to X$ to denote the bundle map over a hyperbolic surface X.

The following construction is based on an argument in [Sap17, Claim 2.3].

Definition 4.1. Given $v \in \mathrm{UT}(X)$ and $\varepsilon > 0$ smaller than the injectivity radius of X, we define the set $N(v,\varepsilon)$ of vectors $w = r_\varphi(g_t(r_\theta(v)))$ such that in these coordinates:

$$|\theta|, |\varphi| \leq \frac{\varepsilon}{2}, 0 \leq t \leq \frac{\varepsilon}{2}.$$

Note that $N(v,\varepsilon) \subseteq B_\varepsilon(v)$, has positive measure and v is the "centre" $(0,0,0)$ of $N(v,\varepsilon)$.

Proposition 4.1. For μ-almost every $v \in \mathrm{UT}(X)$, and every $\varepsilon > 0$, there exist a sequence $(t_i)_{n\in\mathbb{N}}$ of times $t_1 < t_2 < \cdots$ (depending on v and ε) so that for each i, we have a closed loop $\gamma_v(t_i)$ consisting of a concatenation of

- a geodesic arc $g_{[0,t_i]}$ so that $g_{t_i}(v) \in N(v,\varepsilon)$ and
- an arc $\alpha_i \subseteq \pi(B_\varepsilon(v))$
- $\gamma_v(t_i)$ has angle defects smaller than ε

Proof. Up to a set of measure zero $Z \subseteq \mathrm{UT}(X)$ we have that, by Theorem 2.1, for every $\varepsilon > 0$ and for every $v \in M := \mathrm{UT}(X) \setminus Z$ the ball of radius ε, around v returns infinitely often under the geodesic flow g_t, that is, there exists $t_i \in \mathbb{R}$ so that $g_{t_i}(v) \in N(v,\varepsilon)$.

In the neighbourhood $N(v,\varepsilon)$ we let (θ, s, φ) be the coordinates for $w = g_{t_i}(v)$. Then, we can connect $\pi(v)$ to $\pi(w)$ with the geodesic arc α whose lift to the unit tangent bundle is given by:

$$\dot{\alpha}(h) = g_h(r_\theta(v)), \qquad h \in [0,s].$$

Note that $\alpha(0) = \pi(v)$ has angle defect $|\theta| \leq \frac{\varepsilon}{2}$ and $\alpha(s) = \pi(w)$ has angle defect $|\theta - \varphi| \leq \varepsilon$. ∎

In the rest of this section we will show that the curves $\gamma_v(t_i)$

- $\gamma_v(t_i)$ are filling, and
- they can be homotoped to their geodesic representative by a short homotopy.

Figure 4.1 shows a sketch of the random closed geodesic around $\pi(v)$.

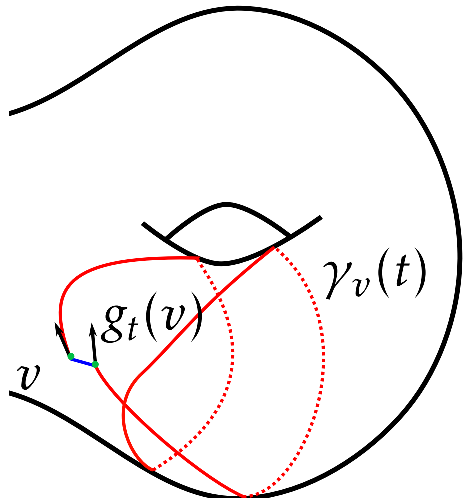

FIGURE 4.1. Sketch of the random closed geodesic.

Remark 4.2. Because periodic orbits of the geodesic flow are dense (see [Dal11, Theorem 3.3]) we have that, in general, the loop $\gamma_v(t)$ will not be primitive. This is because we can find vectors w arbitrarily close to a periodic orbit β, i.e. they track the period orbit for long time t. Thus, if β is filling we can obtain arbitrarily long loops $\gamma_w(t)$ that are homotopic to arbitrarily high powers of β.

Now we will show that our random closed geodesics have small defects, which will play a role in comparing homotopy classes of arcs of $\gamma_v(t)$ to those of its geodesic representative (see Section 5).

Remark 4.3. Consider the arc α as in Proposition 4.1 and let $w := g_t(v)$. Moreover, let v' and w' be the tangent vector at the endpoints of α corresponding to $\pi(v)$ and $\pi(w)$, respectively. Then, note that by the construction in Proposition 4.1 we have that:

$$d(v, v'), d(w, w') < \varepsilon.$$

Definition 4.4. Given a homotopy $H(x,t)$ from γ to $\widehat{\gamma}$ we say that *the height of H is:*

$$\sup_{x\in\gamma}\{\text{length}(H(x,\cdot))\}.$$

In the following Lemma we show that our random loop has a homotopy to its geodesic representative of uniformly bounded height.

Lemma 4.5. If $\ell_X(\alpha) < \varepsilon$, $d(v, v'), d(w, w') < \varepsilon$, and $t > 2\cosh^{-1}\left(\frac{2\pi}{\pi-2\varepsilon}\right)$ there exists a homotopy H from $\gamma_v(t)$ to $\widehat{\gamma}_v(t)$ whose height is at most $K(\varepsilon) := \cosh^{-1}\left(\frac{2\pi}{\pi-2\varepsilon}\right) + 2\varepsilon$.

Proof. For ease of notation we will drop the v from the loop. In the universal cover we have the configuration in Figure 4.2.

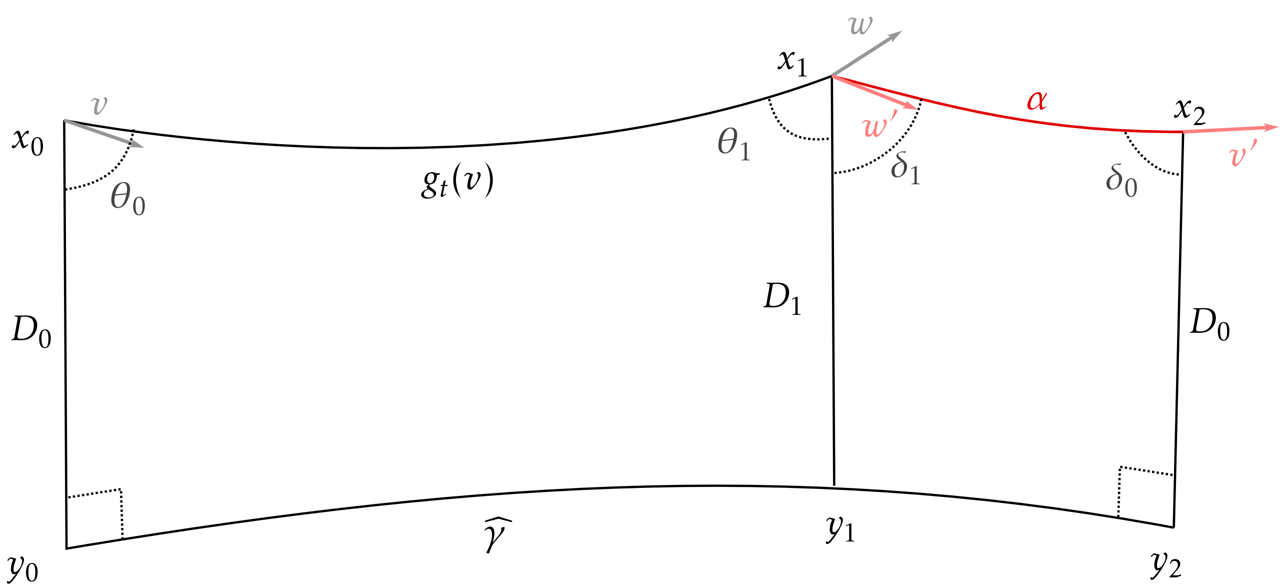

FIGURE 4.2. Lift in the universal cover.

We want to show that the nearest point projection homotopy satisfies our conditions. By convexity of the length function it suffices to show that D_0, D_1 satisfy the required bound. Therefore, we want to show that:

$$\max\{D_0, D_1\} \leq \cosh^{-1}\left(\frac{2\pi}{\pi - 2\varepsilon}\right) + 2\varepsilon$$

Observe that the other possibility for Figure 4.2 is that x_0, x_1 appear on opposite sides of $\widehat{\gamma}$. In that case from $\ell_X(\alpha) < \varepsilon$ is follows that $D_0, D_1 < 2\varepsilon$. Hence, we concentrate on Figure 4.2.

Let $A \subseteq X$ be the annulus with boundary $\widehat{\gamma}$ and γ whose interior angles satisfy:

$$\delta_i + \theta_i \geq \pi - \varepsilon, \qquad i = 0, 1.$$

Since the sum of angles of the convex quadrilateral $[x_1, x_2, y_2, y_1]$ is less than 2π, it follows that $\delta_0 + \delta_1 < \pi$. Hence $\theta_0 + \theta_1 > \pi - 2\varepsilon$, from which we know that at least one of θ_i is larger than $\frac{\pi}{2} - \varepsilon$.

Since the orthogonal projection is 1-Lipschitz, from the geodesic quadrilateral $[x_1, y_1, y_2, x_2]$ we have that:

$$D_1 < D_0 + 2\varepsilon \qquad D_0 < D_1 + 2\varepsilon$$

giving us:

$$D_0 - 2\varepsilon < D_1 < D_0 + 2\varepsilon.$$

Assume that, by the claim, we have $\theta_0 \geq \frac{\pi}{2} - \varepsilon$. We will now show that D_0 is short, which by the above inequality will give us the desired result.

Case 1: Assume that $D_0 > \frac{t}{2}$. Let S be the hyperbolic sector of radius $\frac{t}{2}$ centered at x_0 in $A = [x_0, y_0, x_1, x_1]$. Since the angles at y_0, y_1 are orthogonal, it follows that $S \subseteq A$. Then, we have that:

$$\mathrm{Area}(A) \geq \ \mathrm{Area}(S) = 2\theta_0 \sinh^2(t/4)$$

and:

$$\mathrm{Area}(A) \leq \pi - (\theta_0 + \theta_1) - (\delta_0 - \delta_1) \leq \pi - \theta_0$$

and we assumed that $\theta_0 \geq \frac{\pi}{2} - 2\varepsilon$. Therefore, we have:

$$\begin{aligned} 2\theta_0 \sinh^2(t/4) &\leq \pi - \theta_0 \\ \theta_0(1 + 2\sinh^2(t/4)) &\leq \pi \\ \cosh(t/2) &\leq \frac{\pi}{\theta_0} \leq \frac{2\pi}{\pi - 2\varepsilon} \\ \implies t &\leq 2\cosh^{-1}\left(\frac{2\pi}{\pi - 2\varepsilon}\right) \end{aligned}$$

contradicting our hypothesis.

Case 2: Assume that $D_0 \leq \frac{t}{2}$ and take an angular sector of radius D_0. By the above argument we obtain:

$$D_0 \leq \cosh^{-1}\left(\frac{2\pi}{\pi - 2\varepsilon}\right)$$

concluding the proof. ∎

Remark 4.6. Note that in Lemma 4.5 as $\varepsilon \to 0$ the height of the homotopy goes to $\cosh^{-1}(2) \approx 1.317$.

Finally, we show that our random closed geodesic converge to a filling geodesic current.

Lemma 4.7. Let $\gamma_v(t)$ be the closed geodesic coming from Proposition 4.1, so that its angular defects are at most ε. Then,

$$\lim_{t\to\infty} \frac{\gamma_v(t)}{\ell_X(\gamma_v(t))} \to \mathcal{L}_X$$

as geodesic currents, where $\mathcal{L}_X$ denotes the Liouville current associated to the hyperbolic structure X (see definitions in Subsection 2.4)

Proof. By the Birkhoff Ergodic Theorem 2.2 it suffices to prove that for every continuous function $f\colon \mathrm{UT}(X) \to \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{\ell_X(\gamma_v(t))} \int_{\mathrm{UT}(X)} f\, d\gamma_v(t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(g_s(v))ds.$$

Take a transversal D to the leaf of the geodesic foliation the current $\gamma_v(t)/\ell_X(g_v(t))$ is supported on, intersecting at points $x_1(t), \cdots, x_k(t)$. Let $y_1(t), \cdots, y_k(t)$ be the intersection points of the leaf corresponding to $g_s(v)$ for $s \in (-\infty, \infty)$. We claim that for t long enough, $y_i(t)$ and $x_i(t)$ are close. One way to see this is to look at the endpoints at infinity in the bands model, and note that the endpoints of the closed broken geodesic $\gamma_v(t)$ and the corresponding closed random geodesic are the same, and are in a neighborhood of the boundary at infinity of the endpoints of $\gamma_v(t)$ of radius ε that goes to 0 as t goes to infinity. ∎

Lemma 4.8. For i large enough, the curve $\gamma_v(t_i)$ is filling.

Proof. For a geodesic current μ, let $\mathrm{sys} := \inf\{i(\mu, c) : c \text{ closed curve }\}$ be the systole of μ. By Lemma 4.7, $\lim_{t\to\infty} \gamma_v(t)/\ell(\gamma_v(t)) \to \mathcal{L}_X$. Since $\mathcal{L}_X$ is filling, by [BIPP19, Theorem 1.3]), we have $\mathrm{sys}(\mathcal{L}_X) > 0$. The result then follows since $\mathrm{sys}\colon \mathcal{GC}(S) \to \mathbb{R}_{\geq 0}$ is a continuous function on currents ([BIPP19, Corollary 1.5]). ∎

For μ-almost every $v \in \mathrm{UT}(X)$, we will need to take t_i large enough so that $\gamma_v(t_i)$ is a filling closed geodesic. We will record this time.

Definition 4.2 (Filling time)**.** Let $t^*(v)$ be the smallest t_i for which $\gamma_v(t_i)$ is filling. Without loss of generality we can assume that $t^*(v) > 2\cosh^{-1}\left(\frac{2\pi}{\pi - 2\varepsilon}\right)$.

We now describe how we build our model $\gamma_v(t)$ for $t \geq t^*(v)$. By Proposition 4.1 and Lemma 4.8 we can assume that we have a sequence of times $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i < t_{i+1}$ and $t^*(v) \leq t_1$ such that the closed up loops $\gamma_v(t_i)$ obtained by flowing by time t_i in direction v are:

- filling;
- they are made by two geodesic arcs with angular defects less than ε.

Definition 4.9 (Random geodesic)**.** Then, we define $\gamma_v(t)$, for $t \geq t^*(v)$, by:

$$\gamma_v(t) := \gamma_v(t_i),$$

where $t_i = \max_{t_j < t} t_j$.

Definition 4.10. Given $\gamma_v(t)$ as above we define $\widehat{\gamma_v}(t)$ to be the geodesic representative of $\gamma_v(t)$ and we let $\widehat{\gamma_v}(t)^p$ be the corresponding primitive geodesic representative.

We conclude the section by observing that, by ergodicity, it follows that the volume of our random model goes to infinity.

Lemma 4.11. For μ-almost every $v \in \mathrm{UT}(X)$ we have that $\mathrm{Vol}(\mathrm{UT}(X)\setminus\widehat{\gamma_t(v)}) \to \infty$ as $t \to \infty$.

Proof. If not, by [RM20]'s lower bound this is equivalent to saying that we have $v \in \mathrm{UT}(X)$ such that for t large enough $\gamma_t(v)$ is contained in finitely many U_i^j's. Let U be a U_i^j that $\gamma_t(v)$ never visits. Then, we have that:

$$\int_0^t \chi_U(g_t(v))dt = 0$$

for all $t \in \mathbb{R}$. Thus, by the Birkhoff Ergodic Theorem 2.2 we have that:

$$\mathrm{vol}(U) = 0,$$

which is not true. ■

The main goal now will be to use the stronger mixing properties of the geodesic flow to quantify the asymptotic growth of the volume.

5. Arcs of closure vs arcs of geodesic representative

The aim of this section is to show that given a geodesic pants decomposition $\mathcal{P}$ on $X \cong S_{g,n}$ and our random curve γ_v the number of homotopy classes of arcs does not drop too much after we pull tight to $\widehat{\gamma_v}$. Also note that arcs of $\widehat{\gamma_v}^p$, the primitive representative, are the same as arcs of $\widehat{\gamma_v}$.

Definition 5.1. Given a pants decomposition $\mathcal{P}$ on a hyperbolic surface X and loop γ we define $A_{\mathcal{P}}(\gamma)$ to be:

$$A_{\mathcal{P}}(\gamma) := \sum_{Q \in \mathcal{P}} A_Q(\gamma)$$

where $A_Q(\gamma)$ is the number of free homotopy classes of essential arcs $\gamma \cap Q$, for Q a pant in $\mathcal{P}$.

We now want to study how $A_{\mathcal{P}}$ changes if we go from a loop α to a loop β via a homotopy with short height. Obviously, if one allows arbitrarily long homotopies by point-pushing one can create an arc $\beta \simeq \alpha$ such that $A_{\mathcal{P}}(\beta) = A_{\mathcal{P}}(\alpha) + k$ for k arbitrary, see Figure 5.1.

Note that in the previous figure, because of the point-push, we introduce many bigons with curves of $\partial\mathcal{P}$. This observation will be key in our proof.

Definition 5.2. Let $\alpha, \beta \colon \mathbb{S}^1 \to X$ be homotopic loops via a homotopy $H \colon \mathbb{S}^1 \times [0,1] \to X$ and let $\gamma = \alpha([\theta_0, \theta_1])$ be an arc component of $\alpha \cap Q_i$, for $Q_i \in \mathcal{P}$. Then, we define γ' to be the sub-arc of β obtained by taking the union of:

- $H([\theta_0, \theta_1], 1) \cap Q_i$;
- all components of $H(\mathbb{S}^1, 1) \cap Q_i^C$ that are properly embedded arcs with both end-points on ∂Q_i;

and extending, if needed, along β so that $\partial\gamma' \subseteq \partial Q_i$, see Figure 5.2 for an example.

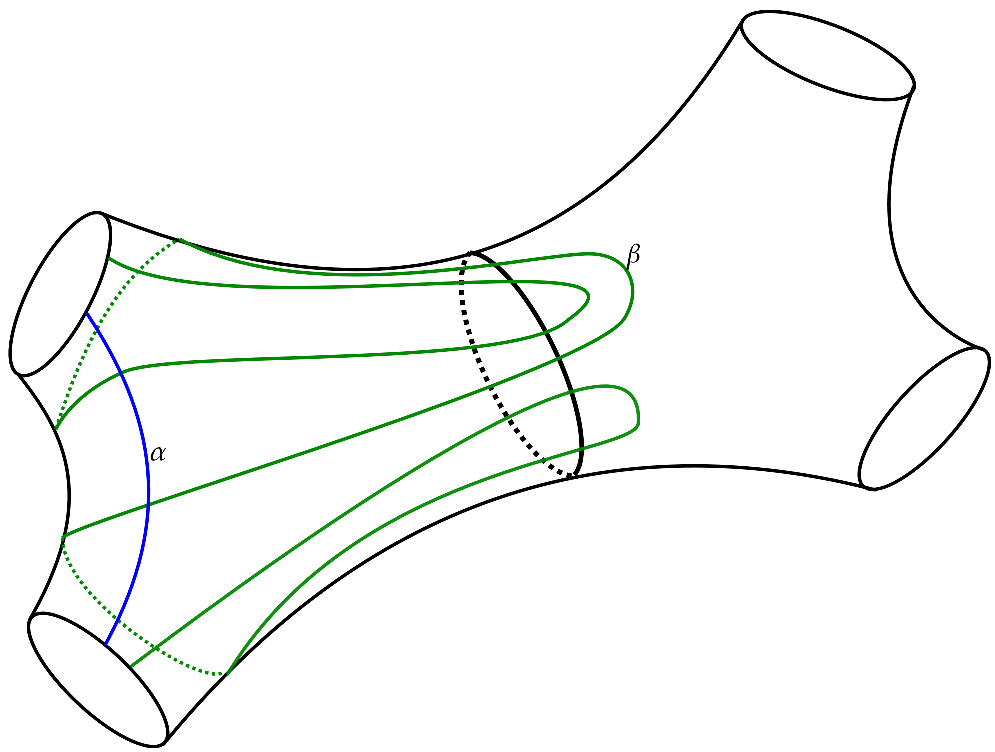

Figure 5.1. An example of how to point push α to an arc β that now has 3 essential arcs in the pair of pants containing α.

We now prove some technical Lemmas about short homotopies of γ-arcs.

Lemma 5.3. Let $\mathcal{P} = \{Q_i\}_{i=1}^{2g-2+n}$ be a pants decomposition of $X \cong \Sigma_{g,n}$, let o be the length of the shortest orthogeodesic on S, and let $\beta \simeq \alpha$ via a homotopy with height $K < \frac{o}{2}$. Then, if γ is an essential arc of $\alpha \cap (X \setminus \mathcal{P})$ contained in the pair of pants Q_i we have the following:

(1) for all $j \neq i$: $Q_j \cap \gamma'$ has no essential components and $Q_i \cap \gamma'$ contains at least an essential component;
(2) if $\gamma' \cap \partial Q_i$ is in minimal position we have that $\gamma' \cap Q_i$ has a unique essential arc.

Proof. We prove them separately.

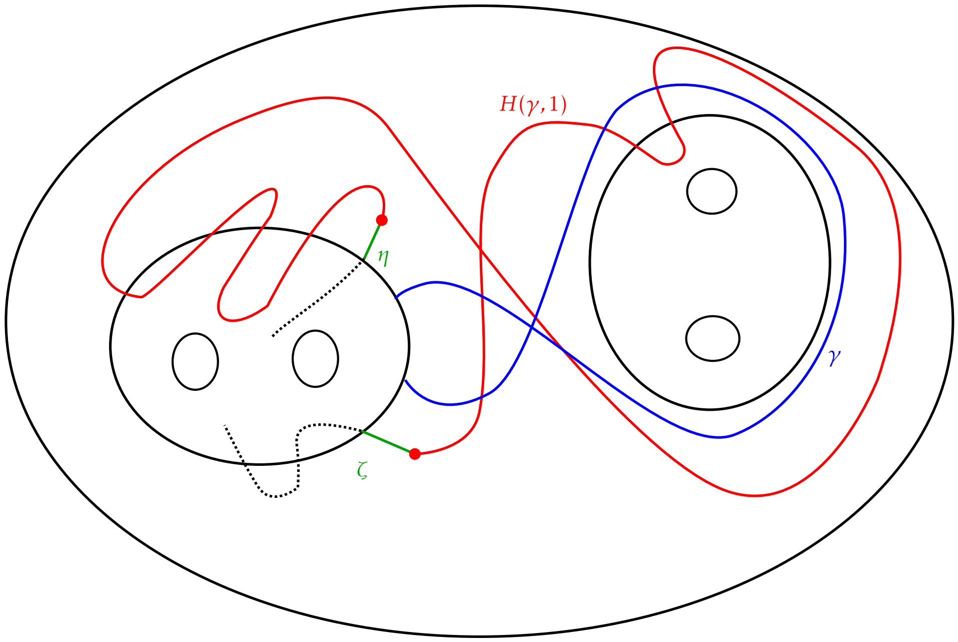

FIGURE 5.2. The arc γ' is obtained by adding η, ζ to $H(\gamma, 1)$.

(1) Assume that there exists $j \neq i$ such that that $\gamma' \cap Q_j$ contains an essential arc δ. Since $\delta \subseteq \gamma' \simeq \gamma$ by playing the homotopy in reverse we can move δ inside Q_i moving each point at most $\frac{\varrho}{2}$. However, $\ell_X(\delta) > \varrho$ giving us a contradiction.[1]
(2) Since we can homotope γ' to have only 2 points of intersection with ∂Q_i we must have that if γ' is in minimal position with ∂Q_i it also has two points of intersections and so $Q_i \cap \gamma'$ has a unique essential arc.

■

We also have the following immediate Corollary:

Corollary 5.4. With the same setup as before. If $\gamma' \cap Q_i$ has at least two essential arcs then γ' is not in minimal position with respect to ∂Q_i.

Remark 5.5. Note that having a short homotopy is not enough to prove what we need. In fact consider a pair of pants Q and label the boundaries A, B, C. Let δ_i, $i = 1, \ldots, n$ be homotopically distinct essential arcs from B to A and let ab be the unique simple arc from A to B. Now consider an arc δ obtained by:

$$\delta := ab \prod_{i=1}^{n} B^m \delta_i A^m ab$$

where m is high enough so that each time we wind m-times around A or B we have a point that is ε close to A or B respectively. Then, by point pushing those points through A or B

[1]The condition that $2K < \varrho$ gives us that each cuff has a collar of length $\frac{\varrho}{2}$.

respectively we decompose, up to free homotopy, δ into the arcs: $ab, \delta_1, \dots, \delta_n$ giving us a small homotopy generating $n+1$ essential arcs from one. However, any such homotopy generates $2n$-bigons between $\delta' \simeq \delta$ and $\partial\mathcal{P}$. Thus, the image of the homotopy cannot be nearly geodesic. Moreover, by taking m high enough and very small ε you can make this construction so that δ' is a (K,R)-quasi-geodesics with constants arbitrarily good, i.e. $K \approx 1$ and $R \approx 0$.

Lemma 5.6. Consider the loop γ_v obtained as in Proposition 4.1 and let $\mathcal{P}$ be a geodesic pants decomposition of X. Then, there exists at most one monogon/bigon between γ_v and $\mathcal{P}$.

Proof. Since γ_v is geodesic everywhere outside the shortcut α^ε we must have that if γ_v and $\mathcal{P}$ are not in minimal position there exists a bigon or a monogon containing α^ε, see [HS94]. The monogon itself must have *feets* on $\mathcal{P}$ so that it can be pushed out to reduce $\iota(\mathcal{P}, \gamma_v)$, see Figure 5.3. Moreover, since $\mathcal{P}$ is simple we must have that the bigon/monogons are embedded. See figure 5.3.

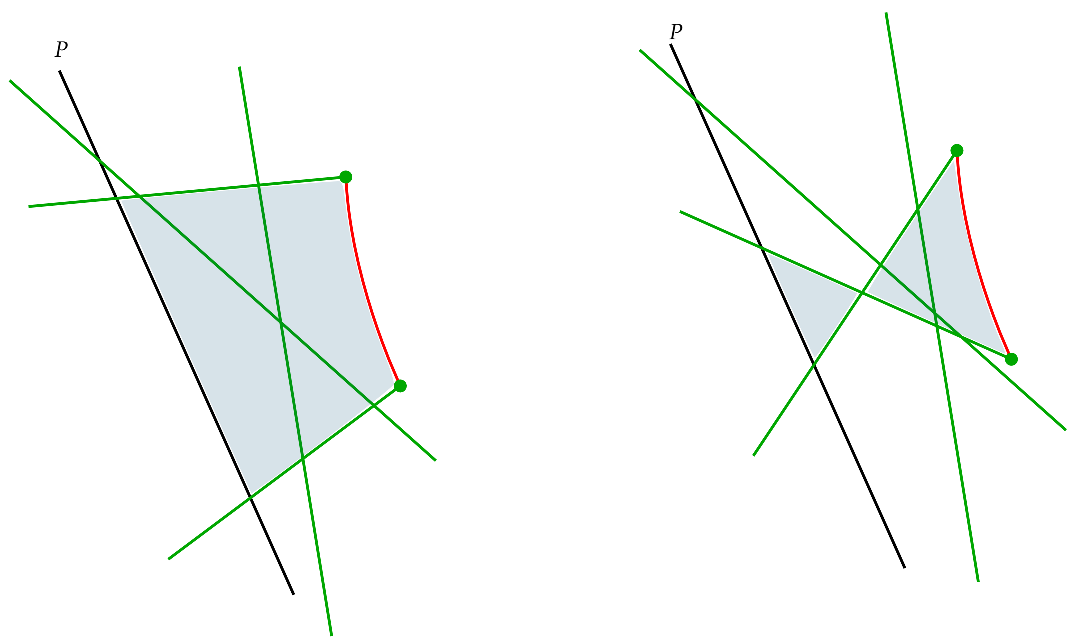

FIGURE 5.3. The bigon/monogons are dashed in blue. The geodesic arc $g_{[0,t]}(v)$ is in green and the geodesic shortcut α^ε is in red.

Assume we have a monogon $\mathscr{M}$ which must contain α^ε and let B be the complementary region containing α^ε. Now, B cannot be a bigon since we said bigons are embedded and ∂B has an intersection point, see figure 5.3.

If B where a monogon $\mathscr{M}'$ then $F = \mathscr{M} \cup \mathscr{M}'$, would be a piece of surface $F \subseteq X$ with boundary a component of $\mathcal{P}$ that is null-homotopic which clearly cannot happen.

We now need to rule out that we have multiple bigons. Assume we have two distinct bigons B_1 and B_2. Both must be embedded and contain α^{ε}. However, that is impossible since then we would have a pants curve α with a boundary parallel arc of γ_v bounding a disk on both sides. Therefore, there is a unique monogon/bigon. ∎

Remark 5.7. By Corollary 5.4 we have that any arc of the geodesic representative $\widehat{\gamma_v}$ that after the homotopy is not in minimal position contributes a bigon/monogon to γ_v and $\mathcal{P}$. However, γ_v and $\mathcal{P}$ have a unique monogon/bigon so we can have at most one 'bad arc'.

Thus, we have the following proposition:

Proposition 5.8. Let $\mathcal{P}$ be a pants decomposition on X and let ε, o be so that

$$\cosh^{-1}\left(\frac{2\pi}{\pi - 2\varepsilon}\right) + 2\varepsilon \leq \frac{o}{2}$$

for o the shortest orthogeodesic of $(\mathcal{P}, X)$. Then, if $\widehat{\gamma}_v^p(t)$ is the primitive representative of the geodesic representative of $\gamma_v(t)$ we have that:

$$A_{\mathcal{P}}(\gamma_v(t)) - 2 \leq A_{\mathcal{P}}(\widehat{\gamma}_v^p(t)) \leq A_{\mathcal{P}}(\gamma_v(t))$$

Proof. Since $A_{\mathcal{P}}(\widehat{\gamma}_v^p(t)) = A_{\mathcal{P}}(\widehat{\gamma_v}(t))$ it suffices to show it for the latter. Any essential arc of $\widehat{\gamma_v}(t)$ that contributes more than once after the homotopy needs to have a bigon/monogon by Corollary 5.4 in $\gamma_v(t)$. However, by Lemma 5.6 we only have one such bigon/monogon. Therefore, we have at most one arc that contributes more than once and since we have a unique bigon/monogon it must contribute at most twice and so the bound follows. ∎

6. Counting arcs of a random ray

In this section we derive our main dynamic estimate which yields a lower bound on the number or arcs appearing in a random geodesic.

6.1. **Notation.** We denote by μ the normalized Liouville (probability) measure on $\mathrm{UT}(X)$, i.e., $\mathcal{L}_X / \mathcal{L}_X(\mathrm{UT}(X))$. Furthermore, we denote by $T\colon \mathrm{UT}(X) \to \mathrm{UT}(X)$ the time-one map of the geodesic flow. Also, we will use the notation

$$\mu(f) = \int_{\mathrm{UT}(X)} f \, d\mu$$

for $f \in L^1(\mathrm{UT}(X), \mu)$. By $\mathcal{S}(\varphi)$ we will denote the Sobolev norm of a smooth function $\varphi \in C_c^{\infty}(\mathrm{UT}(X))$ as defined in section 2.6.

6.2. **Smooth approximations of orthogeodesic sets.** Recall that X is a hyperbolic surface obtained from gluing $|\chi(X)|$-many isometric pairs of pants along their cuffs without twists. In each pair of pants there are infinitely many geodesic arcs that connect any two boundary components. We consider free homotopy classes of these arcs where the endpoints are allowed to glide freely along their respective boundary component. In each of these free homotopy classes there is a unique orthogeodesic arc that meets both boundaries perpendicularly. For such an orthogeodesic arc $o \subseteq X$ we denote its length by $\ell(o)$. Now, we enumerate all orthogeodesic arcs in all pairs of pants $\{o_i\}_{i\in\mathbb{N}}$ by increasing length, i.e. $\ell(o_i) \leq \ell(o_j)$ if $i \leq j$. The length of the i-th orthogeodesic arc o_i will be denoted by $\ell_i = \ell(o_i)$, and for every orthogeodesic arc o_i we denote by P_i the pair of pants that contains it. Moreover, we consider the set B_i of all unit tangent vectors $v \in \mathrm{UT}(P_i)$ whose associated geodesic $\gamma_v \cap P_i$ is in the free homotopy class of o_i. In this way the sets $\{B_i\}_{i\in\mathbb{N}}$ are a certain enumeration of the sets $\{U_k^l\}_{k,l}$ in section 2.4.

Given such an orthogeodesic arc o_i we may lift it to an orthogeodesic arc $\widetilde{o_i}$ in the universal covering $\mathbb{H}^2$, such that $\widetilde{o_i}$ is a subarc of the imaginary axis in the upper half-plane model. Furthermore, we may assume that the lifts of its boundary components are half-circles $C_0(\ell_i)$ and $C_1(\ell_i)$ centered at 0 of radius $e^{-\ell_i/2}$ and $e^{\ell_i/2}$, respectively. Then the universal covering $\pi\colon \mathbb{H}^2 \to X$ induces a bijection between the set B_i and the set $B(\ell_i)$ of all unit tangent vectors $v \in \mathrm{UT}(\mathbb{H}^2)$, whose base point lies between $C_0(\ell_i)$ and $C_1(\ell_i)$ and whose induced geodesic $\gamma_v \subseteq \mathbb{H}^2$ intersect $C_0(\ell_i)$ and $C_1(\ell_i)$.

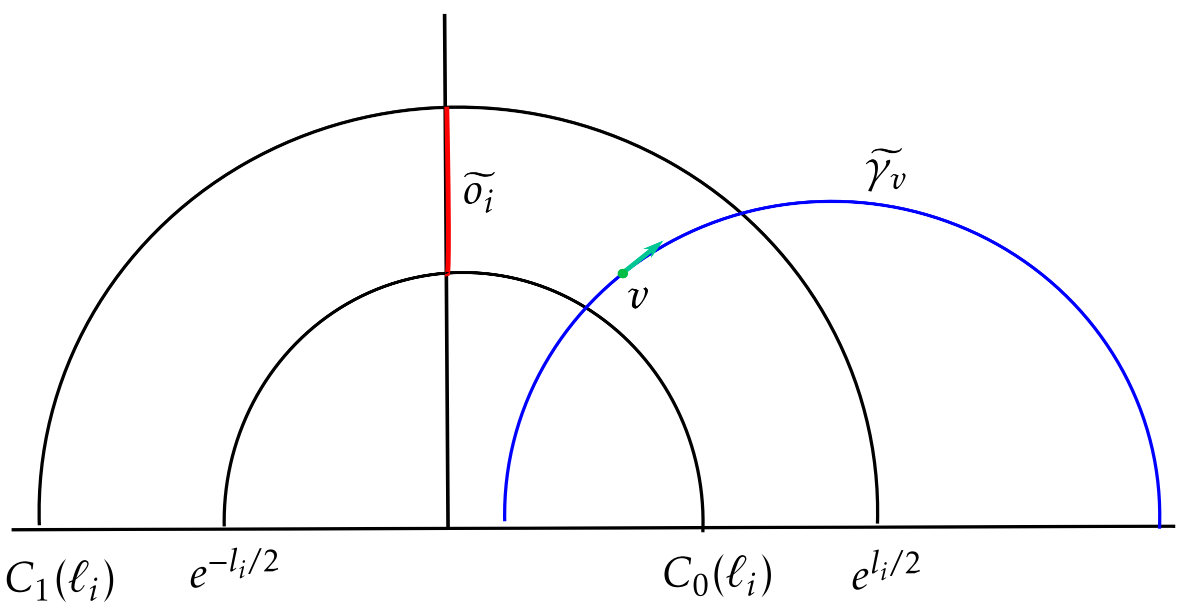

Figure 6.1. The set $B(\ell_i)$ of all unit tangent vectors $v \in \mathrm{UT}(\mathbb{H}^2)$, whose base point lies between $C_0(\ell_i)$ and $C_1(\ell_i)$ and whose induced geodesic $\gamma_v \subseteq \mathbb{H}^2$ intersect $C_0(\ell_i)$ and $C_1(\ell_i)$.

There is a formula for the (normalized) volume of $B(\ell)$:

Lemma 6.1. (1) The measure of $B(\ell)$ is given by

$$\mu(B(\ell)) = \frac{1}{2\pi|\chi(X)|}\cdot\left(Li_2\left(\frac{1}{\cosh^2\left(\frac{\ell}{2}\right)}\right)+\frac{1}{2}\log\left(\frac{1}{\cosh^2\left(\frac{\ell}{2}\right)}\right)\log\left|1-\frac{1}{\cosh^2\left(\frac{\ell}{2}\right)}\right|\right).$$

(2) Asymptotically, this yields

$$\mu(B(\ell)) \sim \frac{1}{\pi|\chi(X)|}\cdot \ell e^{-\ell}$$

as $\ell \to \infty$.

Proof. (1) This follows from Bridgeman's computations in [Bri11, Section 9].
(2) Observe that

$$\begin{aligned}\mu(B(\ell)) &= \frac{1}{2\pi|\chi(X)|}\cdot\left(Li_2\left(\frac{1}{\cosh^2\left(\frac{\ell}{2}\right)}\right)+\frac{1}{2}\log\left(\frac{1}{\cosh^2\left(\frac{\ell}{2}\right)}\right)\log\left|1-\frac{1}{\cosh^2\left(\frac{\ell}{2}\right)}\right|\right)\\ &= \frac{1}{2\pi|\chi(X)|}\cdot\left(Li_2\left(\frac{1}{\cosh^2\left(\frac{\ell}{2}\right)}\right)-\log\left(\cosh\left(\frac{\ell}{2}\right)\right)\log\left(\frac{\sinh^2\left(\frac{\ell}{2}\right)}{\cosh^2\left(\frac{\ell}{2}\right)}\right)\right)\\ &= \frac{1}{2\pi|\chi(X)|}\cdot\left(Li_2\left(\cosh^{-2}\left(\frac{\ell}{2}\right)\right)+2\log\left(\cosh\left(\frac{\ell}{2}\right)\right)\log\left(\frac{\cosh\left(\frac{\ell}{2}\right)}{\sinh\left(\frac{\ell}{2}\right)}\right)\right)\end{aligned}$$

Recall that

$$\log(\cosh(x)) \sim x$$

as $x \to \infty$. Moreover,

$$\log\left(\frac{\cosh(x)}{\sinh(x)}\right) = \log\left(1+e^{-2x}\right)-\log\left(1-e^{-2x}\right) \sim 2e^{-2x}$$

as $x \to \infty$. Thus, with $x = \frac{\ell}{2}$, we have that

$$2\log\left(\cosh\left(\frac{\ell}{2}\right)\right)\log\left(\frac{\cosh\left(\frac{\ell}{2}\right)}{\sinh\left(\frac{\ell}{2}\right)}\right) \sim 2\frac{\ell}{2}\cdot 2e^{-\ell} = 2\ell e^{-\ell} \tag{6.1}$$

as $\ell \to \infty$.

On the other hand

$$Li_2(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^2} = z + O(z^2),$$

and

$$\cosh^{-2}\left(\frac{\ell}{2}\right) \sim 4e^{-\ell}.$$

Thus,

$$Li_2\left(\cosh^{-2}\left(\frac{\ell}{2}\right)\right) \sim 4e^{-\ell}. \tag{6.2}$$

Using (6.1) and (6.2) we get that

$$\mu(B(\ell)) \sim \frac{1}{2\pi|\chi(X)|} \cdot \left(2\ell e^{-\ell} + 4e^{-\ell}\right) \sim \frac{1}{\pi|\chi(X)|} \cdot \ell e^{-\ell}$$

as $\ell \to \infty$.

■

Remark 6.1. It follows from Lemma 6.1 (1) that $\ell_i \leq \ell_{i+1}$ implies $\mu(B(\ell_i)) \geq \mu(B(\ell_{i+1}))$. In particular, the sequence $\{\mu(B(\ell_i))\}_{i\in\mathbb{N}}$ is decreasing.

Using Corollary 2.1 we obtain the following asymptotic for the length of the i-th orthogeodesic arc.

Lemma 6.2. Let $0 < \delta < 1$ denote the Hausdorff dimension of the limit set for (any-) one of the constituent pairs of pants of X. Then

$$\ell_i \sim \frac{1}{\delta} \cdot \log(i).$$

Consequently, there are sequences $(\theta_i)_{i\in\mathbb{N}}, (\theta_i')_{i\in\mathbb{N}}$ converging to 1 as $i \to \infty$, such that

$$\mu(B(\ell_i)) = \frac{1}{\pi|\chi(X)|\delta} \log(i) i^{-\theta_i/\delta} \theta_i'.$$

Proof. Let us fix two boundary curves C_-, C_+ of one of the constituent pairs of pants P and denote by $N_{C_-,C_+}(\ell)$ the number of orthogeodesic arcs in P of length $\leq \ell$ connecting the two given boundary curves. By Corollary 2.1 there is a constant $C_0 > 0$ such that

$$N_{C_-,C_+}(\ell) \sim C_0 \cdot e^{\delta\ell} \tag{6.3}$$

as $\ell \to \infty$, where $\delta > 0$ is the Hausdorff dimension of the limit set of P.

Let us now denote by $N(\ell)$ the *total* number of *all* orthogeodesic arcs in any constituent pair of pants of length $\leq \ell$. It follows from (6.3) that there is a constant $C > 0$ such that

$$N(\ell) \sim C \cdot e^{\delta\ell}.$$

In particular, there are constants $0 < C_1 < C_2$ such that

$$C_1 \cdot e^{\delta\ell} \leq N(\ell) \leq C_2 \cdot e^{\delta\ell}$$

for all $\ell > 0$.

Because we have enumerated the orthogeodesics $\{o_i\}_{i\in\mathbb{N}}$ by increasing length $\{\ell_i\}_{i\in\mathbb{N}}$, there is a constant $D > 0$ such that

$$N(\ell_i) - D \le i \le N(\ell_i) + D$$

for all $i \in \mathbb{N}$. Thus,

$$i \le N(\ell_i) + D \le C_2 e^{\delta \ell_i} + D,$$

equivalently

$$\frac{1}{\delta}\log(i - D) - \frac{1}{\delta}\log(C_2) \le \ell_i,$$

and similarly

$$\ell_i \le \frac{1}{\delta}\log(i + D) - \frac{1}{\delta}\log(C_1).$$

Therefore,

$$\ell_i \sim \frac{1}{\delta}\log(i)$$

as $i \to \infty$. ■

Moreover, we can approximate each of the sets $B(\ell)$ by a smooth bump function $\varphi_\ell \in C_c^\infty(\mathrm{UT}(\mathbb{H}^2))$. In the rest of this section we will explicitly construct for every $\ell > 0$ a smooth function $\varphi_\ell \in C_c^\infty(\mathrm{UT}(\mathbb{H}^2))$, that satisfies $0 \le \varphi_\ell \le 1$ and $\operatorname{supp}\varphi_\ell \subseteq B(\ell)$.

Let $\varepsilon > 0$. For any subset $A \subseteq \mathrm{UT}(X)$ we denote by $N_\varepsilon(A)$ the ε-neighborhood of A, i.e.

$$N_\varepsilon(A) := \{x \in \mathrm{UT}(X) \,|\, d(x, A) < \varepsilon\},$$

where $d(x, A) := \inf\{d(x, y) \,|\, y \in A\}$ denotes the distance from x to the set A. We can define the *ε-interior of A* as

$$\mathrm{int}_\varepsilon(A) := A \setminus N_\varepsilon(A^c),$$

i.e. all points in A that have distance more than ε from its boundary.

Let $\{\psi_\varepsilon\}_{0<\varepsilon<\varepsilon_0} \subseteq C_c^\infty(G)$ be the family of smooth approximate convolutional identities as constructed in Appendix A. In particular, they satisfy

(1) $\psi_\varepsilon \ge 0$, and
(2) $\int_G \psi_\varepsilon \, d\nu = 1$, and
(3) $\mathrm{supp}(\psi_\varepsilon) \subseteq B_\varepsilon(e)$,

for all $\varepsilon > 0$. In addition, for every $d > 0$ there is a constant $K = K(d) > 0$ such that

$$\left\| E_\alpha \cdot \psi_\varepsilon \right\|_1 \le K \cdot \left(\frac{1}{\varepsilon}\right)^d$$

for all $|\alpha| \le d$, where E_α denotes the multi-index derivative by left-invariant vector fields (see section 2.6 and Lemma A.1).

For every $\ell > 0$ and $0 < \varepsilon = \varepsilon(\ell) < \varepsilon_0$, that may depend on ℓ, we set

$$\varphi_\ell := \psi_{\varepsilon/2} * \mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))} \in C_c^\infty(\mathrm{UT}(X)).$$

These smooth approximations satisfy the following lemma.

Lemma 6.3. Let $\varepsilon = \varepsilon(\ell) > 0$. Then there are constants $C_1, C_2, C_3 > 0$, such that

(1) $0 \leq \varphi_\ell \leq 1$;
(2) $\operatorname{supp} \varphi_\ell \subseteq B(\ell)$;
(3) $\|\mathbf{1}_{B(\ell)} - \varphi_\ell\|_1 \leq C_1 \varepsilon(\ell)$;
(4) $\mathcal{S}(\varphi_\ell) \leq C_2 \cdot \mu(B(\ell)) \cdot \left(\frac{1}{\varepsilon(\ell)}\right)^d$, where d denotes the degree of the Sobolev norm;
(5) $\mathcal{S}(\overline{\varphi}_\ell) \leq C_3 \cdot \left(\frac{1}{\varepsilon(\ell)}\right)^d$, where we denote $\overline{\varphi}_\ell := 1 - \varphi_\ell$.

Proof. (1) Follows immediately from the construction.

(2) By definition $\operatorname{supp}(\mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))}) = \overline{\mathrm{int}_{\varepsilon/2}(B(\ell))}$. Thus every point in $\operatorname{supp}(\mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))})$ has distance at least $\varepsilon/2$ from $\partial B(\ell)$. Because $\operatorname{supp}(\psi_{\varepsilon/2}) \subseteq B_{\varepsilon/2}(\mathrm{id})$ and $d_{\mathrm{UT}(X)}(gv, w) \leq d_G(g, \mathrm{id})$ for all $v, w \in \mathrm{UT}(X)$, $g \in G$, it follows by the definition of convolution that $\operatorname{supp}(\varphi_\ell) \subseteq B(\ell)$.

(3) By construction φ_ℓ differs from $\mathbf{1}_{B(\ell)}$ only on the *inner ε-tube neighborhood of* $B(\ell)$

$$T_\varepsilon := B(\ell) \setminus \mathrm{int}_\varepsilon(B(\ell)) \subseteq B(\ell).$$

Because both functions take values in $[0,1]$, one obtains the trivial bound

$$\|\mathbf{1}_{B(\ell)} - \varphi_\ell\|_1 \leq \mu(T_\varepsilon).$$

It can be shown that both the principal curvature of the smooth boundary components of $\partial B(\ell)$ and its area $\mathrm{Area}(\partial B(\ell))$ are uniformly bounded. Thus, there exists $C_1 > 0$ independent of ℓ and $0 < \varepsilon < \varepsilon_0$ such that

$$\mu(T_\varepsilon) \leq C_1 \varepsilon(\ell)$$

by the tube Lemma C.1.

It is well-known that geodesic trajectories in $\mathrm{UT}(\mathbb{H}^2)$ are given unit vector fields of constant slope along geodesic lines in $\mathbb{H}^2$. Note that the unit vectors on $B(\ell)$ have basepoint in a given ideal quadrilateral, such that forward and backwards trajectories intersect two designated opposite sides of the ideal quadrilateral, which we call *opposing sides*.

The boundary of $\partial B(\ell)$ can be describe as the unions of 6 faces: 4 which we call *side faces* and 2 that we call *opposing faces*. Side faces are composed by unit vectors whose basepoints are in an ideal triangle formed by 3 out of the 4 ideal vertex associated to $B(\ell)$ (see Figure 4.2), and their geodesic trajectories end at the ideal vertex that does not pair to form an opposing side in the triangle. Hence, side faces are totally geodesic along the geodesic lines ending at the ideal vertex. In the orthogonal direction we have horocycles tangent at the ideal vertex and the unit vectors are normal to those horocycles. Since $\mathrm{UT}(\mathbb{H}^2)$ is a Riemannian submersion, these horocyclic trajectories have curvature equal to 1. Because of the symmetries of side face, these horocyclic trajectories are lines of curvature, which tells us the the principal curvatures of a side face are 0 (the totally geodesic direction) and 1 (the horocyclic direction).

On the other hand, opposing faces are composed of vectors whose basepoints are on one of the opposing sides of the ideal quadrilateral, such that their geodesic trajectory intersects the other opposing side. Since unit vectors based at a geodesic are a totally geodesic submanifold (by the geodesic description of $\mathrm{UT}(\mathbb{H}^2)$), it follows that opposing faces are totally geodesic in $\mathrm{UT}(\mathbb{H}^2)$, because these faces are open sets of totally geodesic submanifolds.

(4) Recall that

$$\mathcal{S}(\varphi_\ell) = \sum_{|\alpha|=0}^{d} \|E_\alpha \cdot \varphi_\ell\|_2 .$$

By Lemma 2.4 we get

$$\begin{aligned}
\|E_\alpha \cdot \varphi_\ell\|_2 &= \left\|E_\alpha \cdot (\psi_{\varepsilon/2} * \mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))})\right\|_2 \\
&= \left\|(E_\alpha \cdot \psi_{\varepsilon/2}) * \mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))}\right\|_2 \\
&\le \left\|E_\alpha \cdot \psi_{\varepsilon/2}\right\|_1 \cdot \left\|\mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))}\right\|_2 \\
&\le K \cdot \left(\frac{2}{\varepsilon(\ell)}\right)^d \cdot \mu(B(\ell)).
\end{aligned}$$

Hence,

$$\mathcal{S}(\varphi_\ell) = \sum_{|\alpha|=0}^{d} \|E_\alpha \cdot \varphi_\ell\|_2 \le C_2 \cdot \mu(B(\ell)) \cdot \left(\frac{1}{\varepsilon(\ell)}\right)^d$$

for $C_2 = dK2^d > 0$.

(5) Observe that

$$\overline{\varphi}_\ell = 1 - \overline{\varphi}_\ell = 1 - \psi_{\varepsilon/2} * \mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))} = \psi_{\varepsilon/2} * (1 - \mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))}).$$

As before we get

$$\begin{aligned}
\left\|E_\alpha \cdot \overline{\varphi}_\ell\right\|_2 &\le \left\|E_\alpha \cdot \psi_{\varepsilon/2}\right\|_1 \cdot \left\|1 - \mathbf{1}_{\mathrm{int}_{\varepsilon/2}(B(\ell))}\right\|_2 \\
&\le K \cdot \left(\frac{2}{\varepsilon(\ell)}\right)^d .
\end{aligned}$$

Hence,

$$\mathcal{S}(\overline{\varphi}_\ell) = \sum_{|\alpha|=0}^{d} \left\|E_\alpha \cdot \overline{\varphi}_\ell\right\|_2 \le C_3 \cdot \left(\frac{1}{\varepsilon(\ell)}\right)^d$$

for some $C_3 > 0$.

∎

6.3. **Counting the number of visited sets.** The orthogeodesic sets $\{B_i\}_{i\in\mathbb{N}}$ defined in section 6.2 partition the unit tangent bundle $\mathrm{UT}(X)$ (up to a null set). For a given unit tangent vector $v \in \mathrm{UT}(X)$ let $C_N(v)$ denote the number of *different* sets in this partition that the orbit $v, Tv, \ldots, T^{N-1}v$ has visited, i.e.

$$C_N(v) := \#\{i \in \mathbb{N} \,|\, \exists 0 \le n \le N-1 : T^n v \in B_i\}.$$

We would like to find a certain growth rate $m(N)$ for the number of visited sets such that for μ-almost every $v \in \mathrm{UT}(X)$ we have

$$C_N(v) \ge m(N)$$

for N large enough. This is established by the following theorem.

Theorem 6.4. Let $\lambda_1 < 0$ be the first non-zero eigenvalue of the Laplacian of X, let $0 < \alpha \le \frac{1}{2}$ satisfy $\alpha(\alpha-1) \ge \lambda_1$ (as in Theorem B.1), and set $\beta := \frac{\alpha}{1+\alpha}$. Further, let $\eta > 1$ and define

$$F(N) := \frac{\frac{2\beta}{3\eta}N}{W\left(\frac{2\beta}{3\eta}N\right)}$$

for all $N \in \mathbb{N}$, where W denotes the Lambert W function.

Then, for μ-almost every $v \in \mathrm{UT}(X)$ there is an $N_0 = N_0(v) \in \mathbb{N}$ such that

$$C_N(v) \ge F(N)^{\frac{\delta}{2\eta}}$$

for all $N \ge N_0$.

Proof. We need to prove that the set

$$\Omega := \left\{ v \in \mathrm{UT}(X) \,\middle|\, \exists N_0 \in \mathbb{N}\, \forall N \ge N_0 : C_N(v) \ge F(N)^{\frac{\delta}{2\eta}} \right\}$$

has full measure.

Let $N \in \mathbb{N}$ and denote by $\Omega_N \subseteq \mathrm{UT}(X)$ the set of all vectors $v \in \mathrm{UT}(X)$ such that

$$C_N(v) \ge F(N)^{\frac{\delta}{2\eta}}.$$

With this notation we have

$$\Omega = \liminf_{N\to\infty} \Omega_N = \bigcup_{N_0\in\mathbb{N}} \bigcap_{N\ge N_0} \Omega_N.$$

In order to show that Ω has full measure we need to see that its complement

$$\Omega^c = \limsup_{N\to\infty} \Omega_N^c = \bigcap_{N_0\in\mathbb{N}} \bigcup_{N\geq N_0} \Omega_N^c$$

has measure zero. By the Borel–Cantelli Lemma this will follow if the sequence $(\mu(\Omega_N^c))_{N\in\mathbb{N}}$ is summable, i.e.

$$\sum_{N=1}^{\infty} \mu(\Omega_N^c) < +\infty. \tag{6.4}$$

Before we proceed with verifying (6.4) we make some general observations first: if the orbit $v, Tv, \ldots, T^{N-1}v$ has visited less than m sets, i.e. $C_N(v) < m$, then v has not visited all of the first m sets $B_1, \ldots, B_m$ in particular. Thus,

$$\mu(C_N < m) \leq \mu\left(\bigcup_{i=1}^{k} \bigcap_{k=0}^{N-1} T^{-k}(B_i^c)\right) \leq \sum_{i=1}^{m} \mu\left(\bigcap_{k=0}^{N-1} T^{-k}(B_i^c)\right).$$

Notice that

$$\mu\left(\bigcap_{k=0}^{N-1} T^{-k}(B_i^c)\right) = \int_{\mathrm{UT}(X)} \prod_{k=0}^{N-1} \left(\mathbf{1}_{B_i^c} \circ T^k\right) d\mu.$$

We want to apply our exponential mixing theorem B.3 to the right-hand-side. For this we will approximate the sets B_i by smooth functions $\varphi_i \in C_c^\infty(\mathrm{UT}(X))$ as described in section 6.2. For each $i \in \mathbb{N}$ we let $\varepsilon_i > 0$ be a positive number that we will specify later. Given $\varepsilon(\ell_i) = \varepsilon_i$ we obtain smooth functions $\varphi_{\ell_i} \in C_c^\infty(\mathrm{UT}(\mathbb{H}^2))$ that approximate the sets $B(\ell_i) \subseteq \mathrm{UT}(\mathbb{H}^2)$. Via the isometry $B_i \cong B(\ell_i)$ these pull-back to smooth approximations of the set B_i that satisfy the assertions of Lemma 6.3. Finally, we set $\overline{\varphi}_i := 1 - \varphi_i \geq \mathbf{1}_{B_i^c}$.

Thus, we get

$$\int_{\mathrm{UT}(X)} \prod_{k=0}^{N-1} \left(\mathbf{1}_{B_i^c} \circ T^k\right) d\mu \leq \int_{\mathrm{UT}(X)} \prod_{k=0}^{N-1} \left(\overline{\varphi}_i \circ T^k\right) d\mu$$

In order to obtain an error term that tends to zero from Theorem B.3, we will take the product only over an equally spaced subset of $\{0, \ldots, N-1\}$. Let $n \in \mathbb{N}$, which we will also specify later, and set

$$l := \left\lfloor \frac{N-1}{n-1} \right\rfloor.$$

By Theorem B.3 we obtain

$$\begin{aligned}\int_{\mathrm{UT}(X)} \prod_{k=0}^{N-1} \left(\overline{\varphi}_i \circ T^k\right) d\mu &\leq \int_{\mathrm{UT}(X)} \prod_{k=0}^{n-1} \left(\overline{\varphi}_i \circ T^{k\cdot l}\right) d\mu \\ &\leq \mu(\overline{\varphi}_i)^n + L \cdot (n-1) \cdot D^n \cdot \mathcal{S}(\overline{\varphi}_i)^n \cdot e^{-\beta \cdot l}\end{aligned}$$

for all $l \geq -\log(\alpha) - (1+\alpha)\log(2)$, where $\mathcal{S}$ denotes the degree $d = 3$ Sobolev norm.

Therefore,

$$\begin{aligned}
\mu(C_N < m) &\le \sum_{i=1}^{m} \mu\left(\bigcap_{k=0}^{N-1} T^{-k}(B_i^c)\right) \\
&\le \sum_{i=1}^{m} \left(\mu(\overline{\varphi}_i)^n + L\cdot(n-1)\cdot D^n \cdot \mathcal{S}(\overline{\varphi}_i)^n \cdot e^{-\beta\cdot l}\right) \\
&\le \underbrace{\sum_{i=1}^{m} \mu(\overline{\varphi}_i)^n}_{=:A_1(N)} + \underbrace{\sum_{i=1}^{m} L\cdot(n-1)\cdot D^n \cdot \mathcal{S}(\overline{\varphi}_i)^n \cdot e^{-\beta\cdot l}}_{=:A_2(N)}.
\end{aligned}$$

Now, we set

$$\varepsilon_i := \frac{\mu(B_i)}{C_1(2+\log(i))}, \qquad \forall i \in \mathbb{N}, \tag{6.5}$$

$$m(N) := \left\lceil F(N)^{\frac{\delta}{2\eta}} \right\rceil, \qquad \forall N \in \mathbb{N}, \tag{6.6}$$

$$n(N) := \left\lfloor \sqrt{F(N)} \right\rfloor, \qquad \forall N \in \mathbb{N}. \tag{6.7}$$

Notice that with these choices

$$l(N) = \left\lfloor \frac{N-1}{n(N)-1} \right\rfloor \sim \frac{N}{\sqrt{F(N)}} = \sqrt{\frac{2\beta}{d\eta}\cdot N \cdot W\left(\frac{2\beta}{d\eta}N\right)} \to \infty$$

as $N \to \infty$, whence $l(N) \ge -\log(\alpha) - (1+\alpha)\log(2)$ for large N.

Moreover, observe that

$$\mu(\Omega_N^c) = \mu\left(C_N < F(N)^{\frac{\delta}{2\eta}}\right) \le \mu\left(C_N < \left\lceil F(N)^{\frac{\delta}{2\eta}} \right\rceil\right) = \mu(C_N < m(N)).$$

Thus, in order to prove (6.4) it suffices to show that

$$\sum_{N=1}^{\infty} \mu(C_N < m(N)) < +\infty$$

for the above choices of $m = m(N)$, $n = n(N)$, and ε_i. This is equivalent to showing that both

$$\sum_{N=1}^{\infty} A_1(N) < +\infty \quad \text{and} \quad \sum_{N=1}^{\infty} A_2(N) < +\infty.$$

Regarding $A_1(N)$. Observe that

$$\mu(\overline{\varphi}_i) \le 1 - (\mu(B_i) - C_1\varepsilon_i) \le 1 - \tfrac{1}{2}\mu(B_i) \tag{6.8}$$

for every $i \in \mathbb{N}$ by (6.5). Because the sets $\{B_i\}_{i\in\mathbb{N}}$ are enumerated in such a way that the sequence $(\mu(B_i))_{i\in\mathbb{N}}$ is monotonically decreasing, we obtain that

$$A_1(N) \le \sum_{i=1}^{m} \mu(\overline{\varphi}_i)^n \le \sum_{i=1}^{m} \left(1 - \tfrac{1}{2}\mu(B_i)\right)^n \le m\left(1 - \tfrac{1}{2}\mu(B_m)\right)^n.$$

By Lemma 6.2 there are sequences $(\theta_i)_{i\in\mathbb{N}}, (\theta_i')_{i\in\mathbb{N}}$ converging to 1 as $i \to \infty$, such that

$$\mu(B_i) = \mu(B(\ell_i)) = \frac{1}{\pi|\chi(X)|\delta} \log(i) i^{-\theta_i/\delta}\theta_i' \tag{6.9}$$

for all $i \in \mathbb{N}$.

Regarding the logarithm we get

$$\begin{aligned}\log(A_1(N)) &\le \log(m) + n\log\left(1 - \tfrac{1}{2}\mu(B_m)\right)\\ &\overset{(\diamond)}{\le} \log(m) - \frac{n}{2}\mu(B_m)\\ &= \log(m) - \frac{\theta_m'}{2\pi|\chi(X)|\delta} n\log(m) m^{-\theta_m/\delta}\\ &= \log(m)\left(1 - \frac{\theta_m'}{2\pi|\chi(X)|\delta} n m^{-\theta_m/\delta}\right),\end{aligned}$$

where we used the fact that $\log(x+1) \le x$ for all $x \in \mathbb{R}$ in $(\diamond)$ and plugged-in (6.9).

Because $F(N) \to \infty$ as $N \to \infty$, we have

$$m(N) \sim F(N)^{\frac{\delta}{2\eta}} \qquad \text{and} \qquad n(N) \sim \sqrt{F(N)}.$$

Moreover,

$$\log(m(N)) \sim \frac{\delta}{2\eta}\left(\log\left(\frac{2\beta}{\eta d}\right) + \log(N) - \log\left(W\left(\frac{2\beta}{\eta d}N\right)\right)\right) \sim \frac{\delta}{2\eta}\log(N). \tag{6.10}$$

Also, $m(N) \sim n(N)^{\frac{\delta}{\eta}}$, i.e.

$$m(N) = n(N)^{\frac{\delta}{\eta}} \cdot \theta_N'' \tag{6.11}$$

for a sequence $(\theta_N'')_{N\in\mathbb{N}}$ converging to 1 as $N \to \infty$. Hence,

$$\frac{\theta_m'}{2\pi|\chi(X)|\delta} n m^{-\theta_m/\delta} = \frac{\theta_m'}{2\pi|\chi(X)|\delta} (\theta_N'')^{-\theta_m/\delta} n^{1-\frac{\theta_m}{\eta}}$$

Notice that the exponent $1 - \frac{\theta_{m(N)}}{\eta}$ is strictly positive for large N, because $\theta_{m(N)} \to 1$ as $N \to \infty$ and $\eta > 1$, and $n(N) \to \infty$ as $N \to \infty$. Therefore,

$$\left(1 - \frac{\theta_m'}{2\pi|\chi(X)|\delta} n m^{-\theta_m/\delta}\right) \to -\infty \tag{6.12}$$

as $N \to \infty$.

From (6.10) and (6.12) it follows that

$$\log(A_1(N)) \leq -2\log(N)$$

for large $N \in \mathbb{N}$. Hence,

$$A_1(N) \leq \frac{1}{N^2}$$

for large N, such that $(A_1(N))_{N\in\mathbb{N}}$ is summable.

Regarding $A_2(N)$**.** By Lemma 6.3 there is $C_3 > 0$ such that

$$\begin{aligned}\mathcal{S}(\overline{\varphi}_i) = \mathcal{S}(\overline{\varphi}_{\ell_i}) &\leq C_3 \cdot \left(\frac{1}{\varepsilon_i}\right)^d \\ &\leq C_3\,(C_1\pi\,|\chi(X)|\,\delta)^d \underbrace{\left(\frac{1}{\theta_i'} + \frac{2}{\log(i)\theta_i'}\right)^d}_{=:T_i'} i^{\theta_i \frac{d}{\delta}}\end{aligned}$$

for all $i \in \mathbb{N}$. The sequence θ_i and the terms T_i' tend to 1 as $i \to \infty$, whence there is an $i_0 = i_0(\eta) \in \mathbb{N}$ such that $\theta_i \leq \eta$ and $T_i' \leq \eta$ for all $i \geq i_0 + 1$. Thus,

$$\mathcal{S}(\overline{\varphi}_i) \leq \underbrace{C_3\,(C_1\pi\,|\chi(X)|\,\delta)^d\,\eta}_{=:K'=K'(\eta)>0}\, i^{\frac{\eta}{\delta}d} = K' i^{\frac{\eta}{\delta}d} \tag{6.13}$$

for all $i \geq i_0 + 1$. Moreover, there is a constant $K = K(\eta) > 0$ such that

$$\mathcal{S}(\overline{\varphi}_i) \leq K \tag{6.14}$$

for all $i = 1, \ldots, i_0$.

From (6.13) and (6.14) we get

$$\begin{aligned}A_2(N) &= \sum_{i=1}^{m} L(n-1)D^n\,\mathcal{S}(\overline{\varphi}_i)^n e^{-\beta l} \\ &\leq \sum_{i=1}^{i_0} L(n-1)D^n K^n e^{-\beta l} + \sum_{i=i_0+1}^{m} L(n-1)D^n (K')^n i^{\frac{\eta}{\delta}dn} e^{-\beta l} \\ &\leq i_0 \cdot L(n-1)D^n K^n e^{-\beta l} + (m-i_0)\cdot L(n-1)D^n(K')^n m^{\frac{\eta}{\delta}dn} e^{-\beta l} \\ &\leq L(n-1)\widehat{K}^n m^{\frac{\eta}{\delta}dn+1} e^{-\beta l},\end{aligned}$$

where we set $\widehat{K} := \max(DK, DK')$.

Regarding the logarithm we get

$$\log(A_2(N)) \leq \log(L) + \log(n-1) + n\log(\widehat{K}) + \left(\frac{\eta}{\delta}dn + 1\right) \underbrace{\log(m)}_{=\frac{\delta}{\eta}\log(n)+\log(\theta_N'')} - \beta l$$

$$= \log(L) + \log(n-1) + n\log(\widehat{K}) + dn\log(n)\underbrace{\left(1 + \frac{\eta\log(\theta_N'')}{\delta\log(n)} + \frac{\delta}{\eta dn} + \frac{\log(\theta_n'')}{dn\log(n)}\right)}_{=:T_N''\to 1} - \beta l$$

$$= dn\log(n)\underbrace{\left(T_n'' + \frac{\log(L)}{dn\log(n)} + \frac{\log(n-1)}{dn\log(n)} + \frac{\log(\widehat{K})}{d\log(n)}\right)}_{=:T_N'''\to 1} - \beta l$$

$$= dn\log(n)T_N''' - \beta\frac{N}{n}\theta_N'''$$

$$= -\beta\frac{N}{n}\theta_N'''\left(1 - \frac{dn^2\log(n)}{\beta N}\frac{T_N'''}{\theta_N'''}\right),$$

where we used (6.11) and the fact that $l = \left\lfloor \frac{N-1}{n-1} \right\rfloor = \frac{N}{n}\theta_N'''$ for a sequence $(\theta_N''')_{N\in\mathbb{N}}$ converging to 1 as $N \to \infty$.

Asymptotically, we have

$$\frac{dn^2\log(n)}{\beta N}\frac{T_N'''}{\theta_N'''} \sim \frac{d}{2\beta N}n^2\log(n^2) \sim \frac{d}{2\beta N}F(N)\log(F(N)).$$

Observe that the function $g\colon [1,+\infty) \to [0,+\infty), x \mapsto x\log(x)$ is a monotonically increasing bijection with inverse function $g^{-1}\colon [0,+\infty) \to [1,+\infty)$

$$g^{-1}(y) = \frac{y}{W(y)}.$$

By definition $F(N) = g^{-1}\left(\frac{2\beta}{d\eta}N\right)$, with $d = 3$, such that

$$\frac{d}{2\beta N}F(N)\log(F(N)) = \frac{d}{2\beta N}g\left(g^{-1}\left(\frac{2\beta}{d\eta}N\right)\right) = \frac{d}{2\beta N}\cdot\frac{2\beta N}{d\eta} = \frac{1}{\eta} < 1,$$

whence

$$\left(1 - \frac{dn^2\log(n)}{\beta N}\frac{T_N'''}{\theta_N'''}\right) > 1 - \frac{1}{\eta} > 0 \tag{6.15}$$

for large $N \in \mathbb{N}$.

On the other hand,

$$-\beta\frac{N}{n}\theta_N''' \sim -\sqrt{\frac{\beta}{2d\eta}\cdot N\cdot W\left(\frac{2\beta}{d\eta}N\right)} \ll -\sqrt{N}. \tag{6.16}$$

From (6.15) and (6.16) it follows that there is a constant $C > 0$ such that

$$A_2(N) \le \exp(-C\sqrt{N})$$

for large $N \in \mathbb{N}$, whence $(A_2(N))_{N\in\mathbb{N}}$ is summable.

All in all, this shows that the sequence $(\mu(\Omega_N^c))_{N\in\mathbb{N}}$ is summable, and we conclude by the Borel–Cantelli Lemma. ■

7. Proof of the Main Theorem

We now have all the ingredients needed to prove our main volume estimate for volumes of canonical lifts of generic filling geodesics.

We choose a hyperbolic structure X on our surface S_g so that all pair of pants $P \in \mathcal{P}$ are isometric and are glued without twists and all the conditions of Section 4 are satisfied. Moreover, we let $0 < \delta < 1$ denote the Hausdorff dimension of the limit set of P.

Theorem 7.1. Let $\eta > 1 (> \delta > 0)$. There exist positive constants A, B, C, such that for μ-almost every $v \in \mathrm{PT}(X)$ the geodesic $\widehat{\gamma}_v^p := \widehat{\gamma}_v^p(t)$ for all $t \geq t(v)$ large enough satisfies:

$$A \cdot \left(\frac{C \cdot \ell_X(\widehat{\gamma}_v^p)}{W\left(C \cdot \ell_X(\widehat{\gamma}_v^p)\right)} \right)^{\delta/2\eta} - B \leq \mathrm{Vol}(\mathrm{PT}(X)_{\widehat{\gamma}_v^p}),$$

where W denotes the Lambert W function.

Proof. We build our random geodesic $\gamma_v(t)$ with $\varepsilon = \frac{1}{t}$ and $t \geq t(v) := \max\{t^*(v), N_0(v)\}$ (see Definition 4.2 and Theorem 6.4). Let $\widehat{\gamma}_v^p$ be its primitive geodesic representative. Then, $\mathrm{PT}(X)_{\widehat{\gamma}_v^p}$ is hyperbolic and we now want to estimate its volume. We know that by [RM20] we have;

$$\frac{v_3}{2} \sum_{P \in \mathcal{P}} \#\left\{\widehat{\gamma}_v^p\text{-arcs in } P\right\} \leq \mathrm{Vol}(\mathrm{PT}(X)_{\widehat{\gamma}_v^p}).$$

Since the arcs are not counted with multiplicity we have that

$$A_{\mathcal{P}}(\widehat{\gamma}_v^p) = \sum_{P \in \mathcal{P}} \#\left\{\widehat{\gamma}_v^p\text{-arcs in } P\right\} = \sum_{P \in \mathcal{P}} \#\left\{\widehat{\gamma}_v\text{-arcs in } P\right\} = A_{\mathcal{P}}(\widehat{\gamma}_v)$$

where we are now pulling tight γ_v.

By Proposition 5.8 we have that:

$$A_{\mathcal{P}}(\gamma_v) - 2 \leq A_{\mathcal{P}}(\widehat{\gamma}_v)$$

so that:

$$\frac{v_3}{2} A_{\mathcal{P}}(\gamma_v) - v_3 \leq \mathrm{Vol}(\mathrm{PT}(X)_{\widehat{\gamma}_v^p}).$$

by forgetting the shortcut we get that $A_{\mathcal{P}}(\gamma_v) \geq A_{\mathcal{P}}(g_v(t))$ for $g_v(t)$ the geodesic ray of length t originated from v.

We are doing everything in the projective tangent bundle but Theorem 6.4 works for the geodesic flow in the unit tangent bundle. However, the number of arcs in the projective bundle is at least half the number of arcs in the unit tangent bundle. Therefore, Theorem

6.4 gives us that:

$$\frac{1}{2}\left(\frac{\frac{2\beta}{3\eta}t}{W\left(\frac{2\beta}{3\eta}t\right)}\right)^{\delta/2\eta} \leq A_{\mathcal{P}}(g_v(t)).$$

Putting this all together gives us:

$$\frac{v_3}{4}\left(\frac{\frac{2\beta}{3\eta}t}{W\left(\frac{2\beta}{3\eta}t\right)}\right)^{\delta/2\eta} - v_3 \leq \mathrm{Vol}(\mathrm{PT}(X)_{\widehat{\gamma}_v^p}).$$

To finish we need to relate this to the length of $\widehat{\gamma}_v^p$. Notice that $\ell_X(\widehat{\gamma}_v) \leq t+\varepsilon = t+\frac{1}{t}$. Thus, there exists a constant $B>0$ such that:

$$\frac{v_3}{4}\left(\frac{\frac{2\beta}{3\eta}(t+\varepsilon)}{W\left(\frac{2\beta}{3\eta}(t+\varepsilon)\right)}\right)^{\delta/2\eta} - B \leq \frac{v_3}{4}\left(\frac{\frac{2\beta}{3\eta}t}{W\left(\frac{2\beta}{3\eta}t\right)}\right)^{\delta/2\eta} - v_3$$

and we let $A = \frac{v_3}{4}$ and $C = \frac{2\beta}{3\eta}$ so that:

$$A\cdot\left(\frac{C\cdot(t+\varepsilon)}{W(C\cdot(t+\varepsilon))}\right)^{\delta/2\eta} - B \leq \mathrm{Vol}(\mathrm{PT}(X)_{\widehat{\gamma}_v^p}).$$

Finally, since $\ell_X(\widehat{\gamma}_v^p) \leq \ell_X(\widehat{\gamma}_v) \leq t+\varepsilon$ and the function we are considering is increasing we get that:

$$A\cdot\left(\frac{C\cdot\ell_X(\widehat{\gamma}_v^p)}{W\left(C\cdot\ell_X(\widehat{\gamma}_v^p)\right)}\right)^{\delta/2\eta} - B \leq \mathrm{Vol}(\mathrm{PT}(X)_{\widehat{\gamma}_v^p})$$

concluding the proof. ∎

Appendix A. Explicit construction of approximate convolutional identities

We call $\{\psi_\varepsilon\}_{0<\varepsilon<\varepsilon_0} \subseteq L^1(G)$ a *family of approximate convolutional identities* if

(1) $\psi_\varepsilon \geq 0$, and
(2) $\int_G \psi_\varepsilon \, d\nu = 1$, and
(3) $\operatorname{supp}(\psi_\varepsilon) \subseteq B_\varepsilon(e)$,

for all $0 < \varepsilon < \varepsilon_0$.

In the rest of this section we will construct an explicit family of approximate convolutional identities $\{\psi_\varepsilon\}_{0<\varepsilon<\varepsilon_0} \subseteq C_c^\infty(G)$, and estimate the L^1-norms of their derivatives (see Lemma A.1).

There is $\varepsilon_0 > 0$ such that $\exp\colon B_{\varepsilon_0}(0) \subseteq \mathfrak{g} \to G$ is a diffeomorphism on its image. Let us denote by $\log\colon \exp(B_{\varepsilon_0}(0)) \subseteq G \to B_{\varepsilon_0}(0) \subseteq \mathfrak{g}$ its inverse. We may choose a smooth bump function $\widetilde{\psi}_{\varepsilon_0} \in C_c^\infty(\mathfrak{g})$, such that $\widetilde{\psi}_{\varepsilon_0} \geq 0$ and $\emptyset \neq \operatorname{supp}(\widetilde{\psi}_{\varepsilon_0}) \subseteq B_{\varepsilon_0}(0)$. We set

$$\overline{\psi}_{\varepsilon_0}(g) := \begin{cases} \widetilde{\psi}_{\varepsilon_0}(\log(g)), & \text{if } g \in \exp(B_{\varepsilon_0}(0)) \\ 0, & \text{else;} \end{cases}$$

for every $g \in G$. Further, we set

$$M_{\varepsilon_0} := \int_G \overline{\psi}_{\varepsilon_0} \, d\nu > 0,$$

and define

$$\psi_{\varepsilon_0} := \frac{\overline{\psi}_{\varepsilon_0}}{M_{\varepsilon_0}}.$$

Then ψ_{ε_0} satisfies (1) and (2).

Regarding (3) observe that for every $X \in B_{\varepsilon_0}(0)$ the curve $c(t) := \exp(tX), 0 \leq t \leq 1$ connects e and $\exp(X) \in \exp(B_{\varepsilon_0}(0))$. Therefore,

$$d_G(\exp(X), e) \leq \ell(c) = \int_0^1 \|\dot{c}(t)\| \, dt = \|X\| < \varepsilon_0,$$

and $\exp(B_{\varepsilon_0}(0)) \subseteq B_{\varepsilon_0}(e)$. Thus ψ_{ε_0} satisfies (3), too.

We will continue to define the other members of the family $\{\psi_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ by rescaling ψ_{ε_0}. Let $0 < \varepsilon < \varepsilon_0$. We define

$$\widetilde{\psi}_\varepsilon(X) := \widetilde{\psi}_{\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon} X\right)$$

for all $X \in \mathfrak{g}$. By definition, $\operatorname{supp}(\widetilde{\psi}_\varepsilon) \subseteq B_\varepsilon(0)$.

As before, we set

$$\overline{\psi}_\varepsilon(g) := \begin{cases} \widetilde{\psi}_\varepsilon(\log(g)), & \text{if } g \in \exp(B_\varepsilon(0)) \\ 0, & \text{else;} \end{cases}$$

for every $g \in G$,

$$M_\varepsilon := \int_G \overline{\psi}_\varepsilon \, d\nu > 0,$$

and define

$$\psi_\varepsilon := \frac{\overline{\psi}_\varepsilon}{M_\varepsilon}.$$

Then ψ_ε satisfies (1), (2) and (3) as before.

The following L^1-estimates is used in section 6.2.

Lemma A.1. For every $d > 0$ there is a constant $K = K(d) > 0$ such that

$$\left\| E_\alpha \cdot \psi_\varepsilon \right\|_1 \leq K \cdot \left(\frac{1}{\varepsilon} \right)^d$$

holds for every $0 < \varepsilon < \varepsilon_0$ and every multi-index α with $|\alpha| \leq d$.

Proof. Recall that

$$\left\| E_\alpha \cdot \psi_\varepsilon \right\|_1 = M_\varepsilon^{-1} \left\| E_\alpha \cdot \overline{\psi}_\varepsilon \right\|_1 .$$

Let us first estimate $M_\varepsilon = \int_G \overline{\psi}_\varepsilon \, d\nu$, and see that $M_\varepsilon \asymp \varepsilon^3$.

We compute

$$\begin{aligned} M_\varepsilon = \int_G \overline{\psi}_\varepsilon \, d\nu &= \int_{\exp(B_\varepsilon(0))} \widetilde{\psi}_\varepsilon(\log(g)) \, d\nu(g) \\ &= \int_{B_\varepsilon(0)} \widetilde{\psi}_\varepsilon(X) \, d(\log_* \nu)(X) \\ &= \int_{B_\varepsilon(0)} \widetilde{\psi}_\varepsilon(X) \cdot \frac{d(\log_* \nu)}{d\lambda}(X) \, d\lambda(X), \end{aligned}$$

where we denote by λ the Lebesgue measure on $\mathfrak{g}$ and $\frac{d(\log_* \nu)}{d\lambda}$ denotes the Radon-Nikodym derivative of $\log_* \nu$ with respect to λ. This is well-defined since the measure $\log_*(\nu)$ is clearly of Lebesgue class. Moreover, because it is the push-forward of the Haar measure ν via the diffeomorphism $\log\colon \exp(B_{\varepsilon_0}(0)) \to B_{\varepsilon_0}(0)$, the Radon-Nikodym derivative is smooth. Consequently there are uniform upper and lower bounds

$$0 < \inf_X \frac{d(\log_* \nu)}{d\lambda}(X), \qquad \sup_X \frac{d(\log_* \nu)}{d\lambda}(X) < +\infty,$$

where the infimum and the supremum are taken over the compact set $\operatorname{supp}(\psi_{\varepsilon_0}) \subseteq B_{\varepsilon_0}(0)$.

Hence,

$$\begin{aligned} M_\varepsilon \asymp \int_{B_\varepsilon(0)} \widetilde{\psi}_\varepsilon(X) \, d\lambda(X) &= \left(\frac{\varepsilon}{\varepsilon_0} \right)^3 \cdot \int_{B_\varepsilon(0)} \widetilde{\psi}_{\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon} X \right) \cdot \left(\frac{\varepsilon_0}{\varepsilon} \right)^3 d\lambda(X) \\ &= \left(\frac{\varepsilon}{\varepsilon_0} \right)^3 \cdot \int_{B_{\varepsilon_0}(0)} \widetilde{\psi}_{\varepsilon_0} \, d\lambda \asymp \varepsilon^3, \end{aligned}$$

as asserted.

We turn to $\left\|E_\alpha \cdot \overline{\psi}_\varepsilon\right\|_1$. For a single E_i we have

$$(E_i \cdot \overline{\psi}_\varepsilon)(g) = \frac{d}{dt}\bigg|_{t=0} \overline{\psi}_\varepsilon(\exp(-tE_i)g)$$

for every $g \in G$. Thus, if we define by $\widetilde{E}_i$ the *right-invariant* vector field given by $\widetilde{E}_i(e) = -E_i$, then we can rewrite

$$E_\alpha \cdot \overline{\psi}_\varepsilon = \widetilde{E}_\alpha \cdot \overline{\psi}_\varepsilon$$

where the right hand side is understood as the usual derivation with respect to vector fields.

On $\operatorname{supp}(\overline{\psi}_\varepsilon) \subseteq \exp(B_\varepsilon(0))$ we obtain

$$\widetilde{E}_\alpha \cdot \overline{\psi}_\varepsilon = \widetilde{E}_\alpha \cdot (\widetilde{\psi}_\varepsilon \circ \log) = \left(\log_*(\widetilde{E})_\alpha \cdot \widetilde{\psi}_\varepsilon\right) \circ \log.$$

Here we denote by $\log_*(\widetilde{E})_i \in \mathfrak{X}(B_\varepsilon(0))$ the push-forward of the vector field $\widetilde{E}_i$ along the map $\log\colon \exp(B_\varepsilon(0)) \to B_\varepsilon(0) \subseteq \mathfrak{g}$.

We may express $\log_*(\widetilde{E})_i$ with respect to the standard vector fields $\partial_1, \ldots, \partial_3$ of $\mathfrak{g} \cong \mathbb{R}^3$. In this way, we obtain polynomials $P_\alpha = \sum_{0 \le |\beta| \le |\alpha|} c_\alpha^\beta \cdot \partial_\beta \in \mathcal{U}(\mathfrak{X}(B_\varepsilon(0)))$ such that

$$\log_*(\widetilde{E})_\alpha = P_\alpha,$$

where $c_\alpha^\beta\colon B_\varepsilon(0) \to \mathbb{R}$ are smooth coefficient functions. If we apply these to $\widetilde{\psi}_\varepsilon$, we obtain

$$\begin{aligned}
\left(\log_*(\widetilde{E})_\alpha \cdot \widetilde{\psi}_\varepsilon\right)(X) = \left(P_\alpha \cdot \widetilde{\psi}_\varepsilon\right)(X) &= \sum_{0 \le |\beta| \le |\alpha|} c_\alpha^\beta(X) \cdot \partial_\beta \widetilde{\psi}_\varepsilon(X) \\
&= \sum_{0 \le |\beta| \le |\alpha|} c_\alpha^\beta(X) \cdot \partial_\beta \left(\widetilde{\psi}_{\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon} X\right)\right) \\
&= \sum_{0 \le |\beta| \le |\alpha|} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{|\beta|} c_\alpha^\beta(X) \cdot \left(\partial_\beta \widetilde{\psi}_{\varepsilon_0}\right)\left(\frac{\varepsilon_0}{\varepsilon} X\right)
\end{aligned}$$

for every $X \in B_\varepsilon(0)$.

With these preliminary computations we obtain

$$\begin{aligned}
\left\|E_\alpha \cdot \overline{\psi}_\varepsilon\right\|_1 &= \int_G \left|E_\alpha \cdot \overline{\psi}_\varepsilon\right| d\nu \\
&= \int_G \left|\left(\log_*(\widetilde{E})_\alpha \cdot \widetilde{\psi}_\varepsilon\right) \circ \log\right| d\nu \\
&= \int_{B_\varepsilon(0)} \left|\left(\log_*(\widetilde{E})_\alpha \cdot \widetilde{\psi}_\varepsilon\right)(X)\right| \cdot \left|\frac{d \log_*(\nu)}{d\lambda}(X)\right| d\lambda(X) \\
&\ll \int_{B_\varepsilon(0)} \left|\left(\log_*(\widetilde{E})_\alpha \cdot \widetilde{\psi}_\varepsilon\right)(X)\right| d\lambda(X)
\end{aligned}$$

$$\ll \sum_{0\le|\beta|\le|\alpha|} \left|\frac{\varepsilon_0}{\varepsilon}\right|^{|\beta|} \int_{B_\varepsilon(0)} \left|c_\alpha^\beta(X)\right| \cdot \left|\left(\partial_\beta \widetilde{\psi}_{\varepsilon_0}\right)\left(\frac{\varepsilon_0}{\varepsilon}X\right)\right| d\lambda(X)$$

$$= \sum_{0\le|\beta|\le|\alpha|} \left|\frac{\varepsilon_0}{\varepsilon}\right|^{|\beta|-3} \underbrace{\int_{B_{\varepsilon_0}(0)} \left|c_\alpha^\beta\left(\frac{\varepsilon}{\varepsilon_0}X\right)\right| \cdot \left|\left(\partial_\beta \widetilde{\psi}_{\varepsilon_0}\right)(X)\right| d\lambda(X)}_{\text{uniformly bounded}}$$

$$\ll \left(\frac{\varepsilon_0}{\varepsilon}\right)^{|\alpha|-3}.$$

In conjunction with $M_\varepsilon \asymp \varepsilon^3$ we get that

$$\left\|E_\alpha \cdot \psi_\varepsilon\right\|_1 = M_\varepsilon^{-1} \left\|E_\alpha \cdot \overline{\psi}_\varepsilon\right\|_1 \ll \left(\frac{\varepsilon_0}{\varepsilon}\right)^{|\alpha|} \le \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d} \ll \left(\frac{1}{\varepsilon}\right)^{d}.$$

■

Appendix B. Exponential higher-order mixing

Let $Y := \mathrm{UT}(X) \cong \Gamma\backslash G$ denote the unit tangent bundle of a closed hyperbolic surface $X = \Gamma\backslash\mathbb{H}^2$ with $G = PSL_2\,\mathbb{R}$ and Γ a torsion-free cocompact subgroup. We set

$$a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \quad \text{and} \quad u_s := \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$$

for all $t, s \in \mathbb{R}$.

The geodesic flow on Y through $x = \Gamma g \in \Gamma\backslash G$ is given by

$$xa_t := \Gamma g a_t \quad \forall t \in \mathbb{R},$$

and the horocycle flow is given by

$$xu_s := \Gamma g u_s \quad \forall s \in \mathbb{R}.$$

Moreover, we denote by $T\colon Y \to Y$ the time-one map of the geodesic flow, i.e. $T(x) := xa_1$ for all $x \in Y$. Moreover, we will abbreviate integration with respect to the unique invariant probability measure (the normalized Liouville measure) on Y by dx or dy.

The following formula will be useful:

$$a_t u_s a_{-t} = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} = \begin{pmatrix} 1 & -e^t s \\ 0 & 1 \end{pmatrix} = u_{e^t s}$$

for all $t, s \in \mathbb{R}$.

Moreover, we will use the following result due to Burger [Bur90]:

Theorem B.1 ([Bur90, Theorem 2 (C)])**.** Let $\lambda_1 < 0$ be the first non-zero eigenvalue of the Laplacian of X and let $0 < \alpha \le \frac{1}{2}$ satisfy $\alpha(\alpha-1) \ge \lambda_1$. Then, we have for all $f \in C_c^\infty(Y)$

and $T \geq 1$

$$\sup_{x\in Y}\left|\frac{1}{2T}\int_{-T}^{T} f(xu_t)dt - \int_Y f(y)dy\right| \leq c\frac{T^{-\alpha}-T^{\alpha-1}}{1-2\alpha}\mathcal{S}(f),$$

where $\mathcal{S}(f)=\mathcal{S}_3(f)$ denotes the degree $d=3$ Sobolev norm; see section 2.6.

Corollary B.2. In particular, we obtain that there exists $C>0$, $\alpha>0$ such that for all $f\in C_c^\infty(Y)$, for all $T\geq 1/2$, and for all $x\in Y$

$$\left|\frac{1}{T}\int_0^T f(xu_t)dt - \int_Y f(y)dy\right| \leq CT^{-\alpha}\mathcal{S}(f).$$

We will now deduce the following exponential k-mixing result. Note that similar results have been deduced in more generality before; see [BEG17]. However, for our application it is important that the given constants depend neither on the functions nor on the *number of functions*.

Theorem B.3 (Exponential k-mixing)**.** Let α be as in Theorem B.1. There are constants $D,L\geq 1$, such that

$$\left|\int_Y \prod_{j=1}^n \left(f_j\circ T^{-j\cdot k}\right)dx - \prod_{j=1}^n\left(\int_Y f_j\,dx\right)\right| \leq L\cdot(n-1)\cdot D^n\cdot e^{-\frac{\alpha}{1+\alpha}k}\cdot\prod_{j=1}^n\mathcal{S}(f_j)$$

for all $n\in\mathbb{N}$, for all $f_1,\dots,f_n\in C_c^\infty(\mathrm{UT}(S))$ and all integers $k\geq -\log(\alpha)-(1+\alpha)\log(2)$. Here $\mathcal{S}(f)=\mathcal{S}_3(f)$ denotes the degree $d=3$ Sobolev norm as before.

Clearly, the above result for the geodesic flow translates into the following equivalent exponential k-mixing result for the time-reversed geodesic flow.

Corollary B.4. With the same constants as above we have that

$$\left|\int_Y \prod_{j=1}^n \left(f_j\circ T^{j\cdot k}\right)dx - \prod_{j=1}^n\left(\int_Y f_j\,dx\right)\right| \leq L\cdot(n-1)\cdot D^n\cdot e^{-\frac{\alpha}{1+\alpha}k}\cdot\prod_{j=1}^n\mathcal{S}(f_j)$$

for all $n\in\mathbb{N}$, for all $f_1,\dots,f_n\in C_c^\infty(\mathrm{UT}(S))$ and all integers $k\geq -\log(\alpha)-(1+\alpha)\log(2)$.

Proof of Corollary B.4. Indeed, the measure μ is invariant under $T^{-n\cdot k}\colon Y\to Y$. Thus,

$$\begin{aligned}
&\left|\int_Y \prod_{j=1}^n \left(f_j\circ T^{j\cdot l}\right)dx - \prod_{j=1}^n\left(\int_Y f_j\,dx\right)\right| \\
&=\left|\int_Y \prod_{j=1}^n \left(f_j\circ T^{-(n-j)\cdot k}\right)dx - \prod_{j=1}^n\left(\int_Y f_j\,dx\right)\right| \\
&\leq L\cdot(n-1)\cdot D^n\cdot e^{-\frac{\alpha}{1+\alpha}k}\cdot\prod_{j=1}^n\mathcal{S}(f_j)
\end{aligned}$$

after reversing the indices $f_j \leftrightarrow f_{n-j}$. ■

We thank Einsiedler for suggesting the strategy for the following proof of Theorem B.3.

Proof of Theorem B.3. We will prove the theorem by induction on the number of functions n. The induction base $n = 2$ is the classical result that the geodesic flow is exponentially mixing. However, we include a proof here, since it illustrates the idea for the induction step.

Let $f_1, f_2 \in C_c^\infty(Y)$, $k \in \mathbb{N}$, and $T \geq 1/2$. We will choose T appropriately later on. In a first step we split the error into two terms Δ, Δ' that we will then bound separately.

$$\begin{aligned}
&\left|\int_Y f_1(x) f_2(T^{-k}(x))dx - \int_Y f_1(x)dx \int_Y f_2(x)dx\right| \\
&= \left|\int_Y f_1(x) f_2(xa_{-k})dx - \int_Y f_1(x)dx \int_Y f_2(x)dx\right| \\
&= \left|\frac{1}{T}\int_0^T \int_Y f_1(xu_s) f_2(xu_s a_{-k})dxds - \int_Y f_1(x)dx \int_Y f_2(x)dx\right| \\
&\leq \Delta + \Delta',
\end{aligned}$$

where

$$\begin{aligned}
\Delta &:= \left|\frac{1}{T}\int_0^T \int_Y f_1(xu_s) f_2(xu_s a_{-k})dxds - \frac{1}{T}\int_0^T \int_Y f_1(x) f_2(xu_s a_{-k})dx\right|, \\
\Delta' &:= \left|\int_Y f_1(x)\left(\frac{1}{T}\int_0^T f_2(xu_s a_{-k})ds\right)dx - \int_Y f_1(x)\left(\int_Y f_2(y)dy\right)dx\right|.
\end{aligned}$$

Let us first bound Δ. We denote by

$$U := \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} \exp(u_t) \in \mathfrak{sl}_3(\mathbb{R}).$$

By the Sobolev Embedding Theorem 2.2 there is a constant $D_0 > 0$ such that

(B.1)

$$|f(xu_t) - f(x)| = \left|(\lambda_{\exp(tU)} f)(x) - f(x)\right| \leq \int_0^t |(U \cdot f)(xu_s)|\, ds \leq t \cdot \|U \cdot f\|_\infty \leq D_0 \cdot \mathcal{S}(f) \cdot t$$

for all $f \in C_c^\infty(\Gamma\backslash G)$, $t \in \mathbb{R}$, $x \in \Gamma\backslash G$.

Thus, we may estimate

$$\Delta \leq \frac{1}{T}\int_0^T \int_Y \underbrace{|f_1(xu_s) - f_1(x)|}_{\leq D_0 \mathcal{S}(f_1) T} \underbrace{|f_2(xu_s a_{-k})|}_{\leq \|f_2\|_\infty \leq K_{\mathrm{Sob}} \mathcal{S}(f_2)} dxds \leq D_0 K_{\mathrm{Sob}} \mathcal{S}(f_1)\mathcal{S}(f_2) T.$$

Using Corollary B.2 we may estimate Δ' as follows.

$$\begin{aligned}
\Delta' &= \left| \int_Y f_1(x) \left(\frac{1}{T} \int_0^T f_2(x a_{-k} a_k u_s a_{-k}) ds \right) dx - \int_Y f_1(x) \left(\int_Y f_2(y) dy \right) dx \right| \\
&\leq \int_Y \underbrace{|f_1(x)|}_{\leq \|f_1\|_\infty} \underbrace{\left| \frac{1}{T} \int_0^T f_2(x a_k u_{e^k s}) ds}_{= \frac{1}{e^k T} \int_0^{e^k T} f_2(x a_k u_t) dt} - \int_Y f_2(y) dy \right| dx \\
&\overset{(\diamond)}{\leq} K_{\mathrm{Sob}} \mathcal{S}(f_1) \cdot C \cdot \mathcal{S}(f_2) (e^k T)^{-\alpha} \leq C K_{\mathrm{Sob}} \mathcal{S}(f_1) \mathcal{S}(f_2) T^{-\alpha} e^{-\alpha k},
\end{aligned}$$

where we assumed in $(\diamond)$ that $e^k T \geq 1/2$.

Altogether

$$\Delta + \Delta' \leq D^2 \mathcal{S}(f_1) \mathcal{S}(f_2) \underbrace{(T + T^{-\alpha} e^{-\alpha k})}_{=: F_2(T)}$$

for $D := \max(1, C, K_{\mathrm{Sob}}, D_0)$.

The function F_2 attains its global minimum at its unique critical point

$$T_2 := \alpha^{1/(1+\alpha)} e^{k/(1+\alpha)},$$

with

$$F_2(T_2) = \underbrace{\left(\alpha^{1/(1+\alpha)} + \alpha^{-\alpha/(1+\alpha)} \right)}_{\leq L} e^{-\frac{\alpha}{1+\alpha} k},$$

where we set $L := \max(\alpha^{1/(1+\alpha)} + \alpha^{-\alpha/(1+\alpha)}, 1 + 1/(1 - e^{-1}))$. Observe that

$$e^k T_2 = \alpha^{1/(1+\alpha)} e^{k/(1+\alpha)} \geq \frac{1}{2} \quad \Longleftrightarrow \quad k \geq -\log(\alpha) - (1+\alpha)\log(2).$$

All in all,

$$\Delta + \Delta' \leq L D^2 \mathcal{S}(f_1) \mathcal{S}(f_2) e^{-\frac{\alpha}{1+\alpha} k}$$

for all $k \geq -\log(\alpha) - (1+\alpha)\log(2)$. This concludes the proof of the induction base.

We now proceed with the induction step, going from $n-1$ to n functions. Let $f_1, \ldots, f_n \in C_c^\infty(Y)$, $T \geq 1/2$, and $k \in \mathbb{N}$. As for the induction base we will determine T later on. This time we will split the error in three terms $\Delta_1, \Delta_2, \Delta_3$ that we will then bound separately.

$$\begin{aligned}
&\left| \int_Y f_1(x) \cdot \ldots \cdot f_n(x a_{-k(n-1)}) dx - \prod_{j=1}^n \int_Y f_j(x) dx \right| \\
&= \left| \frac{1}{T} \int_0^T \int_Y f_1(x u_s) \cdot \ldots \cdot f_n(x u_s a_{-k(n-1)}) dx ds - \prod_{j=1}^n \int_Y f_j(x) dx \right|
\end{aligned}$$

$$\le \Delta_1 + \Delta_2 + \Delta_3,$$

where we set

$$\begin{aligned}
\Delta_1 &:= \bigg| \frac{1}{T} \int_0^T \int_Y f_1(xu_s) \cdot \ldots \cdot f_n(xu_s a_{-k(n-1)}) dx ds \\
&\qquad - \frac{1}{T} \int_0^T \int_Y f_1(x) \cdot \ldots \cdot f_{n-1}(xa_{-k(n-2)}) \cdot f_n(xu_s a_{-k(n-1)}) dx ds \bigg|, \\
\Delta_2 &:= \bigg| \frac{1}{T} \int_0^T \int_Y f_1(x) \cdot \ldots \cdot f_{n-1}(xa_{-k(n-2)}) \cdot f_n(xu_s a_{-k(n-1)}) dx ds \\
&\qquad - \int_Y f_1(x) \cdot \ldots \cdot f_{n-1}(xa_{-k(n-2)}) \cdot \left(\int_Y f_n(y) dy \right) dx \bigg|, \\
\Delta_3 &:= \bigg| \int_Y f_1(x) \cdot \ldots \cdot f_{n-1}(xa_{-k(n-2)}) \cdot \left(\int_Y f_n(y) dy \right) dx \\
&\qquad - \int_Y f_1(x) dx \cdot \ldots \cdot \int_Y f_n(x) dx \bigg|.
\end{aligned}$$

Let us start with Δ_3. By the induction hypothesis we have that

$$\begin{aligned}
\Delta_3 &\le \left| \int_Y f_1(x) \cdot \ldots \cdot f_{n-1}(xa_{-k(n-2)}) dx - \int_Y f_1(x) dx \cdot \ldots \cdot \int_Y f_{n-1}(x) dx \right| \cdot \int_Y |f_n(x)| dx \\
&\le \|f_n\|_\infty L(n-2) D^{n-2} e^{-\frac{\alpha}{1+\alpha} k} \prod_{j=1}^{n-1} \mathcal{S}(f_j) \\
&\le L(n-2) D^{n-1} e^{-\frac{\alpha}{1+\alpha} k} \prod_{j=1}^{n} \mathcal{S}(f_j).
\end{aligned}$$

Next, we consider Δ_2. By Corollary B.2 we obtain

$$\begin{aligned}
\Delta_2 &\le \int_Y |f_1(x)| \cdot \ldots \cdot \left| f_{n-1}(xa_{-k(n-2)}) \right| \cdot \left| \frac{1}{T} \int_0^T f_n(xa_{-k(n-1)} u_{e^{(n-1)k} s}) ds - \int_Y f_n(y) dy \right| dx \\
&\le K_{\mathrm{Sob}}^{n-1} \prod_{j=1}^{n-1} \mathcal{S}(f_j) \cdot \int_Y \left| \frac{1}{e^{(n-1)k} T} \int_0^{e^{(n-1)k} T} f_n(xa_{-k(n-1)} u_s) ds - \int_Y f_n(y) dy \right| dx \\
&\le C K_{\mathrm{Sob}}^{n-1} e^{-\alpha k(n-1)} T^{-\alpha} \prod_{j=1}^{n} \mathcal{S}(f_j) \\
&\le D^n e^{-\alpha k(n-1)} T^{-\alpha} \prod_{j=1}^{n} \mathcal{S}(f_j),
\end{aligned}$$

where we assumed that $e^{(n-1)k} T \ge 1/2$.

Finally, we turn to Δ_1. Using a telescope sum argument one can show that

$$\left|\prod_{j=1}^{n-1} x_j - \prod_{j=1}^{n-1} y_j\right| \le \sum_{i=1}^{n-1} \left|x_i - y_i\right| \prod_{j\neq i} \max\left(|x_j|, |y_j|\right) \tag{B.2}$$

for all $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1} \in \mathbb{R}$. We will use (B.2) with

$$x_i := f_i(xu_s a_{-k(i-1)}) = f_i(xa_{-k(i-1)} u_{e^{k(i-1)}s}), \qquad y_i := f_i(xa_{-k(i-1)}),$$

for all $i = 1, \dots, n-1$. Notice that in this case $\max\left(|x_j|, |y_j|\right) \le \left\|f_j\right\|_\infty$ for all $j = 1, \dots, n-1$. Moreover,

$$\left|x_i - y_i\right| = \left|f_i(xa_{-k(i-1)} u_{e^{k(i-1)}s}) - f_i(xa_{-k(i-1)})\right| \le D_0 \mathcal{S}(f_i) e^{k(i-1)} T$$

for all $0 \le s \le T$, $i = 1, \dots, n-1$ by (B.1).

With these observations we may estimate Δ_1 as follows.

$$\begin{aligned}
\Delta_1 &\le \frac{1}{T} \int_0^T \int_Y \left|f_1(xu_s) \cdot \ldots \cdot f_{n-1}(xa_{-k(n-2)} u_{e^{k(n-2)}s}) - f_1(x) \cdot \ldots \cdot f_{n-1}(xa_{-k(n-2)})\right| \\
&\qquad \cdot \left|f_n(xu_s a_{-k(n-1)})\right| dx ds \\
&\le \|f_n\|_\infty \frac{1}{T} \int_0^T \int_Y \sum_{i=1}^{n-1} D_0 \mathcal{S}(f_i) e^{k(i-1)} T \prod_{j\neq i} \left\|f_j\right\|_\infty dx ds \\
&\le D^n \frac{e^{k(n-1)} - 1}{e^k - 1} \prod_{j=1}^{n} \mathcal{S}(f_j) \\
&\le D_1 D^n e^{k(n-2)} \prod_{j=1}^{n} \mathcal{S}(f_j),
\end{aligned}$$

where we set $D_1 := 1/(1 - e^{-1}) \ge 1/(1 - e^{-k})$.

Combining these estimates we get

$$\begin{aligned}
&\Delta_1 + \Delta_2 + \Delta_3 \\
&\le D_1 D^n e^{k(n-2)} \prod_{j=1}^{n} \mathcal{S}(f_j) + D^n e^{-\alpha k(n-1)} T^{-\alpha} \prod_{j=1}^{n} \mathcal{S}(f_j) + L(n-2) \underbrace{D^{n-1}}_{\le D^n} e^{-\frac{\alpha}{1+\alpha} k} \prod_{j=1}^{n} \mathcal{S}(f_j) \\
&= D^n \prod_{j=1}^{n} \mathcal{S}(f_j) \cdot \underbrace{\left(D_1 T e^{k(n-2)} + e^{-\alpha(n-1)k} T^{-\alpha} + L(n-2) e^{-\frac{\alpha}{1+\alpha} k}\right)}_{=: F_n(T)}.
\end{aligned}$$

The function F_n attains its global minimum at its unique critical point

$$T_n := \alpha^{\frac{1}{1+\alpha}} \exp\left(-\frac{k}{1+\alpha}((n-2) + \alpha(n-1))\right)$$

with

$$F_n(T_n) = (\underbrace{(1+D_1)}_{=1+1/(1-e^{-1})\le L} + L(n-2))e^{-\frac{\alpha}{1+\alpha}k} \le L(n-1)e^{-\frac{\alpha}{1+\alpha}k}.$$

Also, notice that

$$T_n e^{(n-1)k} = \alpha^{\frac{1}{1+\alpha}} \exp\left(\frac{k}{1+\alpha}\right) \ge \frac{1}{2} \quad \iff \quad k \ge -\log(\alpha) - (1+\alpha)\log(2).$$

All in all,

$$\Delta_1 + \Delta_2 + \Delta_3 \le L(n-1)D^n e^{-\frac{\alpha}{1+\alpha}k} \prod_{j=1}^{n} \mathcal{S}(f_j)$$

for all $k \ge -\log(\alpha) - (1+\alpha)\log(2)$. This concludes the proof.

■

Appendix C. Tube Lemma

Let M^{n+1} be an oriented Riemannian manifold with bounded sectional curvature $|K| \le b$. Let $S^n \subseteq M$ be a codimension 1 orientable hypersurface with normal field $N\colon S \to TS^{\perp} \subseteq TM$. We denote by ω the volume form on M, and by $\overline{\omega}$ the induced volume form on S.

At every point $x \in S$ we can find an orthonormal basis $\mathcal{B} = \{v_1, \dots, v_n\} \subseteq T_xS$ of its tangent space. We consider the symmetric matrix

$$\overline{N}(x) = \left(\langle \nabla_{v_i} N, v_j \rangle\right)_{1\le i,j\le n}.$$

Its eigenvalues $\kappa_1(x), \dots, \kappa_n(x)$ are called the principal curvatures at x of S in M. Note that in this case $\|\overline{N}(x)\|_2 = \max_{i=1,\dots,n} |\kappa_i(x)|$, and this quantity is independent of the basis $\mathcal{B}$.

We can parametrize the ε-tubular neighborhood about S via the map

$$\varphi\colon (-\varepsilon,\varepsilon)\times S \to M, (t,x) \mapsto \mathrm{Exp}_x(t\cdot N(x)).$$

There is a smooth function $\rho \in C^{\infty}((-\varepsilon,\varepsilon)\times S)$ such that

$$\varphi^*\omega = \rho \cdot dt \wedge \overline{\omega}.$$

Lemma C.1 (Tube Lemma)**.** Let $\widehat{b} := \max(1,b)$. There is a constant $C > 0$ such that

$$\left|\rho(t,x)\right| \le C \cdot (1 + \left\|\overline{N}(x)\right\|_2)^n \cdot e^{\widehat{b}nt}$$

for all $(t,x) \in (-\varepsilon,\varepsilon)\times S$.

In particular, if there is a uniform bound $\kappa > 0$ on the principal curvatures, i.e.

$$\left\|\overline{N}(x)\right\|_2 = \max_{i=1,\dots,n} |\kappa_i(x)| \le \kappa$$

for every $x \in S$, then

$$\|\rho\|_\infty \le C \cdot (1+\kappa)^n \cdot e^{\widehat{b}n\varepsilon},$$

such that

$$\mathrm{vol}_M(\varphi(B)) \le C \cdot (1+\kappa^n) \cdot e^{\widehat{b}n\varepsilon} \cdot \mathrm{vol}_{(-\varepsilon,\varepsilon)\times S}(B)$$

for every Borel set $B \subseteq (-\varepsilon,\varepsilon)\times S$ with respect to the volume form $dt \wedge \overline{\omega}$.

Proof. Let $x \in S$ be a point and denote by $c(t) := \mathrm{Exp}_x(t \cdot N(x))$ the normal geodesic at x. Let $v_0 := \dot{c}(0), v_1, \dots, v_n \in T_xS \subseteq T_xM$ be an orthonormal basis and let $E_i(t)$ be the parallel vector field along $c(t)$ such that: $E_i(0) = v_i$. In particular, $E_0(t) = \dot{c}(t)$. Further, let $(t,x) \in (-\varepsilon,\varepsilon)\times S$. Then, $\partial_t \in T_t(-\varepsilon,\varepsilon) \subseteq T_{(t,x)}((-\varepsilon,\varepsilon)\times S) = T_t(-\varepsilon,\varepsilon) \oplus T_xS$, $v_i \in T_xS \subseteq T_{(t,x)}((-\varepsilon,\varepsilon)\times S)$, and $(\partial_t, v_1, \dots, v_n)$ is an orthonormal basis for $T_{(t,x)}((-\varepsilon,\varepsilon)\times S)$. Finally, choose $\gamma_i : \mathbb{R} \to S$ such that $\dot{\gamma}_i(0) = v_i$.

Then

$$\begin{aligned}\rho(t,x) &= \rho(t,x)(dt \wedge \overline{\omega})(\partial_t, v_1, \dots, v_n) = \\ &= (\varphi^*\omega)(\partial_t, v_1, \dots, v_n) = \\ &= \omega(d\varphi(\partial_t), d\varphi(v_1), \dots, d\varphi(v_n)).\end{aligned}$$

Note that

$$d\varphi(\partial_t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Exp}_x(t \cdot N(x)) = \dot{c}(t) = E_0(t), \qquad d\varphi(v_i) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathrm{Exp}_{\gamma_i(s)}(t \cdot N(\gamma_i(s))) =: J_i(t).$$

As a geodesic variation $J_i(t)$ is a Jacobi field, such that it satisfies the Jacobi equation

$$J_i'' = -R(\dot{c}, J_i)\dot{c} = -R(E_0, J_i)E_0$$

with the initial conditions

$$\begin{aligned}J_i(0) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathrm{Exp}_{\gamma_i(s)}(0 \cdot N(\gamma_i(s))) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \gamma_i(s) = v_i = E_i(0), \quad \text{and} \\ J_i'(0) &= \frac{D}{\partial t}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{t,s=0} \mathrm{Exp}_{\gamma_i(s)}(t \cdot N(\gamma_i(s))) \\ &= \frac{D}{\partial s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t,s=0} \mathrm{Exp}_{\gamma_i(s)}(t \cdot N(\gamma_i(s))) = \frac{D}{\partial s}\Big|_{s=0} N(\gamma_i(s)) = \nabla_{v_i} N.\end{aligned}$$

In parallel coordinates $J_i(t) = \sum_j \alpha_i^j(t) \cdot E_j(t)$ for some function $\alpha_i^j(t)$. Therefore, $J_i'' = \sum_j \ddot{\alpha}_i^j E_j$. Then,

$$\begin{aligned}-\langle R(E_0, J_i)E_0, E_j\rangle &= -\langle R(E_0, \sum_k \alpha_i^k E_k)E_0, E_j\rangle = \\ &= -\sum_k \alpha_i^k \langle R(E_0, E_k)E_0, E_j\rangle\end{aligned}$$

and we let $R_{kj} := \langle R(E_0, E_k)E_0, E_j\rangle$. By curvature symmetries we get $R_{kj} = R_{jk}$. Using matrix notation $A_{ij} := \alpha_i^j$ we have:

- $A'' = -AR$;
- $A(0) = \mathrm{id}$;
- $A'(0) = \left(\langle \nabla_{v_i} N, v_j\rangle\right)_{ij} = \overline{N}(x)$.

We reduce this to a first order system as follows:

$$(A' \ \ A'') = (A \ \ A') \cdot \begin{pmatrix} 0 & -R \\ \mathrm{id} & 0 \end{pmatrix}.$$

It follows from ODE theory that

$$\left\|(A(t) \ \ A'(t))\right\|_2 \le \left\|(A(0) \ \ A'(0))\right\|_2 \cdot \exp\left(t \left\| \begin{pmatrix} 0 & -R(t) \\ \mathrm{id} & 0 \end{pmatrix} \right\|_2 \right). \tag{C.1}$$

Notice that

$$\left\| \begin{pmatrix} 0 & -R \\ \mathrm{id} & 0 \end{pmatrix} \right\|_2 = \max(\|\mathrm{id}\|_2, \|R\|_2) = \max(1, \|R\|_2).$$

Since $R^T = R$ we can diagonalise $R_{jk} = \langle R(E_0, E_j)E_0, E_k\rangle$ by $D = QRQ^T$ with Q orthogonal and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Set $F_i := \sum_j Q_i^j E_j$. Then,

$$\begin{aligned}\langle R(E_0, F_i)E_0, F_j\rangle &= \sum_{k,\ell} Q_i^k \langle R(E_0, E_k)E_0, E_\ell\rangle Q_j^\ell = \\ &= \left(QRQ^T\right)_{ij} = \delta_{ij}\lambda_i,\end{aligned}$$

which implies that

$$\lambda_i = \langle R(E_0, F_i)E_0, F_i\rangle = K(\langle E_0, F_i\rangle)$$

for K the sectional curvature.

Therefore, we get that $\|R\|_2 = \|D\|_2 = \max_{i=1,\dots,n} |K(\langle E_0, F_i\rangle)| \le b$ by our curvature bound $|K| \le b$, and consequently

$$\max(1, \|R\|_2) \le \max(1, b) = \widehat{b}.$$

From (C.1) we obtain

$$\|A\|_2 \le \left\|(A(t) \ \ A'(t))\right\|_2 \le \left\|(I \ \ \overline{N}(x))\right\|_2 e^{\widehat{b}t} \le (1 + \left\|\overline{N}(x)\right\|_2) e^{\widehat{b}t}.$$

Hence,

$$\begin{aligned} &\rho(t,x)\omega(d\varphi(\partial_t), d\varphi(v_1), \dots, d\varphi(v_n)) \\ &= \omega(E_0, J_1, \dots, J_n) \end{aligned}$$

$$
\begin{aligned}
&= \det\begin{pmatrix} 1 & \langle J_1, E_0\rangle & \dots & \langle J_n, E_0\rangle \\ 0 & \langle J_1, E_1\rangle & \dots & \langle J_n, E_1\rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \langle J_1, E_n\rangle & \dots & \langle J_n, E_n\rangle \end{pmatrix} \\
&= \det(\langle J_i, E_j\rangle_{1\le i,j\le n}) \\
&= \underbrace{\det(A_{ij})}_{\text{polynomial of degree} \dim S} \\
&\le C\,\|A\|_2^{\dim S} \\
&\le C(1 + \left\|\overline{N}(x)\right\|_2)^n \cdot e^{n\widehat{bt}},
\end{aligned}
$$

for some constant $C > 0$. ■

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