EVERY COUNTABLE GROUP IS A SUBGROUP OF THE MAPPING CLASS GROUP OF THE LOCH NESS MONSTER SURFACE

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Abstract. In this short note we give an elementary proof of the fact that every countable group is a subgroup of the mapping class group of the Loch Ness monster surface.

1. Introduction

The following theorem follows from a deep result of Aougab–Patel–Vlamis [APV21] about hyperbolic isometry groups of infinite type surfaces:

Theorem 1.1. *Every countable group G is a subgroup of* $\mathrm{MCG}(L)$ *where L denotes the Loch Ness monster surface.*

The purpose of this short note is to provide an elementary proof of this fact.

2. Proof of Theorem 1.1

By the Higman–Neumann–Neumann embedding theorem [HNN49, Theorem IV] any countable group G is a subgroup of a finitely generated group H'. Thus, G is also a subgroup of the product $H := H' \times \mathbb{Z} \times \mathbb{Z}$.

Let S' denote a finite generating set for H'. Then $S = \{(s',0,0) \mid s' \in S'\} \cup \{(e,1,0),(e,0,1)\} \subseteq H' \times \mathbb{Z} \times \mathbb{Z}$ is a generating set of H. Observe that the Cayley graph $\Gamma := \mathrm{Cay}_S(H)$ is one-ended and has degree at least 4. Therefore, the surface Σ that is obtained by thickening the Cayley graph Γ of H is one-ended, too. Moreover, one may check that Σ has infinite genus. By the classification of topological surfaces of infinite type [Ker23; Ric63] Σ is thus homeomorphic to the Loch Ness monster L; see Figure 1.

As a subgroup of H the group G acts freely on Γ via homeomorphisms. This action gives rise to a free action of G on $\Sigma \cong L$ by orientation preserving homeomorphisms, $\varphi\colon G \hookrightarrow \mathrm{Homeo}_+(L)$. Moreover, one obtains a continuous G-equivariant surjection $\pi\colon L \to \Gamma$ by projecting the thickening L to Γ.

For the monomorphism φ to pass to the mapping class group $\mathrm{MCG}(L) = \mathrm{Homeo}_+(L)/\mathrm{Homeo}_+^\circ(L)$ we will check that $\varphi(G) \cap \mathrm{Homeo}_+^\circ(L) = \{\mathrm{id}\}$. Assume to the contrary that there is a non-trivial element $g \in G$ such that $\varphi(g)$ is isotopic to the identity. Let e be an edge in the identity coset $\mathbb{Z}^2$ and observe that $ge \neq e$. Note that $\Gamma - \{e, ge\}$ is connected: Both cosets $\mathbb{Z}^2$ and $g \cdot \mathbb{Z}^2$ remain connected (as subgraphs of Γ) after removing the edges e and ge, respectively. As any vertex in Γ can be connected to any $\mathbb{Z}^2$-coset via a path with edges in $S' \subseteq H'$, the graph $\Gamma - \{e, ge\}$ is connected. Let γ be the core curve in $\pi^{-1}(e^\circ) \cong S^1 \times (0,1)$. Then $g\gamma$ needs to be isotopic to the core curve of, and contained in, $\pi^{-1}(ge^\circ)$. However, γ and $g\gamma$ cannot be isotopic: Otherwise, a component of $L - \{\gamma, g\gamma\}$ would be an annulus. But $L - \{\gamma, g\gamma\}$ is connected, since $\Gamma - \{e, ge\}$ is connected; a contradiction. □

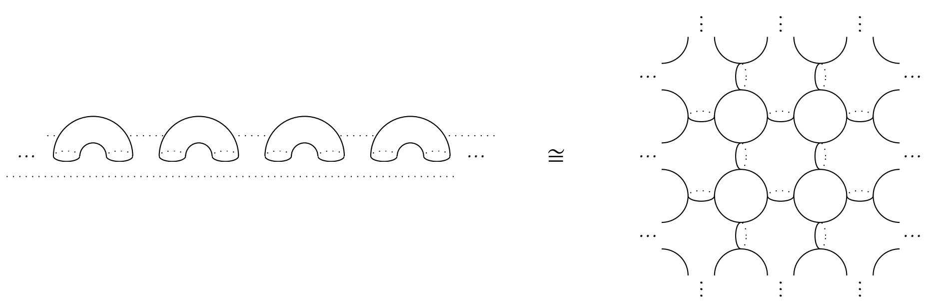

Figure 1. The Loch Ness monster surface L of [Ghy95], also known as the infinite prison window.

Date: December 6, 2021.

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