考试题（A卷）

一、计算下列数列或函数的极限（请从三道题目中任选二道题，多选的话则按照前两道题目给分。每题5分，合计10分）

1. $\lim\limits_{x\to\infty}\left(1+\frac{1}{n}-\frac{1}{n^2}\right)^{\mathrm{n}}$.

解 （方法一）

$$\lim_{x\to\infty}\left(1+\frac{1}{n}-\frac{1}{n^2}\right)^{\mathrm{n}}=\lim_{x\to\infty}\left(1+\frac{n-1}{n^2}\right)^{\mathrm{n}}$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{n-1}{n^2}\right)^{\frac{n^2}{n-1}}\right]^{\frac{\mathrm{n}(n-1)}{n^2}}=e.$$

（方法二）

$$\lim_{x\to\infty}\left(1+\frac{1}{n}-\frac{1}{n^2}\right)^{\mathrm{n}}=\lim_{x\to\infty}e^{\mathrm{n}\ln\left(1+\frac{n-1}{n^2}\right)}$$

$$=e^{\lim\limits_{x\to\infty}\mathrm{n}\ln\left(1+\frac{n-1}{n^2}\right)}=e^{\lim\limits_{x\to\infty}\mathrm{n}\cdot\frac{n-1}{n^2}}=e^1=e.$$

2. $\lim\limits_{x\to 0}\dfrac{\int_0^x (x-t)f(t)dt}{x^2}$，其中 $f(x)$ 是一个连续函数.

解

$$\lim_{x\to 0}\frac{\int_0^x (x-t)f(t)dt}{x^2}=\lim_{x\to 0}\frac{x\int_0^x f(t)dt-\int_0^x tf(t)dt}{x^2}$$

$$=\lim_{x\to 0}\frac{\int_0^x f(t)dt+xf(x)-xf(x)}{2x}$$

$$=\lim_{x\to 0}\frac{f(x)}{2}=\frac{f(0)}{2}.$$

3. **求二元函数** $\lim\limits_{(x,y)\to(0,0)}(x+2y)\ln\left(x^4+y^4\right)$**的极限.**

解（方法一） 平面极坐标为(ρ,θ)。由于$(x,y)\to(0,0)$，不妨设$|x|\le\frac{1}{2},|y|\le\frac{1}{2}$，

于是

$$
\begin{aligned}
&\max(|x|,|y|)\ge\frac{\sqrt{2}}{2}\rho,\\
&x^4+y^4\ge\frac{1}{4}\rho^4,\\
&\left|\ln\left(x^4+y^4\right)\right|=\ln\frac{1}{x^4+y^4}\le\ln\frac{4}{\rho^4}=2\ln 2-4\ln\rho,
\end{aligned}
$$

所以

$$0\le\left|(x+2y)\ln\left(x^4+y^4\right)\right|\le 6(\ln 2-2\ln\rho)\rho\to 0$$

$$\lim_{(x,y)\to(0,0)}(x+2y)\ln\left(x^4+y^4\right)=0$$

解（方法二） **有界量与无穷小量之积是无穷小量**，所以

$$
\begin{aligned}
&\lim_{(x,y)\to(0,0)}(x+2y)\ln\left(x^4+y^4\right)\\
&=\lim_{(x,y)\to(0,0)}\left[\frac{(x+2y)}{\left(x^4+y^4\right)^{\frac{1}{4}}}\cdot\left(x^4+y^4\right)^{\frac{1}{4}}\ln\left(x^4+y^4\right)\right]=0
\end{aligned}
$$

二、 **（8分）过原点作抛物线 $y=f(x)=\sqrt{x-1}$ 的切线，设 D 是该切线与上述抛物线及 x 轴围成的平面区域. 求区域 D 绕 x 轴旋转一周所得旋转体的体积.**

解 设切点为 $\left(x_0,y_0\right)$，则

$$\begin{cases} y_0=\sqrt{x_0-1} \\ \dfrac{y_0}{x_0}=\dfrac{1}{2\sqrt{x_0-1}} \end{cases}$$

解方程组得 $\left(x_0,y_0\right)=\left(2,1\right)$。

所求的旋转体是一个圆锥减去一个旋转抛物面，于是

$$V=\frac{1}{3}\cdot\pi\cdot1^2\cdot2-\int_1^2\pi\cdot f^2(x)dx=\frac{2}{3}\pi-\pi\int_1^2\left(x-1\right)dx=\frac{1}{6}\pi.。$$

三、求下列积分（共**2**小题，每小题**5**分，共**10**分）：

1. $\int \frac{x^2-7}{x^2+x-2}dx$.

解

$$\begin{aligned}
&\int \frac{x^2-7}{x^2+x-2}dx = \int \frac{x^2+x-2-x-5}{x^2+x-2}dx \\
&= \int\left(1-\frac{x+5}{x^2+x-2}\right)dx \\
&= \int\left(1+\frac{1}{x+2}-\frac{2}{x-1}\right)dx \\
&= x+\ln\frac{|x+2|}{(x-1)^2}+C.
\end{aligned}$$

2. $\int \arctan\sqrt{x}dx$.

解 做变量代换 $t=\sqrt{x}$，则

$$\begin{aligned}
&\int \arctan\sqrt{x}dx = \int \arctan t dt^2 \\
&= t^2\arctan t - \int t^2 d\arctan t = t^2\arctan t - \int \frac{t^2}{1+t^2}dt \\
&= t^2\arctan t - \int\left(1-\frac{1}{1+t^2}\right)dt = (t^2+1)\arctan t - t + C \\
&= (x+1)\arctan\sqrt{x} - \sqrt{x} + C
\end{aligned}$$

四、（**7分**）**证明** $\int_0^{\frac{\pi}{2}}\frac{x}{1+\cos x+\sin x}dx=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin x+\cos x}dx$.

解 做变量代换 $x=\frac{\pi}{2}-t$，则

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}\frac{x}{\cos x+1+\sin x}dx\\
&=\int_{\frac{\pi}{2}}^{0}\frac{\frac{\pi}{2}-t}{\cos\left(\frac{\pi}{2}-t\right)+1+\sin\left(\frac{\pi}{2}-t\right)}(-dt),t=\frac{\pi}{2}-x\\
&=\int_0^{\frac{\pi}{2}}\frac{\frac{\pi}{2}-t}{\sin t+1+\cos t}dt\\
&=\int_0^{\frac{\pi}{2}}\frac{\frac{\pi}{2}-x}{\sin x+1+\cos x}dx\\
&=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+1+\cos x}dx-\int_0^{\frac{\pi}{2}}\frac{x}{\cos x+1+\sin x}dx,
\end{aligned}$$

故 $\int_0^{\frac{\pi}{2}}\frac{x}{\cos x+1+\sin x}dx=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{1}{\sin x+1+\cos x}dx.$

五、（**8分**）**求原点到直线**

$$L:\begin{cases} x-z+2=0 \\ -y+2z-1=0 \end{cases}$$

的垂线方程.

解 直线的方向为

$$\begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{vmatrix} = -i-2j-k,$$

设垂足为$P(x,y,z)$，则$\overrightarrow{OP}$与L垂直，且在直线上，所以

$$\begin{cases} x-z+2=0 \\ -y+2z-1=0 \\ -x-2y-z=0 \end{cases},$$

解方程得$x=-\frac{4}{3}, y=\frac{1}{3}, z=\frac{2}{3}$。于是垂线方程为$\frac{x}{-\frac{4}{3}}=\frac{y}{\frac{1}{3}}=\frac{z}{\frac{2}{3}}$，即$\frac{x}{-4}=\frac{y}{1}=\frac{z}{2}$。

六、（**7分**）**设** $f(x)$ **在** $[0,1]$ **连续可微，** $f(0)=0$**，证明在** $(0,1)$ **中存在一点** ξ**，满足**

$$(1-\xi)f'(\xi)=f(\xi).$$

证明 设$F(x)=(1-x)f(x)$，则该函数在闭区间$[0,1]$上连续，在$(0,1)$上可导，且

$F(0)=F(1)=0$。根据罗尔定理，在$(0,1)$中存在一点ξ，满足

$$F'(\xi)=(1-\xi)f'(\xi)-f(\xi)=0,$$

即 $f'(\xi)=\frac{f(\xi)}{1-\xi}$.

七、（**7分**）求 $f(x)=\sqrt{1+x}\sin x$ 在 $x=0$ 点的带皮亚诺余项的3阶泰勒展式，并求

$f^{(3)}(0)$ 的值.

解

$$
\begin{aligned}
f(x)&=\left[1+\frac{1}{2}x+\frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2}x^2+o\left(x^2\right)\right]\left(x-\frac{1}{3!}x^3+o\left(x^4\right)\right)\\
&=\left[1+\frac{1}{2}x-\frac{1}{8}x^2+o\left(x^2\right)\right]\left(x-\frac{1}{6}x^3+o\left(x^4\right)\right)\\
&=x+\frac{1}{2}x^2-\frac{7}{24}x^3+o\left(x^3\right)
\end{aligned}
$$

$$
\therefore \frac{f^{(3)}(0)}{3!}=-\frac{7}{24},\ f^{(3)}(0)=-\frac{7}{4}
$$

八、（**7分**）设 $2n$ 次多项式 $P_{2n}(x)=1+\sum_{k=1}^{2n}(-1)^k\frac{x^k}{k}$，分析多项式的单调性，由此证明该多项式没有零点.

解 $\quad f^{'}(x)=\sum_{k=1}^{2n}(-1)^k x^{k-1}=\dfrac{-1+x^{2n}}{1+x}$， $f^{'}(x)\begin{cases}<0 & x\in(-\infty,-1)\\ <0 & x\in(-1,1)\\ >0 & x\in(1,+\infty)\end{cases}$ 。

于是

$$
f(x)\begin{cases}\text{严格单调下降} & x\in(-\infty,-1]\\ \text{严格单调下降} & x\in[-1,1]\\ \text{严格单调上升} & x\in[1,+\infty)\end{cases},\quad f(x)\begin{cases}\text{严格单调下降} & x\in(-\infty,1]\\ \text{严格单调上升} & x\in[1,+\infty)\end{cases}
$$

$$
\begin{aligned}
&f(x)\ge f(1)=1+\sum_{k=1}^{2n}(-1)^k\frac{x^k}{k}\\
&=(1-1)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right)+\frac{1}{2n}\\
&>0
\end{aligned}
$$

九、（**7分**）**求由方程$e^{xy}-\ln\left(x^2+y\right)=1$所确定的隐函数$y=y(x)$的微分.**

解

$$de^{xy}-d\ln\left(x^2+y\right)=0$$

$$e^{xy}d\left(xy\right)-\frac{1}{x^2+y}d\left(x^2+y\right)=0$$

$$e^{xy}\left(ydx+xdy\right)-\frac{1}{x^2+y}\left(2xdx+dy\right)=0$$

$$\left(xe^{xy}-\frac{1}{x^2+y}\right)dy=\left(-ye^{xy}+\frac{2x}{x^2+y}\right)dx$$

$$dy=\frac{-ye^{xy}+\dfrac{2x}{x^2+y}}{xe^{xy}-\dfrac{1}{x^2+y}}dx=\frac{-ye^{xy}\left(x^2+y\right)+2x}{xe^{xy}\left(x^2+y\right)-1}dx.$$

十、（**7分**）**当$x\to0$时，$\sin x-\tan x$是x的多少阶无穷小量？**

解 因为

$$\lim_{x\to0}\frac{\sin x-\tan x}{x^3}=-\lim_{x\to0}\frac{\tan x\left(1-\cos x\right)}{x^3}=-\lim_{x\to0}\frac{x\cdot\frac{1}{2}x^2}{x^3}=-\frac{1}{2},$$

根据无穷小量阶数的定义，$\sin x-\tan x$**是**x**的3阶无穷小量.**

十一、（**8分**）**讨论函数** $f(x,y)=|x\sin y|$ **在** $(0,0)$ **处的可微性.**

解

$$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0}{x}=0 \qquad 2分$$

$$f_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{x-0}=\lim_{y\to 0}\frac{0}{y}=0 \qquad 4分$$

$$\Delta u=|\Delta x\sin\Delta y| \qquad 5分$$

$$\frac{\Delta u-\left[f_x(0,0)\Delta x+f_y(0,0)\Delta y\right]}{\sqrt{\Delta x^2+\Delta y^2}} \qquad 6分$$

$$=\frac{|\Delta x\sin\Delta y|}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$\to 0 \qquad 8分$$

按照微分的定义，该函数在 $(0,0)$ 处可微。 8分

极限趋于0，两种处理方法

（方法一：夹逼定理）

$$0\le\frac{|\Delta x\sin\Delta y|}{\sqrt{\Delta x^2+\Delta y^2}}\le\frac{|\Delta x\Delta y|}{\sqrt{\Delta x^2+\Delta y^2}}\le\rho\to 0$$

（方法二：无穷小量与有界量相乘是无穷小量）

$$\frac{\Delta u-\left[f_x(0,0)\Delta x+f_y(0,0)\Delta y\right]}{\sqrt{\Delta x^2+\Delta y^2}}=\frac{|\Delta x|}{\sqrt{\Delta x^2+\Delta y^2}}\cdot|\sin\Delta y|\to 0$$

十二、（7分）设$u=e^{xy}\sin\left(x^2+y^2\right)$，求该函数的一阶偏导数与全微分.

解 用微分演算

$$
\begin{aligned}
&du=\sin\left(x^2+y^2\right)de^{xy}+e^{xy}d\sin\left(x^2+y^2\right)\\
&=\sin\left(x^2+y^2\right)e^{xy}d\left(xy\right)+e^{xy}\cos\left(x^2+y^2\right)d\left(x^2+y^2\right)\\
&=\sin\left(x^2+y^2\right)e^{xy}\left(xdy+ydx\right)+e^{xy}\cos\left(x^2+y^2\right)\left(2xdx+2ydy\right)\\
&=e^{xy}\left[x\sin\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\right]dy+e^{xy}\left[y\sin\left(x^2+y^2\right)+2x\cos\left(x^2+y^2\right)\right]dx
\end{aligned}
$$

所以

$$
\begin{aligned}
\frac{\partial u}{\partial x}&=e^{xy}\left[y\sin\left(x^2+y^2\right)+2x\cos\left(x^2+y^2\right)\right],\\
\frac{\partial u}{\partial y}&=e^{xy}\left[x\sin\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\right].
\end{aligned}
$$

十三、（**7分**）**设函数** $f(x,y)$**有连续的二阶偏导数，**

$$z=f\left(x_0+t\left(x_1-x_0\right),y_0+t\left(y_1-y_0\right)\right),$$

求 $\dfrac{d^2y}{dt^2}$.

解 根据复合函数链式法则

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=\left(x_1-x_0\right)\frac{\partial f}{\partial x}+\left(y_1-y_0\right)\frac{\partial f}{\partial y}.$$

同理

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)=\left(x_1-x_0\right)\frac{\partial^2 f}{\partial x^2}+\left(y_1-y_0\right)\frac{\partial^2 f}{\partial x\partial y}$$
$$\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)=\left(x_1-x_0\right)\frac{\partial^2 f}{\partial y\partial x}+\left(y_1-y_0\right)\frac{\partial^2 f}{\partial^2 y}$$

所以

$$\begin{aligned}\frac{d^2z}{dt^2}&=\left(x_1-x_0\right)\left[\left(x_1-x_0\right)\frac{\partial^2 f}{\partial x^2}+\left(y_1-y_0\right)\frac{\partial^2 f}{\partial x\partial y}\right]\\&\quad+\left(y_1-y_0\right)\left[\left(x_1-x_0\right)\frac{\partial^2 f}{\partial y\partial x}+\left(y_1-y_0\right)\frac{\partial^2 f}{\partial^2 y}\right]\\&=\left(x_1-x_0\right)^2\frac{\partial^2 f}{\partial x^2}+2\left(x_1-x_0\right)\left(y_1-y_0\right)\frac{\partial^2 f}{\partial x\partial y}+\left(y_1-y_0\right)^2\frac{\partial^2 f}{\partial^2 y}\\&=\left(\left(x_1-x_0\right)\frac{\partial}{\partial x}+\left(y_1-y_0\right)\frac{\partial}{\partial y}\right)^2 f\end{aligned}$$