

高等数学重要公式手册

三角函数

平方关系:

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\tan^2 \alpha + 1 = \sec^2 \alpha$$

$$\cot^2 \alpha + 1 = \csc^2 \alpha$$

积的关系:

$$\sin \alpha = \tan \alpha \cos \alpha$$

$$\cot \alpha = \cos \alpha \csc \alpha$$

$$\cos \alpha = \cot \alpha \sin \alpha$$

$$\sec \alpha = \tan \alpha \csc \alpha$$

$$\tan \alpha = \sin \alpha \sec \alpha$$

$$\csc \alpha = \sec \alpha \cot \alpha$$

倒数关系:

$$\tan \alpha \cot \alpha = 1$$

$$\sin \alpha \csc \alpha = 1$$

$$\cos \alpha \sec \alpha = 1$$

直角三角形 ABC 中,角 A 的
正弦等于角 A 的对边比斜边
余弦等于角 A 的邻边比斜边
正切等于对边比邻边

三角函数恒等变形公式

两角和与差的三角函数:

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = (\tan \alpha \pm \tan \beta) / (1 \mp \tan \alpha \tan \beta)$$

$$\tan(\alpha - \beta) = (\tan \alpha - \tan \beta) / (1 + \tan \alpha \tan \beta)$$

三角和的三角函数:

$$\sin(\alpha + \beta + \gamma) = \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \sin \gamma$$

$$\cos(\alpha + \beta + \gamma) = \cos \alpha \cos \beta \cos \gamma - \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \cos \gamma$$

$$\tan(\alpha + \beta + \gamma) = (\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma) / (1 - \tan \alpha \tan \beta - \tan \alpha \tan \gamma - \tan \beta \tan \gamma)$$

辅助角公式:

$$A \sin \alpha + B \cos \alpha = \frac{\sin(\alpha + t)}{\sqrt{A^2 + B^2}}, \text{ 其中}$$

$$\sin t = \frac{B}{\sqrt{A^2 + B^2}}$$

$$\cos t = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\tan t = \frac{B}{A}$$

$$A \sin \alpha + B \cos \alpha = \frac{\cos(\alpha - t)}{\sqrt{A^2 + B^2}}, \quad \tan t = \frac{A}{B}$$

倍角公式:

$$\sin(2\alpha) = 2 \sin \alpha \cdot \cos \alpha = 2 / (\tan \alpha + \cot \alpha)$$

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

$$\tan(2\alpha) = 2 \tan \alpha / [1 - \tan^2 \alpha]$$

三倍角公式:

$$\sin(3\alpha) = 3 \sin \alpha - 4 \sin^3 \alpha$$

$$\cos(3\alpha) = 4 \cos^3 \alpha - 3 \cos \alpha$$

半角公式:

$$\sin(\alpha/2) = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos(\alpha/2) = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan(\alpha/2) = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \sin \alpha / (1 + \cos \alpha) = (1 - \cos \alpha) / \sin \alpha$$

降幂公式

$$\sin^2 \alpha = (1 - \cos(2\alpha))/2$$

$$\cos^2 \alpha = (1 + \cos(2\alpha))/2$$

$$\tan^2 \alpha = (1 - \cos(2\alpha)) / (1 + \cos(2\alpha))$$

万能公式:

$$\sin \alpha = 2 \tan(\alpha/2) / [1 + \tan^2(\alpha/2)]$$

$$\cos \alpha = [1 - \tan^2(\alpha/2)] / [1 + \tan^2(\alpha/2)]$$

$$\tan \alpha = 2 \tan(\alpha/2) / [1 - \tan^2(\alpha/2)]$$

积化和差公式:

$$\sin \alpha \cdot \cos \beta = (1/2)[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \cdot \sin \beta = (1/2)[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\cos \alpha \cdot \cos \beta = (1/2)[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin \alpha \cdot \sin \beta = -(1/2)[\cos(\alpha + \beta) - \cos(\alpha - \beta)]$$

和差化积公式:

$$\sin \alpha + \sin \beta = 2 \sin[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2]$$

$$\sin \alpha - \sin \beta = 2 \cos[(\alpha + \beta)/2] \sin[(\alpha - \beta)/2]$$

$$\cos \alpha + \cos \beta = 2 \cos[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2]$$

$$\cos \alpha - \cos \beta = -2 \sin[(\alpha + \beta)/2] \sin[(\alpha - \beta)/2]$$

推导公式

$$\tan \alpha + \cot \alpha = 2 / \sin 2\alpha$$

$$\tan \alpha - \cot \alpha = -2 \cot 2\alpha$$

$$1 + \cos 2\alpha = 2 \cos^2 \alpha$$

$$1 - \cos 2 = 2 \sin^2$$

$$1 + \sin = (\sin \frac{1}{2} + \cos \frac{1}{2})^2$$

其他:

$$\sin + \sin(\frac{1}{2} + \frac{1}{n}) + \sin(\frac{1}{2} + \frac{2}{n}) + \sin(\frac{1}{2} + \frac{3}{n}) + \dots + \sin[\frac{1}{2} + \frac{(n-1)}{n}] = 0$$

$$\cos + \cos(\frac{1}{2} + \frac{1}{n}) + \cos(\frac{1}{2} + \frac{2}{n}) + \cos(\frac{1}{2} + \frac{3}{n}) + \dots + \cos[\frac{1}{2} + \frac{(n-1)}{n}] = 0$$

$$\sin^2(\frac{1}{2}) + \sin^2(\frac{1}{2} - \frac{1}{3}) + \sin^2(\frac{1}{2} + \frac{1}{3}) = \frac{3}{2}$$

$$\tan A \tan B \tan(A+B) + \tan A + \tan B - \tan(A+B) = 0$$

导数公式:

$$(\tan x)' = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \cdot \tan x$$

$$(\csc x)' = -\csc x \cdot \cot x$$

$$(a^x)' = a^x \ln a$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

$$C' = 0 \quad (x^\mu)' = \mu x^{\mu-1}$$

$$(\sin x)^{(n)} = \sin\left(x + n \cdot \frac{\pi}{2}\right)$$

$$(\cos x)^{(n)} = \cos\left(x + n \cdot \frac{\pi}{2}\right)$$

基本积分表:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n} I_{n-2}$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2}) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + C$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$$

$$\int \sqrt{\frac{a+x}{a-x}} dx = a \arcsin \frac{x}{a} - \sqrt{a^2 - x^2} + C \quad (a > 0)$$

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2 \arctan \sqrt{\frac{x-a}{b-x}} + C \quad (a < x < b)$$

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

$$\begin{aligned}
\int \tan x dx &= -\ln|\cos x| + C & \int \frac{dx}{\cos^2 x} &= \int \sec^2 x dx = \tan x + C \\
\int \cot x dx &= \ln|\sin x| + C & \int \frac{dx}{\sin^2 x} &= \int \csc^2 x dx = -\cot x + C \\
\int \sec x dx &= \ln|\sec x + \tan x| + C & \int \sec x \cdot \tan x dx &= \sec x + C \\
\int \csc x dx &= \ln|\csc x - \cot x| + C & \int \csc x \cdot \cot x dx &= -\csc x + C \\
\int \frac{dx}{a^2 + x^2} &= \frac{1}{a} \arctan \frac{x}{a} + C & \int a^x dx &= \frac{a^x}{\ln a} + C \\
\int \frac{dx}{x^2 - a^2} &= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C & \int \operatorname{sh} x dx &= \operatorname{ch} x + C \\
\int \frac{dx}{a^2 - x^2} &= \frac{1}{2a} \ln \frac{a+x}{a-x} + C & \int \operatorname{ch} x dx &= \operatorname{sh} x + C \\
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \arcsin \frac{x}{a} + C & \int \frac{dx}{\sqrt{x^2 \pm a^2}} &= \ln(x + \sqrt{x^2 \pm a^2}) + C
\end{aligned}$$

三角函数的有理式积分:

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad u = \tan \frac{x}{2}, \quad dx = \frac{2du}{1+u^2}$$

一些初等函数:

双曲正弦 : $\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$

双曲余弦 : $\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$

双曲正切 : $\operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

反双曲正弦 :

$\operatorname{arsh} x = \ln(x + \sqrt{x^2 + 1})$

反双曲余弦 :

$\operatorname{arch} x = \pm \ln(x + \sqrt{x^2 - 1})$

反双曲正切 :

$\operatorname{arth} x = \frac{1}{2} \ln \frac{1+x}{1-x}$

两个重要极限:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e = 2.718281828459045...$$

三角函数公式:

• 三角函数值

三角函数 \ 角	0°	30°	45°	60°	90°
$\sin \alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	不存在
$\cot \alpha$	不存在	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

• 诱导公式:

函数 角 A	sin	cos	tan	cot
-	-sin	cos	- tan	- cot
90°-	cos	sin	cot	tan
90°+	cos	-sin	- cot	- tan
180°-	sin	-cos	- tan	-ctg
180°+	-sin	-cos	tan	cot
270°-	-cos	-sin	cot	tan
270°+	-cos	sin	- cot	- tan
360°-	-sin	cos	- tan	- cot
360°+	sin	cos	tan	cot

• 和差角公式:

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\ \tan(\alpha \pm \beta) &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \cdot \tan \beta} \\ \cot(\alpha \pm \beta) &= \frac{\cot \alpha \cdot \cot \beta \mp 1}{\cot \beta \pm \cot \alpha}\end{aligned}$$

• 和差化积公式:

$$\begin{aligned}\sin \alpha + \sin \beta &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\ \sin \alpha - \sin \beta &= 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \\ \cos \alpha + \cos \beta &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\ \cos \alpha - \cos \beta &= 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}\end{aligned}$$

三角函数的角度换算

公式一:

设 α 为任意角, 终边相同的角的同一三角函数的值相等:

$$\sin(2k\pi + \alpha) = \sin \alpha$$

$$\tan(2k\pi + \alpha) = \tan \alpha$$

$$\cos(2k\pi + \alpha) = \cos \alpha$$

$$\cot(2k\pi + \alpha) = \cot \alpha$$

公式二:

设 α 为任意角, $\pi - \alpha$ 的三角函数值与 α 的三角函数值之间的关系:

$$\sin(\pi + \alpha) = -\sin \alpha$$

$$\tan(\pi + \alpha) = \tan \alpha$$

$$\cos(\pi + \alpha) = -\cos \alpha$$

$$\cot(\pi + \alpha) = \cot \alpha$$

公式三：

任意角 α 与 $-\alpha$ 的三角函数值之间的关系：

$$\sin(-\alpha) = -\sin \alpha$$

$$\tan(-\alpha) = -\tan \alpha$$

$$\cos(-\alpha) = \cos \alpha$$

$$\cot(-\alpha) = -\cot \alpha$$

公式四：

利用公式二和公式三可以得到 $\pi - \alpha$ 与 α 的三角函数值之间的关系：

$$\sin(\pi - \alpha) = \sin \alpha$$

$$\tan(\pi - \alpha) = -\tan \alpha$$

$$\cos(\pi - \alpha) = -\cos \alpha$$

$$\cot(\pi - \alpha) = -\cot \alpha$$

公式五：

利用公式一和公式三可以得到 $2\pi - \alpha$ 与 α 的三角函数值之间的关系：

$$\sin(2\pi - \alpha) = -\sin \alpha$$

$$\tan(2\pi - \alpha) = -\tan \alpha$$

$$\cos(2\pi - \alpha) = \cos \alpha$$

$$\cot(2\pi - \alpha) = -\cot \alpha$$

公式六：

$\pi/2 \pm \alpha$ 及 $3\pi/2 \pm \alpha$ 与 α 的三角函数值之间的关系：

$$\sin(\pi/2 + \alpha) = \cos \alpha$$

$$\sin(3\pi/2 + \alpha) = -\cos \alpha$$

$$\cos(\pi/2 + \alpha) = -\sin \alpha$$

$$\cos(3\pi/2 + \alpha) = \sin \alpha$$

$$\tan(\pi/2 + \alpha) = -\cot \alpha$$

$$\tan(3\pi/2 + \alpha) = -\cot \alpha$$

$$\cot(\pi/2 + \alpha) = -\tan \alpha$$

$$\cot(3\pi/2 + \alpha) = -\tan \alpha$$

$$\sin(\pi/2 - \alpha) = \cos \alpha$$

$$\sin(3\pi/2 - \alpha) = -\cos \alpha$$

$$\cos(\pi/2 - \alpha) = \sin \alpha$$

$$\cos(3\pi/2 - \alpha) = -\sin \alpha$$

$$\tan(\pi/2 - \alpha) = \cot \alpha$$

$$\tan(3\pi/2 - \alpha) = -\cot \alpha$$

$$\cot(\pi/2 - \alpha) = \tan \alpha$$

$$\cot(3\pi/2 - \alpha) = \tan \alpha$$

(以上 $k \in \mathbb{Z}$)

• 倍角公式:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

• 半角公式:

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos\alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos\alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} = \frac{1 - \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 + \cos\alpha}$$

$$\cot \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos\alpha}{1 - \cos\alpha}} = \frac{1 + \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 - \cos\alpha}$$

• 正弦定理:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

• 余弦定理:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

• 反三角函数性质:

$$\arcsin x = \frac{\pi}{2} - \arccos x \quad \arctan x = \frac{\pi}{2} - \operatorname{arccot} x$$

高阶导数公式——莱布尼茨公式:

$$(uv)^{(n)} = \sum_{k=0}^n C_n^k u^{(n-k)} v^{(k)}$$

$$= u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v'' + \dots + \frac{n(n-1)\dots(n-k+1)}{k!}u^{(n-k)}v^{(k)} + \dots + uv^{(n)}$$

中值定理与导数应用:

拉格朗日中值定理: $f(b) - f(a) = f'(\xi)(b - a)$

柯西中值定理: $\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)}$

当 $F(x) = x$ 时, 柯西中值定理就是拉格朗日中值定理.

曲率:

弧微分公式: $ds = \sqrt{1 + y'^2} dx$, 其中 $y' = \tan \alpha$

平均曲率: $\bar{K} = \left| \frac{\Delta \alpha}{\Delta s} \right|$. $\Delta \alpha$: 从 M 点到 M' 点, 切线斜率的倾角变化量; Δs : MM' 弧长.

M 点的曲率: $K = \lim_{\Delta s \rightarrow 0} \left| \frac{\Delta \alpha}{\Delta s} \right| = \left| \frac{d\alpha}{ds} \right| = \frac{|y''|}{\sqrt{(1 + y'^2)^3}}$.

直线: $K = 0$.

半径为 a 的圆: $K = \frac{1}{a}$.

定积分的近似计算:

矩形法: $\int_a^b f(x) dx \approx \frac{b-a}{n} (y_0 + y_1 + \cdots + y_{n-1})$

梯形法: $\int_a^b f(x) dx \approx \frac{b-a}{n} \left[\frac{1}{2} (y_0 + y_n) + y_1 + \cdots + y_{n-1} \right]$

抛物线法: $\int_a^b f(x) dx \approx \frac{b-a}{3n} [(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})]$

定积分应用相关公式:

功: $W = F \cdot s$

水压力: $F = p \cdot A$

引力: $F = k \frac{m_1 m_2}{r^2}$, k 为引力系数

函数的平均值: $\bar{y} = \frac{1}{b-a} \int_a^b f(x) dx$

均方根: $\sqrt{\frac{1}{b-a} \int_a^b f^2(t) dt}$

微分方程的相关概念:

一阶微分方程: $y' = f(x, y)$ 或 $P(x, y)dx + Q(x, y)dy = 0$

可分离变量的微分方程: 一阶微分方程可以化为 $g(y)dy = f(x)dx$ 的形式,

解法:

$\int g(y)dy = \int f(x)dx$ 得 $G(y) = F(x) + C$ 称为隐式通解.

齐次方程: 一阶微分方程可以写成 $\frac{dy}{dx} = f(x, y) = \varphi\left(\frac{y}{x}\right)$, 即写成 $\frac{y}{x}$ 的函数,

解法:

设 $u = \frac{y}{x}$, 则 $\frac{dy}{dx} = u + x \frac{du}{dx}$, $u + \frac{du}{dx} = \varphi(u)$, 所以 $\frac{dx}{x} = \frac{du}{\varphi(u) - u}$ 分离变量, 积分后将 $\frac{y}{x}$ 代替 u ,

即得齐次方程通解.

一阶线性微分方程:

1. 一阶线性微分方程: $\frac{dy}{dx} + P(x)y = Q(x)$

$\left\{ \begin{array}{l} \text{当 } Q(x) = 0 \text{ 时, 为齐次方程, } y = Ce^{-\int P(x)dx} \\ \text{当 } Q(x) \neq 0 \text{ 时, 为非齐次方程, } y = \left(\int Q(x)e^{\int P(x)dx} dx + C \right) e^{-\int P(x)dx} \end{array} \right.$

2. 伯努利方程: $\frac{dy}{dx} + P(x)y = Q(x)y^n \quad (n \neq 0, 1)$

全微分方程:

如果 $P(x, y)dx + Q(x, y)dy = 0$ 中左端是某函数的全微分方程, 即:

$du(x, y) = P(x, y)dx + Q(x, y)dy = 0$, 其中 $\frac{\partial u}{\partial x} = P(x, y)$, $\frac{\partial u}{\partial y} = Q(x, y)$

$\therefore u(x, y) = C$ 应该是该全微分方程的通解.

二阶微分方程:

$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = f(x)$, $\left\{ \begin{array}{l} f(x) \equiv 0 \text{ 时为齐次} \\ f(x) \neq 0 \text{ 时为非齐次} \end{array} \right.$

二阶常系数齐次线性微分方程及其解法：

(*) $y'' + py' + qy = 0$, 其中 p, q 为常数；

求解步骤：

1. 写出特征方程： $(\Delta)r^2 + pr + q = 0$, 其中 r^2 , r 的系数及常数项恰好是(*)式中 y'' , y' , y 的系数；
2. 求出(Δ)式的两个根 r_1, r_2
3. 根据 r_1, r_2 的不同情况，按下表写出(*)式的通解：

r_1, r_2 的形式	(*)式的通解
两个不相等实根($p^2 - 4q > 0$)	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
两个相等实根($p^2 - 4q = 0$)	$y = (c_1 + c_2 x) e^{r_1 x}$
一对共轭复根($p^2 - 4q < 0$) $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$ $\alpha = -\frac{p}{2}, \beta = \frac{\sqrt{4q - p^2}}{2}$	$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$

二阶常系数非齐次线性微分方程：

$y'' + py' + qy = f(x)$, p, q 为常数

$f(x) = e^{\lambda x} P_m(x)$ 型, λ 为常数；

$f(x) = e^{\lambda x} [P_l(x) \cos \omega x + P_n(x) \sin \omega x]$ 型

空间解析几何与向量代数:

空间两点的距离: $d = |M_1 M_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

向量在轴上的投影: $\text{Pr } \vec{AB} = |\vec{AB}| \cdot \cos \varphi$, φ 是 \vec{AB} 与 u 轴的夹角.

$\text{Pr } \vec{j}_u (\vec{a}_1 + \vec{a}_2) = \text{Pr } \vec{j}_u \vec{a}_1 + \text{Pr } \vec{j}_u \vec{a}_2$

$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos \theta = a_x b_x + a_y b_y + a_z b_z$, 是一个数量,

两向量之间的夹角: $\cos \theta = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}$

$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$, $|\vec{c}| = |\vec{a}| \cdot |\vec{b}| \sin \theta$. 例: 线速度: $\vec{v} = \vec{\omega} \times \vec{r}$.

向量的混合积: $[\vec{a} \vec{b} \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cos \alpha$, α 为锐角时,

代表平行六面体的体积.

平面的方程:

1. 点法式: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$, 其中 $\vec{n} = \{A, B, C\}$, M_0 坐标为 (x_0, y_0, z_0)

2. 一般方程: $Ax + By + Cz + D = 0$

3. 截距式方程: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

平面外任意一点到该平面的距离: $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

空间直线的方程: $\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p} = t$, 其中 $\vec{s} = \{m, n, p\}$; 参数方程: $\begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$

二次曲面:

1. 椭球面: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

2. 抛物面: $\frac{x^2}{2p} + \frac{y^2}{2q} = z$, (p, q 同号)

3. 双曲面

单叶双曲面: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

双叶双曲面: $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (马鞍面)

多元函数微分法及应用:

$$\text{全微分: } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

全微分的近似计算: $\Delta z \approx dz = f_x(x, y)\Delta x + f_y(x, y)\Delta y$

多元复合函数的求导法:

$$z = f[u(t), v(t)] \quad \frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial t}$$

$$z = f[u(x, y), v(x, y)] \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

当 $u = u(x, y)$, $v = v(x, y)$ 时,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

隐函数的求导公式:

$$\text{隐函数 } F(x, y) = 0, \quad \frac{dy}{dx} = -\frac{F_x}{F_y}, \quad \frac{d^2 y}{dx^2} = \frac{\partial}{\partial x} \left(-\frac{F_x}{F_y} \right) + \frac{\partial}{\partial y} \left(-\frac{F_x}{F_y} \right) \cdot \frac{dy}{dx}$$

$$\text{隐函数 } F(x, y, z) = 0, \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\text{隐函数方程组: } \begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \quad J = \frac{\partial(F, G)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{J} \cdot \frac{\partial(F, G)}{\partial(x, v)} \quad \frac{\partial v}{\partial x} = -\frac{1}{J} \cdot \frac{\partial(F, G)}{\partial(u, x)}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J} \cdot \frac{\partial(F, G)}{\partial(y, v)} \quad \frac{\partial v}{\partial y} = -\frac{1}{J} \cdot \frac{\partial(F, G)}{\partial(u, y)}$$

微分法在几何上的应用:

$$\text{空间曲线} \begin{cases} x = \varphi(t), \\ y = \psi(t), \\ z = \omega(t) \end{cases} \text{在点 } M(x_0, y_0, z_0) \text{ 处的切线方程: } \frac{x-x_0}{\varphi'(t_0)} = \frac{y-y_0}{\psi'(t_0)} = \frac{z-z_0}{\omega'(t_0)}$$

在点 M 处的法平面方程: $\varphi'(t_0)(x-x_0) + \psi'(t_0)(y-y_0) + \omega'(t_0)(z-z_0) = 0$

$$\text{若空间曲线方程为: } \begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0, \end{cases} \text{ 则切向量 } \vec{T} = \left\{ \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}, \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}, \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \right\}$$

曲面 $F(x, y, z) = 0$ 上一点 $M(x_0, y_0, z_0)$, 则:

1. 过此点的法向量: $\vec{n} = \{F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\}$

2. 过此点的切平面方程: $F_x(x_0, y_0, z_0)(x-x_0) + F_y(x_0, y_0, z_0)(y-y_0) + F_z(x_0, y_0, z_0)(z-z_0) = 0$

3. 过此点的法线方程: $\frac{x-x_0}{F_x(x_0, y_0, z_0)} = \frac{y-y_0}{F_y(x_0, y_0, z_0)} = \frac{z-z_0}{F_z(x_0, y_0, z_0)}$

方向导数与梯度:

函数 $z = f(x, y)$ 在一点 $p(x, y)$ 沿任一方向 l 的方向导数为: $\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \varphi + \frac{\partial f}{\partial y} \sin \varphi$

其中 φ 为 x 轴到方向 l 的转角.

函数 $z = f(x, y)$ 在一点 $p(x, y)$ 的梯度: $\text{grad} f(x, y) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$

它与方向导数的关系是: $\frac{\partial f}{\partial l} = \text{grad} f(x, y) \cdot \vec{e}$, 其中 $\vec{e} = \cos \varphi \cdot \vec{i} + \sin \varphi \cdot \vec{j}$, 为 l 方向上的单位向量.

所以 $\frac{\partial f}{\partial l}$ 是 $\text{grad} f(x, y)$ 在 l 上的投影.

多元函数的极值及其求法:

设 $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, 令: $f_{xx}(x_0, y_0) = A$, $f_{xy}(x_0, y_0) = B$, $f_{yy}(x_0, y_0) = C$,

则 (1) $AC - B^2 > 0$ 时, $\begin{cases} A < 0, (x_0, y_0) \text{ 为极大值} \\ A > 0, (x_0, y_0) \text{ 为极小值} \end{cases}$.

(2) $AC - B^2 < 0$ 时, 函数无极值.

(3) $AC - B^2 = 0$ 时, 函数不确定是否有极值.

重积分及其应用:

$$\iint_D f(x, y) dx dy = \iint_{D'} f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$\text{曲面 } z = f(x, y) \text{ 的面积 } A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$$

平面薄片的质心坐标

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_D x \mu(x, y) d\sigma}{\iint_D \mu(x, y) d\sigma}, \quad \bar{y} = \frac{M_x}{M} = \frac{\iint_D y \mu(x, y) d\sigma}{\iint_D \mu(x, y) d\sigma}$$

$$\text{平面薄片的转动惯量: 对于 } x \text{ 轴 } I_x = \iint_D y^2 \mu(x, y) d\sigma, \quad \text{对于 } y \text{ 轴 } I_y = \iint_D x^2 \mu(x, y) d\sigma$$

平面薄片 (位于 xOy 平面) 对 z 轴上质点 $M(0, 0, a) (a > 0)$ 的引力: $F = \{F_x, F_y, F_z\}$, 其中

$$F_x = f \iint_D \frac{\rho(x, y) x d\sigma}{(x^2 + y^2 + a^2)^{\frac{3}{2}}}, \quad F_y = f \iint_D \frac{\rho(x, y) y d\sigma}{(x^2 + y^2 + a^2)^{\frac{3}{2}}}, \quad F_z = -fa \iint_D \frac{\rho(x, y) d\sigma}{(x^2 + y^2 + a^2)^{\frac{3}{2}}}$$

柱面坐标和球面坐标:

$$\text{柱面坐标: } \begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \\ z = z, \end{cases} \quad \iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega} F(r, \theta, z) r dr d\theta dz,$$

其中 $F(r, \theta, z) = f(r \cos \theta, r \sin \theta, z)$

$$\text{球面坐标: } \begin{cases} x = r \sin \varphi \cos \theta, \\ y = r \sin \varphi \sin \theta, \\ z = r \cos \varphi, \end{cases} \quad dv = r d\varphi \cdot r \sin \varphi \cdot d\theta \cdot dr = r^2 \sin \varphi dr d\varphi d\theta$$

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega} F(r, \varphi, \theta) r^2 \sin \varphi dr d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{r(\varphi, \theta)} F(r, \varphi, \theta) r^2 \sin \varphi dr$$

$$\text{质心: } \bar{x} = \frac{1}{M} \iiint_{\Omega} x \rho dv, \quad \bar{y} = \frac{1}{M} \iiint_{\Omega} y \rho dv, \quad \bar{z} = \frac{1}{M} \iiint_{\Omega} z \rho dv, \quad \text{其中 } M = \iiint_{\Omega} \rho dv$$

$$\text{转动惯量: } I_x = \iiint_{\Omega} (y^2 + z^2) \rho dv, \quad I_y = \iiint_{\Omega} (x^2 + z^2) \rho dv, \quad I_z = \iiint_{\Omega} (x^2 + y^2) \rho dv$$

曲线积分:

第一类曲线积分（对弧长的曲线积分）：

设 $f(x, y)$ 在 L 上连续， L 的参数方程为 $\begin{cases} x = \varphi(t), \\ y = \psi(t) \end{cases} \quad (\alpha \leq t \leq \beta)$, 则

$$\int_L f(x, y) ds = \int_{\alpha}^{\beta} f[\varphi(t), \psi(t)] \sqrt{\varphi'^2(t) + \psi'^2(t)} dt \quad (\alpha < \beta)$$

特殊情况 $\begin{cases} x = t, \\ y = \varphi(t). \end{cases}$

第二类曲线积分（对坐标的曲线积分）：

设 L 的参数方程为 $\begin{cases} x = \varphi(t), \\ y = \psi(t), \end{cases}$ 则

$$\begin{aligned} & \int_L P(x, y) dx + Q(x, y) dy \\ &= \int_{\alpha}^{\beta} \{ P[\varphi(t), \psi(t)] \varphi'(t) + Q[\varphi(t), \psi(t)] \psi'(t) \} dt \end{aligned}$$

两类曲线积分之间的关系： $\int_L P dx + Q dy = \int_L (P \cos \alpha + Q \cos \beta) ds$, 其中 α 和 β 分别为 L 上积分起止点处切向量的方向角.

格林公式： $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_L P dx + Q dy$

当 $P = -y, Q = x$, 即 $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$ 时, 得到 D 的面积 $A = \iint_D dx dy = \frac{1}{2} \oint_L x dy - y dx$.

平面上曲线积分与路径无关的条件:

1. G 是一个单连通区域;

2. $P(x, y), Q(x, y)$ 在 G 内具有一阶连续偏导数, 且 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. 注意奇点, 如 $(0,0)$, 应减去对此奇点的积分, 注意方向相反!

二元函数的全微分求积:

在 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ 时, $P dx + Q dy$ 才是二元函数 $u(x, y)$ 的全微分, 其中

$$u(x, y) = \int_{(x_0, y_0)}^{(x, y)} P(x, y) dx + Q(x, y) dy, \text{通常设 } x_0 = y_0 = 0.$$

曲面积分:

对面积的曲面积分

$$\iint_{\Sigma} f(x, y, z) ds = \iint_{D_{xy}} f[x, y, z(x, y)] \sqrt{1 + z_x^2(x, y) + z_y^2(x, y)} dx dy$$

对坐标的曲面积分: $\iint_{\Sigma} P(x, y, z) dy dz + Q(x, y, z) dz dx + R(x, y, z) dx dy$, 其中:

$$\iint_{\Sigma} R(x, y, z) dx dy = \pm \iint_{D_{xy}} R[x, y, z(x, y)] dx dy, \text{ 取曲面的上侧时取正号;}$$

$$\iint_{\Sigma} P(x, y, z) dy dz = \pm \iint_{D_{yz}} P[x(y, z), y, z] dy dz, \text{ 取曲面的前侧时取正号;}$$

$$\iint_{\Sigma} Q(x, y, z) dz dx = \pm \iint_{D_{zx}} Q[x, y(z, x), z] dz dx, \text{ 取曲面的右侧时取正号.}$$

两类曲面积分之间的关系

$$\iint_{\Sigma} P dy dz + Q dz dx + R dx dy = \iint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds$$

高斯公式:

$$\iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv = \oiint_{\Sigma} P dy dz + Q dz dx + R dx dy = \oiint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds$$

高斯公式的物理意义——通量与散度:

散度: $\operatorname{div} \vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$, 即单位体积内所产生的流体质量

$$\text{通量: } \iint_{\Sigma} \vec{A} \cdot \vec{n} ds = \iint_{\Sigma} A_n ds = \iint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds,$$

$$\text{因此, 高斯公式又可写成 } \iiint_{\Omega} \operatorname{div} \vec{A} dv = \oiint_{\Sigma} A_n ds$$

斯托克斯公式——曲线积分与曲面积分的关系:

$$\iint_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{\Gamma} Pdx + Qdy + Rdz$$

上式左端又可写成:
$$\iint_{\Sigma} \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \iint_{\Sigma} \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

空间曲线积分与路径无关的条件:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

旋度:
$$\mathbf{rot} \bar{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

向量场 \bar{A} 沿有向闭曲线 Γ 的环流量:
$$\oint_{\Gamma} Pdx + Qdy + Rdz = \oint_{\Gamma} \bar{A} \cdot \bar{\tau} ds$$

常数项级数:

等比数列:
$$1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}$$

等差数列:
$$1 + 2 + 3 + \cdots + n = \frac{(n+1)n}{2}$$

调和级数:
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
 是发散的

级数审敛法:

1. 正项级数的审敛法——根值审敛法 (柯西判别法):

设:
$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{u_n}, \quad \text{则} \begin{cases} \rho < 1 \text{ 时, 级数收敛} \\ \rho > 1 \text{ 时, 级数发散} \\ \rho = 1 \text{ 时, 不确定} \end{cases}$$

2. 比值审敛法:

设:
$$\rho = \lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n}, \quad \text{则} \begin{cases} \rho < 1 \text{ 时, 级数收敛} \\ \rho > 1 \text{ 时, 级数发散} \\ \rho = 1 \text{ 时, 不确定} \end{cases}$$

3. 定义法:

$$s_n = u_1 + u_2 + \cdots + u_n; \lim_{n \rightarrow \infty} s_n$$
 存在, 则收敛; 否则发散.

交错级数 $u_1 - u_2 + u_3 - u_4 + \dots$ (或 $-u_1 + u_2 - u_3 + \dots, u_n > 0$) 的审敛法——莱布尼茨定理:

如果交错级数满足 $\begin{cases} u_n \geq u_{n+1}, \\ \lim_{n \rightarrow \infty} u_n = 0, \end{cases}$ 那么级数收敛且其和 $s \leq u_1$, 其余项 r_n 的绝对值 $|r_n| \leq u_{n+1}$.

绝对收敛与条件收敛:

(1) $u_1 + u_2 + \dots + u_n + \dots$, 其中 u_n 为任意实数;

(2) $|u_1| + |u_2| + |u_3| + \dots + |u_n| + \dots$

如果 (2) 收敛, 则 (1) 肯定收敛, 且称为绝对收敛级数;

如果 (2) 发散, 而 (1) 收敛, 则称 (1) 为条件收敛级数.

调和级数: $\sum \frac{1}{n}$ 发散, 而 $\sum \frac{(-1)^n}{n}$ 收敛;

级数: $\sum \frac{1}{n^2}$ 收敛;

p 级数: $\sum \frac{1}{n^p}$ $\begin{cases} p \leq 1 \text{ 时发散} \\ p > 1 \text{ 时收敛} \end{cases}$

幂级数:

$$1 + x + x^2 + x^3 + \dots + x^n + \dots \begin{cases} |x| < 1 \text{ 时, 收敛于 } \frac{1}{1-x} \\ |x| \geq 1 \text{ 时, 发散} \end{cases}$$

对于级数 (3) $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$, 如果它不是仅在原点收敛, 也不是在全

数轴上都收敛, 则必存在 R , 使 $\begin{cases} |x| < R \text{ 时收敛} \\ |x| > R \text{ 时发散} \\ |x| = R \text{ 时不定} \end{cases}$, 其中 R 称为收敛半径

求收敛半径的方法: 设 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$, 其中 a_n, a_{n+1} 是 (3) 的系数, 则 $\begin{cases} \rho \neq 0 \text{ 时, } R = \frac{1}{\rho} \\ \rho = 0 \text{ 时, } R = +\infty \\ \rho = +\infty \text{ 时, } R = 0 \end{cases}$

函数展开成幂级数:

函数展开成泰勒级数

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

余项: $R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, $f(x)$ 可以展开成泰勒级数的充要条件是: $\lim_{n \rightarrow \infty} R_n = 0$

$x_0 = 0$ 时即为麦克劳林公式

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

一些函数展开成幂级数:

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots \quad (-1 < x < 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots \quad (-\infty < x < +\infty)$$

欧拉公式:

$$e^{ix} = \cos x + i \sin x \quad \text{或} \quad \begin{cases} \cos x = \frac{e^{ix} + e^{-ix}}{2} \\ \sin x = \frac{e^{ix} - e^{-ix}}{2i} \end{cases}$$

三角级数:

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega t + \varphi_n) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

其中, $a_0 = 2A_0$, $a_n = A_n \sin \varphi_n$, $b_n = A_n \cos \varphi_n$, $\omega t = x$.

正交性: $1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots, \sin nx, \cos nx, \cdots$ 任意两个不同项的乘积在 $[-\pi, \pi]$ 上的积分=0.

傅立叶级数:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad \text{周期} = 2\pi$$

$$\text{其中} \begin{cases} a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx & (n=0,1,2,\cdots) \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx & (n=1,2,3,\cdots) \end{cases}$$

利用函数的傅里叶级数展开式，可以得到一些特殊级数的和：

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24} \quad 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

正弦级数：

$$a_n = 0, \quad b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx \quad n = 1, 2, 3 \cdots \quad f(x) = \sum b_n \sin nx \text{ 是奇函数}$$

余弦级数：

$$b_n = 0, \quad a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx \quad n = 0, 1, 2 \cdots \quad f(x) = \frac{a_0}{2} + \sum a_n \cos nx \text{ 是偶函数}$$

周期为 $2l$ 的周期函数的傅立叶级数：

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right), \quad \text{周期} = 2l$$

$$\text{其中} \begin{cases} a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx \quad (n = 0, 1, 2 \cdots), \\ b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \quad (n = 1, 2, 3 \cdots). \end{cases}$$