These notes are a subset of the lecture notes provided by Prof. Ian Tice, for 21-355, Principles of Real Analysis I, Fall 2025

These are what I figured/took to be important, typically proofs he covered in class; in some cases, the proof may differ from the textbook version (in general, I followed the lectures)

§ Ordered Sets, Ordered Fields, $\mathbb{R}$

- Def: let E be a set, order on E is relation $<$ satisfying <u>trichotomy</u> & <u>transitivity</u>
- Prop: let E be ordered set, $A \subseteq E$, then max/min of A is unique. pf by trichotomy.
- Def: $x \in E$ is <u>supremum</u> of A if x is UB of A and $x \le y$ for every $y \in U(A)$

  unique. $x \in E$ is <u>infimum</u> of A if x is LB of A and $y \le x$ for every $y \in L(A)$.
- Def: let E be ordered set: E satisfies supremum property if every $A \subseteq E$ bounded above has a supremum, and similar for infimum

  HW ↳ Prop: E satisfies sup property $\Leftrightarrow$ E satisfies inf property

  E satisfies sup prop; $A \subseteq E$ bounded below, $L(A) \neq \emptyset$, every $x \in A$ UB of <u>$L(A)$</u> → bounded above.

  $\Rightarrow$ set $z = \sup(L(A))$, show $z = \inf(A)$
- Lemma: F field, $K \subseteq F$, then K subfield $\Leftrightarrow$ (1) $K \neq \emptyset$, $K \neq \{0\}$,

  (2) $x, y \in K \Rightarrow x + (-y) \in K$, (3) $x, y \in K \setminus \{0\} \Rightarrow xy^{-1} \in K$.
- Def: F is ordered field if (& ordered set) $x < y \Rightarrow x + z < y + z$, $x > 0, y > 0 \Rightarrow xy > 0$
- Prop: if F ordered field, $1 > 0$ and $\nexists x \in F$ s.t. $x^2 = -1$.
- Def: F_1, F_2 be OF, say $f : F_1 \to F_2$ is OP homo if.

  1. f is field homo: $f(x+y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$, $f(0) = 0$, $f(1) = 1$. }*
  2. f OP: $x < y$ in $F_1 \Rightarrow f(x) < f(y)$ in F_2.
- Prop: if $f : F_1 \to F_2$ is OP homo, then $f(F_1) \subseteq F_2$ is ordered subfield of F_2
- THM: let F be ordered field, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subseteq F$ ← $\mathbb{Q}$ is smallest OF.

  THM: $\exists!$ OP field hom $h : \mathbb{Q} \to F$ ⇗ $h(\mathbb{Q}) \subseteq F$ ↙ Inductively define on $\mathbb{N}$, then $h(n) = -h(n)$ for $\mathbb{Z}$, $h(a \cdot b) = h(a) \cdot h(b)$ for $\mathbb{Q}$

  idea: if such a map existed, then $h(0) = 0$, $h(1) = 1$, $h(x+y) = h(x) + h(y) \Rightarrow h(n+1) = h(n) + 1$

  1. define $f : \mathbb{N} \to F$, check $(*)$ → 2. $g : \mathbb{Z} \to F$ → 3. $h : \mathbb{Q} \to F$ via $h(p) = \frac{g(m)}{g(n)}$

  uniqueness: assume $\varphi : \mathbb{Q} \to F$ is OP hom, show using idea that $\varphi|_{\mathbb{N}} = f$, $\varphi|_{\mathbb{Z}} = g$, $\varphi = h$.
- Def: let F be an OF, say F is Dedekind complete if F satisfies sup. property $(0<y)$
- THM: let F be a Ded OF, $x \in F$ be s.t. $0 < x$, $1 \le n \in \mathbb{F}$, then $\exists! y \in F$ s.t. $x = y^n$

  note: 1. if F field, $a, b \in F$, $1 \le n \in \mathbb{N}$, $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \dots + a^{n-2}b + a^{n-1})$

  2. if F OF, $0 < x, y \in F$, $1 \le n \in \mathbb{N}$, $x < y \Leftrightarrow x^n < y^n$

  $n=1$ trivial, reduce to $n=2$. define $E = \{y \in F \mid 0 < y \text{ and } y^n < x\}$. ↖ idea: prove this has sup & this is desired sup.

  → $E \neq \emptyset$: $t = \frac{x}{1+x} \in F \Rightarrow t < x \Rightarrow 0 < t^n < t < x$

  → E bounded above: set $s = 1 + x$, then $x < s < s^n$, if $y \in E$, $0 < y \Rightarrow y^n < x < s^n \Rightarrow y < s$ (by 2)

  → since F Ded, deduce that $y = \sup E \in F$, claim that $y^n = x$

§ Section 0: $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$

- THM: $\exists$ a set $\mathbb{N}$ satisfying the following (Peano) axioms:
  1. $0 \in \mathbb{N}$ → then def $1 = S(0)$, $2 = S(1)$, ...
  2. $\exists S: \mathbb{N} \to \mathbb{N}$ called the successor function: guarantees $\{0, 1, \dots\} \subseteq \mathbb{N}$
  3. $S(n) \neq 0$ $\forall n \in \mathbb{N}$ → prevents wrap-around
  4. $S(n) = S(m) \Rightarrow n = m$ (i.e. S injective) → prevents stagnation.
  5. if $P(0)$ is true and $P(n) \Rightarrow P(S(n))$, then $P(n)$ for all $n \in \mathbb{N}$. → $\nexists\, 0' \in \mathbb{N}$ s.t. $0' \notin \{0, 1, 2, \dots\}$
- Def: $m = \min(E)$, $E \subseteq \mathbb{N}$, if $m \in E$ and $m \leq n$ $\forall n \in E$
- THM (well-ordering principle): suppose $\emptyset \neq E \subseteq \mathbb{N}$, then $\exists!\, m \in E$ s.t. $m = \min(E)$.

  pf: $P(n)$ = if $n \in E$ then E has unique min.

  $P(0)$ true: if $0 \in E$, $k \in E \Rightarrow k = 0$ or $0 < k$.

  fix n, suppose $P(k)$ $\forall\, 0 \leq k < n$. if $[n] \cap E \neq \emptyset$, $P(k)$ true, else n is unique min.
- Def: $f: \mathbb{N} \to E$ is an enumeration of E if f bij. order-preserving if $m < n \Rightarrow f(m) < f(n)$.

HW
- THM: $\emptyset \neq E \subseteq \mathbb{N}$ be infinite, then $\exists!$ OP-enumeration of E.

  Cor: E infinite $\Leftrightarrow$ $E \setminus \{x\}$ infinite for each $x \in E$. ← $f: \mathbb{N} \to E$.

  let $E_0 = E$, $f(n) = \min E_n$, $E_{n+1} = E_n \setminus \{f(n)\}$. f is OP $\Leftarrow$ $f(n) < f(n+1)$ $\forall n$

  f surjective: AFSOC $e \in E \setminus f(\mathbb{N})$, then $e \in E_n$ $\forall n$ ($e \in E_0$, $e \in E_n \Rightarrow e \in E_{n+1}$ since $e \neq f(n)$)

  $f(0) = \min E \Rightarrow f(0) < e$, f incr & values in $\mathbb{N} \Rightarrow n \leq f(n)$, then $e \leq f(e)$

  $\Rightarrow n = \max\{k \in \mathbb{N} \mid f(k) < e\}$ is well-defined $\Rightarrow f(n) < e \leq f(n+1)$ →←
- THM: $m, k \in \mathbb{N}$, $m > 0$, $\exists!\, n \in \mathbb{N}$ s.t. $mn \leq k$, $m(n+1) > k$.

  $A = \{n \in \mathbb{N} \mid mn > k\}$ → $k+1 \in A$, so $A \neq \emptyset \Rightarrow$ WOP; $\ell = \min A > 0$. ⇓ (also in $\mathbb{Z}$)
- THM (Euclidean division): $m, k \in \mathbb{N}$, $m > 0$, $\exists!\, n, r \in \mathbb{N}$, $0 \leq r < m$, $k = mn + r$
- THM (Euclid's thm): $\mathbb{P}$ (set of primes) is infinite ← Prop: $\forall n \geq 2$, $\mathbb{P} \cap D(n) \neq \emptyset$

  $q = q_0 q_1 \cdots q_{n-1} + 1$, $r \neq 0 \Rightarrow$ no $q_i \in D(q)$ →← $\mathbb{P} \cap D(q) \neq \emptyset$
- THM (fundamental thm of arithmetic): $n \geq 2$, $\exists k > 0$, $m_i \in \mathbb{N}$, $m_k \geq 1$, s.t. $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$

  factorization is unique: if $n = p_1^{r_1} p_2^{r_2} \cdots p_j^{r_j}$, $r_j \geq 1$, then $j = k$, $m_i = r_i$, $1 \leq i \leq k$.

  pf: let $S = \{$set of $\mathbb{N}$ that admit 2 distinct factorizations$\}$, claim $S = \emptyset$. AFSOC, let $n = \min(S)$.

  then $n = q_1 \cdots q_j = \ell_1 \cdots \ell_k$, $j, k \geq 2$ (otherwise factorization is unique).
  1. none of q_i equals any of ℓ_i (o.w., $n' = q_2 \cdots q_j = \ell_1 \cdots \ell_k$, $n' \in S$ →← $n = \min(S)$, $n' \notin S$ →← unique)
  2. $z := (q_1 - \ell)q_2 \cdots q_j < n$ has uniq. fact. $z = n - \ell_1 q_2 \cdots q_j \Rightarrow \ell_1 \in D(z)$
  3. since ℓ_1 can't divide q_1, $\ell_1 \mid (q_1 - \ell_1) \Rightarrow \ell_1 \mid (q_1 - \ell_1 + \ell_1) \Rightarrow \ell_1 \mid q_1$ →←

A. $y^n < x$ is false: AFSOC $y^n < x$, note $n(y+1)^{n-1} > 0$, then set $h = \frac{1}{2}\min\left\{1, \frac{x - y^n}{n(y+1)^{n-1}}\right\}$

then $0 < (y+h)^n - y^n \overset{(\text{by } 1)}{<} n(y+h-y)(y+h)^{n-1} = hn(y+h)^{n-1} \overset{0<h\leq 1}{<} hn(y+1)^{n-1} < x - y^n$ $h < \boxed{\ }$

↳ $y + h \in E$ →←

B. $y^n > x$ is false: AFSOC, let $k = \frac{y^n - x}{2ny^{n-1}} < \frac{y^n}{ny^{n-1}} < y \Rightarrow 0 < y - k$

$0 < y^n - (y-k)^n \overset{(\text{by } 1)}{<} nky^{n-1} < y^n - x \Rightarrow x < (y-k)^n \Rightarrow y - k$ is UB of E →←

C. (y unique by trichotomy) ← $z^n = x = y^n$

- Def: say OF F is Archimedian if $\forall\, 0 < x \in F$, $\exists n \in \mathbb{N}$ s.t. $x < n$. (note: $\mathbb{Q}$ is Arch)
- Def: F non-Arch OF. say $x \in F$ finite if $|x| < n$ for some $n \in \mathbb{N}$, x infinite if not finite, $x \neq 0$ is infinitesimal if $1/x$ is infinite. (lemma: $\exists$ infinite x, infinitesimal $y \Leftrightarrow 0 < y < 1/n$)

HW
- Prop: F Ded $\Rightarrow$ F Arch

  suppose F Ded & not Arch, then $\exists\, 0 < x \in F$ s.t. $x \geq n$ $\forall n$, then $\mathbb{N}$ bounded $\Rightarrow y = \sup(\mathbb{N}) \in F$

  then $y - 1 < n$ for some $n \Rightarrow y < n + 1 \in \mathbb{N}$ →←

HW
- let F be an OF, then TFAE
  1. F is Arch ↴ AFSOC $\mathbb{N}$ has UB in F, then $\exists x \in F$ s.t. $n \leq x$ $\forall n \in \mathbb{N}$
  2. $\mathbb{N} \subset F$ does not have UB in F
  3. if $A \subseteq \{1/n \in \mathbb{Q} \subseteq F \mid n \in \mathbb{N} \setminus \{0\}\} \subset F$ is infinite, then A has inf & $\inf(A) = 0$
  4. $\forall x \in F$, $B(x) = \{m \in \mathbb{Z} \mid x < m\} \subset F$ has a minimum
  5. $\exists$ a floor function $\lfloor \cdot \rfloor: F \to \mathbb{Z}$ s.t. $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ $\forall x \in F$
  6. if $x, y \in F$ and $x < y$, then $\exists q \in \mathbb{Q} \subseteq F$ s.t. $x < q < y$.

  (2) ⇒ (3): 0 is LB of A, $A^{-1} = \{n \mid 1/n \in A\} \subset \mathbb{N}$ is inf & $\exists f: \mathbb{N} \to A^{-1}$ OP enumeration ← since $A^{-1} \subseteq \mathbb{N}$

  by induction, $n \leq f(n)$. AFSOC $0 < x$ is LB, then $x < 1/f(n) < 1/n \Rightarrow n < 1/x$ is bounded →←

  (3) ⇒ (4): if $x = 0$, $B(x) = \{m \in \mathbb{N} \mid x \geq 1\}$ and $\min(B(x)) = 1$. now suppose $x \neq 0$.

  sps $x > 0$. let $A = \{1/n \mid n \in \mathbb{N} \setminus \{0\}\}$, then $\inf(A) = 0 \Rightarrow \exists n \in \mathbb{N}$ s.t. $1/n < x^{-1}$. (bounded below)

  then $x < n \Rightarrow n \in B(x) \Rightarrow \emptyset \neq B(x) \subseteq \mathbb{N} \Rightarrow \exists$ minimum by WOP. if $x < 0$ use WOP on $\mathbb{Z}$.

  (4) ⇒ (5): define $\lfloor \cdot \rfloor: F \to \mathbb{Z}$ via $\min B(x) - 1 \in \mathbb{Z}$, where $B(x)$ is as in (4). → $x < \min B(x)$ → $\min B(x) - 1 \leq x$

  (5) ⇒ (6): let $x < y$, then $0 < y - x$, $0 < (y-x)^{-1} \Rightarrow$ pick $n = 1 + \lfloor (y-x)^{-1} \rfloor \in \mathbb{Z} \Rightarrow 0 < (y-x)^{-1} < n$

  then $1 < n(y-x) \Rightarrow 1 + nx < ny \overset{(5)}{\Rightarrow} nx < \lfloor nx \rfloor + 1 < ny \Rightarrow$ let $m = \lfloor nx \rfloor + 1$, $nx < m < ny$ $(q = \frac{m}{n})$

  (6) ⇒ (1): let $0 < x \in F$, choose q s.t. $0 < q < x^{-1}$, $q = \frac{m}{n} \to x \leq mx < n$.
- Def: let F be an OF.
  1. $I \subseteq F$ is a closed interval if $I = [x, y] = \{z \in F \mid x \leq z \leq y\}$ for some $x < y$
  2. sps $I_n \subseteq F$ is a closed interval for each $n \in \mathbb{N}$, say $\{I_n\}_{n=0}^{\infty}$ are nested if $I_{n+1} \subseteq I_n$
  3. say F satisfies nested interval property if every sequence of nested intervals $\{I_n\}_{n=0}^{\infty}$ satisfies $\emptyset \neq \bigcap_{n=0}^{\infty} I_n$.

- THM: TFH

THM 0.0.38.

1. let E be a set, TFAE
   (a) E countable, (b) $\exists$ inj. $f: E \to \mathbb{N}$, (c) $\exists$ surj. $\mathbb{N} \to E$
2. $n \in \mathbb{N}$, suppose A_i countable set, $\prod_{i=0}^{n} A_i$ is countable.
3. I countable, A_i countable for each $i \in I$, $\bigcup_{i \in I} A_i$ countable.

- THM (Cantor-Bernstein-Schröder): $\exists$ inj $f: A \to B$, $g: B \to A \Rightarrow$ bij $h: A \to B$ & $\mathrm{card}(A) = \mathrm{card}(B)$
- Define equiv. relation $\simeq$ on $\mathbb{N} \times \mathbb{N}$ s.t. $(m,n) \simeq (m',n') \Leftrightarrow m+n' = m'+n$ → can prove: reflexive, symmetric, transitive.
  let $[(m,n)] = \{(m',n') \mid (m,n) \simeq (m',n')\}$ → equiv. class, then $\mathbb{Z} = \{[(m,n)] \mid (m,n) \in \mathbb{N} \times \mathbb{N}\}$
- THM: exactly one of $\langle x = 0 = [(0,0)],\ x = n = [(n,0)],\ x = -n = [(0,n)] \rangle$ is true.
  Prop: $\varphi: \mathbb{N} \to \mathbb{Z}$ via $\varphi(n) = [(n,0)]$ is inj. homomorphism $(+,\cdot)$
- THM (WOP in $\mathbb{Z}$): assume $\emptyset \neq E \subseteq \mathbb{Z}$, E bounded below ($\exists m' \in \mathbb{Z}$ s.t. $m' \leq x\ \forall x \in E$) $\Rightarrow \exists!\, m \in E$ s.t. $m = \min(E)$
  E bounded above $\Rightarrow \exists!\, M \in E$ s.t. $M = \max(E)$ pf: WOP($\mathbb{N}$) on $F = \{x - m' \mid x \in E\}$.
- Define equiv. relation $\simeq$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ via $(m,n) \simeq (m',n')$ if $mn' = m'n$
  $\mathbb{Q} = \{[m,n] \mid (m,n) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})\}$ key: cancellation property if $m'n' \neq 0$
- THM: let $x \in \mathbb{Q}$, then $x = m/n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, $n \geq 1$

- THM: let F be an OF, then (F is Dedekind complete) $\Leftrightarrow$ (F Arch & satisfies nested int. prop.)
  HW
  ($\Rightarrow$) Ded $\to$ Arch, so only need to verify nested int. let $\{I_n\}_{n=0}^{\infty}$ be seq. of nested closed int.
  for $n \in \mathbb{N}$ write $I_n = [a_n, b_n]$ for $a_n, b_n \in F$, $a_n < b_n$.
  nested $\Rightarrow a_n \leq b_m\ \forall n,m \in \mathbb{N}$, $a_n \leq a_{n+1}$, $b_{n+1} \leq b_n$. let $A = \{a_n\}$, $B = \{b_n\}$ → nonempty & bounded.
  F Ded $\Rightarrow a = \sup(A)$, $b = \inf(B)$, $\forall n, a_n \leq b_m \Rightarrow a \leq b_m\ \forall m \Rightarrow a \leq b \Rightarrow x \in [a,b]$, $x \in \bigcap_{n=0}^{\infty} I_n$.
  ($\Leftarrow$) let $\emptyset \neq E \subseteq F$ be bounded above. let $a_0 \in E$, b_0 be UB.
  recursively do: sps $a_n \in E$, b_n is UB, set $c_{n+1} = \frac{a_n + b_n}{2}$
  - if c_{n+1} is UB, $a_{n+1} = a_n$, $b_{n+1} = c_n$ - otherwise, $a_{n+1} \in E \cap [c_{n+1}, b_n]$, $b_{n+1} = b_n$
  this guarantees $0 \leq b_n - a_n \leq (b_0 - a_0)/2^n$ & $I_n = [a_n, b_n]$ are nested (since $a_n \nearrow, b_n \searrow$)
  claim $\bigcap_{n=0}^{\infty} I_n = \{x\}$ (AFSOC $\exists x,y$ w/ $x < y \Rightarrow 0 < y - x \leq b_n - a_n \Rightarrow n \leq 2^n \leq \frac{b_0 - a_0}{y - x}\ \forall n \to\leftarrow$ $\mathbb{N}$ is bounded)
  then $x = \sup(E)$ (if $x < c$ or $y < x$, similarly $\to\leftarrow$)
  prop: if $b \in F$, F OF, $2 \leq b \Rightarrow n \leq b^n$
- Def: let F be Arch OF, say set $\mathcal{C} \subseteq F$ is (Dedekind) cut if
  (C1) $\mathcal{C} \neq \emptyset$ and $\mathcal{C} \neq F$
  (C2) if $p \in \mathcal{C}$ and $q \in F$ is s.t. $q < p$, then $q \in \mathcal{C}$
  (C3) if $p \in \mathcal{C}$ then $\exists r \in \mathcal{C}$ s.t. $p < r$
- Lem: (C2) $\Rightarrow$ (if $p \in \mathcal{C}$ and $q \notin \mathcal{C}$, then $p > q$), (if $r \notin \mathcal{C}$ and $r < s$, then $s \notin \mathcal{C}$)
- Lem: if $q \in F$, $\{p \in F \mid p < q\}$ is a cut ($q \notin \mathcal{C}$, $q - 1 \in \mathcal{C}$) $\leftarrow$ call this $\mathcal{C}_q$ ← this is an OF field homo
- Def: the dedekind completion of F is $F^* = \{\mathcal{C} \in \mathcal{P}(F) \mid \mathcal{C} \text{ is a cut}\}$
- Prop: sps F is Arch, let $\mathcal{A} \in F^*$ and fix $0 < t \in F^*$, then $\exists n \in \mathbb{Z}$ s.t. $nt \in \mathcal{A}$, $(n+1)t \notin \mathcal{A}$
  $\mathcal{A}$ cut $\Rightarrow p \in \mathcal{A}, q \notin \mathcal{A}$, F Arch $\Rightarrow m \in \mathbb{N}$ s.t. $-p/t < m$, choose k s.t. $q/t < k \Rightarrow kt \notin \mathcal{A}$
  consider $S = \{l \in \mathbb{Z} \mid lt \notin \mathcal{A}\}$, $k \in S \Rightarrow S \neq \emptyset$, w/ $-m$ as LB ($l < -m \Rightarrow lt < (-m)t \Rightarrow (-m)t \notin \mathcal{A}$) $\to\leftarrow$
  by WOP on $\mathbb{Z}$, $\exists!\, s \in \mathbb{Z}$ s.t. $s = \min(S) \Rightarrow$ let $n = s - 1$, $(n+1)t \notin \mathcal{A}$ by construction, if $nt \notin \mathcal{A} \to\leftarrow$
- Lem: let F be an Arch OF, TFH ← sps. $\exists b \in \mathcal{B}$ s.t. $b \notin \mathcal{A}$, then ($a \in \mathcal{A} \overset{C2}{\Rightarrow} a < b \Rightarrow a \in \mathcal{B}$) $\Rightarrow \mathcal{A} \subset \mathcal{B}$
  1. if $\mathcal{A}, \mathcal{B} \in F^*$ then exactly one of the following holds: $\mathcal{A} = \mathcal{B}$, $\mathcal{A} \subset \mathcal{B}$, $\mathcal{B} \subset \mathcal{A}$
  2. if $\mathcal{A}, \mathcal{B}, \mathcal{C} \in F^*$ and $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$
- Def: endow F^* with the order $<$ via: $\mathcal{A} < \mathcal{B}$ if $\mathcal{A} \subset \mathcal{B}$.
- Lem: let $\emptyset \neq E \subseteq F^*$ be bounded above, then $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A} \in F^*$
  (C1) $E \neq \emptyset \Rightarrow \exists \mathcal{A} \in E$, $\mathcal{A} \neq \emptyset \Rightarrow \mathcal{B} \neq \emptyset$ (if $\mathcal{A} \in E$ then $q \notin \mathcal{A}$)
  OTOH, E bounded above $\Rightarrow \exists \mathcal{C} \in F^*$ s.t. $\mathcal{A} \leq \mathcal{C}\ \forall \mathcal{A} \in E$, since $\mathcal{C} \in F^*$, $\exists q \in F^*$ s.t. $c < q\ \forall c \in \mathcal{C}$
  (C2) sps $p \in \mathcal{B}$, i.e. $p \in \mathcal{A}$ for some $\mathcal{A} \in E$, sps $q \in F$ & $q < p$, (C2) $\Rightarrow q \in \mathcal{A} \subseteq \mathcal{B}$
  (C3) sps. $p \in \mathcal{B}$, (C3) $\Rightarrow \exists r \in \mathcal{A}$ s.t. $p < r$, $r \in \mathcal{A} \subseteq \mathcal{B}$

HW

- THM: sps $x, y \in \mathbb{R}$ and $x > 1$, then $E_x(y) = \{x^r \mid r \in \mathbb{Q} \text{ and } r \leq y\}$ is nonempty and bounded above. Consequently, $\sup E_x(y) \in \mathbb{R}$ exists. If $y \in \mathbb{Q}$, $\sup E_x(y) = \max E_x(y) = x^y$

  for rationals, recall $r < s \Rightarrow x^r < x^s$. By Arch prop of $\mathbb{R}$, $\exists q \in \mathbb{Q}$ s.t. $y \leq q$.

  $\Rightarrow$ for any $x^r \in E_x(y)$, $x^r \leq x^q \Rightarrow E_x(y)$ bounded above, $E_x(y) \neq \emptyset$ by Arch $(\exists r < y)$

- Def: let $0 < x \in \mathbb{R}$, $y \in \mathbb{R}$.
  1. assume $x > 1$, define $x^y = \sup E_x(y)$
  2. if $x = 1$, $x^y = 1$, if $0 < x < 1$, $(x^{-1})^{-y}$ $\leftarrow x^{-1} > 1 \Leftrightarrow 0 < x < 1$.

- Prop: let $0 < x \in \mathbb{R}$, $y \in \mathbb{R}$, then $x^y > 0$ (bound w/ rationals)

- THM: sps $x, y, z \in \mathbb{R}$, $x > 0$, then $x^{y+z} = x^y x^z \xrightarrow{\text{Cor.}} (x^y)^{-1} = x^{-y}$

HW

- Lem: sps $1 < x \in \mathbb{R}$, if $y, z \in \mathbb{R}$ with $y < z$, then $x^y < x^z$

  $y < z \Rightarrow E_x(y) \subseteq E_x(z) \Rightarrow x^y = \sup E_x(y) \leq \sup E_x(z) = x^z$, no equality bc. $\exists q \in \mathbb{Q}$, $y < q < z$

HW

- THM: let $b, y \in \mathbb{R}$ with $b > 1$, $y > 0$, then $\exists!\, x \in \mathbb{R}$ s.t. $b^x = y$.
  1. for $z > 1$ and $n \in \mathbb{N}$ positive, $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \dots + 1) \geq n(z-1) \Rightarrow b - 1 \geq n(b^{1/n} - 1)$
  2. if $t > 1$ and $n > (b-1)/(t-1)$ then $b^{1/n} < t$
  3. if $w \in \mathbb{R}$ s.t. $b^w < y$, set $t = yb^{-w} > 1$, then $b^{1/n} < t = yb^{-w} \Rightarrow b^{w+1/n} < y$
  4. if $w \in \mathbb{R}$ s.t. $b^w > y$, let $t = y^{-1}b^w > 1$, then $b^{w-1/n} > y$
  5. set $A = \{w \in \mathbb{R} \mid b^w < y\}$, since $\mathbb{R}$ is Arch, choose $m \in \mathbb{N}$ s.t. $0 < \frac{1}{y(b-1)} < m$.

     (1) w/ $z = b$, $n = m \Rightarrow b^m \geq 1 + m(b-1) > 1/y > 0 \Rightarrow b^{-m} < y \Rightarrow -m \in A \Rightarrow A \neq \emptyset$

     similarly, $\exists k \in \mathbb{N}$ s.t. $0 < \frac{y}{b-1} < k \Rightarrow 0 < y < k(b-1) < b^k \Rightarrow b^w < y < b^k \Rightarrow w < k \Rightarrow A$ bounded above

     $\Rightarrow x = \sup(A) \in \mathbb{R}$, claim $b^x = y$ (if $b^x < y$, $b^{x+1/n} < y$ also $\rightarrow\leftarrow$ x not UB of A, sim. for $b^x > y$)

- Def: let $b, y \in \mathbb{R}$ with $b > 1$ and $y > 0$, define $\log_b : \{t \in \mathbb{R} \mid t > 0\} \to \mathbb{R}$ via $\log_b(y) = x \in \mathbb{R}$, where $b^x = y$ (x exists & is unique by thm ↑), by def. $b^{\log_b(y)} = y$, $\forall y > 0$.

- Lem: sps $x \in \mathbb{R}$, $x > 1$. if $y \in \mathbb{R}$, $x^y = \sup\{x^r \mid r \in \mathbb{Q}, r < y\} = \sup\{x^z \mid z \in \mathbb{R}, z < y\}$.

HW

- Prop: let $b \in \mathbb{R}$ with $b > 1$, $0 < x, y \in \mathbb{R}$
  1. $\log_b(1) = 0$, $\log_b(b) = 1$ $\quad 1 = b^0$, $b = b^1$
  2. $\log_b(xy) = \log_b(x) + \log_b(y)$ $\quad xy = b^{\log_b(x)} b^{\log_b(y)} = b^{\log_b(x) + \log_b(y)}$
  3. $\log_b(x/y) = \log_b(x) - \log_b(y)$ $\quad x/y = b^{\log_b(x) - \log_b(y)}$
  4. $\log_b(x^z) = z\log_b(x)$ $\forall z \in \mathbb{R}$ $\quad$ [Prop: $\forall y, z \in \mathbb{R}$, $x^{yz} = (x^y)^z$] $x^z = b^{z \log_b x}$
  5. $x < y \Leftrightarrow \log_b(x) < \log_b(y)$ $\Leftrightarrow b^{\log_b x} < b^{\log_b y}$
  6. $\log_b : \{t \in \mathbb{R} \mid t > 0\} \to \mathbb{R}$ is a bij. $\quad$ inj. from (5), surj: $\log_b(b^x) = x \log_b b = x$ $\forall x \in \mathbb{R}$
  7. if $c > 1$, $\log_c(x) = \log_c(b)\log_b(x)$ $\quad x = b^{\log_b x} = (c^{\log_c b})^{\log_b x}$

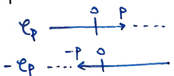
- THM: F^* satisfies the supremum property

  let $\emptyset \neq E \subseteq F^*$ be bounded above, then $\mathcal{B} = \bigcup_{A \in E} A \in F^*$ is UB of E ($A \in E \Rightarrow A \subseteq \mathcal{B}$)

  sps. $\mathcal{C}$ is UB of E, then $A \leq \mathcal{C}$ $\forall A \in E \Rightarrow \mathcal{B} = \bigcup A \leq \mathcal{C} \Rightarrow \mathcal{B} = \min U(E) = \sup E$.

note: the map $F \ni p \mapsto \mathcal{C}_p \in F^*$ is OP ($p < q$ in $F \Rightarrow \mathcal{C}_p < \mathcal{C}_q$ in F^* by transitivity) and hence injective (& bij onto image) $\Rightarrow$ identity $F \cong \{\mathcal{C}_p \mid p \in F\} \subseteq F^*$

[Addition] suppose $A, \mathcal{B} \in F^*$, $A + \mathcal{B} = \{a + b \mid a \in A, b \in \mathcal{B}\}$



$0 := \mathcal{C}_0$, $-A = \{p \in F \mid \exists q > p \text{ s.t. } -q \in A\}$.

↑ & need to exclude $-p$.

[Multiplication] sps initially $A, \mathcal{B} \in F^*$, $A, \mathcal{B} > 0$ → contain elements that are strictly positive

then define $A \cdot \mathcal{B} = \{p \in F \mid p \leq 0\} \cup \{ab \mid 0 < a \in A, 0 < b \in \mathcal{B}\} \in F^*$

in general, define $1 = \mathcal{C}_1$, $A > 0, \mathcal{B} < 0 \overset{\text{def}}{\Rightarrow} A \cdot \mathcal{B} = -[A \cdot (-\mathcal{B})]$ and so forth.

if $A > 0$, set $A^{-1} = \{p \leq 0\} \cup \{0 < p \in F \mid \exists q > p \text{ s.t. } q^{-1} \in A\} \in F^*$ $\quad$ (Def: $\mathbb{R} = \mathbb{Q}^*$)

- THM: given Arch OF F, F^* is Ded (& hence Arch) OF with $F \subseteq F^*$ as ordered subfield.

- THM: let $\mathbb{F}_1, \mathbb{F}_2$ be two Ded OFs, then $\exists!$ OP field isomorphism $\mathbb{F}_1 \underset{\varphi_1^{-1}}{\overset{\varphi_1}{\rightleftarrows}} \mathbb{R} \xrightarrow{\varphi_2} \mathbb{F}_2$

  note it suffices to show if $\mathbb{F}$ is Ded OF, $\exists!\, \varphi : \mathbb{R} \to \mathbb{F}$ that's an OP field iso.

  $\mathbb{F}$ Ded $\Rightarrow$ Arch $\Rightarrow \exists!\, h : \mathbb{Q} \to \mathbb{F}$ that's OP field homo.

  suppose $\mathcal{C} \in \mathbb{R} = \mathbb{Q}^*$, by (C2), $\exists q \in \mathbb{Q}$ s.t. $p < q$ $\forall p \in \mathcal{C}$. since h is OP, $h(p) < h(q)$ $\forall p \in \mathcal{C}$.

  then $h(\mathcal{C}) \subseteq \mathbb{F}$ is bounded above & $\emptyset \neq h(\mathcal{C})$ since $\mathcal{C} \neq \emptyset \Rightarrow$ def $\varphi : \mathbb{R} \to \mathbb{F}$ via $\varphi(\mathcal{C}) = \sup h(\mathcal{C})$

- THM: $\mathbb{R}$ is uncountable, hence irrationals $\mathbb{R} \setminus \mathbb{Q}$ are also uncountable.

  BWOC $\mathbb{R}$ countable, i.e. $\exists$ bij $f : \mathbb{N} \to \mathbb{R}$, let $I_0 = [f(0)+1, f(0)+2] \subseteq \mathbb{R}$, note $f(0) \notin I_0$

  sps for some $n \in \mathbb{N}$ we found $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_0$ s.t. $f(k) \notin I_k$, define I_{n+1} via

  - if $f(n+1) \notin I_n \Rightarrow I_{n+1} = I_n$, else, choose half of I_n that does not incl. $f(n+1)$

  by induction, granted closed int $I_n \subseteq \mathbb{R}$ $\forall n \in \mathbb{N}$, pick $x \in \bigcap_{n=0}^{\infty} I_n$, then $x = f(n)$ for some n $\rightarrow\leftarrow$

- THM: let F be OF, then F is Arch $\Leftrightarrow \mathbb{Q} \subseteq F \subseteq \mathbb{R}$, in which case $\mathbb{R}$ is Ded complete of F

- Def: given $n \in \mathbb{N}$, let $\mathbb{P}_{\mathbb{Q}}^n = \{p : \mathbb{R} \to \mathbb{R} \mid p \text{ is a poly w/ } \mathbb{Q} \text{ coeffs and degree } p \leq n\}$

  write $\mathbb{P}_{\mathbb{Q}} = \bigcup_{n=0}^{\infty} \mathbb{P}_{\mathbb{Q}}^n$, set algebraic real numbers $\mathbb{A} = \{x \in \mathbb{R} \mid p(x) = 0 \text{ for some } p \in \mathbb{P}_{\mathbb{Q}}\}$

- THM: $\mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$: $\mathbb{A} = \bigcup_{p \in \mathbb{P}_{\mathbb{Q}}} R_p$, $R_p = \{x \in \mathbb{R} \mid p(x) = 0\}$, card $\mathbb{P}_{\mathbb{Q}}^n = $ card $\mathbb{Q}^{n+1}$ (coeffs)

  ↑ countable union of countable sets is countable (also $\sqrt{2} \notin \mathbb{Q}$, $x^2 - 2 \in \mathbb{P}_{\mathbb{Q}}$)

- Lem: sps $t \in \mathbb{R}$ transcendental, then t^n transc. $\forall n \in \mathbb{N}^+$, if $t > 0$, t^r transc. $\forall r \in \mathbb{Q} \setminus \{0\}$.

HW

- Prop: let $x, y, z \in \mathbb{R}$ with $0 < x, y$, TFH

  if $z > 0$, $x^z < y^z \Leftrightarrow x < y$, $z < 0$, $y^z < x^z \Leftrightarrow x < y$

  fix $b > 1$, sps $z > 0$, (4)+(5): $x^z < y^z \Leftrightarrow z \log_b x < z \log_b y \Leftrightarrow x < y$

HW

- Prop: let $x, y \in \mathbb{R}$ with $x, y > 0$, let $z \in \mathbb{R}$, then $x^z y^z = (xy)^z$

  $(xy)^z = b^{z \log_b xy} = b^{z \log_b x + z \log_b y} = x^z y^z$

- Def: let $0 \leq p \in \mathbb{R}$, $0^p = 1$ if $p = 0$, 0 if $p > 0$

- Prop: $\forall\, 0 \leq x, y \in \mathbb{R}$, $0 \leq p, q \in \mathbb{R}$.

  (1) $x^p x^q = x^{p+q}$ (2) $(x^p)^q = x^{pq}$ (3) $(xy)^q = x^q y^q$ (4) if $y \neq 0$, $\left(\frac{x}{y}\right)^q = \frac{x^q}{y^q}$

  (5) if $p > 0$, $x^p = 0 \Leftrightarrow x = 0$

§ Sequences

- Def: for $x \in \mathbb{R}$, $r \geq 0$, define

  $B(x, r) = \{y \in \mathbb{R} \mid |x - y| < r\} = (x - r, x + r)$ $B[x, r] = \{y \in \mathbb{R} \mid |x - y| \leq r\} = [x - r, x + r]$

- Lem: $B(x, r) \subseteq B(0, |x| + r)$, $B[x, r] \subseteq B[0, |x| + r]$ (Δ-ineq)

- Prop: let $E \subseteq \mathbb{R}$, E bounded $\Leftrightarrow E \subseteq B(x, r) \Leftrightarrow E \subseteq B[x, r]$ for some $x \in \mathbb{R}$, $r \geq 0$.

  sps $l \leq x \leq u$ $\forall x \Rightarrow l - 1 < x < u + 1 \Rightarrow E \subseteq B\left(\frac{u+l}{2}, \frac{u-l+2}{2}\right)$

- Def: let $X \neq \emptyset$ be a set, a sequence starting at $l \in \mathbb{Z}$ is a function $f: \{n \in \mathbb{Z} : n \geq l\} \to X$

  write $X_n = f(n)$ for $n \geq l$, denote seq. by $\{X_n\}_{n=l}^{\infty} \subseteq X$

- Def: let $\emptyset \neq X$ be a set, seq. $\{x_n\}_{n=l}^{\infty} \subset X$ is <u>eventually constant</u> if $\exists N \geq l$.

  s.t. $x_n = x_N$ $\forall n \geq N$ $\rightsquigarrow$ eventually const. seq encode finitely many elements of X

- Def: let $\{x_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$

  1. let $x \in \mathbb{R}$, say seq. converges to x as $n \to \infty$, i.e. $\lim_{n \to \infty} x_n = x$,

     if $\forall\, 0 < \varepsilon \in \mathbb{R}$, $\exists\, l \leq N \in \mathbb{Z}$ s.t. $n \geq N \Rightarrow |x_n - x| < \varepsilon$.

  2. seq. is Cauchy if $\forall \varepsilon > 0$, $\exists N \geq l$ s.t. $m, n \geq N \Rightarrow |x_n - x_m| < \varepsilon$.

  3. seq. is bounded if $\{x_n \mid n \geq l\} \subseteq \mathbb{R}$ is bounded.

- Prop: if $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$ (i.e. limits are unique)

  AFSOC $x \neq y$, $\varepsilon = \frac{|x-y|}{2}$, $\exists N_1, N_2 \geq l$ s.t. $n \geq$ both $\Rightarrow |x_n - x|, |x_n - y| < \varepsilon$ $\to\leftarrow$ via Δ-ineq.

- THM: seq. convergent $\Rightarrow$ Cauchy $\Rightarrow$ bounded.

  (1) $\exists N \geq l$ s.t. $n \geq N \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$, then $m, n \geq N \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by Δ-ineq.

  (2) $\exists N \geq l$ s.t. $m, n \geq N \Rightarrow |x_n - x_m| < 1$, then $\sup_{n \geq l} |x_n - x_N| \leq \max\{1, \max\{|x_n - x_N| \mid n = l \ldots N\}\} = R < \infty$,

  then $x_n \in B(x_N, R+1)$ $\forall n \geq l$.

- Prop: let $f: \{n \in \mathbb{Z} \mid n \geq l\} \to \mathbb{N} \setminus \{0\}$ be OP, then $\{x_n\}_{n=l}^{\infty}$ via $x_n = \frac{1}{f(n)}$ conv. to 0.

  $l \leq m < n \Rightarrow 0 < x_n < x_m$, and f inj $\Rightarrow A = \{x_n \mid n \geq l\} \subseteq \{\frac{1}{n} \mid 1 \leq n \in \mathbb{N}\} \subseteq \mathbb{R}$ is infinite.

  $\mathbb{R}$ Arch $\Rightarrow$ inf. $A = 0$, let $\varepsilon > 0$, pick $N \geq l$ s.t. $0 < x_N < \varepsilon$

  then, for $n \geq N$, $|x_n - 0| = |x_n| \leq x_N < \varepsilon \Rightarrow x_n \to 0$ as $n \to \infty$.

- Ex: let $0 < r \in \mathbb{R}$, consider $\{\frac{1}{n^r}\}_{n=1}^{\infty} \subseteq \mathbb{R}$, let $\varepsilon > 0$ and pick (Arch!) $N \geq 1$ s.t. $\frac{1}{\varepsilon^{1/r}} < N$.

  then $n \geq N \Rightarrow \left|\frac{1}{n^r} - 0\right| = \frac{1}{n^r} \leq \frac{1}{N^r} < (\varepsilon^{1/r})^r = \varepsilon \Rightarrow \frac{1}{n^r} \to 0$ as $n \to \infty$.

- Ex: $x_n = \frac{n!}{n^n}$ for $n \geq 0$, note $0 < \frac{n!}{n^n} = \frac{n(n-1)\cdots 2 \cdot 1}{n \cdot n \cdots n} \leq 1 \cdot 1 \cdots 1 \cdot \frac{1}{n} = \frac{1}{n}$

  let $\varepsilon > 0$ and pick $N \geq 1$ s.t. $n \geq N \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow \frac{n!}{n^n} \to 0$ as $n \to \infty$.

HW

- Prop: let $\{x_n\}_{n=\ell}^{\infty}, \{\alpha_n\}_{n=\ell}^{\infty}, \{\beta_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, sps. $x_n \to x$, $\alpha_n \to \alpha$, $\beta_n \to \beta$ as $n \to \infty$,
  1. if $\exists j \geq \ell$ s.t. $\alpha_n \leq \beta_n \ \forall n \geq j$, then $\alpha \leq \beta$
  2. if $\exists j \geq \ell$ s.t. $|x_n| \leq \alpha_n \ \forall m \geq j$, then $|x| \leq \alpha$ $\Downarrow$ $|a| \leq b \Leftrightarrow -b \leq a \leq b$

  let $\varepsilon > 0$, choose $K_1, K_2 \geq \ell$ s.t. $k \geq \max\{K_1, K_2\}^{(\& j)} \Rightarrow \alpha - \frac{\varepsilon}{2} \leq \alpha_n \leq \beta_n \leq \frac{\varepsilon}{2} \Rightarrow \alpha - \beta \leq \varepsilon$.

- Def: let $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, monotone if either non-decr or non-incr.
  - non-decreasing if $\ell \leq n < m \Rightarrow x_n \leq x_m$
  - non-increasing if $\ell \leq n < m \Rightarrow x_n \geq x_m$

HW

- THM: sps. $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$ is bounded & monotone, then seq. is convergent.

  sps. seq. is non-decr, consider $E = \{x_n \mid n \geq \ell\}$, $\emptyset \neq E$ is bounded $\Rightarrow x = \sup E$.

  let $\varepsilon > 0$, then $x - \varepsilon$ is not UB of E $\Rightarrow$ $\exists K \geq \ell$ s.t. $x - \varepsilon < x_K \leq x_n \leq x < x + \varepsilon \Rightarrow |x_n - x| < \varepsilon$

- THM (algebra of limits): sps $x_n \to x$, $y_n \to y$ as $n \to \infty$, then $x_n + y_n \xrightarrow{(\Delta\text{-ineq})} x + y$, $x_n y_n \xrightarrow{(2)} xy$, and if $\{x_n\}_{n=\ell}^{\infty} \subset \mathbb{R}\setminus\{0\}$ and $x \in \mathbb{R}\setminus\{0\}$ then $x_n^{-1} \xrightarrow{(3)} x^{-1}$ as $n \to \infty$.

  (2): note that both seq are bounded, so $\exists r > 0$ s.t. $|y_n| < r \ \forall n \geq \ell$. next, bound

  $|x_n y_n - xy| = |x_n y_n - y_n x + y_n x - xy| \leq |y_n||x_n - x| + |x||y_n - y| \leq r|x - x_n| + |x||y_n - y|$

  let $\varepsilon > 0$, since $x_n \to x$, $y_n \to y$, $\exists N \geq \ell$. s.t. $n \geq N \Rightarrow |x - x_n| < \frac{\varepsilon}{2r}$, $|y - y_n| < \frac{\varepsilon}{2(1+|x|)}$

  $\Rightarrow |x_n y_n - xy| \leq r \cdot \frac{\varepsilon}{2r} + |x| \frac{\varepsilon}{2(1+|x|)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \frac{|x|}{1+|x|} = \varepsilon$.

  (3): sps $x_n \neq 0$, $x \neq 0$. note that $\left|\frac{1}{x_n} - \frac{1}{x}\right| = \left|\frac{x - x_n}{x_n x}\right| = |x_n^{-1}||x^{-1}||x - x_n|$

  since $x_n \to x$ as $n \to \infty$ and $|x| > 0$, $\exists N \geq \ell$ s.t. $n \geq N \Rightarrow |x_n - x| < \frac{|x|}{2}$

  then $|x| = |x - x_n + x_n| \leq \frac{|x|}{2} + |x_n| \Rightarrow \frac{|x|}{2} = |x| - \frac{|x|}{2} < |x_n| \Rightarrow 0 < |x_n^{-1}| < \frac{2}{|x|} \ \forall n \geq N$

  let $\varepsilon > 0$, pick $M \geq \ell$ s.t. $n \geq M \Rightarrow |x_n - x| < \frac{\varepsilon}{2} x^2$. then for $n \geq \max\{M, N\}$,

  $\left|\frac{1}{x_n} - \frac{1}{x}\right| \leq 2|x^{-1}|^2 |x_n - x| = \frac{2}{x^2}|x_n - x| < \frac{2}{x^2} \frac{\varepsilon \cdot x^2}{2} = \varepsilon$.

- THM: (1) let $x \in \mathbb{R}$ with $x > 0$, $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $a_n = x^{1/n}$, satisfies $a_n \to 1$ as $n \to \infty$.

  (2) $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ given by $a_n = n^{1/n}$ satisfies $a_n \to 1$ as $n \to \infty$

  (1) sps $x > 1$, set $b_n = a_n - 1$, then $x = (1 + b_n)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}(b_n)^j \geq 1 + n b_n \Rightarrow 0 \leq b_n \leq \frac{x-1}{n}$

  (2) $n = (1 + b_n)^n \geq \frac{n(n-1)}{2}(b_n)^2$, then for $n \geq 2$, $0 \leq b_n \leq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{4}{n}} = \frac{2}{\sqrt{n}} \Rightarrow b_n \to 0$ by SL.

- THM: let $0 < x \in \mathbb{R}$, sps $\{z_n\}_{n=\ell}^{\infty}$ s.t. $z_n \to z$ as $n \to \infty$, then $x^{z_n} \to x^z$ as $n \to \infty$

  [$x > 1$]

  let $m_n = \min\{z, z_n\}$, $M_n = \max\{z, z_n\}$, then $\{m_n\}_{n=\ell}^{\infty}$, $\{M_n\}_{n=\ell}^{\infty}$ bounded.

  $\Rightarrow \exists R > 0$ s.t. $|M_n| \leq R \ \forall n \geq \ell \Rightarrow |x^z - x^{z_n}| = |x^{M_n}(x^{M_n - m_n} - 1)| = x^{M_n}|x^{M_n - m_n} - 1|$

  OTOH, $M_n - m_n = |z - z_n| \geq 0$. let $\varepsilon > 0$, since $x^{1/n} \to 1$, $\exists N \geq 1$ s.t. $|x^{1/N} - 1| < \frac{\varepsilon}{x^R}$,

  $z_n \to z \Rightarrow \exists K, n \geq K \Rightarrow |z_n - z| < 1/N \Rightarrow |x^z - x^{z_n}| \leq x^R(x^{|z - z_n|} - 1) \leq x^R(x^{1/N} - 1) < x^R \cdot \frac{\varepsilon}{x^R} = \varepsilon$.

- Ex: let $0 \leq r \in \mathbb{R}$, set $x_n = \frac{r^n}{n!}$ for $n \geq 0$. $\mathbb{R}$ Arch $\Rightarrow$ pick $r < N \in \mathbb{N}$, then

  $n \geq N+1 \Rightarrow x_n = \frac{r^n}{n!} < \frac{N^n}{n!} = \frac{N^N N^{(n-N)}}{N!\, n(n-1)\cdots(N+1)} = \left(\frac{N^N}{N!}\right)\frac{N}{n}\cdots\frac{N}{N+1} \leq \frac{N^N}{N!}\frac{N}{n}$

  let $\varepsilon > 0$, pick $M \geq 1$ s.t. $n \geq M \Rightarrow \frac{1}{n} < \varepsilon \cdot \frac{N!}{N^{N+1}} \Rightarrow \cdots \Rightarrow \frac{r^n}{n!} \to 0$ as $n \to \infty$

- Ex: let $2 \leq b \in \mathbb{N}$, define $S_n = \sum_{m=0}^{n} \frac{1}{b^m}$ for $n \geq 0$.

  $\frac{S_n}{b} = \sum_{m=0}^{n} \frac{1}{b^{m+1}} = \sum_{m=1}^{n+1} \frac{1}{b^m} \Rightarrow \frac{S_n}{b} - S_n = \frac{1}{b^{n+1}} - 1 \Rightarrow S_n = \frac{b}{b-1}\left(1 - \frac{1}{b^{n+1}}\right) \Rightarrow \left|S_n - \frac{b}{b-1}\right| = \frac{1}{b-1}\frac{1}{b^n}$

  by Arch, $\frac{1}{b^n} \to 0$ as $n \to \infty$, let $\varepsilon > 0$, pick $N \geq 0$ s.t. $n \geq N \Rightarrow \frac{1}{b^n} < (b-1)\varepsilon$.

- Ex: let $2 \leq b \in \mathbb{N}$, sps $\{y_n\}_{n=0}^{\infty} \subseteq B[0,1]$, i.e. $|y_n| \leq 1 \ \forall n$. let $x_n = \sum_{m=0}^{n} \frac{y_m}{b^m}$ $(n \geq 0)$

  for $n > k \geq 0$, $|x_n - x_k| = \left|\sum_{m=k+1}^{n} \frac{y_m}{b^m}\right| \overset{\Delta\text{-ineq}}{\leq} \sum_{m=k+1}^{n} \frac{|y_m|}{b^m} \leq \sum_{m=k+1}^{n} \frac{1}{b^m} = S_n - S_k$

  $\{S_n\}_{n=0}^{\infty}$ is conv $\Rightarrow$ Cauchy $\Rightarrow$ for $\varepsilon > 0$, $\exists N \geq 0$ s.t. $n, k \geq N \Rightarrow |S_n - S_k| < \varepsilon \Rightarrow \{x_n\}_{n=0}^{\infty}$ is Cauchy

- THM: let $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, seq. is conv $\Leftrightarrow$ Cauchy

  sps. seq is Cauchy & hence bounded, $\exists M \geq 0$ s.t. $|x_n| \leq M \ \forall n \geq \ell$.

  define the set $E = \{y \in \mathbb{R} \mid \exists N \geq \ell \text{ s.t. } n \geq N \Rightarrow y < x_n\}$

  $-M \leq x_n \ \forall n \geq \ell \Rightarrow -M-1 \in E$ & if $y \in E$, $y < x_n \leq M$ for $n \geq N$, so M is UB of E.

  since $\mathbb{R}$ Ded, $x = \sup E$. by Cauchyness, $\exists N \geq \ell$ s.t. $m, n \geq N \Rightarrow |x_m - x_n| < \frac{\varepsilon}{2}$.

  in part, $|x_n - x_N| < \frac{\varepsilon}{2} \ \forall n \geq N \Rightarrow x_N - \frac{\varepsilon}{2} < x \Rightarrow x_N - \frac{\varepsilon}{2} \in E \Rightarrow x_N - \frac{\varepsilon}{2} \leq x$

  OTOH, if $y \in E$, then $\exists L \geq \ell$ s.t. $n \geq L \Rightarrow y < x_n$, then $n \geq \max\{L, N\} \Rightarrow$

  $y < x_n \leq |x_n - x_N| + x_N < \frac{\varepsilon}{2} + x_N \Rightarrow x_N + \frac{\varepsilon}{2}$ is an UB of $E \Rightarrow x \leq x_N + \frac{\varepsilon}{2}$

  then $x_N - \frac{\varepsilon}{2} \leq x \leq x_N + \frac{\varepsilon}{2} \Rightarrow |x - x_N| \leq \frac{\varepsilon}{2} \Rightarrow |x_n - x| \leq |x - x_N| + |x_N - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

- Prop: let $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, then $(x_n \to x \Leftrightarrow |x_n - x| \to 0)$, $(x_n \to 0 \Leftrightarrow |x_n| \to 0)$ as $n \to \infty$

- Prop: let $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$ s.t. $x_n \to x$ as $n \to \infty$, then $|x_n| \to |x|$ as $n \to \infty$

  $||a| - |b|| \leq |a - b|$, then for $\varepsilon > 0$, $\exists N \geq \ell$ s.t. $n \geq N \Rightarrow ||x_n| - |x|| \leq |x_n - x| < \varepsilon$.

- Lem (squeeze lemma): let $\{a_n\}_{n=\ell}^{\infty}, \{b_n\}_{n=\ell}^{\infty}, \{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, sps $a_n \to x$ and $b_n \to x$ as $n \to \infty$, if $\exists N \geq \ell$ s.t. $n \geq N \Rightarrow a_n \leq x_n \leq b_n$, then $x_n \to x$ as $n \to \infty$.

  let $\varepsilon > 0$, $\exists M \geq \ell$ s.t. $n \geq M \Rightarrow |a_n - x| < \varepsilon$, $|b_n - x| < \varepsilon$, then for $n \geq \max\{N, M\}$,

  $-\varepsilon < -|a_n - x| \leq a_n - x \leq x_n - x \leq b_n - x \leq |b_n - x| < \varepsilon \Rightarrow |x_n - x| < \varepsilon$.

- Def: let $X \neq \emptyset$ be a set and $\{x_n\}_{n=\ell}^\infty \subseteq X$, let $\varphi : \{n \in \mathbb{Z} \mid n \geq \ell\} \to \{n \in \mathbb{Z} \mid n \geq \ell\}$ be increasing/OP, then call $\{x_{n_k}\}_{k=\ell}^\infty = \{x_{\varphi(k)}\}_{k=\ell}^\infty$ a subsequence of $\{x_n\}_{n=\ell}^\infty$.
- Lem: let $\varphi : \{n \geq \ell\} \to \{n \geq \ell\}$ be increasing, then $n \leq \varphi(n)$ $\forall n \geq \ell$.

  prove $\ell + n \leq \varphi(\ell+n)$ $\forall n \in \mathbb{N}$ by induction: for $n=0$, $\ell \leq \varphi(\ell)$ by def of codomain of φ.
  φ incr $\Rightarrow \varphi(\ell+n) < \varphi(\ell+n+1) \Rightarrow$ by def of order, $\varphi(\ell+n)+1 \leq \varphi(\ell+n+1)$

- Prop: let $\{x_n\}_{n=\ell}^\infty \subseteq \mathbb{R}$, then (1) if $x_n \to x$ and $\{x_{n_k}\}_{k=\ell}^\infty$ is any subseq, $x_{n_k} \to x$ as $k \to \infty$. (2) if $\{x_n\}_{n=\ell}^\infty$ is Cauchy, $\{x_{n_k}\}_{k=\ell}^\infty$ is Cauchy, (3) if $\{x_n\}_{n=\ell}^\infty$ Cauchy and $x_{n_k} \to x$, $x_n \to x$.

  (1) $\Rightarrow$ (2), (1+2) $\Rightarrow$ (3). then (1): let $\varepsilon > 0$, pick $N \geq \ell$ s.t. $n \geq N \Rightarrow |x_n - x| < \varepsilon$.
  $k \leq \varphi(k)$ $\forall k \geq \ell \Rightarrow N \leq k \leq \varphi(k) \Rightarrow |x - x_{\varphi(k)}| < \varepsilon$

- THM (Bolzano-Weierstrass): sps $\{x_n\}_{n=\ell}^\infty \subseteq \mathbb{R}$ is bounded, then $\exists$ a conv. subseq.

  WLOG, assume $\ell = 0$. bounded $\Rightarrow \exists r \geq 0$ s.t. $|x_n| \leq r$ $\forall n \in \mathbb{N}$. define $I_0 = [-r, r]$.
  length of $I_0 = 2r$ and $x_n \in I_0$ $\forall n \in \mathbb{N}$. consider $[-r, 0], [0, r] \subseteq I_0 \Rightarrow$ contain x_n for ∞ n.
  $\Rightarrow I_1 :=$ such interval, I_1 of length $r \Rightarrow I_n \subset I_{n-1} \subset \dots \subset I_0$ s.t. $\{m \in \mathbb{N} \mid x_m \in I_n\}$ is infinite & I_k has length $\frac{2r}{2^k}$. this yields $\{I_n\}_{n=0}^\infty$ with $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n$,
  $b_n - a_n = \frac{r}{2^{n-1}}$, define $\varphi : \mathbb{N} \to \mathbb{N}$ via $\varphi(0) = 0$, given $\varphi(n) \in \mathbb{N}$, set $\varphi(n+1) = \min\{j \mid x_j \in I_{n+1}$ & $j > \varphi(n)\}$.
  then φ is incr; consider $\{x_{\varphi(n)}\}_{n=0}^\infty$, note $x_{\varphi(n)} \in I_n$ $\forall n \in \mathbb{N}$.
  sps $m \geq n \geq 0$, then $x_{\varphi(m)} \in I_m \subseteq I_n$, $x_{\varphi(n)} \in I_n \Rightarrow x_{\varphi(n)}, x_{\varphi(m)} \in [a_n, b_n] \rightsquigarrow$ Cauchy.

§ Extended Reals, Limsup, Liminf.

$A \subseteq B \Rightarrow \inf B \leq \inf A$, $\sup A \leq \sup B$
$\emptyset \neq A \Rightarrow \inf A \leq \sup A$

- Lem: (1) $\overline{\mathbb{R}}$ is ordered set, (2) if $E \subseteq \overline{\mathbb{R}}$, E is bounded (by ∞ and $-\infty$), (3) $\overline{\mathbb{R}}$ satisfies supremum prop.

  Note: $\overline{\mathbb{R}}$ cannot be made into OF: otherwise it would be Ded, then $\exists \psi : \mathbb{R} \to \overline{\mathbb{R}}$ that is OP field iso.
  then $\exists x \in \mathbb{R}$ s.t. $\psi(x) = \infty$, and then $\infty = \psi(x) < \psi(x+1) \in \overline{\mathbb{R}}$ $\rightarrow\leftarrow$

- Def. $\sup_{n \geq N} x_n = \sup\{x_n \mid n \geq N\}$, $\inf_{n \geq N} x_n = \inf\{x_n \mid n \geq N\}$ (i.e. over tail)

  then consider new seqs $\{\sup_{n \geq N} x_n\}_{N=\ell}^\infty$, $\{\inf_{n \geq N} x_n\}_{N=\ell}^\infty \subseteq \overline{\mathbb{R}}$
  define $\limsup_{n \to \infty} x_n = \inf_{N \geq \ell} \sup_{n \geq N} x_n \in \overline{\mathbb{R}}$, $\liminf_{n \to \infty} x_n = \sup_{N \geq \ell} \inf_{n \geq N} x_n$

  as $N \uparrow$, $\sup_{n \geq N} x_n \downarrow$ and $\inf_{n \geq N} x_n \uparrow$

- Prop: let $\{x_n\}_{n=\ell}^\infty \subseteq \overline{\mathbb{R}}$, $\liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n$

  sps $M, N \geq \ell$, $K = \max\{M, N\}$, then $\inf_{n \geq M} x_n \leq \inf_{n \geq K} x_n \leq \sup_{n \geq K} x_n \leq \sup_{n \geq N} x_n$
  i.e. for fixed N, $\inf_{n \geq M} x_n \leq \sup_{n \geq N} x_n$ $\forall M \Rightarrow \liminf_{n \to \infty} x_n = \sup_{M \geq \ell} \inf_{n \geq M} x_n \leq \sup_{n \geq N} x_n$ $\forall N \geq \ell$.
  (then $\liminf_{n \to \infty} x_n \leq \inf_{N \geq \ell} \sup_{n \geq N} x_n$)

- Ex: sps $\{x_n\}_{n=\ell}^\infty$ is bounded, then $\{\sup_{n \geq N} x_n\}_{N=\ell}^\infty$, $\{\inf_{n \geq N} x_n\}_{N=\ell}^\infty$ are bounded & monotone.
  $\Rightarrow$ convergent & $\limsup_{n \to \infty} x_n = \lim_{N \to \infty} \sup_{n \geq N} x_n$, $\liminf_{n \to \infty} x_n = \lim_{N \to \infty} \inf_{n \geq N} x_n$.
- THM: sps $\{a_n\}_{n=\ell}^\infty, \{b_n\}_{n=\ell}^\infty \subseteq \overline{\mathbb{R}}$ are s.t. $\exists K \geq \ell$ s.t. $n \geq K \Rightarrow a_n \leq b_n$, TFH.
  1. if $k \geq K$, $\inf_{n \geq k} a_n \leq \inf_{n \geq k} b_n$ and $\sup_{n \geq k} a_n \leq \sup_{n \geq k} b_n$
  2. $\liminf_{n \to \infty} a_n \leq \liminf_{n \to \infty} b_n$, $\limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n$

  2. $\liminf_{n \to \infty} a_n = \sup_{N \geq \ell}\left[\inf_{n \geq N} a_n\right] \overset{\text{(mono)}}{=} \sup_{N \geq K}\left[\inf_{n \geq N} a_n\right] \leq \sup_{N \geq K}\left[\inf_{n \geq N} b_n\right] \dots$

  1. sps BWOC that $\inf_{n \geq k} a_n > \inf_{n \geq k} b_n$ for $k \geq K$, then $\exists m \geq k$ s.t. $b_m < \inf_{n \geq k} a_n \leq a_m$

- THM: let $\{x_n\}_{n=\ell}^\infty \subseteq \overline{\mathbb{R}}$, TFH ($x \in \overline{\mathbb{R}}$)
  (1) if $\limsup_{n \to \infty} x_n < x$, $\exists N \geq \ell$ s.t. $n \geq N \Rightarrow x_n < x$ → by assumption, $\inf_{N \geq \ell} \sup_{n \geq N} x_n < x \Rightarrow \exists N \geq \ell$ s.t. $\sup_{n \geq N} x_n < x$
  (2) if $x < \liminf_{n \to \infty} x_n$, $\exists N \geq \ell$ s.t. $n \geq N \Rightarrow x < x_n$ → sim: $x < \sup_{N \geq \ell} \inf_{n \geq N} x_n \Rightarrow \exists N \geq \ell$ s.t. $x < \inf_{n \geq N} x_n$
- Lem: let $a, b \in \mathbb{R}$, then $a \leq b \Leftrightarrow a \leq b + \varepsilon$ $\forall \varepsilon > 0$. ($\Leftarrow$) AFSOC $b < a$, $\varepsilon = \frac{a-b}{2} > 0$, $a \leq b + \varepsilon < a$ $\rightarrow\leftarrow$
- THM: sps $\{a_n\}_{n=\ell}^\infty, \{b_n\}_{n=\ell}^\infty \subseteq \overline{\mathbb{R}}$, $a_n + b_n$, $\liminf_{n \to \infty} a_n + \liminf_{n \to \infty} b_n$ well-defined, then TFH
  (1) $\liminf_{n \to \infty}(-a_n) = -\limsup_{n \to \infty} a_n$, $\limsup_{n \to \infty}(-a_n) = -\liminf_{n \to \infty} a_n$
  (2) $\limsup_{n \to \infty}(a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$, $\liminf_{n \to \infty}(a_n + b_n) \geq \liminf_{n \to \infty} a_n + \liminf_{n \to \infty} b_n$

  1. $\forall E \subseteq \overline{\mathbb{R}}$, $-E = \{-x \mid x \in E\}$ satisfies $\inf(-E) = -\sup(E)$, $\inf_{N \geq \ell} \sup_{n \geq N} a_n = \inf_{N \geq \ell} -\inf_{n \geq N}(-a_n) = -\sup_{N \geq \ell} \inf_{n \geq N}(-a_n)$
  2. if $\limsup a_n = \infty$, done, if $\limsup a_n = -\infty$, $\limsup b_n < K$ finite, then $\exists N$, $a_n < -M - K$ $\forall M$
  else $\limsup_{n \to \infty} a_n < \limsup_{n \to \infty} a_n + \frac{\varepsilon}{2} \Rightarrow \exists N$, $a_n < \limsup_{k \to \infty} a_k + \frac{\varepsilon}{2} \Rightarrow a_n + b_n < \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n + \varepsilon$.

- Lem: let $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, $x \in \mathbb{R}$, then $x_n \to x$ as $n \to \infty \Leftrightarrow \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$

($\Rightarrow$) $\exists N$, $|x_n - x| < \varepsilon \Rightarrow x - \varepsilon < x_n < x + \varepsilon \Rightarrow x - \varepsilon \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq x + \varepsilon$.

($\Leftarrow$) $\limsup_{n\to\infty} x_n < x + \varepsilon$, $\liminf_{n\to\infty} x_n > x + \varepsilon \Rightarrow \exists N$, $x - \varepsilon < x_n < x + \varepsilon$

↳ Def: let $\{a_n\}_{n=\ell}^{\infty} \subseteq \overline{\mathbb{R}}$, say seq. is convergent if $\exists a \in \overline{\mathbb{R}}$, $\liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty} a_n$

- Prop: $a_n \to \infty \Leftrightarrow \liminf_{n\to\infty} a_n = \infty \Leftrightarrow \forall M \in \mathbb{R}, \exists N \geq \ell$ s.t. $n \geq N \Rightarrow M < a_n$

  $a_n \to -\infty \Leftrightarrow \limsup_{n\to\infty} a_n = -\infty \Leftrightarrow \forall M > 0, \exists N \in \mathbb{Z}$ s.t. $n \geq N \Rightarrow a_n < -M$

- Prop: let $\{a_n\}_{n=\ell}^{\infty} \subseteq \overline{\mathbb{R}}$, if a_n non-decr, $\lim_{n\to\infty} = \sup_{n\geq\ell} a_n$, if a_n non-incr, $\lim_{n\to\infty} a_n = \inf_{n\geq\ell} a_n$

  in particular, $\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf_{n\geq N} a_n$, $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup_{n\geq N} a_n$

- THM: let $\{a_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, TFH

  1. if $\{a_{n_k}\}_{k=\ell}^{\infty}$ is any subseq. then $\liminf_{n\to\infty} a_n \leq \liminf_{k\to\infty} a_{n_k} \leq \limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$
  2. if $a_n \to a \in \overline{\mathbb{R}}$, $a_{n_k} \to a$ as $k \to \infty$ $\Downarrow$
  3. $\exists$subseq s.t. $a_{n_k} \to \limsup_{n\to\infty} a_n$ / $a_{n_k} \to \liminf_{n\to\infty} a_n$ as $k \to \infty$ — $\limsup(-a_n) = -\liminf a_n$ — $\liminf a_n, \limsup a_n \in E$ by (1): liminf is LB, limsup UB
  5. let $E = \{a \in \overline{\mathbb{R}} \mid \exists \text{subseq } \{a_{n_k}\}_{k=\ell}^{\infty} \text{ s.t. } a_{n_k} \to a\}$, then $\min E = \liminf_{n\to\infty} a_n$, $\max E = \limsup_{n\to\infty} a_n$

↳ Cor (BW for $\overline{\mathbb{R}}$): if $\{a_n\}_{n=\ell}^{\infty} \subseteq \overline{\mathbb{R}}$ then $\exists$ a conv. subseq $\{a_{n_k}\}_{k=\ell}^{\infty}$. (by (3), (4))

(1) $k \leq n_k \; \forall k \geq \ell$, so $\{a_{n_k} \mid k \geq N\} \subseteq \{a_n \mid n \geq N\} \Rightarrow \sup_{k\geq N} a_{n_k} \leq \sup_{n\geq N} a_n$, $\inf_{n\geq N} a_n \leq \inf_{k\geq N} a_{n_k}$

(4) if $\liminf_{n\to\infty} a_n = \infty$, then $a_n \to \infty$ as $n \to \infty$

  if $\liminf_{n\to\infty} a_n = -\infty$, $\inf_{n\geq N} a_n = -\infty \; \forall N \geq \ell \Rightarrow \exists m \geq \ell$ s.t. $a_m < -j$, set $m = n_j$ for $j \geq \ell$. → i.e. $a_{n_j} < -j \; \forall j \Rightarrow a_{n_k} \to -\infty$.

(3) sps $a = \limsup_{n\to\infty} a_n \in \mathbb{R}$, claim $\forall \varepsilon > 0, M \geq \ell, \exists n > M$ s.t. $a - \varepsilon < a_n < a + \varepsilon$

  (pf: $\limsup_{n\to\infty} a_n < a + \varepsilon \Rightarrow \exists K$ s.t. $n \geq K \Rightarrow a_n < a + \varepsilon$. OTOH, $a - \varepsilon < \limsup_{n\to\infty} a_n = \inf_{N\geq\ell} \sup_{n\geq N} a_n \leq \sup_{n\geq\max\{K, M+1\}} a_n$

  then $\exists n \geq \max\{K, M+1\}$ s.t. $a - \varepsilon < a_n < a + \varepsilon$).

  then consider subseq $\{a_{n_k}\}_{k=\ell}^{\infty}$: pick $n_\ell > \ell$ s.t. $a - 2^{-\ell} < a_{n_\ell} < a + 2^{-\ell}$, and given $n_\ell < \dots < n_k$,

  pick $n_{k+1} > n_k$ s.t. $a - 2^{-(k+1)} < a_{n_{k+1}} < a + 2^{-(k+1)}$ $\Rightarrow$ apply squeeze lemma.

§ 4 - Series

- Def: let $S_N = \sum_{n=\ell}^{N} x_n$, if $\{S_N\}_{N=\ell}^{\infty}$ converges in $\mathbb{R}$, say the infinite series $\sum_{n=\ell}^{\infty} x_n$ converges

HW
- Prop: if $\sum_{n=\ell}^{\infty} x_n$ converges, then $x_n \to 0$ as $n \to \infty$ — $x_n = S_n - S_{n-1} \to S - S = 0$ as $n \to \infty$.

HW
- Prop: $\sum_{n=0}^{\infty} x^n$ converges $\Leftrightarrow |x| < 1$. when $|x| < 1$, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ — & sps converge, then $x^n \to 0 \Leftrightarrow |x|^n \to 0$ AFSOC $|x| \geq 1 \Rightarrow |x|^n \geq 1 \rightarrow\leftarrow$

  sps $|x| < 1$, $xS_k = \sum_{n=0}^{k} x^{n+1} = S_k - 1 + x^{k+1} \Rightarrow S_k = \frac{1 - x^{k+1}}{1-x}$ as $k \to \infty$

HW
- Thm (Cauchy criterion): for $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, $\sum_{n=\ell}^{\infty} x_n$ converges $\Leftrightarrow \forall \varepsilon > 0, \exists N$ s.t. $\left|\sum_{k=m}^{n} x_k\right| < \varepsilon$ $(n \geq m \geq N \geq \ell)$

  since $\mathbb{R}$ complete, $\{S_N\}_{N=\ell}^{\infty}$ converges $\Leftrightarrow$ Cauchy

HW
- Prop (Comparison test): let $\{a_n\}_{n=\ell}^{\infty}$, $\{b_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$, if $\exists N \geq \ell$ s.t. $\forall n \geq N$...
  - $|a_n| \leq b_n$ & $\sum_{n=\ell}^{\infty} b_n$ converges $\Rightarrow \sum_{n=\ell}^{\infty} a_n$ converges — Cauchy criterion & $\left|\sum_{n=k}^{m} a_n\right| \leq \sum_{n=k}^{\infty} |a_n| \leq \sum_{n=k}^{\infty} b_n$
  - $0 \leq a_n < b_n$ & $\sum_{n=\ell}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=\ell}^{\infty} b_n$ diverges — (1) $\Rightarrow$ AFSOC $\sum b_n$ conv then $\sum a_n$ conv $\rightarrow\leftarrow$

- Def: $\sum_{n=\ell}^{\infty} x_n$ converges absolutely if $\sum_{n=\ell}^{\infty} |x_n|$ converges — abs conv $\Rightarrow$ conv.

HW
- Prop: let $\{x_n\}_{n=\ell}^{\infty} \subseteq [0, \infty)$, then $\sum_{n=\ell}^{\infty} x_n$ converges $\Leftrightarrow \{S_N\}_{N=\ell}^{\infty}$ is bounded. — ($\Rightarrow$) conv seq bounded ($\Leftarrow$) bounded monotone seq. converges.

- Thm (Cauchy-Schlömilch condensation test): sps $\{x_n\}_{n=1}^{\infty} \subseteq [0, \infty)$ is nonincr, and let $\{u_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ be incr and sps $\sup_{n\geq 1} \frac{u_{n+1} - u_n}{u_n - u_{n-1}} = M < \infty$, then $\sum_{n=1}^{\infty} x_n$ conv $\Leftrightarrow \sum_{n=0}^{\infty} (u_{n+1} - u_n) x_{u_n}$ conv

  since $\{x_n\}_{n=1}^{\infty}$ is nonneg & nonincr, $\forall n \in \mathbb{N}$, $x_{u_{n+1}}(u_{n+1} - u_n) \leq x_{u_n} + \dots + x_{(u_{n+1})-1} \leq x_{u_n}(u_{n+1} - u_n)$

  further, $x_{u_{n+1}} \frac{(u_{n+2} - u_{n+1})}{M} \leq x_{u_{n+1}}(u_{n+1} - u_n) \Rightarrow \frac{1}{M} \sum_{n=0}^{N} x_{u_{n+1}}(u_{n+2} - u_{n+1}) \leq \sum_{n=0}^{N} \sum_{k=u_n}^{u_{n+1}-1} x_k \leq \sum_{n=0}^{N} x_{u_{n+1}}(\dots)$

  then apply the comparison test — $\frac{1}{M} \sum_{n=1}^{N} x_{u_n}(u_{n+1} - u_n)$ — $\sum_{n=u_0}^{u_{N+1}-1}$

  ↳ Ex (Cauchy condensation test): $\sum_{n=1}^{\infty} x_n$ conv $\Leftrightarrow \sum_{n=0}^{\infty} 2^n x_{2^n}$ conv — $u_n = 2^n$ — $\frac{1}{2^{p-1}} < 1$ $\Updownarrow$

- Cor: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv $\Leftrightarrow p > 1$ ← if $p \leq 0$, trivially diverges. $p > 0$, $\sum_{n=0}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$ conv

- Cor: $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ conv $\Leftrightarrow p > 1$. if $p > 0$, $\sum_{n=1}^{\infty} \frac{2^n}{2^n n^p (\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p}$ conv $\Leftrightarrow p > 1$.

- [Algebra of series] Prop: if $\sum_{n=\ell}^{\infty} x_n$, $\sum_{n=\ell}^{\infty} y_n$ conv, $\forall \alpha, \beta \in \mathbb{R}$, $\sum_{n=\ell}^{\infty} (\alpha x_n + \beta y_n)$ conv. — AoL on $\sum^{N} x$, $\sum^{N} y$

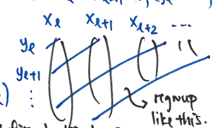
- Thm (series product): sps (1) $X = \sum_{n=\ell}^{\infty} x_n$ conv, (2) $\alpha = \sum_{n=\ell}^{\infty} |x_n|$ conv, (3) $Y = \sum_{n=\ell}^{\infty} y_n$ conv

  for $n \geq \ell$ set $z_n = \sum_{k=\ell}^{n} x_k y_{n+\ell-k} \in \mathbb{R}$, then $\sum_{n=\ell}^{\infty} z_n = X \cdot Y$ conv. — $X \cdot Y = (x_\ell + x_{\ell+1} + \dots)(y_\ell + \dots)$

  $X_n = \sum_{k=\ell}^{n} x_k \nearrow X$, $Y_n = \sum_{k=\ell}^{n} y_k \nearrow Y$, $Z_n = \sum_{k=\ell}^{n} z_k$, $V_n = Y_n - Y \nearrow 0$, regrouping,

  $Z_n = X_\ell Y_n + X_{\ell+1} Y_{n-1} + \dots + X_n Y_\ell = X_\ell(Y + V_n) + X_{\ell+1}(Y + V_{n-1}) + X_n(Y + V_\ell)$

  $= X_n Y + [X_\ell V_n + X_{\ell+1} V_{n-1} + \dots + X_n V_\ell] =: W_n \xrightarrow{\text{want}} 0$ — $X \cdot Y$



  let $\varepsilon > 0$, pick $N \geq \ell + 1$ s.t. $n \geq N \Rightarrow |V_n| < \frac{\varepsilon}{1+\alpha}$ — $\leq \alpha$ — N fixed, $|x_n| \to 0$ since $\sum |x_n|$ conv.

  $\Rightarrow |W_n| \leq \sum_{k=\ell}^{n} |X_{n-k+\ell}||V_k| = \sum_{k=\ell}^{N-1} |x_{n-k+\ell}||V_k| + \sum_{k=N}^{n} |x_{n-k+\ell}||V_k| < \sum_{k=\ell}^{N-1} |x_{n-k+\ell}||V_k| + \varepsilon$

  $\Rightarrow 0 \leq \liminf_{n\to\infty} |W_n| \leq \limsup_{n\to\infty} |W_n| \leq 0$ bc $|x_n|$ conv.

- Thm (root test): set $\alpha = \limsup_{n\to\infty} |x_n|^{1/n} \in [0,\infty]$ (assume $\ell \geq 1$) then TFH
  (1) if $\alpha < 1$, $\sum_{n=\ell}^{\infty}$ converges absolutely, (2) if $a > 1$, $\sum_{n=\ell}^{\infty} x_n$ diverges.

  sps $\alpha \in [0,1)$, pick $\alpha < \gamma < 1$, then $\exists N \geq 1$ s.t. $|x_n|^{1/n} < \gamma \Rightarrow |x_n| < \gamma^n \Rightarrow \left(\sum_{n=0}^{\infty} \gamma^n \text{ conv} \Rightarrow \overset{\text{conv}}{\sum_{n=\ell}^{\infty} |x_n|}\right)$

  sps $1 < \alpha$, pick subseq $\{x_{n_k}\}_{k=\ell}^{\infty}$ s.t. $|x_{n_k}|^{1/n_k} \to \alpha > 1$, then $\exists K$ s.t. $|x_{n_k}|^{1/n_k} \geq 1 \Rightarrow |x_{n_k}| \geq 1$

  AFSOC $\sum x_n$ conv, then $x_n \to 0 \Rightarrow x_{n_k} \to 0 \rightarrow\leftarrow$ req. $\alpha > 1$ so $\exists\, \varepsilon = \frac{\alpha - 1}{2} > 0$ or sth then Δ-ineq to get sth $\geq |\alpha| + |\ldots| \geq 1$ (o.w. it cld approach from below).

HW
- Prop: let $\ell \geq 1$, sps $\{a_n\}_{n=\ell}^{\infty} \subseteq [0,\infty)$, and $\frac{\alpha_{n+1}}{\alpha_n} \in [0,\infty]$ is well-defined, then $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} a_n^{1/n} \underset{(*)}{\leq} \overset{\text{“}\beta}{\limsup_{n\to\infty} \frac{a_{n+1}}{a_n}}$

  for any $\beta < \gamma$, $\exists N \geq \ell$ st. $n \geq N \Rightarrow \frac{a_{n+1}}{a_n} \leq \gamma \Rightarrow a_n > 0 \Rightarrow a_{N+n} \leq \gamma^n a_N \; \forall n \in \mathbb{N}$ pf

  $\Rightarrow a_{N+n}^{1/(N+n)} \leq \gamma^{n/(N+n)} a_N^{1/(N+n)} \; \forall n \Rightarrow \limsup_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} \gamma^{n/(N+n)} a_N^{1/(N+n)} = \gamma$ $\leq \beta$ since this holds $\forall \gamma > \beta$

- Thm (ratio test): let $\frac{|x_{n+1}|}{|x_n|} \in [0,\infty]$ be well-defined, then
  (1) if $\limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \overset{\text{“}\alpha}{\in} [0,1)$, then $\sum_{n=\ell}^{\infty} x_n$ conv. abs. (2) if $\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} > 1$, $\sum_{n=\ell}^{\infty} x_n$ diverges

  $\hookrightarrow$ prop $\Rightarrow \limsup_{n\to\infty} |x_n|^{1/n} \leq \alpha < 1$ by root test. prop $\Rightarrow 1 < \limsup_{n\to\infty} |x_n|^{1/n}$ diverges by root test.

- Lem (summation by parts): define $\{X_n\}_{n=\ell-1}^{\infty} \subseteq \mathbb{R}$ via $X_n = \begin{cases} \sum_{k=\ell}^{n} x_k, & n \geq \ell \\ 0, & n = \ell - 1, \end{cases}$ then for $\ell \leq j \leq k$,
  $\sum_{n=j}^{k} x_n y_n = \sum_{n=j}^{k} (X_n - X_{n-1}) y_n = \sum_{n=j}^{k} X_n y_n - \sum_{n=j-1}^{k-1} X_n y_{n+1} = \sum_{n=j}^{k-1} X_n (y_n - y_{n+1}) + X_k y_k - X_{j-1} y_j$

- Thm (Dirichlet test): if (1) $X_n = \sum_{k=\ell}^{n} x_k$, $\sup_{n \geq \ell} |X_n| = M < \infty$, (2) $0 \leq \alpha_n \; \forall n \geq \ell$, $\{\alpha_n\}_{n=\ell}^{\infty}$ is noninc and converges to 0, then $\sum_{n=\ell}^{\infty} \alpha_n x_n$ converges

  let $\varepsilon > 0$, $N \geq \ell + 1$ s.t. $n \geq N \Rightarrow 0 \leq \alpha_n < \frac{\varepsilon}{2(M+1)}$, then for $N \leq j \leq k$, partial sums Cauchy $\Rightarrow \sum \alpha_n x_n$ conv.

  $\left|\sum_{n=j}^{k} \alpha_n x_n\right| = \left|\sum_{n=j}^{k-1} (\alpha_n - \alpha_{n+1}) X_n + \alpha_k X_k - \alpha_j X_{j-1}\right| \leq M\left(\sum_{n=j}^{k-1} (\alpha_n - \alpha_{n+1}) + \alpha_k + \alpha_j\right) = 2\alpha_j M < \varepsilon$

- Cor (alternating series test): if $\{a_n\}_{n=\ell}^{\infty} \subseteq [0,\infty)$ is noninc & conv to 0, $\sum_{n=\ell}^{\infty} (-1)^n a_n$ conv.

- Cor (Abel's test): sps $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$ s.t. $\sum_{n=\ell}^{\infty} x_n$ conv, $\{a_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$ noninc and $a_n \to a$, then $\sum_{n=\ell}^{\infty} a_n x_n$ conv.

  $\beta_n = a_n - a$, then $\{\beta_n\}_{n=\ell}^{\infty} \subseteq [0,\infty)$ is nondecr, $\beta_n \to 0$ partial sums bounded

  $\Rightarrow \sum_{n=\ell}^{N} a_n x_n = \sum_{n=\ell}^{N} \beta_n x_n^{*} + \beta \sum_{n=\ell}^{N} x_n$. * conv. by Dirichlet.

- Thm (Riemann's rearrangement): let $\{x_n\}_{n=\ell}^{\infty} \subseteq \mathbb{R}$ s.t. $\sum_{n=\ell}^{\infty} x_n$ conv, then
  1. sps series conv. abs. then $\forall$ bijection $\varphi: \{n \geq \ell\} \to \{n \geq \ell\}$, $\sum_{n=\ell}^{\infty} x_{\varphi(n)} = \sum_{n=\ell}^{\infty} x_n$ abs. conv. under permutation.
  2. sps series does not conv. abs, then $\forall L \in \overline{\mathbb{R}}$, $\exists$ bij. φ s.t. $\sum_{n=\ell}^{\infty} x_{\varphi(n)} = L$.

— 4.0 & 4.1 —

§ Topology of $\mathbb{R}$ (Chpt 5)

- Def: $E \subseteq \mathbb{R}$ is open if $\forall x \in E$, $\exists \varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq E$, E closed $\Leftrightarrow E^c = \mathbb{R} \setminus E$ open

- Prop: (1) if $U_\alpha \subseteq \mathbb{R}$ is open $\forall \alpha \in A$, $\bigcup_{\alpha \in A} U_\alpha$ is open $\overset{*}{\leftrightarrow}$ (3) C_α closed ... $\bigcap_{\alpha \in A} C_\alpha$ closed
  (2) if $U_i \subseteq \mathbb{R}$ open for $i = 1, \ldots, k$, $\bigcap_{i=1}^{k} U_i$ is open $\overset{*}{\leftrightarrow}$ (4) C_i closed ... $\bigcup_{i=1}^{k} C_i$ closed

  1. if $x \in \bigcup_{\alpha \in A} U_\alpha$, $x \in U_\beta$ for some $\beta \in A \Rightarrow B(x,\varepsilon) \subseteq U_\beta \subseteq \bigcup_{\alpha \in A} U_\alpha$
  3. sps $x \in \bigcap_{i=1}^{k} U_i$, U_i open $\Rightarrow \exists \varepsilon_i > 0$ s.t. $B(x,\varepsilon_i) \subseteq U_i \Rightarrow$ set $\varepsilon = \min \varepsilon_i$, $B(x,\varepsilon) \subseteq B(x,\varepsilon_i) \; \forall i$

  * $k < \infty$ req is strict: otherwise $U_n = (-\frac{1}{n}, \frac{1}{n}) \rightsquigarrow \bigcap_{i=1}^{\infty} U_n = \{0\}$ (not open)

- Thm: let $C \subseteq \mathbb{R}$, C closed $\Leftrightarrow C$ seq closed (if $\{x_n\}_{n=\ell}^{\infty} \subseteq C$ is conv, $\lim_{n\to\infty} x_n \in C$)
  ($\Rightarrow$) sps $x = \lim_{n\to\infty} x_n \in C^c$, then C^c open $\Rightarrow \exists N \geq \ell$ s.t. $n \geq N \Rightarrow x_n \in C^c$, but $x_n \in C$, $C \cap C^c = \emptyset$ $\rightarrow\leftarrow$
  ($\Leftarrow$) sps C^c not open $\Rightarrow \exists x \in C^c$ s.t. $B(x,\varepsilon) \not\subseteq C^c \; \forall \varepsilon \Rightarrow x_n \in B(x,\frac{1}{n}) \cap C \neq \emptyset \Rightarrow x_n \to x \notin C \rightarrow\leftarrow$

- Thm: $x_n \to x$ as $n \to \infty \Leftrightarrow \forall$ open set $U \subseteq \mathbb{R}$, s.t. $x \in U$, $\exists N \geq \ell$ s.t. $n \geq N \Rightarrow x_n \in U$. if $x_n \to x$
  ($\Leftarrow$) $B(x,\varepsilon)$ open $\Rightarrow x_n \in B(x,\varepsilon) \Rightarrow |x - x_n| < \varepsilon$, ($\Rightarrow$) $x \in U$ open $\Rightarrow B(x,\varepsilon) \subseteq U$, $x_n \in B(x,\varepsilon)$

- Def: E° = set of interior pts $x \in E$ st. $B(x,\varepsilon) \subseteq E$ for some $\varepsilon > 0$ ($U = U^\circ$ if U open)
  ∂E = boundary pts $x \in \mathbb{R}$ st $\forall \varepsilon > 0$, $B(x,\varepsilon) \cap E \neq \emptyset$ and $B(x,\varepsilon) \cap E^c \neq \emptyset$
  closure $\overline{E} = E^\circ \cup \partial E$, limit pt E' if $(B(x,\varepsilon) \cap E) \setminus \{x\} \neq \emptyset \; \forall \varepsilon$, isolated pt $x \in E \setminus E'$

  - for $E = \emptyset, \mathbb{R}$, $E^\circ = \overline{E} = E' = E$ & $\partial E = \emptyset$
    for $E = \mathbb{Q}$, $E^\circ = \mathbb{Q}$, $\overline{E} = E' = \mathbb{R}$, $\partial E = \mathbb{R}$

  also $\partial E \cap E^\circ = \emptyset$, $E^\circ \subseteq E'$, $A \subseteq B \Rightarrow A' \subseteq B'$
  * set closed if $E = \overline{E}$ (contains limit pts)

- Prop: $\partial E = \partial(E^c)$, $E^\circ \subseteq E \cap E'$, $E \setminus E^\circ \subseteq \partial E$ and $E \setminus \partial E \subseteq E^\circ$, $E' \setminus E^\circ \subseteq \partial E$, $\partial E \setminus E'$ = isolated
  $\rightarrow$ sps $x \in \partial E \setminus E'$, $x \notin E' \Rightarrow B(x,\varepsilon) \cap E \subseteq \{x\}$, since $x \in \partial E$, $B(x,\varepsilon) \cap E = \{x\} \Rightarrow x \in E \setminus E'$
  $\leftarrow$ sps $x \in E \setminus E'$, then $\exists \varepsilon > 0$ st. $B(x,\varepsilon) \cap E \subseteq \{x\}$, but $x \in E \Rightarrow B(x,\varepsilon) \cap E = \{x\}$.
  let $r > 0$, $B(x,r) \cap E^c \supseteq B(x,\min(r,\varepsilon)) \cap E^c = B(x,\min(r,\varepsilon)) \setminus \{x\} \neq \emptyset \Rightarrow x \in \partial E \setminus E'$

- Prop: $E^\circ \cup \partial E \cup (E^c)^\circ = \mathbb{R}$ is a disjoint union ($\subseteq$ trivial) (sim for $x \in E^c$)
  let $x \in \mathbb{R} = E \cup E^c$ (disjoint), if $x \in E$, then $x \in E^\circ \cup (E \setminus E^\circ) \subseteq E^\circ \cup \partial E$ & $\partial E = \partial(E^c)$

- Thm: (1) E° is open, $x \in E^\circ$ ($B(x,\varepsilon) \subseteq E$ for $\varepsilon > 0$), if $y \in B(x,\varepsilon)$, $B(y, \varepsilon - |x-y|) \subseteq B(x,\varepsilon) \subseteq E$ $\hookrightarrow B(x,\varepsilon) \subseteq E^\circ$
  (2) $\overline{E}$ closed: $\overline{E} = E^\circ \cup \partial E \Rightarrow (\overline{E})^c = (E^c)^\circ$ is open
  (3) ∂E closed: $(\partial E)^c = E^\circ \cup (E^c)^\circ$ is union of 2 open sets. $r = \min\{\varepsilon - |x-y|, |x-y|\}$
  (4) E' closed: $x \in (E')^c$, then $\exists \varepsilon$ st. $B(x,\varepsilon) \cap E \subseteq \{x\}$, let $y \in B(x,\varepsilon) \setminus \{x\} \Rightarrow B(y,r) \subseteq B(x,\varepsilon) \setminus \{x\}$
  (5) $E \subseteq \overline{E}$ and $E' \subseteq \overline{E}$: $E = E^\circ \cup (E \setminus E^\circ) \subseteq E^\circ \cup \partial E = \overline{E}, \ldots$ $\Rightarrow B(y,r) \cap E = \emptyset \Rightarrow B(x,\varepsilon) \subseteq (E')^c$
  (6) $\overline{E} = E \cup E'$: (5) + $\overline{E} = E^\circ \cup \partial E \subseteq E \cup \partial E \subseteq E \cup (\text{isolated pts} \cup E') \subseteq E \cup E'$

- Thm: (1) E closed $\Leftrightarrow$ (2) $E = \overline{E} \Leftrightarrow$ (3) $\partial E \subseteq E \Leftrightarrow$ (4) $E' \subseteq E$
  (1) $\Leftrightarrow$ (2): E closed $\Leftrightarrow E^c$ open $\Leftrightarrow \mathbb{R} = E^\circ \cup \partial E \cup \underline{E^c}^{=(E^c)^\circ} \Leftrightarrow E = (E^c)^c = E^\circ \cup \partial E = \overline{E}$
  (2) $\Leftrightarrow$ (3) follows from $\overline{E} = E^\circ \cup \partial E$ and $E \subseteq \overline{E}$, (2) $\Leftrightarrow$ (4) follows from $\overline{E} = E \cup E'$

5.1 Limits & Continuity

- Def: let $E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}$, $z \in E'$. for $y \in \mathbb{R}$, say $\lim_{x \to z} f(x) = y$ if $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $x \in E$ and $0 < |x - z| < \delta \Rightarrow |f(x) - y| < \varepsilon$.
- Lem: $E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}$, $z \in E'$. sps $f(x) \to y$ and $f(x) \to w$ as $x \to z$, then $y = w$
  sps $y \neq w$ and $\varepsilon = \frac{|y-w|}{2}$, then $0 < |x - z| < \min(\delta_1, \delta_2) \Rightarrow |y - w| < 2\varepsilon = |y - w| \rightarrow\leftarrow$
- Prop: let $1 < b \in \mathbb{R}$. for $z \in (0, \infty)$, $\lim_{x \to z} \log_b x = \log_b z$.
  let $\varepsilon > 0$, pick $\delta = \frac{z}{2} \min\{1, b^{\varepsilon} - 1\}$. assume $x \in (0, \infty)$, $0 < |x - z| < \delta$.
  - if $0 < z < x$, $|\log_b x - \log_b z| < \varepsilon \Leftrightarrow \log_b \frac{x}{z} < \varepsilon \Leftrightarrow x < z b^{\varepsilon} \Leftrightarrow |x - z| < z(b^{\varepsilon} - 1)$ ✓
  - if $0 < x < z$, $\log_b \frac{z}{x} < \varepsilon \Leftrightarrow |z - x| < (b^{\varepsilon} - 1)x$ ✓ bc $|z - x| < \frac{z}{2} \Leftrightarrow \frac{z}{2} < x < \frac{3z}{2}$
- Thm: (1) $\lim_{x \to z} f(x) = y \Leftrightarrow$ (2) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(E \cap B(z, \delta) \setminus \{z\}) \subseteq B(y, \varepsilon)$ (1) $\Leftrightarrow$ (2) trivial.
  $\Leftrightarrow$ (3) if $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{z\}$ is s.t. $x_n \to z$, then $f(x_n) \to y$ as $n \to \infty$. (char of limits of fn)
  (1) $\Rightarrow$ (3): let $\varepsilon > 0$, pick $\delta > 0$ s.t. $x \in E$ and $0 < |x - z| < \delta \Rightarrow |f(x) - y| < \varepsilon$.
  $\neg$(1) $\Rightarrow \neg$(3): then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0, \exists x \in E$ s.t. $0 < |x - z| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. pick $\delta = 2^{-n}$ for $n \geq \ell$.
  then we get $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{z\}$ s.t. $|x_n - z| < 2^{-n}$ but $|f(x_n) - y| \geq \varepsilon \Rightarrow x_n \to z$ and $f(x_n) \not\to y$.
- Thm (algebra of limits): sps $f(x) \to y$ and $g(x) \to w$, then $f(x) + g(x) \to y + w$, similar for $\cdot$,
  and if $0 \notin f(E)$ and $y \neq 0$, $1/f(x) \to 1/y$ (from corr. result in alg. of seqs) $\rightarrow$ bc. $f(x) \to y \Leftrightarrow f(x_n) \to y$.
- Cor: if $p: E \to \mathbb{R}$ is a polynomial / $p(x) = z^x$ ($z \in \mathbb{R}^+$) / $p = \sin/\cos$, then $\lim_{x \to z} f(x) = f(z)$
- Def: f is <u>continuous</u> at $z \in E$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x \in E$ and $|x - z| < \delta \Rightarrow |f(x) - f(z)| < \varepsilon$.
  ↳ f is vacuously continuous at isolated pts (s.t. $B(z, \delta) \cap E \neq \emptyset$) $\Rightarrow$ only need to verify $E \cap E'$.
- Thm (characterisation of continuity): TFAE: (1) f is cont at z } (1) $\Leftrightarrow$ (2) trivial
  (2) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(E \cap B(z, \delta)) \subseteq B(f(z), \varepsilon)$.
  (3) if $z \in E'$, then $f(x) \to f(z)$ as $x \to z$ (3) $\Leftrightarrow$ (4) from char of limits of fn.
  (4) if $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{x\}$ is s.t. $x_n \to z$ as $n \to \infty$, $f(x_n) \to f(z)$ as $n \to \infty$ $\rightarrow$ (5): ... $\subseteq E$ ...
  (3) $\Rightarrow$ (1): sps $z \in E \cap E'$, then $f(x) \to f(z)$. let $\varepsilon > 0$, then $\exists \delta > 0$ s.t. $x \in E$ and $0 < |x - z| < \delta$ $\Rightarrow |f(x) - f(z)| < \varepsilon$
  (1) $\Rightarrow$ (4): let $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{x\}$ be s.t. $x_n \to z$, then $n \geq N \Rightarrow |x_n - z| < \delta \Rightarrow \ldots$
- Prop: let $A \subseteq E \subseteq \mathbb{R}$, $z \in A$, if f cont at z, restriction of f to A is also cont at z.
- Thm: let $A, B \subseteq \mathbb{R}$, if $f(A) \subseteq B$, f cont at $z \in A$, g cont at $f(z) \in B$, $g \circ f$ is cont at $z \in A$.
- Thm (alg of cont): $f + g$, fg, $1/f$ (if $0 \notin f(E)$) are cont at z
- Def: f is <u>uniformly</u> cont if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x, y \in E$ and $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
  f is Lipschitz cont if $\exists C \geq 0$ s.t. $|f(x) - f(y)| \leq C|x - y|$ $\forall x, y \in E$
- Prop: Lip $\Rightarrow$ uniform cont $\Rightarrow$ cont.
- Prop: if f, g both uniform/Lip cont, $g \circ f$ uniform/Lip cont. (& $\alpha f + \beta g$)

- cont: $\partial(\bar{E}) \subseteq \partial E$ and $\partial(E^\circ) \subseteq \partial E \to E \subseteq \bar{E} \Rightarrow (\bar{E})^c \subseteq E^c$, so $B(x, \varepsilon) \cap (\bar{E})^c \neq \emptyset \Rightarrow B \cap E^c \neq \emptyset$
  also, $B \cap \bar{E} \neq \emptyset \Rightarrow B \cap \partial E \overset{(1)}{\neq} \emptyset$ or $B \cap E^\circ \neq \emptyset \Rightarrow B \cap E \neq \emptyset$ in both cases. (for 1, $B(x, \varepsilon)$ open, so $y \in B(x, \varepsilon) \cap \partial E$ is contained in a ball $B(y, \delta) \subseteq B(x, \varepsilon)$ that must intersect E)
- Prop: $x \in E' \Leftrightarrow \exists$ seq $\{x_n\}_{n=\ell}^{\infty} \subseteq E \setminus \{x\}$ s.t. $\lim_{n \to \infty} x_n = x$.
  ($\Rightarrow$) $\forall \varepsilon > 0$, $(B(x, \varepsilon) \cap E) \setminus \{x\} \neq \emptyset \Rightarrow$ let $x_n \in (B(x, 2^{-n}) \cap E) \setminus \{x\}$ & apply SL.
  ($\Leftarrow$) let $\varepsilon > 0$, $\exists N \geq \ell$. s.t. $n \geq N \Rightarrow (|x - x_n| < \varepsilon \Rightarrow x_n \in (B(x, \varepsilon) \cap E) \setminus \{x\}) \Rightarrow x$ is limit pt.
- Cor: let $x \in E'$ ($E \subseteq \mathbb{R}$), then $x \in (E \cap (x, \infty))'$ or $x \in (E \cap (-\infty, x))'$
  $I_+ = \{n \geq \ell \mid x_n \in (x, \infty)\}$, $I_- = \{n \geq \ell \mid x_n \in (-\infty, x)\}$, trichotomy $\Rightarrow \mathbb{Z} \cap [\ell, \infty) = I_+ \cup I_-$
  either I_+, I_- is infinite $\Rightarrow$ produce subseq $\{x_m\}_{m=\ell}^{\infty}$ with limit x in restricted domain.
- Thm: let $E \subseteq \mathbb{R}$, $\Theta(E) = \{V \subseteq E \mid V \text{ is open}\}$, $\mathscr{C}(E) = \{C \subseteq \mathbb{R} \mid C \text{ closed } \& E \subseteq C\}$
  then (1) $E^\circ = \bigcup_{V \in \Theta(E)} V$ is largest open set in E, (2) $\bar{E} = \bigcap_{C \in \mathscr{C}(E)} C$ is smallest closed set enclosing E.
  1. $E^\circ \subseteq E \Rightarrow E^\circ \in \Theta(E)$, if $V \in \Theta(E)$ then V open $\Rightarrow V = V^\circ \Rightarrow V \subseteq E^\circ$ ($A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$)
  2. $(E^c)^\circ \overset{(1)}{=} \bigcup_{V \in \Theta(E^c)} \Rightarrow \bar{E} = E^\circ \cup \partial E = ((E^c)^\circ)^c = \left(\bigcup_{V \in \Theta(E^c)} V\right)^c = \bigcap_{V \in \Theta(E^c)} V^c = \bigcap_{C \in \mathscr{C}(E)} C$ ← $V \in \Theta(E^c) \Leftrightarrow V^c \in \mathscr{C}(E)$
- Def: let $A, B \subseteq \mathbb{R}$, say A is dense in B if $A \subseteq B$, and $\forall x \in B, \exists \{x_n\}_{n=\ell}^{\infty} \subseteq A$ s.t. $\lim_{n \to \infty} x_n = x$
  ↳ Ex: let $x \in \mathbb{R}$, $\forall n \in \mathbb{N}$, by Arch, $\exists q_n \in \mathbb{Q}$ s.t. $x < q_n < x + 2^{-n}$ & by SL, $\mathbb{Q}$ dense in $\mathbb{R}$.
- Prop: let $A \subseteq B \subseteq \mathbb{R}$. TFAE: (1) A is dense in B, (2) $B \subseteq \bar{A}$, $\rightarrow$ A dense in $\mathbb{R} \Leftrightarrow \bar{A} = \mathbb{R}$.
  (3) $\forall x \in B, \varepsilon > 0, \exists a \in A$ s.t. $|x - a| < \varepsilon$, (4) $\forall x \in B, \varepsilon > 0$, $B(x, \varepsilon) \cap A = \emptyset$.
  recall $\bar{A} = A \cup A'$, then (2) $\Leftrightarrow$ (3) trivial. then for (1) $\Rightarrow$ (2), sps A dense in B, $x \in B$. then either $x \in A \subseteq \bar{A}$, or $\exists \{x_n\}_{n=\ell}^{\infty} \subseteq A \subseteq B \setminus \{x\}$ s.t. $x_n \to x$ and $x \in A' \subseteq \bar{A}$. for (2) $\Rightarrow$ (1), sps $B \subseteq A \cup A'$, for $x \in B$, either $x \in A$ (& $\{x\}_{n=0}^{\infty} \subseteq A$), or if $x \in A' \setminus A$, choose $\{x_n\}_{n=0}^{\infty} \subseteq A \setminus \{x\}$ s.t. $x_n \to x$.
- Cor: sps $A \subseteq B \subseteq \mathbb{R}$, then if A is dense in B, A is dense in $\bar{B}$. $\bar{B} \subseteq \bar{B} \subseteq \bar{A}$.

HW
- Prop: if $Y \subseteq \mathbb{R}$, then Y has a countable dense subset.
  let $f: \mathbb{N} \to \mathbb{Q}$ be bij, $d_n = \inf\{|f(n) - y| \mid y \in Y\}$, then construct $a_n \in Y$ s.t. $|f(n) - a_n| < d_n + 2^{-n}$
  let $y \in Y$. since $\mathbb{Q}$ dense in $\mathbb{R}$, $\exists \{q_n\}_{n=\ell}^{\infty} \subseteq \mathbb{Q}$ s.t. $q_n \to y \Rightarrow (n \geq N \Rightarrow |q_n - y| < \varepsilon)$.
  for each q_n, let n be s.t. $f(n) = q_n$, then $|f(n) - a_n| < \underset{\to \varepsilon}{d_n} + 2^{-n}$, then $|a_n - y| < \varepsilon$ also.

- Prop: let $E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}$ uni cont, if $\{x_n\}_{n=\ell}^{\infty} \subseteq X$ is Cauchy, $\{f(x_n)\}_{n=\ell}^{\infty} \subseteq Y$ is Cauchy.
  pick $N \geq \ell$ s.t. $n, m \geq N \Rightarrow |x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$

- Thm (Intermediate Value Thm): let $a, b \in \mathbb{R}$, $a < b$, $f: [a,b] \to \mathbb{R}$ cont,
  let $y \in (\min(f(a), f(b)), \max(f(a), f(b)))$, then $\exists c \in (a,b)$ s.t. $f(c) = y$.
  sps $f(a) < f(b)$. define $E = \{x \in [a,b] \mid f(x) < y\}$. $f(a) < y \Rightarrow a \in E$, $E \subseteq [a,b]$ bounded
  $\Rightarrow$ set $c = \sup E \in \mathbb{R}$. f cont at a, so set $\varepsilon = y - f(a)$ to get $\delta_1 < b - a$ s.t. $f(x) < y$
  (for $a \leq x \leq a + \delta_1$); similarly, $\exists\, 0 < \delta_2 < b - a$ s.t. $y < f(x)$ for $b - \delta_2 \leq x \leq b$.
  $\rightarrow [a, a+\delta_1] \subseteq E \subseteq [a, b - \delta_2) \Rightarrow a + \delta_1 \leq c \leq b - \delta_2 \Rightarrow c \in (a,b)$, claim $f(c) = y$.
  - BWOC sps $f(c) < y$, cont $\Rightarrow \exists \delta > 0$ s.t. $f(x) < y$ for $c \leq x \leq c + \delta \Rightarrow [c, c+\delta] \subseteq E \rightarrow\leftarrow$ c is UB.
  - AFSOC $y < f(c)$, cont $\Rightarrow \exists \delta > 0$ s.t. $y < f(x)$ for $c - \delta \leq x \leq c \rightarrow\leftarrow$ c is sup $\Rightarrow \exists x \in E \cap [c - \delta, c]$

- Thm: let $p: \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree, i.e. $p(x) = \sum_{n=0}^{2N+1} a_n x^n$. then p has a root in $\mathbb{R}$.

- Thm: sps $f, g: [a,b] \to \mathbb{R}$ continuous, $f([a,b]) \subseteq [c,d]$, $g(\{a,b\}) = \{c,d\}$, then $\exists x$, $f(x) = g(x)$
  sps $c < f(a)$, $f(b) < d$, $h = f - g$, then $h(a) > 0$, $h(b) < 0 \Rightarrow \exists x \in (a,b)$ s.t. $h(x) = 0$.

- Def: $E \subseteq \mathbb{R}$ is compact if every sequence $\{x_n\}_{n=\ell}^{\infty} \subseteq E$ has a subseq converging to $x \in E$.

- Thm (Heine-Borel): E is compact $\Leftrightarrow$ it is closed & bounded.
  $(\Rightarrow)$ AFSOC E not bounded, then $\forall n \in \mathbb{N}$, $B(0, 2^n)^c \cap E \neq \emptyset$, then $\{x_n\}_{n=0}^{\infty} \subseteq E$ s.t. $|x_n| \geq 2^n$ unbounded
  let $\{x_n\}_{n=\ell}^{\infty} \subseteq E$, sps $x_n \to x$, then all subseq $\to x$ (seq. limits agree) $\Rightarrow E$ seq closed $\Rightarrow$ closed.
  $(\Leftarrow)$ $\{x_n\}_{n=\ell}^{\infty} \subseteq E$ bounded, BW $\Rightarrow \exists$ subseq s.t. $x_{n_m} \to x$, then seq closed $\Rightarrow x \in E$.

- Lemma: let $\emptyset \neq K \subseteq \mathbb{R}$ be cpt, then $\exists m, M \in K$ s.t. $m = \min K$, $M = \max K$. $\nearrow m$ (in K) (so min & max.) $\nearrow M$
  K bounded $\Rightarrow$ let $m = \inf K$, $M = \sup K$, pick seqs $m_n, M_n \subseteq K$ s.t. $m \leq m_n < m + 2^{-n}$, $M - 2^{-n} < M_n \leq M$

- Thm: sps $K \subseteq \mathbb{R}$ is cpt, $f: K \to \mathbb{R}$ cont, then $f(K) \subseteq \mathbb{R}$ also cpt. ($K = \emptyset$ trivial)
  $\{y_n\}_{n=\ell}^{\infty} \subseteq f(K)$, select $\{x_n\}_{n=\ell}^{\infty} \subseteq K$ s.t. $y_n = f(x_n)$. cpt so $\exists x_{n_m} \to x$, cont so $y_{n_m} \to f(x)$

- Thm (Extreme Value Thm): let $\emptyset \neq K \subseteq \mathbb{R}$ be cpt, $f: K \to \mathbb{R}$ cont, $\exists x_0, x_1 \in K$ s.t. $f(x_0) \leq f(x) \leq f(x_1)$
  $f(K)$ cpt, then $\exists m, M$ s.t. $m = \min f(K)$, $M = \max f(K)$, pick x_0, x_1 s.t. $m = f(x_0)$ etc.

- Cor: let $f: [a,b] \to \mathbb{R}$ be cont, then $f([a,b]) = [c,d]$ for $c, d \in \mathbb{R}$, $c < d$. $\nearrow c = d \Rightarrow f$ const.
  EVT $\Rightarrow c = \min f([a,b])$, $d = \max f([a,b])$ are well-defined. $c < d \Rightarrow (\text{IVT} \Rightarrow (c,d) \subseteq f([a,b]))$

- Thm: let $\emptyset \neq K \subseteq \mathbb{R}$ be cpt, $f: K \to \mathbb{R}$ cont, then f is unif. cont on K.
  sps not, then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, y \in K$ s.t. $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.
  in part, take $\delta = 2^{-n}$ to generate $x_n, y_n \in K$, by cptness, $\exists$ subseq $\{x_{\varphi(n)}\}_{n=0}^{\infty}$ s.t. $x_{\varphi(n)} \to x$.
  similarly, $\{y_n\}_{n=\ell}^{\infty} \subseteq K$ also has a conv. subseq, $y_{\psi \circ \varphi(n)} \to y$ & $x_{\psi \circ \varphi(n)} \to x$.
  let $n_m = \psi \circ \varphi(m)$, $|x_n - y_n| < 2^{-n} \overset{SL}{\Longrightarrow} x_n - y_n \to 0 \rightarrow x = y$
  then cont of $f \Rightarrow 0 = |f(x) - f(y)| = \lim_{m \to \infty} |f(x_{n_m}) - f(y_{n_m})| \geq \varepsilon$ $\forall \varepsilon \rightarrow\leftarrow$

§6 - Differentiation

- Def: let $E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}$, $x \in E \cap E'$ $\quad \varphi: E \setminus \{x\} \to \mathbb{R}$ $\quad \nearrow$ then this is $f'(x)$
  1. f is differentiable at x if, for $\varphi(t) = \frac{f(t) - f(x)}{t - x}$, $\lim_{t \to x} \varphi(t) \in \mathbb{R}$ exists.
  2. If f is diff at each $x \in S \subseteq E \cap E'$, say f is diff on S.
  3. if $E \subseteq E'$ and f is diff at each $x \in E$, say f is diff.

- Thm: sps $\emptyset \neq E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}$, $x \in E \cap E'$, TFAE.
  (1) f is diff at x and $f'(x) = L$, (2) $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $x, y \in E$ and $|x - y| < \delta \Rightarrow |f(x) - f(y) - L(x-y)| < \varepsilon$
  idea: continuity says $f(x) \approx f(y)$, diff says $f(x) \approx f(y) + L(x - y)$ / affine approx.

- Prop: $f: E \to \mathbb{R}$, $g: F \to \mathbb{R}$, $x \in [E \cap F] \cap [E \cap F]'$, f, g diff at x, then $f = g$ (on $E \cap F$) $\Rightarrow f'(x) = g'(x)$

- Thm: $E \subseteq \mathbb{R}$, $x \in E \cap E'$, f diff at x, then f is cont at x. $\lim_{t \to x} f(t) = f(x) + f'(x) \cdot 0 = f(x)$

- Thm (Algebra of derivatives): sps f, g are diff at x, then
  1. $(\alpha f + \beta g)'(x) = \alpha f'(x) + \beta g'(x) \leftarrow \alpha \frac{f(t) - f(x)}{t - x} + \beta \frac{f(t) - f(x)}{t - x}$ $\quad \nearrow$ & $f(t) \to f(x)$ by continuity.
  2. $(fg)'(x) = f(x) g'(x) + f'(x) g(x) \leftarrow f(t)g(t) - f(x)g(x) = f(t)(g(t) - g(x)) + g(x)(f(t) - f(x))$
  3. sps $0 \notin g(E)$, then $\left(\frac{f}{g}\right)'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{(g(x))^2}$ $\quad \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)} = \frac{g(x)(f(t) - f(x)) - f(x)(g(t) - \ldots)}{g(t) g(x)}$

- Thm (Chain Rule): $f: E \to \mathbb{R}$, $g: F \to \mathbb{R}$, s.t. $f(E) \subseteq F$, sps. $x \in E \cap E'$, $f(x) \in F \cap F'$,
  if f is diff at x and g is diff at $f(x)$, then $g \circ f$ is diff at x, $(g \circ f)'(x) = g'(f(x)) f'(x)$
  for $t \in E \setminus \{x\}$, $M(t) := \frac{f(t) - f(x)}{t - x} - f'(x)$, then extend M to E via $M(t) = 0$ for $t = x$.
  then M is cont at x; similarly, define $N(s) = \frac{g(s) - g(f(x))}{s - f(x)} - g'(f(x))$ for $s \in F \setminus \{f(x)\}$, 0 else.
  now, $\frac{g(f(t)) - g(f(x))}{t - x} = \frac{f(t) - f(x)}{t - x} \cdot \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} = (f'(x) + M(t))(g'(f(x)) + N(f(t)))$ $\forall t \in E$.

- Def: $E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}$, $x \in E$. say f has a local min/max at x if $\exists \delta > 0$ s.t. $\forall y \in \ldots \setminus \{x\}$
  $f(x) \leq f(y)$ / $f(y) \leq f(x)$ $\forall y \in E \cap B(x, \delta)$. say local min/max is strict if $<$ / $>$.

- Thm: sps f is diff at $x \in E \cap E'$, then sps f has a local max at x. $\quad$ extrema = min or max
  - if $x \in [E \cap (x, \infty)]'$ then $f'(x) \leq 0$ $\quad \rightarrow$ for local min... swap $\leq \leftrightarrow \geq$
  - if $x \in [E \cap (-\infty, x)]'$ then $f'(x) \geq 0$ & if $x \in [\ ]' \cap [\ ]'$, $f'(x) = 0$

  $\exists \{t_n\}_{n=\ell}^{\infty} \subseteq E \cap (x, \infty)$ s.t. $t_n \to x$, local max $\Rightarrow \exists \delta > 0$ s.t. $f(x) \geq f(y)$ $\forall y \in E \cap B(x, \delta)$
  pick $N \geq \ell$ s.t. $n \geq N \Rightarrow t_n \in B(x, \delta) \cap E$, then $\frac{f(t_n) - x}{t_n - x} \leq 0 \Rightarrow \lim_{n \to \infty} \ldots \leq 0$ by seq. char of limits

- Cor: sps $f: [a,b] \to \mathbb{R}$ cont on $[a,b]$ and diff on (a,b). write $M = \max(f[a,b])$,
  $m = \min(f[a,b])$, $C = \{x \in (a,b) \mid f'(x) = 0\}$ for the set of critical pts.
  then $E := \{x \in [a,b] \mid f(x) \in \{m, M\}\} \subseteq \{a,b\} \cup C$
  $E \neq \emptyset$ by EVT, thm $\Rightarrow E \cap (a,b) \subseteq C \Rightarrow E = (E \cap (a,b)) \cup (E \cap \{a,b\}) \subseteq C \cup \{a,b\}$.

- Thm (Alg of higher derivs): let $f, g : E \to \mathbb{R}$ be n-times diff at $x \in E \cap E'$ for $n \geq 1$.
  1. $(f+g)^{n}(x) = f^{(n)}(x) + g^{n}(x)$ 2. $(fg)^{(n)}(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)}(x)\, g^{(k)}(x)$ ← by induction

- Thm (higher chain / Faà di Bruno): let $f : E \to \mathbb{R}$ be n-times diff at $x \in E \cap E'$, $f(E) \subseteq F$, $g : F \to \mathbb{R}$ n-times diff at $f(x) \in F \cap F'$, then $g \circ f$ is n-times diff at x.

- Cor (higher quotient): let $E \subseteq \mathbb{R}$, $f : E \to \mathbb{R} \setminus \{0\}$ be n-times diff at $x \in E \cap E'$, then $1/f$ is also n-times diff at x. $1/f = g \circ f$, $g : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ via $g(x) = \frac{1}{x}$ is smooth.

- Thm (Taylor's formula): sps $f : [a,b] \to \mathbb{R}$, let $k \in \mathbb{N}$, sps f k-times diff on $[a,b]$, with all derivs up to order k cont on $[a,b]$, further sps f is $(k+1)$-times diff in (a,b) then for $x, y \in [a,b]$, $x \neq y$, $\exists z \in (\min\{x,y\}, \max\{x,y\})$ s.t.
  $$f(y) = \underline{\sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(x)(y-x)^n} + f^{(k+1)}(z) \frac{(y-x)^{k+1}}{(k+1)!}$$
  $p(t)$ is the Taylor poly of degree k, associated w/ f, centered at t

  sps $x<y$, define $p(t) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(x)(t-x)^n$ and $M = \frac{f(y)-p(y)}{(y-x)^{k+1}}$, then the thm holds if $\exists z \in (x,y)$ s.t. $M = \frac{f^{(k+1)}(z)}{(k+1)!}$. let $g : [a,b] \to \mathbb{R}$ via $g(t) = f(x) - p(t) - M(t-x)^{k+1}$, note that g has same diff & cont properties as f, in part, $g^{(n)}(x) = 0$ $\forall 0 \leq n \leq k$ and $g^{(k+1)}(t) = f^{(k+1)}(t) - M(k+1)!$ for $x<t<y$. then apply MVT inductively.
  - $g(y) = 0$ and $g(x) = 0 \Rightarrow \exists y_1 \in (x,y)$ s.t. $g'(y_1) = 0$
  - then $g'(y_1) = 0 = g'(x) \xrightarrow{MVT} \exists y_2 \in (x,y)$ s.t. $g''(y_2) = 0$

  } $y_{k+1} \in (x,y)$ s.t. $g^{(k+1)}(y_{k+1}) = 0$ $\Rightarrow 0 = f^{(k+1)}(y_{k+1}) - M(k+1)!$

- Cor: sps $f : [a,b] \to \mathbb{R}$ is k-times diff on $[a,b]$ for $k \geq 1$, and $f^{(k)} : [a,b] \to \mathbb{R}$ cont. then for $x \in [a,b]$, $\lim_{y \to x} \frac{1}{|y-x|^k} \left| f(y) - \sum_{n=0}^{k} \frac{f^{(n)}(x)}{n!}(y-x)^n \right| = 0$

  $\forall y \in [a,b] \setminus \{x\}$, $\exists z \in (\min\{x,y\}, \max\{x,y\})$ s.t. $f(y) - \sum_{n=0}^{k} \frac{f^{(n)}(x)}{n!}(y-x)^n = \left(f^{(k)}(z) - f^{(k)}(x)\right)\frac{(y-x)^k}{k!}$ then the limit becomes $\frac{1}{k!}|f^{(k)}(z) - f^{(k)}(x)|$ & $f^{(k)}$ cont at x & $\min\{x,y\} < z < \max\{x,y\}$.

- Thm (Cauchy's MVT): sps $f, g : [a,b] \to \mathbb{R}$ are cont on $[a,b]$, diff on (a,b), then $\exists x \in (a,b)$ s.t. $g'(x)(f(b)-f(a)) = f'(x)(g(b)-g(a))$ (by EVT)
  let $h(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x)$, then $h(a) = h(b)$, then $\exists x \in (a,b)$ s.t. $h'(x) = 0$

- Cor: let $\emptyset \neq E \subseteq \mathbb{R}$ be convex, $f : E \to \mathbb{R}$ diff s.t. $|f'(x)| \leq M$, then $|f(x) - f(y)| \leq M|x-y|$ $\forall x, y \in E$
  sps $x<y$, then MVT $\Rightarrow \frac{f(y)-f(x)}{y-x} = f'(z)$ for some $z \in (x,y) \Rightarrow f$ Lipschitz

- Thm: sps $f : E \to \mathbb{R}$ diff at $x \in E \cap E'$, then f nondecr $\Rightarrow f'(x) \geq 0$ & ... nonincr ... ≤ 0.
  let $t \in E \setminus \{x\}$, if $x<t$, $f(x) \leq f(t)$, $0 \leq \frac{f(t)-f(x)}{t-x}$, sim for $t<x$, thus $0 \leq \lim_{t \to x} \cdots = f'(x)$

- Thm: let $E \subseteq \mathbb{R}$ be convex with $E^\circ \neq \emptyset$, sps f is cont with f diff on E° then
  $\underline{f'(x) > 0}$ $\forall x \in E^\circ \Rightarrow f$ inc, $f'(x) \geq 0$ ... f nondec, $f'(x) = 0$ ... f const, ..., nondec, dec.
  AFSOC f not inc $\Rightarrow \exists x, y \in E$ s.t. $x<y$ but $f(y) \leq f(x)$. since E convex, $[x,y] \subseteq E \Rightarrow (x,y) \subseteq E^\circ$, then MVT applied to $f : [x,y] \to \mathbb{R} \Rightarrow \exists z \in (x,y) \subseteq E^\circ$ s.t. $f'(z) = \frac{f(y)-f(x)}{y-x} \leq 0 \to\leftarrow f'(z) > 0$
  (2) $\exists x, y \in E$ with $x<y$, $f(y) < f(x)$, MVT $\Rightarrow z$ s.t. $0 \leq f'(z) \to\leftarrow$

  i.e. (z,b), $(a,z) \cup (z,b)$, (a,z)

- Thm (L'Hôpital): sps $E \subseteq \mathbb{R}$ is an open set s.t. $z \notin E$ but is a limit pt
  let $f, g : E \to \mathbb{R}$ be diff and sps $g'(x) \neq 0$ $\forall x \in E$, sps $\lim_{x \to z} f(x) = \lim_{x \to z} g(x) = 0$ and $\lim_{x \to z} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $g(x) \neq 0$ $\forall x \in E$ and $\lim_{x \to z} \frac{f(x)}{g(x)} = L$
  convex

  define $F, G : E \cup \{z\} \to \mathbb{R}$ via $F(x) = \begin{cases} f(x) & x \in E \\ 0 & x = z \end{cases}$, similar for G, then F, G cont in $E \cup \{z\}$ and F, G diff on E bc $F|_E = f$, $G|_E = g$. AFSOC $g(x) = 0$ for some $x \in E$, then $G(x) = 0$ then MVT $\Rightarrow \exists y \in E$ s.t. $0 = \frac{G(x)-G(z)}{x-z} = G'(y) = g'(y) \neq 0 \to\leftarrow$. $\therefore \frac{f}{g}$ well-defined.

  let $\{x_n\}_{n=\ell}^{\infty} \subseteq E$ s.t. $x_n \to z$, then Cauchy MVT $\Rightarrow \exists \{y_n\}_{n=\ell}^{\infty} \subseteq E$ s.t. $\min\{z, x_n\} < y_n < \max\{..\}$. then $|y_n - z| < |x_n - z|$ (so $y_n \to z$) and $f'(y_n)g(x_n) = F'(y_n)(G(x_n) - G(z))$ $\overset{CMVT}{=} G'(y_n)(F(x_n) - F(z)) = g'(y_n)f(x_n) \Rightarrow \frac{f(x_n)}{g(x_n)} = \frac{f'(y_n)}{g'(y_n)} \to L$ (then seq char of limits)

- Def: let $E \subseteq \mathbb{R}$, $f : E \to \mathbb{R}$, sps f is diff on E w/ $f' : E \to \mathbb{R}$ its derivative.
  1. if f' is diff at $x \in E \cap E'$, then f is twice diff at x, $f''(x) = \lim_{t \to x} \frac{f'(t) - f'(x)}{t-x}$
  2. sps f is k-times diff for $1 \leq k \in \mathbb{N}$, if $f^{(k)}$ denotes k^{th} derivative, and $f^{(k)}$ is diff at $x \in E \cap E'$, say f is $(k+1)$ diff, $f^{(k+1)}(x) = (f^{(k)})'(x)$ $f^{(0)} = f$.
  3. if f is k-times diff on E $\forall 1 \leq k \leq \mathbb{N}$, say f is smooth or C^∞.

  Ex: sps $g(x) = x|x|$, then $g(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases} \Rightarrow g'(x) = \{ \ldots = 2|x|$ $(g'(0) = 0)$ then g is diff and g' Lip, but g'' does not exist at 0 (1 if $x>0$, -1 if $x<0$)

$\oint$ 7 - Riemann Integration

- Def: let $a,b \in \mathbb{R}$, $a<b$, a partition of $[a,b]$ is a finite set $P = \{x_0, \dots, x_n\} \subseteq [a,b]$ ordered s.t. $a = x_0 < x_1 < \dots < x_n = b$.
  - write $\Pi[a,b] = \{P \mid P \text{ is a partition of } [a,b]\}$.
  - given $P, Q \in \Pi[a,b]$, say Q is a refinement of P if $P \subseteq Q$.
  - given $P = \{p_0, \dots, p_n\}$, $Q = \{q_0, \dots, q_m\} \in \Pi[a,b]$, $P \# Q = P \cup Q \in \Pi[a,b]$, ordered incr.

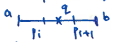
- Def: sps $f : [a,b] \to \mathbb{R}$ bounded.
  1. given $P = \{x_0, \dots, x_n\} \in \Pi[a,b]$, write $m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$, $1 \le i \le n$, $M_i = \sup\{\dots\}$, $\Delta_i = x_i - x_{i-1} > 0$, then $L(P,f) = \sum_{i=1}^n m_i \Delta_i$, $U(P,f) = \sum_{i=1}^n M_i \Delta_i$
     ↖ lower Riemann sum assoc. P, f.
  2. f bounded $\Rightarrow \exists m, M \in \mathbb{R}$ s.t. $m \le f(x) \le M$ $\forall x \in [a,b]$
     $\Rightarrow m \le m_i \le M_i \le M$ $\forall P \in \Pi[a,b]$, then $m(b-a) = \sum_{i=1}^n m\Delta_i \le \sum_{i=1}^n m_i \Delta_i \le \sum_{i=1}^n M_i \Delta_i \le M(b-a)$
     $\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$ $\forall P \in \Pi[a,b]$
     then the upper integral $\overline{\int_a^b} f = \inf\{U(P,f) \mid P \in \Pi[a,b]\} \in \mathbb{R}$, lower int. $\underline{\int_a^b} f = \sup\{L(P,f)\}$.
  3. Say f is Riemann integrable if $\underline{\int_a^b} f = \overline{\int_a^b} f =: \int_a^b f$.
  4. write $\mathcal{R}([a,b]) = \{g : [a,b] \to \mathbb{R} \mid g \text{ is bounded \& integrable}\}$.

regardless of partition, will contain $\mathbb{Q}$ & $\mathbb{R} \setminus \mathbb{Q}$.

Ex: $f : [a,b] \to \mathbb{R}$ via $f(x) = \begin{cases} 0 & x \in [a,b] \cap \mathbb{Q} \\ 1 & \text{else} \end{cases}$, let $P \in \Pi[a,b]$, $m_i = 0$, $M_i = 1$
then $U(P,f) = (b-a)$, $L(P,f) = 0 \Rightarrow \underline{\int_a^b} f = 0$, $\overline{\int_a^b} f = b-a > 0 \Rightarrow f$ is not int

- Thm: let $P, Q \in \Pi[a,b]$ and $P \subseteq Q$ (Q refines P), then for $f : [a,b] \to \mathbb{R}$ bounded,
  $L(P,f) \le L(Q,f) \le U(Q,f) \le U(P,f)$ (restrict this to case where card Q = card $P + 1$)

  $P = \{p_0, \dots, p_n\}$, $Q = \{p_0, \dots, p_i, q, p_{i+1}, \dots, p_n\}$, let $K_0 = \sup\{f(x) \mid p_i \le x \le q\}$,
  $K_1 = \sup\{f(x) \mid q \le x \le p_{i+1}\}$, then writing m_i, M_j for P-quants,



  $U(P,f) - U(Q,f) = M_{i+1}(p_{i+1} - p_i) - K_0(q - p_i) - K_1(p_{i+1} - q) = (M_{i+1} - K_1)(p_{i+1} - q) + (M_{i+1} - K_0)(q - p_i) \ge 0$
  bc. $M_{i+1} = \sup\{f(x) \mid p_i \le x \le p_{i+1}\} \ge K_{1,0} \Rightarrow U(Q,f) \le U(P,f)$.

- Cor: let $f : [a,b] \to \mathbb{R}$ be bounded, then if $P, Q \in \Pi[a,b]$,
  $L(P,f) \le L(P \# Q, f) \le U(P \# Q, f) \le U(Q,f)$. then $\underline{\int_a^b} f \le \overline{\int_a^b} f$

  $\underline{\int_a^b} f = \sup\{L(P,f) \mid P \in \Pi[a,b]\} \le U(Q,f)$.

- Lem: let $f : [a,b] \to \mathbb{R}$ be bounded and $P = \{x_0, \dots, x_n\} \in \Pi[a,b]$, TFH.
  1. if $s_i, t_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$, $\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta_i \le U(P,f) - L(P,f)$
  2. if f is int and $t_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$, $\left| \sum_{i=1}^n f(t_i)\Delta_i - \int_a^b f \right| \le U(P,f) - L(P,f)$.

  1. if $r_i \in [x_{i-1}, x_i]$, $m_i \le f(r_i) \le M_i$, then $m_i \le f(s_i) \le M_i \Rightarrow \sum_{i=1}^n |f(s_i) - f(t_i)| \Delta_i \le \sum_{i=1}^n (M_i - m_i)\Delta_i$
  2. $L(P,f) = \sum_{i=1}^n m_i \Delta_i \le \sum_{i=1}^n f(t_i)\Delta_i \le \sum_{i=1}^n M_i \Delta_i = U(P,f)$, $\int_a^b f$ similarly bounded.

- Thm: sps $f_1, f_2 \in \mathcal{R}([a,b])$, then $\int_a^b (\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \int_a^b f_1 + \alpha_2 \int_a^b f_2$
  let $\varepsilon > 0$, ICT $\Rightarrow \exists P_1, P_2 \in \Pi[a,b]$ s.t. if $P = P_1 \# P_2$, $\forall t_i \in [x_{i-1}, x_i]$, $\left| \sum_{i=1}^n f_1(t_i)\Delta_i - \int_a^b f_1 \right| < \frac{\varepsilon}{1 + |\alpha_1| + |\alpha_2|}$
  then $\left| \sum_{i=1}^n (\alpha_1 f_1(t_i) + \alpha_2 f_2(t_i)) \Delta_i - \alpha_1 \int_a^b f_1 - \alpha_2 \int_a^b f_2 \right| \overset{\Delta\text{-ineq}}{\le} \dots < \varepsilon \Rightarrow$ ICT #6 (& sim for f_2)

- Thm: let f be int, sps $E \subseteq \mathbb{R}$ w/ $f([a,b]) \subseteq E$, $g : E \to \mathbb{R}$ is unif cont & bounded, then $g \circ f$ int
  let $\varepsilon > 0$, pick $0 < \kappa < \frac{\varepsilon}{2(b-a)}$, since g bounded, we can select $K \in \mathbb{R}$ s.t. $|g(y)| \le K$ $\forall y \in E$.
  OTOH, g unif. cont $\Rightarrow \exists \delta > 0$ s.t. $y, z \in E$ and $|y - z| < \delta \Rightarrow |g(y) - g(z)| < \kappa$.
  sps $f \in \mathcal{R}([a,b])$ and $f([a,b]) \subseteq E$, then ICT $\Rightarrow \exists P = \{x_0, \dots, x_n\}$ s.t. $U(P,f) - L(P,f) < \frac{\delta\varepsilon}{4K+1}$
  define $A = \{1 \le i \le n \mid M_i - m_i < \delta\}$, $B = [n] \setminus A$, M_i, m_i the sup/inf. from $U(P,f)$, $L(P,f)$.
  - note $\sum_{i \in B} \Delta_i = \frac{1}{\delta} \sum_{i \in B} \delta \Delta_i \le \frac{1}{\delta} \sum_{i \in B} (M_i - m_i)\Delta_i \overset{*}{\le} \frac{1}{\delta}(U(P,f) - L(P,f)) < \frac{\varepsilon}{4K+1}$ * bc terms missing are in A (i.e. positive).

  now set $n_i = \inf\{g \circ f(x) \mid x \in [x_{i-1}, x_i]\}$, $N_i = \sup\{\dots\}$. if $i \in A$, $x, y \in [x_{i-1}, x_i]$, then
  $f(x) \le M_i < m_i + \delta \le f(y) + \delta$ and $f(y) \le M_i < m_i + \delta \le f(x) + \delta \Rightarrow |f(x) - f(y)| < \delta$.
  so $|g(f(x)) - g(f(y))| < \kappa \Rightarrow N_i - n_i \le \kappa$. OTOH, if $i \in B$, then $|g(y)| \le K \Rightarrow N_i - n_i \le 2K$.
  then $U(P, g \circ f) - L(P, g \circ f) = \sum_{i=1}^n (N_i - n_i)\Delta_i = \sum_{i \in A} (N_i - n_i)\Delta_i + \sum_{i \in B} (N_i - n_i)\Delta_i$
  $\le \sum_{i \in A} \kappa \Delta_i + \sum_{i \in B} 2K\Delta_i \le \kappa(b-a) + 2K \frac{\varepsilon}{4K+1} < \varepsilon$.

- Cor: sps f, g int, then fg int
  f, g bounded $\Rightarrow$ pick m, M s.t. $m \le f(x) + g(x) \le M$ $\forall x \in [a,b]$, let $h : [m,M] \to \mathbb{R}$ via $h(x) = \frac{x^2}{4}$,
  note h is uni cont & bounded, then $fg = \frac{(f+g)^2}{4} - \frac{(f+g)^2}{4} = h(f+g) - h(f-g) \Rightarrow$ int by thm.

- let $f_1, f_2 \in \mathcal{R}([a,b])$, sps $f_1 \le f_2$ on $[a,b] \setminus F$ for F finite, then $\int_a^b f_1 \le \int_a^b f_2$.
  let $P_0 \in \Pi[a,b]$ s.t. $F \subseteq P_0$, let $\varepsilon > 0$, then ICT $\Rightarrow P_1, P_2$ s.t. if $Q = \{x_0, \dots, x_n\}$ refines P_j and $t_i \in [x_{i-1}, x_i]$, $\left| \sum_{i=1}^n f_j(t_i) - \int_a^b f_j \right| < \frac{\varepsilon}{2}$ for $j = 1, 2$.
  let $P = P_0 \# P_1 \# P_2 = \{x_0, \dots, x_n\}$, pick $t_i \in (x_{i-1}, x_i) \rightarrow$ bad pts $F \subseteq P_0$, we never pick endpts/nvr bad.
  then $t_i \notin F \Rightarrow f_1(t_i) \le f_2(t_i)$, then $\int_a^b f_1 < \frac{\varepsilon}{2} + \sum_{i=1}^n f_1(t_i) \le \frac{\varepsilon}{2} + \sum_{i=1}^n f_2(t_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \int_a^b f_2$ $\forall \varepsilon$.

- Cor (Δ-ineq): sps f int, then $|f|$ int and $\left|\int_a^b f\right| \le \int_a^b |f|$.
  f bounded $\Rightarrow$ pick M s.t. $|f(x)| \le M$, then set $g : [-M, M] \to \mathbb{R}$ via $g(z) = z$, g Lip & bounded.
  then $|f| = g \circ f \in \mathcal{R}([a,b])$, OTOH, $-|f| \le f \le |f| \Rightarrow -\int_a^b |f| = \int_a^b -|f| \le \int_a^b f \le \int_a^b |f|$.

- Thm: sps $a < c < b$, then $f|_{[a,c]}$, $f|_{[c,b]}$ int and $\int_a^b f = \int_a^c f + \int_c^b f$
  let $P_c \in [a,b]$ be an incr enum. of $P \cup \{c\}$, in turn, $P_c^\ell \in \Pi[a,c]$ enum $P_c \cap [a,c]$, $P_c^r \in \Pi[c,b]$.
  then $U(P_c, f) = U^\ell(P_c^\ell, f) + U^r(P_c^r, f)$, $L(P_c, f) = L^\ell(P_c^\ell, f) + L^r(P_c^\ell, f)$. pick P s.t. $U(P,f) - L(P,f) < \varepsilon$,
  then P_c refines P, so $(U^\ell(P_c^\ell, f) - L^\ell(P_c^\ell, f)) + (U^r(P_c^r, f) - L^r(P_c^r, f)) = U(P_c, f) - L(P_c, f) \le \dots < \varepsilon$
  $\therefore \begin{cases} U^\ell(P_c^\ell, f) - L^\ell(P_c^\ell, f) < \varepsilon \\ U^r(P_c^r, f) - L^r(P_c^r, f) < \varepsilon \end{cases}$ so $f|_{[a,c]} \in \mathcal{R}([a,c])$, $f|_{[c,b]} \in \mathcal{R}([c,b])$ by ICT.
  note $L(P_c, f) \le \int_a^b f \le U(P_c, f)$ (& sim for r & ℓ), thus $\int_a^b f - \int_b^c f - \int_a^c f \le U(P_c, f) - L^\ell(P_c, f) - L^r(P_c^r, f) < \varepsilon$
  and similarly, $\int_a^b f - \int_c^b f - \int_a^c f \ge L(P_c, f) - U^\ell(P_c^\ell, f) - U^r(P_c^r, f) = L(P_c, f) - U(P_c, f) > -\varepsilon$
  $\therefore \left|\int_a^b f - \int_c^b f - \int_c^a f\right| < \varepsilon$ $\forall \varepsilon > 0$

- Thm (integrability criteria): let $f:[a,b]\to\mathbb{R}$ be bounded. TFAE, 1. f is int

  2. $\forall\varepsilon>0,\ \exists P\in\Pi[a,b]$ s.t. $U(P,f)-L(P,f)<\varepsilon$

  3. $\forall\varepsilon>0,\ \exists P\in\Pi[a,b]$ s.t. if $Q=\{x_0,\dots,x_n\}\in\Pi[a,b]$ refines P and $s_i,t_i\in[x_{i-1},x_i]$ for $1\le i\le n$, then $\sum_{i=1}^n|f(s_i)-f(t_i)|\Delta_i<\varepsilon$ $\Downarrow$ (3): know its int, want part. (4): to check if int.

  4. $\forall\varepsilon>0,\ \exists P=\{x_0,\dots,x_n\}\in\Pi[a,b]$ s.t. if $s_i,t_i\in[x_{i-1},x_i]$, $\sum_{i=1}^n|f(s_i)-f(t_i)|\Delta_i<\varepsilon$

  5. $\exists I\in\mathbb{R}$ s.t. $\forall\varepsilon>0,\ \exists P\in\Pi[a,b]$ s.t. if $Q=\{x_0,\dots,x_n\}\in\Pi[a,b]$ refines P and $t_i\in[x_{i-1},x_i]$, then $\left|\sum_{i=1}^n f(t_i)\Delta_i-I\right|<\varepsilon$ $\Downarrow$ $\to$ there's a partition that works (6), and it still works any time you refine it (5).

  6. $\exists I\in\mathbb{R}$ s.t. $\forall\varepsilon>0,\ \exists P=\{x_0,\dots,x_n\}\in[a,b]$ s.t. if $t_i\in[x_{i-1},x_i]$, $\left|\sum_{i=1}^n f(t_i)\Delta_i-I\right|<\varepsilon$

(1) $\Rightarrow$ (2): since $f\in\mathcal{R}([a,b])$, for $\varepsilon>0$, choose $P_1,P_2\in\Pi[a,b]$ s.t. $\int_a^b f=\underline{\int_a^b}<L(P_1,f)+\frac{\varepsilon}{2}$ and $\int_a^b f=\overline{\int_a^b}f>U(P_2,f)-\frac{\varepsilon}{2}$, then let $P=P_1\# P_2$, $U(P,f)-L(P,f)\le U(P_1,f)-L(P_1,f)<\varepsilon$.

(2) $\Rightarrow$ (1): let $\varepsilon>0$, then $\exists P$ s.t. $0\le\overline{\int_a^b}f-\underline{\int_a^b}f\le U(P,f)-L(P,f)<\varepsilon\ \forall\varepsilon$.

(4) $\Rightarrow$ (2): pick $0<K<\frac{\varepsilon}{4(b-a)}$, $s_i,t_i\in[x_{i-1},x_i]$ s.t. $M_i-K<f(s_i)$, $f(t_i)<m_i+K$.

then $U(P,f)-K(b-a)=\sum_{i=1}^n(M_i-K)\Delta_i<\sum_{i=1}^n f(s_i)\Delta_i$, $L(P,f)+K(b-a)=\sum_{i=1}^n(m_i+K)\Delta_i>\sum_{i=1}^n f(t_i)\Delta_i$

$\Rightarrow U(P,f)-L(P,f)<\sum_{i=1}^n(\underline{f(s_i)-f(t_i)})\Delta_i+2K(b-a)<\varepsilon+\frac{\varepsilon}{2}$ $\to$(4)

(1) + (2) $\Rightarrow$ (5): sps $Q=\{x_0,\dots,x_n\}$ refines P and $t_i\in[x_{i-1},x_i]$, where P s.t. $U(P,f)-L(P,f)<\varepsilon$,

then lemma $\Rightarrow\left|\sum_{i=1}^n f(t_i)\Delta_i-\int_a^b f\right|\le U(Q,f)-L(Q,f)\le U(P,f)-L(P,f)<\varepsilon$.

(6) $\Rightarrow$ (2): let $\varepsilon>0$, $0<K<\frac{\varepsilon}{4(b-a)}$, (6) $\Rightarrow$ pick P s.t. … $|\dots|<\frac{\varepsilon}{4}$, pick $s_i,t_i\in[x_{i-1},x_i]$ s.t.

$M_i-K<f(t_i)$, $f(s_i)<m_i+K$, then $U(P,f)-K(b-a)=\sum_{i=1}^n(M_i-K)\Delta_i<\sum_{i=1}^n f(t_i)<I+\frac{\varepsilon}{4}$

and $L(P,f)+K(b-a)>\sum_{i=1}^n f(s_i)\Delta_i>I-\frac{\varepsilon}{4}\Rightarrow U(P,f)-L(P,f)<\varepsilon$

further, $\left|\int_a^b f-I\right|\le\left|\int_a^b f-\sum_{i=1}^n f(r_i)\Delta_i\right|+\left|\sum_{i=1}^n f(r_i)\Delta_i-I\right|<U(P,f)-L(P,f)+\frac{\varepsilon}{4}\Rightarrow I=\int_a^b f$ $\leftarrow r_i\in[x_{i-1},x_i]$

- Thm: if $f:[a,b]\to\mathbb{R}$ is cont then f is int

  f cont on closed interval $\to$ cpt, so EVT $\Rightarrow f$ bounded & f cont on cpt set $\Rightarrow f$ unif. cont.

  let $\varepsilon>0$, pick $\delta>0$ s.t. $x,y\in[a,b]$ s.t. $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\frac{\varepsilon}{b-a}$ (int. crit. 4).

  let $P=\{x_0,\dots,x_n\}$ s.t. $|x_{i-1}-x_i|<\delta$, let $s_i,t_i\in[x_{i-1},x_i]$, then $|s_i-t_i|<\delta\Rightarrow|f(s_i)-f(t_i)|<\frac{\varepsilon}{b-a}$

  + if f bounded (on $[a,b]$) and cont in $[a,b]\setminus F$ for F finite, then f is int

  + if f bounded and $D=\{x\in[a,b]\mid f\text{ is not cont at }x\}$, then f int $\Leftrightarrow D$ is Lebesque measure 0

- Thm: let f be monotone, then f is int (sps f is nondec) let $P_n=\{x_0,\dots,x_n\}$

  note if $P=\{x_0,\dots,x_n\}$ then $m_i=\inf\{f(x)\mid x_{i-1}\le x\le x_i\}=f(x_{i-1})$, $M_i=f(x_i)$. let $x_i=a+\frac{(b-a)i}{n}$

  then $\Delta_i=\frac{(b-a)}{n}$, hence $U(P_n,f)-L(P_n,f)=\sum_{i=1}^n(M_i-m_i)\Delta_i=\sum_{i=1}^n(f(x_i)-f(x_{i-1}))\frac{(b-a)}{n}=\frac{(b-a)(f(b)-f(a))}{n}$

  then for $\varepsilon>0$, pick $n\ge1$ s.t. $\frac{(\ )(\ )}{n}<\varepsilon$ to see $U(P_n,f)-L(P_n,f)<\varepsilon\Rightarrow f$ int

- Def: let $f\in\mathcal{R}([a,b])$, then $c\in[a,b]\Rightarrow\int_c^c f=0$, $a\le c\le d\le b\Rightarrow\int_d^c f=-\int_c^d f$

- Thm (fund. thm of calc. pt 1): sps $f\in\mathcal{R}([a,b])$, define $F:[a,b]\to\mathbb{R}$ via $F(x)=\int_a^x f$.

  1. F is Lip, and in part, if $|f(x)|\le M\ \forall x\in[a,b]$, then $|F(x)-F(y)|\le M|x-y|$

  2. if f cont at $x\in[a,b]$, then F diff at x and $F'(x)=f(x)$

1. sps $|f(x)|\le M$, then $|F(x)-F(y)|=\left|\int_{\min\{x,y\}}^{\max\{x,y\}}f\right|\le\int_{\min}^{\max}|f|\le\int M=M|x-y|\ \forall x,y\in[a,b]$

2. fix $x\in[a,b]$ s.t. f is cont at x and for $t\in[a,b]\setminus\{x\}$ let $u=\min\{t,x\}$, $v=\min\{..\}$.

   then $\frac{F(t)-F(x)}{t-x}=\frac{1}{v-u}\int_u^v x$, so $\frac{F(t)-F(x)}{t-x}-f(x)=\frac{1}{v-u}\int_u^v(f-\underline{f(x)})$ $\to$ const.

   let $\varepsilon>0$, since f cont at x, $\exists\delta>0$ s.t. $y\in[a,b]$ and $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\frac{\varepsilon}{2}$.

   then for $t\in[a,b]\setminus\{x\}$ s.t. $|x-t|<\delta$, $\left|\frac{F(t)-F(x)}{t-x}-f(x)\right|=\left|\frac{1}{v-u}\int_u^v(f-f(x))\right|\le\frac{1}{v-u}\int_u^v\frac{\varepsilon}{2}$.

- Thm (fund. thm. of calc. pt. 2): sps $F:[a,b]\to\mathbb{R}$ is cont on $[a,b]$ and diff on (a,b) with $F'(x)=f(x)$ for $x\in(a,b)$ where $f\in\mathcal{R}([a,b])$, then $\int_a^b f=F(b)-F(a)$

  let $\varepsilon>0$, pick $P=\{x_0,\dots,x_n\}\in\Pi[a,b]$ s.t. if $t_i\in[x_{i-1},x_i]$ then $\left|\sum_{i=1}^n f(t_i)(x_i-x_{i-1})-\int_a^b f\right|<\varepsilon$

  by MVT, can pick $t_i\in(x_{i-1},x_i)$ s.t. $\underline{f(t_i)}(x_i-x_{i-1})=F(x_i)-F(x_{i-1})$ $\to F'(t_i)$

  then $\sum_{i=1}^n f(t_i)\Delta_i=\sum_{i=1}^n F(x_i)-F(x_{i-1})=F(b)-F(a)\ \therefore\ \left|F(b)-F(a)-\int_a^b f\right|<\varepsilon\ \forall\varepsilon>0$

- Thm (int. by parts): sps $F,G:[a,b]\to\mathbb{R}$ cont on $[a,b]$, diff on (a,b), $F'=f$, $G'=g$ on (a,b) for $f,g\in\mathcal{R}([a,b])$, then $\int_a^b Fg=F(b)G(b)-F(a)G(a)-\int_a^b fG$

  let $H(x):[a,b]\to\mathbb{R}$ via $H(x)=F(x)G(x)$, then H cont on $[a,b]$, diff on (a,b), $H'=fG+Fg$,

  then $FTC_2\Rightarrow F(b)G(b)-F(a)G(a)=H(b)-H(a)\overset{FTC_2}{=}\int_a^b(fG+Fg)=\int_a^b fG+\int_a^b Fg$ by linearity.

- Thm (change of variables): let $a<b$, $A<B$, sps $\varphi:[a,b]\to[A,B]$ is strictly monotone surj. and diff on $[a,b]$ w/ $\varphi':[a,b]\to\mathbb{R}$ cont. let $f:[A,B]\to\mathbb{R}$ be bounded.

  then f int $\Leftrightarrow(f\circ\varphi)\varphi'$ int and $\int_a^b(f\circ\varphi)\varphi'=\int_{\varphi(a)}^{\varphi(b)}f$. sps φ incr, then $\varphi(a)=a$, $\varphi(b)=B$.

  ($\Rightarrow$) sps f int, sps $M>0$ s.t. $|f(y)|\le M$, let $\varepsilon>0$ and $0<K<\frac{\varepsilon}{2M(b-a)}$. since φ' cont on $[a,b]$, its uni cont, so $\exists\delta>0$ s.t. $x,y\in[a,b]$, $|x-y|<\delta\Rightarrow|\varphi'(x)-\varphi'(y)|<K$, since f is int, $\exists P$ s.t. if $Q=\{y_0,\dots,y_n\}$ refines P and $z_i\in[y_{i-1},y_i]$, then $\left|\sum_{i=1}^n f(z_i)(y_i-y_{i-1})-\int_A^B f\right|<\frac{\varepsilon}{2}$ by ICT #5.

  write $P=\{w_0,\dots,w_N\}$, $P'=\{\varphi^{-1}(w_0),\dots,\varphi^{-1}(w_N)\}\in\Pi[a,b]$ bc. φ incr. surj. now let $Q'=\{x_0,\dots,x_n\}$ refine P' s.t. $x_i-x_{i-1}<\delta$, then $Q=\{\varphi(x_0),\dots,\varphi(x_n)\}\in\Pi[a,b]$ bc. φ surj. then Q refines P.

  then MVT $\Rightarrow$ pick $s_i\in(x_{i-1},x_i)$ s.t. $\varphi'(s_i)(x_i-x_{i-1})=\varphi(x_i)-\varphi(x_{i-1})$. now let $t_i\in[x_{i-1},x_i]$,

  ICT #6: $\left|\sum_{i=1}^n f\circ\varphi(t_i)\varphi'(t_i)(x_i-x_{i-1}-1)-\int_A^B f\right|\le\left|\sum_{i=1}^n f\circ\varphi(t_i)(\varphi'(t_i)-\varphi'(s_i))(x_i-x_{i-1})\right|+\mathrm{II}$ (I)

  then $\mathrm{I}\le\sum_{i=1}^n MK(x_i-x_{i-1})<\varepsilon$, $\mathrm{II}=\left|\sum_{i=1}^n f\circ\varphi(t_i)\varphi'(s_i)(x_i-x_{i-1})-\int_A^B f\right|=\left|\sum_{i=1}^n f\circ\varphi(t_i)(\varphi(x_i)-\varphi(x_{i-1}))-\int_A^B f\right|<\frac{\varepsilon}{2}$ (bc. Q ref. P)

  ($\Leftarrow$) sps $(f\circ\varphi)\varphi'$ int, let $\varepsilon>0$ and pick M,K,δ as above.

  pick P' s.t. if Q' refines P' and $Q'=\{x_0,\dots,x_n\}$, $t_i\in[x_{i-1},x_i]$, then (recall $x_i-x_{i-1}<\delta$)

  $\mathrm{II}\to\left|\sum_{i=1}^n f\circ\varphi(t_i)\varphi'(t_i)(x_i-x_{i-1})-\int_a^b(f\circ\varphi)\varphi'\right|<\frac{\varepsilon}{2}$. set $Q=\{\varphi(x_0),\dots,\varphi(x_n)\}$, then use MVT to pick $s_i\in(x_{i-1},x_i)$ s.t. $\varphi'(s_i)(x_i-x_{i-1})=\varphi(x_i)-\varphi(x_{i-1})$. now pick $z_i\in[\varphi(x_{i-1}),\varphi(x_i)]$

  and write $z_i=\varphi(t_i)$, then $\left|\sum_{i=1}^n f(z_i)(\varphi(x_i)-\varphi(x_{i-1}))-\int_a^b(f\circ\varphi)\varphi'\right|\le\sum_{i=1}^n|f(z_i)||\varphi'(t_i)-\varphi'(s_i)|(x_i-x_{i-1})+\mathrm{II}<\varepsilon.$

- Thm (CoV, v2): let $a<b$, $A<B$, sps $\varphi:[a,b]\to[A,B]$ is diff on $[a,b]$ with $\varphi:[a,b]\to\mathbb{R}$ cont. sps $f:[A,B]\to\mathbb{R}$ is cont (& hence int on $[A,B]$), then $\int_a^x (f\circ\varphi)\varphi' = \int_{\varphi(a)}^{\varphi(x)} f$ on $[a,b]$.

note $f\circ\varphi$ is cont, φ' is cont $\Rightarrow (f\circ\varphi)\varphi'$ is cont & thus int. define $F,G:[a,b]\to\mathbb{R}$ via $F(x)=\int_{\varphi(a)}^{\varphi(x)} f$, $G(x)=\int_a^x (f\circ\varphi)\varphi'$, since $f, (f\circ\varphi)\varphi'$ are cont, FTC and chain rule $\Rightarrow$ F,G diff on $[a,b]$ with $G'(x)=f(\varphi(x))\varphi'(x)$, $F'(x)=f(\varphi(x))\varphi'(x)$. write $F(x)=H(\varphi(x))$ for $H:[A,B]\to\mathbb{R}$ via $H(y)=\int_{\varphi(a)}^{y} f$ and FTC $\Rightarrow H'(y)=f(y) \Rightarrow F'(x)=H'(\varphi(x))\varphi'(x)=f(\varphi(x))\varphi'(x)$

OTOH, $F(a)=\int_{\varphi(a)}^{\varphi(a)} f = 0 = \int_a^a (f+\varphi)\varphi' = 0 \Rightarrow (F-G)'=0$ in $[a,b]$ and $(F-G)(a)=0 \Rightarrow F=G$.

Exponential & Trigonometric Series:

series converges absolutely: $\forall x$, $\frac{|x^{n+1}|}{(n+1)!}\frac{n!}{x^n} = \frac{|x|}{n} \to 0$

1. Define $\exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, and $e=\exp(1)\in\mathbb{R}$.
2. $\exp(x+y)=\exp(x)\exp(y)$ series product $\Rightarrow \sum_{n=0}^{\infty} c_n$, $c_n=\sum_{k=0}^{n}\frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!} = \frac{(x+y)^n}{n!}$ by bin. thm.
3. $0<\exp(x)$ $\forall x\in\mathbb{R}$ $1=\exp(0)=\exp(x)\exp(-x) \Rightarrow$ both tve.
4. $x<y \Rightarrow \exp(x)<\exp(y)$ $\exp(y)=\exp(x+(y-x))=\exp(x)\exp(y-x)>\exp(x)$.
5. $\exp(qx)=\exp(x)^q$, and thus $\exp(q)=e^q$ induct on $n\in\mathbb{N}$, then $\mathbb{Z}$, then $q=1/n$ $(1\le n)$, then $q\in\mathbb{Q}$
6. $\exp(x)=e^x$ $\forall x\in\mathbb{R}$. ($\& \ x^y=e^{y\log x}$) let $S=\{\exp(q)\mid q\in\mathbb{Q}$ and $q\le x\}$. $\exp(x)$ is UB of S, then $z:=\sup S$. AFSOC $z<\exp(x)$, then $(0<a<b\Leftrightarrow \log a<\log b)\Rightarrow$ $(0<z<\exp(x)\Leftrightarrow \log z<\log\exp x)$. since $\mathbb{R}$ Arch, pick q s.t. $\log z<q<\log\exp(x) \Leftrightarrow z<e^q<\exp(x)$, a contra, then $z=\exp(x)=\sup S=\sup\{e^q\mid q\in\mathbb{Q}, q\le x\}=e^x$

Further, $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $|e^x-1-x|\le x^2e^{|x|}$ (useful to prove $\exp'(x)=\exp(x)$)

$\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$, $\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$, $\cos(-x)=\cos(x)$, $\sin(-x)=-\sin(x)$

- Thm (CoV, v2): let $a<b$, $A<B$, sps $\varphi:[a,b]\to[A,B]$ is diff on $[a,b]$ with $\varphi:[a,b]\to\mathbb{R}$ cont. sps $f:[A,B]\to\mathbb{R}$ is cont (& hence int on $[A,B]$), then $\int_a^x (f\circ\varphi)\varphi' = \int_{\varphi(a)}^{\varphi(x)} f$ on $[a,b]$.

note $f\circ\varphi$ is cont, φ' is cont $\Rightarrow (f\circ\varphi)\varphi'$ is cont & thus int. define $F,G:[a,b]\to\mathbb{R}$ via $F(x)=\int_{\varphi(a)}^{\varphi(x)} f$, $G(x)=\int_a^x (f\circ\varphi)\varphi'$, since $f, (f\circ\varphi)\varphi'$ are cont, FTC and chain rule $\Rightarrow$ F,G diff on $[a,b]$ with $G'(x)=f(\varphi(x))\varphi'(x)$, $F'(x)=f(\varphi(x))\varphi'(x)$. write $F(x)=H(\varphi(x))$ for $H:[A,B]\to\mathbb{R}$ via $H(y)=\int_{\varphi(a)}^{y} f$ and FTC $\Rightarrow H'(y)=f(y) \Rightarrow F'(x)=H'(\varphi(x))\varphi'(x)=f(\varphi(x))\varphi'(x)$

OTOH, $F(a)=\int_{\varphi(a)}^{\varphi(a)} f = 0 = \int_a^a (f+\varphi)\varphi' = 0 \Rightarrow (F-G)'=0$ in $[a,b]$ and $(F-G)(a)=0 \Rightarrow F=G$.

Exponential & Trigonometric Series:

series converges absolutely: $\forall x$, $\frac{|x^{n+1}|}{(n+1)!}\frac{n!}{x^n} = \frac{|x|}{n} \to 0$

1. Define $\exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, and $e=\exp(1)\in\mathbb{R}$.
2. $\exp(x+y)=\exp(x)\exp(y)$ series product $\Rightarrow \sum_{n=0}^{\infty} c_n$, $c_n=\sum_{k=0}^{n}\frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!} = \frac{(x+y)^n}{n!}$ by bin. thm.
3. $0<\exp(x)$ $\forall x\in\mathbb{R}$ $1=\exp(0)=\exp(x)\exp(-x) \Rightarrow$ both tve.
4. $x<y \Rightarrow \exp(x)<\exp(y)$ $\exp(y)=\exp(x+(y-x))=\exp(x)\exp(y-x)>\exp(x)$.
5. $\exp(qx)=\exp(x)^q$, and thus $\exp(q)=e^q$ induct on $n\in\mathbb{N}$, then $\mathbb{Z}$, then $q=1/n$ $(1\le n)$, then $q\in\mathbb{Q}$
6. $\exp(x)=e^x$ $\forall x\in\mathbb{R}$. ($\& \ x^y=e^{y\log x}$) let $S=\{\exp(q)\mid q\in\mathbb{Q}$ and $q\le x\}$. $\exp(x)$ is UB of S, then $z:=\sup S$. AFSOC $z<\exp(x)$, then $(0<a<b\Leftrightarrow \log a<\log b)\Rightarrow$ $(0<z<\exp(x)\Leftrightarrow \log z<\log\exp x)$. since $\mathbb{R}$ Arch, pick q s.t. $\log z<q<\log\exp(x) \Leftrightarrow z<e^q<\exp(x)$, a contra, then $z=\exp(x)=\sup S=\sup\{e^q\mid q\in\mathbb{Q}, q\le x\}=e^x$

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note $f\circ\varphi$ is cont, φ' is cont $\Rightarrow (f\circ\varphi)\varphi'$ is cont & thus int. define $F, G : [a,b] \to \mathbb{R}$ via $F(x) = \int_{\varphi(a)}^{\varphi(x)} f$, $G(x) = \int_a^x (f\circ\varphi)\varphi'$, since f, $(f\circ\varphi)\varphi'$ are cont, FTC and chain rule $\Rightarrow$ F, G diff on $[a,b]$ with $G'(x) = f(\varphi(x))\varphi'(x)$, $F'(x) = f(\varphi(x))\varphi'(x)$. write $F(x) = H(\varphi(x))$ for $H : [A,B] \to \mathbb{R}$ via $H(y) = \int_{\varphi(a)}^{y} f$ and FTC $\Rightarrow H'(y) = f(y) \Rightarrow F'(x) = H'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x)$

OTOH, $F(a) = \int_{\varphi(a)}^{\varphi(a)} f = 0 = \int_a^a (f\circ\varphi)\varphi' = 0 \Rightarrow (F-G)' = 0$ in $[a,b]$ and $(F-G)(a) = 0 \Rightarrow F = G$.

Exponential & Trigonometric Series:

1. Define $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and $e = \exp(1) \in \mathbb{R}$. series converges absolutely: $\forall x$, $\frac{|x^{n+1}|}{(n+1)!}\frac{n!}{x^n} = \frac{|x|}{n} \to 0$
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Further, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $|e^x - 1 - x| \leq x^2 e^{|x|}$ (useful to prove $\exp'(x) = \exp(x)$)

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