1. Arbitrage: SFP X, $X_0=0$ s.t. $X_T\ge 0$ fs, $X_T>0$ pos.

⇔ Type A: $X_0\le 0$, $X_T\ge 0$ fs, $X_T>0$ pos ← $\exists T>0$

- if $X_T=Y_T$ for $T>0$, $X_t=Y_t$ for $0\le t<T$
- $X_T\le Y_T \Rightarrow X_0\le Y_0$

- hold/buy/deposit: +ve
- short/sell/loan: −ve

2. call = right to buy, $C_T=(S_T-K)^+$    £1 = $E_£^{\$}$(£1)

put = right to sell, $P_T=(K-S_T)^+$

forward contract: obligation to buy/sell at K @ T
- buyer: long pos. receive $S_T$, pay K, $fwd_T=S_T-K$
- seller: short position, receive K, pay $S_T$ at time T
↳ replicate: sell share of S, deposit $K\,d(T)$

3. $d(T)=\frac{1}{1+rT}=\frac{1}{(1+\frac{r[m]}{m})^{mT}}=e^{-r[\infty]\cdot T}=\frac{1}{(1+R)^T}$

forward loan: borrow $A_\eta$, repay F    $A_\eta d(\eta)=F\,d(T)$

↑$A_\eta$ at $\eta$, ↓F at T
- buy ZCB w/ face value $A_\eta$, maturity $\eta$
- sell ZCB w/ face value F, maturity T    ($X_0=0$)

$F=A_\eta(1+R)^{T-\eta}$, effective rate $R=R^{for}_{0,\eta,T}$

4. ZCB: bond pays F at time T, $P_0^{ZCB}=F\,d(T)$

Annuity: m payments of A per year, $P_0^A=\sum_{i=1}^{mT}A\,d(\frac{i}{m})$

Coupon bond: coupon payments of $C=F\cdot\frac{q[m]}{m}$

at times $\frac{1}{m}\cdots\frac{mT}{m}$, $P_0^{CB}=P_0^A+P_0^{ZCB}$    let $\lambda=\frac{1}{1+R_I}$

THM: consider a general security making fixed payments $\{F_1,\dots,F_N\}$ at times $0<T_1<\dots<T_N$, if $F_i>0$ $\forall i$ and $P_0>0$, $\exists$ unique IRR $R_I>-1$

↳ $S=\sum_{i=1}^{mT}\lambda^i=\frac{\lambda(1-\lambda^{mT})}{1-\lambda}$

5. floating rate bond: payment at $\frac{i+1}{m}$ is interest from investment of F over $[\frac{i}{m},\frac{i+1}{m}]$

$=F\left((1+R_{\frac{i}{m},\frac{i+1}{m}})^{1/m}-1\right)=F\cdot\frac{p_i[m]}{m}$ ↳ nominal rate

interest rate swaps: at time $\frac{i}{m}$,
- A pays B floating payment $F\cdot\frac{p_{i-1}[m]}{m}$
- B pays A fixed payment $F\cdot\frac{q[m]}{m}$

at time 0, choose $q[m]$ s.t. ($P_0^{float}=F$)

$\sum_{i=1}^{mT}F\cdot\frac{q[m]}{m}d(\frac{i}{m})=F(1-d(T))$    LHS = val of fixed − $F\,d(T)$

only net payments made: $q^{swap}[m]=\frac{m(1-d(T))}{\sum d(i/m)}$

↳ if $p_{i-1}[m]>q^{swap}[m]$, A pay B: $\frac{F}{m}(p_{i-1}[m]-q^{swap}[m])$

to replicate A, buy CB w/ coupon rate $q^{swap}[m]$, sell float note. (A's payments = float note − ZCB)

* F is notional of contract, neither party makes face value payment. (B's payments: CB − ZCB).

6. forward contract for ZCB: maturity $T_b$, delivery @ $T_d$

$T_d$: ↓$\mathcal{F}_{0,T_d}$; $T_b$: ↑F

replicate: borrow purchase price $F\,d(T_b)$ over $[0,T_d]$, buy ZCB

at $t=0$, $F\,d(T_b)-F\,d(T_b)=0$    forward price for delivery at time $T_d$, set at time 0

at $t=T_d$, repay $F\frac{d(T_b)}{d(T_d)}=\mathcal{F}_{0,T_d}$

forward contract to buy at $T_j$ (just after payment)

$\mathcal{F}=\frac{\sum_{i=j+1}^{N}F_i\,d(T_i)}{d(T_j)}=\frac{P_0-\sum_{i=1}^{j}F_i\,d(T_i)}{d(T_j)}$    no \$, $T_j$, holder of contract receives

7. if stock S pay dividends $\delta_i$ at time $T_i$

$\mathcal{F}=\frac{S_0-\sum_{i=1}^{N}S_i\,d(T_i)}{d(T)}$    idea: $X_T=S_T$, $X_0=S_0-\delta\,d(t)$

borrow $S_0-S_i\,d(T_i)$

replicate: sell ZCB with $F=S_i$, maturity $T_i$, $\forall i$

buy 1 share → at time i, pay off ZCB with dividend

8. known dividend yield: $S_{T^+}=S_{T^-}-\alpha S_{T^-}$, $0<\alpha<1$

replicate: borrow $(1-\alpha)S_0$ over $[0,T]$, buy $(1-\alpha)$ shares (initial capital = 0), at $t=T$, have 1 share, own bank $\mathcal{F}=\frac{(1-\alpha)^N S_0}{d(T)}$, if N dividends paid

9. generally not possible to sell commodities short:
- convenience yield (benefit of keeping C on hand)
- there exists a cost of storing commodities

BUT: if commodity costs $S_0$ today & can be safely stored for $C_{0,T}$ until time T, paid at $t=0$,

$S_0+C_{0,T}\ge\mathcal{F}_{0,T}\,d(T)$

10. futures contract: at $t=0$, value = 0, future price $\mathcal{F}_{0,T}$

$t=1$: receive value $(\mathcal{F}_{1,T}-\mathcal{F}_{0,T})d(T)$ → discount back to day 1

adjust price to $\mathcal{F}_{1,T}$ → in general, $\mathcal{F}_{i,t}$

11. put-call parity: $P_0-C_0=(K-\mathcal{F}_{0,T})d(T)$

if stock pays no dividends, $\mathcal{F}_{0,T}=\frac{S_0}{d(T)}$

$\boxed{\delta}$ $\mathcal{F}_{0,T}\,d(T)=S_0-\delta\,d(T)$, $\boxed{\alpha}$ $\mathcal{F}\,d(T)=(1-\alpha)^N S_0$

12. chooser option: at time $\tau$, choose between put & call

$V_\tau=\max(P_\tau,C_\tau)=C_\tau+\left(\frac{K}{(1+R)^{T-\tau}}-S_\tau\right)^+=C_\tau+\hat{P}_\tau$

⇒ $V_0=C_0+\hat{P}_0$    (both put & call have strike K, exp T)

13. American options: exercise at $t\le T$ to get $g(S_t)$

notation: $V_t^{A,T,K}$ ↳ A/E, exp. date, strike price    intrinsic value function

THM #1: AF value $V_t\ge g(S_t)$

↳ $(V_t-g(S_t))$ = time value of being able to wait

if $V_t<g(S_t)$, buy $V_t$ & immediately exercise to get $g(S_t)$

⇒ risk-free profit of $g(S_t)-V_t$

#2: let $0<T_1<T_2$ be exp. dates, $V_t^{T_1}\le V_t^{T_2}$, $0\le t\le T_1$

construct X: long $V^{T_2}$, short $V^{T_1}$, $X_t=V_t^{T_2}-V_t^{T_1}$

if holder of $V^1$ exercise at t, I exercise $V^2$ to get g

else, I also don't exercise ⇒ $X_{T_2}\ge 0$ fs ⇒ $X_t\ge 0$

#3: if $V^E$ pays $g(S_T)$ at time T, $V_t^{A,T}\ge V_t^{E,T}$

X: long $V^A$, short $V^E$, $X_t=V_t^A-V_t^E$

at T, if $V^E$ exercise, exercise $V^A$, else, exercise $V^A$

if "in the money" ⇒ $X_T\ge 0$ ⇒ $X_t\ge 0$ for $t<T$
- $V_t^A=g(S_t)$ ↳ exercise at t is optimal
- $V_t^A>g(S_t)$ ↳ val. of future exercising > exercising today
- $V_t^E$ = value of exercising at time T
- $V_t^A>V_t^E$ ↳ ∃ some situation where early exercise is optimal

#4: $C_t^E\ge S_t-\frac{K}{(1+R)^{T-t}}$ for $t\le T$    ↳ at $t=T$, $(S_T-K)^+\ge S_T-K$ is true.
- $((\mathcal{F}-K)d(T))^+\le C_0\le\mathcal{F}\,d(T)$
- $((K-\mathcal{F})d(T))^+\le P_0\le K\,d(T)$

#5: if $R>0$, $C_t^A>S_T-K$ ← $\frac{K}{(1+R)^{T-t}}<K$, #3 & #4

#6: if $R>0$, no dividends, $C_t^A=C_t^E$ for $t\le T$

$C_t^A>S_T-K$ (#5) & $C_t^A>0$ (o.w. Type A)

⇒ $C_t^A>g(S_T)$ ⇒ not optimal to exercise at time $t<T$

& $P_t^A\ge P_t^E$

#7: $P_0^A+S_0\le C_0^E+K$

↳ $P_0^A\ge P_0^E$, $C_0^A=C_0^E$ (#6) ⇒ $P_0^A-C_0^A\ge P_0^E-C_0^E$

⇒ $\frac{K}{(1+R)^T}-S_0\le P_0^A-C_0^A\le K-S_0$

X: long $C^E$, short $P^A$, short S, deposit K    (#4)
- if not exercised, $X_T=C_T^E-\cancel{P_T^A}^{\,0}-S_T+K(1+R)^T\ge 0$
- else, receive share of S & pay K ↳ $K((1+R)^t-1)$

at T, $X_T=C_T^E+K((1+R)^t-1)(1+R)^{T-t}\ge 0$.

14. return of a portfolio X: $\frac{X_1-X_0}{X_0}$, r for bank

stock: $\rho^S(H)=\frac{S_1(H)-S_0}{S_0}=u-1$, $\rho^S(T)=\frac{S_1(T)-S_0}{S_0}=d-1$

15. arbitrage: $X_0=0$, $P[X_1\ge 0]=1$, $P[X_1>0]>0$

THM: AF ⇔ $d<(1+r)<u$    (a) $X_0=0$, $\Delta=-1$ (b) $X_0=0$, $\Delta=+1$

① assume $d<u\le(1+r)$ & $(1+r)\le d<u$, show arbitrage

② $X_0=0$, $\Gamma=X_0-\Delta S_0$, if $\Delta=0$, $P[X_1>0]=0$,

if $\Delta>0$ (H), $\Delta<0$ (T), $P[X_1\ge 0]\ne$ ⇒ AF

16. V replicated by Δ shares of stock, initial capital $X_0$

delta-hedging: $\Delta=\frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$ & deposit $X_0-\Delta S_0$

17. $\tilde{P}:\Omega\to[0,1]$ is RNPM: $\sum_{i=1}^{N}\tilde{P}(\omega_i)=1$, $\tilde{P}(\omega_i)>0$

$(1+r)X_0=\tilde{E}[V_1]$, $\tilde{p}=\frac{(1+r)-d}{u-d}$, $\tilde{q}=\frac{u-(1+r)}{u-d}$

$P^1,P^2$ equivalent ⇔ $\forall A\subseteq\Omega$, $P^1[A]=0$ ⇔ $P^2[A]=0$

18. 1st fundamental thm. of AP: 1 period finite model is AF ⇔ there is a RNPM    (bc. $\tilde{E}[X_1]>0$)

* if $\tilde{P}$ equiv real-world P, $\tilde{P}[X_1\ge 0]=1$, $\tilde{P}[X_1>0]>0$ ⇒ $X_0\ne 0$

- law of 1 price: replicating portfolio ⇒ unique AF price

19. 2nd fund. thm: complete ⇔ ∃ unique RNPM

complete: AF & every derivative security has replicating p

20. utility function $u:(0,\infty)\to\mathbb{R}$ s.t. $u'(x)>0$, $u''(x)<0$

- Jensen's: $E[u(X)]\le u(E[X])$    $E[u(\hat{X}_1)]\ge E[u(X_1)]$ $\forall X_1\in\mathcal{X}$

THM: $\hat{X}_1$ is optimal portfolio (wrt u) (assume complete model)

⇔ $\exists\lambda\in\mathbb{R}$ s.t. $u'(\hat{X}_1(\omega))=\lambda\frac{\tilde{P}(\omega)}{P(\omega)}$ $\forall\omega\in\Omega$

21. mean variance analysis (initial capital $X_0$, $\Delta^i$ of $S^i$)

$\mu^i=E[\rho^i]$, bank rate r, expected return $\hat{r}=E[\rho]$

constraint: $\sum_{i=1}^{k}x^i(\mu^i-r)=(\hat{r}-r)X_0$, $x^i=\Delta^i S_0^i$

objective: minimize $Var(X_1)=\sum_{i=1}^{k}\sum_{j=1}^{k}x^i x^j\sigma_{ij}$

THM: $(\hat{x}^1,\dots,\hat{x}^k)\in\mathcal{X}$ optimal ⇔ ∃λ s.t. $\sum_{j=1}^{k}\sigma_{ij}\hat{x}^j=\lambda(\mu^i-r)$

set $\hat{x}^j=\lambda\hat{y}^j$, then $\sum_{j=1}^{k}\sigma_{ij}\hat{y}^j=(\mu^i-r)$, $\lambda=\frac{(\hat{r}-r)X_0}{\sum_{j=1}^{k}(\mu^j-r)\hat{y}^j}$ (>0)

* $\hat{y}$ doesn't depend on investor pref (only λ)

if $\sigma_{ij}=0$ $\forall i\ne j$ ($\rho^i$ uncorrelated), $\hat{x}^i=\frac{\lambda(\mu^i-r)}{\sigma_{ii}}$

⇒ $\lambda=\frac{(\hat{r}-r)X_0}{\sum_{j=1}^{k}\frac{(\mu^j-r)^2}{\sigma_{jj}}}$ (>0) → if $\mu^i>r$, $i^{th}$ stock bought in optimal portfolio, sold if $\mu^i<r$ → if $\mu^i=r$, portfolio avoids $i^{th}$ stock.

market index portfolio

1-fund thm: ∃ portfolio M s.t. every m-v. investor can have an optimal portfolio of bank inv. + some amt of M

where # shares of $S^i$ in M, $\Delta_0^i=\hat{y}^i/\hat{s}^i$    $\hat{s}^i(\omega)=S_1^i(\omega)/S_0^i$ "scaled stock price"

- $\sigma_{ij}=Cov(\hat{S}_1^i,\hat{S}_1^j)=\frac{1}{S_0^iS_0^j}Cov(S_1^i,S_1^j)=Cov(\rho^i,\rho^j)$
- $Cov(X,Y)=E[(X-E[X])(Y-E[Y])]=E[XY]-E[X]E[Y]$