1. Arbitrage: SFP X, $X_0 = 0$ s.t. $\exists T > 0$ s.t. $X_T \geq 0$ for sure, $X_T > 0$ possible

$\Leftrightarrow$ Type A portfolio: $X_0 \leq 0$, $\exists T > 0$ s.t. $X_T \geq 0$ for sure & $X_T > 0$ possible.

$\rightarrow$ if $X_T = Y_T$ for some $T > 0$, then $X_t = Y_t$ for $0 \leq t < T$

$\rightarrow$ if $X_T \leq Y_T$, then $X_0 \leq Y_0$

2. call: right to buy, $C_T = (S_T - K)^+$

put: right to sell, $P_T = (K - S_T)^+$

} hold/buy/deposit +ve; short/sell/loan −ve

$\rightarrow$ forward contract: obligation to buy/sell at delivery price on delivery date

buyer: long position
- receives S_T, pays K
- $\text{fwd}_T = S_T - K$

seller: short position
- receives K, pays S_T
- $\text{fwd}_T = K - S_T$

} replicate: sell share of S, deposit $K\,d(T)$

3. $d(T) = \dfrac{1}{1+rT} = \dfrac{1}{\left(1+\frac{r[m]}{m}\right)^{mT}} = e^{-r[\infty]\cdot T} = \dfrac{1}{(1+R)^T}$ → effective rate

simple interest; m compounding periods/yr; continuous compounding

4. ZCB: bond pays F at time T, $P_0^{ZCB} = F\,d(T)$ $\qquad S = \sum_{i=1}^{mT} \lambda^i = \dfrac{\lambda(1-\lambda^{mT})}{1-\lambda}$

Annuity: m payments of A per year: $P_0^A = \sum_{i=1}^{mT} A\,d\left(\frac{i}{m}\right)$

Coupon Bond: coupon payments of $C = F \cdot \frac{q[m]}{m}$ at times $\frac{1}{m} \cdots \frac{mT}{m}$, $P_0^{CB} = P_0^A + P_0^{ZCB}$

THM: consider a general security making fixed payments $\{F_1, \ldots, F_N\}$ at times $0 < T_1 < \ldots < T_N$, if $F_i > 0\ \forall i$ and $P_0 > 0$, then $\exists$ a unique IRR $R_I > -1$

5. floating rate bond: payment at $\frac{i+1}{m}$ is interest from investment of F over $\left[\frac{i}{m}, \frac{i+1}{m}\right]$

interest payment @ $\frac{i+1}{m}$: $F\left(\left(1+R_{\frac{i}{m},\frac{i+1}{m}}\right)^{1/m} - 1\right) = F \cdot \frac{p_i[m]}{m}$ ⇒ nominal rate

$P^{float} = F$

6. interest rate swaps: at time $\frac{i}{m}$,

A pays B floating payment $F \cdot \frac{p_{i-1}[m]}{m}$

B pays A fixed payment $F \cdot \frac{q[m]}{m}$

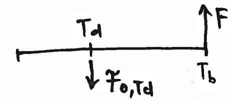
} at time 0, choose $q[m]$ s.t. $\underbrace{\sum_{i=1}^{mT} F \cdot \frac{q[m]}{m}\, d\left(\frac{i}{m}\right)}_{\text{val. of fixed}} = \underbrace{F(1-d(T))}_{\text{val. of floating}}$

only net payments made: $q^{swap}[m] = \dfrac{m(1-d(T))}{\sum d(i/m)}$

if $p_{i-1}[m] > q^{swap}[m]$, A pays B: $\frac{F}{m}\left(p_{i-1}[m] - q^{swap}[m]\right)$ } F is notional of contract: neither party makes face value payment.

to replicate A: buy coupon bond with coupon rate $q^{swap}[m]$, sell float note

7. forward contract for ZCB: maturity T_b, delivery date $T_d < T_b$



replicate: borrow purchase price $F\,d(T_b)$ over $[0, T_d]$, buy bond at $t=0$, $F\,d(T_b) - F\,d(T_b) = 0$, at $t = T_d$, repay $F\dfrac{d(T_b)}{d(T_d)} = F_{0,T_d}$

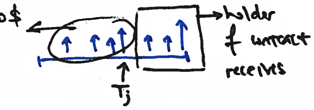
↳ looks like forward deposit! $F = F_{0,T_d}\left(1+R^{for}_{0,T_d,T_b}\right)^{T_b - T_d}$

$F_{0,t}$ = forward price for delivery at time t

$R^{for}_{0,t,s}$ = rate for investing over $[t,s]$ determined at time 0

$R_{t,s}$ = rate for investing over $[t,s]$ determined at time t ↔ S_t = spot price at time t

8. forward contract to buy at T_j (just after payment is made)



$F = \dfrac{\sum_{i=j+1}^{N} F_i\, d(T_i)}{d(T_j)} = \dfrac{P_0 - \sum_{i=1}^{j} F_i\, d(T_i)}{d(T_j)}$ } val. of payment received by holder of long position → borrow over $[0, T_j]$

9. if stock S pay dividends $\delta_1, \ldots, \delta_N$ at times $T_1, \ldots, T_N < T$, $F = \dfrac{S_0 - \sum_{i=1}^{N} \delta_i\, d(T_i)}{d(T)}$

↳ replicate: sell ZCB with $F = \delta_i$, maturity T_i, borrow $S_0 - \delta_i\, d(T_i)$, buy 1 share ⇒ at time t, pay off ZCB with dividend, at $t=T$, have 1 share

10. known dividend yield: factor α of share price at $t = \tau$, $S_{\tau+} = S_{\tau-} - \alpha S_{\tau-}$, $0 < \alpha < 1$

replicate: borrow $(1-\alpha)S_0$ over $[0,T]$, buy $(1-\alpha)$ shares (initial capital is 0)

at $t = T$, have 1 share, own the bank $F = \dfrac{(1-\alpha)^N S_0}{d(T)}$, if N dividends paid

11. generally not possible to sell commodities short: "convenience yield" (benefit of keeping commodity on hand) + $\exists$ cost of storing commodities

BUT: if a commodity costs S_0 today & can be safely stored for $C_{0,T}$ until time T (paid at $t=0$), then $S_0 + C_{0,T} \geq F_{0,T}\, d(T)$

12. futures contract: at $t=0$, value of contract is 0, future price $F_{0,T}$

$t=1$: receive the value $(F_{1,T} - F_{0,T})\, d(T)$, adjust price to $F_{1,T}$

$t=2$: receive the value $(F_{2,T} - F_{1,T})\, d(T)$, adjust price to $F_{2,T}$

13. put-call parity: $P_0 - C_0 = (K - F_{0,T})\, d(T)$ ⇒ if no dividends, $F_{0,T} = \dfrac{S_0}{d(T)}$

↳ valid if can take long/short position in put & call/fwd position, even if not in underlying S

at times $\tau > 0$, $P_\tau - C_\tau = (K - F_{\tau,T})\, d(T-\tau)$ $\qquad \delta$: $F_{0,T} = \dfrac{S_0 - d(\tau)\delta}{d(T)}$, α: $(1-\alpha)S_0 = F_{0,T}\, d(T)$

14. chooser option: at time τ, choose between put & call with strike K, exp. T.

$V_\tau = \max(P_\tau, C_\tau) = C_\tau + \left(\dfrac{K}{(1+R)^{T-\tau}} - S_\tau\right)^+ = C_\tau + \hat{P}_\tau \Rightarrow V_0 = C_0 + \hat{P}_0$

1. forward contract: obligation to buy/sell at delivery price on delivery date

buyer: long position
- receives S_T, pays K
- $fwd_T = S_T - K$

seller: short position
- receives K, pays S_T
- $fwd_T = K - S_T$

} replicate: sell a share of S + deposit $K/(1+r)$

2. call: right to buy on expiry date (put: ... sell ...)

3. capital: hold/buy/long (deposit) is positive, short/sell/loan is negative

4. self-financing portfolio: no capital added/removed after initial time

→ arbitrage: SFP X, $X_0 = 0$, s.t. $\exists T > 0$ s.t. $X_T \geq 0$ for sure, $X_T > 0$ possible

→ suppose $\exists$ bank, if $X_T = Y_T$ for some $T > 0$, then $X_t = Y_t$ for $0 \leq t < T$

→ free of arbitrage ⇔ no "Type A" portfolio

↳ $X_0 \leq 0$, $\exists T > 0$ s.t. $X_T \geq 0$ for sure & $X_T > 0$ is possible

→ if $X_T \geq Y_T$ for sure, $X_T > Y_T$ possible, then $X_0 \leq Y_0$ ⇒ arbitrage

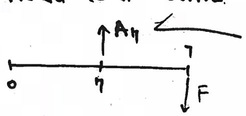
5. replicating portfolio: buy Δ shares of stock, deposit A_0 in bank, $X_0 = \Delta S_0 + A_0$

6. Zero Coupon Bond: makes payment of F (face value) at time T (maturity)

→ discount factor $d(T) = \frac{B_0}{F}$ (B_t = price at time t)

→ $F = \underbrace{B_0(1+rT)}_{\text{simple interest}} = \underbrace{B_0\left(1+\frac{r[m]}{m}\right)^{mT}}_{m \text{ compounding periods/year}} = \underbrace{B_0 e^{r[\infty]\cdot T}}_{\text{continuous compounding}} = \underbrace{B_0(1+R)^T}_{\text{effective rate.}} = \overbrace{\frac{1}{d(T)}}^{\text{amt owed at } T}$

7. forward loan (of A_η): borrow A_η, repay F



- buy ZCB with face value A_η, maturity η
- sell ZCB with face value F, maturity F

} $X_0 = A_\eta d(\eta) - F d(T) = 0$!

$(R^{for}_{0,\eta,T} = \hat{R})$ ↳ effective rate $\hat{R}$: $F = A_\eta(1+\hat{R})^{T-\eta}$

8. $£1 = E^{\$}_{£}(\$1)$, $P_T = (K - S_T)^+$, $C_T = (S_T - K)^+$, forward contract $= (S_T - F)$

9. $R_{T_1 T_2}$ = effective rate at T_1 for bonds/loans/etc maturing at T_2 (spot rate at T_1)

forward price for delivery of 1 share of stock $= \frac{S_0}{d(1)} = \frac{S_0}{1/(1+r)}$

* if $A < V_T < B$, then $\frac{A}{(1+R)^T} < V_0 < \frac{B}{(1+R)^T}$ (borrow V_0, buy V | sell V, deposit V_0)

* $X_0 > Y_0$ ⇒ $X_T > Y_T$ is possible, but not $X_T \geq Y_T$ for sure

else, $Y_T \geq X_T$ for sure. if $Y_T > X_T$ possible, $Y_0 > X_0$. else if $X_T = Y_T$, then $X_0 = Y_0$ | $X_0 = S_0$, $X_T = S_T < S_0$, $Y_T = (1+R)^T Y_0 > S_0$ then $X_T < Y_T$ (e.g. $S_0 = 50$, $S_1 \in \{25, 100\}$, $R = \frac{1}{4}$)

Ex: if $X_T \geq Y_T$, $X_0 < Y_0$, construct Z = long X, short Y, deposit $(Y_0 - X_0)$

then $Z_0 = 0$, $Z_T = \underbrace{X_T - Y_T}_{\geq 0} + \underbrace{(1+r)(Y_0 - X_0)}_{> 0} > 0$ for sure

Ex: pay \$K at $T=1$, receive £1000 at $t=2$ } r is 1-period interest rate.

⇔ borrow $\frac{K}{1+r^{\$}}$ dollars, invest $\frac{1000}{(1+r^{£})^2}$ pounds at $t=0$

Ex: V: pays holder $S_2 - S_1$ at $t=3$ ⇔ pays S_1, receive S_2

⇔ $\boxed{S_1}$ sell $\frac{1}{1.1^2}$ of stock at $t=0$, borrow $\frac{S_1}{1.1^2}$ from bank at $t=1$ (\$ to return stock)

$\boxed{S_2}$ buy $\frac{1}{1.1}$ shares of stock at $t=0$, deposit (from sale) $\frac{S_2}{1.1}$ in bank.

Ex: forward deposit between $t=2$, $t=5$: borrow $\frac{A}{(1+R_0(2))^2}$, deposit $\frac{F}{(1+R_0(5))^5}$

* $P[A] = \sum_{\omega \in A} P(\omega)$