Scalars, Vectors, Matrices (Chapter 1)

- a scalar is an element of a field ($\mathbb{R}$, $\mathbb{C}$, etc.) $\mathbb{F}$: binary field (mod 2)
- a field is a set closed under addition and multiplication that has

  additive & multiplicative: identity: $x + 0 = x$, $x \cdot 1 = x$

  inverse: $x + (-x) = 0$, $x \cdot x^{-1} = 1$

  \+ commutative & associative for addition & multiplication

  \+ distributive law: $x(y+z) = xy + xz$

- vectors: finite list of scalars, e.g. $\begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^2$ ← field; 2: # components

  ↳ vector addition is commutative & associative.

  \+ scalar multiplication is distributive over vector addition.

$(AB)_{*j} = AB_{*j}$

$AB = (AB_{*1} \mid AB_{*2} \mid \dots)$

$= \begin{pmatrix} A^T_{1*}B \\ \hline A^T_{2*}B \\ \hline \dots \end{pmatrix}$ ← $(AB)^T_{1*} = A^T_{1*}B$

- matrices: A, B, C with m rows, n columns, A_{ij} is value of row i and column j

  $A_{m\times n}$, $B_{n\times p}$

  matrix-vector multiplication: for $1 \le i \le m$, $(A\vec{x})_i = (A_{i*} \cdot \vec{x})$

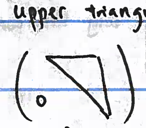
  matrix-matrix multiplication: for $1 \le i \le m$, $1 \le j \le p$, $(AB)_{ij} = A_{i*} \cdot B_{*j} = \sum_{k=1}^{n} A_{ik}B_{kj}$

  $(\ )\cdot(\ )$

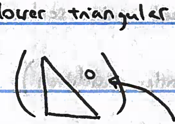
  ↳ matrix mult. is associative but not commutative.

$$((AB)C)_{ij} = \sum_{k=1}^{p} (AB)_{ik} C_{kj} = \sum_{k=1}^{p} \left(\sum_{\ell=1}^{n} A_{i\ell} B_{\ell k} \right) C_{kj}$$
$$= \sum_{k=1}^{p} \sum_{\ell=1}^{n} A_{i\ell} B_{\ell k} C_{kj} = \sum_{\ell=1}^{n} \sum_{k=1}^{p} A_{i\ell} B_{\ell k} C_{kj}$$
$$= \sum_{\ell=1}^{n} A_{i\ell} \left(\sum_{k=1}^{p} B_{\ell k} C_{kj} \right) = \sum_{\ell=1}^{n} A_{i\ell} (BC)_{\ell j} = (A(BC))_{ij}$$

- a function f is linear if it respects linear combinations: $f(x+y) = f(x) + f(y)$, $f(cx) = cf(x)$
- upper triangular, lower triangular, diagonal (UT & LT), identity I_n



$A_{ij} = 0,\ i > j$



$A_{ij} = 0,\ i < j$

$\begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \ a_{nn} \end{pmatrix}$

$A_{ij} = 0$ if $i \neq j$

↳ diagonal matrix where $a_{ii} = 1$, $1 \le i \le n$

permutation P

↳ if A, B is UT, AB is UT.

(all other entries 0) → ↳ exactly one 1 in each row & col.

- $m \times n$ matrix A: function/mapping/transformation from domain $\mathbb{R}^n$ to codomain $\mathbb{R}^m$

Gaussian elimination, LU decomposition (Chapter 2)

- Solutions/sets for linear equations are lines in $\mathbb{R}^2$, planes in $\mathbb{R}^3$, hyperplanes in $\mathbb{R}^n$
- row operations preserve solutions; GOAL, convert to UT, solve via back substitution
  ↳ matrix mult. on left by elimination matrices $E_{ij} = \begin{pmatrix} 1 & & \\ & \ddots & \\ -\ell_{ij} & & 1 \end{pmatrix}$

  $A \xrightarrow{R_i + (-\ell_{ij})R_j} E_{ij}A$, to eliminate entry in row i col. j (a_{ij}), set $-\ell_{ij} = -\left(\frac{a_{ij}}{a_{jj}}\right)$ $\neq 0$ pivot.

  ↳ insight for E: $\underset{1\times m}{\vec{y}^T}\underset{m\times n}{A}$ is a lin. comb. of rows of A: $\underset{1\times n}{y_1 A^T_{1*} + y_2 A^T_{2*} + \ldots}$
- $E_a E_b E_c A = U$, want to write $A = \underset{LT}{L}\underset{UT}{U} = E_c^{-1}E_b^{-1}E_a^{-1}U$, $E_{ij}^{-1} = \begin{pmatrix} 1 & & \\ & \ddots & \\ \ell_{ij} & & 1 \end{pmatrix}$
- then $A\vec{x} = \vec{b} \equiv L\underbrace{U\vec{x}}_{\vec{c}} = \vec{b}$. ① solve $L\vec{c} = \vec{b}$ by forward sub.

  ② solve $U\vec{x} = \vec{c}$ by back sub.
- do row swaps FIRST $PA = LU$, $P = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$ swaps $R_2 \leftrightarrow R_3$ matrix mult on left: row ops

Gauss-Jordan algorithm (for finding inverses)

- augmented matrix $(A \mid I_n) \xrightarrow{\text{row ops}} (I_n \mid A^{-1})$
- all elimination matrices are invertible & $P^{-1} = P^T$

Vector Spaces (Chapter 3)

- vector space: set of vectors V (and associated field of scalars $\mathbb{F}$) satisfying $\mathbb{R}^n, \mathbb{C}^n, \mathbb{F}_2^n$

key properties

  ↳ $\vec{0} \in V$ (V nonempty),

  → if $a \in \mathbb{F}$, $\vec{v} \in V$, then $a\vec{v} \in V$ (V closed under scalar mult.)

  → if $\vec{v}, \vec{u} \in V$, then $\vec{v} + \vec{u} \in V$ (V closed under addition)

  equivalently: V nonempty & if $\vec{v}, \vec{u} \in V$ then any lin. comb. $a\vec{v} + b\vec{u} \in V$.
- span $\{\vec{v}_1, \ldots, \vec{v}_n\} = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n : a_1, \ldots, a_n \in \mathbb{R}\}$

  ↳ span of nonzero vector in $\mathbb{R}^2$ is a line through $\vec{0}$, span of 2 independent nonzero vectors in $\mathbb{R}^2$ is a plane through $\vec{0}$
- if W is a vector space contained in V, W is a subspace of V

  ↳ if $W \subseteq V$, check key properties $(\vec{0}, \cdot, +)$

  → the span of any set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$

- $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent if $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0} \Rightarrow a_1 = a_2 = \dots = a_n$
- basis for vector space: lin. indp. spanning set
  - ↳ if $\vec{v}_1, \dots, \vec{v}_n$ is a basis for V, then for all $\vec{v} \in V$,
    there is exactly 1 way to write $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$
- $A\vec{x} = \vec{b}$ has a solution $\Leftrightarrow \vec{b} \in C(A)$, $C(A) = \text{span}\{\text{column vectors}\}$
- null space: set of vectors $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{0}$ (e.g. $N(A) = \{a(\;) + b(\;) : a, b \in \mathbb{R}\}$)
  - ↳ let R be RREF of A, then $A\vec{x} = 0 \Leftrightarrow R\vec{x} = 0$, ie., $N(A) = N(R)$
- RREF for $m \times n$ matrix with rank r (r = # pivot rows = # pivot cols)

$$\begin{array}{r c} & \begin{array}{cc} r & (n-r) \\ \text{pivot cols} & \text{free cols} \end{array} \\ \begin{array}{r} r \text{ pivot rows} \\ (m-r)\ 0 \text{ rows} \end{array} & \left(\begin{array}{c|c} I_r & F \\ \hline 0_{m-r,r} & 0_{m-r,n-r} \end{array}\right) \end{array}$$

then basis for $N(R)$ is cols of $n \times (n-r)$ matrix

$$N = \left(\frac{-F}{I_{n-r}}\right)$$

& $R\vec{x} = \vec{0}$ when $\vec{x}_{pivot} = -F\vec{x}_{free}$

- if R does not have block form:
  - → PR permutes rows of R
  - → RP permutes cols of R

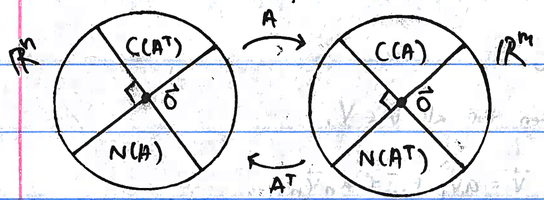
  basis for $N(RP)$ is col. vectors of $N = \left(\frac{-F}{I}\right)$ & basis for $N(R)$ is col. vectors of PN
- complete solution $\vec{x} = \vec{x}_p + \vec{x}_n$, where $\vec{x}_p$ is a particular soln and $\vec{x}_n \in N(A)$
  - ↳ to find $\vec{x}_p$: set free variables to 0, solve $R\vec{x} = \vec{b}'$.
  - → all solutions to $A\vec{x} = \vec{b}$ are $\{(\vdots) + a(\vdots) + b(\vdots) : a, b \in \mathbb{R}\}$ ← affine set in $\mathbb{R}^n$

① $r = n < m$ ("tall")	② $r = m < n$ ("wide")	③ $r = m = n$ ("invertible")	④ $r < m$, $r < n$
$\left(\frac{I_r}{0}\right)$ → full col. rank	$(I_r \mid F)$ → full row rank.	(I_r) → full row & col. rank.	$N(A)$ has basis w/ $n-r$ vectors
no free cols/vars → $N(A) = \{0\}$	no zero rows → $C(A) = \mathbb{R}^m$	$N(A) = \{0\}$, $C(A) = \mathbb{R}^m$	⇒ either 0 or ∞ soln.
$\left(\frac{I_r}{0}\right)\vec{x} = \left(\frac{\star}{0 \text{ or } \neq 0}\right)$ → 0/1 sol.	$A\vec{x} = \vec{b}$ always has a soln; $n-r$ free cols → ∞ soln. (pivot rows & cols).	$A\vec{x} = \vec{b}$ exactly 1 soln.	

→ every matrix has an invertible $r \times r$ submatrix and other dependent rows & cols

- basis for a given vector space is unique: (2 implies 3rd) (lin. indp., spanning, n vectors).

· 4 fundamental subspaces



dimension: # basis vectors in any basis.

$\dim C(A^T) = r$ $\qquad$ $\dim C(A) = r$

$\dim N(A) = n-r$ $\qquad$ $\dim N(A^T) = m-r$

where $N(A^T) = \{\vec{y} \in \mathbb{R}^m : A^T\vec{y} = \vec{0}\} = \{\vec{y} \in \mathbb{R}^m : \vec{y}^T A = \vec{0}^T\}$.

↳ vectors to multiply @ left of A to get zero.

① row space of A = row space of R (row ops do not change span of pivot rows)

↳ in R: pivot rows are a basis for row space, lin. indp, spanning

② same cols. form basis for $C(A)$ & $C(R)$: pivot cols (row ops change col. space but not dependence)

↳ cols. are lin. indp. before row ops ⟺ cols. are lin. indp. after.

Rank-nullity theorem: for an $m \times n$ matrix, $\dim C(A) + \dim N(A) = n$.

Geometry of $\mathbb{R}^n$ (Chapter 4).

· dot product: $\vec{v}^T \vec{w}$ (1st entry $\vec{v}$, 2nd entry $\vec{w}$)

↳ linear in 1st & 2nd entries (bilinear) and symmetric

→ length of a vector in Euclidean geometry $\|\vec{v}\| = \sqrt{\vec{v}^T\vec{v}}$

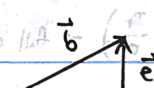
→ $\vec{u}^T\vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta$ $\quad$ ($\vec{v} = \ell \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, θ is ∡ with x-axis)

· V and W are orthogonal ($V \perp W$) if $\forall\, \vec{v} \in V, \vec{w} \in W$, $\vec{v}^T\vec{w} = 0$

· orthogonal complement of $V \subseteq \mathbb{R}^n$ is $V^\perp = \{\vec{w} \in \mathbb{R}^n : \vec{v}^T\vec{w} = 0 \;\forall\, \vec{v} \in V\}$

↳ orthogonal subspaces ⊆ orthogonal complements

· $N(A) \perp C(A^T) \Rightarrow N(A) \subseteq C(A^T)^\perp \wedge C(A^T) \subseteq N(A)^\perp$



· projection of $\vec{b}$ onto $C(A)$: vector $\vec{p} \in C(A)$ closest to $\vec{b}$

↳ if $A\vec{x} = \vec{b}$ unsolvable, solve $A\hat{x} = \vec{p}$ / $A^TA\hat{x} = A^T\vec{b}$ or $\hat{x} = (A^TA)^{-1}A^T\vec{b}$

→ $\vec{p} = A(A^TA)^{-1}A^T\vec{b}$, where $P = A(A^TA)^{-1}A^T$ is projection matrix onto $C(A)$

→ A^TA is invertible when A has indp. cols b.c. $N(A^TA) = N(A) = \{0\}$

↳ square matrix ⟺ trivial nullspace

- projection onto vector: $proj_{\vec{v}} \vec{u}$ is projection of $\vec{u}$ onto 1D subspace spanned by $\vec{v} \neq \vec{0}$
  - ↳ $A = (\vec{v})$, $proj_{\vec{v}} \vec{u} = \frac{\vec{v}^T \vec{u}}{\vec{v}^T \vec{v}} \vec{v}$
- least squares approximation: $b = C + Dt$ $(1, 2): 1 \cdot C + D \cdot 1 = 2 \ \vdots \rightarrow A = \begin{pmatrix} 1 & 1 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2 \\ \vdots \end{pmatrix}$
  - ↳ $\vec{p} = A\hat{x}$, error $\vec{e} = \vec{b} - \vec{p}$ ↳ find best approx. sol. $\hat{x} = \begin{pmatrix} \hat{C} \\ \hat{D} \end{pmatrix}$
- sum of projections of vector onto standard/orthogonal basis vectors equals original vector.
- $\{\vec{q}_1, \dots, \vec{q}_n\}$ is orthonormal if $\vec{q}_i^T \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$
  - ↳ $m \times n$ matrix Q has orthonormal columns iff. $Q^T Q = I_n$
    - ↳ ⚠ in general, $QQ^T \neq I_m$ if Q not square (if Q square, $Q^T = Q^{-1}$)
  - → $\vec{x} = proj_{\vec{q}_1} \vec{x} + \dots + proj_{\vec{q}_n} \vec{x} = (\vec{q}_1^T \vec{x}) \vec{q}_1 + \dots + (\vec{q}_n^T \vec{x}) \vec{q}_n$] sum of its projections.
  - → projections onto $C(Q)$: $\hat{x} = Q^T \vec{b}$, $\vec{p} = QQ^T \vec{b}$, $P = QQ^T$

Gram - Schmidt algorithm & QR decomposition

- INPUT: a basis $\{\vec{a}_1, \dots, \vec{a}_n\}$ for V ⟶ OUTPUT: orthonormal basis $\{\vec{q}_1, \dots, \vec{q}_n\}$.

$\vec{A}_1 = \vec{a}_1$ $\quad \vec{A}_2 = \vec{a}_2 - proj_{\vec{A}_1} \vec{a}_2$ $\quad A_3 = a_3 - proj_{\vec{A}_1} \vec{a}_3 - proj_{\vec{A}_2} \vec{a}_3$ $\quad \vec{A}_n = \vec{a}_n - \sum_{i=1}^{n-1} proj_{\vec{A}_i} \vec{a}_n$

$\vec{q}_1 = \frac{\vec{A}_1}{\|A_1\|}$ $\quad \vec{q}_2 = \frac{\vec{A}_2}{\|A_2\|}$ $\quad \vec{q}_3 = \frac{\vec{A}_3}{\|A_3\|}$ $\quad \vec{q}_n = \frac{\vec{A}_n}{\|A_n\|}$

- if matrix Q has orthonormal columns, then it leaves lengths & angles unchanged (dot product unchanged)
- matrix Q is orthogonal if it is square and has orthonormal columns.

$R = Q^T A$

$$\underbrace{(\vec{a}_1 | \vec{a}_2 | \cdots | \vec{a}_n)}_{\text{lin. indp. cols.}}^{m \times n \ (m \geq n)} = \underbrace{(\vec{q}_1 | \vec{q}_2 | \cdots | \vec{q}_n)}_{\text{orthonormal cols.}}^{m \times n} \begin{pmatrix} \vec{q}_1^T \vec{a}_1 & \vec{q}_1^T \vec{a}_2 & \cdots & \vec{q}_1^T \vec{a}_n \\ & \vec{q}_2^T \vec{a}_2 & \cdots & \vec{q}_2^T \vec{a}_n \\ & & \ddots & \vdots \\ & & & \vec{q}_n^T \vec{a}_n \end{pmatrix}^{n \times n} \leftarrow A = QR.$$

basis → lin. indp. cols. UT, invertible. ← diagonal $\neq 0$ ($proj_{\vec{q}_i} \vec{a}_i \neq 0$)

↳ $\vec{a}_j$ = lin. comb. of $\vec{q}_1, \dots, \vec{q}_j, \cancel{\dots \vec{q}_n}$ = sum of proj onto $\underline{\vec{q}_1, \dots, \vec{q}_j}$ = $(\vec{q}_1^T \vec{a}_j) \vec{q}_1 + \dots + (\vec{q}_j^T \vec{a}_j) \vec{q}_j$ $= QR_{*j}$

span $\{\vec{a}_1, \dots \vec{a}_j\}$ = span $\{\vec{q}_1, \dots, \vec{q}_j\}$ orthogonal

- $E_{ij} = \begin{pmatrix} 1 & & -\ell_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}$, $-\ell_{ij} = -\frac{\vec{A}_i^T \vec{a}_j}{\vec{A}_i^T \vec{A}_i}$, $E_j = \begin{pmatrix} 1 & & \\ & x_j & \\ & & 1 \end{pmatrix}$, $x_j = \frac{1}{\|\vec{A}_j\|}$
  - ↳ every step of GS corresponds to multiplying A by an invertible UT matrix on the right ← col. op.
    - ↳ $Q = A E_{12} E_{13} E_{23} E_{norm} \Rightarrow Q = AR^{-1} \Rightarrow A = QR$ ← UT: $\vec{q}_i$ is orthogonal to every a_j, $j < i$ (call prev. considered vectors)

$C_2 - \_ C_1$ $\quad C_3 - \_ C_2$