AP Calculus BC

Squeeze Theorem: Let l be an open interval such that $a \in l$, if $g(x) \leq f(x) \leq h(x)$ for all $x \in l \backslash \{a\}$ and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.

Important Limit: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Intermediate Value Theorem: If f is continuous on a closed interval $[a, b]$, and $k \in \mathbb{R}$ where $f(a) < k < f(b)$, then there exist $c \in (a, b)$ such that $f(c) = k$.

Extreme Value Theorem: If a function f is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and absolute minimum value on $[a, b]$.

First Derivative Test: Let $(c, f(c))$ be a critical point of f. If f is continuous at $x = c$, and differentiable on some open interval containing c (except possibly at c).
(i) If f' changes sign from negative to positive at c, then $(c, f(c))$ is a local minimum point.
(ii) If f' changes sign from positive to negative at c, then $(c, f(c))$ is a local maximum point.

Inverse Functions: $\left(f^{-1}\right)'(b) = \frac{1}{f'\left[f^{-1}(b)\right]}$

Second Derivative Test: Suppose that $f'(c) = 0$,
(i) if $f''(c) > 0$, then f has a relative minimum at c.
(i) if $f''(c) > 0$, then f has a relative maximum at c.
(i) if $f''(c) = 0$, then the test is inconclusive.

Concavity of a Function: for an open interval l,
(i) If $f'' > 0 \; \forall x \in l$, then f concave upwards on l.
(i) If $f'' < 0 \; \forall x \in l$, then f concave downwards on l.

Inflection Points: If f is continuous at $x = a$, and changes concavity in the vicinty of $x = a$, then $(a, f(a))$ is a point of inflection.

Rolle's Theorem: If f is continuous on the closed bounded interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exist at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem: If f is continuous on the closed bounded interval $[a, b]$ and differentiable on the open interval (a, b), then there exist at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

First Fundamental Theorem of Calculus:

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Mean Value Theorem If f is continuous on $[a, b]$, then (i) $m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$, where m and M are the absolute minimum and absolute maximum values of f on $[a, b]$ respectively.
(ii) By Mean Value Theorem, there is at least one point $c \in [a, b]$ such that

$$\int_a^b f(x)\, dx = f(c)(b - a)$$

The average value of f on $[a, b]$ is: $\frac{1}{b-a} \int_a^b f(x)\, dx$

The average rate of change of f over $[a, b]$: $\frac{f(b) - f(a)}{b - a}$

Second Fundamental Theorem of Calculus:

If f is continuous on an interval l, then

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)\, dt\right) = f(g(x)) \cdot g'(x)$$

Exponential growth: $\frac{dy}{dx} = ky \to y(t) = y_o e^{kt}$

Logistic growth: $\frac{dy}{dx} = k(L - y)y$

Arc lengths:

$$\begin{aligned} L &= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \end{aligned}$$

Polar coordinates:

$$A = \frac{1}{2} \int r^2\, d\theta$$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r \sin\theta)}{\frac{d}{d\theta}(r \cos\theta)} = \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\sin\theta + \cos\frac{dr}{d\theta}}$$

1 Improper Integrals

A definite integral $\displaystyle\int_a^b f(x)\,dx$ is an improper integral if at least one of the conditions is true

(a) either $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to b^-} f(x)$ does not exist

- the improper integral of f over the interval $[a,b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{m\to a^+}\int_m^b f(x)\,dx \quad \text{or} \quad \lim_{k\to b^-}\int_a^k f(x)\,dx$$

(b) there exist a number $c\in(a,b)$ such that either $\lim\limits_{x\to c^+} f(x)$ or $\lim\limits_{x\to c^-} f(x)$ does not exist

- $\displaystyle\int_a^b f(x)\,dx$ converges only if both improper integrals $\displaystyle\int_a^c f(x)\,dx$ and $\displaystyle\int_c^b f(x)\,dx$ converges

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(c) either $a=-\infty$ or $b=\infty$

$$\int_a^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \quad \text{or} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx$$

Definitions and results

1. The improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ diverges if either $\displaystyle\int_{-\infty}^{p} f(x)\,dx$ or $\displaystyle\int_{p}^{\infty} f(x)\,dx$ diverges.
2. $\displaystyle\int_1^{\infty}\frac{1}{x^p}$ is convergent if $p>1$ and divergent if $p\le 1$
3. **Linearity**: let $\displaystyle\int_a^b f(x)\,dx$ and $\displaystyle\int_a^b g(x)\,dx$ be improper integrals, then
   - if $\displaystyle\int_a^b f(x)\,dx$ and $\displaystyle\int_a^b g(x)\,dx$ are both convergent, then $\displaystyle\int_a^b [f(x)\pm g(x)]\,dx$ is convergent
   - if $\displaystyle\int_a^b f(x)\,dx$ is convergent, then $\displaystyle\int_a^b k\cdot f(x)\,dx$ is convergent
4. **Comparison Theorem**: suppose f and g are continuous functions with $f(x)\ge g(x)\ge 0$ on (a,b), then
   - if $\displaystyle\int_a^b f(x)\,dx$ is convergent, then $\displaystyle\int_a^b g(x)\,dx$ is convergent

2 Limits

1. If $\lim\limits_{x\to\infty} f(x)=L$, then $\lim\limits_{n\to\infty} f(n)=L$
   - if $\lim\limits_{n\to\infty}|a_n|=0$, then $\lim\limits_{n\to\infty} a_n=0$
2. If $\lim\limits_{n\to\infty} a_n=L$ and f is continuous at L, then $\lim\limits_{n\to\infty} f(a_n)=f\left(\lim\limits_{n\to\infty} a_n\right)$
3. The sequence $(a_n)_{n=1}^{\infty}$ converges to L if and only if $(a_{2n})_{n=1}^{\infty}$ and $(a_{2n-1})_{n=1}^{\infty}$ both converge to L
4. If $\lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty} c_n=L$, and $a_n\le b_n\le c_n$ for all integers $n\ge N$, then $\lim\limits_{n\to\infty} b_n=L$
5. If $-1<r<1$, then $\lim\limits_{n\to\infty} r^n=0$
6. The number e is defined to be the value of $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

3 Infinite Series

Telescoping series: $S = \lim_{n\to\infty} s_n$, where the n^{th} partial sum $s_n = \sum_{k=1}^{n}(a_{k+1} - a_k)$

Geometric series: $\sum_{k=1}^{\infty} ar^{k-1}$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$

3.1 Convergence Tests

(a) **Divergence Test** If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{k=1}^{\infty} a_k$ diverges

(b) **Integral Test**: Let f be a continuous, positive, decreasing function on $[N, \infty)$ such that $a_k = f(k)$ for all integers $k \geq N$. Then $\sum_{k=1}^{\infty} a_k$ and $\int_N^{\infty} f(x)\, dx$ either both converge or both diverge

- Let $f(x) = \frac{1}{x}$. Then f is continuous and positive on the interval $[1, \infty)$.
  Since $f'(x) = -\frac{1}{x^2} < 0$, the function f is decreasing for all $x \geq 1$. Finally, since
  $$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x}\, dx\right) = \lim_{b\to\infty}\,[\ln x]_1^b = \lim_{b\to\infty}(\ln b - \ln 1) = \infty$$
  the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the Integral Test

(c) **p-series** $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges is $p > 1$ and diverges if $p \leq 1$

(d) **Comparison Test**: Suppose there exist an integer N such that $0 \leq a_k \leq b_k$ for all integers $k \geq N$

- If the series $\sum_{k=1}^{\infty} b_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges
- If the series $\sum_{k=1}^{\infty} a_k$ diverges, then the series $\sum_{k=1}^{\infty} b_k$ diverges
- Since $0 \leq \frac{1}{k^2+5} \leq \frac{1}{k^2}$ for all positive integers k, and the 2-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the series $\sum_{k=1}^{\infty} \frac{1}{k^2+5}$ converges by the Comparison Test

(e) **Limit Comparison Test**: Suppose a_k and b_k are positive for all positive integers k. Let $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$

- If $0 \leq \rho < \infty$ and if the series $\sum_{k=1}^{\infty} b_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges
- If $0 < \rho \leq \infty$ and if the series $\sum_{k=1}^{\infty} b_k$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges
- Note that $\frac{1}{\sqrt{k}+2} > 0$ and $\frac{1}{\sqrt{k}} > 0$ for all positive integers k.
  Since $\lim_{k\to\infty} \frac{\frac{1}{\sqrt{k}+2}}{\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k}+2} = 1$ (which is a finite positive number) and $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent (p-series with $p = \frac{1}{2}$), the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+2}$ diverges by the Limit Comparison Test

(f) **Absolute Convergence**: If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (i.e. $\sum_{k=1}^{\infty} |a_k|$ converges), then it is convergent.

(g) **Ratio Test**: Let $\sum_{k=1}^{\infty} a_k$ be a series with nonzero terms and suppose that $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- If $\rho < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely
- If $\rho > 1$ or $\rho = \infty$, the series $\sum_{k=1}^{\infty} a_k$ diverges
- If $\rho = 1$ or if the limit does not exist, no conclusion can be drawn
- Since $\lim_{n\to\infty} \left|\frac{\left(\frac{(-1)^{n+1}2^{n+1}}{(n+1)!}\right)}{\left(\frac{(-1)^n 2^n}{n!}\right)}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$ converges absolutely by the Ratio Test, hence it converges.

(h) **Root Test**: Let $\sum_{k=1}^{\infty} a_k$ be a series and let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- If $\rho < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely
- If $\rho > 1$ or $\rho = \infty$, the series $\sum_{k=1}^{\infty} a_k$ diverges
- If $\rho = 1$ or if the limit does not exist, no conclusion can be drawn
- Since $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{4n-1}{2n-1}\right)^n\right|} = \lim_{n\to\infty} \left(\frac{4n-1}{2n-1}\right) = 2 > 1$, $\sum_{k=1}^{\infty} \left(\frac{4k-1}{2k-1}\right)^k$ diverges by the Root Test.

(i) **Alternating Series Test**: If $0 \le a_{n+1} \le a_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ converges. *Note: inverse/converse may not be true, AST cannot be used to prove divergence.*

- Let $a_n = \frac{1}{3n+1}$ for $n = 1, 2, 3, \ldots$ Since $0 \le a_{n+1} = \frac{1}{3n+4} \le \frac{1}{3n+1} = a_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3n+1} = 0$, the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{3n+1}$ converges by the AST.
- **Error bound**: $|S - s_n| \le a_{n+1}$. Moreover, if $0 < a_{n+1} < a_n$, $|S - s_n| < a_{n+1}$.
- A **conditionally convergent** series is convergent but not absolutely convergent
  - terms of a conditionally convergent series cannot be rearranged

4 Power Series

A **power series** centered at $x = a$ is a series of the form $\sum_{k=0}^{\infty} c_k (x-a)^k$. Then one of the conditions hold

(a) The power series converges when $x = a$
(b) The series converges absolutely when $|x - a| < R$ and diverges when $|x - a| > R$

- First, we observe that the power series $\sum_{k=0}^{\infty} \frac{(x-2)^k}{k}$ converges if $x = 2$.

  Now we suppose $x \ne 2$. Since

$$\lim_{k\to\infty} \left|\frac{\frac{(x-2)^{k+1}}{k+1}}{\frac{(x-2)^k}{k}}\right| = \lim_{k\to\infty} \frac{k|x-2|}{k+1} = |x-2| \lim_{k\to\infty} \frac{k}{k+1} = |x-2|$$

  By Ratio Test, the given power series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$.

  Therefore, the radius of convergence is 1.

When $x=1$, we have $\displaystyle\sum_{k=0}^{\infty}\frac{(x-2)^k}{k}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k}$, which converges by the Alternating Series Test.

When $x=3$, we have $\displaystyle\sum_{k=0}^{\infty}\frac{(x-2)^k}{k}=\sum_{k=0}^{\infty}\frac{1}{k}$, which is divergent (Harmonic series).

In conclusion, the interval of convergence of the power series is $[1,3)$.

(c) The series converges for all real values of x

The **Taylor series** for f about $x=a$ is given by $\displaystyle\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$

$$
\begin{aligned}
e^x &= \sum_{k=0}^{\infty}\frac{x^k}{k!} && = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots && x\in\mathbb{R}\\
\sin x &= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} && = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots && x\in\mathbb{R}\\
\cos x &= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} && = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots && x\in\mathbb{R}\\
\ln(1+x) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k} && = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots && -1<x\le 1\\
(1+x)^n &= \sum_{k=0}^{\infty}\binom{n}{k}x^k && = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots && -1<x<1
\end{aligned}
$$

Taylor's formula with remainder: $f(x)=P_n(x)+R_n(x)$, where $P_n(x)$ is the n^{th}-degree Taylor polynomial.

Lagrange error bound: If a function f is $(n+1)$ times differentiable on an open interval containing a and x, and if $\left|f^{(n+1)}(t)\right|\le M$ for all $t\in[a,x]$, then $|R_n(x)|\le\dfrac{M}{(n+1)!}|x-a|^{n+1}$

Suppose a Taylor series $\displaystyle\sum_{k=0}^{\infty}c_k(x-a)^k$ has a nonzero radius of convergence R. Then,

(a) The function $f(x)=\displaystyle\sum_{k=0}^{\infty}c_k(x-a)^k$ is differentiable on the interval $(a-R,a+R)$

(b) $\dfrac{d}{dx}[f(x)]=\displaystyle\sum_{k=0}^{\infty}c_k\frac{d}{dx}\left[(x-a)^k\right]=\sum_{k=1}^{\infty}c_k k(x-a)^{k-1}$ for $a-R<x<a+R$

- $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)=\dfrac{d}{dx}\left(\displaystyle\sum_{k=0}^{\infty}x^k\right)=\sum_{k=0}^{\infty}\frac{d}{dx}x^k=\frac{d}{dx}(1)+\sum_{k=1}^{\infty}x^k=\sum_{k=1}^{\infty}kx^{k-1}$

(c) $\displaystyle\int f(x)\,dx=\sum_{k=0}^{\infty}\left[c_k\int(x-a)^k\,dx\right]$ for $a-R<x<a+R$

- $\ln(1+x)=\displaystyle\int_0^x\frac{1}{1+t}\,dt=\int_0^x\left(\sum_{k=1}^{\infty}(-1)^{k-1}t^{k-1}\right)dt=\sum_{k=1}^{\infty}\left(\int_0^x(-1)^{k-1}t^{k-1}\,dt\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k}$

5 Numerical Methods for Differential Equations

Euler's method: $y_{n+1}=y_n+f(x_n,y_n)\Delta x,\quad f(x,y)=\dfrac{dy}{dx}$

Percentage error $=\dfrac{\text{exact value}-\text{approximation}}{|\text{exact value}|}$