266 Exam

- for $\vec{b} \neq \vec{0}$, $\text{proj}_{\vec{b}} \vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$, $\text{comp}_{\vec{b}} \vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} \in \mathbb{R}$ → +ve for acute ∡s.

- every quadratic polynomial $q(\vec{x})$ can be expressed uniquely as $q(\vec{x}) = \vec{x}^T A \vec{x} + \vec{b}\cdot\vec{x} + c$

  ↳ by Principal Axis Theorem, $q(\vec{x}) = \lambda_1 u_1^2 + \dots + (\vec{v}_1 \cdot \vec{b}) u_1 + \dots + c$

  → $\underbrace{\{\vec{v}_1, \dots, \vec{v}_n\}}_{=B}$ is orthonormal eigenbasis for A → $S = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$, $\vec{u} = S^T \vec{x}$, $u_i = \vec{v}_i \cdot \vec{x}$

  $\vec{u}$ is the B-coordinate vector of $\vec{x}$ ⟺ $\vec{x} = S\vec{u}$, where S is change of basis matrix for B

  → $\chi_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- quadrics in $\mathbb{R}^2$ (conic sections)

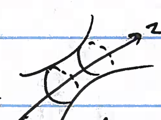
  → $a(x-x_0)^2 + b(y-y_0)^2 = 1$ → if $a, b > 0$, ellipse
  if $a > 0$, $b < 0$, hyperbola with asymptote at $y = \pm\sqrt{\frac{a}{b}}\, x$

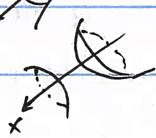
  → $y - y_0 = a(x-x_0)^2$, $a \neq 0$ → parabola

- quadrics in $\mathbb{R}^3$ (quadric surfaces)

  → $\square x^2 + \square y^2 + \square z^2 = 1$

+	+	+	→ ellipsoid
+	+	−	→ hyperboloid of 1 sheet
+	−	−	→ hyperboloid of 2 sheets

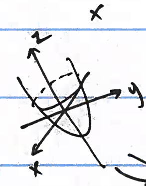


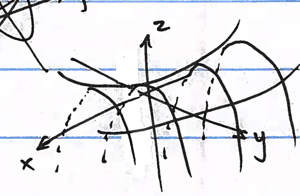


  → $\square x^2 + \square y^2 + \square z^2 = 0$ → double cone

  → $z = \square x^2 + \square y^2$

−/+	−/+	→ elliptic paraboloid
+	−	→ hyperbolic paraboloid





  → elliptic, hyperbolic and parabolic cylinders

- for $f: \mathbb{R}^n \to \mathbb{R}$, graph of f is in $\mathbb{R}^{n+1}$, level set $f(\vec{x}) = k$ is in $\mathbb{R}^n$

- cartesian (x, y, z) ⟷ cylindrical (r, θ, z) ⟷ spherical (ρ, ϕ, θ)

  $r^2 = x^2 + y^2$, $y = x\tan\theta$

  $r = \rho\sin\phi$, $z = \rho\cos\phi$

  $\rho^2 = r^2 + z^2$, $r = z\tan\phi$

  ...) $\rho^2 = x^2 + y^2 + z^2$

- $f: \mathbb{R}^n \to \mathbb{R}^m$, Jacobian Df : $(m \times n)$ matrix whose (i,j)-component is $\frac{\partial f_i}{\partial x_j}$

  gradient vector $\nabla f = (Df)^T \rightsquigarrow$ if $\vec{g}(x,y) = (g_1, g_2)$, $D\vec{g} = \begin{pmatrix} \partial_x g_1 & \partial_y g_1 \\ \partial_x g_2 & \partial_y g_2 \end{pmatrix}$

- tangent space to graph of f ($z = f(x,y)$) at $(\vec{a}, f(\vec{a}))$ is $x_{n+1} = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$
- Hessian matrix of $f: \mathbb{R}^n \to \mathbb{R}$: $n \times n$ matrix where $[Hf]_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j}$, $Hf(\vec{x}) = D\nabla f(\vec{x})$
- $Q(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \frac{1}{2}(\vec{x} - \vec{a})^T Hf(\vec{a})(\vec{x} - \vec{a}) \longrightarrow$ if all $d_k > 0$, all $\lambda_i > 0$

  if d_k is $\begin{cases} \text{neg for odd } k \\ \text{pos for even } k \end{cases}$, all $\lambda_i < 0$

  else, mixed λ_i's.
- chain rule: $D(\vec{g} \circ \vec{f})(\vec{a}) = \overset{k \times n}{} \; \overset{k \times m}{(D\vec{g}(\vec{f}(\vec{a})))} \overset{m \times n}{(D\vec{f}(\vec{a}))}$
- directional derivative $D_{\vec{v}} \vec{f}(\vec{a}) = D\vec{f}(\vec{a}) \frac{\vec{v}}{\|\vec{v}\|}$ (0 along $\vec{v} = \vec{x} - \vec{a}$ when $\vec{x} \in$ tangent space)
- tangent space to level set $f(\vec{x}) = c \rightarrow \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$
- finding global ext. — ① $\nabla f = \vec{0}$, parametrize boundary, endpoints.

  ② if $\vec{a}$ is a global extremum of f on $D = \{\vec{x} \in \mathbb{R}^n \mid g(\vec{x}) = k\}$ s.t. $\nabla g(\vec{a}) \neq \vec{0}$

  $\rightarrow \nabla f(\vec{a}) = \lambda \nabla g(\vec{a})$ for some $\lambda \in \mathbb{R}$ (if $\vec{a}$ is a critical pt, $\lambda = 0$).
- $\vec{F}$ conservative $\Leftrightarrow \vec{F} = \nabla f \Rightarrow D\vec{F}$ symmetric / $\nabla \times \vec{F} = 0$ ($\Leftarrow$ if domain sc)

  $\vec{F}$ curl field / incompressible / solenoidal $\Leftrightarrow \vec{F} = \nabla \times \vec{G} \Leftrightarrow \nabla \cdot \vec{F} = 0$
- flow line of $\vec{F}$: $\vec{x}\,'(t) = \vec{F}(\vec{x}(t))$
- arc length of $\vec{r}(t) = \int_{\vec{r}} 1 \, ds = \int_a^b \|\vec{r}\,'(t)\| \, dt$

  $\Lsh \int_{\vec{r}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$

  $\Lsh$ $d\vec{s} = \vec{T} \, ds$, $\vec{T}(t) = \frac{\vec{x}\,'(t)}{\|\vec{x}\,'(t)\|}$, $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$, $\kappa(t) = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\|\vec{T}'(t)\|}{\|\vec{x}\,'(t)\|}$
- FTLI: $\int_C \nabla f(\vec{x}) \cdot d\vec{s} = f(\vec{r}(\vec{b})) - f(\vec{r}(\vec{a}))$

  $\nabla \cdot \nabla f \neq 0$, $\nabla \cdot (\nabla \times \vec{F}) = 0$, $\nabla \times \nabla f = \vec{0}$

$\rightarrow$ Riemann Sum of f : $\sum_{R_{ij} \in P} f(\vec{x}_{ij}^{\,*}) \Delta A_{ij}$, where $\vec{x}_{ij}^{\,*} \in R_{ij}$, $\Delta A_{ij} = \text{area}(R_{ij})$

mesh$(P) = \max\{\Delta A_{ij}\}$, as mesh$(P) \to 0$, val. of Riemann sum $\to \iint_R f(x,y) \, dA$.

$\nabla \cdot \vec{F} \to$ div

$\nabla \times \vec{F} \to$ curl.

- $\int \cdots \int_D f(\vec{x})\, dV(\vec{x}) = \int \cdots \int_{D^*} f(\vec{T}(\vec{u})) \cdot |\det(D\vec{T}(\vec{u}))|\, dV(\vec{u})$
- $\vec{N} = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix} = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$ $\rightsquigarrow$ for graph of a function

  $\vec{N} = \nabla f$ $\rightsquigarrow$ for level set $f(\vec{x}) = c$

  $\vec{N} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ $\rightsquigarrow$ for surface S (parametrization $\vec{r}$ of S is smooth $\Leftrightarrow$ $\vec{r}_u$ & $\vec{r}_v$ L.I $\Leftrightarrow$ $\vec{r}_u \times \vec{r}_v \neq \vec{0}$)
- $SA(S) = \iint_D \underbrace{\|\vec{r}_u \times \vec{r}_v\|\, dA(u,v)}_{dS} = \iint_S 1\, dS$
- flux of $\vec{F}$ across S : (vector) surface integral $\iint_{\vec{r}} \vec{F} \cdot d\vec{S} = \iint_{\vec{r}} (\vec{F} \cdot \vec{N})\, dS = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$

  circulation of $\vec{F}$ around C : (vector) line integral $\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot \vec{T}\, ds$

	Circulation	Flux
Green	$\oint_C \vec{F} \cdot d\vec{s} = \iint_D (\nabla \times \vec{F}) \cdot \hat{k}\, dA$	$\oint_C \vec{F} \cdot d\vec{n} = \iint_D (\nabla \cdot \vec{F})\, dA$
	$\oint_C P\, dx + Q\, dy = \iint_D (Q_x - P_y)\, dA$	$\oint_C (-Q)\, dx + P\, dy = \iint (P_x + Q_y)\, dA$
Stokes	$\underbrace{\oint_C \vec{F} \cdot d\vec{s}}_{\text{circulation}} = \iint_S \underbrace{(\nabla \times \vec{F})}_{\text{circulation density}} \cdot d\vec{S}$	Gauss $\underbrace{\oiint_S \vec{F} \cdot d\vec{S}}_{\text{flux}} = \iiint_E \underbrace{(\nabla \cdot \vec{F})}_{\text{flux density.}}\, dV$