4.(a) $x_t - 1.1x_{t-1} + 0.5x_{t-2} = 5 + w_t$.

$\phi(B) = 1 - 1.1B + 0.5B^2$ gives two roots. $r_1 = 1.1 + 0.89i$ and $r_2 = 1.1 - 0.89i$.

Both roots are outside unit circle, so AR(2) is causal.

(b) $x_{2005} = 9$, $x_{2006} = 11$, $x_{2007} = 10$.

$$\tilde{x}_{2008} = 5 + 1.1 \cdot x_{2007} \cancel{+} - 0.5 \cdot x_{2006} = 10.5$$

$$\tilde{x}_{2009} = 5 + 1.1 \cdot \tilde{x}_{2008} - 0.5\, x_{2007} = 11.55$$

(c) For year 2008, $m=1$, $SE = \sigma_w = \sqrt{2}$

$$PI = 10.5 \pm 1.96 \cdot \sqrt{2} = (7.73, 13.27)$$

For year 2009, $m=2$, $SE = \sigma_w \sqrt{1+\psi_1^2}$.

$$\psi(z) = \frac{1}{1 - 1.1z + 0.5z^2} = \sum_{j=0}^{\infty} (1.1z - 0.5z^2)^j = 1 + 1.1z - 0.5z^2 + \cdots$$ , so $\psi_1 = 1.1$ and

$$SE = \sqrt{2} \cdot \sqrt{1 + 1.1^2} = \sqrt{4.42} = 2.1$$

$$PI = 11.55 \pm 1.96 \cdot 2.1 = (7.43, 15.67)$$

(d) This is because at the early stage, as $m$ increases, the standard error increases. In this question, standard error increases from $\sigma_w \psi_0$ to $\sigma_w \sqrt{\psi_0^2 + \psi_1^2}$. Thus, PI becomes wider

(e) It would not surprise me because 12 is within 95% PI.

(f) $\tilde{x}_{2009} = 5 + 1.1 \cdot 12 - 0.5 \cdot 10 = 13.2$ million