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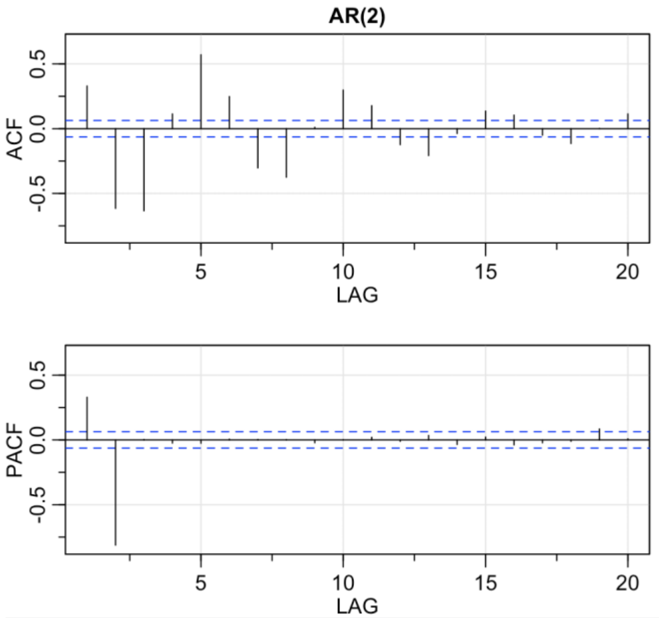
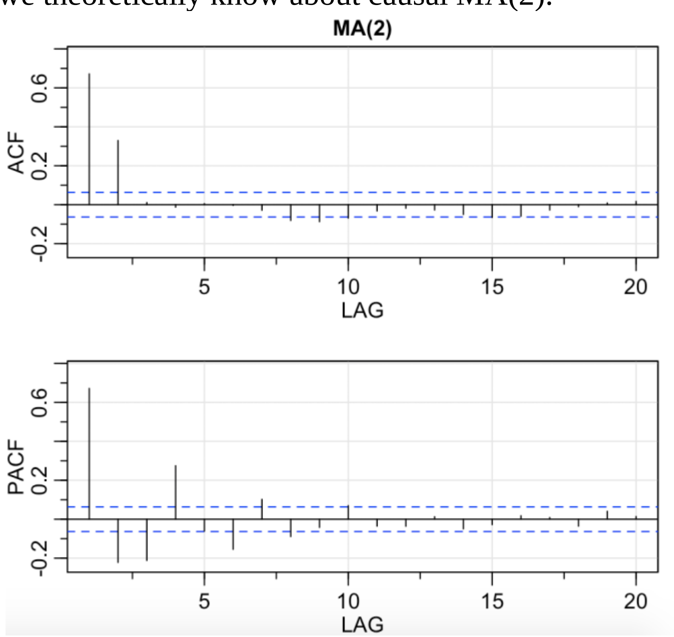
STAT 5170
Homework 5

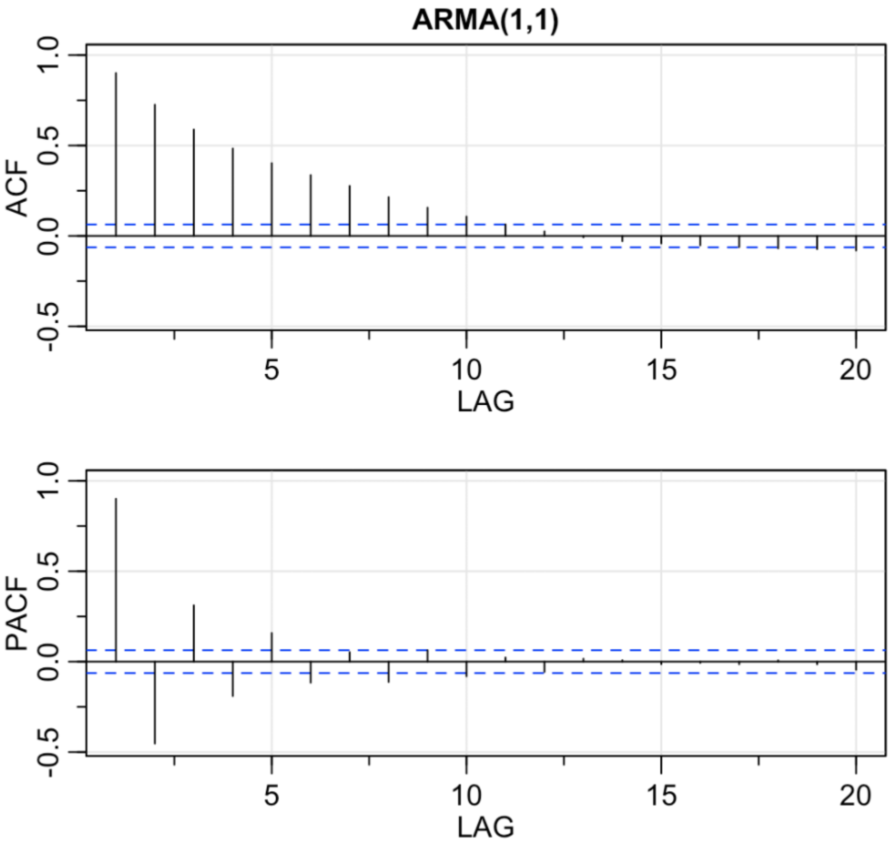
Question 1
(a) The ACP plot decays and PACF plot shows 0 correlation after lag 2. The plots match what we theoretically know about causal AR(2).



(b) The ACF plot shows 0 correlation after lag 2 and PACF plot decays. The plots match what we theoretically know about causal MA(2).

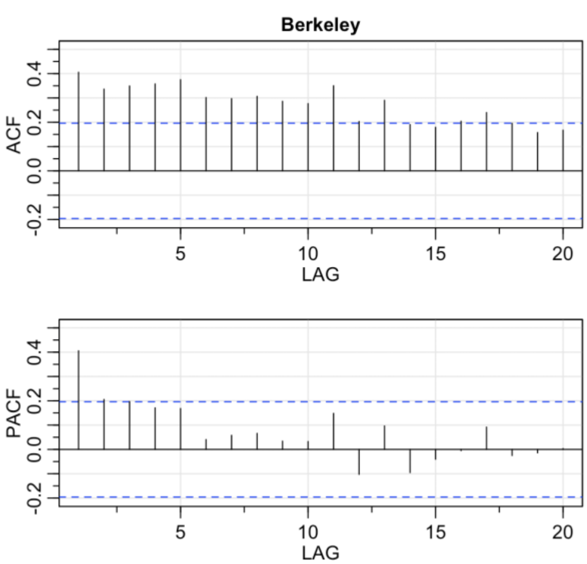


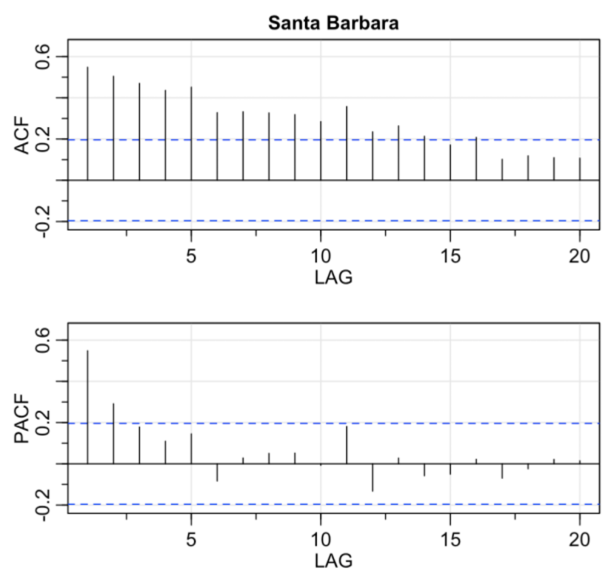
(c) The ACF plot and the PACF plot both decay. The plots match what we theoretically know about causal ARMA(1, 1).



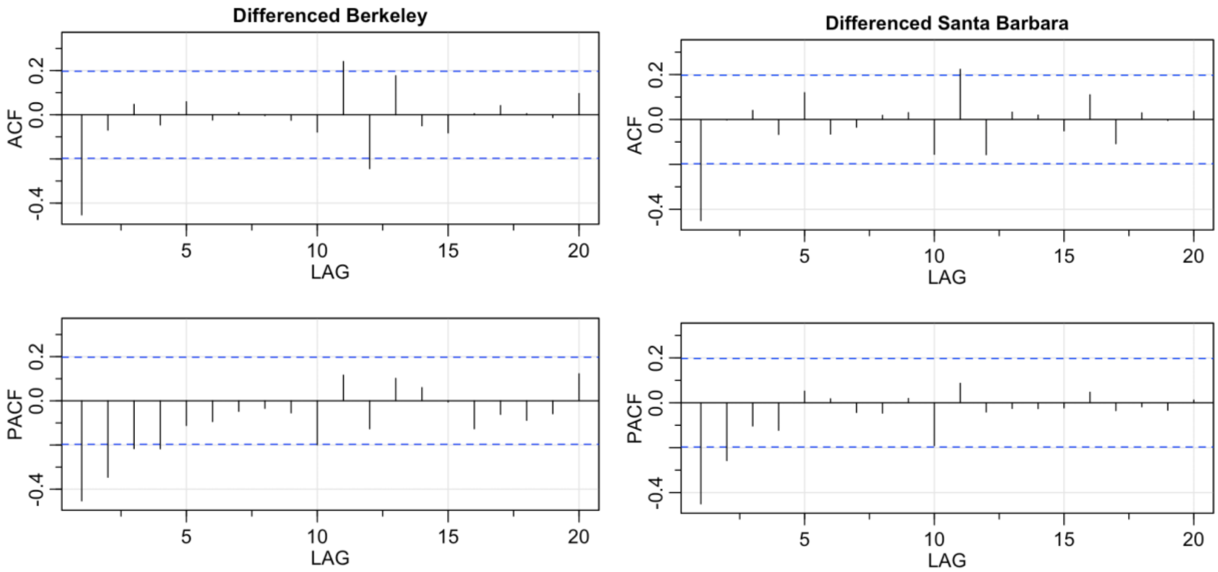
Question 2

(a) Both ACF plots and PACF plots decay, so I would suggest that both Berkeley and Santa Barbara are ARMA model.





(b) Both differenced ACF plots show 0 correlatiobn after lag 1 and differenced PACF plots decays, so I would suggest that both Differenced Berkeley and Differenced Santa Barbara are MA(1) model.



3. (a) Best Linear Predicator is $X_{n+m}^n = \sum_{j=1}^{n} \alpha_j X_j$

The prediction equation is

$$E[(X_{n+m} - X_{n+m}^n) X_k] = E[(X_{n+m} - \sum_{j=1}^{n} \alpha_j X_j) X_k] = 0$$

(b) By further rearranging the prediction equation, we have

$$E(X_{n+m} \cdot X_k) = \sum_{j=1}^{n} \alpha_j E(X_j \cdot X_k)$$

$$\gamma(n+m-k) = \sum_{j=1}^{n} \alpha_j \gamma(j-k)$$

$$\phi^{|n+m-k|} = \sum_{j=1}^{n} \alpha_j \phi^{|j-k|}$$ since $\rho(n+m-k) = \phi^{|n+m-k|}$ and $\rho(j-k) = \phi^{|j-k|}$

Then, we express the equation in matrix form.

$$\begin{pmatrix} \phi^{n+m-1} \\ \cdots \\ \phi^m \end{pmatrix} = \begin{pmatrix} 1 & \phi^1 & \phi^2 \cdots & \phi^{n-1} \\ \phi^1 & 1 & \phi^1 \cdots & \phi^{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} \cdots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_n \end{pmatrix}$$

From the matrix, we can see the solution is

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ \phi^m \end{pmatrix}$$

$\alpha_s = 0$ for $1 \leq s < n$ and $\alpha_n = \phi^m$.

Then $X_{n+m}^n = \sum_{j=1}^{n-1} \alpha_j X_j + \alpha_n X_n = \phi^m X_n$

(c) $$E(X_{n+m} - X_{n+m}^n)^2 = E(X_{n+m} - \phi^m X_n)^2$$

$$= E(X_{n+m}^2) - 2\phi^m E(X_{n+m} X_n) + \phi^{2m} E(X_n^2)$$

$$= \gamma(0) - 2\phi^m \gamma(m) + \phi^{2m} \gamma(0)$$

$$= \frac{\sigma_w^2}{1-\phi^2} - 2\phi^m \cdot \frac{\sigma_w^2}{1-\phi^2} \cdot \phi^m + \phi^{2m} \cdot \frac{\sigma_w^2}{1-\phi^2}$$

$$= \frac{\sigma_w^2}{1-\phi^2} - \phi^{2m} \cdot \frac{\sigma_w^2}{1-\phi^2}$$

$$= \sigma_w^2 \cdot \left(\frac{1-\phi^{2m}}{1-\phi^2} \right)$$

4.(a) $x_t - 1.1x_{t-1} + 0.5x_{t-2} = 5 + w_t$.

$\phi(B) = 1 - 1.1B + 0.5B^2$ gives two roots. $r_1 = 1.1 + 0.89i$ and $r_2 = 1.1 - 0.89i$.

Both roots are outside unit circle, so AR(2) is causal.

(b) $X_{2005} = 9$, $X_{2006} = 11$, $X_{2007} = 10$.

$$\hat{X}_{2008} = 5 + 1.1 \cdot X_{2007} - 0.5 \cdot X_{2006} = 10.5$$

$$\hat{X}_{2009} = 5 + 1.1 \cdot \hat{X}_{2008} - 0.5\, X_{2007} = 11.55$$

(c) For year 2008, $m=1$, $SE = \sigma_W = \sqrt{2}$

$PI = 10.5 \pm 1.96 \cdot \sqrt{2} = (7.73, 13.27)$

For year 2009, $m=2$, $SE = \sigma_W \sqrt{1+\psi_1^2}$.

$\psi(z) = \frac{1}{1-1.1z+0.5z^2} = \sum_{j=0}^{\infty} (1.1z - 0.5z^2)^j = 1 + 1.1z - 0.5z^2 + \cdots$, so $\psi_1 = 1.1$ and

$SE = \sqrt{2} \cdot \sqrt{1+1.1^2} = \sqrt{4.42} = 2.1$

$PI = 11.55 \pm 1.96 \cdot 2.1 = (7.43, 15.67)$

(d) This is because at the early stage, as m increases, the standard error increases. In this question, standard error increases from $\sigma_w \psi_0$ to $\sigma_w \sqrt{\psi_0^2 + \psi_1^2}$. Thus, PI becomes wider

(e) It would not surprise me because 12 is within 95% PI.

(f) $\hat{x}_{2009} = 5 + 1.1 \cdot 12 - 0.5 \cdot 10 = 13.2$ million