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STAT 5170
Homework 8

1.(a) $\gamma(0) = Var(x_t) = 1+\theta_1^2+\theta_2^2$

$\gamma(-1) = \gamma(1) = E(x_t x_{t-1}) = \theta_1 + \theta_1\theta_2$

$\gamma(-2) = \gamma(2) = E(x_t x_{t-2}) = \theta_2$

$\gamma(h) = 0$ otherwise

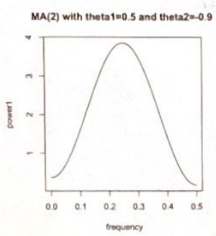
(b) $f(w) = \sum_{h=-\infty}^{\infty} \gamma(h)\cdot e^{-2\pi i w h} = \gamma(0)+\gamma(1)e^{-2\pi i w}+\gamma(2)e^{-4\pi i w}+\gamma(-1)e^{2\pi i w}+\gamma(-2)e^{4\pi i w}$

$= \gamma(0)+\gamma(1)\cdot(e^{-2\pi i w}+e^{2\pi i w}) + \gamma(2)(e^{-4\pi i w}+e^{4\pi i w})$

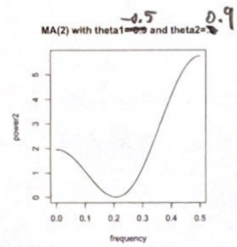
$= \gamma(0)+\gamma(1)\cdot 2\cos(2\pi w) + \gamma(2)\cdot 2\cdot \cos(4\pi w)$

$= (1+\theta_1^2+\theta_2^2)+2(\theta_1+\theta_1\theta_2)\cos(2\pi w)+2\theta_2\cos(4\pi w)$

(c) i.



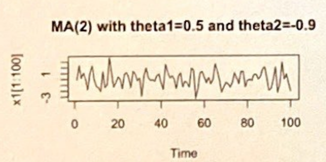
ii.

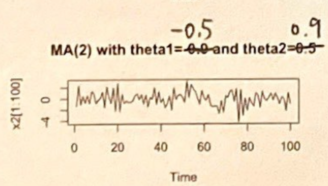


(d) For (i), f(w) increases as w increases and reaches the highest point when w=0.25. Afterwards, f(w) decreases as w increases.
For (ii), decreases as w increases and reaches the lowest point when w=0.25. Afterwards, f(w) increases as w increases.
MA(2) with theta1=~~0.9~~ -0.5 and theta2=~~0.5~~ 0.9 should result in a faster-changing process, because f(w) is high when w is large.

(e) Due to symmetry of the function and its repeating pattern for frequencies outside the range -1/2 to +1/2, we only need to be concerned with frequencies between 0 and +1/2.

(f) As we can see from two plots below, MA(2) with theta1=~~0.9~~ -0.5 and theta2=~~0.5~~ 0.9 result in a faster-changing process.





2. $x_t = -0.5x_{t-1} + w_t + 0.4w_{t-1}$

The ARMA(1,1) is causal because $\phi(z)=0 \Rightarrow z=-2$.

$\phi(z) = 1+0.5z$, $\theta(z) = 1+0.4z$

$$|\phi(e^{-2\pi i w})|^2 = |(1+0.5e^{-2\pi i w})|^2$$

$$= |(1+0.5\cos(-2\pi w) + i\cdot 0.5\cdot \sin(-2\pi w))|^2$$

$$= [1+0.5\cos(2\pi w)]^2 + \frac{1}{4}\sin^2(2\pi w)$$

$$= 1 + \cos(2\pi w) + \frac{1}{4}$$

$$= \frac{5}{4} + \cos(2\pi w)$$

$$|\theta(e^{-2\pi i w})|^2 = |1+0.4e^{-2\pi i w}|^2$$

$$= |1+0.4\cos(-2\pi w) + i\cdot 0.4\cdot \sin(-2\pi w)|^2$$

$$= [1+0.4\cos(2\pi w)]^2 + \frac{4}{25}\sin^2(2\pi w)$$

$$= 1 + \frac{4}{5}\cos(2\pi w) + \frac{4}{25}$$

$$= \frac{29}{25} + \frac{4}{5}\cos(2\pi w)$$

$$f(w) = \sigma_w^2 \cdot \left|\frac{\theta(e^{-2\pi i w})}{\phi(e^{-2\pi i w})}\right|^2 = \frac{\frac{29}{25} + \frac{4}{5}\cos(2\pi w)}{\frac{5}{4} + \cos(2\pi w)}$$

3. $\Phi(z) = 1 - \Phi z^{12}, \quad \theta(z) = 1$

$$|\Phi(e^{-2\pi i w})|^2 = |1 - \Phi(e^{-2\pi i w})^{12}|^2$$

$$= |1 - \Phi e^{-24\pi i w}|^2$$

$$= |1 - \Phi\cos(24\pi w) - i \cdot \Phi \cdot \sin(24\pi w)|^2$$

$$= [1 - \Phi\cos(24\pi w)]^2 + \Phi^2 \sin^2(24\pi w)$$

$$= 1 - 2\Phi\cos(24\pi w) + \Phi^2.$$

Since we assume the process causal, we have

$$f(w) = \sigma_w^2 \cdot \left| \frac{\theta(e^{-2\pi i w})}{\Phi(e^{-2\pi i w})} \right|^2$$

$$= \sigma_w^2 \frac{1}{|\Phi(e^{-2\pi i w})|^2}$$

$$= \frac{\sigma_w^2}{1 - 2\Phi\cos(24\pi w) + \Phi^2}$$

4. ARMA $(1,0)\times(1,0)_{12}$: $X_t = \phi X_{t-1} + \Phi X_{t-12} - \Phi\phi X_{t-13} + W_t$

We redefine our $\phi(z) = 1 - \phi z - \Phi z^{12} + \Phi\phi z^{13}$

$$= (1-\phi z)(1-\Phi z^{12})$$

$$|\phi(e^{-2\pi i w})|^2 = |(1-\phi e^{-2\pi i w})(1-\Phi e^{-24\pi i w})|^2$$

using result from 3

$$= |1-\phi e^{-2\pi i w}|^2 \cdot |1-\Phi e^{-24\pi i w}|^2$$

$$= |1-\phi\cos(2\pi w) - i\phi\sin(2\pi w)|^2 \cdot (1-2\Phi\cos(24\pi w)+\Phi^2)$$

$$= \left[1-\phi\cos(2\pi w)\right]^2 + \phi^2\sin^2(2\pi w) \cdot (1-2\Phi\cos(24\pi w)+\Phi^2)$$

$$= (1-2\phi\cos(2\pi w)+\phi^2)(1-2\Phi\cos(24\pi w)+\Phi^2).$$

Since we assume ARMA$(1,0)\times(1,0)_{12}$ causal, we have

$$f(w) = \sigma_w^2 \cdot \left|\frac{\theta(e^{-2\pi i w})}{\phi(e^{-2\pi i w})}\right|^2$$

$$= \sigma_w^2 \frac{1}{|\phi(e^{-2\pi i w})|^2}$$

$$= \frac{\sigma_w^2}{(1-2\phi\cos(2\pi w)+\phi^2)(1-2\Phi\cos(24\pi w)+\Phi^2)}$$