3. (a) Best Linear Predicator is $X_{n+m}^n = \sum_{j=1}^{n} \alpha_j X_j$

The prediction equation is

$$E[(X_{n+m} - X_{n+m}^n) X_k] = E[(X_{n+m} - \sum_{j=1}^{n} \alpha_j X_j) X_k] = 0$$

(b) By further rearranging the prediction equation, we have

$$E(X_{n+m} \cdot X_k) = \sum_{j=1}^{n} \alpha_j E(X_j \cdot X_k)$$

$$\gamma(n+m-k) = \sum_{j=1}^{n} \alpha_j \gamma(j-k)$$

$$\phi^{|n+m-k|} = \sum_{j=1}^{n} \alpha_j \phi^{|j-k|}$$ since $\rho(n+m-k) = \phi^{|n+m-k|}$ and $\rho(j-k) = \phi^{|j-k|}$

Then, we express the equation in matrix form.

$$\begin{pmatrix} \phi^{n+m-1} \\ \cdots \\ \phi^{m} \end{pmatrix} = \begin{pmatrix} 1 & \phi^1 & \phi^2 & \cdots & \phi^{n-1} \\ \phi^1 & 1 & \phi^1 & \cdots & \phi^{n-2} \\ \cdots & \cdots & \cdots & & \cdots \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_n \end{pmatrix}$$

From the matrix, we can see the solution is

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ \phi^m \end{pmatrix}$$

$\alpha_s = 0$ for $1 \le s < n$ and $\alpha_n = \phi^m$.

Then $X_{n+m}^n = \sum_{j=1}^{n-1} \alpha_j X_j + \alpha_n X_n = \phi^m X_n$

(c)
$$\begin{aligned}
E(X_{n+m} - X_{n+m}^n)^2 &= E(X_{n+m} - \phi^m X_n)^2 \\
&= E(X_{n+m}^2) - 2\phi^m E(X_{n+m} X_n) + \phi^{2m} E(X_n^2) \\
&= \gamma(0) - 2\phi^m \gamma(m) + \phi^{2m} \gamma(0) \\
&= \frac{\sigma_w^2}{1-\phi^2} - 2\phi^m \cdot \frac{\sigma_w^2}{1-\phi^2} \cdot \phi^m + \phi^{2m} \cdot \frac{\sigma_w^2}{1-\phi^2} \\
&= \frac{\sigma_w^2}{1-\phi^2} - \phi^{2m} \cdot \frac{\sigma_w^2}{1-\phi^2} \\
&= \sigma_w^2 \cdot \left(\frac{1-\phi^{2m}}{1-\phi^2} \right)
\end{aligned}$$