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STAT 5170

HW 4

(a) $x_t = \frac{8}{3}x_{t-1} + x_{t-2} + w_t + \frac{7}{6}w_{t-1} + \frac{1}{3}w_{t-2}$

$(1 - \frac{8}{3}B - B^2)x_t = (1 + \frac{7}{6}B + \frac{1}{3}B^2)w_t$

$\phi(z) = 1 - \frac{8}{3}z - z^2 = (1+\frac{1}{3}z)(1-3z)$

$\theta(z) = 1 + \frac{7}{6}z + \frac{1}{3}z^2 = (1+\frac{1}{2}z)(1+\frac{2}{3}z)$

- There is no common factor among $\phi(z)$ and $\theta(z)$, so no parameter redundancy.
- $p=2$, $q=2$
- $\phi(z)=0$ implies $z=\frac{1}{3}$ or -3. Since $\frac{1}{3}<1$, it's not causal.
  $\theta(z)=0$ implies $z=-1.5$ or -2. Since $|-1.5|>1$ and $|-2|>1$, it's invertible
- $\pi(z) = \frac{\phi(z)}{\theta(z)} = (1+\frac{1}{3}z)(1-3z)\cdot\frac{1}{(1+\frac{1}{2}z)}\cdot\frac{1}{(1+\frac{2}{3}z)}$

$= (1-\frac{8}{3}z-z^2)\cdot\sum_{j=0}^{\infty}(-\frac{1}{2}z)^j\cdot\sum_{k=0}^{\infty}(-\frac{2}{3}z)^k$

$= (1-\frac{8}{3}z-z^2)(1-\frac{1}{2}z+\frac{1}{4}z^2-\frac{1}{8}z^3+\frac{1}{16}z^4-\cdots)(1-\frac{2}{3}z+\frac{4}{9}z^2-\frac{8}{27}z^3+\frac{16}{81}z^4-\cdots)$

$= 1 + (-\frac{8}{3}-\frac{1}{2}-\frac{2}{3})z + (\frac{4}{9}+\frac{1}{4}+\frac{4}{3}+\frac{16}{9}+\frac{1}{3}-1)z^2 + (-\frac{8}{27}-\frac{2}{9}-\frac{1}{6}-\frac{1}{8}-\frac{32}{27}-\frac{8}{9}-\frac{2}{3}+\frac{1}{2}+\frac{2}{3})z^3$

$+ (\frac{16}{81}+\frac{4}{27}+\frac{1}{9}+\frac{1}{12}+\frac{1}{16}+\frac{64}{81}+\frac{16}{27}+\frac{4}{9}+\frac{1}{3}-\frac{4}{9}-\frac{1}{3}-\frac{1}{4})z^4+\cdots$

$= 1 - 3.84z + 3.14z^2 - 2.38z^3 + 1.74z^4$

$w_t = x_t - 3.84x_{t-1} + 3.14x_{t-2} - 2.38x_{t-3} + 1.74x_{t-4}$

(b) $x_t = \frac{2}{3}x_{t-1} + w_t + \frac{5}{2}w_{t-1} + w_{t-2}$

$(1-\frac{2}{3}B)x_t = (1+\frac{5}{2}B+B^2)w_t$

$\phi(z) = 1-\frac{2}{3}z$

$\theta(z) = 1+\frac{5}{2}z+z^2 = (1+2z)(1+\frac{1}{2}z)$

- There is no common factor, so no parameter redundancy.
- $p=1$, $q=2$
- $\phi(z)=0$ implies $z=\frac{3}{2}>1$. It's causal
- $\theta(z)=0$ implies $z=-\frac{1}{2}$ or -2. Since $|-\frac{1}{2}|<1$, it's not invertible

$\psi(z) = \frac{\theta(z)}{\phi(z)} = (1+\frac{5}{2}z+z^2)\cdot\frac{1}{1-\frac{2}{3}z}$

$= (1+\frac{5}{2}z+z^2)\cdot\sum_{j=0}^{\infty}(\frac{2}{3}z)^j$

$= 1 + (\frac{2}{3}+\frac{5}{2})z + (\frac{4}{9}+\frac{5}{3}+1)z^2 + (\frac{8}{27}+\frac{10}{9}+\frac{2}{3})z^3 + (\frac{16}{81}+\frac{20}{27}+\frac{4}{9})z^4+\cdots$

$= 1 + 3.17z + 3.11z^2 + 2.07z^3 + 1.38z^4 + \cdots$

$x_t = w_t + 3.17w_{t-1} + 3.11w_{t-2} + 2.07w_{t-3} + 1.38w_{t-4} + \cdots$

(c) $x_t = \frac{9}{4}x_{t-1} + \frac{9}{4}x_{t-2} + w_t - 3w_{t-1} + \frac{1}{9}w_{t-2} - \frac{1}{3}w_{t-3}$

$(1-\frac{9}{4}B - \frac{9}{4}B^2)x_t = (1-3B+\frac{1}{9}B^2-\frac{1}{3}B^3)w_t$

$\phi(z) = 1-\frac{9}{4}z-\frac{9}{4}z^2 = (1-3z)(1+\frac{3}{4}z)$

$\theta(z) = 1-3z+\frac{1}{9}z^2-\frac{1}{3}z^3 = (1-3z)(\frac{1}{3}z-i)(\frac{1}{3}z+i)$

- common factor is $1-3z$, so there is parameter redundancy
- now $\phi(z) = 1+\frac{3}{4}z$, $\theta(z) = \frac{1}{9}z^2+1$, so $p=1$, $q=2$
- $\phi(z)=0$ implies $z=-\frac{4}{3}$. Since $|-\frac{4}{3}|>1$, it's causal

  $\theta(z)=0$ implies $z=\pm 3i$, outsider unit circle, so it's invertible
- $\psi(z) = \frac{\theta(z)}{\phi(z)} = (\frac{1}{9}z^2+1)\cdot\frac{1}{1+\frac{3}{4}z}$

$$= (\frac{1}{9}z^2+1)\cdot\sum_{j=0}^{\infty}(-\frac{3}{4}z)^j$$

$$= (\frac{1}{9}z^2+1)(1-\frac{3}{4}z+\frac{9}{16}z^2-\frac{27}{64}z^3+\frac{81}{256}z^4-\dots)$$

$$= 1-\frac{3}{4}z+(\frac{9}{16}+\frac{1}{9})z^2+(-\frac{27}{64}-\frac{1}{12})z^3+(\frac{81}{256}+\frac{1}{16})z^4+\dots$$

$$= 1-0.75z+0.67z^2-0.51z^3+0.38z^4+\dots$$

$x_t = w_t - 0.75w_{t-1} + 0.67w_{t-2} - 0.51w_{t-3} + 0.38w_{t-4}$

$\pi(z) = \frac{\phi(z)}{\theta(z)} = (1+\frac{3}{4}z)\cdot\sum_{j=0}^{\infty}(-\frac{1}{9}z^2)^j$

$$= 1+\frac{3}{4}z-\frac{1}{9}z^2-\frac{1}{12}z^3+\frac{1}{81}z^4$$

$w_t = x_t + 0.75x_{t-1} - 0.11z^2 - 0.083z^3 + 0.012z^4$