Stat 5170
Homework 1
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Question 3

(a) $Var(X) = E(X^2) - [E(X)]^2 \Rightarrow E(X^2) = Var(X) + [E(X)]^2 = 4 + 0 = 4.$

$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \Rightarrow Cov(X,Y) = Corr(X,Y) \cdot \sqrt{Var(X)Var(Y)} = 0.5 \cdot 2 = 1$

$Cov(X,Y) = E(XY) - E(X)E(Y) \Rightarrow E(X,Y) = Cov(X,Y) + E(X)E(Y) = 1 + 0 = 1$

$Cov(X, X+Y) = E(X^2+XY) - E(X)E(X+Y)$
$= E(X^2) + E(XY) - [E(X)]^2 - E(X)E(Y)$
$= 4 + 1 - 0 - 0 = 5$

(b) $Var(X+Y) = Var(X) + Var(Y) + 2cov(X,Y) = 4 + 1 + 2 \cdot 0.5 = 6$

$Var(X-Y) = Var(X) + Var(Y) - 2cov(X,Y) = 4 + 1 - 2 \cdot 0.5 = 4$

$Cov(X+Y, X-Y) = E(X^2 - Y^2) - E(X+Y)E(X-Y)$
$= E(X^2) - E(Y^2) - [E(X)]^2 + [E(Y)]^2$
$= Var(X) - Var(Y)$
$= 4 - 1 = 3$

$Corr(X+Y, X-Y) = \frac{Cov(X+Y, X-Y)}{\sqrt{Var(X+Y)Var(X-Y)}} = \frac{3}{\sqrt{24}}$

Question 4

$E(y_t) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$, $E(z_t) = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$

$E(y_t^2) = 0 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} = \frac{1}{2}$, $E(z_t^2) = (-1)^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} = 1$

$E(x_t) \overset{\leftarrow \text{independence}}{=} E(y_t)(1 - E(y_t))E(z_t) = 0$

$cov(x_t, x_s) = E((x_t - E(x_t))(x_s - E(x_s)))$
$= E(x_t \cdot x_s)$
$= E(y_t(1-y_{t-1}) z_t \cdot y_s(1-y_{s-1}) z_s)$
$= E(y_t(1-y_{t-1}) y_s(1-y_{s-1})) \cdot E(z_t \cdot z_s)$ ~~← independence~~

when $s \neq t$, $E(z_t \cdot z_s) = E(z_t) \cdot E(z_s) = 0$ ← independence.

thus, $cov(x_t, x_s) = 0$ if $s \neq t$.

when $s = t$, $var(x_t) = cov(x_t, x_t) = E(y_t^2(1-y_{t-1})^2) \cdot E(z_t^2)$
$= \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$

Thus, $\{x_t\}$ is a white noise.

Then, when $x_{t-1}=1$, this implies $y_{t-1}=1$. This further implies

$$x_t = y_t(1-y_{t-1})z_t = 0.$$

We can say $P(x_{t-1}=1, x_t=1)=0$.

However, $P(x_{t-1}=1)\cdot P(x_t=1) = (\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2})(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2})$

$$= \frac{1}{64} \neq P(x_{t-1}=1, x_t=1)$$

Thus, $\{x_t\}$ is not independent.