Neural Operator for Nonlinear Filter

BY ZHIJUN ZENG

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1 Problem setting

Consider the signal based model

$$\begin{cases} d\,x_t = f(x_t, t)\,d\,t + g(x_t, t)\,d\,v_t \\ d\,y_t = h(x_t, t)\,d\,t + d\,w_t \end{cases}$$

where $x_t \in \mathbb{R}^n$ is the state of system at time t, the initial state $x_0 \sim p_0$ satisfying some initial distribution. The observation $y_t \in \mathbb{R}^m$, initialize at $y_0 = 0$. The noise is vector based Brownian motion $\mathbb{E}[\mathrm{dv}_t \mathrm{dv}_t^{\mathrm{T}}] = Q(t)\mathrm{dt} \in \mathbb{R}^{n \times n}$, and $\mathbb{E}[\mathrm{dw}_t \mathrm{dw}_t^{\mathrm{T}}] = S(t)\mathrm{dt} \in \mathbb{R}^{n \times n}$, here $S(t) > 0$.

The unnormalized conditional density function of the states x_t, denoted by $\sigma(x, t)$ satisfies the following SPDE (DMZ)

$$\begin{cases} d\,\sigma(x, t) = \mathcal{L}\,\sigma(x, t)\,d\,t + \sigma(x, t)\,h^T(x, t)\,S^{-1}\,d\,y_t \\ \sigma(x, 0) = \sigma_0(x) \end{cases}$$

where $\sigma(x, 0) = p_0(x)$, and

$$\mathcal{L}(\cdot) := \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i\,\partial x_j}((g\,Q\,g^T)_{ij}\,\cdot\,) - \sum_{i=1}^{n} \frac{\partial\,(f_i\,\cdot\,)}{\partial x_i}$$

Here we call f the drift term, g the diffusion term, h the observation term.

By an invertible exponential transformation

$$\sigma(x, t) = \exp\,(h^T(x, t)\,S^{-1}(t)\,y_t)\,u(x, t)$$

We obtain an PDE with stochastic coefficients

$$\begin{cases} \begin{aligned} \frac{\partial}{\partial t}u(x, t) + \frac{\partial}{\partial t}(h^T S^{-1})\,y_t\,u(x, t) = \\ \exp\,(-h^T\,(x, t)\,S^{-1}(t)\,y_t)\left(\mathcal{L} - \frac{1}{2}h^T S^{-1}\,h\right) \\ \cdot(\exp\,(-h^T\,(x, t)\,S^{-1}(t)\,y_t)\,u(x, t)) \\ u(x, 0) = \sigma_0(x) \end{aligned} \end{cases}$$

This equation is called pathwise robust DMZ equation.

Assume that the observation time sequences $0 = t_0 < \cdots < t_{N_t} = T$ are given. Then the observation $\{y_{t_j}\}$ at each observation time t_j are unknown until the online experiment. Therefore, one can freeze the y_t to $y_{t_{j-1}}$ in time $t_{j-1} \leqslant t < t_j$, making the exponential transformation

$$u_j(x, t) = \exp\,(h^T(x, t)\,S^{-1}(t)\,y_{t_{j-1}})\,u(x, t).$$

One obtain that u_j satisfies the FKE

$$\frac{\partial}{\partial t}u_j(x, t) = \left(\mathcal{L} - \frac{1}{2}h^T S^{-1}\,h\right)u_j(x, t)$$

The finite difference solver:

M. Yueh, W. Lin, and S. T. Yau, “An efficient num-erical method

for solving high-dimensional nonlinear filtering problems,” Comˆ

mun. Inf. Syst., vol. 14, no. 4, pp. 243-262, 2014.

2 Numerical Method for NLS

For

$$\begin{cases} d\,x_t = f(x_t,t)\,d\,t + d\,v_t \\ d\,y_t = h(x_t,t)\,d\,t + d\,w_t \end{cases}$$

The FKE becomes

$$\frac{\partial}{\partial t}u_j(x,t) = \frac{1}{2}\Delta\,u_j(x,t) - f(x)\cdot\nabla u_j(x,t) - \left(\nabla\cdot f(x) + \frac{1}{2}\|h(s)\|_2^2\right)u_j(x,t)$$

The initial condition is $u(0,s)=\sigma_0(s)$, once the new measurement $y(t_k)$ arrive we correct the solution $(u_j(x,t_k))$ at t_k by

$$\exp\{[y(t_k)-y(t_{k-1})]\cdot h(x)\}u_j(x,t_k)$$

Initially, we discretize the time interval $[t_{k-1},t_k]$ by taking a uniform partition $[t_{k-1},t_{k-1}+\Delta t,\ldots t_{k-1}+N\Delta t = t_k]$.

For the sake of computations, we restrict the domain $\mathbb{R}^D$ to $[-R,R]^D$ cells and partition it uniformly.

2.1 2D example

$$\begin{cases} d\,x_1 = \cos(x_1)d\,v_1 \\ d\,x_2 = \cos(x_2)d\,v_2 \\ d\,y_1 = x_1^3\,d\,t + d\,w_1 \\ d\,y_2 = x_2^3\,d\,t + d\,w_2 \end{cases}$$

Then

$$f(x) = \begin{bmatrix} \cos(x_1) \\ \cos(x_2) \end{bmatrix}, h(x) = \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix}$$

The strong form of equation will be

$$\frac{\partial}{\partial t}u_j = \frac{1}{2}\Delta\,u_j - \cos(x_1)\partial_{x_1}u_j - \cos(x_2)\partial_{x_2}u_j + \left(\sin(x_1)+\sin(x_2) - \frac{1}{2}(x_1^6+x_2^6)\right)u_j$$

The weak form of equation will be

$$\frac{<u_j(t+\Delta t),v> - <u_j(t),v>}{\Delta t} = -\frac{1}{2}<\nabla u_j,\nabla v> - <f\cdot\nabla u_j, v> + <a(x)u_j, v>$$

where $a(x)=\sin(x_1)+\sin(x_2)-\frac{1}{2}(x_1^6+x_2^6)$. The boundary condition is dirichlet boundary condition.

Here is some discretization parameter: $\Delta t = 0.001, \sigma_0(s) = \exp\{-10|x|^2\}, h = 0.1, \Omega = [-5, 5]^2$.

3 Numerical Example

3.1 1D

Consider the problem of

$$\begin{cases} \mathrm{dx}_1 = \mathrm{dv}_1 \\ \mathrm{dy}_1 = x_1(1 + 0.2\cos(x_1))\mathrm{dt} + \mathrm{dw}_1 \end{cases}$$

where $\mathbb{E}[\mathrm{dw}_t \mathrm{dw}_t] = I_1 \mathrm{dt}, \mathbb{E}[\mathrm{dv}_t \mathrm{dv}_t] = 0.1 I_1 \mathrm{dt}$. For filter problem, $x(0) = [1.0]$.

For operator learning problem, $g(t) = 1, h(x, t) = x(1 + 0.2\cos(x)), Q(t) = 0.1, S(t) = 1$, the FKE becomes

$$\frac{\partial}{\partial t} u_j(x, t) = \left(\frac{1}{2} 0.1 \frac{\partial^2 u_j}{\partial x^2} - \frac{1}{2} x^2 (1 + 0.2\cos(x))^2 u_j(x) \right)$$

The neural operator problem can be defined as predicting $u_j(x, t + \Delta t)$ from $u_j(x, t)$.

3.2 2D

Consider the problem of

$$\begin{cases} d\,x_1 = d\,v_1 \\ d\,x_2 = d\,v_2 \\ d\,y_1 = x_1\,(1 + 0.2\cos(x_2))\,d\,t + d\,w_1 \\ d\,y_2 = x_2\,(1 + 0.2\cos(x_1))\,d\,t + d\,w_2 \end{cases}$$

where $\mathbb{E}[\mathrm{dw}_t \mathrm{dw}_t] = I_2 \mathrm{dt}, \mathbb{E}[\mathrm{dv}_t \mathrm{dv}_t] = 0.1 I_2 \mathrm{dt}$. For filter problem, $x(0) = [1.0, 1.2]^\top$.

For operator learning problem, $g(t) = I_2, h(x, t) = \begin{bmatrix} x_1\,(1 + 0.2\cos(x_2)) \\ x_2\,(1 + 0.2\cos(x_1)) \end{bmatrix}, Q(t) = 0.1 I_2, S(t) = I_2$, the FKE becomes

$$g\,Q\,g^T = 0.1 I_2, h^T S^{-1} h = h^T h =$$

$$\frac{\partial}{\partial t} u_j(x_1, x_2, t) = \frac{1}{2} 0.1 \left(\frac{\partial^2 u_j}{\partial x_1{}^2} + \frac{\partial^2 u_j}{\partial x_2{}^2} \right) - [x_1^2\,(1 + 0.2\cos(x_2))^2 + x_2^2\,(1 + 0.2\cos(x_1))^2] u_j(x_1, x_2, t)$$

The neural operator problem can be defined as predicting $u_j(x, t + \Delta t)$ from $u_j(x, t)$.

3.2.1 Example with explicit solution

First consider a heat equation in $[0, \pi]^2$

$$u_t = \frac{1}{2}(u_{x_1 x_1} + u_{x_2 x_2})$$

The solution $u(x) = e^{-t}\sin(x_1)\sin(x_2)$ can be verified by

$$-u(x, t) = \frac{1}{2}(-e^{-t}\sin(x_1)\sin(x_2) - e^{-t}\sin(x_1)\sin(x_2))$$

with initial condition $u(x,0)=\sin(x_1)\sin(x_2)$. The verification code is update in Heat_equation. ipynb

Suppose that $u=e^{-2t}\sin(x)\sin(y)$, then

$$u_t=-2u, \Delta u=2u$$

$$u_t=\frac{1}{2}\Delta u-\left[\begin{array}{c}1\\1\end{array}\right]\cdot\left[\begin{array}{c}u_{x_1}\\u_{x_2}\end{array}\right]-u+g(x,t)$$

To determine $g(x,t)=\left[\begin{array}{c}1\\1\end{array}\right]\cdot\left[\begin{array}{c}u_{x_1}\\u_{x_2}\end{array}\right]$,

$$g(x)=u_{x_1}+u_{x_2}=e^{-2t}(\cos(x_1)\sin(x_2)+\cos(x_2)\sin(x_1))$$

The boundary condition is $u(x,t)=0$.

3.3 Post-process

Obtained $u_j(x,t+\Delta t)$, we can compute

$$u(x,t+\Delta t)=\exp\left(-h^T(x,t)\,S^{-1}(t)\,y_{t_{j-1}}\right)u_j(x,t+\Delta t)$$

And the prediction of $\sigma(x,t)$ is

$$\sigma(x,t+\Delta t)=\exp\left(h^T(x,t)\,S^{-1}(t)\,y_{t_j}\right)u(x,t+\Delta t)$$